ON THE SUMMABILITY OF THE SOLUTIONS OF THE INHOMOGENEOUS HEAT EQUATION WITH A POWER-LAW NONLINEARITY AND VARIABLE COEFFICIENTS

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Abstract. In this article, we investigate the summability of the formal power series solutions in time of the inhomogeneous heat equation with a power-law nonlinearity of degree two, and with variable coefficients. In particular, we give necessary and sufficient conditions for the 1-summability of the solutions in a given direction. These conditions generalize the ones given for the linear heat equation by W. Balser and M. Loday-Richaud in a 2009 article [5].

1. Introduction

For several years, various works have been done on the divergent solutions of some classes of linear partial differential equations or integro-differential equations in two variables or more, allowing thus to formulate many results on Gevrey properties, summability or multisummability (e.g. [1, 3, 5, 7–9, 11–13, 20, 22, 24, 25, 30–38, 43, 44, 48–50, 59, 60]).

In the case of the nonlinear partial differential equations, the situation is much more complicated. The existing results concern mainly Gevrey properties, especially the convergence (e.g. [10, 14, 16, 17, 21, 26, 39–41, 47, 51–58]), and there are very few results about the summation (see [15, 23, 27, 42, 45]).

In this article, we are interested in the summability of the formal solutions of the inhomogeneous semilinear heat equation

$$\begin{cases}\partial_t u - a(x)\partial_x^2 u - b(x)u^2 = \widetilde{f}(t,x)\\ u(0,x)=\varphi(x)\end{cases} \tag{1.1}$$

in two variables $(t,x)\in\mathbb{C}^2$, where the coefficients $a(x)$ and $b(x)$, and the initial condition $\varphi(x)$ are analytic on a disc D_ρ with center $0\in\mathbb{C}$ and radius $\rho>0$, and where the inhomogeneity $\widetilde{f}(t,x)$ is a formal power series in t with analytic coefficients in D_ρ (denoted in the sequel by $\widetilde{f}(t,x)\in\mathcal{O}(\mathcal{D}_\rho)[[t]]$) which may be smooth, or not[1]. Observe that an important particular case of Eq. (1.1) is the inhomogeneous linear heat equation

$$\begin{cases}\partial_t u - a(x)\partial_x^2 u = \widetilde{f}(t,x)\\ u(0,x)=\varphi(x)\end{cases} \tag{1.2}$$

2000 *Mathematics Subject Classification.* 35C10, 35C20, 35K05,35K55, 40B05.

Key words and phrases. Summability,Heat equation,Inhomogeneous partial differential equation,Nonlinear partial differential equation,Formal power series,Divergent power series.

[1]We denote $\widetilde{f}$ with a tilde to emphasize the possible divergence of the series $\widetilde{f}$.

obtained for $b(x) = 0$.

Equation (1.1) arises in many physical, chemical, biological, and ecological problems involving diffusion and nonlinear growth such as heat and mass transfer, combustion theory, and spread theory of animal of plant populations. For example, if a chemical reaction generates heat at a reat depending on the temperature u, then u satisfies Eq. (1.1). In biological and ecological problems, the nonlinear term u^2 represents the growth of animal or plant population.

Proposition 1.1. *Equation (1.1) admits a unique solution* $\widetilde{u}(t,x) \in \mathcal{O}(\mathcal{D}_\rho)[[t]]$.

Proof. Writing the inhomogeneity $\widetilde{f}(t,x)$ in the form

$$\widetilde{f}(t,x) = \sum_{j\geqslant 0} f_{j,*}(x)\frac{t^j}{j!} \quad \text{with } f_{j,*}(x) \in D_\rho,$$

and looking for $\widetilde{u}(t,x)$ on the same type, one easily checks that the coefficients $u_{j,*}(x) \in D_\rho$ are uniquely determined for all $j \geqslant 0$ by the initial condition $u_{0,*}(x) = \varphi(x)$ and by the recurrence relations

$$u_{j+1,*}(x) = f_{j,*}(x) + a(x)\partial_x^2 u_{j,*}(x) + b(x)\sum_{k=0}^{j}\binom{j}{k}u_{k,*}(x)u_{j-k,*}(x).$$

□

In 1999, D. A. Lutz, M. Miyake and R. Schäfke considered the case of Eq. (1.2) with $a(x) = 1$ and $\widetilde{f}(t,x) = 0$. Using an approach based on the definition of the 1-summability in terms of the Borel transformation, they gave necessary and sufficient conditions on $\varphi(x)$ for $\widetilde{u}(t,x)$ be 1-summable in a fixed direction $\arg(t) = \theta$ [20]. Afterwards, and using the same approach, various authors have extended this result in the case where, either $a(x) = a \in \mathbb{C}^*$, or $\widetilde{f}(t,x) = 0$ [1, 4, 9] (see also [6, 30, 31] for an extension in higher spatial dimensions). The general case of Eq. (1.2) was treated by W.Balser and M. Loday-Richaud in [5], but with a different approach based on the definition of the 1-summability in terms of the successive derivatives.

In this article, we propose to extend the results of [5] to the general equation (1.1). In Section 2, we recall some basic definitions and properties about the 1-summable formal series and we state the main result of our article (Theorem 2.4). This result is proved in Section 3.

2. 1-summability of $\widetilde{u}(t,x)$

All along the article, we consider t as the variable and x as a parameter. Thereby, to define the notion of 1-summability of formal power series in $\mathcal{O}(\mathcal{D}_\rho)[[t]]$, one extends the classical notions of 1-summability of elements in $\mathbb{C}[[t]]$ to families parametrized by x in requiring similar conditions, the estimates being however uniform with respect to x. Doing that, any formal power series in $\mathcal{O}(\mathcal{D}_\rho)[[t]]$ can be seen as a formal power series in t with coefficients in a convenient Banach space defined as the space of functions that are holomorphic on a disc D_r $(0 < r < \rho)$ and continuous up to its boundary, equipped with the usual supremum norm. For a general study of series with coefficients in a Banach space, we refer for instance to [2].

2.1. 1**-summable formal series.** Among the many equivalent definitions of 1-summability in a given direction $\arg(t)=\theta$ at $t=0$, we choose in this article a generalization of Ramis' definition which states that a formal series $\widetilde{g}(t,x)\in\mathbb{C}[[t]]$ is 1-summable in direction θ if there exists a holomorphic function g which is 1-Gevrey asymptotic to $\widetilde{g}$ in an open sector $\Sigma_{\theta,>\pi}$ bisected by θ and with opening larger than π [46, Def. 3.1]. To express the 1-Gevrey asymptotic, there also exist various equivalent ways. We choose here the one which sets conditions on the successive derivatives of g (see [28, p. 171] or [46, Thm. 2.4] for instance).

Definition 2.1 (1-summability)**.** A formal series $\widetilde{u}(t,x)\in\mathcal{O}(\mathcal{D}_\rho)[[t]]$ is said to be 1-*summable in the direction* $\arg(t)=\theta$ if there exist a sector $\Sigma_{\theta,>\pi}$, a radius $0<r\leqslant\rho$ and a function $u(t,x)$ called 1-*sum of* $\widetilde{u}(t,x)$ *in direction* θ such that

(1) u is defined and holomorphic on $\Sigma_{\theta,>\pi}\times D_r$;
(2) For any $x\in D_r$, the map $t\mapsto u(t,x)$ has $\widetilde{u}(t,x)=\displaystyle\sum_{j\geqslant 0}u_{j,*}(x)\frac{t^j}{j!}$ as Taylor series at 0 on $\Sigma_{\theta,>\pi}$;
(3) For any proper[2] subsector $\Sigma\Subset\Sigma_{\theta,>\pi}$, there exist constants $C>0$ and $K>0$ such that, for all $\ell\geqslant 0$, all $t\in\Sigma$ and all $x\in D_r$,
$$\left|\partial_t^\ell u(t,x)\right|\leqslant CK^\ell\Gamma(1+2\ell).$$

We denote by $\mathcal{O}(\mathcal{D}_\rho)\{t\}_{1;\theta}$ the subset of $\mathcal{O}(\mathcal{D}_\rho)[[t]]$ made of all the 1-summable formal series in the direction $\arg(t)=\theta$.

Note that, for any fixed $x\in D_r$, the 1-summability of $\widetilde{u}(t,x)$ coincides with the classical 1-summability. Consequently, Watson's lemma implies the unicity of its 1-sum, if any exists.

Note also that the 1-sum of a 1-summable formal series $\widetilde{u}(t,x)\in\mathcal{O}(\mathcal{D}_\rho)\{t\}_{1;\theta}$ may be analytic with respect to x on a disc D_r smaller than the common disc D_ρ of analyticity of the coefficients $u_{j,*}(x)$ of $\widetilde{u}(t,x)$.

Denote by $\partial_t^{-1}\widetilde{u}$ (resp. $\partial_x^{-1}\widetilde{u}$) the anti-derivative of $\widetilde{u}$ with respect to t (resp. x) which vanishes at $t=0$ (resp. $x=0$). Proposition 2.2 below specifies the algebraic structure of $\mathcal{O}(\mathcal{D}_\rho)\{t\}_{1;\theta}$.

Proposition 2.2. *Let $\theta\in\mathbb{R}/2\pi\mathbb{Z}$. Then, $(\mathcal{O}(\mathcal{D}_\rho)\{t\}_{1;\theta},\partial_t,\partial_x)$ is a $\mathbb{C}$-differential algebra stable under the anti-derivatives ∂_t^{-1} and ∂_x^{-1}.*

We refer for instance to [5, Prop. 3.2] for a proof of this result.

With respect to t, the 1-sum $u(t,x)$ of a 1-summable series $\widetilde{u}(t,x)\in\mathcal{O}(\mathcal{D}_\rho)\{t\}_{1;\theta}$ is analytic on an open sector for which there is no control on the angular opening except that it must be larger than π (hence, it contains a closed sector $\overline{\Sigma}_{\theta,\pi}$ bisected by θ and with opening π) and no control on the radius except that it must be positive. Thereby, the 1-sum $u(t,x)$ is well-defined as a section of the sheaf of analytic functions in (t,x) on a germ of closed sector of opening π (that is, a closed interval $\overline{I}_{\theta,\pi}$ of length π on the circle S^1 of directions issuing from 0; see [29, 1.1] or [18, I.2]) times $\{0\}$ (in the plane $\mathbb{C}$ of the variable x). We denote by $\mathcal{O}_{\overline{I}_{\theta,\pi}\times\{0\}}$ the space of such sections.

[2]A subsector Σ of a sector Σ' is said to be *a proper subsector* and one denotes $\Sigma\Subset\Sigma'$ if its closure in $\mathbb{C}$ is contained in $\Sigma'\cup\{0\}$.

Corollary 2.3. *The operator of* 1*-summation*

$$\begin{array}{rccc} \mathcal{S}_{1;\theta}: & \mathcal{O}(\mathcal{D}_\rho)\{t\}_{1;\theta} & \longrightarrow & \mathcal{O}_{\overline{I}_{\theta,\pi}\times\{0\}} \\ & \widetilde{u}(t,x) & \longmapsto & u(t,x) \end{array}$$

is a homomorphism of differential $\mathbb{C}$*-algebras for the derivations* ∂_t *and* ∂_x*. Moreover, it commutes with the anti-derivations* ∂_t^{-1} *and* ∂_x^{-1}*.*

Let us now turn to the study of the 1-summability of the formal solution $\widetilde{u}(t,x)\in\mathcal{O}(\mathcal{D}_\rho)[[t]]$ of Eq. (1.1).

2.2. **Main result.** Before stating our main result, let us start with a preliminary remark. Write the coefficients $a(x)$ and $b(x)$ in the form

$$a(x)=\sum_{n\geqslant 0}a_n\frac{x^n}{n!},\quad b(x)=\sum_{n\geqslant 0}b_n\frac{x^n}{n!}$$

and the formal series $\widetilde{u}(t,x)$ and $\widetilde{f}(t,x)$ in the form

$$\widetilde{u}(t,x)=\sum_{n\geqslant 0}\widetilde{u}_{*,n}(t)\frac{x^n}{n!},\quad \widetilde{f}(t,x)=\sum_{n\geqslant 0}\widetilde{f}_{*,n}(t)\frac{x^n}{n!}.$$

By identifying the terms in x^n in Eq. (1.1), we get the identities

$$\left\{\begin{aligned} &a_0\widetilde{u}_{*,2}(t)=\partial_t\widetilde{u}_{*,0}(t)-b_0\widetilde{u}_{*,0}(t)\widetilde{u}_{*,0}(t)-\widetilde{f}_{*,0}(t)\\ &a_0\widetilde{u}_{*,3}(t)+a_1\widetilde{u}_{*,2}(t)=\partial_t\widetilde{u}_{*,1}(t)-2b_0\widetilde{u}_{*,0}(t)\widetilde{u}_{*,1}(t)-b_1\widetilde{u}_{*,0}(t)\widetilde{u}_{*,0}(t)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\widetilde{f}_{*,1}(t),\end{aligned}\right. \tag{2.1}$$

for $n=0$ and $n=1$, and the identities

$$a_0\widetilde{u}_{*,n+2}(t)+na_1\widetilde{u}_{*,n+1}(t)=\\ \partial_t\widetilde{u}_{*,n}(t)-\sum_{n_1+n_2+n_3=n}\frac{n!}{n_1!n_2!n_3!}b_{n_1}\widetilde{u}_{*,n_2}(t)\widetilde{u}_{*,n_3}(t)-\widetilde{f}_{*,n}(t)$$

for $n\geqslant 2$. Consequently, each formal series $\widetilde{u}_{*,n}(t)$ is uniquely determined from $\widetilde{u}_{*,0}(t)$, $\widetilde{u}_{*,1}(t)$ and $\widetilde{f}(t,x)$.

In the case of the linear heat equation (1.2), W. Balser and M. Loday-Richaud proved, under the assumption that $(a_0,a_1)\neq(0,0)$, that the terms $\widetilde{u}_{*,0}(t)$, $\widetilde{u}_{*,1}(t)$ and $\widetilde{f}(t,x)$ allow to fully characterize the 1-summability of the formal solution $\widetilde{u}(t,x)$ in a given direction [5].

In the case of our semilinear heat equation (1.1), Theorem (2.4) below tells us that this characterization remains valid. More precisely, we have:

Theorem 2.4. *Let* $\arg(t)=\theta\in\mathbb{R}/2\pi\mathbb{Z}$ *be a direction issuing from* 0*. Assume that either* $a(0)\neq 0$*, or* $a(0)=0$ *and* $a'(0)\neq 0$*. Then,*

1. *The unique formal series solution* $\widetilde{u}(t,x)\in\mathcal{O}(\mathcal{D}_\rho)[[t]]$ *of Eq. (1.1) is* 1*-summable in the direction* θ *if and only if the inhomogeneity* $\widetilde{f}(t,x)$ *and the coefficients* $\widetilde{u}_{*,0}(t)$ *and* $\widetilde{u}_{*,1}(t)$ *are* 1*-summable in the direction* θ*.*
2. *Moreover, the* 1*-sum* $u(t,x)$*, if any exists, satisfies Eq. (1.1) in which* $\widetilde{f}(t,x)$ *is replaced by its* 1*-sum* $f(t,x)$ *in the direction* θ*.*

When $a(x) = O(x^2)$, Theorem (2.4) fails: the formal solution $\widetilde{u}(t,x)$ may not be 1-summable in a given direction, while $\widetilde{u}_{*,0}(t)$, $\widetilde{u}_{*,1}(t)$ and $\widetilde{f}(t,x)$ are 1-summable. Such a situation occurs for example in the case where $a(x) = x^2$, $b(x) = 0$, $\widetilde{f}(t,x) = 0$ and $\varphi(x) = \frac{1}{1-x}$. We refer to [5, Counter example 3.5] for the details of the calculations.

Let us now turn to the proof of Theorem 2.4.

3. Proof of Theorem 2.4

3.1. **Case** $a(0) \neq 0$.

$\lhd$ Point 1 (necessary condition). This is straightforward from Proposition 2.2. We have indeed $\widetilde{u}_{*,0}(t) = \widetilde{u}(t,0)$, $\widetilde{u}_{*,1}(t) = \partial_x \widetilde{u}(t,x)_{|x=0}$, and

$$\widetilde{f}(t,x) = \partial_t u(t,x) - a(x)\partial_x^2 u(t,x) - b(x)u(t,x)^2.$$

$\lhd$ Point 1 (sufficient condition). To prove that the condition is sufficient, we shall proceed in a similar way as the proof of [5, Thm. 3.4] (see also [48–50]).

By assumption, we have $a(0) \neq 0$. Hence, the functions $A(x) = 1/a(x)$ and $B(x) = b(x)/a(x)$ are both well-defined and holomorphic on a convenient disc $D_{\rho'}$ with $0 < \rho' < \rho$.

Let us set $\widetilde{u}(t,x) = \widetilde{v}(t,x) + \partial_x^{-2}\widetilde{w}(t,x)$ with $\widetilde{v}(t,x) = \widetilde{u}_{*,0}(t) + \widetilde{u}_{*,1}(t)x$. With these notations, Eq. (1.1) becomes

$$\widetilde{w} - A(x)\partial_t\partial_x^{-2}\widetilde{w} + 2B(x)\widetilde{v}(t,x)\partial_x^{-2}\widetilde{w} + B(x)(\partial_x^{-2}\widetilde{w})^2 = \widetilde{g}(t,x) \tag{3.1}$$

with

$$\widetilde{g}(t,x) = A(x)(\partial_t\widetilde{v}(t,x) - b(x)\widetilde{v}(t,x)^2 - \widetilde{f}(t,x)).$$

Let us now assume that $\widetilde{u}_{*,0}(t)$, $\widetilde{u}_{*,1}(t)$ and $\widetilde{f}(t,x)$ are 1-summable in a given direction θ. Then, $\widetilde{v}(t,x)$ and $\widetilde{g}(t,x)$ are both 1-summable in the direction θ (see Proposition 2.2) and identity (3.1) above tells us it suffices to prove that it is the same for $\widetilde{w}(t,x)$. To this end, we shall proceed similarly as [5, 48–50] through a fixed point method. Of course, as we shall see below, the nonlinear term $(\partial_x^{-2}\widetilde{w})^2$ induces much more complicated calculations.

Let us set $\widetilde{w}(t,x) = \sum_{m\geqslant 0} \widetilde{w}_m(t,x)$ and let us choose the solution of Eq. (3.1) recursively determined for all $m \geqslant 0$ by the system

$$\begin{cases} \widetilde{w}_0 = \widetilde{g} \\ \widetilde{w}_{m+1} = A(x)\partial_t\partial_x^{-2}\widetilde{w}_m - 2B(x)\widetilde{v}\partial_x^{-2}\widetilde{w}_m - B(x)\displaystyle\sum_{k=0}^{m}(\partial_x^{-2}\widetilde{w}_k)(\partial_x^{-2}\widetilde{w}_{m-k}) \end{cases} \tag{3.2}$$

Observe that $\widetilde{w}_m(t,x) \in \mathcal{O}(D_{\rho'})[[t]]$ for all $m \geqslant 0$. Observe also that the $\widetilde{w}_m(t,x)$'s are of order $O(x^{2m})$ in x for all $m \geqslant 0$, and, consequently, the series $\widetilde{w}(t,x)$ itself makes sense as a formal series in t and x.

Let us now respectively denote by $w_0(t,x)$ and $v(t,x)$ the 1-sums of $\widetilde{w}_0 = \widetilde{g}$ and $\widetilde{v}$ in direction θ and, for all $m \geqslant 0$, let $w_m(t,x)$ be determined as the solution of System (3.2) in which $\widetilde{v}$ is replaced by v and all the $\widetilde{w}_m$ are replaced by w_m. By construction, all the functions $w_m(t,x)$ are defined and holomorphic on a common domain $\Sigma_{\theta,>\pi} \times D_{\rho''}$ with a convenient radius $0 < \rho'' \leqslant \rho'$.

To end the proof, it remains to prove that the series $\sum_{m\geqslant 0} w_m(t,x)$ is convergent, and that its sum $w(t,x)$ is the 1-sum of $\widetilde{w}(t,x)$ in direction θ.

According to Definition (2.1), the 1-summability of $\widetilde{w}_0$ and $\widetilde{v}$ implies that there exists $0 < r' < \min(1,\rho'')$ such that, for any proper subsector $\Sigma \Subset \Sigma_{\theta,>\pi}$, there exist two positive constants $C > 0$ and $K \geqslant 1$ such that, for all $\ell \geqslant 0$ and all $(t,x) \in \Sigma \times D_{r'}$, the functions w_0 and v satisfy the inequalities

$$\left|\partial_t^\ell w_0(t,x)\right| \leqslant CK^\ell\Gamma(1+2\ell) \quad\text{and}\quad \left|\partial_t^\ell v(t,x)\right| \leqslant CK^\ell\Gamma(1+2\ell). \tag{3.3}$$

Let us now fix a proper subsector $\Sigma \Subset \Sigma_{\theta,>\pi}$ and let us denote by α (resp. β) the maximum of $|A(x)|$ (resp. $|B(x)|$) on the closed disc $|x| \leqslant r'$. Proposition (3.1) below provides us some estimates on the derivatives $\partial_t^\ell w_m$.

Proposition 3.1. *The following inequalities*

$$\left|\partial_t^\ell w_m(t,x)\right| \leqslant C(\alpha+2\pi^2C\beta)^m K^{\ell+m}\Gamma(1+2(\ell+m))\frac{|x|^{2m}}{(2m)!} \tag{3.4}$$

hold for all $\ell, m \geqslant 0$ and all $(t,x) \in \Sigma \times D_{r'}$.

Proof. The proof proceeds by recursion on m. The case $m = 0$ is straightforward from the first inequality of (3.3). Let us now suppose that the inequalities (3.4) hold for all $0 \leqslant k \leqslant m$ for a certain $m \geqslant 0$. According to the relations (3.2), we deduce from the Leibniz Formula that

$$\begin{aligned}\partial_t^\ell w_{m+1}(t,x) &= A(x)\partial_t^{\ell+1}\partial_x^{-2}w_m(t,x)\\ &\qquad - 2B(x)\sum_{j=0}^{\ell}\binom{\ell}{j}\partial_t^j v(t,x)\partial_t^{\ell-j}\partial_x^{-2}w_m(t,x)\\ &\qquad\qquad - B(x)\sum_{k=0}^{m}\sum_{j=0}^{\ell}\binom{\ell}{j}\partial_t^j\partial_x^{-2}w_k(t,x)\partial_t^{\ell-j}\partial_x^{-2}w_{m-k}(t,x)\end{aligned}$$

for all $\ell \geqslant 0$ and $(t,x) \in \Sigma \times D_{r'}$. Hence, applying the second inequality of (3.3) and the inequalities (3.4) for all the w_k's with $k = 0, ..., m$, and using the fact that $K \geqslant 1$ and $r' < 1$, we get the inequalities

$$\begin{aligned}\left|\partial_t^\ell w_{m+1}(t,x)\right| &\leqslant C(\alpha+2\pi^2C\beta)^m K^{\ell+m+1}\Gamma(1+2(\ell+m+1))\frac{|x|^{2m+2}}{(2m+2)!}\\ &\qquad\times(\alpha+2C\beta S_{m,\ell}+C\beta S'_{m,\ell})\end{aligned}$$

for all $\ell \geqslant 0$ and $(t,x) \in \Sigma \times D_{r'}$, where $S_{m,\ell}$ and $S'_{m,\ell}$ are respectively defined by

$$S_{m,\ell} = \sum_{j=0}^{\ell}\binom{\ell}{j}\frac{\Gamma(1+2j)\Gamma(1+2(\ell-j+m))}{\Gamma(1+2(\ell+m+1))} \text{ and}$$

$$S'_{m,\ell} = \sum_{k=0}^{m}\sum_{j=0}^{\ell}\binom{\ell}{j}\frac{(2m+2)!\Gamma(1+2(j+k))\Gamma(1+2(\ell-j+m-k))}{(2k+2)!(2m-2k+2)!\Gamma(1+2(\ell+m+1))}.$$

Inequalities (3.4) follow then from Lemmas (3.2) and (3.3) below and from the fact that $2 \leqslant \pi^2$. This ends the proof. □

Lemma 3.2. $S_{m,\ell} \leqslant 1$ *for all* $m, \ell \geqslant 0$.

Proof. Lemma (3.2) stems obvious from the identity

$$S_{m,\ell} = \frac{1}{(2\ell+2m+2)(2\ell+2m+1)} \sum_{j=0}^{\ell} \frac{\binom{\ell}{j}}{\binom{2\ell+2m}{2j}}$$

and the combinatorial inequalities $\binom{2\ell+2m}{2j} \geqslant \binom{\ell}{j}\binom{\ell+2m}{j} \geqslant \binom{\ell}{j}$. □

Lemma 3.3. *$S'_{m,\ell} \leqslant \pi^2$ for all $m, \ell \geqslant 0$.*

Proof. First of all, let us observe that

$$\begin{aligned}
S'_{m,\ell} &\leqslant \sum_{k=0}^{m}\sum_{j=0}^{\ell} \binom{\ell}{j} \frac{(2m+2)!\Gamma(1+2(j+k))\Gamma(1+2(\ell-j+m-k+1))}{(2k+2)!(2m-2k+2)!\Gamma(1+2(\ell+m+1))} \\
&= \sum_{k=0}^{m}\sum_{j=0}^{\ell} \frac{\binom{\ell}{j}\binom{2m+2}{2k}}{(2k+2)(2k+1)\binom{2\ell+2m+2}{2j+2k}} \\
&\leqslant \sum_{k=0}^{m} \left(\frac{1}{(k+1)^2} \sum_{j=0}^{\ell} \frac{\binom{\ell}{j}\binom{2m+2}{2k}}{\binom{2\ell+2m+2}{2j+2k}} \right).
\end{aligned}$$

Applying then the combinatorial inequality $\binom{2\ell+2m+2}{2j+2k} \geqslant \binom{\ell}{j}^2\binom{2m+2}{2k}$, we finally get

$$S'_{m,\ell} \leqslant \sum_{k=0}^{m} \left(\frac{1}{(k+1)^2} \sum_{j=0}^{\ell} \frac{1}{\binom{\ell}{j}} \right).$$

Let us now observe that

$$\sum_{j=0}^{\ell} \frac{1}{\binom{\ell}{j}} \leqslant 6$$

for all $\ell \geqslant 0$: the inequality is clear for $\ell \in \{0, 1, 2, 3\}$, and, for $\ell \geqslant 4$, we have

$$\sum_{j=0}^{\ell} \frac{1}{\binom{\ell}{j}} = 2 + \frac{2}{\ell} + \sum_{j=2}^{\ell-2} \frac{1}{\binom{\ell}{j}} \leqslant 2 + \frac{2}{\ell} + \sum_{j=2}^{\ell-2} \frac{1}{\binom{\ell}{2}} = 2 + \frac{2}{\ell} + \frac{2(\ell-3)}{\ell(\ell-1)} \leqslant 6.$$

Hence,

$$S'_{m,\ell} \leqslant 6 \sum_{k=0}^{m} \frac{1}{(k+1)^2} \leqslant 6 \sum_{k=1}^{+\infty} \frac{1}{k^2} = \pi^2,$$

which proves Lemma (3.3). □

From Proposition (3.1), we next derive the inequalities

$$\left|\partial_t^{\ell} w_m(t,x)\right| \leqslant C K'^{\ell}\Gamma(1+2\ell)(c\,|x|^2)^m \tag{3.5}$$

for all $\ell, m \geqslant 0$ and all $(t,x) \in \Sigma \times D_{r'}$, where K' and c are the two positive constants defined by $K' = 4K$ and $c = 4K(\alpha + 2\pi^2 C\beta)$. Indeed, applying the inequalities (3.4), we easily have

$$\begin{aligned}\left|\partial_t^\ell w_m(t,x)\right| &\leqslant C(\alpha + 2\pi^2 C\beta)^m K^{\ell+m}\Gamma(1+2\ell)\,|x|^{2m} \times \binom{2\ell+2m}{2m} \\ &\leqslant C2^{2\ell+2m}(\alpha + 2\pi^2 C\beta)^m K^{\ell+m}\Gamma(1+2\ell)\,|x|^{2m}.\end{aligned}$$

Let us now choose for Σ a sector containing a proper subsector Σ' bisected by the direction θ and opening larger than π (such a choice is already possible by definition of a proper subsector, see Footnote 2). Let us also choose a radius $0 < r < \min(r', 1/\sqrt{c})$ and let us set $C' := C \sum_{m\geqslant 0} (cr^2)^m \in \mathbb{R}_+^*$.

Thanks to the inequalities (3.5), the series $\sum_{m\geqslant 0} \partial_t^\ell w_m(t,x)$ are normally convergent on $\Sigma \times D_r$ for all $\ell \geqslant 0$ and satisfy the inequalities

$$\sum_{m\geqslant 0} \left|\partial_t^\ell w_m(t,x)\right| \leqslant C'K'^\ell\Gamma(1+2\ell)$$

for all $(t,x) \in \Sigma \times D_r$. In particular, the sum $w(t,x)$ of the series $\sum_{m\geqslant 0} w_m(t,x)$ is well-defined, holomorphic on $\Sigma \times D_r$ and satisfies the inequalities

$$\left|\partial_t^\ell w(t,x)\right| \leqslant C'K'^\ell\Gamma(1+2\ell)$$

for all $\ell \geqslant 0$ and all $(t,x) \in \Sigma \times D_r$. Hence, Conditions 1 and 3 of Definition 2.1 hold.

To prove the second condition of Definition 2.1, we proceed as follows. The removable singularities theorem implies the existence of $\lim_{\substack{t\to 0\\ t\in\Sigma'}} \partial_t^\ell w(t,x)$ for all $x \in D_r$ and, thereby, the existence of the Taylor series of w at 0 on Σ' for all $x \in D_r$ (see for instance [28, Cor. 1.1.3.3]; see also [19, Prop. 1.1.11]). On the other hand, considering recurrence relations (3.2) with w_m and the 1-sums $v(t,x)$ and $g(t,x)$ instead of $\widetilde{w}_m$, $\widetilde{v}(t,x)$ and $\widetilde{g}(t,x)$, it is clear that $w(t,x)$ satisfies equation (3.1) with $v(t,x)$ in place of $\widetilde{v}(t,x)$ and right-hand side $g(t,x)$ in place of $\widetilde{g}(t,x)$ and, consequently, so does its Taylor series. Then, since equation (3.1) has a unique formal series solution $\widetilde{w}(t,x)$, we then conclude that the Taylor expansion of $w(t,x)$ is $\widetilde{w}(t,x)$. Hence, Condition 2 of Definition 2.1 holds.

This achieves the proof of the 1-summability of $\widetilde{w}(t,x)$ and, thereby, the fact that the condition is sufficient.

$\lhd$ Point 2. The fact that the 1-sum $u(t,x)$ of $\widetilde{u}(t,x)$ in direction θ satisfies Eq. (1.1) with right-hand side the 1-sum $f(t,x)$ of $\widetilde{f}(t,x)$ in direction θ in place of $\widetilde{f}(t,x)$ is a direct consequence of Corollary 2.3. This completes the proof of Theorem 2.4 in the case $a(0) \neq 0$.

3.2. **Case $a(0) = 0$ and $a'(0) \neq 0$.** The necessary condition of the first point and the second point result as before from Proposition 2.2 and Corollary 2.3. We sketch here below the proof of the sufficient condition of the first point.

Denote $a(x) = xa_1(x)$. By assumption, $a_1(0) \neq 0$. Then, the functions $A_1(x) = 1/a_1(x)$ and $B_1(x) = b(x)/a_1(x)$ are both well-defined and holomorphic on a convenient common disc centered at the origin $0 \in \mathbb{C}$.

Setting as before $\widetilde{u}(t,x) = \widetilde{v}(t,x) + \partial_x^{-2}\widetilde{w}(t,x)$ with $\widetilde{v}(t,x) = \widetilde{u}_{*,0}(t) + \widetilde{u}_{*,1}(t)x$, Eq. (1.1) becomes

$$\widetilde{w} - \frac{A_1(x)}{x}\partial_t\partial_x^{-2}\widetilde{w} + 2\frac{B_1(x)}{x}\widetilde{v}(t,x)\partial_x^{-2}\widetilde{w} + \frac{B_1(x)}{x}(\partial_x^{-2}\widetilde{w})^2 = \widetilde{g}(t,x) \tag{3.6}$$

with

$$\widetilde{g}(t,x) = A_1(x)\frac{\partial_t\widetilde{v}(t,x) - b(x)\widetilde{v}(t,x)^2 - \widetilde{f}(t,x)}{x}.$$

By assumption, we have $a(0) = a_0 = 0$; hence, due to the first equality of (2.1), the constant term in x of $\partial_t\widetilde{v}(t,x) - b(x)\widetilde{v}(t,x)^2 - \widetilde{f}(t,x)$ is zero, and, consequently, $\widetilde{g}(t,x)$ is again a formal power series in t and x. Assuming then $\widetilde{v}(t,x)$ and $\widetilde{g}(t,x)$ to be 1-summable in the direction θ, we can prove as previously that $\widetilde{w}(t,x)$ is also 1-summable in the direction θ.

Observe that the $\widetilde{w}_m(t,x)$ are now recursively determined for all $m \geqslant 0$ by the system

$$\begin{cases} \widetilde{w}_0 = \widetilde{g} \\ \widetilde{w}_{m+1} = \dfrac{A_1(x)}{x}\partial_t\partial_x^{-2}\widetilde{w}_m - 2\dfrac{B_1(x)}{x}\widetilde{v}\partial_x^{-2}\widetilde{w}_m - \dfrac{B_1(x)}{x}\displaystyle\sum_{k=0}^{m}(\partial_x^{-2}\widetilde{w}_k)(\partial_x^{-2}\widetilde{w}_{m-k}) \end{cases}$$

In particular, the operator $\dfrac{1}{x}\partial_x^{-2}$ in place of ∂_x^{-2} implies that the $\widetilde{w}_m(t,x)$'s are now of order $O(x^m)$ in x for all $m \geqslant 0$, instead of $O(x^{2m})$ as in the previous case. Still, $\widetilde{w}(t,x)$ is again a formal power series in t and x.

The estimates on the derivatives $\partial_t^\ell w_m$ given in Proposition (3.1) are modified as follows: for all $m, \ell \geqslant 0$ and all $(t,x) \in \Sigma \times D_{r'}$,

$$\left|\partial_t^\ell w_m(t,x)\right| \leqslant C(\alpha_1 + 2\pi^2 C\beta_1)^m K^{\ell+m}\Gamma(1 + 2(\ell+m))\frac{|x|^m}{(m!)^2},$$

where α_1 (resp. β_1) stands for the maximum of $|A_1(x)|$ (resp. $|B_1(x)|$) on the closed disc $|x| \leqslant r'$. Consequently, the inequalities (3.5) obtained in the case $a(0) \neq 0$ become

$$\left|\partial_t^\ell w_m(t,x)\right| \leqslant CK'^\ell\Gamma(1+2\ell)(c\,|x|)^m$$

for all $\ell, m \geqslant 0$ and all $(t,x) \in \Sigma \times D_{r'}$, where K' and c are the two positive constants defined by $K' = 4K$ and $c = 16K(\alpha_1 + 2\pi^2 C\beta_1)$.

The end of the proof is similar to the one of the case $a(0) \neq 0$ and is left to the reader. This completes the proof of Theorem 2.4.

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