A two-stage robust approach for the weighted number of tardy jobs with objective uncertainty

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Abstract

Minimizing the weighted number of tardy jobs is a classical and intensively studied scheduling problem. In this paper, we develop a two-stage robust approach, where exact weights are known after accepting to perform the jobs, and before sequencing them on the machine. This assumption allows diverse recourse decisions to be taken in order to better adapt one's tactical plan.

The contribution of this paper is twofold: first, we introduce a new scheduling problem and model it as a min-max-min optimization problem with mixed-integer recourse by extending existing models proposed for the classical problem where all the costs are assumed to be known. Second, we take advantage of the special structure of the problem to propose two solution approaches based on results from the recent robust optimization literature, namely finite adaptability (Bertsimas and Caramanis, 2010) and a convexification-based approach (Arslan and Detienne, 2020). We also study the cost of finding anchored solutions, where the sequence of jobs has to be decided before the uncertainty is revealed. Computational experiments to analyze the effectiveness of our approaches are reported.

Keywords: One-machine scheduling, robust optimization, two-stage optimization, mixed-integer recourse, exact approach, integer programming

1. Introduction

Historically, scheduling optimization problems used to be tackled in a deterministic fashion, assuming that all input data were perfectly known at decision time. These problems have been, and still are, extensively studied in the literature. For an overview of the broad scheduling literature, the reader may refer to [1], which introduced the widely used notation for scheduling problem, and to [2], which covers important theoretical models and significant scheduling problems occuring in the real world.

More recently, researchers have started to focus on scheduling problems where the input data are no longer considered to be known in advance. Rather, input data are often considered to be random variables for which one knows a probability distribution (e.g., stochastic scheduling) or as a support of its density function (e.g., robust scheduling). While the first situation yields solutions that are good on average (i.e., leading to an optimal expected objective value), the second approach gives solutions that are never too bad (worst-case objective value). In the static framework, one has to choose the value of all decision variables before the realization of the random variable is revealed, whereas two-stage models ask the decider to fix only a part of the solution before knowing the uncertain parameters, at the *first stage*. At the *second stage*, he is given the opportunity to react after the revelation of the true input data, by determining *recourse* decisions.

A number of approaches have been proposed in the literature to deal with two-stage robust optimization problems. Exact solution approaches mainly focus on problems with continuous recourse decisions and are

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based on mixed-integer programming large-scale reformulations and dynamic generation of the obtained formulation (see for example [3, 4, 5, 6]). Due to the discrete nature of scheduling problems, the natural mathematical models involve mixed-integer variables in both the first and second stages, which invalidates many solution algorithms. In [7], the authors propose a general framework for exactly solving two-stage mixed-integer robust problems with interdiction binary linking constraints (i.e., constraints of the form $y \leq x$ where both y and x are binary, x denoting the first-stage decision variable and y the recourse decision variable). Most of the other suitable methods are approximate in the sense that they restrict the set of possible recourse decisions. The *decision rules*-based approaches (see *e.g.* [8]) restrict the second-stage variables to simple functions of the random parameters. In [9], the authors present another type of conservative approximation, known as finite adaptability or K-adaptability, which implies limiting the number of recourse actions in the second stage. These possible recourse policies are determined at the first stage, and the second stage only selects the best action to implement in reaction to the uncertain parameters revealed.

Regarding robust scheduling approaches, [10] and [11] present different complexity results for single machine problems. They show that even simple scheduling problems become $\mathcal{NP}$-hard as soon as the uncertainty set contains more than one scenario. In [12], the authors provide approximation algorithms for the problem of minimizing the weighted and unweighted sum of completion times on a single machine where the uncertainty lies in the processing time of the tasks. Problems with stochastic breakdowns on one machine (resp. two machines) are studied in [13] (resp. [14]). Affine decision rules are proposed for two-stage robust batch process scheduling under polyhedral uncertainty in [15], based on continuous-time models oriented towards chemistry applications. In [16], a variant of $1||\sum U_j$ is studied, where the processing times are uncertain. Given a discrete scenario-based uncertainty set, one has to determine an initial sequence of jobs, feasible for nominal processing times. At the second stage, once the scenario of actual processing times is revealed, the sequence must be adapted to the modified instance by rejecting some jobs. The objective is to minimize the expected cost of the repaired solution. The authors propose dynamic programming, a branch-and-bound algorithm and a branch-and-price algorithm to solve the problem exactly.

In this paper, we introduce a new scheduling problem as an extension to the well-known $1|r_j|\sum w_j U_j$ problem where the weighted sum of tardy jobs has to be minimized. The problem we are addressing is one in which the jobs are subject to failures, which lead to additional costs. Once the uncertain parameters are revealed, the decision maker is allowed to take discrete recourse actions: determining the sequence of jobs, outsourcing or spending more time on the jobs to fix them. We address this problem with a robust approach.

This problem has several practical applications. Consider the following example, which arises in the astronomical field when one needs to allocate observatory time. At planning time, a number of sessions have to be reserved for observation purposes, yet many factors which can alter the quality of the observation are not known, *e.g.*, weather, air quality, etc. As a result, the sessions may lead to lower-quality observations, a problem which can be fixed by increasing the time allocated to it, or outsourcing it to another facility. The costs involved in such situations are typically high, so a worst-case-type optimization approach is appropriate.

From an operational point of view, rescheduling jobs might be difficult or costly, and decision-makers may favor recourse solutions where the modifications are easier to handle. In this case, one may seek *anchored solutions*, which involve minor modifications to the original plan (see *e.g.*, [17]). The advantages of such solutions are well understood: they reduce the operational costs, reduce the possibility of error in the process, and are generally better accepted by the operators. We therefore propose an anchored version of the problem where the sequence of jobs cannot be modified after the uncertainty is revealed.

To the best of our knowledge, these problems have never been studied before. Our scheduling problems are special cases of robust optimization with mixed-integer recourse. In general, such problems are Σ_2^P-hard, which means that even verifying that a solution is feasible is an $\mathcal{NP}$-hard problem. We show that they belong to the subclass of robust problems identified by [7], and are thus $\mathcal{NP}$-complete.

We rely on [7] to propose an exact branch-and-price algorithm, and on [18] to obtain a deterministic MILP model for the K-adaptable version of the problems. The computational experiments confirm that the practical difficulty of the problem is considerable: even with state-of-the art methodologies, some instances with 25 jobs remain open after one hour of computing time. We also show that designing anchored solutions increases the cost only marginally.

In Section 2, we recall some useful results on the deterministic $1|r_j|\sum w_j U_j$, which will be used in the remainder of the paper. In Section 3, we formally describe a first robust version of this problem, before

proposing solution methods in Section 4. Section 5 is devoted to the anchored version of the problem. We report our computational experiments in Section 6 before concluding.

2. Minimizing the weighted number of tardy jobs: literature review

Minimizing the weighted number of tardy jobs on a single machine, denoted $1|r_j|\sum w_jU_j$ in the literature, is a well- known $\mathcal{NP}$-hard scheduling problem (see [1]) and can be stated as follows.

> PROBLEM : $1|r_j|\sum w_jU_j$ (DECISION)
> INPUT DATA : $(V, \mathcal{J}, (r, d, w, p))$, where V is a positive value, $\mathcal{J}$ a set of jobs, each of which are characterized by the following data: r_j: a release date (i.e., the time before which the job cannot start); d_j: a due date (i.e., the time after which the job is considered tardy; w_j: a weight (i.e., the fixed cost for executing the job tardily) ; p_j: a processing time (i.e., the time needed to execute the job).
> QUESTION : Is there a permutation σ of the tasks whose cost (i.e., the weighted number of tardy jobs) is smaller than V?

This problem has been extensively studied in the literature. In particular, [19] proposes a dominance rule for cases with equal release dates, known as the Earliest Deadline First rule. Heuristic approaches and lower bounds are given in [20, 21] while exact approaches are given in [22, 23, 24, 25]. To our knowledge, the best exact results are described in [26], where up to 500 job instances are solved in less than one hour.

Since it will be of particular interest for the problem we are addressing in this paper, we formally describe here a mixed integer linear programming (MILP) formulation introduced in [26] for solving the $1|r_j|\sum w_jU_j$ problem. The approach is based on two distinct decisions: (1) decide which jobs are to be executed tardily and (2) in what order will the on-time jobs be executed. This is possible since late jobs can be postponed arbitrarily without incurring additional costs. Moreover, we know [27] that if jobs have *agreeable time windows* (i.e., the tasks can be ordered in such a way that $i < j$ implies $r_i \leq r_j$ and $d_i \leq d_j$), then a feasible sequence of on-time jobs exists if and only if the earliest due-date first rule yields a feasible solution. The main idea of [26] is to reformulate $1|r_j|\sum w_jU_j$ into the selection of jobs on a single machine with agreeable time windows, so that the dominance rule can still be exploited. Formally, for any pair of jobs $J_i \in \mathcal{J}$ and $J_j \in \mathcal{J}$ that can be both on time in some solutions and with non-agreeable time windows (i.e., $r_i + p_i + p_j \leq d_j$, $r_i < r_j$ and $d_i > d_j$), a job occurrence $J_k \in \widetilde{\mathcal{J}}$ is created, which represents scheduling J_i before J_j. It has a hard deadline $\bar{d}_k = d_j$, and $r_k = r_i, p_k = p_i, w_k = 0$. The original job J_i is also added to the set of job occurrences $\widetilde{\mathcal{J}}$, with a null weight as well. For every job $J_j \in \mathcal{J}$, let $\mathcal{G}_j$ be the set gathering all job occurrences related to J_j.

The following dominance rule extends the earliest deadline first rule to the general $1|r_j|\sum w_jU_j$ problem.

Dominance rule 1 ([26]). *There is at least one optimal solution to $1|r_j|\sum w_jU_j$ in which the selected job occurrences of the on-time jobs are ordered according to a non-decreasing order of their deadlines with ties being broken in non-decreasing order of release dates.*

In the following, we assume that the job occurrences are ordered according to Proposition 1. Moreover, $\bullet_k$ denotes the data $\bullet$ of the kth job occurrence in that order. For instance, p_k denotes the processing time of the kth job occurrence in that order.

We now describe in detail an MILP model, which is based on the following consideration: assume that, having fixed the sequencing of the on-time tasks, a straightforward way to check the feasibility of the corresponding schedule is to plan every task as soon as possible. In [26], it is shown that the quality of the formulation is improved if reversed time windows are assigned to each job occurrence $J_j \in \widetilde{\mathcal{J}}$ as $[\hat{r_j}, \hat{d_j}] = [d_{\max} - d_j, d_{\max} - r_j]$ where $d_{\max} = \max\left\{d_j : J_j \in \widetilde{\mathcal{J}}\right\}$. Trivially, the timing problem for the tasks is equivalent to its reverse counterpart and if the job occurrences are sorted according to a non-decreasing order of their original deadlines, they are sorted according to a non-decreasing order of their reversed release dates.

This allows [26] to derive an efficient MILP model for $1|r_j|\sum w_jU_j$, close to the one proposed in [21] as follows: for every job $J_j \in \mathcal{J}$, let U_j be a binary variable equal to 1 if J_j is tardy, 0 otherwise. Then, for

every job occurrence $J_k \in \mathcal{G}_j$, x_k is the binary variable equal to 1 if J_k is selected, 0 otherwise, and t_k is a variable equal to its completion time in the reversed time. The following MILP model solves $1|r_j|\sum w_jU_j$.

$$\text{minimize}$$

$$\sum_{j|J_j\in\mathcal{J}} w_jU_j \tag{1}$$

$$\text{subject to}$$

$$\sum_{k|J_k\in\mathcal{G}_j} x_k = 1 - U_j \quad \forall j|J_j \in \mathcal{J} \tag{2}$$

$$t_k \geq \hat{r}_k \quad \forall k|J_k \in \widetilde{\mathcal{J}} \tag{3}$$

$$t_{k-1} - t_k - p_k x_k \geq 0 \quad \forall k|k > 1, J_k \in \widetilde{\mathcal{J}} \tag{4}$$

$$t_k + p_k x_k - M_k(1 - x_k) \leq \hat{d}_k \quad \forall k|J_k \in \widetilde{\mathcal{J}} \tag{5}$$

$$U_j \in \{0,1\} \quad \forall j|J_j \in \mathcal{J} \tag{6}$$

$$x_k \in \{0,1\} \quad \forall k|J_k \in \widetilde{\mathcal{J}} \tag{7}$$

$$t_k \geq 0 \quad \forall k|J_k \in \widetilde{\mathcal{J}} \tag{8}$$

Objective function (1) minimizes the weighted number of tardy jobs. Constraints (2) ensure that exactly one job occurrence is selected for an on-time job while no job occurrence should be selected if the job is tardy. Constraints (3) ensure that no job starts before its release date while constraints (4) ensure that jobs do not overlap. Finally, constraints (5) force each job to finish before its deadline. A suitable value for constant M_k is given by $\max(0, \max_{l>k}\{d_l - d_k\})$.

3. Robust $1|r_j|\sum w_jU_j$

In this section, we introduce a problem where the weighted number of tardy jobs has to be minimized and where the weight associated with the execution of a task is subject to uncertainty. In $1|r_j|\sum w_jU_j$, if a job is not processed on time, it can be arbitrarily delayed. So, deciding that a job will be processed late can be seen as the decision not to accept the order at all. Thus the cost associated with the late job can be seen as a loss of income.

In the two-stage robust version of $1|r_j|\sum w_jU_j$ that we are considering, we assume that the precise weights w_j are known only after deciding which jobs will be on time, and before actually processing them. Such a hypothesis holds for example when orders are accepted at a tactical decision level without precise knowledge of the future technical difficulties that may arise from specific jobs, incurring extra production costs.

Our approach is also a conservative approximation of multi-stage decision processes, where the actual production costs would be revealed along time. To the best of our knowledge, uncertain multi-stage integer problems are far out of reach of existing optimization methodologies (see for example [28] or [29] for algorithms dedicated to multi-stage continuous optimization problems), so that such an approximation can still be useful.

3.1. Problem description

We now consider that the executed jobs may fail in such a way that the output of the task is deteriorated, leading to an additional cost. When a deterioration is detected, the decision maker is allowed to take recourse decisions of three types: (1) accept the output of the job as it is, thus incurring a penalty for producing faulty goods; (2) spend more time on the job to perform it correctly or repair what has been damaged and thus avoid the additional cost; or (3) outsource the execution of jobs. Note that it is possible to outsource any job J_j, even if its weight w_j has not been modified, since it gives additional time for other jobs to be executed or repaired.

We assume that the maximum possible penalty of each job $J_j \in \mathcal{J}$ is known, and denoted by $\bar{\delta}_j$. The penalty to pay for a given task $J_j \in \mathcal{J}$ is computed as $\bar{\delta}_j\xi_j$, where ξ_j is the (uncertain) ratio of the penalty

incurred by J_j. Following the robust optimization framework, vector $\xi \in \mathbb{R}^{|\mathcal{J}|}$ belongs to the *uncertainty set* $\Xi \subset \mathbb{R}^{|\mathcal{J}|}$. In this paper, we consider the *budgeted uncertainty set*

$$\Xi = \left\{ \xi \in \mathbb{R}_+^{|\mathcal{J}|} \middle| \xi_j \leq 1, \forall J_j \in \mathcal{J} \text{ and } \sum_{j|J_j \in \mathcal{J}} \xi_j \leq \Gamma \right\}$$

where Γ is referred to as the uncertainty budget or uncertainty parameter. This uncertainty set was introduced in [30] and is conservative in the following sense: if Γ increases, the size of Ξ increases. A given vector $\xi \in \Xi$ can be interpreted as a job failure scenario. In any scenario, at most Γ jobs can incur their maximum penalty, but more of them can be partially impacted since vector x_i does not have to take integer values, and the worst-case scenario is generally not an extreme point of Ξ in the two-stage robust optimization context. Remark that when $\Gamma \geq |\mathcal{J}|$, the uncertainty set embeds all possible job failure scenarios.

The decision flow goes as follows: in a here-and-now phase (or first stage), the decision maker decides on a set of jobs to be executed on time, then, as the jobs fail, the decision maker is allowed, in a wait-and-see phase (or second stage), to take recourse actions. Note that the optimal solution may be to decide to execute more tasks on time than what is feasible and to finally outsource some of these jobs so as to restore the feasibility of the final schedule.

The problem can now be enunciated formally as follows. We first describe the (second-stage) repairing problem.

PROBLEM : REPAIRING PROBLEM (DECISION)
INPUT DATA : $(V, \mathcal{J}, (r, d, w, p, \bar{\delta}, \tau, f), A, \xi)$, where $V \in \mathbb{R}$ is a target value, $\mathcal{J}$, a set of jobs characterized by data (r, d, w, p), and for each job j, a maximum additional cost $\bar{\delta}_j$ if j fails, a fixed extra time τ_j needed to repair j, a fixed cost f_j for outsourcing j, a set of initially on-time jobs A, and $\bar{\xi}$ is a failure scenario.
QUESTION : Is there a partition (B, C, D) of A and a permutation σ of $B \cup C$, where B is the set of jobs to be scheduled without modification, C is the set of jobs to be fixed and scheduled, and D is the set of jobs to be outsourced, such that all jobs of $B \cup C$ are on time and $\sum_{j \in \mathcal{J} \setminus A} w_j + \sum_{j \in B} \bar{\delta}_j \xi_j + \sum_{j \in D} f_j \leq V$?

Using this definition, one can enunciate the two-stage problem formally.

PROBLEM : ROBUST $1|r_j|\sum w_j U_j$ (DECISION)
INPUT DATA : $(V, \mathcal{J}, (r, d, w, p, \bar{\delta}, \tau, f), \Xi)$, where $V \in \mathbb{R}$ is a target value, $\mathcal{J}$, a set of jobs characterized by data (r, d, w, p), and for each job j, a maximum additional cost $\bar{\delta}_j$ if j fails, a fixed extra time τ_j needed to fix j, a fixed cost f_j for outsourcing j, and Ξ is an uncertainty set.
QUESTION : Is there a subset $A \subseteq \mathcal{J}$ of on-time jobs such that for any scenario $\xi \in \Xi$, REPAIRING PROBLEM with data $(V, \mathcal{J}, (r, d, w, p, \bar{\delta}, \tau, f), A, \xi)$ has the answer yes?

Our scheduling problem consists in seeking the minimum value of V such that the decision problem has the answer yes. This optimization problem will be further referred to as problem $(\mathcal{P})$.

Note that the solution representation we use above does not directly provide a certificate for the decision problem whose size is polynomial in the size of the input data, and whose validity can be checked in polynomial time, since one would need an algorithm to verify that for any scenario, there are recourse actions that lead to a cost that is less than value V.

When $\bar{\delta}_j = 0$ for any job J_j, the problem becomes the classical $1|r_j|\sum w_j U_j$ and is therefore $\mathcal{NP}$-hard.

3.2. Formulation

In this section, we show that model (1)-(8) can be extended to model problem $(\mathcal{P})$. To do so, we use an approach similar to [26], as outlined in section 2. Its validity is based on dominance rule 1, so we need to ensure that it still holds for the robust case.

We create a set of job occurrences $\widetilde{\mathcal{J}}$ from the original set of jobs $\mathcal{J}$ in order to turn the problem into a job occurrence selection problem with agreeable time windows. Formally, consider a job $J_i \in \mathcal{J}$, for any job $J_j \in \mathcal{J}$ whose time window is not agreeable with that of J_i (i.e., $r_i < r_j$, $d_i > d_j$, and $r_i + p_i + p_j \leq d_j$), we create a job occurrence $J_k \in \widetilde{\mathcal{J}}$ such that $r_k = r_i, p_k = p_i, w_k = 0, f_k = f_i, \bar{\delta}_k = \bar{\delta}_j, \tau_k = \tau_j$ and $d_k = d_j$. Again, the original job J_i is also added to $\widetilde{\mathcal{J}}$ and we introduce the set $\mathcal{G}_j$ as the set of job occurrences related to a given job J_j.

We can now extend the dominance rule 1 in the following sense:

Dominance rule 2. *There is at least one optimal solution $((B^*, C^*, D^*), \sigma^*)$ for REPAIRING PROBLEM such that jobs of $B^* \cup C^*$ are scheduled according to a non-decreasing order of their deadlines with ties being broken in non-decreasing order of release dates.*

Proof. Let $((B, C, D), \sigma)$ be a feasible solution for REPAIRING PROBLEM, and consider two jobs J_i and J_j such that $J_i, J_j \in B \cup C$. Assume moreover that J_i is before J_j in σ.

If $d_i < d_j$ (or $d_i = d_j$ and $r_i < r_j$), then J_i and J_j are already scheduled in the desired order. Otherwise, for any job J_ℓ, let $\hat{p}_\ell = p_\ell$ if $J_\ell \in B$ and $\hat{p}_\ell = p_\ell + \tau_\ell$ if $J_\ell \in C$. Note that since (B, C, D) is feasible, it holds that

$$r_i + \hat{p}_i + \hat{p}_j \leq d_j. \tag{9}$$

Two cases remain:

- $d_i > d_j$ and $r_i < r_j$: (9) implies that $r_i + p_i + p_j \leq d_j$. Therefore, there is a job occurrence $J_k \in \mathcal{G}_i$ such that $d_k = d_j$ and with which we can replace J_i. In doing so, we end up in the desired order and the objective value remains unchanged.
- $d_i \geq d_j$ and $r_i \geq r_j$: swapping J_i and J_j leads to the desired order, without modifying the cost of the solution. Since $r_j \leq r_i$ and $d_i \geq d_j$, it holds from (9) that $r_j + \hat{p}_j + \hat{p}_i \leq d_i$, which shows that the solution remains feasible. □

In the exact same way as in [26] and as outlined in section 2, we sort the job occurrences according to a non-decreasing order of their deadlines, and we assign reversed time windows to each job occurrence given by $[\hat{r}_j, \hat{d}_j] = [d_{\max} - d_j, d_{\max} - r_j]$.

We now propose a characterization of valid recourse decisions. Schedule-feasibility of a selection of jobs has been studied in the static case by [26] (described in section 2) and can be extended to our case.

For any $J_k \in \widetilde{\mathcal{J}}$, binary variable y_k takes value 1 if J_k is selected, 0 otherwise; variable z_k takes value 1 if J_k is repaired, 0 otherwise. For each occurrence $J_k \in \widetilde{\mathcal{J}}$, variable $\rho_k \in \mathbb{R}_+$ is equal to the processing time of the job (including possible repairing).

We define the set $\bar{\mathcal{Y}} \subset \{0,1\}^{2|\widetilde{\mathcal{J}}|} \times \mathbb{R}_+^{2|\widetilde{\mathcal{J}}|}$, which contains every feasible schedule, as follows.

$$\bar{\mathcal{Y}} = \begin{cases} \rho_k = p_k y_k + \tau_k z_k \quad \forall k | J_k \in \widetilde{\mathcal{J}} & (10) \\ z_k \leq y_k \quad \forall k | J_k \in \widetilde{\mathcal{J}} & (11) \\ \sum_{k | J_k \in \mathcal{G}_j} y_k \leq 1 \quad \forall J_j \in \mathcal{J} & (12) \\ t_k \geq \hat{r}_k \quad \forall k | J_k \in \widetilde{\mathcal{J}} & (13) \\ t_{k-1} - t_k - \rho_k \geq 0 \quad \forall k \neq 1 | J_k \in \widetilde{\mathcal{J}} & (14) \\ t_k + \rho_k - M_k(1 - y_k) \leq \hat{d}_k \quad \forall k | J_k \in \widetilde{\mathcal{J}} & (15) \\ t_k \geq 0 \quad \forall k | J_k \in \widetilde{\mathcal{J}} & (16) \\ y_k, z_k \in \{0,1\} \quad \forall k | J_k \in \widetilde{\mathcal{J}} & (17) \\ \rho_k \geq 0 \quad \forall k | J_k \in \widetilde{\mathcal{J}} & (18) \end{cases}$$

Constraints (10) ensure that each value ρ_k is equal to the processing time of J_k with respect to the recourse action. Constraints (11) ensure that a job may be fixed only if it is scheduled while constraints (12)-(17) are understood exactly as (1)-(8) where the constant processing times p have been substituted by

decision variables ρ. Recall that a job can be outsourced, which explains why constraints (12) are inferiority constraints whereas (2) are equality constraints.

Let us also denote the set of second-stage decisions that admit a feasible timing of the jobs as

$$\mathcal{Y} = \{(y, z) | \exists t, \rho : (y, z, t, \rho) \in \bar{\mathcal{Y}}\}.$$

To enforce the non-anticipation property, which stipulates that the decided recourse action may not contradict the first-stage decision, we add so-called linking constraints between the first-stage decisions and the second-stage decisions. These linking constraints are expressed as

$$\sum_{k|J_k \in \mathcal{G}_j} y_k \leq 1 - U_j \quad \forall J_j \in \mathcal{J} \tag{19}$$

We also introduce set $\mathcal{Y}(U)$, which contains every admissible recourse decision respecting both the non-anticipation and the schedule-feasibility property. It is therefore given by $\mathcal{Y}(U) = \{(y, z) \in \mathcal{Y} \mid (19)\}$.

We now turn to the objective value. Let $J_j \in \mathcal{J}$ be a job to be scheduled, then:

- if $U_j = 1$, then J_j is executed tardily and we have $\forall J_k \in \mathcal{G}_j, y_k = z_k = 0$ (i.e., no recourse action)
- if $U_j = 0$, then J_j is accepted in the first stage. In the second stage (i.e., once the uncertainty is revealed) the sequencing of the jobs has to be decided as well as the recourse actions. The following cases may arise:
  - there is $J_k \in \mathcal{G}_j$ such that $y_k = z_k = 1$: the job is executed and fixed
  - there is $J_k \in \mathcal{G}_j$ such that $y_k = 1$ and $z_k = 0$: the job is executed
  - for all $J_k \in \mathcal{G}_j, y_k = z_k = 0$: the job is outsourced.

The problem can finally be cast as:

$$(\mathcal{P}) : \min_{U \in \{0,1\}^{|\mathcal{J}|}} \sum_{j|J_j \in \mathcal{J}} w_j U_j + f_j(1 - U_j) + \max_{\xi \in \Xi} \min_{(y,z) \in \mathcal{Y}(U)} R(\xi, y, z)$$

where $R(\xi, y, z)$ denotes the cost of recourse action (y, z) corresponding to scenario ξ given by:

$$R(\xi, y, z) = \sum_{j|J_j \in \mathcal{J}} \sum_{k|J_k \in \mathcal{G}_j} \left[(\bar{\delta}_k \xi_j - f_k) y_k - \bar{\delta}_k \xi_j z_k \right].$$

Note that the outsourcing cost appears both in the first-stage and second-stage objective functions. This technical manipulation is required to preserve their linearity, and can be interpreted as always paying outsourcing, unless the job is actually scheduled in the second stage.

4. Solution approaches

In this section, we develop two solution approaches for solving this min-max-min problem based on two recent studies on robust optimization. First, we present a conservative approximation known as the K-adaptability, introduced in [9], which uses restrictive assumptions on the recourse set. Next, an exact approach is developed based on polyhedral results introduced in [7], which are then briefly outlined for the sake of completeness.

4.1. Finite adaptability

One of the most effective approaches to two-stage robust problems is the so-called finite adaptability introduced in [9], also referred to as K-adaptability. To obtain a tractable problem, the idea is to restrict the set of possible recourse actions. The derivation of an exact MILP model of the finite adaptability approximation for robust problems with recourse where the uncertainty is confined in the objective function has been summarized in [18] and is well established. By limiting the number of used recourses to K and denoting by (y^q, z^q), $q = 1, \ldots, K$, the q^{th} selected recourse solution, it serves to determine these K recourse

solutions at the first stage, keeping only the choice among these K solutions in the inner minimization problem. We get the following model:

$$\text{minimize}$$
$$\sum_{j|J_j\in\mathcal{J}} w_j U_j + f_j(1-U_j) + \max_{\xi\in\Xi} \min_{q=1,\dots,K} R(\xi, y^q, z^q) \tag{20}$$
$$\text{subject to}$$
$$(y^q, z^q) \in \mathcal{Y}(U) \quad q = 1,\dots,K \tag{21}$$
$$U \in \{0,1\}^{|\mathcal{J}|} \tag{22}$$

The reformulation process proposed in [18] involves writing the $max - min$ problem in (20) as a single-stage maximization linear program (LP), making use of an epigraph formulation of the inner finite minimum. Using LP duality, an equivalent minimization LP model is derived to express the cost of the second-stage $max - min$ sub-problem. Integrated into the first-stage model, this yields a bilinear model which is further linerarized with the help of additional decision variables.

More precisely, let us focus on the inner maximization problem in (20) and employ an epigraph formulation of the finite minimum; the result is that
$\max_{\xi\in\Xi} \min_{q=1,\dots,K} R(\xi, y^q, z^q)$ is equivalent to the following problem:

$$\text{maximize } \theta \tag{23}$$
$$\text{subject to } \theta \le R(\xi, y^q, z^q) \quad q = 1,\dots,K \qquad (\beta_q \ge 0) \tag{24}$$
$$\sum_{j|J_j\in\mathcal{J}} \xi_j \le \Gamma \qquad (u \ge 0) \tag{25}$$
$$\xi_j \le 1 \quad \forall j|J_j \in \mathcal{J} \qquad (v_j \ge 0) \tag{26}$$
$$\xi_j \ge 0 \quad \forall j|J_j \in \mathcal{J} \tag{27}$$
$$\theta \in \mathbb{R} \tag{28}$$

where constraints (25)-(27) ensure that $\xi \in \Xi$. Observe that model (23)-(28) is a feasible and bounded linear program. We can then use the strong duality theorem in linear programming to obtain the following equivalent dual linear program, where β, u and v are the vectors of dual variables respectively associated with constraints (24), (25) and (26), of conforming dimensions, and $R(\xi, y^q, z^q)$ is replaced by its expression.

$$\text{minimize}$$
$$\Gamma u + \sum_{j|J_j\in\mathcal{J}} v_j - \sum_{j|J_j\in\mathcal{J}} \sum_{k|J_k\in\mathcal{G}_j} \sum_{q=1}^{K} f_k y_k^q \beta_q \tag{29}$$
$$\text{subject to}$$
$$\sum_{k|J_k\in\mathcal{G}_j} \sum_{q=1}^{K} \bar{\delta}_k \left(z_k^q - y_k^q\right) \beta_q + u + v_j \ge 0 \quad \forall j|J_j \in \mathcal{J} \qquad (\xi \ge 0) \tag{30}$$
$$\sum_{q=1}^{K} \beta_q = 1 \qquad (\theta \in \mathbb{R}) \tag{31}$$
$$\beta_q \ge 0, q = 1,\dots,K \tag{32}$$
$$v_j \ge 0, \forall j|J_j \in \mathcal{J} \tag{33}$$
$$u \ge 0 \tag{34}$$

This equivalent formulation contains a bilinear term in (y, z) and β which can be linearized using standard techniques and introducing auxiliary variables, such that $\psi_k^q = y_k^q \beta_q$ and $\zeta_k^q = z_k^q \beta_q$ for all $q = 1,\dots,K$ and $k|J_k \in \widetilde{\mathcal{J}}$. In doing so, we obtain the following MILP finite adaptability formulation:

minimize

$$\sum_{j|J_j\in\mathcal{J}} [w_j U_j + f_j(1-U_j) + v_j] + \Gamma u - \sum_{q=1}^{K} \sum_{k|J_k\in\widetilde{\mathcal{J}}} f_k \psi_k^q \tag{35}$$

subject to

$$t_k^q \geq \hat{r}_k \quad \forall k|J_k \in \widetilde{\mathcal{J}}, q = 1,\ldots,K \tag{36}$$

$$\rho_k^q = p_k y_k^q + \tau_k z_k^q \quad \forall k|J_k \in \widetilde{\mathcal{J}}, q = 1,\ldots,K \tag{37}$$

$$t_{k-1}^q - t_k^q - \rho_k^q \geq 0 \quad \forall k \neq 1|J_k \in \widetilde{\mathcal{J}}, q = 1,\ldots,K \tag{38}$$

$$t_k^q + \rho_k^q - M_k(1-y_k^q) \leq \hat{d}_k \quad \forall k|J_k \in \widetilde{\mathcal{J}}, q = 1,\ldots,K \tag{39}$$

$$z_k^q \leq y_k^q \quad \forall k|J_k \in \widetilde{\mathcal{J}}, q = 1,\ldots,K \tag{40}$$

$$\sum_{k|J_k\in\mathcal{G}_j} y_k^q \leq 1 - U_j \quad \forall j|J_j \in \mathcal{J}, q = 1,\ldots,K \tag{41}$$

$$\sum_{k|J_k\in\mathcal{G}_j} \sum_{q=1}^{K} \bar{\delta}_k \left(\zeta_k^q - \psi_k^q\right) + u + v_j \geq 0 \quad \forall j|J_j \in \mathcal{J} \tag{42}$$

$$\psi_k^q \leq y_k^q \quad \forall k|J_k \in \widetilde{\mathcal{J}}, q = 1,\ldots,K \tag{43}$$

$$\psi_k^q \leq \beta_q \quad \forall k|J_k \in \widetilde{\mathcal{J}}, q = 1,\ldots,K \tag{44}$$

$$\psi_k^q \geq \beta_q - 1 + y_k^q \quad \forall k|J_k \in \widetilde{\mathcal{J}}, q = 1,\ldots,K \tag{45}$$

$$\zeta_k^q \leq z_k^q \quad \forall k|J_k \in \widetilde{\mathcal{J}}, q = 1,\ldots,K \tag{46}$$

$$\zeta_k^q \leq \beta_q \quad \forall k|J_k \in \widetilde{\mathcal{J}}, q = 1,\ldots,K \tag{47}$$

$$\zeta_k^q \geq \beta_q - 1 + z_k^q \quad \forall k|J_k \in \widetilde{\mathcal{J}}, q = 1,\ldots,K \tag{48}$$

$(31) - (34)$

$$t_k^q \geq 0 \quad \forall k|J_k \in \widetilde{\mathcal{J}}, q = 1,\ldots,K \tag{49}$$

$$U_j \in \{0,1\} \quad \forall j|J_j \in \mathcal{J} \tag{50}$$

$$y_k^q \in \{0,1\} \quad \forall k|J_k \in \widetilde{\mathcal{J}}, q = 1,\ldots,K \tag{51}$$

$$z_k^q \in \{0,1\} \quad \forall k|J_k \in \widetilde{\mathcal{J}}, q = 1,\ldots,K \tag{52}$$

$$\psi_k^q \geq 0 \quad \forall k|J_k \in \widetilde{\mathcal{J}}, q = 1,\ldots,K \tag{53}$$

$$\zeta_k^q \geq 0 \quad \forall k|J_k \in \widetilde{\mathcal{J}}, q = 1,\ldots,K \tag{54}$$

Here, equation (35) defines the objective function to be minimized. Constraints (36)-(39) correspond to scheduling constraints (10),(13)-(15) for each $q = 1,\ldots,K$, derived from [26], ensuring that the final decision must yield a feasible schedule. Constraints (40) make sure that a job can be fixed only if it is scheduled, while constraints (41) ensure that one may only process a job which was chosen to be on time in the first stage. Constraint s(42) corresponds to the dualized cost corresponding to the uncertain event, obtained from (30) through linearization of the bilinear terms. Finally, constraint (31) comes from the dual of the epigraph formulation and constraints (43)-(48) correspond to the linearization of $y_k^q\beta_q$ and $z_k^q\beta_q$.

This problem will be referred to as problem $(\mathcal{P}_K)$ and its model as model KAdapt1.

4.2. Convexification of the recourse set

In [7], the authors introduce an exact single-stage MILP formulation for two-stage robust problems with mixed recourse decisions and binary variables in linking constraints between the first and second stages. In problem $(\mathcal{P})$, the inner $max - min$ problem involves continuous decision variables for the max part, and mixed-binary decision variables for the min part. Thanks to the special structure of the linking constraints (19), one can use Proposition 1 described below and employ the following reformulation process. First, replace the feasible space of the inner min sub-problem with its convex hull **(step 1)**. It is then possible to swap the max and min operators **(step 2)**. This second step leads to a static robust MILP model. The

third step applies the classical LP duality-based reformulation [30] to obtain a single-stage deterministic model **(step 3)**. To write a proper MILP model, the final step expresses $\mathcal{Y}(U)$ in terms of its extreme points **(step 4)**. This step implies an exponential growth of the model, which, at solution time, is taken care of with the help of a column generation algorithm.

We first detail the reformulation process applied to problem $(\mathcal{P})$, and then show how the large-scale MILP model obtained can be solved.

4.2.1. Reformulation

To perform **step 1**, observe that the recourse cost function R is affine in (y, z). It follows that minimizing R over $\mathcal{Y}(U)$ and $\text{conv}(\mathcal{Y}(U))$ is equivalent. Problem $(\mathcal{P})$ is then equivalent to:

$$\min_{U\in\{0,1\}^{|\mathcal{J}|}} \sum_{j|J_j\in\mathcal{J}} [w_j U_j + f_j(1-U_j)] + \max_{\xi\in\Xi} \min_{(y,z)\in\text{conv}(\mathcal{Y}(U))} R(\xi, y, z)$$

Step 2: since function R is affine in both (y, z) and ξ, it is convex in (y, z) and concave in ξ. Moreover, thanks to step 1, both *max* and inner *min* operators are performed over compact convex sets, so that we can use the well-known minimax theorem [31] to swap them. Grouping both *min* operators yields:

$$\min_{\substack{U\in\{0,1\}^{|\mathcal{J}|}\\(y,z)\in\text{conv}(\mathcal{Y}(U))}} \left[\sum_{j|J_j\in\mathcal{J}} [w_j U_j + f_j(1-U_j)] \; + \max \left\{ R(\xi, y, z) : \begin{array}{ll} \sum\limits_{j|J_j\in\mathcal{J}} \xi_j \le \Gamma & (u \ge 0) \\ \xi_j \le 1, \forall j|J_j \in \mathcal{J} & (v_j \ge 0) \\ \xi_j \ge 0, \forall j|J_j \in \mathcal{J} & \end{array} \right\} \right]$$

Step 3 relies on the fact that the inner maximization problem is a feasible and bounded LP. Using the strong LP duality theorem, one can replace it with its dual problem to get the following formulation, where u and v are dual variables associated with the constraints imposing $\xi \in \Xi$:

$$\begin{aligned}
&\text{minimize} \\
&\sum_{J_j\in\mathcal{J}} [w_j U_j + f_j(1-U_j) + v_j] + \Gamma u - \sum_{k|J_k\in\widetilde{\mathcal{J}}} f_k y_k \\
&\text{subject to} \\
&(y, z) \in \text{conv}(\mathcal{Y}(U)) \qquad (55) \\
&u + v_j \ge \sum_{k|J_k\in\mathcal{G}_j} \bar{\delta}_k (y_k - z_k) \quad \forall j|J_j \in \mathcal{J} \\
&U_j \in \{0,1\} \quad \forall j|J_j \in \mathcal{J} \\
&y_k, z_k \ge 0 \quad \forall k|J_k \in \widetilde{\mathcal{J}} \\
&v_j \ge 0 \quad \forall j|J_j \in \mathcal{J} \\
&u \ge 0
\end{aligned}$$

This model is linear except for constraint (55). In order to write a linear system for these conditions, **step 4** alleviates a key obstacle: considering a fixed vector $\bar{U}$, it is easy to express the set $\text{conv}(\mathcal{Y}(\bar{U}))$ in terms of the extreme points of $\mathcal{Y}(\bar{U})$ since it is a bounded mixed-integer set. However, the set of extreme points to consider depends on the value of $\bar{U}$. In a general setting, this naturally leads to a disjunctive formulation whose numerical solution seems to be very challenging (the reader may refer to [7] for details about this technical difficulty and approaches to cope with it). Problem $(\mathcal{P})$ enjoys a convenient structure that allows us to use the convex hull of $\mathcal{Y}$ instead of $\mathcal{Y}(U)$, and impose the restrictions over y, z and U independently. This means that a single set of extreme points, independent of U, can be considered in the model. To this end, we use the following key result:

Proposition 1 (Arslan and Detienne [7]).
Consider the following two-stage robust mixed-integer linear problem with objective uncertainty:

$$\min_{x\in\mathcal{X}} \left\{ c^T x + \max_{\xi\in\Xi} \min_{y\in\tilde{\mathcal{Y}}(x)} \xi^T Q x \right\}$$

where $\mathcal{X} \subset \{0,1\}^N \times \mathbb{R}^M$ *denotes the set of feasible first-stage decisions,* Ξ *represents the uncertainty polyhedron and* $\tilde{\mathcal{Y}}(x)$ *denotes the set of eligible second-stage decisions defined as* $\{y \in \mathcal{Y} | y_1 \leq x_1\}$ *with* $\mathcal{Y} \subset \{0,1\}^N \times \mathbb{R}^M$ *and* $y_1 \in \{0,1\}^N$, $x_1 \in \{0,1\}^N$. $\text{conv}(\tilde{\mathcal{Y}}(x)) = \text{conv}(\mathcal{Y}) \cap \{y | y_1 \leq x_1\}$ *holds for any* $x \in \mathcal{X}$.

From which we easily derive the following corollary:

Corollary 1.

$$\text{conv}(\mathcal{Y}(U)) = \text{conv}(\mathcal{Y}) \cap \left\{ \begin{array}{l} y \in \mathbb{R}^{|\tilde{\mathcal{J}}|} \\ z \in \mathbb{R}^{|\tilde{\mathcal{J}}|} \\ t \in \mathbb{R}_+^{|\tilde{\mathcal{J}}|} \end{array} \middle| \sum_{k|J_k \in \mathcal{G}_j} y_k \leq 1 - U_j, \forall j | J_j \in \mathcal{J} \right\}$$

Let us denote the set of extreme points of $\mathcal{Y}$ by $(\mathbf{y}^e, \mathbf{z}^e), e \in E$ (E being a list for their indices). Problem $(\mathcal{P})$ is finally modeled by this deterministic equivalent program:

$$[DEP]: \text{ minimize } F(U,u,v,\alpha) = \sum_{j|J_j \in \mathcal{J}} [w_j U_j + f_j(1-U_j) + v_j] + \Gamma u - \sum_{k|J_k \in \tilde{\mathcal{J}}} \left[f_k \sum_{e \in E^R} \mathbf{y}_k^e \alpha_e \right] \tag{56}$$

subject to

$$\sum_{e \in E} \alpha_e = 1 \tag{57}$$

$$\sum_{k|J_k \in \mathcal{G}_j} \sum_{e \in E} \mathbf{y}_k^e \alpha_e \leq 1 - U_j \quad \forall j | J_j \in \mathcal{J} \tag{58}$$

$$u + v_j \geq \sum_{k|J_k \in \mathcal{G}_j} \left[\bar{\delta}_k \sum_{e \in E} (\mathbf{y}_k^e - \mathbf{z}_k^e) \alpha_e \right] \quad \forall j | J_j \in \mathcal{J} \tag{59}$$

$$U_j \in \{0,1\} \quad \forall j | J_j \in \mathcal{J}$$

$$\alpha_e \geq 0 \quad \forall e \in E$$

$$u \geq 0$$

$$v_j \geq 0 \quad \forall j | J_j \in \mathcal{J}$$

Here, decision vector α represents the convex combination multipliers from the reformulation of $\text{conv}(\mathcal{Y})$. Again, u and v are the dual variables associated with the constraint $\xi \in \Xi$. Constraints (58) link the recourse action with the first-stage decision. Constraint (57) ensures that the recourse actions are convex combinations of the extreme points of $\text{conv}(\mathcal{Y})$. Finally, constraints (59) embed the dualized cost associated with the worst-case scenario. This model will be referred to as ColGen1.

Unlike the typical application of Dantzig-Wolfe decomposition, there are no integrality requirements over the reformulated variables (here, the second-stage variables y and z). This stems from the nature of $(\mathcal{P})$, which involves integer variables in the second stage, but is mathematically equivalent to a problem where the second-stage variables belong to the convex hull of the second-stage feasibility set. Note that, in optimal solutions of $[DEP]$, second-stage variables are non-integer most of the time. Intuitively, several different second-stage solutions are required to prevent the adversary that maximizes the cost of the solution from increasing the value of the first-stage solution by moving the uncertain parameters slightly.

Problem $(\mathcal{P})$ is trivially $\mathcal{NP}$-hard, since considering null penalties yields a problem equivalent to the deterministic $1|r_i|\sum w_i U_i$. However, it is often unclear whether two-stage robust problems lie higher in the polynomial hierarchy or not (see [32] for example). As a by-product, this reformulation shows that problem $(\mathcal{P})$ is not harder than $\mathcal{NP}$-complete problems (the result is proven in a more general setting in [7]).

Corollary 2. *Problem* $(\mathcal{P})$ *is* $\mathcal{NP}$*-complete.*

4.2.2. Column generation-based solution algorithm

Model $[DEP]$ has an exponential number of variables. A classical approach to solve such problems is to use the column generation algorithm to compute its linear relaxation. In this section, we formally present the master program and the pricing problem that has to be solved for this purpose. We then describe the column generation procedure. Finally, we depict the so-called branch-and-price algorithm, which is a tree search embedding the column generation routine to compute the optimal feasible solution of $[DEP]$.

The column generation procedure solves the linear relaxation of model $[DEP]$. Its basic idea is to consider only a subset E^R of the α-variables and to optimally solve the so-called restricted master program (RMP) $[DEP]^R$, using for example the simplex algorithm. The linear relaxation of RMP can be stated as follows (constraints $U_j \leq 1$ are dropped since they are implied by constraints (62)).

$$[DEP]^R: \text{ minimize } F^R(U,u,v,\alpha) = \sum_{j|J_j\in\mathcal{J}} [w_jU_j + f_j(1-U_j) + v_j] + \Gamma u - \sum_{k|J_k\in\widetilde{\mathcal{J}}} \left[f_k \sum_{e\in E^R} \mathbf{y}_k^e \alpha_e \right] \tag{60}$$

subject to

$$\sum_{e\in E^R} \alpha_e = 1 \tag{61}$$

$$\sum_{e\in E^R} \mathbf{y}_k^e \alpha_e + U_j \leq 1 \quad \forall k|J_k \in \mathcal{G}_j, \forall j|J_j \in \mathcal{J} \tag{62}$$

$$u + v_j \geq \sum_{k|J_k\in\mathcal{G}_j} \left[\bar{\delta}_k \sum_{e\in E^R} (\mathbf{y}_k^e - \mathbf{z}_k^e)\, \alpha_e \right] \quad \forall j|J_j \in \mathcal{J} \tag{63}$$

$$\alpha_e \geq 0 \quad \forall e \in E^R \tag{64}$$

$$U_j \geq 0, v_j \geq 0 \quad \forall j|J_j \in \mathcal{J} \tag{65}$$

$$u \geq 0 \tag{66}$$

Basic LP theory tells us that the solution obtained is optimal for the linear relaxation of $[DEP]$ if the reduced costs of all the α-variables are non-negative. Let λ, μ and π be the dual variables respectively associated with constraints (61), (62) and (63). Given an optimal dual solution $(\lambda^*, \mu^*, \pi^*)$ to $[DEP]^R$, the so-called pricing problem that seeks a minimum reduced cost α-variable can be cast as:

$$\begin{aligned}
&[Pricing(\lambda^*,\mu^*,\pi^*)]: \text{ minimize } G(\lambda^*,\mu^*,\pi^*,y,z) \\
&= -\lambda^* + \sum_{j|J_j\in\mathcal{J}} \sum_{k|J_k\in\mathcal{G}_j} \left[(-f_k - \mu_k^* + \bar{\delta}_k\pi_j^*)y_k - \bar{\delta}_k\pi_j^* z_k \right] \\
&\text{subject to } (y,z,t,\rho) \in \bar{\mathcal{Y}}
\end{aligned}$$

This problem can be interpreted as a variant of $1|r_i|\sum w_iU_i$ where each job comes in two possible modes (related with variables y or z) having different processing times and weights.

When the optimal solution $(U^*, u^*, v^*, \alpha^*)$ of the linear relaxation of $[DEP]$ satisfies the integrality requirements (*i.e.* $U^* \in \{0,1\}^{|\mathcal{J}|}$), then it provides an optimal first-stage solution for $(\mathcal{P})$. Otherwise, one has to branch in order to exclude the current fractional solution and explore the feasibility set. Algorithm 1 summarizes the branch-and-price procedure proposed to solve problem $(\mathcal{P})$ through its formulation $[DEP]$. Line 1 initializes the set of columns so that the restricted master problem is feasible. The best primal bound found, $PrimalBound$, and the best feasible solution found, S^*, are initialized in Line 2. Each node is encoded as the set of branching constraints, $\mathcal{B}$, defining the set of solutions of that node. The list of open nodes, $\mathcal{Q}$, is thus initialized in Line 2 with the root node, which has no branching constraints. Loop 3-14 processes the open nodes. The solution of the relaxation at the current node is computed in Line 5. If the solution satisfies the integrality requirements (Line 10), $PrimalBound$ and S^* are updated (line 11). When U^* is not an integer, branching is performed in Lines 13 and 14.

Algorithm 2 depicts the column generation procedure used to compute the relaxation at each node of the search tree in Line 5 of Algorithm 1. Loop 1-8 adds new columns to the restricted master $[DEP]^R$ until no negative reduced cost column is found. Model $[DEP]^R$ is solved in Line 2, providing optimal dual variables

Algorithm 1: Branch-and-price algorithm for solving model $[DEP]$.

1 *Initialize the set of columns so that* $[DEP]^R$ *is feasible:* $(\bar{\mathbf{y}}^1, \bar{\mathbf{z}}^1) \leftarrow \{(\mathbf{0}, \mathbf{0})\}$, $E^R \leftarrow \{1\}$
2 $PrimalBound \leftarrow \infty$, $S^* \leftarrow \emptyset$, $\mathcal{Q} \leftarrow \{\emptyset\}$
3 **while** $\mathcal{Q} \neq \emptyset$ **do**
4 Pop a node/set of branching constraints $\mathcal{B}$ from $\mathcal{Q}$
5 $(U^*, u^*, v^*, \alpha^*) \leftarrow optimizeRelaxation(\mathcal{B}, E^R)$
6 $DualBound \leftarrow F(U^*, u^*, v^*, \alpha^*)$
7 **if** $DualBound \geq PrimalBound$ **then**
8 *Current node is pruned by bound*
9 **else**
10 **if** $U^* \in \{0,1\}^{|J|}$ **then**
11 Update $PrimalBound$ and S^* with $DualBound$ and $(U^*, u^*, v^*, \alpha^*)$
12 **else**
13 Choose $i \in \{1, \ldots, |J|\}$ such that $U_i^* \in]0,1[$
14 Add two nodes $\mathcal{B}_0 = \mathcal{B} \cup \{U_i = 0\}$ and $\mathcal{B}_1 = \mathcal{B} \cup \{U_i = 1\}$ to $\mathcal{Q}$
15 **return** S^*, an optimal solution of $[DEP]$

that are used as inputs to the pricing problem in Line 4. Lines 6-7 add a new column to $[DEP]^R$ if the pricing problem returns a column with a negative reduced cost.

Algorithm 2: $optimizeRelaxation(\mathcal{B}, E^R)$: column generation algorithm for computing the dual bound at each node of the search tree when solving $[DEP]$.

Input: $\mathcal{B}$: set of branching constraints, E^R: set of indices of columns
1 **repeat**
2 Solve $[DEP]^R$ with additional branching constraints $\mathcal{B}$
3 Let $(U^*, u^*, v^*, \alpha^*)$ be the optimal solution and λ^*, μ^* and π^* be the optimal dual values associated with constraints (61), (62) and (63)
4 Solve $[Pricing(\lambda^*, \mu^*, \pi^*)]$
5 Let (y^*, z^*, t^*, ρ^*) be the optimal solution
6 **if** $G(\lambda^*, \mu^*, \pi^*, y, z) < 0$ **then**
7 $E^R \leftarrow E^R \cup \{|E^R| + 1\}$, $(\bar{\mathbf{y}}, \bar{\mathbf{z}})^{|E^R|} \leftarrow (y^*, z^*)$
8 **until** $G(\lambda^*, \mu^*, \pi^*, y, z) \geq 0$
9 **return** $(U^*, u^*, v^*, \alpha^*)$

5. An anchored variant of the problem

In this section, we study an anchored variant of problem $(\mathcal{P})$, denoted $(\widetilde{\mathcal{P}})$, where the first-stage decisions include not only the selection of jobs to process, but also their order. In a wait-and-see phase, the recourse actions to be decided for each accepted job are: process the job (possibly with a decreased cost), outsource the job, or repair the job. In problem $(\mathcal{P})$, one can decide on the actual processing order of the jobs after their true weight is known, while in $(\widetilde{\mathcal{P}})$ the sequence of accepted jobs is decided at the first stage, and can only be amended by removing some elements from the sequence in the second stage.

We first formalize this variant, characterize its relation with problem $(\mathcal{P})$ and derive MILP formulations. We finally show that, unlike problem $(\mathcal{P})$, it is possible to recast problem $(\widetilde{\mathcal{P}})$ using a more precise uncertainty set, affecting the job occurrences instead of the jobs.

The same two solution approaches, namely finite adaptability and convexification, can be applied. Since their application to this variant uses the same mathematical results as the first problem, we report them in Appendix A.

5.1. Formulation

We formally state problem $(\widetilde{\mathcal{P}})$. Similarly to problem $(\mathcal{P})$, we first define the recourse problem.

PROBLEM : ANCHORED REPAIRING PROBLEM (DECISION)
INPUT DATA : $(V, \mathcal{J}, (r, d, w, p, \bar{\delta}, \tau, f), A, \xi, \sigma)$, where $V \in \mathbb{R}$ is a target value, $\mathcal{J}$, a set of jobs characterized by data (r, d, w, p), and for each job j, a maximum additional cost $\bar{\delta}_j$ if j fails, a fixed extra time τ_j needed to repair j, a fixed cost f_j for outsourcing j, a set of initially on-time jobs A, $\bar{\xi}$ a failure scenario, and σ a permutation of the elements of A.
QUESTION : Is there a partition (B, C, D) of A, where B is the set of jobs to be scheduled without modification, C is the set of jobs to be fixed and scheduled, and D is the set of jobs to be outsourced, σ restricted to $B \cup C$ is feasible, and
$\sum_{j \in \mathcal{J} \setminus A} w_j + \sum_{j \in B} \bar{\delta}_j \xi_j + \sum_{j \in D} f_j \leq V$?

The anchored problem can now be defined formally.

PROBLEM : ANCHORED ROBUST $1|r_j|\sum w_j U_j$ (DECISION)
INPUT DATA : $(V, \mathcal{J}, (r, d, w, p, \bar{\delta}, \tau, f), \Xi)$, where $V \in \mathbb{R}$ is a target value, $\mathcal{J}$, a set of jobs characterized by data (r, d, w, p), and for each job j, a maximum additional cost $\bar{\delta}_j$ if j fails, a fixed extra time τ_j needed to fix j, a fixed cost f_j for outsourcing j, and Ξ is an uncertainty set.
QUESTION :
Is there a subset $A \subseteq \mathcal{J}$ of on-time jobs, and a permutation σ of the elements of A such that for any scenario $\xi \in \Xi$, REPAIRING PROBLEM with data $(V, \mathcal{J}, (r, d, w, p, \bar{\delta}, \tau, f), A, \xi, \sigma)$ has the answer yes?

Problem $(\widetilde{\mathcal{P}})$ can be formulated in a similar way to what has been done for problem $(\mathcal{P})$. Just like in section 2, let us denote, for any job occurrence $J_k \in \widetilde{\mathcal{J}}$, by x_k the selection of the kth job occurrence in the non-decreasing order of their deadlines (i.e., 1 if the kth job occurrence is used, 0 otherwise). Variables y_k, z_k for job occurrences and U_j will keep the same meaning as in the previous section.

Again, regarding the set of admissible recourses, the schedule-feasibility property is dealt with by set $\mathcal{Y}$ introduced in section 3. Concerning the non-anticipativity property, which stipulates that the recourse action should not contradict a first-stage decision, we impose that $y_k \leq x_k, \forall k | J_k \in \widetilde{\mathcal{J}}$. That is, that one may confirm the execution of a job, or fix a job, only if it were actually accepted in the first stage. The set of admissible recourses is then given by the following set: $\widetilde{\mathcal{Y}}(x) = \{(y, z) \in \mathcal{Y} \mid y_k \leq x_k \quad \forall k | J_k \in \widetilde{\mathcal{J}}\}$.

We now detail the different possible decisions. Let $J_j \in \mathcal{J}$ be a job:

- if $U_j = 1$, then the job is executed tardily and no recourse action may be taken (i.e., $y_k = z_k = 0$)
- if there is k such that $J_k \in \mathcal{G}_j$ and $x_k = 1$, the job is to be scheduled on time in the first stage:
  - if $y_k = z_k = 1$, job J_j is executed and fixed on time
  - if $y_k = 1$ and $z_k = 0$, job J_j is executed on time and a penalty is paid
  - if $y_k = z_k = 0$, job J_j is outsourced

The objective function can therefore be expressed as follow:

$$\min_{(x,U) \in \widetilde{\mathcal{X}}} \sum_{j | J_j \in \mathcal{J}} [w_j U_j + f_j (1 - U_j)] + \max_{\xi \in \Xi} \min_{(y,z) \in \widetilde{\mathcal{Y}}(x)} R(\xi, y, z)$$

where $\widetilde{\mathcal{X}}$ denotes the set of feasible first-stage solutions (i.e., the set of solutions which define a sequence of tasks), that is:

$$\widetilde{\mathcal{X}} = \left\{ (x, U) \in \{0,1\}^{|\widetilde{\mathcal{J}}|} \times \{0,1\}^{|\mathcal{J}|} \;\middle|\; \sum_{k | J_k \in \mathcal{G}_j} x_k + U_j = 1 \quad \forall j | J_j \in \mathcal{J} \right\}$$

Again, note that the first-stage decision does not have to be physically feasible in that the optimal solution may be to decide a sequence of jobs in the first stage and to outsource some of them in the second stage so as to make the schedule feasible.

5.2. Relation with problem $(\mathcal{P})$

This section is devoted to showing that the first problem yields a lower bound for the second problem, which is an intuitive result: compared to $(\widetilde{\mathcal{P}})$, in $(\mathcal{P})$ some decisions are postponed. That means that, for different realizations of the uncertainty, those decisions can be different in $(\mathcal{P})$ but must be identical in $(\widetilde{\mathcal{P}})$. In that sense, $(\mathcal{P})$ relaxes some of the non-anticipativity constraints of $(\widetilde{\mathcal{P}})$. The following proposition provides formal proof of this observation.

Proposition 2. *Denoting by* $(\bullet)^*$ *the optimal value of problem* $\bullet$*, the following relation holds:*

$$(\mathcal{P})^* \leq (\widetilde{\mathcal{P}})^*$$

Proof. Let $(x,U) \in \widetilde{\mathcal{X}}$. We first formally show that $\widetilde{\mathcal{Y}}(x) \subseteq \mathcal{Y}(U)$, *i.e.* the set of recourse actions available when selecting specific occurrences is a part of the possible decisions that can be taken when only choosing the corresponding jobs. Let $(y,z) \in \widetilde{\mathcal{Y}}(x)$. Then by definition of $\widetilde{\mathcal{Y}}(x)$, $y_k \leq x_k, \forall k | J_k \in \tilde{J}$. Summing up over the occurrences of a given job, we get $\sum_{k|J_k \in G_j} y_k \leq \sum_{k|J_k \in G_j} x_k, \forall j | J_j \in J$. By definition of $\widetilde{\mathcal{X}}$, we obtain $\sum_{k|J_k \in G_j} y_k \leq 1 - U_j, \forall j | J_j \in J$. Thus, $(y,z) \in \mathcal{Y}(U)$.

It successively follows that:

$$\begin{aligned}
& \forall (x,U) \in \widetilde{\mathcal{X}}, \xi \in \Xi, \min_{(y,z)\in\mathcal{Y}(U)} R(\xi,y,z) \leq \min_{(y,z)\in\widetilde{\mathcal{Y}}(x)} R(\xi,y,z) \\
\Rightarrow \quad & \forall (x,U) \in \widetilde{\mathcal{X}}, \max_{\xi\in\Xi} \min_{(y,z)\in\mathcal{Y}(U)} R(\xi,y,z) \leq \max_{\xi\in\Xi} \min_{(y,z)\in\widetilde{\mathcal{Y}}(x)} R(\xi,y,z) \\
\Rightarrow \quad & \forall U \in \{0,1\}^{|J|}, \max_{\xi\in\Xi} \min_{(y,z)\in\mathcal{Y}(U)} R(\xi,y,z) \leq \min_{x:(x,U)\in\tilde{\mathcal{X}}} \max_{\xi\in\Xi} \min_{(y,z)\in\widetilde{\mathcal{Y}}(x)} R(\xi,y,z) \\
\Rightarrow \quad & \forall U \in \{0,1\}^{|J|}, \sum_{j|J_j\in\mathcal{J}} [w_j U_j + f_j(1-U_j)] + \max_{\xi\in\Xi} \min_{(y,z)\in\mathcal{Y}(U)} R(\xi,y,z) \\
& \qquad \leq \sum_{j|J_j\in\mathcal{J}} [w_j U_j + f_j(1-U_j)] + \min_{x:(x,U)\in\tilde{\mathcal{X}}} \max_{\xi\in\Xi} \min_{(y,z)\in\widetilde{\mathcal{Y}}(x)} R(\xi,y,z) \\
\Rightarrow \quad & \min_{U\in\{0,1\}^{|J|}} \sum_{j|J_j\in\mathcal{J}} [w_j U_j + f_j(1-U_j)] + \max_{\xi\in\Xi} \min_{(y,z)\in\mathcal{Y}(U)} R(\xi,y,z) \\
& \qquad \leq \min_{(x,U)\in\tilde{\mathcal{X}}} \sum_{j|J_j\in\mathcal{J}} [w_j U_j + f_j(1-U_j)] + \max_{\xi\in\Xi} \min_{(y,z)\in\widetilde{\mathcal{Y}}(x)} R(\xi,y,z) \\
\Leftrightarrow \quad & (\mathcal{P})^* \leq (\widetilde{\mathcal{P}})^*
\end{aligned}$$

□

We now provide an example showing that $(\widetilde{\mathcal{P}})$ is a strict relaxation of $(\widetilde{\mathcal{P}})$.

Proposition 3. *Given one problem instance, optimal solutions to* $(\mathcal{P})$ *may attain a strictly lower objective value than the optimal solutions to* $(\widetilde{\mathcal{P}})$*.*

Proof. Consider the following instance:

J_j	r_j	d_j	p_j	τ_j	w_j	$\bar{\delta}_j$	f_j
J_i	0	6	1	4	100	6	∞
J_j	5	8	2	2	100	4	∞
J_k	1	9	2	3	100	5	∞

, $\Gamma = 1$

where the outsourcing of a task is never considered (unaffordable) for simplicity. The three jobs can be scheduled on time and it is never optimal to execute a job tardily, even after the penalty is known (i.e., it is always better to pay the penalty than to execute the job tardily). That being said, it is clear that the optimal sequence is either (i, k, j) or (i, j, k). Since the uncertainty parameter Γ is set to one, exactly one job will be affected by the uncertainty (note that, because we are in a two-stage robust context, the uncertainty can be spread among different random parameters in the worst case, but this does not happen for this instance).

Let us first consider problem $(\widetilde{\mathcal{P}})$ where one decides on the sequencing of the jobs before the uncertainty is known. Figure 1 depicts the two solutions detailed below. If the decision, in the first stage, implies using sequence i, k, j, then it is only possible to fix J_k. Thus, the cost of the worst case is given by $\max(\bar{\delta}_i, \bar{\delta}_j, 0) = \max(6, 4, 0) = 6$. If one were to choose sequence i, j, k however, the only fixable task is J_i, which means that the worst-case scenario costs $\max(0, \bar{\delta}_j, \bar{\delta}_k) = \max(0, 4, 5) = 5$. The optimal solution to the overall problem minimizes the worst-case cost, hence $(\widetilde{\mathcal{P}})^* = 5$.

If we consider $(\mathcal{P})$ where one only selects on-time jobs in the first stage, however, one can react better to the uncertainty in the second stage. Indeed, if the uncertainty affects job J_j one is forced to pay $\bar{\delta}_j = 4$ since it is never possible to fix it. However, if the uncertainty affects job J_i, one can react to that scenario by choosing the sequence i, j, k under which job J_j can be fixed. If, on the contrary, the uncertainty affects job J_k, the optimal recourse decision is realized by using the sequence i, k, j under which one can fix job J_k. This shows easily that, for this problem, the worst case is realized when the uncertainty hits job J_j. In any event, $(\widetilde{\mathcal{P}})^* > (\mathcal{P})^*$ $(5 > 4)$. □

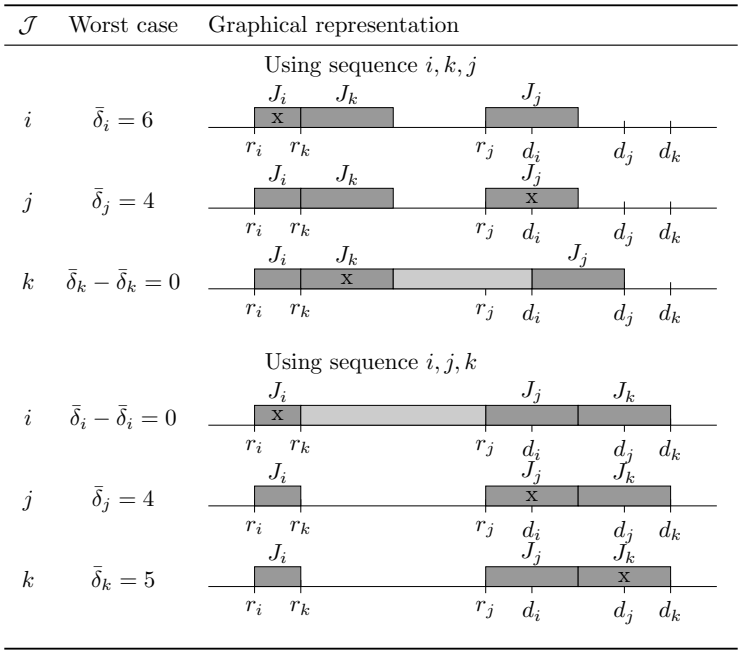


Figure 1: Worst-case scenarios if the sequence of jobs is fixed before the uncertainty is revealed

From left to right : the failing job, the maximum cost, a graphical representation of the schedule

5.3. An alternative uncertainty set

In this section, we show that substituting the uncertainty set Ξ with the following set

$$\hat{\Xi} = \left\{ \xi \in \mathbb{R}_+^{|\widetilde{\mathcal{J}}|} \middle| \xi_k \leq 1, \forall k | J_k \in \widetilde{\mathcal{J}} \text{ and } \sum_{k | J_k \in \widetilde{\mathcal{J}}} \xi_k \leq \Gamma \right\}$$

where the uncertainty directly affects job occurrences rather than actual jobs is invariant for problem $(\widetilde{\mathcal{P}})$. Note that the same is untrue for problem $(\mathcal{P})$. We define problem $(\widetilde{\mathcal{P}}_{\hat{\Xi}})$ from $(\widetilde{\mathcal{P}})$ by substituting Ξ with $\hat{\Xi}$ and the recourse cost function R with the adapted function

$$\hat{R}(\xi, y, z) = \sum_{j|J_j \in \mathcal{J}} \sum_{k|J_k \in \mathcal{G}_j} \left[(\bar{\delta}_k \xi_k - f_k) y_k - \bar{\delta}_k \xi_k z_k \right].$$

The following proposition holds:

Proposition 4. *Denoting by* $(\bullet)^*$ *the optimal value of problem* $\bullet$*, the following relation holds:*

$$(\widetilde{\mathcal{P}})^* = (\widetilde{\mathcal{P}}_{\hat{\Xi}})^*$$

for a fixed uncertainty budget Γ*.*

Proof. The proof involves showing that, given a feasible first-stage solution, from an uncertain vector of one of the two problems, we can always build an uncertain vector of the other problem so that the recourse cost functions are identical for feasible second-stage solutions. Let $(x, U) \in \widetilde{\mathcal{X}}$.

First, for any $\xi \in \Xi$, let us define $\hat{\xi} \in [0,1]^{|\widetilde{\mathcal{J}}|}$ such that $\forall j | J_j \in \mathcal{J}$ and $k | J_k \in \mathcal{G}_j$, $\hat{\xi}_k = 1$ if $x_k = \xi_j = 1$, and $\hat{\xi}_k = 0$ otherwise. Note that $\hat{\xi} \in \hat{\Xi}$. Then we have, for all $(y, z) \in \{0,1\}^{2|\widetilde{\mathcal{J}}|}$, $R(\xi, y, z) = \hat{R}(\hat{\xi}, y, z)$. It follows that

$$\max_{\xi \in \Xi} \min_{(y,z) \in \tilde{\mathcal{Y}}(x)} R(\xi, y, z) \leq \max_{\hat{\xi} \in \hat{\Xi}} \min_{(y,z) \in \tilde{\mathcal{Y}}(x)} \hat{R}(\hat{\xi}, y, z)$$

and finally $(\widetilde{\mathcal{P}})^* \leq (\widetilde{\mathcal{P}}_{\hat{\Xi}})^*$.

Second, for any $\hat{\xi} \in \hat{\Xi}$, let us define vector $\xi \in [0,1]^{|\mathcal{J}|}$ such that $\forall j | J_j \in \mathcal{J}$, $\xi_j = 1$ if there exists $k \in \mathcal{G}_j$ such that $\hat{\xi}_k = 1$, and $\xi_j = 0$ otherwise. Besides, for all $(y, z) \in \tilde{\mathcal{Y}}(x)$, we have for all j, $\sum_{k|J_k \in \mathcal{G}_j} y_k \in \{0,1\}$ and $\sum_{k|J_k \in \mathcal{G}_j} z_k \in \{0,1\}$. It follows that

$$\sum_{k|J_k \in \mathcal{G}_j} \left[(\bar{\delta}_k \xi_j - f_k) y_k - \bar{\delta}_k \xi_j z_k \right] = \sum_{k|J_k \in \mathcal{G}_j} \left[(\bar{\delta}_k \hat{\xi}_k - f_k) y_k - \bar{\delta}_k \hat{\xi}_k z_k \right]$$

and so $R(\xi, y, z) = \hat{R}(\hat{\xi}, y, z)$. Similarly to the first point, this implies $(\widetilde{\mathcal{P}})^* \geq (\widetilde{\mathcal{P}}_{\hat{\Xi}})^*$, which yields the result. □

The model resulting from such a substitution and using the convexification of the recourse set will be referred to as ColGen3.

6. Computational experiments

This section reports the main computational results for the two problems we are addressing. We first give some details about our implementation, and then explain how random instances were generated. We also describe our protocol to compare the exact approach with the finite adaptability method, which is exact only if the input parameter K is large enough.

6.1. Implementation details and experimental setting

All mixed integer linear programs, as well as linear programs inside the column generation procedures, are solved using IBM ILOG Cplex 12.9, through the C callable library, using default parameters and four threads. The generic implementation BapCod [33] of the branch-and-price Algorithm 1 is used to optimize models ColGen1, ColGen2 and ColGen3. At each node of the search tree, the linear relaxation of the problem is computed using column generation (Algorithm 2). The pricing sub-problem is solved using the MILP solver. At most one column is added to the master program $[DEP]^R$ at each iteration. To improve the convergence of the column generation procedure, we use stabilization by automatic smoothing of the dual variables of the master program, as described in [34]. When the optimal solution of the corresponding relaxation does not satisfy the integrality requirements of first-stage variables, one fractional variable is chosen and two child

nodes are created in order to exclude its current value from the search space. This variable is chosen to be the closest to 0.5. The open nodes are processed according to the best first rule. The implementation of the branch-and-price algorithm is sequential.

All our experiments are conducted using a 2 Dodeca-core Haswell Intel Xeon E5-2680 v3 2.5 GHz machine with 128Go RAM running Linux OS, part of the PlaFRIM[4] experimental platform. The resources of this machine are strictly partitioned using Slurm Workload Manager[5] to run several tests in parallel. The resources available for each run (algorithm-instance) are set to 4 threads and a 20 GB RAM limit (we remark that our branch-and-price algorithm does not benefit from parallel processing). This virtually creates six independent machines, each running one single instance at a time.

6.2. Instances

The test bed was randomly generated based on the technique used in [20] in which the authors generate a random test bed for the deterministic $1|r_j|\sum w_jU_j$ problem. Their approach takes as input three parameters: the number of jobs N, a factor for the dispersion of the release dates R_1 and a factor controlling the dispersion of the deadlines R_2. Having fixed these parameters, we generate, for each of the N jobs, random characteristics defined as follows:[6]

$$\begin{aligned} p_j &\sim \mathcal{U}(1,100) & w_j &\sim \mathcal{U}(1,100) \\ \bar{\delta}_j &\sim \mathcal{U}(1,100) & f_j &\sim \mathcal{U}(1,100) \\ r_j &\sim \mathcal{U}(0, N\times R_1) & \Delta_j &\sim \mathcal{U}(0, N\times R_2) \\ d_j &= r_j + p_j + \Delta_j & \tau_j &\sim \mathcal{U}(0, \lambda\Delta_j) \end{aligned}$$

Here, Δ_j denotes the slack time of jobs J_j within its time window. Note that the extra time needed to fix one job τ_j is generated depending on an extra parameter λ, fixed, for our experiments, to $\frac{5}{4}$. This implies that it is *generally* feasible to fix a job if its whole time window was to be scheduled (i.e., if no other job interferes with it). The parameters which were used are combinations of $N \in \{5, 10, 15, 20, 25\}$, $R_1 \in \{5, 10, 20, 30\}$ and $R_2 \in \{5, 10, 20, 30\}$. In total, 480 instances were generated. However, each of these instances are parameterized by the uncertainty budget Γ. Throughout our experiments, the value for Γ varied, from 1 to 3 for 5-job instances, from 1 to 7 for 10-job instances, from 1 to 10 for 15, 20 and 25-job instances. Therefore, we compared the two approaches over 3200 instances.

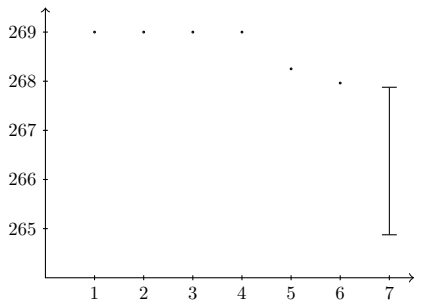
6.3. Protocol for comparing the two solution methods

The finite adaptability method solves the problem exactly only if parameter K is large enough; otherwise it solves an approximation which is tighter and tighter as its parameter K grows. For this reason, we must be careful when comparing its numerical performance with that of the convexification approach. For a given problem $(\mathcal{P})$, let $(\mathcal{P}_K)$ be its approximation as a K-adaptability problem. We denote by $(\bullet)^*$ the optimal solution of problem $\bullet$ (or the best known upper bound in case the time limit ois reached) and by $t(\bullet)$ the computation time to solve problem $\bullet$. An attractive case to compare the finite adaptability and the exact approach is when $(\mathcal{P})^* = (\mathcal{P}_K)^*$, since it amounts to comparing two exact methods. However, large values of K lead to intractable models, so we need to find the smallest value for K which fulfills this condition. More formally, we are interested in the following problem: $K^* = \min\{K : (\mathcal{P}_K)^* = (\mathcal{P})^*, K \in \mathbb{N}^*\}$. This problem is feasible and has an upper bound equal to $\dim \Xi + 1 = |\mathcal{J}| + 1$ ([18]). Note that the a priori knowledge of the optimal value $(\mathcal{P})^*$ is assumed. If this assumption is not satisfied, we do not have a practical method to find K^*, since a local minimum of function $K \mapsto (\mathcal{P}_K)^*$ sometimes fails to be global. This is illustrated in Figure 2, which reports the optimal objective value for a specific 15-job instance of $(\widetilde{\mathcal{P}})$ for various values of K. We can see that from the 1-adaptability to the 4-adaptability, the optimal value does not change while it does for greater values of K. This shows that guessing the value K^*, for which the approximation is, in fact, an exact solution, is hard and to our knowledge there is no straightforward stopping criterion for the search

[4]PlaFRIM: Plateforme Fédérative pour la Recherche en Informatique et Mathématiques (`https://www.plafrim.fr/fr/accueil/`)

[5]`https://slurm.schedmd.com/` (accessed June 2020)

[6]$\mathcal{U}$ denotes the discrete uniform distribution law



K	Objective	Gap
1	269.000	
2	269.000	
3	269.000	
4	269.000	
5	268.252	
6	267.963	
7	267.888	1.129

Figure 2: A K-adaptability plateau for a 15-jobs instances : the optimal objective value does not change between the 1-adaptability and the 4-adaptability, we were unable to solve the instance using the 7-adaptability within one hour (the vertical bar represents the optimality gap)

for K^*. As a matter of fact, for that specific instance, we are unable to conclude if $K^* = 7$ or if $K^* > 7$, since we were unable to solve it within one hour. This particular observation holds for the two problems $(\mathcal{P})$ and $(\widetilde{\mathcal{P}})$.

We run our algorithms under a given time limit and not all instances are solved to optimality, thus $(\mathcal{P})^*$, $(\mathcal{P}_K)^*$ and their anchored counterparts are sometimes approximated. To overcome this difficulty, we focus on a slightly different problem: find the smallest value for K which, under a given time limit T, yields an objective function at least as good as the solution of the exact approach. Formally, we estimate K^* by

$$\widehat{K}^* = \min \left\{ K : \begin{array}{l} (\mathcal{P}_K)^* \leq (\mathcal{P})^*, \\ t(\mathcal{P}) \leq T, \\ t(\mathcal{P}_K) \leq T, \\ K \in \mathbb{N}^* \end{array} \right\}$$

In our experiments, the search for $\widehat{K}^*$ is done iteratively starting from $K = 1$ and increasing K by one unit until one of the four conditions is reached:

- $(\mathcal{P}_K)^* \leq (\mathcal{P})^*, t(\mathcal{P}) \leq T$ and $t(\mathcal{P}_K) \leq T$: the two problems were solved optimally and we set $\widehat{K}^* = K$ (in this case, the equality of the objectives holds and the approximation is tight: $\widehat{K}^* = K^*$);
- $(\mathcal{P}_K)^* \leq (\mathcal{P})^*, t(\mathcal{P}) > T$ and $t(\mathcal{P}_K) \leq T$: the exact approach could not achieve and/or prove optimality, and $(\mathcal{P})^*$ equals the best upper bound found. We set $\widehat{K}^* = K$ and we know that $\widehat{K}^* \leq K^*$;
- $t(\mathcal{P}) \leq T$ and $t(\mathcal{P}_K) > T$: the finite adaptability approach could not achieve and/or prove optimality, the search for $\widehat{K}^*$ is stopped since increasing K typically increases the computation time to solve $(\mathcal{P}_K)$. We set $\widehat{K}^* = K$ and we know that $\widehat{K}^* \leq K^*$;
- $t(\mathcal{P}) > T$ and $t(\mathcal{P}_K) > T$: none of the two problems could be solved to be proven optimality, the search for $\widehat{K}^*$ is stopped and the two methods are considered to perform as badly as each other. We set $\widehat{K}^* = K$ and we know that $\widehat{K}^* \leq K^*$.

Note that, as K^* is typically unknown, comparing $(\mathcal{P})$ with $(\mathcal{P}_{K^*})$ or $(\mathcal{P}_{\widehat{K}^*})$ in fact gives an advantage to the finite adaptability.

6.4. Comparison of the approaches for problem $(\mathcal{P})$

In Table 1, we present a comparison between finite adaptability (columns KAdapt1) and the exact approach ColGen1. We use the experimental protocol described above with a time limit $T = 1$ hour for each run. The first two columns describe the main characteristics of the instances considered (number of jobs and value of Γ). The left-hand part of the table gathers the percentages of instances that were not solved to optimality within the time limit. In the right-hand part of the table, the average computing times are reported (instances for which the time limit is reached count for 3600 seconds).

Finite adaptability solves all five-job instances, but fails to solve 11.25% of the instances for 10 jobs and $\Gamma = 2$. Less than 15% of the 25-job instances instances are solved by this method when $\Gamma \leq 4$. Indeed, we have noticed that the hardest instances correspond to those having a value of Γ such that the ratio $\Gamma/|\mathcal{J}|$ is around 0.3-0.4. For very small values of Γ the problem often becomes easy since a small number of critical jobs have to be found. For large values of Γ, however, the problem almost reduces to the deterministic problem where all the jobs are penalized. The in-between instances are the most challenging. The branch-and-price algorithm ColGen1 is able to solve all instances up to 20 jobs to optimality, and more than 88% of the 25-job instances. When finite adaptability is able to find the optimal solution, it is generally faster than ColGen1. Note that the reported computing times are those of the *last* run of KAdapt1 during the search for $\hat{K}^*$ (they can be obtained only if one is able to "guess" value of $\hat{K}^*$ beforehand, which is not the case in a practical context).

Table 2 shows the percentage of 25-job instances for which each method could find a feasible solution within the time limit T, although without being able to prove its optimality. KAdapt1 always finds a feasible solution while this is clearly not the case for the convexification approach. Once again, note that the results for KAdapt1 are obtained with the first value of K for which the execution time exceeded T, which may not be equal to K^*. This means that the cost reported is an upper bound of the actual cost of the first-stage solution found by K-adaptability (since the recourse used is heuristic if K is not large enough). For these instances, KAdapt1 always finds a feasible solution but with a very large optimality gap: the gap between the lower and upper bounds of the MILP solver at the time limit is larger than 70% on average. This is partly explained by the poor linear relaxation of the MILP model. Conversely, the branch-and-price algorithm ColGen1 based on the convexification approach does not always find a feasible solution, but when it does, the optimality gap is often small (4.5% on average).

In table 3, we study the values of K that are needed to obtain the optimal solution with the $K-$adaptability method. For this purpose, we report, for every value of K from 1 to K^*, the average gap between the value found by KAdapt1 and the exact method ColGen1 (the gap is always 0 when $K = K^*$). We also compare the computation times of KAdapt1 and ColGen1, according to the different values of K. An important item of information gathered from the table is that a very large proportion of instances can be solved to optimality within one hour with a value of $K = 1$. This means that for many instances, the so-called static model where the recourse actions are decided a priori is sufficient to solve the problem.

It can also be noted that $K-$adaptability with a small value of K can be a good heuristic: for the 16 instances where $\hat{K}^* = 5$, setting K to 1 produces a gap of less than 7%, for computation times often two orders of magnitude smaller than the time required to solve the problem optimally using the branch-and-price algorithm. This table also shows the high sensitivity of the $K-$adaptability approach to parameter K. For example, let us consider instances with $\hat{K}^* = 2$. When $K = 1$, 696 instances can be solved within one hour, whereas incrementing the value of K to 2 allows only 110 instances to be solved within the same time limit. KAdapt1 appears, at first glance, to perform surprisingly better when $\hat{K}^*$ increases. Indeed, when $\hat{K}^* = 5$ and $K = 3$, and when $\hat{K}^* = 6$ and $K = 4$, its computation time looks significantly smaller than the one for ColGen1. But this anomaly is explained by the fact that only the instances that could be solved using KAdapt1 with $K = 5$ and $K = 6$, respectively, are reported in these sections of the table. They are very likely to be well-suited to this approach, which explains the very good results when the value of K is smaller. Note that we could determine the value of K^* for 2231 out of the 3200 instances, leaving this question open for the 969 others.

Finally, looking at the last column of table 3, one can see that KAdapt1, when $K = 1$, is significantly faster than ColGen1. In this case, the MILP model simplifies to a static robust optimization model (thanks to the structure of constraint (31)), which explains the very good performance. However, for the more complex settings, ColGen1 outperforms KAdapt1 by one to two orders of magnitude.

6.5. Comparison of the approaches for problem $(\widetilde{\mathcal{P}})$

Table 4 compares computation times for approaches KAdapt2, ColGen2 and ColGen3. The values reported for KAdapt2 are obtained with parameter $K = \hat{K}^*$ for each instance.

We can see that problem $(\widetilde{\mathcal{P}})$ is much more challenging to solve than $(\mathcal{P})$, since the $K-$adaptability method KAdapt2 reaches limitations for 10-job instances while the branch-and-price algorithm ColGen2 is unable to solve some 15-job instances. This is mainly explained by the relatively large number of variables in the models. Indeed, while the number of first-stage variables was $\mathcal{O}(|\mathcal{J}|)$ for problem $(\mathcal{P})$, it is now

$\mathcal{O}(|\widetilde{\mathcal{J}}|) = \mathcal{O}(|\mathcal{J}|^2)$. For the same reasons, ColGen3 performs very badly (i.e., the uncertainty set's dimension is $\mathcal{O}(|\mathcal{J}|^2)$).

Table 5 shows the computation time ratio between KAdapt2 and ColGen2. As for problem $(\mathcal{P})$, we can see that the closer K gets to $\hat{K}^*$, the faster the branch-and-price algorithm becomes compared to K-adaptability.

We also studied the cost of obtaining anchored solutions, *i.e.* the increase in the objective function when one has to decide on the sequence of the selected jobs before uncertainty is revealed. In Table 6, for several sizes of instances and several values of Γ, we report the average costs related to problem $(\mathcal{P})$ (column Free) and $(\widetilde{\mathcal{P}})$ (column Anchored), respectively.

It transpires from our experiments that anchoring solutions only leads to a marginal increase in the cost on average. The largest gap we obtained was 0.26% for instances with ten jobs.

As a conclusion, it appears that for this problem, the cost of obtaining anchored solutions is not the cost of the solutions itself, but the practical difficulty of solving the optimization problem with state-of-the-art algorithms.

7. Conclusion

In this paper, we have described a robust version of the classical one-machine scheduling problem where one minimizes the weighted number of tardy jobs. Although solving general robust integer programs with integer recourse is typically Σ_P^2-hard, we were able to show that this problem is $\mathcal{NP}$-complete, and proposed two solution approaches: an exact reformulation which can be solved by means of the branch-and-price algorithmic procedure, and a (MILP) conservative approximation. Our computational experiments show that this problem is hard to solve in practice, since state-of-the-art methods may fail to solve 25-job instances in one hour.

Regarding the exact method we proposed, we think that the development of good heuristic procedures for solving the pricing problem involved may lead to substantial improvements in the method in terms of computing times. As for the conservative approximation, its main drawback is its poor linear relaxation, which is a known issue in the literature. We have also investigated another version of the problem where the sequencing decisions from the first stage cannot be modified. It appears that this version of the problem, which serves to find so-called anchored solutions, is harder than the first: some 15-job instances are left unsolved by both approaches.

8. Acknowledgments

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Appendix A. Solution approaches for problem $(\widetilde{\mathcal{P}})$

Appendix A.1. Finite adaptability

In this section, we give a K-adaptability formulation of the second problem. The main constraints are identical to those derived in section 4.1. Only the modified constraints and variables are explained below:

minimize

$$\sum_{j|J_j\in\mathcal{J}} [w_j U_j + (1-U_j)f_j + v_j] + \Gamma u - \sum_{q=1}^{K} \sum_{k|J_k\in\widetilde{\mathcal{J}}} f_k \psi_k^q \tag{A.1}$$

subject to

$(36) - (39)$

$$y_k^q \leq x_k \quad \forall k|J_k \in \widetilde{\mathcal{J}}, q = 1,\dots,K \tag{A.2}$$

$$z_k^q \leq y_k^q \quad \forall k|J_k \in \widetilde{\mathcal{J}}, q = 1,\dots,K \tag{A.3}$$

$$\sum_{k|J_k\in\mathcal{G}_j} x_k = 1 - U_j \quad \forall J_j \in \mathcal{J} \tag{A.4}$$

$(42) - (48)$

$$t_k^q \geq 0 \quad \forall k|J_k \in \widetilde{\mathcal{J}}, q = 1,\dots,K \tag{A.5}$$

$$y_k^q \in \{0,1\} \quad \forall k|J_k \in \widetilde{\mathcal{J}}, q = 1,\dots,K \tag{A.6}$$

$$z_k^q \in \{0,1\} \quad \forall k|J_k \in \widetilde{\mathcal{J}}, q = 1,\dots,K \tag{A.7}$$

$$\psi_k^q \geq 0 \quad \forall k|J_k \in \widetilde{\mathcal{J}}, q = 1,\dots,K \tag{A.8}$$

$$\zeta_k^q \geq 0 \quad \forall k|J_k \in \widetilde{\mathcal{J}}, q = 1,\dots,K \tag{A.9}$$

$$U_j \in \{0,1\} \quad \forall j|J_j \in \mathcal{J} \tag{A.10}$$

$$u \geq 0 \tag{A.11}$$

$$v_j \geq 0 \quad \forall j|J_j \in \mathcal{J} \tag{A.12}$$

$$\beta_q \geq 0 \quad q = 1,\dots,K \tag{A.13}$$

Here, the binary (first-stage) decision variable x_k represents the selection of the kth job occurrence in the non-decreasing order of the deadlines while U_j denotes the variable indicating whether a job is executed tardily or not. Constraint (A.2) links the first- and second-stage decisions, constraint (A.3) ensures that a job is repaired only if it is scheduled and constraint (A.4) ensures that exactly one job occurrence is selected for on-time jobs. The other variables and constraints have the same meaning as in KAdapt1. This model will be referred to as KAdapt2.

Appendix A.2. Convexification of the recourse set

In a very similar way to what has been done for problem $(\mathcal{P})$, we can derive an exact formulation for this problem variant by using proposition 1 on the set of eligible second-stage solutions $\widetilde{\mathcal{Y}}$. Then, by enumerating the extreme points of the convex hull of $\mathcal{Y}$, we can derive the following model:

minimize

$$\sum_{j|J_j\in\mathcal{J}} [w_j U_j + f_j(1-U_j)] - \sum_{k|J_k\in\widetilde{\mathcal{J}}} \sum_{e\in E} f_k \mathbf{y}_k^e \alpha_e + \Gamma u + \sum_{j|J_j\in\mathcal{J}} v_j$$

subject to

$$\sum_{e\in E} \alpha_e = 1 \tag{A.14}$$

$$\sum_{e\in E} \mathbf{y}_k^e \alpha_e \leq x_k \quad \forall k|J_k \in \widetilde{\mathcal{J}} \tag{A.15}$$

$$\sum_{k|J_k\in\mathcal{G}_j} x_k = 1 - U_j \quad \forall j|J_j \in \mathcal{J} \tag{A.16}$$

$$u + v_j \geq \sum_{k|J_k\in\mathcal{G}_j} \left[\bar{\delta}_k \sum_{e\in E} (\mathbf{y}_k^e - \mathbf{z}_k^e)\, \alpha_e \right] \quad \forall j|J_j \in \mathcal{J} \tag{A.17}$$

$$x_k \in \{0,1\} \quad \forall k|J_k \in \widetilde{\mathcal{J}} \tag{A.18}$$

$$U_j \in \{0,1\} \quad \forall j|J_j \in \mathcal{J} \tag{A.19}$$

$$\alpha_e \geq 0 \quad \forall e \in E \tag{A.20}$$

$$u \geq 0 \tag{A.21}$$

$$v_j \geq 0 \quad \forall j|J_j \in \mathcal{J} \tag{A.22}$$

Again, decision vector α represents the convex combination multipliers from the inner description of $\text{conv}(\mathcal{Y})$ and u and v are the dual variables associated with the constraint $\xi \in \Xi$. Constraint (A.14) ensures that the second-stage variables must be convex combinations of some extreme points. Constraint (A.15) links the first-stage variables with the second-stage variables while constraint (A.16) ensures that exactly one job occurrence per job is selected in the first stage. Finally, constraint (A.17) corresponds to the dualized cost implied by the venue of a scenario $\xi \in \Xi$.

This model will be referred to as ColGen2.

We solve this large-scale MILP model using a simple adaptation of the branch-and-price algorithm presented in section 4.2.2. Algorithm 1 is modified by checking, at line 10, the integrality of both U^* and x^*, and lines 13 and 14 are adapted to branch either on a U- or an x-variable. Surprisingly, the pricing problem differs only in the value of the dual variable μ in input, which is now associated with constraint (A.15) instead of (58).

Table 1: CPU execution times for solving problem ($\mathcal{P}$)

# Jobs	Γ	Unsolved within $T = 1$ hour (%) KAdapt1	ColGen1	Average CPU time (s.) KAdapt1	ColGen1
5	1	**0.00**	**0.00**	**0.14**	1.92
	2	**0.00**	**0.00**	**0.16**	1.74
	3	**0.00**	**0.00**	**0.12**	1.53
10	1	6.25	**0.00**	304.43	**13.43**
	2	11.25	**0.00**	448.22	**15.01**
	3	5.00	**0.00**	195.92	**10.52**
	4	1.25	**0.00**	45.32	**8.12**
	5	1.25	**0.00**	51.47	**7.50**
	6	0.00	**0.00**	**0.17**	7.09
	7	0.00	**0.00**	**0.15**	7.00
15	1	35.00	**0.00**	1548.00	**43.10**
	2	57.50	**0.00**	2262.48	**68.78**
	3	46.25	**0.00**	1673.86	**64.35**
	4	28.75	**0.00**	1074.45	**47.08**
	5	12.50	**0.00**	450.27	**28.67**
	6	10.00	**0.00**	360.25	**23.32**
	7	2.50	**0.00**	90.19	**20.65**
	8	**0.00**	**0.00**	**0.15**	17.44
	9	**0.00**	**0.00**	**0.17**	16.27
	10	**0.00**	**0.00**	**0.17**	16.48
20	1	66.25	**0.00**	2619.71	**117.91**
	2	87.50	**0.00**	3172.19	**189.76**
	3	86.25	**0.00**	3147.91	**272.85**
	4	71.25	**0.00**	2603.77	**331.63**
	5	55.00	**0.00**	1989.44	**345.51**
	6	35.00	**0.00**	1260.50	**269.09**
	7	20.00	**0.00**	720.37	**188.32**
	8	5.00	**0.00**	180.26	**127.12**
	9	**0.00**	**0.00**	**0.22**	66.69
	10	**0.00**	**0.00**	**0.21**	46.46
25	1	82.50	**8.75**	3037.96	**657.10**
	2	96.25	**12.50**	3469.86	**995.30**
	3	91.25	**16.25**	3286.36	**1111.68**
	4	77.50	**18.75**	2790.83	**1173.04**
	5	66.25	**21.25**	2385.73	**1189.17**
	6	57.50	**20.00**	2070.69	**1147.60**
	7	38.75	**17.50**	1395.58	**994.75**
	8	31.25	**12.50**	1125.50	**836.73**
	9	17.50	**10.00**	**630.40**	743.91
	10	6.25	**5.00**	**225.32**	452.04

From left to right : the number of jobs, the uncertainty budget, the average computation time (when less than $T = 1$ hour) for the K-adaptability approach, the convexification-based branch-and-price algorithm, the percentage of times one method was found to be the most efficient and the percentage of instances which could not be solved within $T = 1$ hour.

Table 2: Feasible solutions found for $(\mathcal{P})$, over instances that could not be solved to optimality by the method within the time limit $T = 1$ hour

# Jobs	Γ	Feasible solutions found (%) KAdapt1	ColGen1
25	1	**100**	28.57
	2	**100**	30
	3	**100**	15.38
	4	**100**	6.67
	5	**100**	5.88
	6	**100**	6.25
	7	**100**	14.29
	8	**100**	10
	9	**100**	37.5
	10	**100**	25

From left to right : the number of jobs, the uncertainty budget and the percentage of instances for which a feasible solution could be found within $T = 1$ hour over the instances which could not be solved optimally within $T = 1$ hour

Table 3: The cost of approximating with finite adaptability for problem $(\mathcal{P})$

$\hat{K}^*$	K	# Instances	Approximation gap (%)	Time ratio
1	1	1994	0	0.03
2	1	696	6.6	0.01
	2	110	0	5.23
3	1	368	5.86	0.01
	2	368	1.43	6.00
	3	90	0	14.59
4	1	123	6.3	0.01
	2	123	2.06	0.09
	3	123	0.6	14.23
	4	32	0	13.27
5	1	16	6.85	0.01
	2	16	2.25	0.04
	3	16	0.67	0.82
	4	16	0.24	34.76
	5	3	0	29.28
$\geq$6	1	3	1.92	0.01
	2	3	1.7	0.02
	3	3	0.58	0.05
	4	3	0.13	0.31
	5	3	0.02	5.87
	6	2	0	2.06

From left to right : the value of $\hat{K}^*$ (*i.e.* the value of K required for K-adaptability to be equivalent to $(\mathcal{P})$ or a lower bound on this value), the value of K which was used, the number of instances with this value of $\hat{K}^*$ which were solved using Kadapt1 with that value of K within $T = 1$ hour, the approximation gap computed as $|KAdapt1^* - ColGen1^*|/|ColGen1^*|$, the computation time ratio computed as $t(KAdapt1)/t(ColGen1)$.

Table 4: Computation times for solving problem ($\widetilde{\mathcal{P}}$)

# Jobs	Γ	Average CPU time (s.)			Fastest approach (%)			Unsolved within 1 hour (%)		
		Kadapt2	ColGen2	ColGen3	Kadapt2	ColGen2	ColGen3	Kadapt2	ColGen2	ColGen3
5	1	**0.13**	1.12	1.23	**80**	11.25	3.75			
	2	**0.13**	0.97	1.05	**86.25**	2.5	7.5			
	3	**0.1**	0.76	0.79	**87.5**	5	5			
10	1	**13.93**	27.98	68.22	**93.75**	3.75				1.25
	2	**22.13**	24.42	90.13	**85**	8.75	1.25	1.25		3.75
	3	**9.36**	10.91	76.6	**93.75**	5		1.25		2.5
	4	42.91	8.34	50.27	**97.5**	1.25				
	5	36.78	**3.72**	36.68	**98.75**	1.25				
	6	**0.14**	3.2	24.35	**100**					
	7	**0.13**	3.22	14.43	**100**					
15	1	**199.43**	243.14	392.33	**56.25**	17.5	2.5	16.25	**7.5**	18.75
	2	**214.32**	219.46	379.51	**40**	25	7.5	46.25	16.25	26.25
	3	180.51	**158.42**	279.5	**55**	20	3.75	36.25	**12.5**	28.75
	4	**135.01**	168.64	174.61	**71.25**	10	1.25	21.25	**7.5**	28.75
	5	**0.15**	76.6	218.33	**87.5**	6.25	1.25	12.5	**3.75**	25
	6	**0.14**	47.46	196.13	**92.5**	3.75		7.5	**3.75**	22.5
	7	**0.14**	60.57	187.76	**97.5**	1.25	1.25	**2.5**	**2.5**	17.5
	8	**0.15**	50.22	232.61	**100**					11.25
	9	**0.15**	14.76	122.86	**100**					10
	10	**0.15**	10.48	71.99	**100**					10

From left to right : the number of jobs, the uncertainty budget, the average computation time (when less than $T = 1$ hour) for each method, the percentage of times one method was found to be the most efficient and the percentage of instances which could not be solved within the time limit $T = 1$ hour.

Table 5: The cost of approximating with finite adaptability for problem $(\widetilde{\mathcal{P}})$

$\hat{K}^*$	K	# Instances	Approximation gap (%)	Time ratio
1	1	1251	0	0.22
2	1	95	3.71	0.06
	2	88	0	0.11
3	1	93	4.21	0.01
	2	93	-0.8	0.1
	3	66	0	3.68
4	1	100	6.85	0.04
	2	100	1.16	0.09
	3	80	-0.22	1.87
	4	59	0	5.79
5	1	44	8.65	0.01
	2	44	2.63	0.02
	3	44	0.8	0.5
	4	44	0.15	7.99
	5	14	0	14.17
6	1	15	5.89	0.01
	2	15	2.85	0.02
	3	15	1.06	0.09
	4	15	0.39	0.76
	5	15	0.08	12.39
	6	6	0	35.19
≥ 7	1	2	0.37	0.01
	2	2	0.37	0.02
	3	2	0.37	0.14
	4	2	0.37	0.65
	5	2	0.14	5.52
	6	2	0.03	78.13
	7	0	—	—

From left to right : the value of $\hat{K}^*$ (*i.e.* the value of K required for K-adaptability to be equivalent to $(\mathcal{P})$ or a lower bound on this value), the value of K which was used, the number of instances with this value of $\hat{K}^*$ which were solved using Kadapt2 with that value of K within $T = 1$ hour, the approximation gap computed as $|KAdapt2^* - ColGen2^*|/|ColGen2^*|$, the computation time ratio computed as $t(KAdapt2)/t(ColGen2)$.

Table 6: Anchored solutions cost analysis

# Jobs	Γ	Objective cost Free (ColGen1)	Objective cost Anchored (ColGen2)	Gap (%)	N. Instance
5	1	70.39	70.40	0.00	80
5	2	75.27	75.28	0.01	80
5	3	75.94	75.94	0.00	80
10	1	144.76	145.14	0.26	80
10	2	165.92	166.21	0.18	80
10	3	171.73	171.80	0.04	80
10	4	173.24	173.26	0.01	80
10	5	173.61	173.61	0.00	80
10	6	173.70	173.70	0.00	80
10	7	173.70	173.70	0.00	80
15	1	192.83	193.32	0.25	74
15	2	232.46	233.06	0.26	67
15	3	248.97	249.57	0.24	70
15	4	253.39	253.78	0.15	74
15	5	254.77	254.91	0.05	77
15	6	255.35	255.37	0.01	77
15	7	255.74	255.74	0.00	78
15	8	256.98	256.98	0.00	80
15	9	256.98	256.98	0.00	80
15	10	256.98	256.98	0.00	80

From left to right: the number of jobs, the uncertainty budget, the average objective costs of free solutions and anchored solutions, the relative gap between the two, the number of instances which where accounted for in the computation (i.e., instances which could be solved within the time limit $T = 1$ hour for both problems).