

# PRIMITIVE LIE ALGEBRAS OF RATIONAL VECTOR FIELDS

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ABSTRACT. A transitive Lie algebra  $\mathfrak{g}$  of rational vector fields on a projective manifold which do not preserve any foliation determines a rational map to an algebraic homogenous space  $G/H$  which maps  $\mathfrak{g}$  to  $\mathrm{lie}(G)$ .

## 1. INTRODUCTION

**1.1. Algebraic normalization.** Let  $X$  be a projective manifold and let  $\mathfrak{g}$  be a finite dimensional Lie subalgebra of the Lie algebra of rational vector fields on  $X$ . If  $x \in X$  is any sufficiently general point then we define the isotropy subalgebra of  $\mathfrak{g}$  at  $x$ , denoted by  $\mathfrak{h}_x$ , as the subalgebra of  $\mathfrak{g}$  consisting of vector fields vanishing at  $x$ . We assume  $x$  sufficiently general in order to avoid the polar locus of vector fields in  $\mathfrak{g}$ .

A Lie algebra  $\mathfrak{g}$  as above is called transitive if it contains a basis of the  $\mathbb{C}(X)$ -vector space of rational vector fields on  $X$ . When  $\mathfrak{g}$  is transitive, two sufficiently general points  $x, x' \in X$  give rise to isomorphic (through an isomorphism of  $\mathfrak{g}$ ) isotropy subalgebras  $\mathfrak{h}_x$  and  $\mathfrak{h}_{x'}$ . Thus when  $\mathfrak{g}$  is transitive we can safely refer to the isotropy subalgebra  $\mathfrak{h}$ , meaning the isotropy subalgebra for a sufficiently general point  $x \in X$ . We will denote the normalizer of  $\mathfrak{h}$  in  $\mathfrak{g}$  by  $N_{\mathfrak{g}}(\mathfrak{h})$ , i.e.

$$N_{\mathfrak{g}}(\mathfrak{h}) = \{v \in \mathfrak{g} \mid \mathrm{ad}(v)(\mathfrak{h}) \subset \mathfrak{h}\},$$

where  $\mathrm{ad}(v)(\cdot) = [v, \cdot]$  stands for the adjoint action.

We will say that Lie algebra is complete, following [4, Chapter I, Section 3], if all its derivations are inner derivations and if it has trivial center.

The result below is an alternative version of [1, Theorem 4.3], which although less general, is formulated in terms of intrinsic properties of the Lie algebra  $\mathfrak{g}$  and of its isotropy subalgebra  $\mathfrak{h}$ .

**Theorem 1.1.** *Let  $\mathfrak{g}$  be a finite dimensional Lie algebra of rational vector fields on a projective manifold  $X$  with isotropy algebra equal to  $\mathfrak{h}$ . If  $\mathfrak{g}$  is transitive, complete and  $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$  then there exists a rational map  $\varphi : X \dashrightarrow G/H$  to an algebraic homogeneous space such that  $\mathfrak{g}$  coincides with the pull-back under  $\varphi$  of the Lie algebra of left invariant vector fields on  $G/H$ .*

The result above is not optimal. As shown by Proposition 3.3, there exists non-complete transitive Lie algebras of rational vector fields (e.g. the Lie algebra of the group of affine transformations of  $\mathbb{C}^n$  for  $n \geq 2$ ) for which the conclusion of Theorem 1.1 still holds true. Nevertheless, we point out that the assumption  $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$  cannot be completely disregarded. Indeed there are transitive actions of the complete Lie algebra  $\mathfrak{sl}(2, \mathbb{C})$  on

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*Date:* July 23, 2020.

*Key words and phrases.* Foliation, Transverse Structure, Birational Geometry.

We thank CNRS, Université de Rennes 1, Henri Lebesgue Center, ANR-16-CE40-0008 project “Foliage” for financial support and CAPES-COFECUB Ma 932/19 project. The third author was supported by Cnpq and FAPERJ..

$\mathbb{C}^3$  which are transitive but not algebraically conjugated to the Lie algebra of left invariant vector fields on  $\mathrm{SL}(2, \mathbb{C})$ , see [1, Section 2.5] for examples. Another explicit examples of Lie algebras of polynomial vector fields on  $\mathbb{C}^3$  isomorphic to  $\mathfrak{sl}(2, \mathbb{C})$  are thoroughly studied in [3].

**1.2. Primitive Lie algebras.** A transitive Lie algebra of rational vector fields  $\mathfrak{g}$  is called primitive if its isotropy subalgebra  $\mathfrak{h}$  is maximal among all Lie subalgebras of  $\mathfrak{g}$ .

**Theorem A.** *Let  $\mathfrak{g}$  be a finite dimensional Lie algebra of rational vector fields on a projective manifold  $X$ . If  $\mathfrak{g}$  is transitive and primitive then there exists a rational map  $\varphi : X \dashrightarrow G/H$  to an algebraic homogeneous space such that  $\mathfrak{g}$  coincides with the pull-back under  $\varphi$  of the Lie algebra of left invariant vector fields on  $G/H$ .*

## 2. TRANSITIVE AND COMPLETE LIE ALGEBRAS

**2.1. Transitive Lie subalgebras of  $\mathfrak{X}(\mathbb{C}^n, 0)$ .** Throughout the text  $\mathfrak{X}(\mathbb{C}^n, 0)$  will stand for the Lie algebra of formal vector fields at the formal completion of the  $n$ -dimensional affine space at a point. We have deliberately omitted the usual *hat* ( $\hat{\phantom{x}}$ ) from the notation to avoid overburdening it.

Fix  $n \geq 1$ . In this paragraph we present some simple facts about transitive Lie subalgebras of  $\mathfrak{X} = \mathfrak{X}(\mathbb{C}^n, 0)$ . More specifically, we want to provide sufficient conditions in order to guarantee that a given Lie subalgebra  $\mathfrak{g} \subset \mathfrak{X}$  has normalizer

$$N_{\mathfrak{X}}(\mathfrak{g}) = \{x \in \mathfrak{X} \mid [x, \mathfrak{g}] \subset \mathfrak{g}\}$$

in  $\mathfrak{X}$  equal to itself.

**Lemma 2.1.** *Let  $\mathfrak{g} \subset \mathfrak{X}$  be a Lie algebra of formal vector fields. The normalizer  $N_{\mathfrak{X}}(\mathfrak{g})$  fits into the exact sequence*

$$0 \rightarrow C_{\mathfrak{X}}(\mathfrak{g}) \longrightarrow N_{\mathfrak{X}}(\mathfrak{g}) \longrightarrow \mathrm{Der}(\mathfrak{g})$$

where  $C_{\mathfrak{X}}(\mathfrak{g}) = \{x \in \mathfrak{X} \mid [x, \mathfrak{g}] = 0\}$  is the centralizer of  $\mathfrak{g}$  in  $\mathfrak{X}$ , and  $\mathrm{Der}(\mathfrak{g})$  is the algebra of derivations of  $\mathfrak{g}$ .

*Proof.* One has just to observe that for any element  $x \in N_{\mathfrak{X}}(\mathfrak{g})$  the restriction of  $\mathrm{ad}_x(\cdot) = [x, \cdot]$  to  $\mathfrak{g}$  is a derivation of  $\mathfrak{g}$  since  $[x, \mathfrak{g}] = \mathrm{ad}_x(\mathfrak{g}) \subset \mathfrak{g}$ . The kernel of the morphism  $x \mapsto (\mathrm{ad}_x)|_{\mathfrak{g}}$  is, by definition,  $C_{\mathfrak{X}}(\mathfrak{g})$ .  $\square$

**Lemma 2.2.** *Let  $\mathfrak{g} \subset \mathfrak{X}(\mathbb{C}^n, 0)$  be a finite dimensional transitive subalgebra. The centralizer of  $\mathfrak{g}$  in  $\mathfrak{X} = \mathfrak{X}(\mathbb{C}^n, 0)$  is isomorphic to the normalizer of the isotropy subalgebra  $\mathfrak{h}$  in  $\mathfrak{g}$ , i.e.*

$$C_{\mathfrak{X}}(\mathfrak{g}) \simeq \frac{N_{\mathfrak{g}}(\mathfrak{h})}{\mathfrak{h}}.$$

*Proof.* Let  $G$  be a Lie group with Lie algebra  $\mathfrak{g}$ . The isotropy subalgebra  $\mathfrak{h}$  is the Lie algebra of a (non necessarily closed) subgroup  $H$  of  $G$ . The orbits of the left action of  $H$  on  $G$  defines a smooth foliation  $\mathcal{H}$ . When  $H$  is a closed subgroup of  $G$  then the leaf space of  $\mathcal{H}$  is naturally identified with the space of left  $H$ -cosets  $G/H$ . Notice that the left action of  $G$  on itself preserves the foliation  $\mathcal{H}$ . In contrast, the right action of  $G$  on itself does not preserve the foliation  $\mathcal{H}$  in general. If an element  $g \in G$  maps left  $H$ -cosets to left cosets through right multiplication then it must map the coset  $g^{-1}H$  to  $H$  since the set  $g^{-1}Hg$  contains the identity. It follows that the set of elements in  $G$  preserving  $\mathcal{H}$  through right multiplication is formed precisely by  $N_G(H)$  the normalizer of  $H$  in  $G$ .

Restrict now to a sufficiently small neighborhood  $U$  of the identity in  $G$ . Since the foliation  $\mathcal{H}$  is smooth we can consider the local leaf space of  $\mathcal{H}$ . Denote it by  $U/\mathcal{H}$ . The left invariant vector fields on  $G$  descend to vector fields on  $U/\mathcal{H}$ . Let us denote its image by  $\mathfrak{g}_{\text{left}}$ . The right invariant vector fields are not, in general, infinitesimal automorphisms of  $\mathcal{H}$ . Only those in the normalizer of  $\mathfrak{h}$  in  $\mathfrak{g}$  will be infinitesimal automorphisms and will descend to the leaf space  $U/\mathcal{H}$ . Of course, the vector fields in  $\mathfrak{h}$  descend to zero and the quotient  $N_{\mathfrak{g}}(\mathfrak{h})/\mathfrak{h}$  can be identified with the centralizer of  $\mathfrak{g}_{\text{left}}$  in the Lie algebra of formal vector fields on the completion of  $U/\mathcal{H}$  at the image of the identity.

If  $\hat{G}$  denotes the completion of  $G$  at the identity then a subalgebra  $\mathfrak{g} \subset \mathfrak{X}(\mathbb{C}^n, 0)$  determines a formal action of  $\hat{G}$  on  $(\mathbb{C}^n, 0)$

$$\hat{G} \times (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$$

which identifies  $(\mathbb{C}^n, 0)$  with the completion of  $U/\mathcal{H}$  at the image of the identity and identifies the subalgebra  $\mathfrak{g}$  with  $\mathfrak{g}_{\text{left}}$  as defined in the previous paragraph. The result follows.  $\square$

**2.2. Proof of Theorem 1.1.** As already mentioned at the Introduction, Theorem 1.1 is an easy consequence of [1, Theorem 4.3]. In accordance with the conventions of Section 2.1, if  $X$  is a complex manifold and  $p \in X$  is a (closed) point we will denote the formal vector fields at the formal completion of  $X$  at  $p$  by  $\mathfrak{X}(X, p)$ .

**Theorem 2.3.** [1, Theorem 4.3] *Let  $X$  be a smooth irreducible complex algebraic variety and let  $\mathfrak{g}$  be a transitive, centerless, finite dimensional Lie subalgebra of rational vector fields on  $X$ . If for a general  $p \in X$  the normalizer of  $\mathfrak{g}$  in  $\mathfrak{X}(X, p)$  coincides with  $\mathfrak{g}$  then there exists a dominant rational map  $f : X \dashrightarrow G/H$  such that  $f_*\mathfrak{g}$  coincides with the Lie algebra of left invariant vector fields on  $G/H$ .*

Assumptions as in Theorem 1.1. Let  $\mathfrak{X} = \mathfrak{X}(X, p)$  be the Lie algebra of formal vector fields at a sufficiently general point  $p$  of  $X$ . We may, and will, consider  $\mathfrak{g}$  as a Lie subalgebra of  $\mathfrak{X}$ .

If every derivation of  $\mathfrak{g}$  is an inner derivation then the morphism  $N_{\mathfrak{X}}(\mathfrak{g}) \rightarrow \text{Der}(\mathfrak{g})$  considered in Lemma 2.1 is surjective. If furthermore  $\mathfrak{g}$  is centerless then the image of this morphism is isomorphic to  $\mathfrak{g}$ . Hence for  $\mathfrak{g} \subset \mathfrak{X}$  equal to a complete Lie algebra, we have that  $N_{\mathfrak{X}}(\mathfrak{g})$  fits into the exact sequence

$$0 \rightarrow C_{\mathfrak{X}}(\mathfrak{g}) \longrightarrow N_{\mathfrak{X}}(\mathfrak{g}) \longrightarrow \mathfrak{g} \rightarrow 0.$$

Finally, the assumption  $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$  guarantees, through Lemma 2.2, that the centralizer  $C_{\mathfrak{X}}(\mathfrak{g})$  is trivial. Thus  $N_{\mathfrak{X}}(\mathfrak{g}) = \mathfrak{g}$  and we can apply Theorem 2.3 to conclude.  $\square$

### 3. PRIMITIVE LIE ALGEBRAS

**3.1. Structure of primitive Lie Algebras [after Morozov].** Let  $\mathfrak{g}$  be a primitive Lie algebra of vector fields. It turns out that for a primitive Lie algebra  $\mathfrak{g}$  there are no  $\mathfrak{g}$ -ideals contained in the isotropy subalgebra  $\mathfrak{h}$ . This remark lead Guillemin-Sternberg to introduce the concept of effective primitive pair  $(\mathfrak{g}, \mathfrak{h})$  —  $\mathfrak{g}$  and  $\mathfrak{h}$  are abstract Lie algebras with  $\mathfrak{h} \subset \mathfrak{g}$  and  $\mathfrak{h}$  no non-trivial  $\mathfrak{g}$ -ideal — and to prove that effective primitive pairs can be realized as Lie algebras of formal vector fields at  $\text{Spec}(\mathbb{C}[[x_1, \dots, x_n]])$  where  $n = \dim \mathfrak{g} - \dim \mathfrak{h}$ . For more on these results see [2] and references therein. In particular, one can find there the following result by Morozov, and its proof, which clarifies the structure of effective primitive pairs.

**Theorem 3.1.** *Suppose that  $(\mathfrak{g}, \mathfrak{h})$  is an effective primitive pair. Then either  $\mathfrak{g}$  is simple, or else we are in one of the following two situations:*

- (1)  *$\mathfrak{g}$  is the direct sum  $\mathfrak{l} \oplus \mathfrak{l}$  of two isomorphic simple Lie algebras and  $\mathfrak{h}$  is the diagonal subalgebra; or*
- (2)  *$\mathfrak{g}$  is the semi-direct product  $\mathfrak{h} \ltimes \mathfrak{m}$  where  $\mathfrak{h}$  is the direct sum of a semisimple lie algebra with a lie algebra of dimension at most 1; and  $\mathfrak{m}$  is an irreducible and faithful  $\mathfrak{h}$ -module equipped with trivial Lie bracket:  $[\mathfrak{m}, \mathfrak{m}] = 0$ .*

*Conversely, in the latter two cases  $(\mathfrak{g}, \mathfrak{h})$  is primitive and effective.*

The primitive Lie algebras fitting the description given by item (1), respectively item (2), of Morozov's Theorem will be called primitive Lie algebras of diagonal type, respectively primitive Lie algebras of affine type. The primitive Lie algebras with  $\mathfrak{g}$  simple will be called simple primitive Lie algebras.

The following result was extracted from the proof of [5, Theorem 6.5].

**Proposition 3.2.** *Let  $C$  be a projective curve and  $\mathfrak{g}$  a Lie algebra of rational vector fields on  $C$ . If  $\dim \mathfrak{g} \geq 2$  then there exists a morphism  $\varphi : C \rightarrow \mathbb{P}^1$  such that  $\mathfrak{g}$  coincides with a subalgebra of the pull-back under  $\varphi$  of the Lie algebra of holomorphic vector fields on  $\mathbb{P}^1$ , i.e.  $\mathfrak{g} \subset \varphi^* \mathfrak{sl}(2, \mathbb{C})$ .*

*Proof.* A classical result of Lie (cf. [2]) says that a finite dimensional Lie subalgebra of  $\mathbb{C}(z) \frac{\partial}{\partial z}$  has dimension at most three. Moreover, if its dimension is two then it is isomorphic to the affine Lie algebra  $\mathfrak{aff}(\mathbb{C})$ , and if its dimension is three then, it is isomorphic to the projective Lie algebra  $\mathfrak{sl}(2, \mathbb{C})$ . Therefore  $\mathfrak{g}$  has dimension at most three.

In both cases there exists  $v_1, v_2 \in \mathfrak{g}$  satisfying  $[v_1, v_2] = v_1$ . Consider the morphism  $\varphi : C \rightarrow \mathbb{P}^1$  defined by the quotient  $-\frac{v_2}{v_1}$ .

At an arbitrary point of  $C$  choose a local analytic coordinate  $w$ . We can write  $v_1 = a(w) \frac{\partial}{\partial w}$ ,  $v_2 = b(w) \frac{\partial}{\partial w}$ ,  $\varphi(w) = -b(w)/a(w)$  locally. On the one hand, the relation  $[v_1, v_2] = v_1$  implies  $ab' - ba' = a$ . On the other hand,

$$\varphi^* \frac{\partial}{\partial z} = -\frac{1}{(b/a)'(w)} \frac{\partial}{\partial w} = -\frac{a^2}{(a'b - ab')} \frac{\partial}{\partial w} = a \frac{\partial}{\partial w} = v_1.$$

Similarly  $\varphi^* z \frac{\partial}{\partial z} = v_2$ . This proves the proposition when  $\simeq \mathfrak{aff}(\mathbb{C})$ .

To conclude the proof when  $\mathfrak{g} \simeq \mathfrak{sl}(2, \mathbb{C})$  it suffices to verify that  $\tilde{v}_3 = \varphi^* z^2 \frac{\partial}{\partial z}$  belongs to  $\mathfrak{g}$ . Let  $v_3 \in \mathfrak{g}$  be such that  $[v_1, v_3] = 2v_2$ . Clearly  $[v_1, \tilde{v}_3]$  is also equal to  $2v_2$ . Thus  $[v_1, v_3 - \tilde{v}_3] = 0$  and  $\tilde{v}_3$  must be linear combination with constant coefficients of  $v_1$  and  $v_3$ .  $\square$

**3.2. Algebraic normalization of primitive Lie algebras of affine type.** The algebraic normalization of primitive Lie algebras of affine type can be carried out using a natural generalization of the argument above.

**Proposition 3.3.** *Let  $\mathfrak{g}$  be a primitive Lie algebra of rational vector fields on a projective manifold  $X$  of dimension  $n$ . If  $\mathfrak{g}$  is of affine type then there exists a rational map  $\varphi : X \dashrightarrow \mathbb{C}^n$  which conjugates  $\mathfrak{g}$  to a Lie algebra of polynomial vector fields of degree at most one.*

*Proof.* As case  $n = 1$  has already been treated in Proposition 3.2, we will assume  $n \geq 2$ . Since  $\mathfrak{g}$  is of affine type, we can write  $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{m}$  with  $[\mathfrak{m}, \mathfrak{m}] = 0$ .

In order to establish the proposition it suffices to produce a  $\mathbb{C}$ -vector space  $V \subset \mathbb{C}(X)$  of dimension  $n + 1$ , containing the constants  $\mathbb{C}$ , and invariant by the action of  $\mathfrak{g}$  by derivations.

To construct  $V$  let us consider  $\Theta_0 \in \det T_X \otimes \mathbb{C}(X)$  defined by the determinant of  $\mathfrak{m}$ , i.e. if  $v_1, \dots, v_n$  is a basis for  $\mathfrak{m}$  then  $\Theta_0 = v_1 \wedge \dots \wedge v_n$ . Notice that for any  $v \in \mathfrak{g}$ ,

$$(3.1) \quad [v, \Theta_0] = -\text{Tr}(\text{ad}(v))\Theta_0$$

where  $\text{ad}(v) \in \text{End}(\mathfrak{m})$  is the endomorphism  $\text{ad}(v)(m) = [v, m]$ , and  $\text{Tr}$  is the trace of an endomorphism.

Let us consider the  $\mathfrak{g}$ -module  $W = \wedge^{n-1} \mathfrak{m} \otimes \mathfrak{h}_0$  where  $v \in \mathfrak{h}_0$  if and only if  $v \in \mathfrak{h}$  and  $\text{Tr}(\text{ad}(v)) = 0$ . Consider the map

$$\begin{aligned} \psi : W &\longrightarrow \mathbb{C}(X) \\ \theta \otimes w &\mapsto \frac{\theta \wedge w}{\Theta_0} \end{aligned}$$

and let  $V_0$  denote its image. Observe that the non-zero functions in  $V_0$  are non-constant as they vanish at the point fixed by the isotropy subalgebra  $\mathfrak{h}$ . Equation (3.1) implies  $\psi$  is a morphism of  $\mathfrak{h}_0$ -modules and therefore  $V_0$  is invariant by the action of  $\mathfrak{h}_0$ . Moreover  $V_0$  is invariant by the action  $\mathfrak{h}$  and is mapped to  $\mathbb{C}$  by  $\mathfrak{m}$ .

Let us prove that  $\dim V_0 = n$ . On the one hand, the dimension of  $V_0$  is at most  $n$  since otherwise we would produce a non-constant function in  $V_0$  in the intersection of the kernel of the derivations in  $\mathfrak{m}$ . This would contradict the transitivity of  $\mathfrak{m}$ . On the other hand, the dimension of  $V_0$  is at least  $n$  as otherwise the functions in  $V_0$  would define a foliation invariant by the Lie algebra  $\mathfrak{g}$  contradicting the primitiveness of  $\mathfrak{g}$ .

It is now clear rational map  $\varphi : X \dashrightarrow \mathbb{C}^n$  with entries given by a basis of  $V_0$  sends  $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{m}$  to a Lie algebra of vector fields of degree at most one.  $\square$

**3.3. Proof of Theorem A.** If  $\mathfrak{g}$  is a primitive Lie algebra with isotropy  $\mathfrak{h}$  then  $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$  as otherwise there would exist a Lie algebra strictly containing  $\mathfrak{h}$  and strictly contained in  $\mathfrak{g}$ . Thus if  $\mathfrak{g}$  happens to be a complete Lie algebra then it satisfies the assumptions of Theorem 1.1 and the result follows. Theorem 3.1 tell us that  $\mathfrak{g}$  is semi-simple (and hence complete) unless  $\mathfrak{g}$  is a primitive Lie algebra of affine type. To conclude we apply Proposition 3.3.  $\square$

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