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Algebraic intersection for translation surfaces in the stratum $\mathcal{H}(2)$

Intersection algébrique dans la strate $\mathcal{H}(2)$

Smaïl Cheboui, Arezki Kessi, Daniel Massart

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Abstract

Nous étudions la quantité KVol définie par l'équation (1) sur la strate $\mathcal{H}(2)$ des surfaces de translation de genre 2, avec une singularité conique. Nous donnons une suite explicite de surfaces $L(n, n)$ telles que $\text{KVol}(L(n, n)) \rightarrow 2$ quand n tend vers l'infini, 2 étant l'infimum-conjectural-de KVol sur $\mathcal{H}(2)$.

We study the quantity KVol defined in Equation (1) on the stratum $\mathcal{H}(2)$ of translation surfaces of genus 2, with one conical point. We provide an explicit sequence $L(n, n)$ of surfaces such that $\text{KVol}(L(n, n)) \rightarrow 2$ when n goes to infinity, 2 being the conjectured infimum for KVol over $\mathcal{H}(2)$.

1 Introduction

Let X be a closed surface, that is, a compact, connected manifold of dimension 2, without boundary. Let us assume that X is oriented. Then the algebraic intersection of closed curves in X endows the first homology $H_1(X, \mathbb{R})$ with an antisymmetric, non degenerate, bilinear form, which we denote $\text{Int}(\cdot, \cdot)$.

Now let us assume X is endowed with a Riemannian metric g . We denote $\text{Vol}(X, g)$ the Riemannian volume of X with respect to the metric g , and for any piecewise smooth closed curve α in X , we denote $l_g(\alpha)$ the length of α with respect to g . When there is no ambiguity we omit the reference to g .

We are interested in the quantity

$$\text{KVol}(X, g) = \text{Vol}(X, g) \sup_{\alpha, \beta} \frac{\text{Int}(\alpha, \beta)}{l_g(\alpha)l_g(\beta)} \quad (1)$$

where the supremum ranges over all piecewise smooth closed curves α and β in X . The $\text{Vol}(X, g)$ factor is there to make KVol invariant to re-scaling of the metric g . See [6] as to why KVol is finite. It is easy to make KVol go to infinity, you just need to pinch a non-separating closed curve α to make its length go to zero. The interesting surfaces are those (X, g) for which KVol is small.

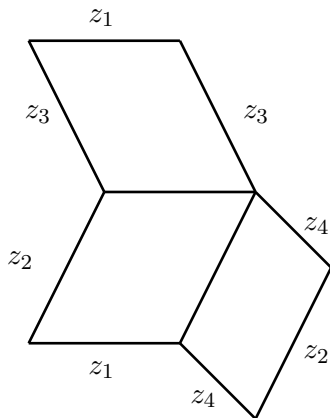


Figure 1: Unfolding an element of $\mathcal{H}(2)$

When X is the torus, we have $\text{KVVol}(X, g) \geq 1$, with equality if and only if the metric g is flat (see [6]). Furthermore, when g is flat, the supremum in (1) is not attained, but for a negligible subset of the set of all flat metrics. In [6] KVVol is studied as a function of g , on the moduli space of hyperbolic (that is, the curvature of g is -1) surfaces of fixed genus. It is proved that KVVol goes to infinity when g degenerates by pinching a non-separating closed curve, while KVVol remains bounded when g degenerates by pinching a separating closed curve.

This leaves open the question whether KVVol has a minimum over the moduli space of hyperbolic surfaces of genus n , for $n \geq 2$. It is conjectured in [6] that for almost every (X, g) in the moduli space of hyperbolic surfaces of genus n , the supremum in (1) is attained (that is, it is actually a maximum).

In this paper we consider a different class of surfaces : translation surfaces of genus 2, with one conical point. The set (or stratum) of such surfaces is denoted $\mathcal{H}(2)$ (see [3]). By [7], any surface X in the stratum $\mathcal{H}(2)$ may be unfolded as shown in Figure 1, with complex parameters z_1, z_2, z_3, z_4 . The surface is obtained from the plane template by identifying parallel sides of equal length.

It is proved in [4] (see also [2]) that the systolic volume has a minimum in $\mathcal{H}(2)$, and it is achieved by a translation surface tiled by six equilateral triangles. Since the systolic volume is a close relative of KVVol , it is interesting to keep the results of [4] and [2] in mind.

We have reasons to believe that KVVol behaves differently in $\mathcal{H}(2)$, both from the systolic volume in $\mathcal{H}(2)$, and from KVVol itself in the moduli space of hyperbolic surfaces of genus 2 ; that is, KVVol does not have a minimum over $\mathcal{H}(2)$.

We also believe that the infimum of KVVol over $\mathcal{H}(2)$ is 2. This paper is a first step towards the proof : we find an explicit sequence $L(n, n)$ of surfaces in $\mathcal{H}(2)$, whose KVVol tends to 2 (see Proposition 2.5). These surfaces are obtained from very thin, symmetrical, L-shaped templates (see Figure 2).

In the companion paper [1] we study KVVol as a function on the Teichmüller disk (the $SL_2(\mathbb{R})$ -orbit) of surfaces in $\mathcal{H}(2)$ which are tiled by three identical parallelograms (for instance $L(2, 2)$), and prove that KVVol does have a minimum there, but is not bounded from above. Therefore KVVol is not bounded from above as a function on $\mathcal{H}(2)$. In [1] we also compute KVVol for the translation surface tiled by six equilateral triangles, and find it equals 3, so it does not minimize KVVol , neither in $\mathcal{H}(2)$, nor even in its own Teichmüller disk.

2 $L(n, n)$

2.1 Preliminaries

Following [8], for any $n \in \mathbb{N}$, $n \geq 2$, we call $L(n+1, n+1)$ the $(2n+1)$ -square translation surface of genus two, with one conical point, depicted in Figure 2, where the upper and rightmost rectangles are made up with n unit squares. We call A (resp. B) the region in $L(n+1, n+1)$ obtained, after identifications, from the uppermost (resp. rightmost) rectangle, and C the region in $L(n+1, n+1)$ obtained, after identifications, from the bottom left square. Both A and B are annuli with a pair of points identified on the boundary, while C is a square with all four corners identified. We call e_1, e_2 , (resp. f_1, f_2) the closed curves in $L(n+1, n+1)$ obtained by gluing the endpoints of the horizontal (resp. vertical) sides of A and B . The closed curve which sits on the opposite side of C from e_1 (resp. f_1) is called e'_1 (resp. f'_1), it is homotopic to e_1 (resp. f_1) in $L(n+1, n+1)$. The closed curves in $L(n+1, n+1)$ which correspond to the diagonals of the square C are called g and h .

Figure 3 shows a local picture of $L(n+1, n+1)$ around the singular (conical) point S , with angles rescaled so the 6π fit into 2π .

Since e_1, e_2, f_1, f_2 do not meet anywhere but at S , the local picture yields the algebraic intersections between any two of e_1, e_2, f_1, f_2 , summed up in the following matrix:

$$\begin{array}{ccccc}
 \text{Int} & e_2 & f_1 & e_1 & f_2 \\
 e_2 & 0 & 1 & 0 & -1 \\
 f_1 & -1 & 0 & 0 & 0 \\
 e_1 & 0 & 0 & 0 & 1 \\
 f_2 & 1 & 0 & -1 & 0
 \end{array} \tag{2}$$

We call T_A (resp. T_B) the flat torus obtained by gluing the opposite sides of the rectangle made with the $n+1$ leftmost squares (resp. with the $n+1$ bottom squares), so the homology of T_A (resp. T_B) is generated by e_1 and the concatenation of f_1 and f_2 (resp. f_1 and the concatenation of e_1 and e_2).

Lemma 2.1. *The only closed geodesics in $L(n+1, n+1)$ which do not intersect e_1 nor f_1 are, up to homotopy, e_1, f_1, g , and h .*

Proof. Let γ be such a closed geodesic. It cannot enter, nor leave, A, B , nor C . If it is contained in A , and does not intersect e_1 , then it must be homotopic to e_1 , which is the

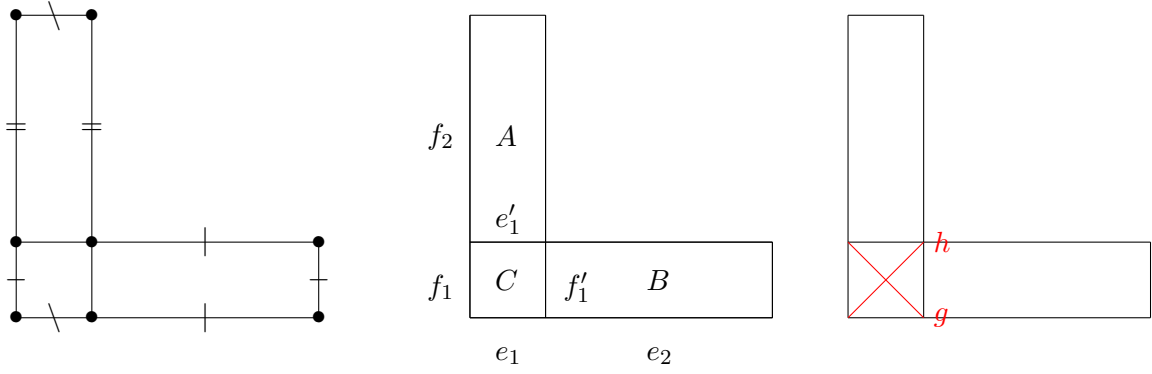


Figure 2: $L(n+1, n+1)$

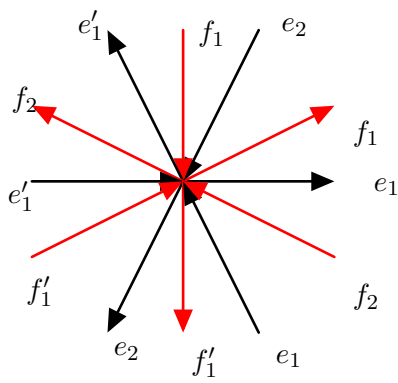


Figure 3: Local picture around the conical point

soul of the annulus from which A is obtained by identifying two points on the boundary. Likewise, if it is contained in B , and does not intersect f_1 , then it must be homotopic to f_1 . Finally, if γ is not contained in A nor in B , it must be contained in C . The only closed geodesics contained in C are the sides and diagonals of the square from which C is obtained, which are e_1, e'_1, f_1, f'_1, g , and h . \square

Lemma 2.2. *For any closed geodesic γ in $L(n+1, n+1)$, we have $l(\gamma) \geq n|\text{Int}(\gamma, e_1)|$.*

Proof. For each intersection with e_1 , γ must go through A , from boundary to boundary. \square

Obviously a similar lemma holds with f_1 instead of e_1 . For g and h the proof is a bit different :

Lemma 2.3. *For any closed geodesic γ in $L(n+1, n+1)$, we have $l(\gamma) \geq n|\text{Int}(\gamma, g)|$.*

Proof. First, observe that between two consecutive intersections with g , γ must go through either A or B , unless γ is g itself, or h : indeed, the only geodesic segments contained in C with endpoints on g are segments of g , or h . Obviously $\text{Int}(g, g) = 0$, and from the intersection matrix (2), knowing that $[g] = [e_1] - [f_1]$, $[h] = [e_1] + [f_1]$, we see that $\text{Int}(g, h) = 0$.

Thus, either $\text{Int}(\gamma, g) = 0$, or each intersection must be paid for with a trek through A or B , of length at least n . \square

Obviously a similar lemma holds with h instead of g . Note that Lemmata 2.1, 2.2, 2.3 imply that the only geodesics in $L(n+1, n+1)$ which are shorter than n are e_1, f_1, g, h , and closed geodesics homotopic to e_1 or f_1 .

Lemma 2.4. *Let I, J be positive integers, take $a_{ij}, i = 1, \dots, I, j = 1, \dots, J$ in \mathbb{R}_+ , and $b_1, \dots, b_I, c_1, \dots, c_J$ in \mathbb{R}_+^* . Then we have*

$$\frac{\sum_{i,j} a_{ij}}{\left(\sum_{i=1}^I b_i\right) \left(\sum_{j=1}^J c_j\right)} \leq \max_{i,j} \frac{a_{ij}}{b_i c_j}.$$

Proof. Re-ordering, if needed, the a_{ij}, b_i, c_j , we may assume

$$\frac{a_{ij}}{b_i c_j} \leq \frac{a_{11}}{b_1 c_1} \quad \forall i = 1, \dots, I, j = 1, \dots, J.$$

Then $a_{ij} b_1 c_1 \leq a_{11} b_i c_j \quad \forall i = 1, \dots, I, j = 1, \dots, J$, so

$$b_1 c_1 \sum_{i,j} a_{ij} \leq a_{11} \sum_{i,j} b_i c_j = a_{11} \left(\sum_{i=1}^I b_i\right) \left(\sum_{j=1}^J c_j\right).$$

\square

2.2 Estimation of $\text{KVol}(L(n, n))$

Proposition 2.5.

$$\lim_{n \rightarrow +\infty} \text{KVol}(L(n+1, n+1)) = 2.$$

Proof. First observe that $\text{Vol}(L(n+1, n+1)) = 2n+1$, $l(e_1) = 1$, $l(f_2) = n$, $\text{Int}(e_1, f_2) = 1$, so

$$\text{KVol}(L(n+1, n+1)) \geq 2 + \frac{1}{n}.$$

To bound $\text{KVol}(L(n+1, n+1))$ from above, we take two closed geodesics α and β ; by Lemmata 2.2, 2.3, if either α or β is homotopic to e_1 , f_1 , g , or h , then

$$\frac{\text{Int}(\alpha, \beta)}{l(\alpha)l(\beta)} \leq \frac{1}{n},$$

so from now on we assume that neither α or β is homotopic to e_1 , f_1 , g , h . We cut α and β into pieces using the following procedure : we consider the sequence of intersections of α with e_1, e'_1, f_1, f'_1 , in cyclical order, and we cut α at each intersection with e_1 or e'_1 which is followed by an intersection with f_1 or f'_1 , and at each intersection with f_1 or f'_1 which is followed by an intersection with e_1 or e'_1 . We proceed likewise with β . We call $\alpha_i, i = 1, \dots, I$, and $\beta_j, j = 1, \dots, J$, the pieces of α and β , respectively.

Note that

$$l(\alpha) = \sum_{i=1}^I l(\alpha_i), \quad l(\beta) = \sum_{j=1}^J l(\beta_j), \quad \text{and}$$

$$|\text{Int}(\alpha, \beta)| \leq \sum_{i,j} |\text{Int}(\alpha_i, \beta_j)|,$$

so Lemma 2.4 says that

$$\frac{|\text{Int}(\alpha, \beta)|}{l(\alpha)l(\beta)} \leq \max_{i,j} \frac{|\text{Int}(\alpha_i, \beta_j)|}{l(\alpha_i)l(\beta_j)}.$$

We view each piece α_i (resp. β_j) as a geodesic arc in the torus T_A (resp. T_B), with endpoints on the image in T_A (or T_B) of f_1 or f'_1 (resp. e_1 or e'_1), which is a geodesic arc of length 1, so we can close each α_i (resp. β_j) with a piece of f_1 or f'_1 (resp. e_1 or e'_1), of length ≤ 1 . We choose a closed geodesic $\hat{\alpha}_i$ (resp. $\hat{\beta}_j$) in T_A (resp. T_B) which is homotopic to the closed curve thus obtained. We have $l(\hat{\alpha}_i) \leq l(\alpha_i) + 1$, $l(\hat{\beta}_j) \leq l(\beta_j) + 1$, so

$$\frac{1}{l(\hat{\alpha}_i)l(\hat{\beta}_j)} \geq \frac{1}{(l(\alpha_i) + 1)(l(\beta_j) + 1)}.$$

Now recall that $l(\alpha_i), l(\beta_j) \geq n$, so $l(\alpha_i) + 1 \leq (1 + \frac{1}{n})l(\alpha_i)$, whence

$$\frac{1}{l(\hat{\alpha}_i)l(\hat{\beta}_j)} \geq \frac{1}{l(\alpha_i)l(\beta_j)} \left(\frac{n}{n+1} \right)^2.$$

Next, observe that $|\text{Int}(\alpha_i, \beta_j)| \leq |\text{Int}(\hat{\alpha}_i, \hat{\beta}_j)| + 1$, because $\hat{\alpha}_i$ (resp. $\hat{\beta}_j$) is homologous to a closed curve which contains α_i (resp. β_j) as a subarc, and the extra arcs cause at

most one extra intersection, depending on whether or not the endpoints of α_i and β_j are intertwined. So,

$$\frac{|\text{Int}(\alpha_i, \beta_j)|}{l(\alpha_i)l(\beta_j)} \leq \frac{|\text{Int}(\hat{\alpha}_i, \hat{\beta}_j)| + 1}{l(\hat{\alpha}_i)l(\hat{\beta}_j)} \left(\frac{n+1}{n}\right)^2 \leq \left(\frac{|\text{Int}(\hat{\alpha}_i, \hat{\beta}_j)|}{l(\hat{\alpha}_i)l(\hat{\beta}_j)} + \frac{1}{n^2}\right) \left(\frac{n+1}{n}\right)^2,$$

where the last inequality stands because $l(\hat{\alpha}_i) \geq n$, $l(\hat{\beta}_j) \geq n$, since $\hat{\alpha}_i$ and $\hat{\beta}_j$ both have to go through a cylinder A or B at least once. Finally, since $\hat{\alpha}_i$ and $\hat{\beta}_j$ are closed geodesics on a flat torus of volume $n+1$, we have (see [6])

$$\frac{|\text{Int}(\hat{\alpha}_i, \hat{\beta}_j)|}{l(\hat{\alpha}_i)l(\hat{\beta}_j)} \leq \frac{1}{n+1}, \text{ so}$$

$$\frac{|\text{Int}(\alpha_i, \beta_j)|}{l(\alpha_i)l(\beta_j)} \leq \left(\frac{1}{n+1} + \frac{1}{n^2}\right) \left(\frac{n+1}{n}\right)^2 = \frac{1}{n} + o\left(\frac{1}{n}\right),$$

which yields the result, recalling that $\text{Vol}(L(n+1, n+1)) = 2n+1$. □

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