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Nicolas Bouleau

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# Fascination and exemplarity of arithmetic

Nicolas BOULEAU

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*To Gustave Choquet, generous professor,  
and profound philosopher of the question  
"to discover or to invent".*

The history of mathematics for centuries, as well as its recent developments, clearly show that arithmetic and number theory arouse a passion of a very particular nature in the mathematician, whether he is an amateur or a high-level researcher.

This is the phenomenon that we are exploring here, starting with concrete examples where this fascination is apparent and analyzable, and then discussing a more philosophical answer through the exemplary value of arithmetic.

The formulas which relate to whole numbers have a very particular attractiveness. They give the impression of a remarkable, eternal crystallization, a find revealing an incontestable reality and moreover understandable by the Boeotian. Ramanujan's combinatorial formulas are a famous example of this, which shows that in some minds creativity in this area does not require the prerequisite of a vast mathematical culture. We start the discussion with the psychological side.

## I. Fascination.

In his famous *Essai sur la théorie des nombres* Legendre writes "It is also likely that Euler had a particular penchant for this kind of research, and that he engaged in it with a kind of passion, as happens to all who do likewise". Polite euphemism ! I would go so far as to say that for the researcher, number theory should be consumed with moderation. This is the most addictive field there is. For many reasons, in particular by the fact that we often meet there curious formulas that give us the impression of leading to a French formal garden but where the apparent ordering quickly changes, alas, into an inextricable copse.

There is of course the pleasure of knowing *libido sciendi* : conjectures accompany the theory and some have acquired from their age the respectability of vestiges that have stood the test of time, such as famous old bottles. The Riemann hypothesis is the biggest challenge right now. It shines like a distant mirage. It is *almost* proven. Millions of zeros of the  $\zeta$  function have  $1/2$  abscissa. This is not the result of numerical approximation, but of rigorous proof.<sup>1</sup> But of course it happens in mathematics that a regularity be

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<sup>1</sup>In the logical hierarchy this hypothesis is a  $\Pi_1^0$  statement (G. Kreisel), its negation is a recursively enumerable statement. This *might* suggest that his refutation will come about by chance.

verified up to a very distant bound and nevertheless turn out to be false.<sup>2</sup> It fascinates in itself much more than its equivalence in terms of the enumeration of prime numbers and is accompanied today by a cluster of very fine works and generalizations.

To reason concretely I have chosen a simple example already revealing of a typical form of dialectic. We can establish the formula

$$\zeta(t) = \sum_n \frac{1}{n^s} \frac{1}{Q_s^t(n)}.$$

where  $\zeta$  is the Riemann function,  $s$  and  $t$  are here strictly positive real numbers,  $t > 1$ ,  $Q_s^t(n) = \prod_{p^s-1} \frac{p^t-1}{p^s-1}$  the product being taken over prime divisors  $p$  of  $n$ , and  $n$  runs over natural numbers.<sup>3</sup>

The above formulas have the property of modifying the terms of the series  $\sum \frac{1}{n^t}$  by arithmetic functions so that the sum remains unchanged.

For example we have

$$\sum_n \frac{1}{n^2 Q_2^4(n)} = \zeta(4).$$

$Q_2^4(n) = \prod (1 + p^2)$  the product being for  $p$  prime dividing  $n$ . This whole number varies a lot, sometimes larger sometimes smaller than  $n^2$ . The ratio  $Q_2^4(n)/n^2$  has no limit. For instance  $Q_2^4(94) = 11050$ , and  $Q_2^4(96) = 50$ .

We can legitimately wonder that the values of  $Q_2^4(n)$  for large  $n$  exactly compensate the differences for small  $n$  so that the sum be exactly  $\zeta(4)$ . There appears a kind of complicity between the large values of  $n$  and the small ones through the prime numbers that intervene in the product  $Q_2^4$ , the integer  $Q_2^4(n)$  playing the role of  $n^2$ . And this pretender seems to play his role very well because we also have

$$\sum_n \frac{Q_2^4(n)}{n^4} = \zeta(2).$$

On the other hand both the series  $\sum (Q_2^4(n))^s$  and  $\sum (Q_2^4(n))^{-s}$  diverge.

The formulas expressing regularities of integers seem to provide properties of a *non-contingent* reality. In general, advances in functional analysis or in other abstract branches do not reach the same degree of *realism*. They are more like artistic creation where one can admire the talented use of a language.<sup>4</sup> And as the language of contemporary mathematics is very rich, so rich that it repels the layman, it is not surprising that with such freedom of language new things may be obtained.<sup>5</sup>

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<sup>2</sup>A famous example concerns the functions  $\pi(x)$  and  $li(x)$  for counting prime numbers smaller than  $x$ . Gauss had verified the inequality  $\pi(x) < li(x)$  for  $x$  up to 3 million. We now know (J. E. Littlewood 1914) that inequality reverses infinitely many times.

<sup>3</sup>Several proofs are available. For instance by using the fact that if the series  $\sum f(n)$  converges and if  $f$  is multiplicative then  $\sum f(n) = \prod_p (1 + f(p) + f(p^2) + \dots)$ .

<sup>4</sup>Typically, as Renaissance artists used the language of Greek architecture, enriching it with new combinations, such as the Serlian motif attributed to Sebastiano Serlio.

<sup>5</sup>Note that however Palladio [13] emphasizes the importance of the composition being such that nothing can be added nor taken away. See also [4].

In arithmetic one has the impression that reality speaks by itself. And somehow, on things as “obvious” as whole numbers, one can be surprised that reality has not already said everything, and that it reveals its secrets to us in droplets.

## II. Exemplarity.

A point that deserves further study is whether this keen interest to integers is not linked to a desire to know more deeply than the enjoyment of handling simple entities and arriving at formulas that everyone understands. Does not the fascination with arithmetic come from the fact that it is an exemplary of properties of physical and living nature?

There is an ambiguity in the world of culture concerning mathematics, which arithmetic illustrates particularly well.

For many philosophers, rationality is a reducing frame of reality unable to account for the most important things in life. This is particularly clear from the beginning of the 19th century when, as several observers have noted, the “two cultures” separated with on the one hand the impressive development of mathematized physics and on the other the emergence of more committed philosophies than Kantianism with the ambition to tackle what gives meaning to human existence. Hegel and then Marx, for example, tackle the question of becoming, of what makes history move forward. Some even go so far as to denigrate deductive rigor, such as Schopenhauer who mocks mathematicians who are looking for a demonstration of Euclid’s axiom of parallels and who did not understand that it was obvious. This passage from his great treatise deserves to be quoted:

Die Eukleidische Demonstrirmethode hat aus ihrem eigenen Schooß ihre treffendste Parodie und Karikatur geboren, an der berühmten Streitigkeit über die Theorie der Parallelen und den sich jedes Jahr wiederholenden Versuchen, das elfte Axiom zu beweisen. Dieses nämlich besagt, und zwar durch das mittelbare Merkmal einer schneidenden dritten Linie, daß zwei sich gegen einander neigende (denn dies eben heisst kleiner als zwei rechte seyn), wenn genugsam verlängert, zusammentreffen müssen; welche Wahrheit nun zu komplicirt seyn soll, um für selbstevident zu gelten, daher sie eines Beweises bedarf, der nun aber nicht aufzubringen ist; eben weil es nichts Unmittelbareres giebt. Mich erinnert dieser Gewissenskrupel an die Schillersche Rechtsfrage:

“Jahre lang schon bedien’ ich mich meiner Nase zum Riechen:

Hab’ ich denn wirklich an sie auch ein erweisliches Recht?”

ja, mir scheint, daß die logische Methode sich hiedurch bis zur Niaiserie steigere.<sup>6</sup>

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<sup>6</sup>The Euclidean method of demonstration has brought forth from its own womb its most striking parody and caricature in the famous controversy over the theory of parallels, and in the attempts, repeated every year, to prove the eleventh axiom. This axiom asserts, and that indeed through the indirect criterion of a third intersecting line, that two lines inclined to each other (for this is the precise meaning of “less than two right angles”), if produced far enough, must meet. Now this truth is supposed to be too complicated to pass as self-evident, and therefore needs a proof; but no such proof can be produced, just because there is nothing more immediate. This sample of conscience reminds me of Schiller’s question of law : “For years I have already made use of my nose for smelling: Then have I actually a right to it that can be demonstrated ?” In fact, it seems to me that the logical method is in this way reduced to an absurdity. [*Die Welt als Wille und Vorstellung*, Band 2, Kap. 13]

That Schopenhauer does not know the non-Euclidean geometries developed by Gauss, Lobachevsky and Bolyai, and their fruitfulness, is one thing. These works were recent in his time. But it's a shame he didn't realize how fascinating it is that Euclid felt the need to make this statement an axiom. The treaty is written as if we were going to try to do without this statement. The first 28 propositions of the Elements are established without resorting to this axiom, it is admitted, it seems, for lack of having been able to demonstrate it. If Euclid is a group of Greek geometers who lived after Plato and before Archimedes, as is generally believed today, after how many attempts at demonstration have they resolved to confess their ignorance and to throw at the sagacity of scholars this "obvious" which remained unresolved for more than twenty centuries? It is that these Greeks did not compromise with rigor. Demonstrated meant demonstrated, and not obvious. They invented true mathematics. That is no mean achievement!

Nietzsche goes even further. Science is for him a descriptive language that explains nothing.

Aber eine essentiell mechanische Welt wäre eine essentiell sinnlose Welt! Gesetzt, man schätzte den Werth einer Musik darnach ab, wie viel von ihr gezählt, berechnet, in Formeln gebracht werden könne – wie absurd wäre eine solche "wissenschaftliche" Abschätzung der Musik! Was hätte man von ihr begriffen, verstanden, erkannt! Nichts, geradezu Nichts von dem, was eigentlich an ihr "Musik" ist!<sup>7</sup>

Other philosophers consider the two cultures as a matter of fact, as two immiscible modes of thought, which do not concern the same areas.

And yet, if we can get an impression from the practice of number theory, it is because the natural numbers show a nature that is not at all apodictic but rather mischievous, always showing us how irregular numbers are, whimsical, elusive. We are faced with this paradox that in arithmetic everything is perfectly rigorous and yet the result of all this rigor is a jumble which seems to tell us that beyond the small numbers exists an unknown landscape whose difficulty is revealed by innumerable conjectures that we leave unsolved.

The arithmetic shows us a sample of nature which seems to thwart all the stratagems that would ensure the rule of law. A capricious, varied, hidden nature, with secret gifts that appear through sublime connections :  $\pi^2/6$ , etc.

In recent works [2] Beliakov and Matiyasevich study, through computer-driven numerical calculations, approximations of the zeros of  $\zeta(s)$  by finite Dirichlet series. Their calculations are compatible to this degree of approximation with the Riemann hypothesis, but they insist on *the structures* that these approximations reveal and present them as new conjectures. It seems there, and in the same way in [11], that the epistemological scene which is played out attributes a new role to arithmetic conceived as a resource of novelty and not as an automated production.

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<sup>7</sup>A mostly mechanical world, but it would be a mostly stupid world! If we measured the value of music by what it can be calculated and counted, by what it can be translated into numbers, how absurd would this scientific assessment be! What would we have understood, appreciated, known of a melody thus gauged? Nothing, and literally nothing, of what exactly makes this "music"! [*The Gay Science*, book V, §373]

The case of Henri Bergson is characteristic. He poses the problem very well by trying to think why living things, plants, animals would be more creative than what science can be with its rationality which according to him limits it to automatic things. For this, he forged the concepts of 'cinematographic mechanism' and 'mechanistic illusion'. This is precisely where the only assessment error of his approach lies. Deduction at the arithmetic level is not automatic. Bergson misplaced the automatic/creative cutoff. It would be completely anachronistic to blame him for it since at that time David Hilbert himself had deemed it relevant to launch his *program* in an attempt to demonstrate the consistency of arithmetic (cf. [6]) .

Let us refer to the terms used by Hilbert in his penetrating introduction to the famous 23 problems. After having explained what an interesting mathematical problem is, and that its solution must proceed through a finite number of purely logical steps, he defends the thesis that every mathematical problem has a solution:

This conviction of the solvability of every mathematical problem is a powerful incentive to the worker. We hear within us the perpetual call : There is the problem. Seek its solution. You can find it by pure reason, for in mathematics there is no *ignorabimus*.<sup>8</sup>

The existence of Hilbert's program corroborates the validity of Bergson's vision for thinking about nature in his era.

However, the "automatic vs creative" cutoff would have had to put the arithmetic on the creative side.<sup>9</sup> We can say today that – contrary to what Bergson thought – arithmetic carries a very fundamental exemplarity for us to understand living things precisely. This is due to two *epistemological revolutions* separated by about thirty years.

First, the proof of the impossibility of Hilbert's program, the incompleteness of arithmetic and the phenomenology of the undecidable, which we must attribute to Kurt Gödel, Alonzo Church and Alan Turing.

Then the transformation of biology. The 20th century brought us a real epistemological revolution by explaining the combinatorial nature of living things. With the discovery of the double helix in the years 1950-60 we know that living beings are based precisely on a combination, which radically changed the landscape in which all the philosophies of nature were in dialogue. What is quite disturbing in this case is that nature also proceeds *more geometrico*. Since the discovery of the role of DNA and its generality in the living world, it appears that nature is written like arithmetic. So that to understand the specificity of genomic modifications it is quite enlightening to look at mathematics because they are very similar to biology and synthetic chemistry. Of course, the stakes are not the same, but biology also presents the phenomenology of the undecidable, a fundamental trait from the point of view of knowledge. The main difference is that in mathematics the context of combinatorics is simple and *ne varietur* while in biology it is very rich, evolving, and poorly understood.

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<sup>8</sup>Translated for the *Bulletin of the A.M.S.*, with the author's permission, by Dr. Mary Winston Newson. The original appeared in the *Göttinger Nachrichten*, 1900, pp. 253-297.

<sup>9</sup>The cutoff is between Presburger's arithmetic and arithmetic.

The most convincing argument regarding the presence of the undecidable in natural or artificial genomic innovation is based on the study of the word problem for Thue systems (cf. [5] chap. XV): we consider the words written, say, with two letters, we set the equivalence of a finite number of pairs of words and we propagate this equivalence by the operations of concatenation, of simplification on the left and on the right. And we ask the question of knowing if there is an algorithm making it possible to know if two given words are equivalent. In 1947 Markov and Post showed that this problem was in general undecidable. Since then, Tsejtin and Matiyasevich clarified this undecidability for increasingly simple systems up to a Thue system with an alphabet of two letters and three relations (cf. [12]).

In comparison in the living, the alphabet has four letters (adenine, thymine, cytosine, guanine) and the natural mutations – or artificial with transgenesis – which proceed to splits and reattachment are of such complexity that we do not see how they could fail to lead to undecidability.

We can express the results of incompleteness on the hierarchy of theories that include arithmetic by saying that there is no formalized axiomatic providing all the properties of integers.

A misunderstanding concerns the significance of Gödel's own result. For the sake of rigor, Gödel did not insist enough on the new vision of arithmetic that appears. There is undecidability, Gödel proves it for certain particular statements but these statements are there only because on them *the proof* of the undecidability is possible, obviously the landscape is now that of a zone between the provable and the refutable whose frontier is unknown. Lady Arithmetic keeps all kinds of properties in her attics including, perhaps, the many conjectures that we leave unsolved.

So the epistemic lesson is not just that we can only gradually discover the combinatorial properties, there being no automatism in this process, but moreover there will always remain an unknown residue whose presence we will have to get used to. Gödel said "*ignorabimus*, we will never know, that exists." Now it's up to us to live with it and take it into account.

Nature is now arithmetic, but by no means automatic, on the contrary the bearer of a creativity that largely escapes strictly nomological thought.<sup>10</sup>

There are obviously important differences. The context of mathematical theories, namely the language of first-order predicates, is not affected by the progress of proofs at the heart of those theories, while living things evolve in a context which they both take account of and modify. But these differences only increase the reasons for realizing that we must get used to an epistemology in which there is an element of definitive ignorance.<sup>11</sup>

This ignorance in chemistry or synthetic biology means that we do not know in advance what a new molecule will do on all existing living things. And since we do not know the

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<sup>10</sup>The decision problem, that is whether there is an algorithm to solve a question, has been guessed by synthetic chemists and biologists intuitively long before a logical approach. Roald Hoffmann remarked [7] from the problem of the synthesis of cubane that there is no a priori method for chemical synthesis, and that when the synthesis succeeds it takes paths through molecules more complex than the one we ultimately want to obtain.

<sup>11</sup>As the mathematician Pierre Samuel wrote half a century ago "The forgetting of the limitations of science is the direct cause of many of the myths which constitute the credo of scientism." (*Survivre* n10 oct-nov 1971).

details of attempts nature made during evolution to reach the present state, it may be that the purely combinatorial artificial game fabricates entities that have never been encountered by nature, neither used nor tried. The required caution was recognized and made explicit on containment issues at the Asilomar and Cartagena conferences, but international institutions to monitor their application have yet to be established.

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