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Algebraic intersection for translation surfaces in a family of Teichmüller disks

S. Cheboui, A. Kessi, D. Massart

July 22, 2020

Abstract

We give a hyperbolic-geometric construction to compute the quantity KVol defined in Equation (1) in a family of Teichmüller disks of square-tiled surfaces.

1 Introduction

1.1 Definitions

Let X be a closed surface, that is, a compact, connected manifold of dimension 2, without boundary. Let us assume that X is oriented. If two C^1 closed curves α and β in X intersect transversally at a point $P \in X$, we set $\text{Int}_P(\alpha, \beta) = 1$ if β crosses α from right to left, and $\text{Int}_P(\alpha, \beta) = -1$ otherwise. Then the algebraic intersection $\text{Int}(\alpha, \beta)$ of α and β is the sum over all intersection points P of $\text{Int}_P(\alpha, \beta)$. The algebraic intersection endows the first homology $H_1(X, \mathbb{R})$ with a symplectic bilinear form. In particular $\text{Int}(\alpha, \beta)$ is finite, and only depends on the homology classes of α and β .

Now let us assume X is endowed with a Riemannian metric g . We denote $\text{Vol}(X, g)$ the Riemannian volume of X with respect to the metric g , and for any piecewise smooth closed curve α in X , we denote $l_g(\alpha)$ the length of α with respect to g . When there is no ambiguity we omit the reference to g .

We are interested in the quantity

$$\text{KVol}(X, g) = \text{Vol}(X, g) \sup_{\alpha, \beta} \frac{\text{Int}(\alpha, \beta)}{l_g(\alpha)l_g(\beta)} \quad (1)$$

where the supremum ranges over all piecewise smooth closed curves α and β in X . The $\text{Vol}(X, g)$ factor is there to make KVol invariant to re-scaling of the metric g . See [8] as to why KVol is finite. The quantity KVol comes up naturally when you want to compare the stable norm (a norm which measures the length of a homology class, with respect to the metric g) with the Hodge norm (or L^2 -norm) on $H_1(M, \mathbb{R})$ (see [8]). It is easy to make KVol go to infinity, you just need to pinch a non-separating closed curve α to make its length go to zero. The interesting surfaces are those (X, g) for which KVol is small.

When X is the torus, we have $\text{KVol}(X, g) \geq 1$, with equality if and only if the metric g is flat (see [8]). Furthermore, when g is flat, the supremum in (1) is not attained, but for a negligible subset of the set of all flat metrics. In [8] KVol is studied as a function of g , on the moduli space of hyperbolic (that is, the curvature of g is -1) surfaces of fixed genus. It is proved that KVol goes to infinity when g degenerates by pinching a non-separating closed curve, while KVol remains bounded when g degenerates by pinching a separating closed curve.

This leaves open the question whether KVol has a minimum over the moduli space of hyperbolic surfaces of genus n , for $n \geq 2$. It is conjectured in [8] that for almost every (X, g) in the moduli space of hyperbolic surfaces of genus n , the supremum in (1) is attained (that is, it is actually a maximum).

In this paper we consider a different class of surfaces : translation surfaces of genus s , with one conical point. The set (or stratum) of such surfaces is denoted $\mathcal{H}(2s - 2)$ (see [5]). We consider the family of translation surfaces $St(2s - 1)$ (so named after [10]) depicted in Figure 1, for $s \in \mathbb{N}, s \geq 2$, obtained by gluing the opposite sides of a staircase-shaped template made of $2s - 1$ squares (see Figure 1).

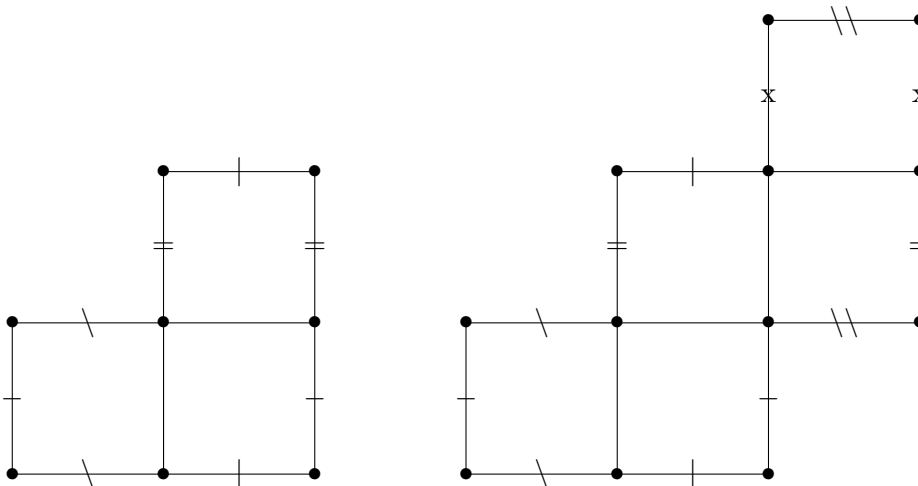


Figure 1: $St(3)$ and $St(5)$

Our first result is (see Corollary 3.3)

$$\forall s \geq 2, \text{KVol}(St(2s - 1)) = 2s - 1.$$

This is the first exact computation of KVol , outside of flat tori.

Before stating our next result we need to elaborate a little bit on the Teichmüller disk of $St(2s - 1)$.

1.2 The Teichmüller disc \mathcal{T} of $St(2s - 1)$

Let us explain the terminology. Every translation surface may be viewed as a plane template with parallel sides of equal length pairwise identified. The group $GL_2^+(\mathbb{R})$ acts linearly on templates, preserving identifications, so it acts on translation surfaces. It may happen that for some template T and some element A of $GL_2^+(\mathbb{R})$, both T and $A.T$ are templates of the same translation surface X . Then we say that A lies in the Veech group of X , that is, the subgroup of $GL_2^+(\mathbb{R})$ which preserves X . Since the Veech group must preserve volumes, it is a subgroup of $SL_2(\mathbb{R})$, and it turns out to be a Fuchsian group (see [12]). The orbit of X under $GL_2^+(\mathbb{R})$ is called the Teichmüller disk of X . For the purpose of studying $KVol$, it is convenient to identify templates which are related by a similitude transformation (an isometry composed with a re-scaling). Recall that $GL_2^+(\mathbb{R})$ quotiented by similitudes is the hyperbolic plane \mathbb{H}^2 . The Teichmüller disk of X may thus be seen as the quotient of \mathbb{H}^2 by the Veech group of X , which is a Fuchsian group.

We denote by $\mathcal{T}(St(2s - 1))$, or just \mathcal{T}_s if there is no ambiguity, the Teichmüller disc \mathcal{T} of $St(2s - 1)$. The Veech group Γ of $St(2s - 1)$ is generated, for any $s \geq 2$, by

$$T = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \text{ and } R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

it is a subgroup of index 3 in $SL_2(\mathbb{Z})$ (see [11], or [3]). It has a fundamental domain \mathcal{D} in \mathbb{H}^2 comprised between the straight lines $x = \pm 1$, and the unit half-circle centered at the origin. The two vertical straight lines are identified by T , and the two halves of the unit half-circle are identified by R , with i as fixed point (see Figure 2).

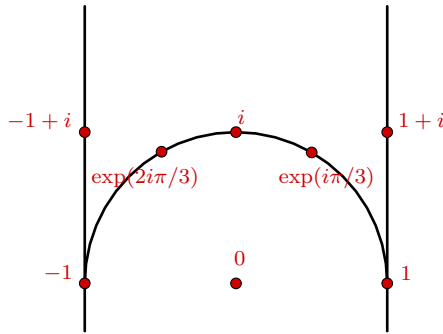


Figure 2: A fundamental domain for the Teichmüller disk of $St(2s - 1)$

Here is a convenient way to view the Teichmüller disk \mathcal{T}_s of $St(2s - 1)$. Any element X of \mathcal{T}_s , such as the one depicted on the right in Figure 3, has a template made of $2s - 1$ congruent parallelograms. Modulo some similitude, we may assume $b=1$. Then β lies in the fundamental domain depicted above, and completely determines X , together with the number $2s - 1$ of squares. We shall often label the surface X by the complex number α , or the corresponding vector in \mathbb{R}^2 , and the integer s . For instance, $St(3)$ is

$X_2(i) = X_2(0, 1)$, and $St(2s - 1) = X_s(i)$ for all s . There are other interesting surfaces in \mathcal{T}_2 : $X_2(\pm 1 + i) = X_2(\pm 1, 1)$ is the only (other than $St(3)$) three-square surface of genus 2, while $X_2(\exp(i\pi/3)) = X_2(\exp(2i\pi/3))$ is the only translation surface of genus

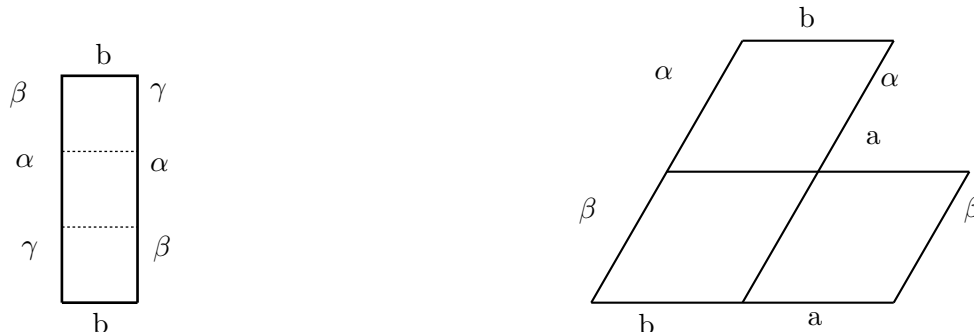


Figure 3: $X_2(1, 1)$ and $X_2(\exp(i\pi/3))$

2 tiled by 6 equilateral triangles. It turns out that $X_s(1, 1)$ and $X_s(\exp(i\pi/3))$ also have $KVol = 2s - 1$.

For $(x, y) \in \mathbb{H}^2$, we denote $T(x, y)$ the flat torus

$$\mathbb{R}^2/\mathbb{Z}(1, 0) \oplus \mathbb{Z}(x, y).$$

The translation surface $X_s(x, y)$ is a ramified, $(2s - 1)$ -fold, Riemannian cover of $T(x, y)$. For $(p, q) \in \mathbb{Z}^2$ with $p \wedge q = 1$, we say a geodesic segment in $T(x, y)$ has direction (p, q) if its lifts to the universal cover \mathbb{R}^2 of $T(x, y)$ are parallel to the vector $p(1, 0) + q(x, y)$. We say a geodesic segment in $X_s(x, y)$ has direction (p, q) if it projects to a geodesic segment of direction (p, q) in $T(x, y)$.

Beware this may cause a bit of confusion at first, because a geodesic segment in $T(x, y)$ with direction (p, q) may not lift, in \mathbb{R}^2 , to a segment parallel to (p, q) , unless $(x, y) = (0, 1)$. The upside is that this definition is independent of (x, y) . For instance, the algebraic intersection of two simple closed geodesics in $T(x, y)$, of respective directions (p, q) and (p', q') , is $pq' - p'q$, regardless of (x, y) .

This allows a neat formulation of the following property: given $p, q, p', q' \in \mathbb{Z}$ with $p \wedge q = p' \wedge q' = 1$, and $p/q \neq p'/q'$ in $\mathbb{Q} \cup \{\infty\}$, the hyperbolic geodesic with endpoints p/q and p'/q' is the locus of the flat tori in which the geodesics with respective directions (p, q) and (p', q') are orthogonal. Since the covering $X(x, y) \rightarrow T(x, y)$ is Riemannian, the hyperbolic geodesic with endpoints p/q and p'/q' is also the locus of surfaces $X(x, y)$ in which the geodesics with respective directions (p, q) and (p', q') are orthogonal.

In the torus all directions are alike, because the group $SL_2(\mathbb{Z})$ of orientation-preserving, affine diffeomorphisms of \mathbb{T}^2 acts transitively on the set of directions. However (see [5]), the Veech group Γ of $St(2s - 1)$ has two orbits, that of $(0, 1)$, which comprises all directions (p, q) with $p \neq q \pmod{2}$, and that of $(1, 1)$, which comprises all directions (p, q) with $p = q \pmod{2}$. Note that $(p, q) = (1, 0)$ gives $p/q = \infty$, which is one of the

points at infinity of the Teichmüller disk \mathcal{T} , while $(1, 1)$ gives 1, which is the other point at infinity of \mathcal{T} .

Simple closed geodesics on a translation surface in $\mathcal{H}(2s - 2)$ are of two kinds (see [5]): saddle connections, which go through the conical point, and non-singular closed geodesics, which do not. Since our surfaces have only one conical point, every closed curve is homologous to a linear combination of closed saddle connections, so in the definition of $KVol$ we may consider only closed saddle connections. Non-singular closed geodesics come in cylinders of parallel geodesics of equal length. In a translation surface in $\mathcal{H}(2s - 2)$, for any direction, there are at most s such cylinders. If the translation surface lies in \mathcal{T}_s , the s -cylinder directions are precisely those in the orbit, under the Veech group Γ , of $(0, 1)$, and the other directions have only one cylinder.

For any direction (p, q) , denote $r = p/q$ and $l_r(x, y)$ the length, in the torus $T(x, y)$, of simple closed geodesics of direction (p, q) . The horocycles with point at infinity r are the level sets of the function l_r . We shall see (Lemma 2.5) that the saddle connections, in $X(x, y)$, of direction (p, q) , have the same length $l_r(x, y)$.

Therefore there are two ways of going to infinity in the Teichmüller disk $\mathcal{T}_s : (x, y)$ may converge to $(1, 0)$, in which case there is a one-cylinder direction in $X(x, y)$ whose saddle connections become arbitrarily short ; or $y \rightarrow \infty$, in which case there is an s -cylinder direction in $X(x, y)$ whose saddle connections become arbitrarily short.

1.3 Statement and discussion of the results

Let us call $V_{\pm 1}$ the open horocyclic neighborhood of the lower cusp depicted in Figure 4. Denote \mathcal{Z} the union in \mathbb{H}^2 of all hyperbolic geodesics with endpoints in $\mathbb{Z} \cup \{\infty\}$, and their images under the Veech group Γ . Let us denote $\text{End}(\mathcal{Z})$ the set of pair of points at infinity of \mathbb{H}^2 which are endpoints of elements of \mathcal{Z} .

Theorem 1.1. *For every $s \geq 2$, the minimum of $KVol$ over the Teichmüller disk \mathcal{T}_s is $(2s - 1)\sqrt{\frac{143}{144}}$. It is achieved at the two points $(\pm \frac{9}{14}, \frac{\sqrt{143}}{14})$. For $(x, y) \in V_{\pm 1}$, we have $KVol(X_s(x, y)) > 2s - 1$, and $KVol(X_s(x, y))$ goes to infinity when (x, y) tends to $(\pm 1, 0)$. Outside $V_{\pm 1}$, we have $KVol(X_s(x, y)) \leq 2s - 1$, and $KVol(X_s(x, y)) = 2s - 1$ if and only if $(x, y) \in \mathcal{Z}$. Furthermore $KVol(X_s(x, y))$ tends to $2s - 1$ when $y \rightarrow \infty$ while (x, y) remains in \mathcal{D} .*

The idea of the proof of Theorem 1.1 is that $KVol(X_s(x, y))$, has a simple expression as a function of the hyperbolic distance between (x, y) and \mathcal{Z} . The minimum is achieved by those points of \mathcal{D} which are furthest away from \mathcal{Z} . Then the problem becomes a simple exercise in hyperbolic geometry.

Here is why the set \mathcal{Z} enters the picture : given two directions (p, q) and (p', q') , the saddle connections of respective directions (p, q) and (p', q') only have so much algebraic intersection to share between them all since they are lifted from closed curves on the torus. Thus, to realize $KVol$ we must look for pairs of directions which have saddle connections which take up all the possible intersection. Those are the pairs of directions for which the invariant probability measures supported on the saddle connections are

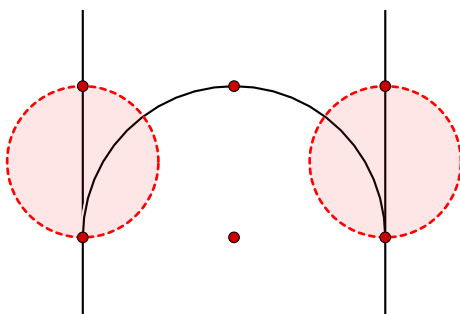


Figure 4: The horocycle neighborhood $V_{\pm 1}$ of Theorem 1.1

least equidistributed. By Theorem 1.2 of [9], those pairs of directions correspond to those geodesics in \mathcal{T}_s which are the least recurrent, that is, the geodesics which exit the fastest towards the cusps of \mathcal{T}_s . Those are precisely the geodesics in \mathcal{Z} . The precise meaning of exiting fastest towards the cusp is that they bounce at most twice on the lower boundary of \mathcal{D} -that is, the semi-circle between 1 and -1 - before going straight to a cusp. Another way of defining \mathcal{Z} would be to say that the geodesic between p/q and p'/q' is in \mathcal{Z} if and only if $(p/q, p'/q')$ is in the orbit, under the Veech group, of $(\infty, p''/q'')$, where the continued fraction expansion of p''/q'' has length ≤ 2 . A quantitative version of Theorem 1.2 of [9] would be useful to generalize this work. In our proof, we use pedestrian arguments (Lemmata 2.6 and 2.7) instead.

It is proved in [6] (see also [4] and [1]) that $X_2(\exp(i\pi/3))$ minimizes the systolic volume in $\mathcal{H}(2)$. In the companion paper [2], we prove that $\inf \text{KVol} \leq 2$ over $\mathcal{H}(2)$. Since the systolic volume is a close relative of KVol , it is interesting to contrast the results of [6], [1] and [4] with ours. It is mildly surprising that the minimizers of KVol in \mathcal{T}_s look, at first glance, pretty dull, while the interesting surfaces are (degenerate) local maxima in \mathcal{T}_s . This prompts several questions.

Question 1.2. *Is KVol , as a function on $\mathcal{H}(2)$, differentiable at $St(3)$? Is it critical at $St(3)$?*

Question 1.3. *Is every square-tiled surface (ramified Riemannian covering of the square flat torus) in $\mathcal{H}(2)$, a local maximum of KVol in its own Teichmüller disk? Are they critical points of KVol in $\mathcal{H}(2)$?*

An easy consequence of Theorem 1.1 is that for every $X \in \mathcal{T}$, the supremum in (1) is actually a maximum, and furthermore we identify the maximizers. We speculate this might be the case for every square-tiled surface; since the union of the Teichmüller disks of all square-tiled surfaces is dense in $\mathcal{H}(2)$, could it be true in the whole $\mathcal{H}(2)$?

Question 1.4. *Is it true that for every $X \in \mathcal{H}(2)$, the supremum in (1) is actually a maximum?*

This would be in sharp contrast, both with the known behavior for flat tori, and with the expected behavior for hyperbolic surfaces.

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2 Preliminaries

2.1 $St(2s - 1)$

We call $e_1, \dots, e_s, e'_2, \dots, e'_s$, (resp. $f_1, \dots, f_s, f'_1, \dots, f'_{s-1}$) the horizontal (resp. vertical) closed curves of length 1 in $St(2s - 1)$ obtained by gluing the endpoints of the sides of the squares (see Figure 5). Note that e_i and e'_i (resp. f_i and f'_i) are homotopic since they bound a cylinder.

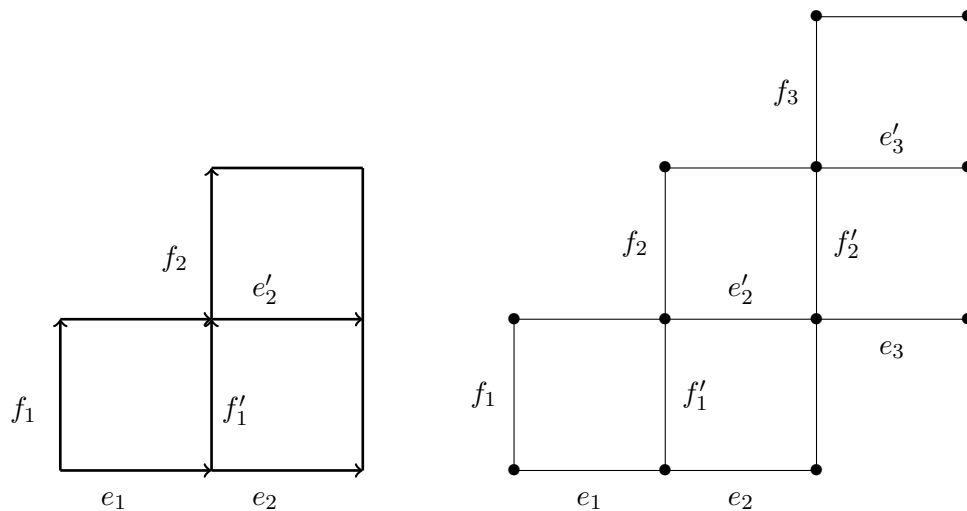


Figure 5: the closed curves e_i, e'_i, f_i, f'_i

Figure 6 shows a local picture of $St(3)$ around the singular (conical) point S , with angles rescaled so the 6π fit into 2π .

The local picture at S in $St(2s - 1)$ may be obtained by induction : start from the local picture in $St(2s - 3)$, split f_{s-1} into f_{s-1} and f'_{s-1} , and insert, between f_{s-1} and f'_{s-1} , f_s, e_s , and e'_s , as shown in Figure 7.

Since e_1, e_2, f_1, f_2 do not meet anywhere but at S , the local picture yields the algebraic intersections between any two of e_1, e_2, f_1, f_2 , summed up in the following matrix:

Int	e_1	f_1	e_2	f_2
e_1	0	1	0	-1
f_1	-1	0	0	0
e_2	0	0	0	1
f_2	1	0	-1	0

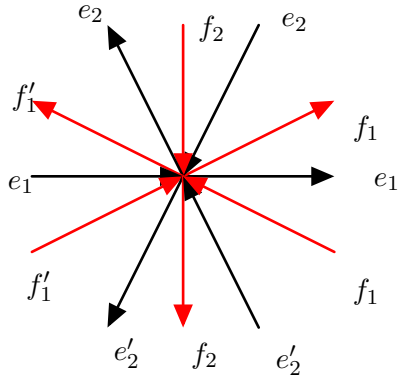


Figure 6: Local picture around the conical point

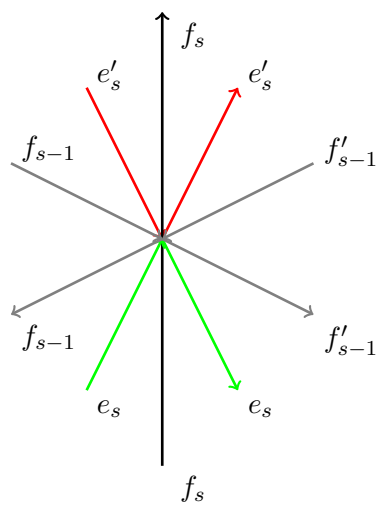


Figure 7: Insertion of f_s, e_s, e'_s

Let us call $\alpha_1, \dots, \alpha_s$ (resp. β_1, \dots, β_s) the closed, non-singular horizontal (resp. vertical) geodesics depicted in Figure 8.

Lemma 2.1. For $i, j = 1, \dots, s$ we have :

$$Int(e_i, f_j) = \begin{cases} (-1)^{j-i} & \text{if } j \geq i \\ 0 & \text{if } j < i \end{cases}$$

and

$$Int(e_i, e_j) = Int(f_i, f_j) = 0$$

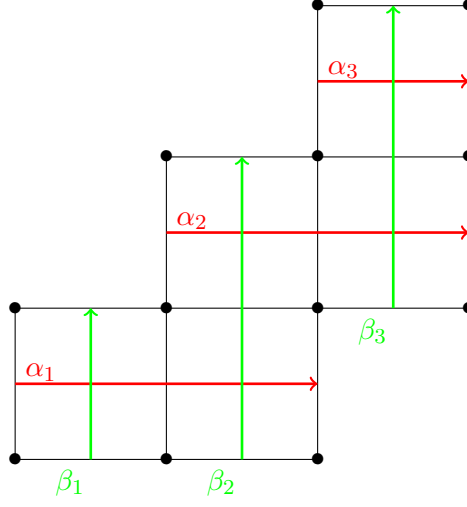


Figure 8: The non-singular geodesics $\alpha_i, \beta_i, i = 1, 2, 3$

Proof. Observe that the closed, non-singular vertical geodesic β_1 is homotopic to f_1 and for each $j = 2, \dots, s$, β_j are homotopic to $f_j + f_{j-1}$. We observe that

$$\text{Int}(e_i, \beta_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Then we have :

For $j = 1$, $\text{Int}(e_1, f_1) = 1$ and for $i > 1$, $\text{Int}(e_i, f_1) = 0$.

For $j \geq 2$ we have three cases:

1) if $i > j$ we have $\text{Int}(e_i, \beta_j) = 0$ which implies $\text{Int}(e_i, f_j) = -\text{Int}(e_i, f_{j-1})$, by induction we deduce

$$\text{Int}(e_i, f_j) = (-1)^{i-j} \text{Int}(e_i, f_1) = 0$$

2) if $i = j$ we have

$$\text{Int}(e_i, f_i) + \text{Int}(e_i, f_{i-1}) = 1$$

but by the first case $\text{Int}(e_i, f_{i-1}) = 0$, so

$$\text{Int}(e_i, f_i) = 1$$

3) if $i < j$ we have

$$\text{Int}(e_i, f_j) = -\text{Int}(e_i, f_{j-1})$$

By induction and using the second case, we obtain

$$\begin{aligned} \text{Int}(e_i, f_j) &= (-1)^{j-i} \text{Int}(e_i, f_i) \\ &= (-1)^{j-i}. \end{aligned}$$

Observe that the closed, non-singular horizontal geodesic α_s , is homotopic to e_s and for each $j = 1, \dots, s-1$, α_j are homotopic to $e_j + e_{j+1}$. Then for $i = 1, \dots, s$, $\text{Int}(e_i, e_s) = 0$ and for $i = 1, \dots, s$, $j = 1, \dots, s-1$ $\text{Int}(e_i, \alpha_j) = 0$. Finally, we obtain

$$\begin{aligned} \text{Int}(e_i, e_j) &= -\text{Int}(e_i, e_{j+1}) \\ &= (-1)^{s-j} \text{Int}(e_i, e_s) = 0. \end{aligned}$$

In an analogous way we have $\text{Int}(f_i, f_j) = 0$. \square

For $s > 2$, the intersection matrix is, in the basis $e_1, f_1, \dots, e_s, f_s$ (in this example $s = 4$):

$$\left(\begin{array}{cc|cc|cc|cc} 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \end{array} \right).$$

Since the determinant of the intersection matrix is not zero, we get for free the fact that e_i, f_i , $i = 1, \dots, s$ form a basis of $H_1(\text{St}(2s-1), \mathbb{R})$. From now on we always refer to homology classes in $H_1(\text{St}(2s-1), \mathbb{R})$ by their coordinates $\epsilon_1, \dots, \epsilon_s, \phi_1, \dots, \phi_s$ in the basis e_i, f_i .

Observe that the homology classes of $\alpha_1, \dots, \alpha_{s-1}, \alpha_s$ are $[e_1] + [e_2], \dots, [e_{s-1}] + [e_s], [e_s]$, while the homology classes of $\beta_1, \beta_2, \dots, \beta_s$ are $[f_1], [f_1] + [f_2], \dots, [f_{s-1}] + [f_s]$.

Thus, using the intersection matrix, we see that, for any homology class h with coordinates $\epsilon_1, \dots, \epsilon_s, \phi_1, \dots, \phi_s$, we have

$$\begin{aligned} \epsilon_i &= \text{Int}(h, [\beta_i]) \\ \phi_i &= -\text{Int}(h, [\alpha_i]). \end{aligned}$$

2.2 A short excursion into the Teichmüller space of flat tori

Recall that the Teichmüller space of flat tori is the hyperbolic plane ; one way to see that is to fix a homology class of simple closed curves, then every flat torus is biholomorphic to

$$T(x, y) = \mathbb{R}^2 / \mathbb{Z}(1, 0) \oplus \mathbb{Z}(x, y)$$

where $(x, y) \in \mathbb{R} \times \mathbb{R}_+^*$ and the biholomorphism sends a simple closed curve in the homology class h to the image in $T(x, y)$ of $(0, 1)$.

Now take $p, q, p', q' \in \mathbb{Z}$ such that $p \wedge q = p' \wedge q' = 1$, set $r = p/q, r' = p'/q'$, r (resp. r') being understood as the point at infinity if $q = 0$ (resp. $q' = 0$). Let $\gamma_{r, r'}$ be the hyperbolic geodesic with endpoints r and r' . Recall that $\gamma_{r, r'}$ is the locus, in the Teichmüller space, of the flat tori in which the closed geodesics with respective directions (p, q) and (p', q') are orthogonal.

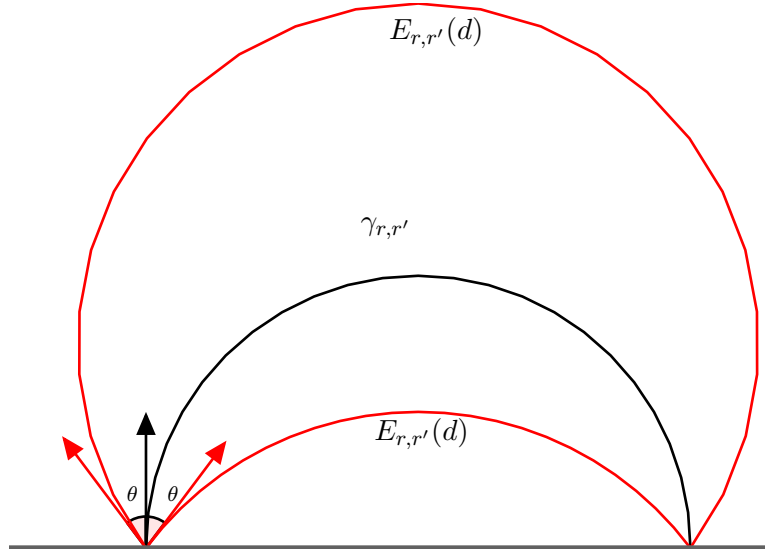


Figure 9: A banana neighborhood

Denote by $d_{r,r'} : \mathbb{H}^2 \rightarrow \mathbb{R}$ the distance function to the geodesic $\gamma_{r,r'}$. Call $E_{r,r'}(d)$ the d -level set of $d_{r,r'}$, for any $d \geq 0$. Pick (x, y) in \mathbb{H}^2 , let $d = d_{r,r'}(x, y)$, and let $\theta_{r,r'}(x, y)$ be the angle between $E_{r,r'}(d)$ and $\gamma_{r,r'}$ (see Figure 9).

Denote $l_r(x, y)$ the length, in the flat torus $T(x, y)$, of the closed geodesics with direction (p, q) . Set

$$K_{r,r'}(x, y) = \text{Vol}(T(x, y)) \frac{pq' - p'q}{l_r(x, y)l_{r'}(x, y)}.$$

This definition is tailored so

$$\text{KVVol}(T(x, y)) = \sup_{r,r'} K_{r,r'}(x, y).$$

Lemma 2.2. *For any $g \in SL_2(\mathbb{Z})$, we have*

$$K_{r,r'}(g(x, y)) = K_{g(r),g(r')}(x, y).$$

Proof. First, $pq' - p'q$ and $\text{Vol}(T(x, y))$ are invariant under the orientation- and volume-preserving diffeomorphism g . Second, $l_r(g(x, y)) = l_{g(r)}(x, y)$. \square

Lemma 2.3. *For any $(x, y) \in \mathbb{H}^2$ and $p, q, p', q' \in \mathbb{Z}$ such that $p \wedge q = p' \wedge q' = 1$, we have, setting $r = p/q, r' = p'/q'$,*

$$K_{r,r'}(x, y) = \cos \theta_{r,r'}(x, y).$$

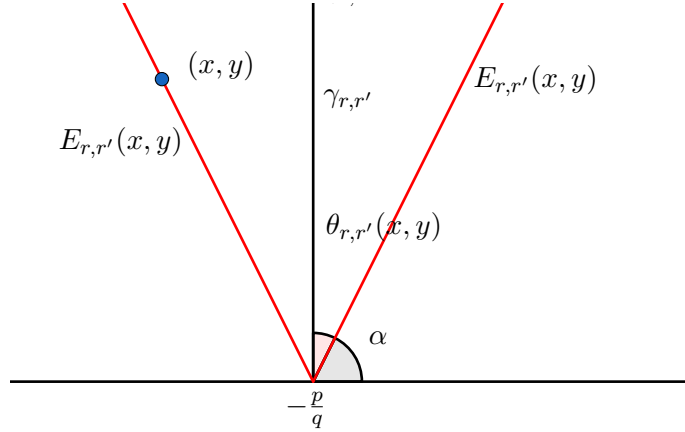


Figure 10: $\alpha = \pi/2 - \theta_{r,r'}(x, y)$

Proof. By Lemma 2.2, acting by some element of $SL_2(\mathbb{Z})$ if we need to, we may assume that $p' = 1, q' = 0$, that is, $\gamma_{r,r'}$ is vertical. Then $l_{r'}(x, y) = 1$, $l_r(x, y) = \sqrt{(p + qx)^2 + (qy)^2}$, $pq' - p'q = q$, and $\text{Vol}(T(x, y)) = y$, so

$$K_{r,r'}(g(x, y)) = \frac{qy}{\sqrt{(p + qx)^2 + (qy)^2}}.$$

Take radial coordinates (ρ, α) with origin at $-r$, so $x + p/q = \rho \cos \alpha$ and $y = \rho \sin \alpha$. Then

$$K_{r,r'}(g(x, y)) = \frac{\rho \sin \alpha}{\rho} = \sin \alpha.$$

Now observe on Figure 10 that $\alpha = \pi/2 - \theta_{r,r'}(x, y)$. This concludes the proof. \square

2.3 Saddle connections and intersection

Recall that a saddle connection is a closed geodesic which contains a cone point. Saddle connections on surfaces in the Teichmüller disk of $St(2s - 1)$ have a special property :

Lemma 2.4. *If $X \in \mathcal{T}_s$ and γ is a saddle connection on X , then the projection $\Pi : X \rightarrow \mathbb{T}^2$, restricted to γ , is 1-to-1.*

Proof. Parametrize γ at unit speed by $[0, l(\gamma)]$, so that $\gamma(0) = \gamma(l(\gamma)) = S$, S being the cone point of X . Assume $\Pi(\gamma(t_1)) = \Pi(\gamma(t_2))$ for $t_1, t_2 \in [0, l(\gamma)]$, with $t_1 \neq t_2$. Then $\Pi(\gamma)([t_1, t_2])$ is a closed subarc of $\Pi(\gamma)([0, l(\gamma)])$, parametrized at unit speed, so it must contain all of $\Pi(\gamma)([0, l(\gamma)])$. Therefore there exists $t_3 \in [t_1, t_2]$ such that $\Pi(\gamma)(t_3) = (0, 0) = \Pi(S)$. Now recall that S is the only pre-image of $(0, 0)$ (this is where surfaces in \mathcal{T}_s are special), and the only pre-images of S by the chosen parametrization of γ are 0 and $l(\gamma)$, so $t_3 = 0$ or $t_3 = l(\gamma)$. Then $\Pi(\gamma)(t_1) = \Pi(\gamma)(t_2) = (0, 0)$, whence $\gamma(t_1) = \gamma(t_2) = S$, which proves that $t_1 = 0, t_2 = l(\gamma)$, and the lemma. \square

The conical point is the only pre-image of the ramification point, unlike the case of the surface $L(n, n)$ (see [2]), so by Lemma 2.4 a closed geodesic of direction (p, q) projects 1-to-1 to a closed geodesic in the flat torus. Therefore a saddle connection of direction (p, q) is exactly $\sqrt{(p + qx)^2 + (qy)^2}$ -long. For future reference we state this fact as a lemma.

Lemma 2.5. *Given $(x, y) \in \mathbb{H}^2$ and $(p, q) \in \mathbb{Z}^2$ with $p \wedge q = 1$, for any saddle connection α with direction (p, q) in the translation surface $X_s(x, y)$, we have, using the notation of Subsection 2.2, $l(\alpha) = l_r(x, y)$.*

Here is why Subsection 2.2 is relevant to the determination of KVol in \mathcal{T}_s . For $(x, y) \in \mathbb{H}^2$, and for $p, q, p', q' \in \mathbb{Z}$ such that $p \wedge q = p' \wedge q' = 1$, setting $r = p/q, r' = p'/q'$, and calling $\alpha_i, i = 1, \dots, 2s - 1$ (resp. $\beta_i, i = 1, \dots, 2s - 1$) the saddle connections with direction (p, q) (resp. (p', q')) in the translation surface $X_s(x, y)$, we have $l(\alpha_i) = l_r(x, y)$ and $l(\beta_i) = l_{r'}(x, y)$ for $i = 1, \dots, 2s - 1$, and $\text{Vol}(X_s(x, y)) = (2s - 1)\text{Vol}(T(x, y))$, since $X_s(x, y)$ is a $(2s-1)$ -fold ramified Riemannian cover of $T(x, y)$. Hence

$$\text{KVol}(X_s(x, y)) \geq \sup_{r \neq r'} \sup_{i, j} \text{Vol}(X_s(x, y)) \frac{\text{Int}(\alpha_i, \beta_j)}{l(\alpha_i)l(\beta_j)} \quad (2)$$

$$= (2s - 1) \sup_{r \neq r'} \sup_{i, j} \frac{\text{Int}(\alpha_i, \beta_j)}{pq' - p'q} K_{r, r'}(x, y). \quad (3)$$

The reason why there is a \geq instead of $=$ in the inequality above is that there could be saddle connections with the same direction, i.e. $r = r'$, and non-zero intersection, in which case $pq' - p'q = 0$ and the formula above does not apply ; more on that in subsection 2.3.1. For the time being, we set

$$I_{r, r'}(x, y) = \sup_{i, j} \frac{\text{Int}(\alpha_i, \beta_j)}{pq' - p'q}$$

and study $I_{r, r'}(x, y)$ as a function of r and r' . Note that actually $I_{r, r'}(x, y)$ does not depend on (x, y) , so from now on we shall denote it $I_{r, r'}$.

2.3.1 Intersections and $I_{r, r'}$

Recall from [5] that the action on \mathbb{Z}^2 of the Veech group Γ of $St(2s-1)$ has two orbits, that of $(1, 0)$, which consists of vectors whose coordinates are not equal modulo 2, and that of $(1, 1)$, which consists of vectors whose coordinates are equal modulo 2. Therefore, since Γ acts by orientation-preserving diffeomorphisms, for $(p, q) \in \mathbb{Z}^2$, with $p \not\equiv q \pmod{2}$, the saddle connections in $St(2s - 1)$ with direction (p, q) are the images under some diffeomorphism of $e_1, e_i, e'_i, i = 2, \dots, s$, so they have zero mutual intersection.

On the other hand, for $p, q \in \mathbb{Z}$ both odd, the saddle connections in $St(2s - 1)$ with direction (p, q) are the images under some diffeomorphism of $g_i, g'_i, g_s, i = 1, \dots, s - 1$ depicted in Figure 11.

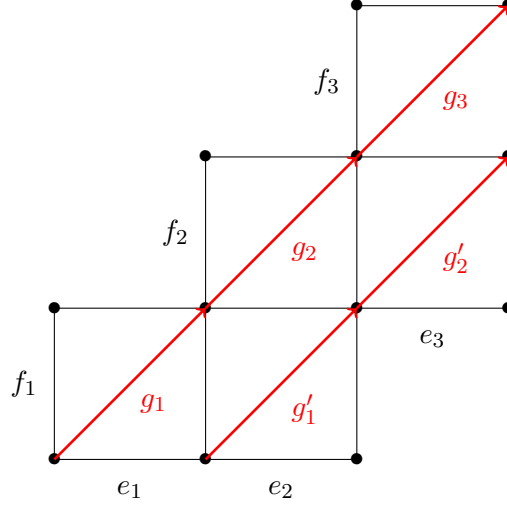


Figure 11: the saddle connections of direction $(1,1)$

The homology classes of $g_1, g_i, g'_{i-1}, i = 2, \dots, s$ are, in the basis $\{e_1, \dots, e_s, f_1, \dots, f_s\}$, $g_1 = e_1 + f_1$, and $g_i = e_i + f_i, g'_{i-1} = e_i + f_{i-1}$, so using the intersection matrix we see that

$$\text{Int}(g_i, g_j) = \text{Int}(g'_{j-1}, g'_{i-1}) = \begin{cases} (-1)^{j-i} & \text{if } j > i \\ -(-1)^{i-j} & \text{if } j < i \end{cases}$$

and for any $i = 1, \dots, s, j = 2, \dots, s, \text{Int}(g_i, g'_{j-1}) = -(-1)^{j-i}$.

Since the game is to find curves which intersect a lot, our next task is to maximize $I_{r,r'}$ as a function of r, r' . The next two lemmas are the reason why the geodesics in \mathcal{Z} are special.

Lemma 2.6. *For any $(r, r') \in \text{End}(\mathcal{Z})$, we have $I_{r,r'} = 1$.*

Proof. Let $(p, q), (p', q') \in \mathbb{Z}^2$ be such that $p \wedge q = p' \wedge q' = 1$, and $p/q = r, p'/q' = r'$. We want to prove that for any surface $X_s(x, y) \in \mathcal{T}_s$, there exist saddle connections γ, γ' , with respective directions (p, q) and (p', q') , such that $\text{Int}(\gamma, \gamma') = pq' - p'q$. By the definition of $\text{End}(\mathcal{Z})$, there exists $V \in \Gamma$ such that $(V(r), V(r')) \in \mathbb{Z}^2 \cup \{\infty\}$, where $V(r)$ is understood as the projective action of Γ on the projective line. Since Γ has two orbits in \mathbb{Q} , that of ∞ (or 0), which corresponds to $p \not\equiv q \pmod{2}$, and that of 1, which corresponds to $p \equiv q \pmod{2}$, we may assume $V(r) = \infty$ (or 0) or $V(r) = 1$.

First case : $V(r) = 0$, that is, $V(p, q) = (0, 1)$ (where $V(p, q)$ is understood as the linear action of Γ on \mathbb{R}^2). Since $V(r') \in \mathbb{Z}$, there exists $n \in \mathbb{Z}$ such that $V(p', q') = (-n, 1)$. Then (recall that $R^3 = -R$), $R^3 \circ V(p', q') = (1, n)$ and $R^3 \circ V(p, q) = (1, 0)$. Since Γ is a subgroup of $SL_2(\mathbb{R})$, we have $n = pq' - p'q$. Since $I_{r,r'}$ does not depend on (x, y) , we might as well assume $(x, y) = (0, 1)$, that is, $X_s = St(2s - 1)$. Observe (in

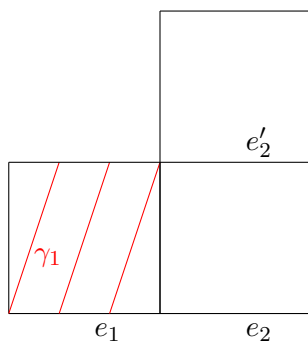


Figure 12: the saddle connection γ_1 , with $s = 2$ and $n = 3$, in the two-cylinder case

Figure 12) that there exists a saddle connection γ_1 with direction $(1, n)$, whose homology class is $[e_1] + n[f_1]$, so $\text{Int}(e_1, \gamma_1) = n$, hence $I_{r,r'} = 1$.

Second case : $V(r) = 1$, that is, $V(p, q) = (1, 1)$. Since $V(r') \in \mathbb{Z}$, there exists $n \in \mathbb{Z}$ such that $V(p', q') = (-n, 1)$. Since $I_{r,r'}$ does not depend on (x, y) , we might as well assume $(x, y) = (-1, 1)$. The surface $X_s(-1, 1)$ has a $(2s - 1)$ -square template depicted in Figure 13, and there exists a saddle connection γ_1 with direction $(-n, 1)$, whose homology class is $-n[e_s] + [g_s]$, so $\text{Int}(g_s, \gamma_1) = n$, hence $I_{r,r'} = 1$. \square

Lemma 2.7. For $r, r' \in \mathbb{Q} \cup \{\infty\}$, with $(r, r') \notin \text{End}(\mathcal{Z})$, we have $I_{r,r'} < 9/10$.

Proof. Let $p, q, p', q' \in \mathbb{Z}$ be such that $p \wedge q = p' \wedge q' = 1$, $p/q = r$, $p'/q' = r'$.

First case : p, q, p', q' are not all odd.

Multiplying (p, q) and (p', q') by some element of the Veech group, and swapping (p, q) and (p', q') if necessary, we may assume $(p', q') = (1, 0)$, that is, $r' = \infty$, and $|p| < |q|$, so $pq' - p'q = q$. Furthermore by the symmetries of $St(2s - 1)$, we only need consider the case when $0 < p < q$. Note that we then have $q \geq 2$, otherwise p is 0 and in that case we have $(r, r') \in \text{End}(\mathcal{Z})$. Let γ be a saddle connection with direction (p, q) . We want to show that

$$\begin{aligned} \text{Int}(e_i, \gamma) &= \text{Int}(e'_i, \gamma) < \frac{9q}{10} \text{ for } i = 2, \dots, s \\ \text{Int}(e_1, \gamma) &< \frac{9q}{10}. \end{aligned}$$

The projection Π restricted to γ is injective by Lemma 2.4, so the $q - 1$ intersections of (p, q) with $(1, 0)$ in $\mathbb{T}^2 \setminus \{(0, 0)\}$ lift to $q - 1$ intersections of γ with the singular cycle $e_1 + e_2 + e'_2 + \dots + e_s + e'_s$. Let us consider the sequence S_{int} of all intersections of γ with e_1, e_i, e'_i , for $i = 2, \dots, s$, in cyclical order along γ , with the intersection at the conical point S set apart. Denote $\#S_{int}(e_i)$ (resp. $\#S_{int}(e'_i)$) the number of e_i 's (resp. e'_i 's) in the sequence S_{int} . We have

$$\text{Int}(e, \gamma) \leq \#S_{int}(e) + 1 \text{ for } e = e_1, \dots, e_s, \text{ and } e = e'_2, \dots, e'_s,$$

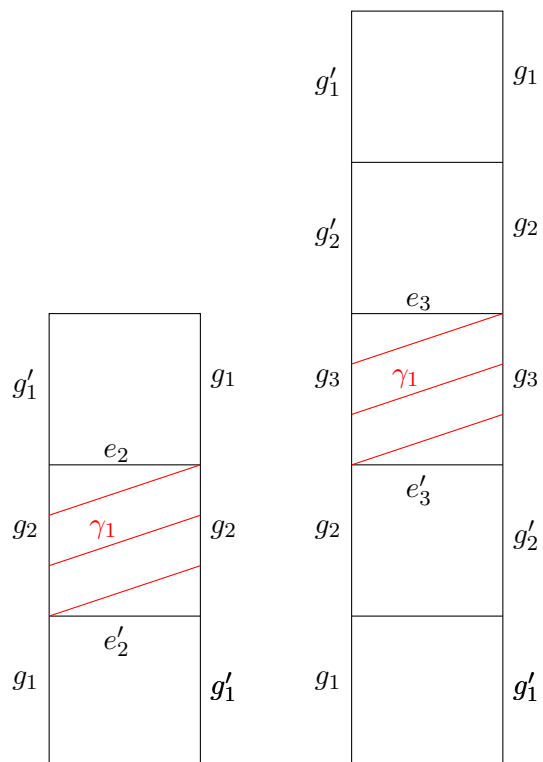


Figure 13: the saddle connection γ_1 , for $s = 2$, and $s = 3$, and $n = 3$, in the one-cylinder case

where the +1 accounts for the intersection at S . Observe that in the sequence S_{int} there are never two consecutive e_i or e'_i , for $i = 2, \dots, s$: each e'_i is followed by e_i or e'_{i+1} , and each e_i is followed by e'_i or e_{i-1} . So the proportion of e_i or e'_i in S_{int} , for $i = 2, \dots, s$ is at most $1/2$ (only possible when $s = 2$). Therefore, since $q \geq 2$,

$$\text{Int}(e_i, \gamma) = \text{Int}(e'_i, \gamma) \leq \frac{q-1}{2} + 1 \leq \frac{3}{4}q.$$

Intersections with e_1 must be treated separately, with a case-by-case analysis :

Case 1.1: $p < q < 2p$. Observe that each block of e_1 has length at most 2, and is followed by at least an e'_2 , so the proportion of e_1 in the sequence S_{int} is at most $2/3$, so

$$\text{Int}(e_1, \gamma) \leq \frac{2(q-1)}{3} + 1 \leq \frac{9q}{10}.$$

Case 1.2: $2p < q < 3p$ (see Figure 14). Then a block of e_1 has length at most 3, and is followed by at least an e'_2 and an e_2 , so as previously, the proportion of e_1 in the sequence S_{int} is at most $3/5$, so

$$\text{Int}(e_1, \gamma) \leq \frac{3(q-1)}{5} + 1 \leq \frac{9q}{10}.$$

Case 1.3: $3p < q$. Intersections with e_1 come in blocks of length at most $\lceil q/p \rceil$, because two consecutive (along γ) intersections are exactly p/q apart along e_1 (see Figure 14). Each block of e_1 's is followed by a block of $e'_2 e_2$ (this is where we use the fact that $(r, r') \notin \text{End}(\mathcal{Z})$), of length at least $\lfloor q/2p \rfloor$. Recall that $2\lfloor q/2p \rfloor \geq \lceil q/p \rceil - 2$. So the proportion of e_1 in S_{int} is at most

$$\frac{\lceil q/p \rceil}{\lceil q/p \rceil + 2\lfloor q/2p \rfloor} \leq \frac{\lceil q/p \rceil}{2\lceil q/p \rceil - 2} \leq \frac{3}{4}$$

where the last inequality stands because $3 < q/p$. Then

$$\text{Int}(e_1, \gamma) \leq \frac{3(q-1)}{4} + 1 \leq \frac{9q}{10}.$$

This finishes the first case.

Remark 2.8. *There is numerical evidence that $I_{r,\infty} \leq 2/3$ for all $r \in \mathbb{Q}$ such that $(r, \infty) \notin \text{End}(\mathcal{Z})$, except when $r = 3/7$, in which case $I_{r,\infty} = 5/7$.*

Second case : p, q, p', q' are all odd. Multiplying (p, q) and (p', q') by some element of the Veech group, and swapping (p, q) and (p', q') if necessary, we may assume $(p', q') = (1, 1)$, that is, $r' = 1$, so $pq' - p'q = p - q$. Again multiplying by an element of the Veech group which stabilizes $(1, 1)$, we may assume $(p, q) = (p - q)(1, 0) + q(1, 1)$ satisfies $|p - q| > |q|$. Furthermore by the symmetries of $St(2s - 1)$, we only need consider the case when $0 < q < p - q$. Note that we then have $p - q \geq 2$, otherwise $p = q$ and in that

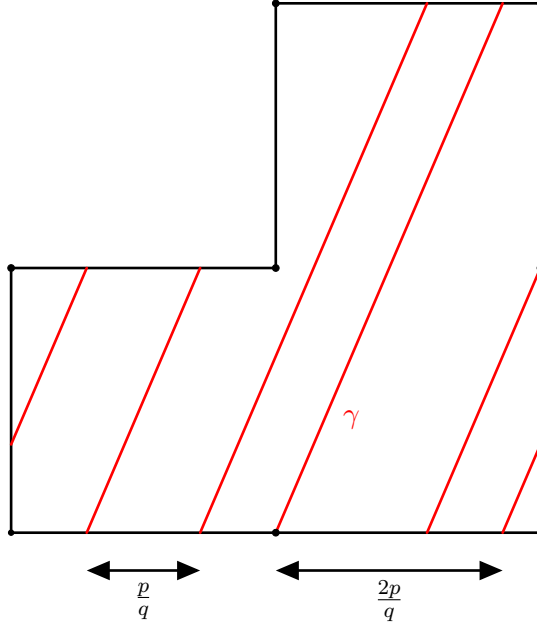


Figure 14: $(p, q) = (3, 7)$

case we have $(r, r') \in \text{End}(\mathcal{Z})$. Let γ be a saddle connection with direction (p, q) . We want to show that

$$\begin{aligned} \text{Int}(\gamma, g_i) &< \frac{9(p-q)}{10} \text{ for } i = 1, \dots, s \\ \text{Int}(\gamma, g'_i) &< \frac{9(p-q)}{10} \text{ for } i = 1, \dots, s-1. \end{aligned}$$

The projection Π restricted to γ is injective by Lemma 2.4, so the $p-q-1$ intersections of (p, q) with $(1, 1)$ in $\mathbb{T}^2 \setminus \{(0, 0)\}$ lift to $p-q-1$ intersections of γ with the singular cycle $g_1 + g'_1 + g_2 + g'_2 + \dots + g_s$. Let us consider the sequence S_{int} of all intersections of γ with g_1, g'_1, \dots, g_s , in cyclical order along γ , with the intersection at the conical point S set apart. Denote $\#S_{int}(g_i)$ (resp. $\#S_{int}(g'_i)$) the number of g_i 's (resp. g'_i 's) in the sequence S_{int} . We have

$$\text{Int}(g, \gamma) \leq \#S_{int}(g) + 1 \text{ for } g = g_1, \dots, g_s, \text{ and } g = g'_1, \dots, g'_{s-1},$$

where the $+1$ accounts for the intersection at S . Observe that in the sequence S_{int} there are never two consecutive g_i , for $i = 1, \dots, s-1$, or g'_i , for $i = 2, \dots, s-1$: each g'_i ($2 \leq i \leq s-1$) is followed by g_i or g_{i-1} , and each g_i ($1 \leq i \leq s-1$) is followed by g'_i or g'_{i+1} (see Figure 13). So the proportion of g_i or g'_i in S_{int} , for $i = 2, \dots, s$ is at most $1/2$

(only possible when $s = 2$). Therefore, since $p - q \geq 2$,

$$\text{Int}(g, \gamma) \leq \frac{p - q - 1}{2} + 1 \leq \frac{3}{4}(p - q) \text{ for } g = g_1, \dots, g_{s-1}, \text{ or } g = g'_2, \dots, g'_{s-1}.$$

Intersections with g'_1 and g_s must be treated separately, with a case-by-case analysis. First we deal with g_s :

Case 2.1: $q < p - q < 2q$. Observe that each block of g_s has length at most 2, and is followed by at least a g_{s-1} , so the proportion of g_s in the sequence S_{int} is at most $2/3$, whence $\text{Int}(g_s, \gamma) \leq \frac{9(p-q)}{10}$.

Case 2.2: $2q < p - q < 3q$. Then a block of g_s has length at most 3, and is followed by at least a g_{s-1} and a g'_{s-1} , so so the proportion of g_s in the sequence S_{int} is at most $3/5$, whence $\text{Int}(g_s, \gamma) \leq \frac{9(p-q)}{10}$.

Case 2.3: $3q < p - q$. Intersections with g_s come in blocks of length at most $\lceil (p - q)/q \rceil$, because two consecutive (along γ) intersections are exactly $q/(p - q)$ apart along g_s (see Figure 13). Each block of g_s 's is followed by a block of $g_{s-1}g'_{s-1}$, of length at least $\lfloor (p - q)/2q \rfloor$. Recall that $2\lfloor (p - q)/2q \rfloor \geq \lceil (p - q)/q \rceil - 2$. So the proportion of g_s in S_{int} is at most

$$\frac{\lceil (p - q)/q \rceil}{\lceil (p - q)/q \rceil + 2\lfloor (p - q)/2q \rfloor} \leq \frac{\lceil (p - q)/q \rceil}{2\lceil (p - q)/q \rceil - 2} \leq \frac{3}{4}$$

where the last inequality stands because $3 < (p - q)/q$. Again, $\text{Int}(g_s, \gamma) \leq \frac{9(p-q)}{10}$.

Now we deal with g'_1 . In that case it is more convenient to write $(p, q) = (p - q)(2, 1) + (2q - p)(1, 1)$. Note that $2q - p < 0$ because $p - q > q$.

Case 2.4: $p - q < 2|2q - p|$, that is, $2q < p - q$. Observe that each block of g'_1 has length at most 2, and is followed by at least a g_1 , so the proportion of g'_1 in the sequence S_{int} is at most $2/3$, whence $\text{Int}(g'_1, \gamma) \leq \frac{9(p-q)}{10}$.

Case 2.5: $2|2q - p| < p - q < 3|2q - p|$, that is, $3q/2 < p - q < 3q$. Then a block of g'_1 has length at most 3, and is followed by at least a g_1 and a g'_2 , so so the proportion of g_s in the sequence S_{int} is at most $3/5$, whence $\text{Int}(g_s, \gamma) \leq \frac{9(p-q)}{10}$.

Case 2.6: $p - q > 3|2q - p|$, that is, $q < p - q < 3q/2$. Intersections with g'_1 come in blocks of length at most $\lceil (p - q)/|2q - p| \rceil$, because two consecutive (along γ) intersections are exactly $|2q - p|/(p - q)$ apart along g'_1 (see Figure 13). Each block of g'_1 's is followed by a block of g'_2g_1 , of length at least $\lfloor (p - q)/2|2q - p| \rfloor$. Recall that $2\lfloor (p - q)/2|2q - p| \rfloor \geq \lceil (p - q)/|2q - p| \rceil - 2$. So the proportion of g'_1 in S_{int} is at most

$$\frac{\lceil (p - q)/|2q - p| \rceil}{\lceil (p - q)/|2q - p| \rceil + 2\lfloor (p - q)/2|2q - p| \rfloor} \leq \frac{\lceil (p - q)/|2q - p| \rceil}{2\lceil (p - q)/|2q - p| \rceil - 2} \leq \frac{3}{4}$$

where the last inequality stands because $3 < (p - q)/|2q - p|$. Again, $\text{Int}(g'_1, \gamma) \leq \frac{9(p-q)}{10}$. \square

3 Proof of Theorem 1.1

Lemma 3.1. *Take $s \in \mathbb{N}^*$ and $X \in \mathcal{H}(2s - 2)$. Assume X is an n -fold ramified Riemannian cover of a flat torus. Then $\text{KVol}(X) \leq n$, unless there exists a pair of closed geodesics α and β on X , which have the same direction, non-zero intersection, and such the product of their lengths is $< \text{Vol}(X)/n$.*

Proof. Assume, up to isometry and re-scaling, that the flat torus which X covers is $\mathbb{R}^2/(1, 0)\mathbb{Z} \oplus (a, b)\mathbb{Z}$, with $|a| \leq 1/2$ and $a^2 + b^2 \geq 1$. Then any closed geodesic on the torus is at least 1 long, and the volume of the torus is b , so the volume of X is nb .

Let α and β be simple closed geodesics in X . Since X has but one conical point, both α and β have a well-defined direction, say $\frac{p}{q}$ and $\frac{p'}{q'}$ in irreducible terms.

Remark 3.2. *If X had several conical points, α or β could be made up of several saddle connections of distinct directions.*

Let $\Pi : X \rightarrow \mathbb{T}^2$ be the projection of the ramified cover.

Then Π maps α (resp. β) to a closed geodesic of the torus, with homology class (p, q) (resp. (p', q')).

First case : $pq' - p'q \neq 0$.

Observe that $\Pi(\alpha)$ and $\Pi(\beta)$ intersect exactly $|pq' - p'q|$ times, and their intersections are equidistributed along $\Pi(\alpha)$, since the first return map to $\Pi(\alpha)$ of the linear flow with direction (p', q') , is a rotation of $\Pi(\alpha)$. Therefore, two consecutive intersections are exactly $\frac{\sqrt{(p+aq)^2 + b^2q^2}}{|pq' - p'q|}$ apart along $\Pi(\alpha)$.

Thus, two consecutive intersections of α and β cannot be less than $\frac{\sqrt{(p+aq)^2 + b^2q^2}}{|pq' - p'q|}$ apart along α , since the restriction to α of Π is a local isometry. Hence denoting $l(\alpha)$ the length of α ,

$$\text{Int}(\alpha, \beta) \leq \frac{|pq' - p'q|}{\sqrt{(p+aq)^2 + b^2q^2}} l(\alpha).$$

Besides, since Π is 1-Lipschitz, we have $l(\beta) \geq \sqrt{(p' + aq')^2 + b^2q'^2}$, whence

$$\frac{\text{Int}(\alpha, \beta)}{l(\alpha)l(\beta)} \leq \frac{|pq' - p'q|}{\sqrt{(p+aq)^2 + b^2q^2} \sqrt{(p' + aq')^2 + b^2q'^2}} \leq K(\mathbb{T}^2) = \frac{1}{b}$$

where the last equality stems from [8].

Second case : $pq' - p'q = 0$. Then since α et β are simple, we have $p = p', q = q'$. So the closed curves α and β , if they are distinct, cannot meet anywhere but at the conical point. Hence their algebraic intersection is 0 or ± 1 . Therefore

$$\frac{\text{Int}(\alpha, \beta)}{l(\alpha)l(\beta)} \leq \frac{1}{(p+aq)^2 + b^2q^2} \leq 1.$$

So we have $\text{KVol}(X) \leq \frac{1}{b}nb = n$, unless there exists a pair of closed geodesics α and β on X , which have the same direction (that is, $(p, q) = (p', q')$), non-zero intersection, and such that the product of their lengths is $< b$. \square

Corollary 3.3. *We have $\text{KVol}(\text{St}(2s-1)) = 2s-1$.*

Proof. Lemma 3.1 and the facts that any closed curve on $\text{St}(2s-1)$ is at least 1 long, and that $\text{Vol}(\text{St}(2s-1)) = 2s-1$, entail that $\text{KVol}(\text{St}(2s-1)) \leq 2s-1$. On the other hand, we have $\text{Int}(e_1, f_2) = 1$, $l(e_1) = l(f_2) = 1$. \square

Now we want to know for which elements $X = X_s(x, y)$ of \mathcal{T}_s we have $\text{KVol}(X_s(x, y)) > 2s-1$.

Lemma 3.4. *For $(x, y) \in V_{\pm 1} \cap \mathcal{D}$, where $V_{\pm 1} = \{(x, y) : (x \pm 1)^2 + (y - 1/2)^2 < 1/4\}$, we have $\text{KVol}(X_s(x, y)) > 2s-1$. For any other (x, y) in \mathcal{D} , we have $\text{KVol}(X_s(x, y)) \leq 2s-1$. Furthermore, $\text{KVol}(X_s(x, y))$ goes to infinity when y goes to zero, and goes to $2s-1$ when y goes to ∞ while (x, y) remains in \mathcal{D} .*

Proof. By Lemma 3.1 and Subsubsection 2.3, finding which elements $X_s(x, y)$ of \mathcal{T}_s have $\text{KVol}(X_s(x, y)) > 2s-1$ amounts to finding all (x, y) in the fundamental domain \mathcal{D} such that there exist $p, q \in \mathbb{Z}$, coprime and both odd, such that

$$\frac{y}{(p+qx)^2 + (qy)^2} > 1 \quad (4)$$

$$\iff \left(x + \frac{p}{q}\right)^2 + \left(y - \frac{1}{2q^2}\right)^2 < \frac{1}{4q^4} \quad (5)$$

that is, (x, y) lies inside the open disk $D(p, q)$ of radius $1/2q^2$, centered at $(-p/q, 1/2q^2)$. As we shall see this only happens when $|p| = |q| = 1$. Indeed if $|p| > |q|$, the center of $D(p, q)$ lies at least $1/q$ apart from the vertical boundaries of \mathcal{D} , and since the radius of $D(p, q)$ is $< 1/q$, the whole $D(p, q)$ lies outside of \mathcal{D} . If $|p| < |q|$, the center of $D(p, q)$ lies below the half-circle $x^2 + y^2 = 1$, and the distance between the center of $D(p, q)$ and the half-circle $x^2 + y^2 = 1$ is greater than the radius of $D(p, q)$, because

$$1 - \sqrt{\frac{p^2}{q^2} + \frac{1}{4q^4}} > \frac{1}{2q^2},$$

so the whole $D(p, q)$ lies below the half-circle $x^2 + y^2 = 1$. Since p and q are coprime, the only remaining possibility is $p = \pm 1$, $q = 1$. So for any $X_s(x, y)$ in \mathcal{T}_s , we have $\text{KVol}(X_s(x, y)) \leq 2s-1$, unless (x, y) lies in the horocyclic neighborhood $V_{\pm 1}$ of the lower cusp of \mathcal{D} , bounded by the dotted circles depicted in Figure 4.

For (x, y) inside $V_{\pm 1}$, we have

$$\text{KVol}(X_s(x, y)) \geq \max\left(\frac{(2s-1)y}{\sqrt{(1+x)^2 + y^2}}, \frac{(2s-1)y}{\sqrt{(1-x)^2 + y^2}}\right) > 2s-1$$

and

$$\lim_{y \rightarrow 0} \text{KVol}(X_s(x, y)) = +\infty.$$

Since $\text{KVol}(X_s(1, y)) = 2s-1$, KVol is continuous as a function of (x, y) , and every (x, y) in \mathcal{D} is within distance $1/y$ of $(1, y)$, it follows that $\text{KVol}(X_s(x, y))$ tends to $2s-1$ when y goes to ∞ . \square

For $p, q \in \mathbb{Z}$, $p \wedge q = 1$, setting $r = p/q$, and for $(x, y) \in \mathbb{H}^2$, define

$$J_r(x, y) = \frac{y}{(p + qx)^2 + (qy)^2}.$$

Now, thanks to Lemmata 3.1 and 3.4, we can give a more precise version of Equation (3) :

$$\forall (x, y) \in \mathcal{D}, \text{KVVol}(X_s(x, y)) = (2s-1) \max\{J_1(x, y), J_{-1}(x, y), \sup_{r \neq r' \in \mathbb{Q}} I_{r,r'} K_{r,r'}(x, y)\}. \quad (6)$$

Lemma 3.5. *For $(x, y) \in \mathcal{D}$, for all $(r, r') \in \text{End}(\mathcal{Z})$, we have*

$$K_{r,r'}(x, y) \geq \sqrt{\frac{143}{144}}$$

and equality occurs if and only if $(x, y) = (\pm 9/14, \sqrt{143}/14)$.

Proof. By Lemma 2.6, we have $I_{r,r'} = 1$ whenever $(r, r') \in \text{End}(\mathcal{Z})$, and by Lemma 2.7, we have $I_{r,r'} \leq 9/10 < \sqrt{\frac{143}{144}}$ whenever $(r, r') \notin \text{End}(\mathcal{Z})$.

Elementary calculations show that the bisectors of the triangle T delimited by the geodesics $\gamma_{-1,1}$, $\gamma_{-2,1}$ and $\gamma_{0,2}$ intersect at the point $k = (\frac{9}{14}, \frac{\sqrt{143}}{14})$. For each $\gamma_{r,r'}$ different from the sides of T , $\gamma_{r,r'}$ does not pass through the interior of T , therefore the distance between the point k and $\gamma_{r,r'}$ is greater than the distance between k and the sides of the triangle T . Thus, for each $\gamma_{r,r'}$ different from $\gamma_{-1,1}$, $\gamma_{-2,1}$ and $\gamma_{0,2}$, we have $\theta_{r,r'}(k) > \theta_{-1,1}(k)$, so $\cos \theta_{r,r'}(k) < \cos \theta_{-1,1}(k)$, which entails:

$$\sup_{(r,r')} K_{r,r'}(k) = K_{-1,1}(k) = \sqrt{\frac{143}{144}}.$$

In all that follows, we call $V_{r,r'}$ the banana neighbourhood of the geodesic $\gamma_{r,r'}$ such that

$$\theta_{r,r'}(x, y) = \theta_{-1,1}\left(\frac{9}{14}, \frac{\sqrt{143}}{14}\right).$$

Let

$$C_n = C\left(\left(\frac{1-n}{2}, -\frac{1}{\sqrt{143}} \frac{1+n}{2}\right), \frac{1+n}{2} \sqrt{\frac{144}{143}}\right)$$

and

$$C^n = C\left(\left(\frac{1-n}{2}, \frac{1}{\sqrt{143}} \frac{1+n}{2}\right), \frac{1+n}{2} \sqrt{\frac{144}{143}}\right)$$

be the circles with centre respectively

$$\left(\frac{1-n}{2}, -\frac{1}{\sqrt{143}} \frac{1+n}{2}\right) \text{ and } \left(\frac{1-n}{2}, \frac{1}{\sqrt{143}} \frac{1+n}{2}\right)$$

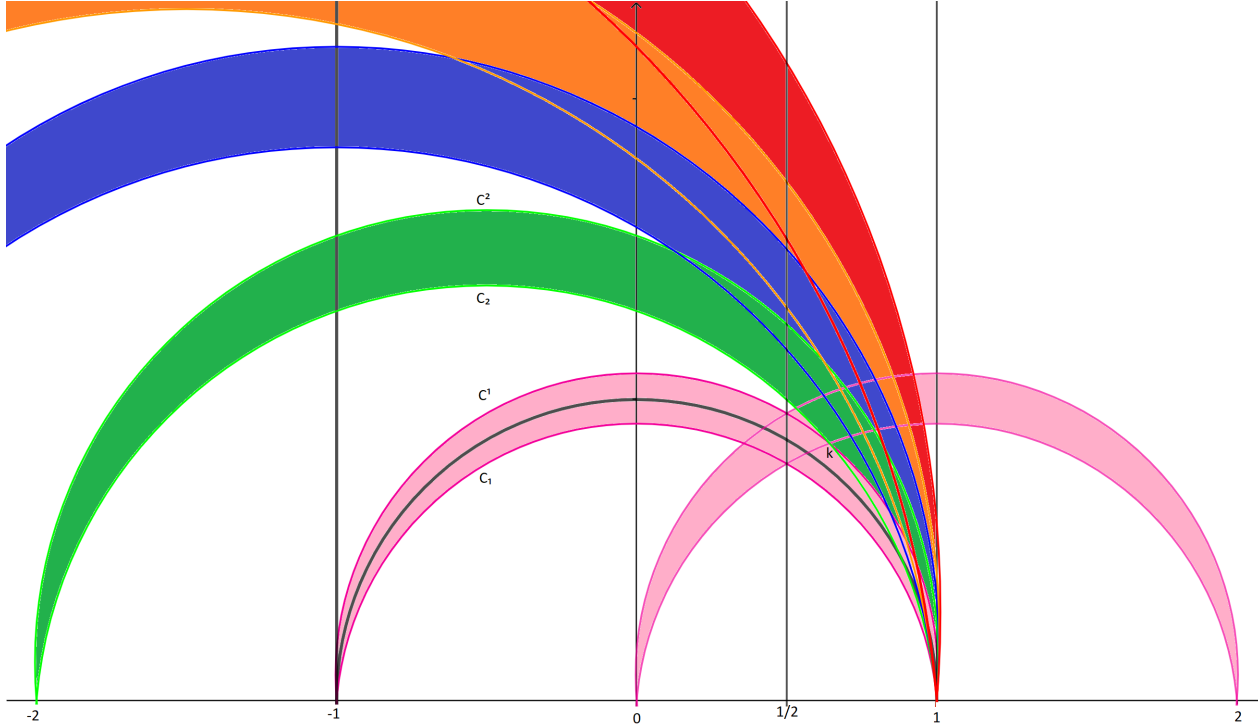


Figure 15: covering by banana neighbourhoods

and the same radius $\frac{1+n}{2}\sqrt{\frac{144}{143}}$, $n \geq 1$. The banana neighbourhood $V_{-n,1}$, for $n \geq 1$, is delimited by the maximal arcs of C_n and C^n , respectively, which are contained in \mathbb{H}^2 ; and the banana neighbourhood $V_{\frac{1}{2},\infty}$ is delimited by the two straight lines of equation $y = \sqrt{143}x + 1/2$ and $y = -\sqrt{143}x + 1/2$, respectively.

Let \mathcal{A} be the region delimited by $x = 1/2$, $x = 1$ and $\gamma_{-1,1}$. For all $n \geq 3$, $C_n \cap \mathcal{A}$, $C^n \cap \mathcal{A}$ is contained in the interior of, respectively, $V_{-n+1,1}$, and $V_{-n-1,1}$; and $C^2 \cap \mathcal{A}$ is contained in the interior of $V_{-3,1}$ (see Figure 15).

Also, the interiors of $V_{0,2}$, $V_{-1,1}$ and $V_{\frac{1}{2},\infty}$ cover the region delimited by C_2 , $\gamma_{-1,1}$ and $\gamma_{\frac{1}{2},\infty}$ except the point k (see Figure 15), so the interior of $V_{0,2}$, $V_{\frac{1}{2},\infty}$ and $V_{-n,1}$, $n \geq 1$, cover all the region \mathcal{A} except the point k .

Thus for $(x, y) \in \mathcal{A}$, $(x, y) \neq k$, we have $K_{r,r'}(x, y) > \sqrt{\frac{143}{144}}$. By symmetry with respect to $x = 1/2$, and afterwards by $x = 0$, we then deduce the lemma. \square

We are now set to finish the proof of Theorem 1.1. It is straightforward to check that

$$\max\{J_1\left(\frac{9}{14}, \frac{\sqrt{143}}{14}\right), J_{-1}\left(\frac{9}{14}, \frac{\sqrt{143}}{14}\right)\} = \max\{J_1\left(-\frac{9}{14}, \frac{\sqrt{143}}{14}\right), J_{-1}\left(-\frac{9}{14}, \frac{\sqrt{143}}{14}\right)\} = \sqrt{\frac{143}{144}}$$

so we have

$$\text{KVVol}(X_s(\pm \frac{9}{14}, \frac{\sqrt{143}}{14})) = (2s - 1)\sqrt{\frac{143}{144}},$$

and for every $(x, y) \in \mathcal{D}$, we have $\text{KVVol}(X_s(x, y)) \geq (2s - 1)\sqrt{\frac{143}{144}}$, with equality if and only if $(x, y) = (\pm 9/14, \sqrt{143}/14)$.

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