On log-algebraic identities for Anderson t-modules and characteristic p multiple zeta values

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ON LOG-ALGEBRAIC IDENTITIES FOR ANDERSON t-MODULES AND CHARACTERISTIC p MULTIPLE ZETA VALUES

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Abstract. Based on the notion of Stark units we present a new approach that obtains refinements of log-algebraic identities for Anderson t-modules. As a consequence we use our techniques to recover many earlier results and prove stronger results in some cases. Further, we devise a direct and conceptual way to get logarithmic interpretations for multiple zeta values in positive characteristic. This generalizes the work of Anderson and Thakur for Carlitz zeta values.

1. Introduction

1.1. **Background.** The power-series $\sum_{n\geq 1}\frac{z^n}{n}$ is log-algebraic:

$$\sum_{n\geq 1}\frac{z^n}{n}=-\log(1-z).$$

This identity allows one to obtain the value of a Dirichlet L-series at $s=1$ as an algebraic linear combination of logarithms of circular units.

By a well-known analogy between the arithmetic of number fields and that of global function fields, conceived of in the 1930s by Carlitz, we now switch to the function field setting. We briefly recount some of the many advances which have been made in function field arithmetic. In particular we will focus on the study of special values of Goss L-functions and their generalizations, like Thakur's characteristic p multiple zeta values (MZV's for short). Especially, we wish to highlight the reliance many of these results have on log-algebraic identities.

We let $A=\mathbb{F}_q[\theta]$ with θ an indeterminate over a finite field $\mathbb{F}_q$. In the 1930's Carlitz [15] introduced the Carlitz zeta values $\zeta_A(n)$ for $n\in\mathbb{N}$, which are analogues of positive special values of the Riemann zeta function, $\zeta(n)$. He then related the zeta value $\zeta_A(1)$ to the so-called Carlitz module C. One of his fundamental theorems gave a log-algebraic identity

$$\exp_C(\zeta_A(1))=1$$

where $\exp_C$ is the exponential series attached to the Carlitz module. We mention that Goss [27] introduced a new type of L-functions in the arithmetic of function fields over finite fields and showed that Carlitz zeta values can be realized as special values of such L-functions (see [28, Chapter 8]).

In the 1970's Drinfeld [25, 26] made a breakthrough and defined Drinfeld modules even for a more general ring A. It turned out that the Carlitz module is the simplest

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example of a Drinfeld module. Several years later Anderson [1] developed the theory of t-modules which are higher dimensional generalizations of Drinfeld modules.

Since the introduction of t-modules, several additional log-algebraic identities for Anderson t-modules have been discovered. The theory began with the seminal paper of Anderson and Thakur [5] where they proved log-algebraic identities for tensor powers $C^{\otimes n}$ ($n \in \mathbb{N}$) of the Carlitz module. The latter result implies logarithmic interpretations for Carlitz zeta values $\zeta_A(n)$ at positive integers n generalizing the aforementioned result of Carlitz. Combining the above result with his transcendence theory, Yu [45] proved that $\zeta_A(n)$ is transcendental for all positive integers n. Based on the criteria for linear and algebraic independence developed by Jing Yu [46], Anderson-Brownawell-Papanikolas [4] and Papanikolas [35], Chang and Yu [24, 46] determined all algebraic relations among the Carlitz zeta values. These results are very striking when compared to the extremely limited knowledge we have about the transcendence of odd Riemann zeta values in the classical setting.

In recent years various works have revealed the importance of log-algebraicity on Anderson t-modules in function field arithmetic. On the one hand, following the pioneering work of Anderson [3] in which he introduced the analogue of cyclotomic units for the Carlitz module, Anglès, Tavares Ribeiro and the second author have developed the theory of Stark units for Anderson modules which turns out to be a powerful tool for investigating log-algebraicity. Roughly speaking, they are units in the sense of Taelman [37, 38] coming from the canonical deformation of Drinfeld modules in Tate algebras in the sense of Pellarin [36]. Note that the concept of Stark units appeared implicitly in [11, 12]. The notion was formalized in [13] for Drinfeld modules over $\mathbb{F}_q[\theta]$ and then further developed in more general settings in [7, 8, 9]. Recently, combining Stark units and the class formula à la Taelman, Anglès, Tavares Ribeiro and the second author [9] obtained various log-algebraicity results for tensor powers of the Carlitz module, generalizing the work of Anderson-Thakur [5] and recovering that of Papanikolas [34]. On the other hand, log-algebraicity has been successfully applied to the study of Goss's zeta values and Thakur's characteristic p multiple zeta values (MZV's). For example, using log-algebraicity Chang [17] gave an effective algorithm to determine linear relations among depth-two MZV's, the authors [30] generalized the work of Chang and Yu [24] by completely determining algebraic relations among Goss's zeta values on function fields of elliptic curves, and Chang and Mishiba [20] proved a conjecture of Furusho concerning MZV's and their ν-adic variants over function fields.

In the present paper, inspired by the notion of Stark units, we introduce a new approach to obtain refinements of log-algebraic identities for Anderson t-modules. One of the main benefits of our approach is that it provides a concise general theory on the existence of log-algebraic identities, and thus it gives a unifying framework to many such previous results which have been previously proven in a somewhat ad-hoc fashion. To demonstrate the unification our new techniques allow, we use them to recover many previously known results in a straightforward way, and in some cases our techniques even lead to stronger results. We also apply our techniques to prove new formulas relating to characteristic p MZV's.

For applications of our new techniques, we first investigate the dual t-motives introduced by Anderson and Thakur [6] and developed further by many people (see [16, 17, 18, 19, 20, 23]). Our main result yields log-algebraic identities for the t-modules attached to these dual t-motives. Next we obtain a generalization of one

of the main theorems in [17] where simple and elegant logarithmic interpretations for special cases of MZV's were presented. Along the way, we clarify connections between these t-modules and MZV's and recover many results in [18, 19, 20, 23]. Finally we devise new dual t-motives called star motives which provide direct logarithmic interpretations for both MZV's and ν-adic MZV's in the same spirit of the original work of Anderson and Thakur. This generalizes completely the work of Anderson and Thakur [5] and answers positively to a problem raised by Chang and Mishiba [20].

1.2. **Statement of the main result.** Let us give now more precise statements of our results.

Let $A = \mathbb{F}_q[\theta]$ be the polynomial ring in the variable θ over a finite field $\mathbb{F}_q$ of q elements of characteristic $p > 0$. Let $K = \mathbb{F}_q(\theta)$ be the fraction field of A equipped with the infinity place ∞ of degree 1. Let K_∞ be the completion of K at ∞ and $\mathbb{C}_\infty$ be the completion of a fixed algebraic closure $\overline{K}$ of K at ∞. Letting t be another independent variable, we denote by $\mathbb{T}$ the Tate algebra in the variable t with coefficients in $\mathbb{C}_\infty$ and by $\mathbb{L}$ the fraction field of $\mathbb{T}$.

Let $\overline{K}[\tau]$ (resp. $\overline{K}[\sigma]$) denote the non-commutative skew-polynomial ring with coefficients in $\overline{K}$, subject to the relation for $c \in \overline{K}$,

$$\tau c = c^q \tau \quad (\text{resp. } \sigma c = c^{1/q} \sigma).$$

We define Frobenius twisting on $\overline{K}[t]$ by setting for $i \in \mathbb{Z}$ and $g = \sum_j c_j t^j \in \overline{K}[t]$,

$$g^{(i)} = \sum_j c_j^{q^i} t^j.$$

We extend twisting to matrices in $\mathrm{Mat}_{i\times j}(\overline{K}[t])$ by twisting coordinatewise.

We will work with effective dual t-motives and Anderson t-modules introduced by Anderson (see [1, 14, 32]). In what follows, we let $\mathcal{M}'$ denote an effective dual t-motive in the sense of [32, §2.4], which is a left $\overline{K}[t,\sigma]$-module that is free and finitely generated over $\overline{K}[t]$ such that for $\ell \gg 0$ we have $(t-\theta)^\ell(\mathcal{M}'/\sigma\mathcal{M}') = \{0\}$. Letting $\mathbf{m} = (m_1, \dots, m_r)^\top$ be a $\overline{K}[t]$-basis of $\mathcal{M}'$, then there exists a unique matrix $\Phi' \in \mathrm{Mat}_r(\overline{K}[t]) \cap \mathrm{GL}_r(\overline{K}(t))$ such that

$$\sigma\mathbf{m} = \Phi'\mathbf{m}.$$

We suppose further that $\mathcal{M}'$ is free and finitely generated over $\overline{K}[\sigma]$ and that $\mathcal{M}'$ is uniformizable or rigid analytically trivial, which means that there exists a matrix $\Psi' \in \mathrm{GL}_r(\mathbb{L})$ satisfying $\Psi'^{(-1)} = \Phi'\Psi'$.

Anderson associated to $\mathcal{M}'$ an Anderson t-module E' defined over $\overline{K}$ (see [32, §2.5.2]). This is an $\mathbb{F}_q$-algebra homomorphism $E' : A \longrightarrow \mathrm{Mat}_d(\overline{K})[\tau]$ (a skew polynomial ring - see §2.2) for $d = \mathrm{rank}_{\overline{K}[\sigma]}(\mathcal{M}') \in \mathbb{N}$ (called the dimension of E') such that for all $a \in A$, if we write

$$E'_a = d[a] + E'_{a,1}\tau + \dots,$$

then we have $(d[a] - aI_d)^d = 0$. Note that for any $\overline{K}$-algebra B, we can define two A-module structures on B^d: the first one is denoted by $E'(B)$ where A acts on B^d via E', and the second one is denoted by $\mathrm{Lie}_{E'}(B)$ where A acts on B^d via $d[\cdot]$.

We recall the definition of the maps

$$\delta_0 : \mathcal{M}' \to \overline{K}^d, \quad \delta_1 : \mathcal{M}' \to \overline{K}^d,$$

from [32, Proposition 2.5.8]. Letting $m \in \mathcal{M}'$, if $\{w_j\}_{1\leq j\leq d}$ is a $\overline{K}[\sigma]$-basis for $\mathcal{M}'$, then we write

$$m = c_{0,1}w_1 + \cdots + c_{0,d}w_d + c_{1,1}\sigma(w_1) + \cdots + c_{1,d}\sigma(w_d) + \ldots, \quad c_{i,j} \in \overline{K},$$

and we set

$$\delta_0(m) = \begin{pmatrix} c_{0,1} \\ \vdots \\ c_{0,d} \end{pmatrix}, \quad \delta_1(m) = \begin{pmatrix} c_{0,1} \\ \vdots \\ c_{0,d} \end{pmatrix} + \begin{pmatrix} c_{1,1} \\ \vdots \\ c_{1,d} \end{pmatrix}^{(1)} + \cdots . \tag{1.1}$$

The map δ_0 extends to $\mathcal{M}' \otimes_{\overline{K}[t]} \mathbb{T}_\theta \simeq \mathrm{Mat}_{r\times 1}(\mathbb{T}_\theta)$ in the natural way, where $\mathbb{T}_\theta$ is a Tate algebra of functions which converge at $t = \theta$. One can show that there exists a unique exponential series $\mathrm{Exp}_{E'} \in I_d + \mathrm{Mat}_d(\overline{K})[[\tau]]\tau$ associated to E' such that

$$\mathrm{Exp}_{E'}\, d[a] = E'_a \,\mathrm{Exp}_{E'}, \quad a \in A.$$

The logarithm function $\mathrm{Log}_{E'}$ is then defined as the formal power series inverse of $\mathrm{Exp}_{E'}$. We note that as functions on $\mathbb{C}_\infty^d$ the function $\mathrm{Exp}_{E'}$ is everywhere convergent, whereas $\mathrm{Log}_{E'}$ converges on some finite polydisc in $\mathbb{C}_\infty^d$.

Let $\mathcal{M}$ be the effective dual t-motive given by the matrix

$$\Phi = \begin{pmatrix} \Phi' & 0 \\ \mathbf{f} & 1 \end{pmatrix}, \quad \text{with } \mathbf{f} = (f_1, \ldots, f_r) \in \mathrm{Mat}_{1\times r}(\overline{K}[t]).$$

Let Ψ be a rigid analytic trivialization for $\mathcal{M}$ such that we can write (see §3.1 for more details)

$$\Psi = \begin{pmatrix} \Psi' & 0 \\ \Psi_{\mathbf{f}} & 1 \end{pmatrix} \in \mathrm{GL}_{r+1}(\mathbb{L}), \quad \text{with } \Psi_{\mathbf{f}} \in \mathrm{Mat}_{1\times r}(\mathbb{L}),$$

and

$$\Upsilon := \Psi^{-1} = \begin{pmatrix} \Upsilon' & 0 \\ \Upsilon_{\mathbf{f}} & 1 \end{pmatrix} \in \mathrm{Mat}_{r+1}(\mathbb{T}), \quad \text{with } \Upsilon_{\mathbf{f}} \in \mathrm{Mat}_{1\times r}(\mathbb{T}).$$

Note that by [35, Proposition 3.3.9] there exists a polynomial $F \in \mathbb{F}_q[t]$ such that $F\Psi \in \mathrm{Mat}_{r+1}(\mathbb{T})$. Inspired by [23] we construct a point $\delta_1(\mathbf{f}^\top) = \mathbf{v}_{\mathcal{M}} \in E'(\overline{K})$, where we make the implicit identification $\mathcal{M}' \cong \overline{K}[t]^r$ for the basis $\mathbf{m}$ (see (3.3)).

We now introduce some notion inspired by that of units and Stark units. We mention that the former was introduced by Taelman in [37] and the latter has been introduced and developed in [7, 8, 9, 13] following the pioneering work of Anderson [3] in which he introduced the analogue of cyclotomic units for the Carlitz module. In recent years the notion of Stark units has been successfully applied to achieve important results related to special values of the Goss L-functions, characteristic p multiple zeta values, Anderson's log-algebraicity identities, Taelman's units, and Drinfeld modular forms in Tate algebras (see [7, 8, 9, 11, 12, 13, 29, 30, 31]). Let z be an indeterminate with $\tau z = z\tau$ and let $\mathbb{T}_z(\mathbb{C}_\infty)$ be the Tate algebra in the variable z with coefficients in $\mathbb{C}_\infty$. We define the canonical z-deformation $\widetilde{E}'$, which is called the t-module defined over a Tate algebra in the sense of Pellarin [36] (see also [11, 12]). It is the homomorphism of $\mathbb{F}_q[z]$-algebras $\widetilde{E}' : A[z] \to \mathrm{Mat}_d(\overline{K}[z])[\tau]$ such that

$$\widetilde{E}'_a = d[a] + \sum_{k\geq 1} E'_{a,k} z^k \tau^k, \quad a \in A.$$

Then there exists a unique series $\mathrm{Exp}_{\widetilde{E}'} \in I_d + \tau \operatorname{Mat}_d(\overline{K}[z])[[\tau]]$ such that

$$\mathrm{Exp}_{\widetilde{E}'} d[a] = \widetilde{E}'_a \mathrm{Exp}_{\widetilde{E}'}, \quad a \in A.$$

One can show that if we write $\mathrm{Exp}_{E'} = \sum_{i\geq 0} Q_i \tau^i$, then $\mathrm{Exp}_{\widetilde{E}'} = \sum_{i\geq 0} Q_i z^i \tau^i$. Thus $\mathrm{Exp}_{\widetilde{E}'}$ converges on $\mathrm{Lie}_{\widetilde{E}'}(\mathbb{T}_z(\mathbb{C}_\infty))$ and induces a homomorphism of $A[z]$-modules

$$\mathrm{Exp}_{\widetilde{E}'} : \mathrm{Lie}_{\widetilde{E}'}(\mathbb{T}_z(\mathbb{C}_\infty)) \to \widetilde{E}'(\mathbb{T}_z(\mathbb{C}_\infty)).$$

We denote by $\mathrm{Log}_{\widetilde{E}'} \in I_d + \tau \operatorname{Mat}_d(\overline{K}[z])[[\tau]]$ the inverse of $\mathrm{Exp}_{\widetilde{E}'}$.

Definition 1.1 (Definitions 2.2 and 3.3)**.** Letting $\mathbf{z} \in \mathbb{C}_\infty^d$, we put $\mathbf{Z} := \mathrm{Exp}_{E'}(\mathbf{z})$.

1) We say that $\mathbf{z}$ is *a log-algebraic point for* E' if $\mathbf{Z} = \mathrm{Exp}_{E'}(\mathbf{z}) \in \overline{K}^d$. The latter equality is also called *a log-algebraic identity for* E'.

2) We say that $\mathbf{z}$ is *a special log-algebraic point for* E' if we can write $\mathbf{z} = \mathbf{x}\big|_{z=1}$ for some $\mathbf{x} \in \mathrm{Lie}_{\widetilde{E}'}(\mathbb{T}_z(\mathbb{C}_\infty))$ satisfying $\mathrm{Exp}_{\widetilde{E}'}(\mathbf{x}) \in \overline{K}[z]^d$. We also say that we have *a Stark logarithmic identity for* $\mathbf{z}$, and we write

$$\mathrm{Log}^{St}_{E'}(\mathbf{Z}) = \mathbf{z}.$$

3) Suppose that there exists some finite collection of pairs $\{(a_i, \mathbf{u}_i)\} \subset A \times \overline{K}^d$ where each $\mathbf{u}_i$ is in the domain of convergence of $\mathrm{Log}_{E'}$, such that

$$\mathbf{Z} = \sum_i E'_{a_i} \mathbf{u}_i, \quad \text{and} \quad \mathbf{z} = \sum_i d[a_i] \mathrm{Log}_{E'}(\mathbf{u}_i),$$

we will say that we have *a split-logarithmic identity for* $\mathbf{z}$, and we write

$$\mathrm{Log}^{!}_{E'}(\mathbf{Z}) = \mathbf{z}.$$

We mention that Stark logarithmic identities and split-logarithmic identities are much stronger than log-algebraic identities and could be viewed as refinements thereof. We emphasize that compared to log-algebraic identities, Stark logarithmic identities allows one to bypass convergence issues arising from logarithmic series and to "deal directly" with logarithms.

Proposition 1.2. *Having a split-logarithmic identities implies having a Stark logarithmic identities.*

Proof. Let the notation be as above. Since the $\mathbf{u}_i$ are in the domain of convergence of $\mathrm{Log}_{E'}$, we may set $\mathbf{x} = \sum_i d[a_i] \mathrm{Log}_{\widetilde{E}'}(\mathbf{u}_i)$, and a quick calculation shows that $\mathbf{x}|_{z=1} = \mathbf{z}$ (for $\mathbf{x}$ from Definition 1.1) and that

$$\mathrm{Exp}_{\widetilde{E}'}(\mathbf{x}) = \sum_i \widetilde{E}'_{a_i} \mathbf{u}_i \in \overline{K}[z]^d.$$

□

Split-logarithmic identities are common when one discovers a log-algebraic identity of the form $\mathrm{Exp}_{E'}(\mathbf{z}) = \mathbf{Z}$, but $\mathbf{Z}$ is not inside the domain of convergence of $\mathrm{Log}_{E'}$. Thakur [40] suggested that one can decompose $\mathbf{Z}$ into a sum of terms $E'_{a_i}\mathbf{u}_i$, such that each $\mathbf{u}_i$ is inside the domain of convergence of $\mathrm{Log}_{E'}$. Such a decomposition is the motivation for the above definition of split-logarithmic identity. This is the case in the celebrated log-algebraicity theorem of Anderson and Thakur for tensor powers of the Carlitz module [5, Theorem 3.8.3] (see also [40, §II]).

We are now ready to state the main result of our paper (see §3 for more details).

Theorem 1.3 (Theorem 3.5)**.** *Let Ψ, Υ and $\mathbf{f}$ be defined as above. Let $F \in \mathbb{F}_q[t]$ such that $F\Psi \in \mathrm{Mat}_{r+1}(\mathbb{T})$. We suppose that $F\Psi_{\mathbf{f}}^{(k)}$ converges to 0 as $k \to \infty$ as a vector of functions in $\mathrm{Mat}_{1\times r}(\mathbb{T})$.*

Then the point $\delta_0(\mathbf{f}^\top - \Upsilon_{\mathbf{f}}^\top)$ is a special log-algebraic point for E' and we have a Stark logarithmic identity

$$\mathrm{Log}_{E'}^{St}(\mathbf{v}_{\mathcal{M}}) = \delta_0(\mathbf{f}^\top - \Upsilon_{\mathbf{f}}^\top).$$

Further, if the point $\mathbf{v}_{\mathcal{M}}$ satisfies some mild convergence conditions (see Theorem 3.5 (b) for details), then we have a split-logarithmic identity

$$\mathrm{Log}_{E'}^{!}(\mathbf{v}_{\mathcal{M}}) = \delta_0(\mathbf{f}^\top - \Upsilon_{\mathbf{f}}^\top).$$

We now sketch the main ideas of the proof of Theorem 1.3.

(1) We explicitly compute the coefficient matrices of the logarithm series $\mathrm{Log}_{E'}$ of E'. Our method is based on a joint work of the second author with Anglès and Tavares Ribeiro [9] and is different from the approach of Anderson and Thakur [5]. It also differs from the logarithm computations of the first author, Chang and Mishiba in [18, 29].

(2) Then we consider the canonical z-deformation $\widetilde{E'}$ of the Anderson t-module E' and also the canonical z-deformation $\mathbf{v}_{\mathcal{M},z}$ of $\mathbf{v}_{\mathcal{M}}$. Using Step (1) we compute the value $\mathrm{Log}_{\widetilde{E'}}(\mathbf{v}_{\mathcal{M},z})$ as a formal series in z.

(3) Using the hypothesis on $\Psi_{\mathbf{f}}$, we show that the formal series $\mathrm{Log}_{\widetilde{E'}}(\mathbf{v}_{\mathcal{M},z})$ belongs to the Tate algebra $\mathrm{Lie}_{\widetilde{E'}}(\mathbb{T}_z(\mathbb{C}_\infty))$ in the variable z and with coefficients in $\mathbb{C}_\infty$. Hence we obtain the desired result.

We again emphasize that although the techniques used in the proof of the main theorem are original, much of the structure wherein these techniques exist is due to Anderson (see [1]). Notably, the definition of t-motives and their identification with t-modules under the maps δ_0 and δ_1 are present in unpublished notes of Anderson, where he also gives technique for finding log-algebraic identities (see Section 3.5 for details). However, the reader should note that our calculations for the logarithm and the proof of Stark and split logarithmic identities differ substantially from Anderson's original ideas.

1.3. **Applications of the main result.** We present several applications of our main result to Carlitz zeta values and characteristic p multiple zeta values. We briefly recall the definitions of these values. In [15] Carlitz introduced the Carlitz zeta values $\zeta_A(n)$ $(n \in \mathbb{N})$ given by

$$\zeta_A(n) := \sum_{a \in A_+} \frac{1}{a^n} \in K_\infty$$

which are analogues of classical special zeta values in the function field setting. Here A_+ denotes the set of monic polynomials in A. For any tuple of positive integers $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$, Thakur [41] defined the characteristic p multiple zeta value (MZV for short) $\zeta_A(\mathfrak{s})$ or $\zeta_A(s_1, \dots, s_r)$ by

$$\zeta_A(\mathfrak{s}) := \sum \frac{1}{a_1^{s_1} \dots a_r^{s_r}} \in K_\infty$$

where the sum runs through the set of tuples $(a_1, \dots, a_r) \in A_+^r$ with $\deg a_1 > \cdots > \deg a_r$. We call r the depth of $\zeta_A(\mathfrak{s})$. We note that Carlitz zeta values are exactly depth one MZV's.

In [6], for $\mathfrak{s}=(s_1,\dots,s_r)\in\mathbb{N}^r$ as above, Anderson and Thakur used Anderson-Thakur polynomials to construct an effective dual t-motive which is rigid analytically trivial such that $\zeta_A(\mathfrak{s})$ appears in the entries of the attached rigid analytic trivialization. It has been generalized to the so-called Anderson-Thakur (effective) dual t-motives $\mathcal{M}'_{\mathfrak{s},\mathfrak{Q}}$ indexed by more general tuples of polynomials $\mathfrak{Q}=(Q_1,\dots,Q_r)\in\overline{K}[t]^r$ and notably developed by Chang [16] and Chang, Papanikolas and Yu [23] in recent years (see also [17, 18, 19, 20]).

Section 4 is devoted to the applications of the main result (see Theorem 1.3) to the setting of the Anderson-Thakur dual t-motives. Inspired by [23] we define the t-module $E'_{\mathfrak{s},\mathfrak{Q}}$ associated to the dual t-motive $\mathcal{M}'_{\mathfrak{s},\mathfrak{Q}}$ and the associated special point $\mathbf{v}_{\mathfrak{s},\mathfrak{Q}}\in E'_{\mathfrak{s},\mathfrak{Q}}(\overline{K})$. We then establish a split-logarithmic identity for $E'_{\mathfrak{s},\mathfrak{Q}}$:

Theorem 1.4 (Theorem 4.3)**.** *We have a split-logarithmic identity*

$$\mathrm{Log}^{!}_{E'}(\mathbf{v}_{\mathfrak{s},\mathfrak{Q}})=\delta_0\begin{pmatrix}(-1)^{r-1}\mathfrak{L}^{\star}(s_r,\dots,s_1)\Omega^{-(s_1+\cdots+s_r)}\\(-1)^{r-2}\mathfrak{L}^{\star}(s_r,\dots,s_2)\Omega^{-(s_2+\cdots+s_r)}\\\vdots\\\mathfrak{L}^{\star}(s_r)\Omega^{-s_r}\end{pmatrix},$$

where the series $\mathfrak{L}^{\star}$ are defined in (4.4) following Chang [16].

When we specialize $\mathfrak{Q}$ to Anderson-Thakur polynomials (see §4.8), the dual t-motives are intimately related to MZV's and are well studied in the aforementioned works. In this setting Chang gave very simple and elegant logarithmic interpretations for some special MZV's (see [17, Theorem 1.4.1]). However, as Chang and Mishiba [21] explained to us, the relations among Chang's theorem and the works of Chang-Papanikolas-Yu [23] and other works [19, 20] are still mysterious. The aim of Theorem 4.7 is twofold. It presents a generalization of Chang's theorem to the general setting, i.e., for any tuple $\mathfrak{Q}$ and also clarifies the connections with the previous works [19, 20, 23].

Theorem 1.5 (Theorem 4.7)**.** *Let $\mathfrak{s}=(s_1,\dots,s_r)\in\mathbb{N}^r$ with $r\geq 2$. Assume that, for $1\leq\ell<j\leq r+1$, the values $\mathfrak{L}(s_\ell,\dots,s_{j-1})(\theta)$ do not vanish (see (4.5) for a precise definition). We further suppose that $\mathfrak{L}(s_2,\dots,s_r)(\theta)\in K$. Then there exist $a_{\mathfrak{s}}\in A$, an integral point $\mathbf{Z}_{\mathfrak{s}}\in C^{\otimes(s_1+\cdots+s_r)}(A)$ and a point $\mathbf{z}_{\mathfrak{s}}\in\mathbb{C}_\infty^{s_1+\cdots+s_r}$ such that*

1) the last coordinate of $\mathbf{z}_{\mathfrak{s}}$ equals $a_{\mathfrak{s}}\mathfrak{L}(s_1,\dots,s_r)(\theta)\widetilde{\pi}^{s_1+\cdots+s_r}$,

2) $\mathrm{Exp}_{C^{\otimes(s_1+\cdots+s_r)}}(\mathbf{z}_{\mathfrak{s}})=\mathbf{Z}_{\mathfrak{s}}$.

Next we apply Theorem 1.4 to the dual t-motives studied by Chang, Papanikolas and Yu in [23] in §4.8 and to those connected to multiple polylogarithms studied by Chang, the first author and Mishiba in [18, 19, 20] in §4.9. We recover some earlier results (see Proposition 4.4 and Theorem 4.14) and discover new results, which we state briefly below (see §4.8 for precise definitions of Γ_i and $\zeta_A^{\star}$).

Theorem 1.6 (Theorem 4.13)**.** *For $\mathfrak{s}=(s_1,\dots,s_r)\in\mathbb{N}^r$, we put $d_\ell:=s_\ell+\cdots+s_r$ for $1\leq\ell\leq r$. Let the polynomials $\mathfrak{Q}$ of Theorem 1.4 be specialized to be Anderson-Thakur polynomials (see §4.8). Then the $(d_1+\cdots+d_\ell)$th coordinate of $\mathrm{Log}^{!}_{E'}(\mathbf{v}_{\mathfrak{s},\mathfrak{Q}})$ of Theorem 1.4 equals $(-1)^{r-\ell}\Gamma_{s_\ell}\dots\Gamma_{s_r}\zeta_A^{\star}(s_r\dots,s_\ell)$.*

Section 5 is devoted to proving new logarithmic interpretations for MZV's and for certain ν-adic MZV's in the same spirit of the original work of Anderson and Thakur

[5] for Carlitz zeta values. We note that the entries of $\Upsilon(\theta) = \Psi^{-1}(\theta)$ attached to the above Anderson-Thakur dual t-motives are not MZV's except in the depth one case as in [5]. This may explain some of the difficulties encountered when one wishes to extend the work of Anderson and Thakur via this setting (see [20, Introduction] for a detailed discussion). To bypass this issue, for $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$, we devise a new dual t-motive $\mathcal{M}'^\star$ called the star dual t-motive whose entries of the associated matrix $\Upsilon^\star(\theta)$ naturally contain MZV's. We explicitly construct an Anderson t-module $E'^\star$ defined over A and an integral point $\mathbf{v}^\star_{\mathfrak{s}} \in E'^\star(A)$. Finally, we apply our main result to obtain the desired logarithmic interpretation for MZV's (see §5.4 for related definitions and §5.5 for the ν-adic setting).

Theorem 1.7 (Theorem 5.5). *For $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$, we put $d_\ell := s_\ell + \cdots + s_r$ for $1 \leq \ell \leq r$. Then we have*

$$\mathrm{Log}^!_{E'^\star}(\mathbf{v}^\star_{\mathfrak{s}}) = \delta_0 \begin{pmatrix} -\mathfrak{L}(s_r, \dots, s_1)\Omega^{-(s_1+\cdots+s_r)} \\ -\mathfrak{L}(s_r, \dots, s_2)\Omega^{-(s_2+\cdots+s_r)} \\ \vdots \\ -\mathfrak{L}(s_r)\Omega^{-s_r} \end{pmatrix}$$

In particular, for $1 \leq \ell \leq r$, the $(d_1 + \cdots + d_\ell)$th coordinate of the $\mathrm{Log}^!_{E'^\star}(\mathbf{v}^\star_{\mathfrak{s}})$ equals $-\Gamma_{s_\ell} \dots \Gamma_{s_r} \zeta_A(s_r \dots, s_\ell)$.

In §6 we provide examples to illustrate our results and compare our work with the works of Anderson-Thakur [5] and Chang-Mishiba [20]. Compared to Chang-Mishiba's construction, ours is much more direct, has smaller dimension (see Proposition 6.3) and is in the same spirit of [5] as illustrated in §6.2. In §6.3 we present further examples inspired by those given in [5].

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2. Anderson t-modules and dual t-motives

In this section we briefly review the basic theory of Anderson t-modules and dual t-motives and the relation between them. We refer the reader to [32, §2.5] for more details.

2.1. Notation.

In this paper we will use the following notation.

- $\mathbb{N} = \{1, 2, \dots\}$: the set of positive integers.
- $\mathbb{Z}^{\geq 0} = \{0, 1, \dots\}$: the set of non-negative integers.
- $\mathbb{Z}$: the set of integers.
- $\mathbb{F}_q$: a finite field having q elements.
- p: the characteristic of $\mathbb{F}_q$.

- θ, t: independent variables over $\mathbb{F}_q$.
- A: the polynomial ring $\mathbb{F}_q[\theta]$.
- A_+: the set of monic polynomials in A.
- $K = \mathbb{F}_q(\theta)$: the fraction field of A.
- ∞: the unique place of K which is a pole of θ.
- v_∞: the discrete valuation on K corresponding to the place ∞ normalized such that $v_\infty(\theta) = -1$.
- $|\cdot|_\infty = q^{-v_\infty}$: an absolute value on K.
- $K_\infty = \mathbb{F}_q((\frac{1}{\theta}))$: the completion of K at ∞.
- $\mathbb{C}_\infty$: the completion of a fixed algebraic closure $\overline{K}_\infty$ of K_∞. The unique valuation of $\mathbb{C}_\infty$ which extends v_∞ will still be denoted by v_∞.
- $\overline{K}$: the algebraic closure of K inside $\mathbb{C}_\infty$.

2.2. **Review of Anderson t-modules.** Let R be an $\mathbb{F}_q$-algebra and let $R[\tau]$ denote the (non-commutative) skew-polynomial ring with coefficients in R, subject to the relation for $r \in R$,

$$\tau r = r^q \tau.$$

We similarly define $R[\sigma]$, but we require additionally that R must be a perfect ring, now subject to the relation

$$\sigma r = r^{1/q} \sigma.$$

We define Frobenius twisting on $R[t]$ by setting for $i \in \mathbb{Z}$ and $g = \sum_j c_j t^j \in R[t]$,

$$g^{(i)} = \sum_j c_j^{q^i} t^j.$$

We extend twisting to matrices in $\mathrm{Mat}_{i\times j}(R[t])$ by twisting coordinatewise.

Definition 2.1. Let R be an $\mathbb{F}_q$-algebra equipped with an injective $\mathbb{F}_q$-algebra homomorphism $i : A \to R$ (recall $A = \mathbb{F}_q[\theta]$).

1) A d-dimensional Anderson t-module over R is an $\mathbb{F}_q$-algebra homomorphism $E : A \to \mathrm{Mat}_d(R)[\tau]$, such that for each $a \in A$,

$$E_a = d[a] + E_{a,1}\tau + \ldots, \quad E_{a,i} \in \mathrm{Mat}_d(R)$$

where $d[a] = i(a)I_d + N$ for some nilpotent matrix $N \in \mathrm{Mat}_d(R)$ (depending on a).

2) A Drinfeld module over R is a (non-trivial) one-dimensional Anderson t-module $\rho : A \to R[\tau]$.

For the rest of this paper, we will drop i when no confusion results. Anderson t-modules will sometimes be called t-modules.

The map $d[\cdot] : A \longrightarrow \mathrm{Mat}_d(R)$ is a ring homomorphism, and if R is a field, then it extends naturally to $d[\cdot] : K \longrightarrow \mathrm{Mat}_d(R)$ and describes the Lie action of E. Note that there is an implicit dependence of the map $d[\cdot]$ on the t-module E which we omit, since it does not cause any confusion. Let E be an Anderson t-module of dimension d over R as above and let B be an R-algebra. We can define two A-module structures on B^d. The first one is denoted by $E(B)$ where A acts on B^d via E:

$$a \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_d \end{pmatrix} = d[a] \begin{pmatrix} b_1 \\ \vdots \\ b_d \end{pmatrix} + \sum_{k\geq 1} E_{a,k} \begin{pmatrix} b_1^{q^k} \\ \vdots \\ b_d^{q^k} \end{pmatrix}, \quad \text{for } a \in A, \ \begin{pmatrix} b_1 \\ \vdots \\ b_d \end{pmatrix} \in B^d.$$

The second one is denoted by $\mathrm{Lie}_E(B)$ where A acts on B^d via $d[\cdot]$:

$$a \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_d \end{pmatrix} = d[a] \begin{pmatrix} b_1 \\ \vdots \\ b_d \end{pmatrix}, \quad \text{for } a \in A, \begin{pmatrix} b_1 \\ \vdots \\ b_d \end{pmatrix} \in B^d.$$

From now on, we will always work with Anderson t-modules over R such that $R \subset \mathbb{C}_\infty$ is an A-subalgebra. Let $E : A \to \mathrm{Mat}_d(\mathbb{C}_\infty)[\tau]$ be an Anderson module of dimension d over $\mathbb{C}_\infty$. We define Exp_E to be the exponential series associated to E, which is the unique function on $\mathbb{C}_\infty^d$ such that as an $\mathbb{F}_q$-linear power series we can write

$$\mathrm{Exp}_E(\mathbf{z}) = \sum_{i=0}^{\infty} Q_i \mathbf{z}^{(i)}, \quad Q_i \in \mathrm{Mat}_d(\mathbb{C}_\infty), \mathbf{z} \in \mathbb{C}_\infty^d,$$

with $Q_0 = I_d$ and such that for all $a \in A$ and $\mathbf{z} \in \mathbb{C}_\infty^d$,

$$\mathrm{Exp}_E(d[a]\mathbf{z}) = E_a(\mathrm{Exp}_E(\mathbf{z})).$$

The logarithm function Log_E is then defined as the formal power series inverse of Exp_E. We denote its power series as

$$\mathrm{Log}_E(\mathbf{z}) = \sum_{i=0}^{\infty} P_i \mathbf{z}^{(i)}, \quad P_i \in \mathrm{Mat}_d(\mathbb{C}_\infty), \mathbf{z} \in \mathbb{C}_\infty^d.$$

We note that as functions on $\mathbb{C}_\infty^d$ the function Exp_E is everywhere convergent, whereas Log_E has some finite radius of convergence.

2.3. **Log-algebraic identities.** We define the Tate algebra $\mathbb{T}$ over $\mathbb{C}_\infty$ as the space of power series in t which converge on the disc of radius 1, in other words,

$$\mathbb{T} := \left\{ \sum_{i=0}^{\infty} b_i t^i \in \mathbb{C}_\infty[[t]] \;\middle|\; |b_i|_\infty \to 0 \right\}.$$

We denote by $\mathbb{L}$ the fraction field of $\mathbb{T}$.

Define the Gauss norm $\|\cdot\|$ on $\mathbb{T}$ by setting

$$\|f\| := \max_i \{|b_i|_\infty\}$$

for $f = \sum_{i \geq 0} b_i t^i \in \mathbb{T}$. We then extend the Gauss norm to $\mathrm{Mat}_{\ell \times m}(\mathbb{L})$ by setting

$$\|B\| = \max_{i,j} \{\|B_{ij}\|\}$$

for $B = (B_{ij}) \in \mathrm{Mat}_{\ell \times m}(\mathbb{L})$.

Similarly, we denote by $\mathbb{T}_\theta$ the Tate algebra over $\mathbb{C}_\infty$ as the space of power series in t which converge on the disc of radius $|\theta|_\infty$, in other words,

$$\mathbb{T}_\theta := \left\{ \sum_{i=0}^{\infty} b_i t^i \in \mathbb{C}_\infty[[t]] \;\middle|\; q^i |b_i|_\infty \to 0 \right\}.$$

We also define different norms $\|.\|_\theta$. In particular, if $\sum_{i=0}^{\infty} b_i t^i \in \mathbb{T}_\theta$, then

$$\|f\|_\theta = \max_i \left\{ q^i |b_i|_\infty \right\}.$$

We note that $\mathbb{T}_\theta \subset \mathbb{T}$.

In what follows we fix an Anderson t-module $E : A \to \mathrm{Mat}_d(\overline{K})[\tau]$ of dimension d over $\overline{K}$. Let z be an indeterminate with $\tau z = z\tau$ and let $\mathbb{T}_z(\mathbb{C}_\infty)$ be the Tate

algebra $\mathbb{T}$ with the variable t replaced by the variable z. We define the canonical z-deformation of the t-module E denoted by $\widetilde{E}$ to be the homomorphism of $\mathbb{F}_q[z]$-algebras $\widetilde{E} : A[z] \to \mathrm{Mat}_d(\overline{K}[z])[\tau]$ such that

$$\widetilde{E}_a = \sum_{k\geq 0} E_{a,k} z^k \tau^k, \quad a \in A.$$

Then there exists a unique series $\mathrm{Exp}_{\widetilde{E}} \in I_d + \tau\,\mathrm{Mat}_d(\overline{K}[z])[[\tau]]$ such that

$$\mathrm{Exp}_{\widetilde{E}}\, d[a] = \widetilde{E}_a \,\mathrm{Exp}_{\widetilde{E}}, \quad a \in A,$$

(see [11, §3] for more details). One can show that if $\mathrm{Exp}_E = \sum_{i\geq 0} Q_i \tau^i$, then $\mathrm{Exp}_{\widetilde{E}} = \sum_{i\geq 0} Q_i z^i \tau^i$. In particular, $\mathrm{Exp}_{\widetilde{E}}$ converges on $\mathrm{Lie}_{\widetilde{E}}(\mathbb{T}_z(\mathbb{C}_\infty))$ and induces a homomorphism of $A[z]$-modules

$$\mathrm{Exp}_{\widetilde{E}} : \mathrm{Lie}_{\widetilde{E}}(\mathbb{T}_z(\mathbb{C}_\infty)) \to \widetilde{E}(\mathbb{T}_z(\mathbb{C}_\infty)).$$

We denote by $\mathrm{Log}_{\widetilde{E}} \in I_d + \tau\,\mathrm{Mat}_d(\overline{K}[z])[[\tau]]$ the inverse of $\mathrm{Exp}_{\widetilde{E}}$. Similarly, if $\mathrm{Log}_E = \sum_{i\geq 0} P_i \tau^i$, then $\mathrm{Log}_{\widetilde{E}} = \sum_{i\geq 0} P_i z^i \tau^i$.

We denote by $\mathrm{ev} : \mathrm{Lie}_{\widetilde{E}}(\mathbb{T}_z(\mathbb{C}_\infty)) \to \mathrm{Lie}_E(\mathbb{C}_\infty)$ the evaluation map at $z = 1$. If $\mathbf{x} \in \mathrm{Lie}_{\widetilde{E}}(\mathbb{T}_z(\mathbb{C}_\infty))$, then we also write $\mathbf{x}\big|_{z=1}$ for $\mathrm{ev}(\mathbf{x})$. Following [7, 12, 13, 37] we introduce some notion of log-algebraic points and of logarithmic identities for Anderson t-modules.

Definition 2.2. Letting $\mathbf{z} \in \mathbb{C}_\infty^d$, we put $\mathbf{Z} := \mathrm{Exp}_E(\mathbf{z})$.

1) We say that $\mathbf{z}$ is *a log-algebraic point for* E if $\mathbf{Z} = \mathrm{Exp}_E(\mathbf{z}) \in \overline{K}^d$. The latter equality is also called *a log-algebraicity identity for* E.

2) We say that $\mathbf{z}$ is *a special log-algebraic point for* E if we can write $\mathbf{z} = \mathbf{x}\big|_{z=1}$ for some $\mathbf{x} \in \mathrm{Lie}_{\widetilde{E}}(\mathbb{T}_z(\mathbb{C}_\infty))$ satisfying $\mathrm{Exp}_{\widetilde{E}}(\mathbf{x}) \in \overline{K}[z]^d$. We also say that we have *a Stark logarithmic identity for* $\mathbf{z}$, and we write

$$\mathrm{Log}_E^{St}(\mathbf{Z}) = \mathbf{z}.$$

Remark 2.3. We note that the notion of log-algebraic points and special log-algebraic points is related to that of Taelman's units and Stark units. We refer the reader to [37, 38] for more details about arithmetic of these units. The first example of Stark units appeared in the pioneering work of Anderson [3] in which he introduced the analogue of cyclotomic units for the Carlitz module. Recently, based on the fundamental work of Pellarin in [36], Anglès, Tavares Ribeiro and the second author have introduced and developed the theory of Stark units for Anderson modules (see [7, 9, 13]). This notion turns out to be a powerful tool for investigating log-algebraic identities [7, 9, 12] as well as the class formula à la Taelman in full generality [10].

We note that if $\mathbf{z}$ is a special log-algebraic point for E, then it is also a log-algebraic point for E. In fact, we set $\mathbf{Z} := \mathrm{Exp}_E(\mathbf{z})$. By Definition 2.2 there exists $\mathbf{x} \in \mathrm{Lie}_{\widetilde{E}}(\mathbb{T}_z(\mathbb{C}_\infty))$ such that $\mathrm{Exp}_{\widetilde{E}}(\mathbf{x}) \in \overline{K}[z]^d$ and $\mathbf{z} = \mathbf{x}\big|_{z=1}$. It follows that

$$\mathbf{Z} = \mathrm{Exp}_E(\mathbf{z}) = \mathrm{Exp}_{\widetilde{E}}(\mathbf{x})\big|_{z=1} \in \overline{K}^d.$$

Hence, $\mathbf{z}$ is also a log-algebraic point for E.

Remark 2.4. 1) We continue with the above notation. If we write the polynomial $\mathrm{Exp}_{\widetilde{E}}(\mathbf{x}) = \sum_{i=0}^{m} \mathbf{Z}_i z^i$ with $\mathbf{Z}_i \in \overline{K}^d$ $(0 \leq i \leq m)$, then the fact that $\mathbf{x} \in \mathrm{Lie}_{\widetilde{E}}(\mathbb{T}_z(\mathbb{C}_\infty))$ is equivalent to the following condition

$$P_k \mathbf{Z}_0^{(k)} + \cdots + P_{k-m} \mathbf{Z}_m^{(k-m)} \to 0 \quad \text{when } k \to +\infty.$$

Here we understand that $P_{k-i} = 0$ if $k - i < 0$. And we get

$$\mathbf{z} = \sum_{k \geq 0} \left(P_k \mathbf{Z}_0^{(k)} + \cdots + P_{k-m} \mathbf{Z}_m^{(k-m)} \right).$$

In other words, $\mathbf{z}$ is a kind of re-indexed logarithm as already observed in [29, 31, 40].

2) If the polynomial $\mathrm{Exp}_{\widetilde{E}}(\mathbf{x})$ is a monomial, then we express $\mathrm{Exp}_{\widetilde{E}}(\mathbf{x}) = \mathbf{Z}_i z^i$ for some $i \geq 0$. It is clear that $\mathbf{Z}_i$ lies in the domain of convergence of Log_E and $\mathbf{z}$ is a logarithm:

$$\mathbf{z} = \mathrm{Log}_E(\mathbf{Z}_i).$$

2.4. **Review of dual t-motives.**

We briefly review the notion of dual t-motives and explain the relation with t-modules thanks to Anderson (see [14, §4] and [32, §2.5] for more details).

Definition 2.5. An effective dual t-motive is a left $\overline{K}[t, \sigma]$-module $\mathcal{M}$ which is free and finitely generated over $\overline{K}[t]$ such that for $\ell \gg 0$ we have

$$(t - \theta)^\ell (\mathcal{M}/\sigma\mathcal{M}) = \{0\}.$$

Remark 2.6. 1) We mention that effective dual t-motives are special cases of Frobenius modules considered in [23, §2.2].
2) Note that Hartl and Juschka [32, §2.4] introduced a more general notion of dual t-motives. In particular, effective dual t-motives are always dual t-motives.

Throughout this paper we will always work with effective dual t-motives. Therefore, we will sometimes drop the word "effective" where there is no confusion.

Let $\mathcal{M}$ and $\mathcal{M}'$ be two effective dual t-motives. Then a morphism of effective dual t-motives $\mathcal{M} \to \mathcal{M}'$ is just a homomorphism of left $\overline{K}[t, \sigma]$-modules. We denote by $\mathcal{F}$ the category of effective dual t-motives equipped with the trivial object $\mathbf{1}$, which is simply the module $\overline{K}[t]$ with σ-action given by the first Frobenius twist.

We say that an object $\mathcal{M}$ of $\mathcal{F}$ is given by a matrix $\Phi \in \mathrm{Mat}_r(\overline{K}[t])$ if $\mathcal{M}$ is a $\overline{K}[t]$-module free of rank r and the action of σ is represented by the matrix Φ on a given $\overline{K}[t]$-basis for $\mathcal{M}$.

Recall that $\mathbb{L}$ denotes the fraction field of the Tate algebra $\mathbb{T}$. We say that an object $\mathcal{M}$ of $\mathcal{F}$ is uniformizable or rigid analytically trivial if there exists a matrix $\Psi \in \mathrm{GL}_r(\mathbb{L})$ satisfying $\Psi^{(-1)} = \Phi\Psi$. The matrix Ψ is called a rigid analytic trivialization of $\mathcal{M}$. By [35, Proposition 3.3.9], if $\mathcal{M}$ is uniformizable, then there exists a rigid analytic trivialization Ψ_0 of $\mathcal{M}$ with $\Psi_0 \in \mathrm{GL}_r(\mathbb{T})$. Further, if Ψ is a rigid analytic trivialization of $\mathcal{M}$, then $\Psi = \Psi_0 B$ with $B \in \mathrm{GL}_r(\mathbb{F}_q(t))$.

In what follows, let $\mathcal{M}$ be an effective dual t-motive of rank r over $\overline{K}[t]$ which is also free and finitely generated of rank d over $\overline{K}[\sigma]$. Let $\mathbf{m} = \{m_1, \ldots, m_r\}$ denote a $\overline{K}[t]$-basis for $\mathcal{M}$ and let $\mathbf{w} = \{w_1, \ldots, w_d\}$ denote a $\overline{K}[\sigma]$-basis for $\mathcal{M}$. Using the basis $\mathbf{m} = \{m_1, \ldots, m_r\}$, we identify $\overline{K}[t]^r$ with $\mathcal{M}$ by the map

$$\iota_{\mathbf{m}} : \overline{K}[t]^r \to \mathcal{M}, \quad (g_1, \ldots, g_r)^\top \mapsto g_1 m_1 + \cdots + g_r m_r. \tag{2.1}$$

We extend $\iota_{\mathbf{m}}$ to Tate algebras still denoted by $\iota_{\mathbf{m}} : \mathbb{T}^r \to \mathcal{M} \otimes_{\overline{K}[t]} \mathbb{T}$.

Similarly, using the basis $\mathbf{w} = \{w_1, \dots, w_d\}$, we also identify $\overline{K}[\sigma]^d$ with $\mathcal{M}$

$$\iota_{\mathbf{w}} : \overline{K}[\sigma]^d \to \mathcal{M}, \quad (h_1, \dots, h_d)^\top \mapsto h_1 w_1 + \cdots + h_d w_d. \tag{2.2}$$

Letting $\iota = \iota_{\mathbf{w}}^{-1} \circ \iota_{\mathbf{m}}$, we get the map

$$\iota : \overline{K}[t]^r \to \overline{K}[\sigma]^d \tag{2.3}$$

which is an isomorphism of (left) $\overline{K}[t, \sigma]$-modules.

Once we fix the $\overline{K}[t]$-basis $\mathbf{m}$, then there exists some matrix $\Phi' \in \mathrm{Mat}_r(\overline{K}[t])$ such that σ acts on $\overline{K}[t]^r$ by inverse twisting and right multiplication by Φ' — or we may transpose to get a left multiplication:

$$\sigma \begin{pmatrix} g_1 \\ \vdots \\ g_r \end{pmatrix} = {\Phi'}^\top \begin{pmatrix} g_1 \\ \vdots \\ g_r \end{pmatrix}^{(-1)}, \quad g_i \in \overline{K}[t].$$

We note that this σ-action extends to $\mathbb{T}^r \cong \mathcal{M} \otimes_{\overline{K}[t]} \mathbb{T}$ in the natural way.

Recall the definition of the maps $\delta_0 : \mathcal{M} \to \overline{K}^d$ and $\delta_1 : \mathcal{M} \to \overline{K}^d$ from (1.1). We observe that the kernel of δ_1 equals $(\sigma - 1)\mathcal{M}$, and thus by [32, Proposition 2.5.8], we can write the commutative diagram

$$\begin{array}{ccc} \mathcal{M}/(\sigma-1)\mathcal{M} & \xrightarrow{\delta_1} & \overline{K}^d \\ {\scriptstyle a(t)}\downarrow & & \downarrow{\scriptstyle E'_a} \\ \mathcal{M}/(\sigma-1)\mathcal{M} & \xrightarrow{\delta_1} & \overline{K}^d \end{array}$$

where the left vertical arrow is multiplication by $a(t)$ and the right vertical arrow is the map induced by multiplication by a, which we denote by E_a. Then E defines an Anderson t-module over $\overline{K}$, and we call this the Anderson t-module associated with $\mathcal{M}$. Thus we have canonical isomorphisms of $\mathbb{F}_q[t]$-modules

$$\mathcal{M}/\sigma\mathcal{M} \xrightarrow{\sim} \mathrm{Lie}_E(\overline{K}),$$

induced by δ_0 and

$$\mathcal{M}/(\sigma-1)\mathcal{M} \xrightarrow{\sim} E(\overline{K}),$$

induced by δ_1. By [33, Lemma 3.4.1], δ_0 extends to $\mathcal{M} \otimes \mathbb{T}_\theta$, where $\mathbb{T}_\theta$ is a Tate algebra of functions with radius of convergence $|\theta|$ (see also [32, Proposition 2.5.8]). Maintaining that notation, letting z be a variable, we define the z-version $\delta_{1,z} : \mathcal{M} \to \overline{K}[z]^d$ of the map δ_1 by

$$\delta_{1,z}(m) = \begin{pmatrix} c_{0,1} \\ \vdots \\ c_{0,d} \end{pmatrix} + \begin{pmatrix} c_{1,1} \\ \vdots \\ c_{1,d} \end{pmatrix}^{(1)} z + \begin{pmatrix} c_{2,1} \\ \vdots \\ c_{2,d} \end{pmatrix}^{(2)} z^2 + \cdots,$$

which is an $\mathbb{F}_q[t]$-module homomorphism.

Remark 2.7. Anderson showed that the functor $\mathcal{M} \mapsto E$ gives rise to an equivalence from the category of effective dual t-motives $\mathcal{M}$ that are free and finitely generated as $\overline{K}[\sigma]$-modules onto the full subcategory of so-called A-finite Anderson t-modules (see for example [32, Theorem 2.5.11]).

3. The main result

This section aims to prove the main result of this paper (see Theorem 3.5). We establish refinements of log-algebraic identities for Anderson t-modules which provide a general framework for many earlier results which have been proven in a somewhat ad-hoc fashion. Finally we discuss relations with Anderson's analytic theory of A-finite t-modules and emphasize the advantage of special log-algebraic points compared to log-algebraic points.

3.1. **Ext^1-modules and t-modules.** In this section we explain a deep connection due to Anderson between some Ext^1-modules and Anderson t-modules. We follow closely the presentation given in [23, §5.2].

In what follows, we let $\mathcal{M}'$ be an effective dual t-motive of rank r over $\overline{K}[t]$ and rank d over $\overline{K}[\sigma]$, and let E' be its associated t-module. Recall the definitions of $\mathbf{m} = \{m_1, \dots, m_r\}$, $\mathbf{w} = \{w_1, \dots, w_d\}$, $\iota_{\mathbf{m}}$, $\iota_{\mathbf{w}}$, ι, δ_0 and δ_1 from §2.4. Composing with the map $\iota_{\mathbf{m}}$ defined in (2.1), we get three maps $\delta_0 \circ \iota_{\mathbf{m}} : \overline{K}[t]^r \to \overline{K}^d$, $\delta_1 \circ \iota_{\mathbf{m}} : \overline{K}[t]^r \to \overline{K}^d$ and $\delta_{1,z} \circ \iota_{\mathbf{m}} : \overline{K}[t]^r \to \overline{K}[z]^d$. From now on, to avoid heavy notation, we still denote these maps by

$$\delta_0 : \overline{K}[t]^r \to \overline{K}^d$$

and

$$\delta_1 : \overline{K}[t]^r \to \overline{K}^d, \quad \delta_{1,z} : \overline{K}[t]^r \to \overline{K}[z]^d.$$

We denote by $\Phi' \in \mathrm{Mat}_r(\overline{K}[t])$ the matrix defining the σ-action on $\mathcal{M}'$. If $\mathbf{m} = (m_1, \dots, m_r)$ is the $\overline{K}[t]$-basis of $\mathcal{M}'$ on which the σ-action is represented by the matrix Φ', then let $\mathcal{M}$ be the dual t-motive given by the matrix

$$\Phi = \begin{pmatrix} \Phi' & 0 \\ \mathbf{f} & 1 \end{pmatrix}, \quad \text{with } \mathbf{f} = (f_1, \dots, f_r) \in \mathrm{Mat}_{1\times r}(\overline{K}[t]),$$

on the basis $(m_1, \dots, m_r, m_{r+1})$. We note that $\mathcal{M}$ fits into an exact sequence of the form

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathbf{1} \to 0,$$

and so is an extension of the trivial dual t-motive $\mathbf{1}$ (with trivial σ-action) by $\mathcal{M}'$ in the sense of [23, §5.2] (although the idea goes back to Anderson originally), i.e., $\mathcal{M}$ represents a class in $\mathrm{Ext}^1_{\mathcal{F}}(\mathbf{1}, \mathcal{M}')$. Note that $\mathrm{Ext}^1_{\mathcal{F}}(\mathbf{1}, \mathcal{M}')$ has a natural $\mathbb{F}_q[t]$-module structure defined as follows. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two objects of $\mathrm{Ext}^1_{\mathcal{F}}(\mathbf{1}, \mathcal{M}')$ defined by the matrices

$$\Phi_1 = \begin{pmatrix} \Phi' & 0 \\ \mathbf{v}_1 & 1 \end{pmatrix} \in \mathrm{Mat}_{r+1}(\overline{K}[t]), \quad \mathbf{v}_1 \in \mathrm{Mat}_{1\times r}(\overline{K}[t]),$$

and

$$\Phi_2 = \begin{pmatrix} \Phi' & 0 \\ \mathbf{v}_2 & 1 \end{pmatrix} \in \mathrm{Mat}_{r+1}(\overline{K}[t]), \quad \mathbf{v}_2 \in \mathrm{Mat}_{1\times r}(\overline{K}[t]).$$

Then for any $a_1, a_2 \in \mathbb{F}_q[t]$, $a_1 * \mathcal{M}_1 + a_2 * \mathcal{M}_2$ is defined to be the class in $\mathrm{Ext}^1_{\mathcal{F}}(\mathbf{1}, \mathcal{M}')$ represented by

$$\begin{pmatrix} \Phi' & 0 \\ a_1\mathbf{v}_1 + a_2\mathbf{v}_2 & 1 \end{pmatrix} \in \mathrm{Mat}_{r+1}(\overline{K}[t]).$$

By abuse of notation, since $\mathrm{Ext}^1_{\mathcal{F}}(\mathbf{1}, \mathcal{M}')$ parameterizes classes of extensions of $\mathbf{1}$ by $\mathcal{M}'$, we will refer to such extensions as being in $\mathrm{Ext}^1_{\mathcal{F}}(\mathbf{1}, \mathcal{M}')$.

Theorem 5.2.1 of [23] then shows

$$\alpha : \mathrm{Ext}^1_{\mathcal{F}}(\mathbf{1}, \mathcal{M}') \to \mathcal{M}'/(\sigma-1)\mathcal{M}' \qquad \mathcal{M} \mapsto f_1 m_1 + \cdots + f_r m_r \tag{3.1}$$

is an isomorphism of $\mathbb{F}_q[t]$-modules. We refer the reader to the citation for details.

For such an extension $\mathcal{M} \in \mathrm{Ext}^1_{\mathcal{F}}(\mathbf{1}, \mathcal{M}')$, we know that $\mathcal{M}$ is uniformizable by [32, Lemma 2.4.21]. We suppose that there exists a rigid analytic trivialization $\Psi \in \mathrm{Mat}_{r+1}(\mathbb{L})$ of Φ, i.e., $\Psi^{(-1)} = \Phi\Psi$, such that if we set $\Upsilon := \Psi^{-1}$, then $\Upsilon \in \mathrm{Mat}_{r+1}(\mathbb{T})$. We set

$$\Psi = \begin{pmatrix} \Psi' & 0 \\ \Psi_{\mathbf{f}} & 1 \end{pmatrix} \in \mathrm{GL}_{r+1}(\mathbb{L}),$$

and

$$\Upsilon = \begin{pmatrix} \Upsilon' & 0 \\ \Upsilon_{\mathbf{f}} & 1 \end{pmatrix} \in \mathrm{Mat}_{r+1}(\mathbb{T}), \tag{3.2}$$

where

$$\Psi_{\mathbf{f}} = (\Psi_{\mathbf{f},1}, \dots, \Psi_{\mathbf{f},r}) \in \mathrm{Mat}_{1\times r}(\mathbb{L}),$$

and

$$\Upsilon_{\mathbf{f}} = (\Upsilon_{\mathbf{f},1}, \dots, \Upsilon_{\mathbf{f},r}) \in \mathrm{Mat}_{1\times r}(\mathbb{T}).$$

Remark 3.1. (1) By [35, Proposition 3.3.9, §4.1.6], there exist a matrix $U \in \mathrm{GL}_{r+1}(\mathbb{F}_q(t))$ and a rigid analytic trivialization $\Psi_0 \in \mathrm{GL}_{r+1}(\mathbb{T})$ of Φ such that $\Psi U = \Psi_0$. By [33, Definition 3.4.5], one can show that $\Psi_0 \in \mathrm{GL}_{r+1}(\mathbb{T}_\theta)$. Thus it follows that there exists $F \in \mathbb{F}_q[t]$ such that $F\Psi \in \mathrm{Mat}_{r+1}(\mathbb{T}_\theta)$.

(2) Suppose now that for some nonzero $F' \in \mathbb{F}_q[t]$ such that $F'\Psi \in \mathrm{Mat}_{r+1}(\mathbb{T})$. By the previous discussion, we know that there exists a nonzero $F \in \mathbb{F}_q[t]$ such that $F\Psi \in \mathrm{Mat}_{r+1}(\mathbb{T}_\theta)$. We deduce that $F'\Psi \in \mathrm{Mat}_{r+1}(\mathbb{T}_\theta)$.

(3) By similar arguments, one can show that $\Upsilon \in \mathrm{Mat}_{r+1}(\mathbb{T}_\theta)$. In particular, $\Upsilon_{\mathbf{f}} = (\Upsilon_{\mathbf{f},1}, \dots, \Upsilon_{\mathbf{f},r}) \in \mathrm{Mat}_{1\times r}(\mathbb{T}_\theta)$.

Inspired by [23] we define the point $\mathbf{v}_{\mathcal{M}} \in E'(\overline{K})$ by the image of $\mathcal{M}$ via the composition of isomorphisms

$$\delta_1 \circ \alpha : \mathrm{Ext}^1_{\mathcal{F}}(\mathbf{1}, \mathcal{M}') \xrightarrow{\sim} \mathcal{M}'/(\sigma-1)\mathcal{M}' \xrightarrow{\sim} E'(\overline{K}). \tag{3.3}$$

Thus $\mathbf{v}_{\mathcal{M}} = \delta_1(\mathbf{f}^\top) \in E'(\overline{K})$. We also set

$$\mathbf{v}_{\mathcal{M},z} := \delta_{1,z}(\mathbf{f}^\top) \in \widetilde{E}'(\overline{K}[z])$$

where $\widetilde{E}'$ is the z-deformation t-module attached to E' (see §2.3).

Remark 3.2. Let $\mathcal{M}$ be a class in $\mathrm{Ext}^1_{\mathcal{F}}(\mathbf{1}, \mathcal{M}')$. Let E' be the t-module attached to $\mathcal{M}'$ and $\mathbf{v}_{\mathcal{M}} \in E'(\overline{K})$ be the special point attached to $\mathcal{M}$ as above. We observe that $\mathcal{M}$ is a torsion class in $\mathrm{Ext}^1_{\mathcal{F}}(\mathbf{1}, \mathcal{M}')$ if and only if $\mathbf{v}_{\mathcal{M}}$ is a torsion point in $E'(\overline{K})$.

3.2. **Statement of the Main Result.** We keep the above notation. We give a definition which simplifies notation throughout the paper.

Definition 3.3. Given a d-dimensional t-module E over $\mathbb{C}_\infty$ with logarithm function Log_E and two points $\mathbf{z}, \mathbf{Z} \in \mathbb{C}_\infty^d$, we say that we have a split-logarithmic identity (for $\mathbf{z}$)

$$\mathrm{Log}_E^!(\mathbf{Z}) = \mathbf{z}$$

if there exists some finite collection of pairs $\{(a_i, \mathbf{u}_i)\} \subset A \times \overline{K}^d$ where each $\mathbf{u}_i$ is in the domain of convergence of Log_E, such that

$$\mathbf{z} = \sum_i d[a_i] \mathrm{Log}_E(\mathbf{u}_i),$$
$$\mathbf{Z} = \sum_i E_{a_i} \mathbf{u}_i.$$

Remark 3.4.
(1) We note that if $\mathrm{Log}_E^!(\mathbf{Z}) = \mathbf{z}$, then $\mathrm{Log}_E^{\mathrm{St}}(\mathbf{Z}) = \mathbf{z}$ and $\mathrm{Exp}_E(\mathbf{z}) = \mathbf{Z}$. Further, each $\mathrm{Log}_E(\mathbf{u}_i)$ is a special log-algebraic point for E. This implies that $\mathbf{z}$, which is a linear combination of special log-algebraic points with coefficients in A (via the action $a \mapsto d[a]$), is also a special log-algebraic point for E.
(2) Split-logarithmic identities are common when one discovers a log-algebraic identity of the form $\mathrm{Exp}_E(\mathbf{z}) = \mathbf{Z}$, but $\mathbf{Z}$ is not inside the domain of convergence of Log_E. In some cases one can decompose $\mathbf{Z}$ into a sum of terms $E_{a_i}\mathbf{u}_i$ as above, such that each $\mathbf{u}_i$ is inside the domain of convergence of Log_E. Such is the case in the celebrated log-algebraicity theorem of Anderson and Thakur for tensor powers of the Carlitz module [5, Theorem 3.8.3] (see also [40, §II]).

We are ready to state the main result of this paper which provides log-algebraic identities for Anderson t-modules.

Theorem 3.5. *We keep the above notation and let* $\mathrm{Log}_{E'}^{St}$ *and* $\mathrm{Log}_{E'}^!$ *be defined as in Definitions 2.2 and 3.3, respectively.*

Let $F \in \mathbb{F}_q[t]$ *such that* $F\Psi \in \mathrm{Mat}_{r+1}(\mathbb{T})$ *(see Remark 3.1, Parts 1 and 2). We suppose that* $F\Psi_{\mathbf{f}}^{(k)}$ *converges to* 0 *as a vector of functions in* $\mathrm{Mat}_{1\times r}(\mathbb{T})$ *as* k *tends to* $+\infty$.

(a) Let $\Upsilon_{\mathbf{f}}$ *be defined as in (3.2) (see Remark 3.1, Part 3). Then*
(a1) The point $\delta_0(\mathbf{f}^\top - \Upsilon_{\mathbf{f}}^\top)$ *is a special log-algebraic point for* E'.
(a2) We have a Stark logarithmic identity

$$\mathrm{Log}_{E'}^{St}(\mathbf{v}_{\mathcal{M}}) = \delta_0(\mathbf{f}^\top - \Upsilon_{\mathbf{f}}^\top).$$

(b) Let α *be the map defined in (3.1). Suppose that there exists some finite collection of triples* $\{(\ell_i, n_i, \mathbf{u}_i = (u_{i,1}, \dots, u_{i,d})^\top)\} \subset \mathbb{Z}^{\geq 0} \times \mathbb{Z}^{\geq 0} \times \mathbb{C}_\infty^d$ *where each* $\mathbf{u}_i$ *is in the domain of convergence of* $\mathrm{Log}_{E'}$, *such that*

$$\alpha(\mathcal{M}) = \sum_i t^{n_i} \sigma^{\ell_i} \left(\sum_{j=1}^d u_{i,j} w_j \right),$$

where w_j *are elements of the* $\overline{K}[\sigma]$*-basis* $\mathbf{w}$. *Then we have a split-logarithmic identity*

$$\mathrm{Log}_{E'}^!(\mathbf{v}_{\mathcal{M}}) = \delta_0(\mathbf{f}^\top - \Upsilon_{\mathbf{f}}^\top).$$

If we additionally have that $\delta_0(\mathbf{f}^\top)=0$, then the right-hand side of the main equations in Parts (a) and (b) above is simply given by $\delta_0(-\Upsilon_{\mathbf{f}}^\top)$.

Remark 3.6. 1) By Remark 3.4, Part (b) could be considered as a refinement of Part (a).

2) It is clear that the condition that $F\Psi_{\mathbf{f}}^{(k)}$ converges to 0 as a vector of functions in $\mathrm{Mat}_{1\times r}(\mathbb{T})$ does not depend on the choice of $F\in\mathbb{F}_q[t]$. In particular, when $\Psi\in\mathrm{GL}_{r+1}(\mathbb{T})$, we could take $F=1$ as we will see in the next sections.

Remark 3.7. We mention below some known examples of Theorem 3.5.

1) As mentioned before, Anderson and Thakur [5] gave split-logarithmic identities for Carlitz zeta values.

2) Chang, Mishiba and the first author gave split-logarithmic identities for Carlitz multiple star polylogarithms (see [18, 20] and §4.9 for more details).

3) For higher genus curves, Thakur studied special zeta values associated to rings A such that A is principal. For such rings, he obtained both Stark logarithmic identity and split-logarithmic identity for special zeta values at 1 (see [40, §II]).

4) For elliptic curves, Stark logarithmic identities for special zeta values can be obtained using minor adjustments to [29, 30, 31]. However, it seems very difficult to obtain split-logarithmic identities for these values (see [29, Remark 6.4]).

3.3. **Proof of the main theorem: Part (a).** In this section we prove Theorem 3.5, Part (a). The proof is divided into several steps.

Step 0. Since $F\Psi\in\mathrm{Mat}_{r+1}(\mathbb{T})$ with $F\in\mathbb{F}_q[t]$, by Remark 3.1, Part 2, we know that $F\Psi\in\mathrm{Mat}_{r+1}(\mathbb{T}_\theta)$.

We claim that $F\Psi_{\mathbf{f}}^{(k)}$ converges to 0 as a vector of functions in $\mathrm{Mat}_{1\times r}(\mathbb{T}_\theta)$ as k tends to $+\infty$. In fact, we see that $\|F\Psi_{\mathbf{f}}^{(k)}\|=\|F\Psi_{\mathbf{f}}\|^{q^k}$. Thus the condition that $F\Psi_{\mathbf{f}}^{(k)}$ converges to 0 as a vector of functions in $\mathrm{Mat}_{1\times r}(\mathbb{T})$ as k tends to $+\infty$ implies $\|F\Psi_{\mathbf{f}}\|<1$. Next, since $F\Psi\in\mathrm{Mat}_{r+1}(\mathbb{T}_\theta)$, one can show that

$$\|F\Psi_{\mathbf{f}}^{(k)}\|_\theta\le\|F\Psi_{\mathbf{f}}\|_\theta\times\|F\Psi_{\mathbf{f}}\|^{q^k-1}.$$

Since $\|F\Psi_{\mathbf{f}}\|<1$, it follows that $\|F\Psi_{\mathbf{f}}^{(k)}\|_\theta\to 0$ as $k\to\infty$. The claim is proved.

Step 1. We compute the coefficients of $\mathrm{Log}_{E'}$. We set

$$\Theta:=(\Phi^{-1})^\top\in\mathrm{Mat}_{r+1}(\overline{K}(t)),$$

and

$$\Theta':=(\Phi'^{-1})^\top\in\mathrm{Mat}_r(\overline{K}(t)).$$

Now if we write

$$\mathrm{Log}_{E'}=\sum_{n\ge0}P_n\tau^n,$$

then

$$\mathrm{Log}_{\widetilde{E'}}=\sum_{n\ge0}P_nz^n\tau^n.$$

By [9, Proposition 2.2], for $n\ge0$, the nth coefficient of the logarithm series of E' is given as follows. Let $\mathbf{v}=(v_1,\dots,v_d)^\top\in\overline{K}^d$. Letting $m:=\iota_{\mathbf{w}}(\mathbf{v})=v_1w_1+\cdots+v_dw_d$, we see that m belongs to $\mathcal{M}'$. Thus we can express it in the $\overline{K}[t]$-basis $\{m_1,\dots,m_r\}$ using the map ι from (2.3)

$$\iota^{-1}(\mathbf{v})=\iota^{-1}(v_1,\dots,v_d)^\top=(g_1,\dots,g_r)^\top\in\overline{K}[t]^r.$$

In other words, $m = v_1w_1+\cdots+v_dw_d = g_1m_1+\cdots+g_rm_r$. Then by [9, Proposition 2.2] (see also [18, Lemma 4.2.1] for an explicit example of this) we have

$$P_n\mathbf{v}^{(n)} = \delta_0(\Theta'^{(1)}\dots\Theta'^{(n)}\iota^{-1}(\mathbf{v})^{(n)}).$$

Remark 3.8. We include a remark clarifying the relationship between the above formula and that of [9, Proposition 2.2], since our notation and setup here differs slightly. For fixed $k \in \mathbb{Z}^{\geq 0}$ we define a linear operator $\alpha_k : \overline{K}[t]^r \to \overline{K}[t]^r$ for $\mathbf{h} \in \overline{K}[t]^r$ by setting $\alpha_k(\mathbf{h}) = (\Theta'^{(1)}\dots\Theta'^{(k)}\mathbf{h}^{(k)})$. We see that $\alpha_k(\sigma^\ell(\mathbf{h})) = \alpha_{k-\ell}$ for $k \geq \ell$ and thus α_k satisfies the property given in [9, Lemma 2.1.iii]. A short calculation, together with the fact that $\sigma : \overline{K}[t]^r \to \overline{K}[t]^r$ is an injection, shows that $\alpha_k = \iota_{\mathbf{m}}^{-1}\circ\varphi_k\circ\iota_{\mathbf{m}}$ for φ_k, the inverse Frobenius mapping of [9, Proposition 2.2].

Step 2. We recall that

$$\mathbf{v}_{\mathcal{M},z} := \delta_{1,z}(\mathbf{f}^\top) \in \widetilde{E}'(\overline{K}[z]),$$

and

$$\mathbf{v}_{\mathcal{M}} = \mathbf{v}_{\mathcal{M},z}\big|_{z=1}.$$

This means that if we write

$$f_1m_1+\dots f_rm_r = v_{0,1}w_1+\cdots+v_{0,d}w_d+v_{1,1}\sigma(w_1)+\cdots+v_{1,d}\sigma(w_d)+\dots, \quad \text{with } v_{i,j}\in\overline{K},$$

and set $\mathbf{v}_i = (v_{i,1},\dots,v_{i,d})^\top$, then we get

$$\mathbf{v}_{\mathcal{M},z} = \delta_{1,z}(\mathbf{f}^\top) = \mathbf{v}_0 + \mathbf{v}_1^{(1)}z + \mathbf{v}_2^{(2)}z^2 + \dots.$$

Let $\mathbf{v} = (v_1,\dots,v_d)^\top \in \mathrm{Mat}_{d\times 1}(\overline{K})$. By (2.2), $\mathbf{v}$ can be identified as an element $\iota_{\mathbf{w}}(\mathbf{v}) = v_1w_1+\cdots+v_dw_d$ of $\mathcal{M}'$. We recall that $\iota_{\mathbf{m}}^{-1}(\sigma(\iota_{\mathbf{w}}(\mathbf{v}))) = \Phi'^{\top}\iota^{-1}(\mathbf{v})^{(-1)}$. Then we get an equality of formal series in z (we will interpret this identity in a Tate algebra under certain conditions in the Step 3 of the proof)

$$\begin{aligned}
\mathrm{Log}_{\widetilde{E}'}(\mathbf{v}z) &= \sum_{n\geq 0}\delta_0(\Theta'^{(1)}\dots\Theta'^{(n)}\iota^{-1}(\mathbf{v})^{(n)})z^{n+1}\\
&= \sum_{n\geq 0}\delta_0(\Theta'^{(1)}\dots\Theta'^{(n)}\Theta'^{(n+1)}\Phi'^{\top(n+1)}\iota^{-1}(\mathbf{v})^{(n)})z^{n+1}\\
&= \sum_{n\geq 0}\delta_0(\Theta'^{(1)}\dots\Theta'^{(n)}\Theta'^{(n+1)}\iota_{\mathbf{m}}^{-1}(\sigma(\iota_{\mathbf{w}}(\mathbf{v})))^{(n+1)})z^{n+1}\\
&= \sum_{n\geq 0}\delta_0(\Theta'^{(1)}\dots\Theta'^{(n)}\iota_{\mathbf{m}}^{-1}(\sigma(\iota_{\mathbf{w}}(\mathbf{v})))^{(n)})z^{n}.
\end{aligned}$$

Here the second equality comes from the fact that $\Theta' := ((\Phi')^{-1})^\top$, and the last one holds since $\delta_0(\sigma(\iota_{\mathbf{w}}(\mathbf{v}))) = 0$.

More generally, by similar arguments we show that for $j \in \mathbb{N}$,

$$\mathrm{Log}_{\widetilde{E}'}(\mathbf{v}z^j) = \sum_{n\geq 0}\delta_0(\Theta'^{(1)}\dots\Theta'^{(n)}\iota_{\mathbf{m}}^{-1}(\sigma^j(\iota_{\mathbf{w}}(\mathbf{v})))^{(n)})z^n. \tag{3.4}$$

We claim that

$$\mathrm{Log}_{\widetilde{E}'}(\mathbf{v}_{\mathcal{M},z}) = \sum_{n\geq 0}\delta_0(\Theta'^{(1)}\dots\Theta'^{(n)}(\mathbf{f}^\top)^{(n)})z^n.$$

In fact, by (3.4) we obtain

$$\begin{aligned}\mathrm{Log}_{\widetilde{E}'}(\mathbf{v}_{M,z}) &= \mathrm{Log}_{\widetilde{E}'}\left(\mathbf{v}_0+\mathbf{v}_1^{(1)}z+\mathbf{v}_2^{(2)}z^2+\dots\right)\\ &= \sum_{n\geq 0}\sum_{j\geq 0}\delta_0\left(\Theta'^{(1)}\dots\Theta'^{(n)}\iota_{\mathbf{m}}^{-1}\left(\sigma^j\left(\iota_{\mathbf{w}}(\mathbf{v}_j^{(j)})\right)\right)^{(n)}\right)z^n\\ &= \sum_{n\geq 0}\delta_0\left(\Theta'^{(1)}\dots\Theta'^{(n)}\iota_{\mathbf{m}}^{-1}\left(\sum_{j\geq 0}\sigma^j\left(\iota_{\mathbf{w}}(\mathbf{v}_j^{(j)})\right)\right)^{(n)}\right)z^n.\end{aligned}$$

We analyze now the sum $\sum_{j\geq 0}\sigma^j\left(\iota_{\mathbf{w}}(\mathbf{v}_j^{(j)})\right)$ viewed as an element of $\mathcal{M}'$. We have

$$\begin{aligned}\sum_{j\geq 0}\sigma^j\left(\iota_{\mathbf{w}}(\mathbf{v}_j^{(j)})\right) &= \sum_{j\geq 0}\sigma^j\left(v_{j,1}^{(j)}w_1+\dots+v_{j,d}^{(j)}w_d\right)\\ &= \sum_{j\geq 0}(v_{j,1}\sigma^j(w_1)+\dots+v_{j,d}\sigma^j(w_d))\\ &= f_1m_1+\dots+f_rm_r.\end{aligned}$$

This implies

$$\iota_{\mathbf{m}}^{-1}\left(\sum_{j\geq 0}\sigma^j\left(\iota_{\mathbf{w}}(\mathbf{v}_j^{(j)})\right)\right)=\iota_{\mathbf{m}}^{-1}(f_1m_1+\dots+f_rm_r)=\mathbf{f}^\top$$

and the claim follows immediately.

Step 3. We recall that $\mathbf{f}=(f_1,\dots,f_r)$ and $\Psi_{\mathbf{f}}=(\Psi_{\mathbf{f},1}\dots,\Psi_{\mathbf{f},r})$. Since $\Upsilon'=\Psi'^{-1}$, we get

$$\Upsilon_{\mathbf{f}}^\top=-\Upsilon'^\top\Psi_{\mathbf{f}}^\top. \tag{3.5}$$

The equality $\Psi^{(-1)}=\Phi\Psi$ implies

$$\begin{pmatrix}\Psi'^{(-1)} & 0\\ \Psi_{\mathbf{f}}^{(-1)} & 1\end{pmatrix}=\begin{pmatrix}\Phi' & 0\\ \mathbf{f} & 1\end{pmatrix}\begin{pmatrix}\Psi' & 0\\ \Psi_{\mathbf{f}} & 1\end{pmatrix}=\begin{pmatrix}\Phi'\Psi' & 0\\ \mathbf{f}\Psi'+\Psi_{\mathbf{f}} & 1\end{pmatrix}.$$

Thus

$$\Psi_{\mathbf{f}}^{(-1)}=\mathbf{f}\Psi'+\Psi_{\mathbf{f}}.$$

Note that $\Upsilon'=\Psi'^{-1}$. We then get

$$\mathbf{f}^\top=\Upsilon'^\top(\Psi_{\mathbf{f}}^{\top(-1)}-\Psi_{\mathbf{f}}^\top). \tag{3.6}$$

Next, since $\Psi'^{(-1)}=\Phi'\Psi'$, we deduce

$$\Theta'\Upsilon'^\top=(\Phi'^{-1})^\top\Upsilon'^\top=\Upsilon'^{\top(-1)}.$$

Thus for $n\geq 1$, we have

$$\Theta'^{(1)}\dots\Theta'^{(n)}\Upsilon'^{\top(n)}=\Upsilon'^\top. \tag{3.7}$$

Combining Equations (3.6) and (3.7), we get

$$\begin{aligned}\Theta'^{(1)}\dots\Theta'^{(n)}(\mathbf{f}^\top)^{(n)} &= \Theta'^{(1)}\dots\Theta'^{(n)}\Upsilon'^{\top(n)}(\Psi_{\mathbf{f}}^{\top(n-1)}-\Psi_{\mathbf{f}}^{\top(n)})\\ &= \Upsilon'^\top(\Psi_{\mathbf{f}}^{\top(n-1)}-\Psi_{\mathbf{f}}^{\top(n)})\\ &= \Upsilon'^\top F^{-1}(F\Psi_{\mathbf{f}}^{\top(n-1)}-F\Psi_{\mathbf{f}}^{\top(n)}).\end{aligned}$$

Thus

$$\begin{aligned}\text{(3.8)}\qquad \mathrm{Log}_{\widetilde{E}'}(\mathbf{v}_{\mathcal{M},z}) &= \delta_0(\mathbf{f}^\top) + \sum_{n\geq 1}\delta_0(\Theta'^{(1)}\dots\Theta'^{(n)}(\mathbf{f}^\top)^{(n)})z^n \\ &= \delta_0(\mathbf{f}^\top) + \sum_{n\geq 1}\delta_0(\Upsilon'^\top F^{-1}(F\Psi_{\mathbf{f}}^{\top(n-1)} - F\Psi_{\mathbf{f}}^{\top(n)}))z^n.\end{aligned}$$

Since $F\Psi_{\mathbf{f}}^{\top(k)}$ converges to 0 with respect to $\|.\|_\theta$ by Step 0, it follows that $\mathrm{Log}_{\widetilde{E}'}(\mathbf{v}_{M,z}) \in \mathbb{T}_z(\overline{K})^d$.

By evaluating Equation (3.8) at $z = 1$, we obtain a telescoping series on the right-hand side and get

$$\begin{aligned}\text{(3.9)}\qquad \mathrm{Log}_{\widetilde{E}'}(\mathbf{v}_{\mathcal{M},z})\big|_{z=1} &= \delta_0(\mathbf{f}^\top) + \sum_{n\geq 1}\delta_0(\Upsilon'^\top(\Psi_{\mathbf{f}}^{\top(n-1)} - \Psi_{\mathbf{f}}^{\top(n)})) \\ &= \delta_0(\mathbf{f}^\top) + \delta_0(\Upsilon'^\top\Psi_{\mathbf{f}}^\top) \\ &= \delta_0(\mathbf{f}^\top - \Upsilon_{\mathbf{f}}^\top).\end{aligned}$$

Here the last line holds by (3.5).

We conclude that $\delta_0(\mathbf{f}^\top - \Upsilon_{\mathbf{f}}^\top)$ is a special log-algebraic point for E' and get a Stark logarithmic identity

$$\mathrm{Log}_{E'}^{\mathrm{St}}(\mathbf{v}_{\mathcal{M}}) = \delta_0(\mathbf{f}^\top - \Upsilon_{\mathbf{f}}^\top)$$

which finishes Part (a).

3.4. **Proof of the main theorem: Part (b).** In this section we prove Theorem 3.5, Part (b).

By (3.9) we write

$$\begin{aligned}\mathbf{v}_{\mathcal{M},z} = \delta_{1,z}(\mathbf{f}^\top) &= \delta_{1,z}\left(\sum_i t^{n_i}\sigma^{\ell_i}\left(\sum_{j=1}^d u_{i,j}w_j\right)\right) \\ &= \sum_i \widetilde{E}'_{\theta^{n_i}}\delta_{1,z}\left(\sigma^{\ell_i}\left(\sum_{j=1}^d u_{i,j}w_j\right)\right).\end{aligned}$$

Here the last equality follows from the construction of t-modules associated to dual t-motives as explained in §2.4.

We then get the following equality between formal series in z:

$$\begin{aligned}\mathrm{Log}_{\widetilde{E}'}(\mathbf{v}_{\mathcal{M},z}) &= \sum_i d[\theta^{n_i}]\,\mathrm{Log}_{\widetilde{E}'}\left(\delta_{1,z}\left(\sigma^{\ell_i}\left(\sum_{j=1}^d u_{i,j}w_j\right)\right)\right) \\ &= \sum_i d[\theta^{n_i}]\,\mathrm{Log}_{\widetilde{E}'}((u_{i,1},\dots,u_{i,d})^\top z^{\ell_i}) \\ &= \sum_i d[\theta^{n_i}]z^{\ell_i}\,\mathrm{Log}_{\widetilde{E}'}(\mathbf{u}_i).\end{aligned}$$

Since all $\mathbf{u}_i$ are in the domain of convergence of $\mathrm{Log}_{E'}$, the above equality holds in the Tate algebra $\mathbb{T}_z(\mathbb{C}_\infty)$.

By Part (a) we apply the evaluation map ev to obtain

$$\begin{aligned}\delta_0(\mathbf{f}^\top - \Upsilon_{\mathbf{f}}^\top) &= \operatorname{Log}_{\widetilde{E}'}(\mathbf{v}_{\mathcal{M},z})\big|_{z=1}\\ &= \sum_i d[\theta^{n_i}] z^{\ell_i} \operatorname{Log}_{\widetilde{E}'}(\mathbf{u}_i)\Big|_{z=1}\\ &= \sum_i d[\theta^{n_i}] \operatorname{Log}_{E'}(\mathbf{u}_i),\end{aligned}$$

and finishes the proof of Part (b).

3.5. **Relations with Anderson's analytic theory of A-finite t-modules.** In this section we will apply the elaborate analytic theory of A-finite t-modules developed by Anderson (see [32, §2.5.3]) to obtain a result which is similar to Theorem 3.5. A similar analysis appeared in [30, §3.4], which was the starting point of this paper.

Theorem 3.9. *We keep the above notation. Then $\delta_0(\mathbf{f}^\top - \Upsilon_{\mathbf{f}}^\top)$ is a log-algebraic point for E'. Further, we have*

$$\operatorname{Exp}_{E'}(\delta_0(\mathbf{f}^\top - \Upsilon_{\mathbf{f}}^\top)) = \mathbf{v}_{\mathcal{M}}.$$

Remark 3.10. We give some comments to compare Theorems 3.5 and 3.9.
1) In Theorem 3.9 we do not require any restrictions. Consequently, we can only conclude that $\delta_0(\mathbf{f}^\top - \Upsilon_{\mathbf{f}}^\top)$ is a log-algebraic point, which is weaker than showing it is a special log-algebraic point as is done in Theorem 3.5 (see Remark 2.3). Roughly speaking, Theorem 3.9 allows us to use the machinery of special log-algebraic points and to bypass the convergence issue of logarithm series. This point of view turns out to be very powerful and has already led to several arithmetic applications (for example, compare [9] to [5, 34], also [7] to [2, 38]).
2) In addition, we mention again that the proof of Theorem 3.9 makes use of Anderson's analytic theory of A-finite t-modules which is much more complicated than the ingredients given in the proof of Theorem 3.5.

Proof of Theorem 3.9. Since $\Psi'^{(-1)} = \Phi'\Psi'$, we have

$$\Phi'^{\top}((\Psi'^{-1})^\top)^{(-1)} = (\Psi'^{-1})^\top.$$

Similarly, since $\Psi^{(-1)} = \Phi\Psi$, we have

$$\Phi^\top((\Psi^{-1})^\top)^{(-1)} = (\Psi^{-1})^\top.$$

It follows that

$$\Phi'^{\top}(\Upsilon_{\mathbf{f}}^\top)^{(-1)} + \mathbf{f}^\top = \Upsilon_{\mathbf{f}}^\top.$$

Recall that by Anderson's analytic theory of A-finite t-modules (see [32, Theorem 2.5.21, Corollaries 2.5.23 and 2.5.24]), if $\mathbf{v} \in \mathbb{T}_\theta^r$ and $\mathbf{z} \in \overline{K}[t]^r$ satisfy

$$\Phi'^{\top}\mathbf{v}^{(-1)} - \mathbf{v} = \mathbf{z},$$

then

$$\operatorname{Exp}_{E'}(\delta_0(\mathbf{v}+\mathbf{z})) = \delta_1(\mathbf{z}).$$

We apply the above result for $\mathbf{v} = -\Upsilon_{\mathbf{f}}^\top \in \mathbb{T}_\theta^r$ (see Remark 3.1, Part 3) and $\mathbf{z} = \mathbf{f}^\top$ to obtain

$$\operatorname{Exp}_{E'}(\delta_0(-\Upsilon_{\mathbf{f}}^\top + \mathbf{f}^\top)) = \delta_1(\mathbf{f}^\top) = \mathbf{v}_{\mathcal{M}}$$

as required. □

4. Application to the Anderson-Thakur dual t-motives

4.1. **Some history.** We investigate the Anderson-Thakur dual t-motives which were first introduced by Anderson and Thakur in [6]. Shortly thereafter, Chang [16] studied the Anderson-Thakur dual t-motives in a general setting and proved many fundamental properties and results. In [23] Chang, Papanikolas and Yu revisited the dual t-motives connected to multiple zeta values. They introduced the associated t-modules and the corresponding special points and gave an effective criterion for Eulerian MZV's in positive characteristic. Further, Chang, Mishiba and the first author investigated the dual t-motives connected to multiple polylogarithms at algebraic points with important applications to ∞-adic and ν-adic multiple zeta values in positive characteristic (see [17, 18, 19, 20]).

In this section we apply our main result to obtain log-algebraic identities for the t-modules attached to the Anderson-Thakur dual t-motives. Then we generalize Chang's theorem in [17] where he gave elegant logarithmic interpretations for special cases of MZV's. We also recover many previously known results in a straightforward way.

4.2. **Anderson-Thakur dual t-motives and periods.** In what follows, let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ be a tuple for $r \geq 1$ and $\mathfrak{Q} = (Q_1, \dots, Q_r) \in \overline{K}[t]^r$ satisfying the condition

$$\|Q_i\| < |\theta|_\infty^{\frac{s_i q}{q-1}} \quad \text{for all } 1 \leq i \leq r. \tag{4.1}$$

We should mention that this condition, inspired by [20, Remark 4.1.3], is slightly stronger than that given in [23, (2.3.1)], but is enough for applications to multiple zeta values and Carlitz star multiple polylogarithms.

We set the Anderson-Thakur dual t-motives $\mathcal{M}'_{\mathfrak{s},\mathfrak{Q}}$ and $\mathcal{M}_{\mathfrak{s},\mathfrak{Q}}$ attached to $\mathfrak{s}$ and $\mathfrak{Q}$ to be given by the matrices

$$\Phi'_{\mathfrak{s},\mathfrak{Q}} = \begin{pmatrix} (t-\theta)^{s_1+\cdots+s_r} & 0 & \dots & 0 \\ Q_1^{(-1)}(t-\theta)^{s_1+\cdots+s_r} & (t-\theta)^{s_2+\cdots+s_r} & \dots & 0 \\ 0 & Q_2^{(-1)}(t-\theta)^{s_2+\cdots+s_r} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & (t-\theta)^{s_r} \end{pmatrix} \in \mathrm{Mat}_r(\overline{K}[t]),$$

$$\Phi_{\mathfrak{s},\mathfrak{Q}} = \begin{pmatrix} (t-\theta)^{s_1+\cdots+s_r} & 0 & 0 & \dots & 0 \\ Q_1^{(-1)}(t-\theta)^{s_1+\cdots+s_r} & (t-\theta)^{s_2+\cdots+s_r} & 0 & \dots & 0 \\ 0 & Q_2^{(-1)}(t-\theta)^{s_2+\cdots+s_r} & \ddots & & \vdots \\ \vdots & & \ddots & (t-\theta)^{s_r} & 0 \\ 0 & \dots & 0 & Q_r^{(-1)}(t-\theta)^{s_r} & 1 \end{pmatrix} \in \mathrm{Mat}_{r+1}(\overline{K}[t]).$$

From now on, to simplify the notation, we will drop the subscripts $\mathfrak{s}$ and $\mathfrak{Q}$ whenever no confusion results. For example, we will write Φ instead of $\Phi_{\mathfrak{s},\mathfrak{Q}}$ and so on.

Throughout this paper, we work with the Carlitz period $\widetilde{\pi}$ which is a fundamental period of the Carlitz module (see [28, 41]). We make a choice of $(q-1)$st root of $(-\theta)$ and set

$$\Omega(t) := (-\theta)^{-q/(q-1)} \prod_{i \geq 1} \left(1 - \frac{t}{\theta^{q^i}}\right) \in \mathbb{T}^\times$$

so that $\Omega^{(-1)} = (t-\theta)\Omega$ and

$$\frac{1}{\Omega(\theta)} = \widetilde{\pi}. \tag{4.2}$$

Given $\mathfrak{s}$ and $\mathfrak{Q}$ satisfying (4.1) as above, Chang introduced the following series (see [16, Lemma 5.3.1] and also [23, Equation (2.3.2)]):

$$\mathfrak{L}_{\mathfrak{s},\mathfrak{Q}} := \sum_{i_1>\cdots>i_r\geq 0} (\Omega^{s_r}Q_r)^{(i_r)}\dots(\Omega^{s_1}Q_1)^{(i_1)}. \tag{4.3}$$

We also need the star series

$$\mathfrak{L}^{\star}_{\mathfrak{s},\mathfrak{Q}} := \sum_{i_1\geq\cdots\geq i_r\geq 0} (\Omega^{s_r}Q_r)^{(i_r)}\dots(\Omega^{s_1}Q_1)^{(i_1)}. \tag{4.4}$$

If we denote $\mathcal{E}$ the ring of series $\sum_{n\geq 0} a_n t^n \in \overline{K}[[t]]$ such that $\lim_{n\to+\infty} \sqrt[n]{|a_n|_\infty} = 0$ and $[K_\infty(a_0,a_1,\dots):K_\infty] < \infty$, then any $f\in\mathcal{E}$ is an entire function. It is proved that $\mathfrak{L}_{\mathfrak{s},\mathfrak{Q}} \in \mathcal{E}$ (see [16, Lemma 5.3.1]).

More generally, for $1\leq \ell < j \leq r+1$, we define the series

$$\begin{aligned}\mathfrak{L}(s_\ell,\dots,s_{j-1}) &:= \sum_{i_\ell>\cdots>i_{j-1}\geq 0} (\Omega^{s_{j-1}}Q_{j-1})^{(i_{j-1})}\dots(\Omega^{s_\ell}Q_\ell)^{(i_\ell)},\\ \mathfrak{L}^{\star}(s_\ell,\dots,s_{j-1}) &:= \sum_{i_\ell\geq\cdots\geq i_{j-1}\geq 0} (\Omega^{s_{j-1}}Q_{j-1})^{(i_{j-1})}\dots(\Omega^{s_\ell}Q_\ell)^{(i_\ell)},\end{aligned} \tag{4.5}$$

which are the series in (4.3) and (4.4) attached to $(s_\ell,\dots,s_{j-1})$ and $(Q_\ell,\dots,Q_{j-1})$. We should mention that we omit the subscript $\mathfrak{Q}$ from the definition of the above series to avoid heavy notation.

Lemma 4.1. *For $1\leq\ell\leq j\leq r$, we have*

$$(-1)^\ell\mathfrak{L}^{\star}(s_j,\dots,s_\ell) = \sum_{k=\ell+1}^{j}(-1)^{k-1}\mathfrak{L}(s_\ell,\dots,s_{k-1})\mathfrak{L}^{\star}(s_j,\dots,s_k) + (-1)^j\mathfrak{L}(s_\ell,\dots,s_j).$$

and

$$(-1)^j\mathfrak{L}^{\star}(s_j,\dots,s_\ell) = \sum_{k=\ell+1}^{j}(-1)^{k}\mathfrak{L}(s_k\dots,s_j)\mathfrak{L}^{\star}(s_{k-1},\dots,s_\ell) + (-1)^\ell\mathfrak{L}(s_\ell,\dots,s_j).$$

Proof. The proof follows similarly to the proof of [19, 4.2.1] and is a straightforward exercise in the inclusion/exclusion principal. We leave the details to the reader. □

The matrix given by

$$\Psi = \begin{pmatrix} \Omega^{s_1+\cdots+s_r} & 0 & 0 & \dots & 0\\ \mathfrak{L}(s_1)\Omega^{s_2+\cdots+s_r} & \Omega^{s_2+\cdots+s_r} & 0 & \dots & 0\\ \vdots & \mathfrak{L}(s_2)\Omega^{s_3+\cdots+s_r} & \ddots & & \vdots\\ \vdots & & \ddots & \ddots & \vdots\\ \mathfrak{L}(s_1,\dots,s_{r-1})\Omega^{s_r} & \mathfrak{L}(s_2,\dots,s_{r-1})\Omega^{s_r} & \dots & \Omega^{s_r} & 0\\ \mathfrak{L}(s_1,\dots,s_r) & \mathfrak{L}(s_2,\dots,s_r) & \dots & \mathfrak{L}(s_r) & 1 \end{pmatrix} \in \mathrm{GL}_{r+1}(\mathbb{T})$$

satisfies

$$\Psi^{(-1)} = \Phi\Psi.$$

Thus Ψ is a rigid analytic trivialization associated to the dual t-motive $\mathcal{M}$.

Using Lemma 4.1 we see that the periods of $\mathcal{M}$ are given by the matrix $\Upsilon = \Psi^{-1}$:

$$\Upsilon = \begin{pmatrix} \Omega^{-(s_1+\cdots+s_r)} & 0 & 0 & \dots & 0 \\ -\mathfrak{L}^\star(s_1)\Omega^{-(s_1+\cdots+s_r)} & \Omega^{-(s_2+\cdots+s_r)} & 0 & \dots & 0 \\ \vdots & -\mathfrak{L}^\star(s_2)\Omega^{-(s_2+\cdots+s_r)} & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ (-1)^{r-1}\mathfrak{L}^\star(s_{r-1},\dots,s_1)\Omega^{-(s_1+\cdots+s_r)} & (-1)^{r-2}\mathfrak{L}^\star(s_{r-1},\dots,s_2)\Omega^{-(s_2+\cdots+s_r)} & \dots & \Omega^{-s_r} & 0 \\ (-1)^{r}\mathfrak{L}^\star(s_r,\dots,s_1)\Omega^{-(s_1+\cdots+s_r)} & (-1)^{r-1}\mathfrak{L}^\star(s_r,\dots,s_2)\Omega^{-(s_2+\cdots+s_r)} & \dots & -\mathfrak{L}^\star(s_r)\Omega^{-s_r} & 1 \end{pmatrix}.$$

Note that $\Upsilon \in \mathrm{GL}_{r+1}(\mathbb{T})$.

Lemma 4.2. *For $1 \le \ell \le j \le r$, we have*

$$\mathfrak{L}^\star(s_{j-1},\dots,s_\ell)^{(-1)}$$
$$= \mathfrak{L}^\star(s_{j-1},\dots,s_\ell) + \mathfrak{L}^\star(s_{j-1},\dots,s_{\ell+1})Q_\ell^{(-1)}[(t-\theta)\Omega]^{s_\ell} + \cdots + Q_\ell^{(-1)}\dots Q_{j-1}^{(-1)}[(t-\theta)\Omega]^{(s_\ell+\cdots+s_{j-1})},$$

where $\mathfrak{L}^\star(s_{j-1},\dots,s_\ell) = 1$ if $\ell = j$.

Proof. Since $\Psi^{(-1)} = \Phi\Psi$, we get $\Upsilon^{(-1)} = \Upsilon\Phi^{-1}$. Using the above formulas we deduce the required equality by direct calculations. We leave the details to the reader. □

4.3. **The associated t-modules and special points.** We maintain the notation of the previous section where $\mathcal{M}'$ was defined in terms of $\mathfrak{s}$ and $\mathfrak{Q}$. We now present the associated t-modules and special points inspired by the work of Chang, Papanikolas and Yu [23]. Let $\mathbf{m} = \{m_1,\dots,m_r\}$ be the $\overline{K}[t]$-basis of $\mathcal{M}'$ with respect to the action of σ represented by Φ'. It is not hard to check that $\mathcal{M}'$ is a free left $\overline{K}[\sigma]$-module of rank $d = (s_1+\cdots+s_r)+(s_2+\cdots+s_r)+\cdots+s_r$ and that

(4.6)
$$\mathbf{w} := \{w_1,\dots,w_d\} := \{(t-\theta)^{s_1+\cdots+s_r-1}m_1,\dots,m_1,\dots,(t-\theta)^{s_r-1}m_r,\dots,m_r\}$$

is a $\overline{K}[\sigma]$-basis of $\mathcal{M}'$. We further observe that $(t-\theta)^\ell\mathcal{M}'/\sigma\mathcal{M}' = (0)$ for $\ell \gg 0$.

For such $\mathcal{M}'$, we recall that we can identify $\mathcal{M}'/(\sigma-1)\mathcal{M}'$ with the direct sum of d copies of $\overline{K}$ as follows. Fixing a $\overline{K}[\sigma]$-basis $\mathbf{w} = \{w_1,\dots,w_d\}$ of $\mathcal{M}'$ given as above, we can express any $m \in \mathcal{M}'$ as

$$m = \sum_{i=1}^{d} u_i w_i, \quad u_i \in \overline{K}[\sigma],$$

and then can write down $\delta_1 : \mathcal{M}' \to \mathrm{Mat}_{d\times 1}(\overline{K})$ from §2.4 by

$$\delta_1(m) := (\delta(u_1),\dots,\delta(u_d))^\top = \begin{pmatrix} \delta(u_1) \\ \vdots \\ \delta(u_d) \end{pmatrix}$$

where

$$\delta\left(\sum_i c_i\sigma^i\right) = \sum_i c_i^{q^i}.$$

It follows that δ_1 is a map of $\mathbb{F}_q$-vector spaces with kernel $(\sigma-1)\mathcal{M}'$. We note that if $(b_1,\dots,b_d)^\top \in \mathrm{Mat}_{d\times 1}(\overline{K})$, then there is a natural lift to $\mathcal{M}'$, since

$$\delta_1(b_1w_1+\cdots+b_dw_d) = (b_1,\dots,b_d)^\top.$$

We denote by E' the Anderson t-module defined over $\overline{K}$ with $E'(\overline{K})$ identified with $\mathrm{Mat}_{d\times 1}(\overline{K})$ on which the $\mathbb{F}_q[t]$-module structure given by

$$E' : \mathbb{F}_q[\theta] \to \mathrm{Mat}_d(\overline{K})[\tau]$$

so that

$$\delta_1(t(b_1w_1 + \cdots + b_dw_d)) = E'_\theta(b_1, \ldots, b_d)^\top = E'_\theta \begin{pmatrix} b_1 \\ \vdots \\ b_d \end{pmatrix}.$$

Then E' is the t-module associated the dual t-motive $\mathcal{M}'$ as explained in §2.4.

We can write down explicitly the map $\delta_0 : \mathcal{M}' \to \mathrm{Mat}_{d\times 1}(\overline{K})$. Let $m \in \mathcal{M}' = \overline{K}[t]m_1 + \cdots + \overline{K}[t]m_r$. Then we can write (recall the definition of d_ℓ from Thm. 1.7)

$$m = \sum_{\ell=1}^{r} (c_{d_\ell-1,\ell}(t-\theta)^{d_\ell-1} + \cdots + c_{0,\ell} + F_\ell(t)(t-\theta)^{d_\ell})m_\ell,$$

with $c_{i,\ell} \in \overline{K}$ and $F_\ell(t) \in \overline{K}[t]$. Then

$$\delta_0(m) := (c_{d_1-1,1}, \ldots, c_{0,1}, \ldots, c_{d_r-1,r}, \ldots, c_{0,r})^\top. \tag{4.7}$$

Inspired by Chang-Papanikolas-Yu (see [23, §5.3]) we define the point

$$\mathbf{v}_{\mathfrak{s},\mathfrak{Q}} := \mathbf{v}_{\mathcal{M}} := \delta_1(Q_r^{(-1)}(t-\theta)^{s_r}m_r) \in E'(\overline{K}). \tag{4.8}$$

4.4. **Logarithm series.** The coefficients of the logarithm series can be calculated following [9]. In this particular case, it was also done in [18, §4.2].

We set

$$\Theta = (\Phi^{-1})^\top = \begin{pmatrix} \Theta_{1,1} & \cdots & \Theta_{1,r+1} \\ & \ddots & \vdots \\ & & \Theta_{r+1,r+1} \end{pmatrix} \in \mathrm{Mat}_{r+1}(\overline{K}(t))$$

where for $1 \le i \le j \le r+1$,

$$\Theta_{i,j} = (-1)^{j-i} \frac{\prod_{i\le k<j} Q_k^{(-1)}}{(t-\theta)^{s_j+\cdots+s_r}}.$$

We set

$$\Theta' := ((\Phi')^{-1})^\top = \begin{pmatrix} \Theta_{1,1} & \cdots & \Theta_{1,r} \\ & \ddots & \vdots \\ & & \Theta_{r,r} \end{pmatrix} \in \mathrm{Mat}_r(\overline{K}(t)).$$

If we write

$$\mathrm{Log}_{E'} = \sum_{n\ge 0} P_n\tau^n,$$

then by [9, Proposition 2.2], for $n \ge 0$, the nth coefficient of the logarithm series of E' evaluated at $\mathbf{v} \in \overline{K}^d$ is given by

$$P_n\mathbf{v}^{(n)} = \delta_0(\Theta'^{(1)} \ldots \Theta'^{(n)}\iota^{-1}(\mathbf{v})^{(n)}).$$

4.5. **Log-algebraic identities for t-modules associated to Anderson-Thakur dual t-motives.** In this section we apply Theorem 3.5 to obtain log-algebraic identities for the t-module E' associated to the Anderson-Thakur dual t-motive $\mathcal{M}'$.

Theorem 4.3. *We have a split-logarithmic identity*

$$\mathrm{Log}^{!}_{E'}(\mathbf{v}_{\mathfrak{s},\mathfrak{Q}}) = \delta_0 \begin{pmatrix} (-1)^{r-1}\mathfrak{Q}^{\star}(s_r,\dots,s_1)\Omega^{-(s_1+\cdots+s_r)} \\ (-1)^{r-2}\mathfrak{Q}^{\star}(s_r,\dots,s_2)\Omega^{-(s_2+\cdots+s_r)} \\ \vdots \\ \mathfrak{Q}^{\star}(s_r)\Omega^{-s_r} \end{pmatrix}.$$

Proof. Note that

$$\mathbf{f} = (0,\dots,0,Q_r^{(-1)}(t-\theta)^{s_r}),$$
$$\Psi_{\mathbf{f}} = (\mathfrak{Q}(s_1,\dots,s_r),\mathfrak{Q}(s_2,\dots,s_r),\dots,\mathfrak{Q}(s_r)).$$

So $\Psi_{\mathbf{f}}^{(k)} \to 0$ as $k \to \infty$ by [16, Lemma 5.3.1]. Recall that $\mathbf{v}_{\mathfrak{s},\mathfrak{Q}} = \delta_1(Q_r^{(-1)}(t-\theta)^{s_r}m_r)$, and we need to express $Q_r^{(-1)}(t-\theta)^{s_r}m_r$ in the $\overline{K}[\sigma]$-basis

$$\mathbf{w} = \{(t-\theta)^{s_1+\cdots+s_r-1}m_1,\dots,(t-\theta)m_1,m_1,\dots,(t-\theta)^{s_r-1}m_r,\dots,(t-\theta)m_r,m_r\}$$

of $\mathcal{M}'$ (see (4.6)). By definition we have $\sigma m_1 = (t-\theta)^{s_1+\cdots+s_r}m_1$ and for $1 < \ell \leq r$,

$$\sigma m_\ell = Q_{\ell-1}^{(-1)}(t-\theta)^{s_{\ell-1}+\cdots+s_r}m_{\ell-1} + (t-\theta)^{s_\ell+\cdots+s_r}m_\ell.$$

It follows that

$$\begin{aligned} Q_r^{(-1)}(t-\theta)^{s_r}m_r &= Q_r^{(-1)}(\sigma m_r - Q_{r-1}^{(-1)}(t-\theta)^{s_{r-1}+s_r}m_{r-1}) \\ &= Q_r^{(-1)}\sigma m_r - Q_r^{(-1)}Q_{r-1}^{(-1)}(\sigma m_{r-1} - Q_{r-2}^{(-1)}(t-\theta)^{s_{r-2}+\cdots+s_r}m_{r-2}) \\ &= \dots \\ &= \sum_{\ell=1}^{r}(-1)^{r-\ell}Q_r^{(-1)}\dots Q_\ell^{(-1)}\sigma m_\ell. \end{aligned}$$

Then we write

$$\begin{aligned} Q_r\cdots Q_\ell &= (b_{r,0}+b_{r,1}t+\cdots+b_{r,\beta_r}t^{\beta_r})\dots(b_{\ell,0}+b_{\ell,1}t+\cdots+b_{\ell,\beta_\ell}t^{\beta_\ell}) \\ &= \sum_{(i_\ell,\dots,i_r)\in\prod_{\ell\leq j\leq r}\{0,\dots,\beta_j\}} (b_{r,i_r}\cdot b_{r-1,i_{r-1}}\cdots b_{\ell,i_\ell})t^{i_r+\cdots+i_\ell}. \end{aligned}$$

We then substitute the above expression into the preceding expression for $Q_r^{(-1)}(t-\theta)^{s_r}m_r$ to obtain an expression of the form

$$Q_r^{(-1)}(t-\theta)^{s_r}m_r = \sum_i t^{n_i}\sigma^{\ell_i}\left(\sum_{j=1}^{d}u_{i,j}w_j\right),$$

for triples $(\ell_i, n_i, \mathbf{u}_i = (u_{i,1},\dots,u_{i,d})^\top) \in \mathbb{N}\times\mathbb{Z}^{\geq 0}\times\mathbb{C}_\infty^d$, where i is indexed over some finite set. As the coefficients $b_{i,j}$ are coefficients of the polynomials Q_i, by (4.1), we know that $\|Q_i\| < |\theta|_\infty^{\frac{s_i q}{q-1}}$ for $1 \leq i \leq r$. Therefore

$$|b_{r,i_r}\cdot b_{r-1,i_{r-1}}\cdots b_{\ell,i_\ell}|_\infty < |\theta|_\infty^{\frac{(s_r+\cdots+s_\ell)q}{q-1}}.$$

Then by [18, Lemma 4.2.1] each $\mathbf{u}_i$ is inside the radius of convergence of $\mathrm{Log}_{E'}$. Thus the t-module E', the point $\Psi_{\mathbf{f}}$ and the point $\mathbf{v}_{\mathfrak{s},\mathfrak{Q}}$ satisfy the conditions of

Theorem 3.5 (b), which we apply. The final observation is that by the above calculations, $Q_r^{(-1)}(t-\theta)^{s_r}m_r \in \sigma(\mathcal{M}')$ and hence $\delta_0(\mathbf{f}^\top)=0$, which allows us to apply the last statement of Theorem 3.5 and finishes the proof. □

4.6. **Relations with a theorem of Chang-Papanikolas-Yu.** We now apply Theorem 4.3 to obtain another proof of [23, Theorem 2.5.2 (a) ($\Leftarrow$) and (b)] in our setting.

Proposition 4.4. *Suppose that $\mathcal{M}$ represents a torsion class in $Ext^1_{\mathcal{F}}(\mathbf{1},\mathcal{M}')$. Then all the values $\mathfrak{L}(s_1,\dots,s_r)(\theta),\dots,\mathfrak{L}(s_r)(\theta)$ are in K.*

Proof. By Remark 3.2, $\mathbf{v}_{\mathfrak{s},\mathfrak{Q}}$ is a torsion point in $E'(\overline{K})$ since $\mathcal{M}$ represents a torsion class in $\mathrm{Ext}^1_{\mathcal{F}}(\mathbf{1},\mathcal{M}')$. It follows that $d[a]\log^!_{E'}(\mathbf{v}_{\mathfrak{s},\mathfrak{Q}})$ is a period of E' for some nonzero $a\in A$. Thus we can write

$$\log^!_{E'}(\mathbf{v}_{\mathfrak{s},\mathfrak{Q}}) = d[a_1]\lambda_1 + d[a_2]\lambda_2 + \cdots + d[a_r]\lambda_r, \quad a_i\in K, \tag{4.9}$$

where λ_i are the A-basis of the period lattice $\Lambda_{E'}$ given by the map δ_0 applied to the column vectors of $\Upsilon'^\top$ (see [32, Cor. 2.5.24] and also [30, Lemma 3.7]).

For $1\le \ell\le r$, we set $d_\ell := s_\ell+\cdots+s_r$ and $d:=d_1+\cdots+d_r$. For $1\le\ell\le r$, we consider the $(d_1+\cdots+d_\ell)$th coordinate of both sides in (4.9). Then

(1) By Theorem 4.3 and (4.2), the $(d_1+\cdots+d_\ell)$th coordinate of $\log^!_{E'}(\mathbf{v}_{\mathfrak{s},\mathfrak{Q}})$ equals $(-1)^{r-\ell}\mathfrak{L}^\star(s_r,\dots,s_\ell)(\theta)\cdot\widetilde{\pi}^{s_\ell+\cdots+s_r}$.
(2) The $(d_1+\cdots+d_\ell)$th coordinate of λ_ℓ equals $\widetilde{\pi}^{s_\ell+\cdots+s_r}$.
(3) The matrices $d[a_i]$ are upper triangular and equal a_i along the main diagonal. This can be seen quickly from the definition of the $\overline{K}[\sigma]$-basis (4.6) and from the definition of δ_0 in (4.7).
(4) For $1\le j<\ell$, the $(d_1+\cdots+d_\ell)$th coordinate of λ_j is zero and for $\ell\le j\le r$ it equals $(-1)^{j-\ell}\mathfrak{L}^\star(s_{j-1},\dots,s_\ell)(\theta)\cdot\widetilde{\pi}^{s_\ell+\cdots+s_r}$. Thus the $(d_1+\cdots+d_\ell)$th coordinate of $\sum_{j=1}^r d[a_j]\lambda_j$ is $\sum_{j=\ell}^r(-1)^{j-\ell}a_j\mathfrak{L}^\star(s_{j-1},\dots,s_\ell)(\theta)\cdot\widetilde{\pi}^{s_\ell+\cdots+s_r}$.
(5) Thus $(-1)^{r-\ell}\mathfrak{L}^\star(s_r,\dots,s_\ell)(\theta)=\sum_{j=\ell}^r(-1)^{j-\ell}a_j\mathfrak{L}^\star(s_{j-1},\dots,s_\ell)(\theta)$.

Thus, by applying the above equality together with Lemma 4.1 we get

$$\begin{aligned}\mathfrak{L}(s_\ell,\dots,s_r)(\theta) &= \sum_{k=\ell}^r(-1)^{r-k}\mathfrak{L}(s_\ell,\dots,s_{k-1})(\theta)\mathfrak{L}^\star(s_r,\dots,s_k)(\theta)\\ &= \sum_{k=\ell}^r\sum_{j=k}^r(-1)^{j-k}a_j\mathfrak{L}(s_\ell,\dots,s_{k-1})(\theta)\mathfrak{L}^\star(s_{j-1},\dots,s_k)(\theta)\\ &= a_\ell.\end{aligned}$$

Since $a_\ell\in K$ for all $1\le\ell\le r$, we deduce that all the values

$$\mathfrak{L}(s_1,\dots,s_r)(\theta),\mathfrak{L}(s_2,\dots,s_r)(\theta),\dots,\mathfrak{L}(s_r)(\theta)$$

are in K. □

Remark 4.5. We explain briefly how to extend the above result to the more general setting considered in [23] and leave the interested reader to fill in the details.

We put $w:=\sum_{i=1}^r s_i$ and let $Q\in\overline{K}[t]$ such that $\|Q\|<|\theta|_\infty^{wq/(q-1)}$. We consider the effective dual t-motive $\mathcal{N}\in\mathrm{Ext}^1_{\mathcal{F}}(\mathbf{1},\mathcal{M}')$ defined by the matrix

$$\begin{pmatrix}\Phi' & 0\\ \mathbf{u}_{w,Q} & 1\end{pmatrix}\in\mathrm{Mat}_{r+1}(\overline{K}[t]),$$

with $\mathbf{u}_{w,Q} = (Q^{(1)}(t-\theta)^w, 0, \dots, 0) \in \mathrm{Mat}_{1\times r}(\overline{K}[t])$. Note that $\mathcal{N}$ admits a rigid analytic trivialization given by

$$\begin{pmatrix} \Psi' & 0 \\ (\mathfrak{L}_{w,Q}, 0, \dots, 0) & 1 \end{pmatrix} \in \mathrm{Mat}_{r+1}(\overline{K}[t]),$$

where $\mathfrak{L}_{w,Q}$ is the series in (4.3) attached to (w) and (Q).

We apply our method to obtain log-algebraic identities for the t-module attached to $\mathcal{N}$. Consequently, we get [23, Theorem 2.5.2 (a) ($\Leftarrow$) and (b)] which states that if the classes of $\mathcal{M}$ and $\mathcal{N}$ are $\mathbb{F}_q[t]$-linearly dependent in $\mathrm{Ext}^1_{\mathcal{F}}(\mathbf{1}, \mathcal{M}')$, then all the values $\mathfrak{L}(s_2, \dots, s_r)(\theta), \dots, \mathfrak{L}(s_r)(\theta)$ are in K.

Remark 4.6. We should mention that by using the powerful ABP criterion [4] and also [35], the converse was also proved in [23, Theorem 2.5.2(a) ($\Rightarrow$)] under the mild conditions that the values $\mathfrak{L}(s_\ell, \dots, s_{j-1})(\theta)$ do not vanish for $1 \le \ell < j \le r+1$.

4.7. **A generalization of a theorem of Chang.** In the fundamental work [5] Anderson and Thakur gave logarithmic interpretations for Carlitz zeta values, i.e., depth one multiple zeta values. In [17] Chang presented very simple and elegant logarithmic interpretations for some special MZV's (see [17, Theorem 4.1.1]) and deduced an effective criterion for the dimension of depth-two multiple zeta values. However, as Chang and Mishiba [21] explained to us, to their knowledge, the relations among Chang's theorem and the works of Chang-Papanikolas-Yu [23] and Chang-Mishiba [19, 20] are still mysterious.

The aim of this section is to present a generalization of Chang's theorem as an application of our main result (see Theorem 4.7). As a consequence, we clarify the connection between the work of Chang [17] and that of Chang-Papanikolas-Yu [23]. We close this section by deducing an unusual formula of Thakur from Chang's theorem (see Remark 4.9).

Theorem 4.7. *Let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ with $r \ge 2$. Assume that, for $1 \le \ell < j \le r+1$, the values $\mathfrak{L}(s_\ell, \dots, s_{j-1})(\theta)$ do not vanish. We further suppose that $\mathfrak{L}(s_2, \dots, s_r)(\theta) \in K$. Then there exist nonzero $a_\mathfrak{s} \in A$, an algebraic point $\mathbf{Z}_\mathfrak{s} \in C^{\otimes(s_1+\cdots+s_r)}(\overline{K})$ (the $s_1+\cdots+s_r$-tensor power of the Carlitz module) and a point $\mathbf{z}_\mathfrak{s} \in \mathbb{C}_\infty^{s_1+\cdots+s_r}$ such that*

1) the last coordinate of $\mathbf{z}_\mathfrak{s}$ equals $a_\mathfrak{s}\mathfrak{L}(s_1, \dots, s_r)(\theta)\widetilde{\pi}^{s_1+\cdots+s_r}$,

2) $\mathrm{Exp}_{C^{\otimes(s_1+\cdots+s_r)}}(\mathbf{z}_\mathfrak{s}) = \mathbf{Z}_\mathfrak{s}$.

Proof. Since the values $\mathfrak{L}(s_\ell, \dots, s_{j-1})(\theta)$ do not vanish for $1 \le \ell < j \le r+1$, the hypothesis of [23, Theorem 2.5.2] holds. Thus this theorem implies that $\mathfrak{L}(s_3, \dots, s_r)(\theta), \dots, \mathfrak{L}(s_r)(\theta)$ are also in K since $\mathfrak{L}(s_2, \dots, s_r)(\theta) \in K$.

For $2 \le \ell \le r$, we set

$$a_\ell = \mathfrak{L}(s_\ell, \dots, s_r)(\theta) \in K.$$

We take $a_\mathfrak{s} \in A$ such that $a_\mathfrak{s} a_\ell \in A$ for all $2 \le \ell \le r$.

We denote by $\lambda_1, \dots, \lambda_r$ the A-basis of the period lattice $\Lambda_{E'}$ given by the map δ_0 applied to the column vectors of $\Upsilon'^{\top}$ (see [32, Cor. 2.5.24] and also [30, Lemma 3.7]).

For $1 \le \ell \le r$, we consider the $(d_1 + \cdots + d_\ell)$th coordinate of $\mathrm{Log}^!_{E'}(\mathbf{v}_{\mathfrak{s},\mathfrak{Q}})$ and $\lambda_1, \dots, \lambda_r$. Then

(1) By Theorem 4.3 and (4.2), the $(d_1 + \cdots + d_\ell)$th coordinate of $\mathrm{Log}^!_{E'}(\mathbf{v}_{\mathfrak{s},\mathfrak{Q}})$ equals $(-1)^{r-\ell}\mathfrak{L}^\star(s_r, \dots, s_\ell)(\theta) \cdot \widetilde{\pi}^{s_\ell+\cdots+s_r}$.

(2) For $1 \leq j < \ell$, the $(d_1+\cdots+d_\ell)$th coordinate of λ_j is zero and for $\ell \leq j \leq r$ it equals $(-1)^{j-\ell}\mathfrak{L}^\star(s_{j-1},\dots,s_\ell)(\theta)\cdot\widetilde{\pi}^{s_\ell+\cdots+s_r}$.
(3) The $(d_1+\cdots+d_\ell)$th coordinate of λ_ℓ equals $\widetilde{\pi}^{s_\ell+\cdots+s_r}$.
(4) The matrices $d[a_i]$ are upper triangular and equal a_i along the main diagonal. This can be seen quickly from the definition of the $\overline{K}[\sigma]$-basis (4.6) and from the definition of δ_0 in (4.7).

We consider

$$\mathbf{z}'_{\mathfrak{s}} = d[a_{\mathfrak{s}}]\,\mathrm{Log}^!_{E'}(\mathbf{v}_{\mathfrak{s},\mathfrak{Q}}) - d[a_{\mathfrak{s}}a_2]\lambda_2 - \cdots - d[a_{\mathfrak{s}}a_r]\lambda_r.$$

Then we deduce

(1) The d_1th coordinate of $\mathbf{z}'_{\mathfrak{s}}$ equals $a_{\mathfrak{s}}\mathfrak{L}(s_1,\dots,s_r)(\theta)\widetilde{\pi}^{s_1+\cdots+s_r}$ by Lemma 4.1.
(2) For $d_1 < j \leq d_1+\cdots+d_r$, the jth coordinate of $\mathbf{z}'_{\mathfrak{s}}$ is zero by Lemma 4.1 (see also Proposition 4.4, Remark 4.6 and [18, Lemma 3.4.5]).

Thus to conclude it suffices to choose $\mathbf{z}_{\mathfrak{s}}$ to be the first d_1 coordinates of $\mathbf{z}'_{\mathfrak{s}}$. This finishes the proof. □

Remark 4.8. 1) The proof presented above grew out of many discussions of the second author and F. Pellarin to whom he would like to express his gratitude.

2) Chang [21] informed us that Y.-T. Chen and R. Harada are working on generalizing Chang's result to the case where $\mathfrak{Q} = (u_1,\dots,u_r) \in \overline{K}^r$ satisfying $|u_r|_\infty < q^{\frac{sq}{q-1}}$ and $|u_i|_\infty \leq q^{\frac{s_iq}{q-1}}$ for $1 \leq i \leq r-1$.

Remark 4.9. If we write the Carlitz logarithm attached to the Carlitz module C as

$$\log_C = \sum_{i\geq 0}\frac{1}{\ell_i}\tau^i, \quad \ell_i \in A,$$

then in [42, Theorem 6] Thakur gave the following "strange" formula

$$\zeta_A(1,q^3-1) = \left(\frac{1}{\ell_3}+\frac{1}{\ell_2}+\frac{\theta}{\ell_2}\right)\zeta_A(q^3) - \frac{1}{\ell_2}\left(\log_C(\theta^{1/q})\right)^{q^3}. \tag{4.10}$$

We claim that this identity can be seen as an explicit example of the above Theorem. In fact, we put $\mathfrak{s} = (1, q^3-1)$ and consider the tensor power $C^{\otimes q^3}$. We know that, by [5], the last row of the logarithm associated to $C^{\otimes q^3}$ denoted by $\iota^{-1}(\mathrm{Log}_{C^{\otimes q^3}})$ is given by

$$\iota^{-1}(\mathrm{Log}_{C^{\otimes q^3}}(0,\dots,0,x)^\top) = \sum_{i\geq 0}\frac{1}{\ell_i^{q^3}}\tau^i(x).$$

Thus

$$\left(\log_C(\theta^{1/q})\right)^{q^3} = \sum_{i\geq 0}\frac{1}{\ell_i^{q^3}}\tau^{i+3}(\theta^{1/q}) = \sum_{i\geq 0}\frac{1}{\ell_i^{q^3}}\tau^i(\theta^{q^2}) = \iota^{-1}(\mathrm{Log}_{C^{\otimes q^3}}(0,\dots,0,\theta^{q^2})^\top).$$

The celebrated Anderson-Thakur theorem [5, Theorem 3.8.3] shows that $\zeta_A(q^3)$ can be interpreted as the last coordinate of $\mathrm{Log}_{C^{\otimes q^3}}$. We conclude that (4.10) gives an explicit interpretation for the MZV $\zeta_A(1,q^3-1)$ as the last coordinate of a split-logarithmic identity involving $\mathrm{Log}_{C^{\otimes q^3}}$ as is implied by Chang's theorem.

F. Pellarin has informed us that, in an ongoing project with O. Gezmis, they construct more examples of such explicit identities for MZV's.

4.8. **Log-algebraic identities for Chang-Papanikolas-Yu's t-modules.** In this section we specialize $\mathfrak{Q} = (Q_1, \dots, Q_r)$ to Anderson-Thakur polynomials and study the corresponding t-modules considered in the work of Chang, Papanikolas and Yu [23] (see also [6]). Then we apply Theorem 3.5 to obtain several applications to this case.

These dual t-motives are related to the multiple zeta values defined by Thakur [41] as follows. For any tuple of positive integers $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$, we introduce

$$\zeta_A(\mathfrak{s}) = \zeta_A(s_1, \dots, s_r) := \sum \frac{1}{a_1^{s_1} \cdots a_r^{s_r}} \in K_\infty$$

where the sum runs through the set of tuples $(a_1, \dots, a_r) \in A_+^r$ with $\deg a_1 > \cdots > \deg a_r$; r is called the depth and $w := s_1 + \cdots + s_r$ the weight of $\zeta_A(\mathfrak{s})$. depth one MZV's are also called Carlitz zeta values (see [15]). It is proved that $\zeta_A(\mathfrak{s})$ are nonzero by Thakur [42]. We refer the reader to the excellent surveys [43, 44] for more details about MZV's.

We briefly review Anderson-Thakur polynomials introduced in [5]. For $k \geq 0$, we set

$$[k] := \theta^{q^k} - \theta,$$

$$D_k := \prod_{\ell=1}^{k} [\ell]^{q^{k-\ell}} = [k][k-1]^q \cdots [1]^{q^{k-1}}.$$

For $n \in \mathbb{N}$, we write

$$n - 1 = \sum_{j \geq 0} n_j q^j, \quad 0 \leq n_j \leq q - 1,$$

and define

$$\Gamma_n := \prod_{j \geq 0} D_j^{n_j}.$$

We set

$$\gamma_0(t) := 1,$$

$$\gamma_j(t) := \prod_{\ell=1}^{j} (\theta^{q^j} - t^{q^\ell}), \quad j \geq 1.$$

Then Anderson-Thakur polynomials $\alpha_n(t) \in A[t]$ are given by the generating series

$$\sum_{n \geq 1} \frac{\alpha_n(t)}{\Gamma_n} x^n := x \left(1 - \sum_{j \geq 0} \frac{\gamma_j(t)}{D_j} x^{q^j} \right)^{-1}.$$

Finally, we define $H_n(t)$ by switching θ and t (notice that in this definition, the index is shifted by one from the original definition):

$$H_n(t) = \alpha_n(t)\big|_{t=\theta, \theta=t}.$$

By [5, 3.7.3] we get that $\|H_n\| < |\theta|_\infty^{\frac{nq}{q-1}}$. Thus the polynomials $(Q_1, \dots, Q_r) = (H_{s_1}, \dots, H_{s_r})$ satisfy (4.1).

In what follows, we will specialize the t-motives $\mathcal{M}$ and $\mathcal{M}'$ from the previous sections to $(Q_1, \dots, Q_r) = (H_{s_1}, \dots, H_{s_r})$ and get logarithmic interpretations for multiple zeta star values.

We wish to study the point $\mathbf{v}_{\mathfrak{s}} \in E'(\overline{K})$ which corresponds to $H_{s_r}^{(-1)}(t-\theta)^{s_r} m_r \in \mathcal{M}'/(\sigma-1)\mathcal{M}'$. Note that this point was first introduced by Chang, Papanikolas and Yu in [23] and played an important role in their effective criterion to determine whether the corresponding multiple zeta value $\zeta_A(\mathfrak{s})$ is Eulerian. Further, they proved the following integrality result:

Theorem 4.10 ([23], Theorem 5.3.4)**.** *1) The t-module E' is defined over A.*
2) The point $\mathbf{v}_{\mathfrak{s}}$ is an integral point in $E'(A)$.

The following examples were given in [23, §6.1.2]. We refer the reader there for more examples.

Example 4.11. We consider $q=3$ and $\mathfrak{s}=(s_1=2,s_2=4)$. Then

$$E'_\theta = \left(\begin{array}{cccccc|cccc} \theta & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \theta & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \theta & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \theta & 1 & 0 & 0 & 0 & 0 \\ \tau & 0 & 0 & 0 & 0 & \theta & -\tau & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \theta & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \tau & 0 & 0 & \theta \end{array}\right)$$

and

$$\mathbf{v}_{\mathfrak{s}} = (0,0,1,0,1,(\theta+2\theta^3),2,0,2,(2\theta+\theta^3))^\top.$$

Example 4.12. We consider $q=3$ and $\mathfrak{s}=(s_1=4,s_2=2)$. Then

$$E'_\theta = \left(\begin{array}{cccccc|cc} \theta & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \theta & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \theta & 1 & 0 & 0 & \tau & 0 \\ 0 & 0 & 0 & \theta & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \theta & 1 & \tau & 0 \\ \tau & 0 & 0 & 0 & 0 & \theta & (\theta+2\theta^3)\tau & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \theta & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \tau & \theta \end{array}\right)$$

and

$$\mathbf{v}_{\mathfrak{s}} = (0,0,1,0,1,(\theta+2\theta^3),0,1)^\top.$$

For $1 \le \ell < j$, we have defined the series

$$\mathfrak{L}(s_\ell,\dots,s_{j-1}) := \sum_{i_\ell > \cdots > i_{j-1} \ge 0} (\Omega^{s_{j-1}} H_{s_{j-1}})^{(i_{j-1})} \dots (\Omega^{s_\ell} H_{s_\ell})^{(i_\ell)},$$

$$\mathfrak{L}^\star(s_\ell,\dots,s_{j-1}) := \sum_{i_\ell \ge \cdots \ge i_{j-1} \ge 0} (\Omega^{s_{j-1}} H_{s_{j-1}})^{(i_{j-1})} \dots (\Omega^{s_\ell} H_{s_\ell})^{(i_\ell)}.$$

By [16, 5.5.3] we have

$$\left[\mathfrak{L}(s_\ell,\dots,s_{j-1})\Omega^{-(s_\ell+\cdots+s_{j-1})}\right](\theta) = \Gamma_{s_\ell}\dots\Gamma_{s_{j-1}}\zeta_A(s_\ell,\dots,s_{j-1}). \tag{4.11}$$

We define the multiple zeta star values by

$$\zeta_A^\star(s_1,\dots,s_r) := \sum \frac{1}{a_1^{s_1}\dots a_r^{s_r}} \in K_\infty$$

where the sum runs through the set of tuples $(a_1, \dots, a_r) \in A_+^r$ with $\deg a_1 \geq \cdots \geq \deg a_r$. Note that by [6, Eq. (1)] we have

$$\Gamma_{s_\ell} \dots \Gamma_{s_{j-1}} \zeta_A^\star(s_\ell, \dots, s_{j-1}) = \left[\mathfrak{L}^\star(s_\ell, \dots, s_{j-1}) \Omega^{-(s_\ell + \cdots + s_{j-1})} \right](\theta).$$

We observe that these quantities can be completely determined by the relations

$$\zeta_A^\star(s_r, \dots, s_1) = \sum_{\ell=2}^{r} (-1)^\ell \zeta_A(s_1, \dots, s_{\ell-1}) \zeta_A^\star(s_r, \dots, s_\ell) + (-1)^{r-1} \zeta_A(s_1, \dots, s_r).$$

We apply Theorem 4.3 to this situation and obtain

Theorem 4.13. *Recall that for $1 \leq \ell \leq r$, we put $d_\ell = s_\ell + \cdots + s_r$. Then we have*

$$\mathrm{Log}_{E'}^!(\mathbf{v}_\mathfrak{s}) = \delta_0 \begin{pmatrix} (-1)^{r-1} \mathfrak{L}^\star(s_r, \dots, s_1) \Omega^{-(s_1 + \cdots + s_r)} \\ (-1)^{r-2} \mathfrak{L}^\star(s_r, \dots, s_2) \Omega^{-(s_2 + \cdots + s_r)} \\ \vdots \\ \mathfrak{L}^\star(s_r) \Omega^{-s_r} \end{pmatrix}.$$

In particular, for $1 \leq \ell \leq r$, the $(d_1 + \cdots + d_\ell)$th coordinate of $\mathrm{Log}_{E'}^!(\mathbf{v}_\mathfrak{s})$ equals $(-1)^{r-\ell} \Gamma_{s_\ell} \dots \Gamma_{s_r} \zeta_A^\star(s_r \dots, s_\ell)$.

4.9. **Log-algebraic identities for t-modules connected to multiple polylogarithms at algebraic points.** For $\mathbf{u} = (u_1, \dots, u_r) \in \overline{K}^r$ satisfying $|u_i|_\infty < q^{\frac{s_i q}{q-1}}$ for $1 \leq i \leq r$, Chang and Mishiba specialize the dual t-motives $\mathcal{M}$ and $\mathcal{M}'$ from the previous section to $(Q_1, \dots, Q_r) = (u_1, \dots, u_r)$ and thus get logarithmic interpretations for Carlitz star multiple polylogarithms (see [20, Theorem 4.2.3] and also [18, Theorem 3.3.5]). They then use these polylogarithmic interpretations to get a logarithmic interpretation for MZV's; we will present a more direct way to recover MZV's using our techniques in §5. In this section we show how our techniques recover Chang and Mishiba's result [20, Theorem 4.2.3] (which only gives a certain coordinate of the logarithm) and that they also include the extra information given in Chang, Mishiba and the first author's result [18, Theorem 3.3.7] (which gives all the coordinates of the logarithm).

We now define Carlitz (star) multiple polylogarithms, following as in [20, §3.1]. For any index $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ we define the series

$$\mathrm{Li}_\mathfrak{s}(z_1, \dots, z_r) := \sum_{i_1 > \cdots > i_r \geq 0} \frac{z_1^{(i_1)} \dots z_r^{(i_r)}}{L_{i_1}^{s_1} \dots L_{i_r}^{s_r}} \in \mathbb{C}_\infty[[z_1, \dots, z_r]],$$

$$\mathrm{Li}_\mathfrak{s}^\star(z_1, \dots, z_r) := \sum_{i_1 \geq \cdots \geq i_r \geq 0} \frac{z_1^{(i_1)} \dots z_r^{(i_r)}}{L_{i_1}^{s_1} \dots L_{i_r}^{s_r}} \in \mathbb{C}_\infty[[z_1, \dots, z_r]],$$

where $L_0 := 1$ and $L_i := (\theta - \theta^q) \cdots (\theta - \theta^{q^i})$ for $i \in \mathbb{N}$. The following formula is shown in [19, Lemma 4.2.1]:

$$\begin{aligned} &\mathrm{Li}_{(s_r, \dots, s_1)}^\star(z_r, \dots, z_1) \\ &= \sum_{\ell=2}^{r} (-1)^\ell \mathrm{Li}_{(s_1, \dots, s_{\ell-1})}(z_1, \dots, z_{\ell-1}) \mathrm{Li}_{(s_r, \dots, s_\ell)}^\star(z_r, \dots, z_\ell) + (-1)^{r+1} \mathrm{Li}_{(s_1, \dots, s_r)}(z_1, \dots, z_r). \end{aligned}$$

In particular, for $r=2$, we obtain

$$\begin{aligned}\mathrm{Li}^\star_{(s_2,s_1)}(z_2,z_1) =&\mathrm{Li}_{s_1}(z_1)\mathrm{Li}^\star_{s_2}(z_2)-\mathrm{Li}_{(s_1,s_2)}(z_1,z_2)\\ =&\mathrm{Li}_{s_1}(z_1)\mathrm{Li}_{s_2}(z_2)-\mathrm{Li}_{(s_1,s_2)}(z_1,z_2).\end{aligned}$$

We then define t-deformed versions of L_i as

$$\mathbb{L}_0:=1 \text{ and } \mathbb{L}_i:=(t-\theta^q)\cdots(t-\theta^{q^i}) \text{ for } i\in\mathbb{N}. \tag{4.12}$$

We also define t-deformations of the Li and $\mathrm{Li}^\star$ series as is done in [18, §3.1] by setting

$$\mathfrak{Li}_{\mathfrak{s}}(t;z_1,\dots,z_r):=\mathfrak{Li}_{\mathfrak{s}}(z_1,\dots,z_r):=\sum_{i_1>\cdots>i_r\geq 0}\frac{z_1^{q^{i_1}}\cdots z_r^{q^{i_r}}}{\mathbb{L}_{i_1}^{s_1}\cdots\mathbb{L}_{i_r}^{s_r}}\in\mathbb{C}_\infty[[t,z_1,\dots,z_r]],$$

$$\mathfrak{Li}^\star_{\mathfrak{s}}(t;z_1,\dots,z_r):=\mathfrak{Li}^\star_{\mathfrak{s}}(z_1,\dots,z_r):=\sum_{i_1\geq\cdots\geq i_r\geq 0}\frac{z_1^{q^{i_1}}\cdots z_r^{q^{i_r}}}{\mathbb{L}_{i_1}^{s_1}\cdots\mathbb{L}_{i_r}^{s_r}}\in\mathbb{C}_\infty[[t,z_1,\dots,z_r]].$$

Observe that if we set $\mathfrak{Q}=(Q_1,\dots,Q_r)$ from §4.2 to be equal $\mathbf{u}=(u_1,\dots,u_r)\in\overline{K}^n$, then we have the equalities

$$\mathfrak{L}(s_1,\dots,s_r)=\Omega^{s_1+\cdots+s_r}\,\mathfrak{Li}_{\mathfrak{s}}(t;u_1,\dots,u_r),$$

$$\mathfrak{L}^\star(s_1,\dots,s_r)=\Omega^{s_1+\cdots+s_r}\mathfrak{Li}^\star_{\mathfrak{s}}(t;u_1,\dots,u_r).$$

We set $\Phi_{\mathbf{u}}$ equal to Φ from §4.2 with $(Q_1,\dots,Q_r)=(u_1,\dots,u_r)$, and similarly for $\mathcal{M}_{\mathbf{u}}$ and $\mathcal{M}'_{\mathbf{u}}$. Then, using the above equations we quickly deduce that the rigid analytic trivialization given by

$$\Psi_{\mathbf{u}}=\begin{pmatrix}\Omega^{s_1+\cdots+s_r} & 0 & 0 & \dots & 0\\ \mathfrak{Li}_{s_1}(u_1)\Omega^{s_1+\cdots+s_r} & \Omega^{s_2+\cdots+s_r} & 0 & \dots & 0\\ \vdots & \mathfrak{Li}_{s_2}(u_2)\Omega^{s_2+\cdots+s_r} & \ddots & & \vdots\\ \vdots & & \ddots & \ddots & \vdots\\ \mathfrak{Li}_{(s_1,\dots,s_{r-1})}(u_1,\dots,u_{r-1})\Omega^{s_1+\cdots+s_r} & \mathfrak{Li}_{(s_2,\dots,s_{r-1})}(u_2,\dots,u_{r-1})\Omega^{s_2+\cdots+s_r} & \dots & \Omega^{s_r} & 0\\ \mathfrak{Li}_{(s_1,\dots,s_r)}(u_1,\dots,u_r)\Omega^{s_1+\cdots+s_r} & \mathfrak{Li}_{(s_2,\dots,s_r)}(u_2,\dots,u_r)\Omega^{s_2+\cdots+s_r} & \dots & \mathfrak{Li}_{s_r}(u_r)\Omega^{s_r} & 1\end{pmatrix}.$$

satisfies $\Psi_{\mathbf{u}}\in\mathrm{GL}_{r+1}(\mathbb{T})$ and

$$\Psi_{\mathbf{u}}^{(-1)}=\Phi_{\mathbf{u}}\Psi_{\mathbf{u}}.$$

The periods of $\mathcal{M}_{\mathbf{u}}$ are given by the matrix $\Upsilon_{\mathbf{u}}=\Psi_{\mathbf{u}}^{-1}$:

$$\Upsilon_{\mathbf{u}}=\begin{pmatrix}\Omega^{-(s_1+\cdots+s_r)} & 0 & \dots & 0\\ -\mathfrak{Li}^\star_{s_1}(u_1)\Omega^{-(s_2+\cdots+s_r)} & \Omega^{-(s_2+\cdots+s_r)} & \dots & 0\\ \vdots & & \ddots & \vdots\\ (-1)^r\,\mathfrak{Li}^\star_{(s_r,\dots,s_1)}(u_r,\dots,u_1) & \dots & -\mathfrak{Li}^\star_{s_r}(u_r) & 1\end{pmatrix}.$$

Note that $\Upsilon_{\mathbf{u}}\in\mathrm{GL}_{r+1}(\mathbb{T})$ and

$$\mathbf{f}_{\mathbf{u}}=(0,\dots,0,u_r^{(-1)}(t-\theta)^{s_r}),$$

$$(\Psi_{\mathbf{u}})_{\mathbf{f}_{\mathbf{u}}}=(\mathfrak{Li}_{(s_1,\dots,s_r)}(u_1,\dots,u_r)\Omega^{s_1+\cdots+s_r},\mathfrak{Li}_{(s_2,\dots,s_r)}(u_2,\dots,u_r)\Omega^{s_2+\cdots+s_r},\dots,\mathfrak{Li}_{s_r}(u_r)\Omega^{s_r}).$$

In particular, they verify the hypothesis of Theorem 3.5 (a) by [16, Lemma 5.3.1 and Theorem 5.5.2].

We can define the point $\mathbf{v}_{\mathbf{u}} \in E'_{\mathbf{u}}(\overline{K})$ as before. By the same calculations given in the proof of Theorem 4.3 we see that it coincides with the point given in [20, Equation (4.1.6)] (see also [19, Equation (3.3.1)]):

$$\mathbf{v}_{\mathbf{u}} = (0, \dots, 0, (-1)^{r-1}(u_r \dots u_1), \dots, (-1)^{r-2}(u_r \dots u_2), \dots, u_r)^{\top}.$$

Here for $1 \leq \ell \leq r$, the $(d_1 + \dots + d_\ell)$th coordinate of $\mathbf{v}_{\mathbf{u}}$ equals $(-1)^{r-\ell} u_r \dots u_\ell$ and the other coordinates of $\mathbf{v}_{\mathbf{u}}$ vanish. Applying Theorem 3.5 in this situation gives a refinement of [20, Theorem 4.2.3], (see also [19]), and it recovers [18, Theorem 3.3.5].

Theorem 4.14. *Recall that for $1 \leq \ell \leq r$, we put $d_\ell = s_\ell + \dots + s_r$. Then we have*

$$\mathrm{Log}_{E'_{\mathbf{u}}}(\mathbf{v}_{\mathbf{u}}) = \delta_0 \begin{pmatrix} (-1)^{r-1}\, \mathfrak{Li}^{\star}_{(s_r,\dots,s_1)}(u_r, \dots, u_1) \\ (-1)^{r-2}\, \mathfrak{Li}^{\star}_{(s_r,\dots,s_2)}(u_r, \dots, u_2) \\ \vdots \\ \mathfrak{Li}^{\star}_{s_r}(u_r) \end{pmatrix}.$$

In particular, for $1 \leq \ell \leq r$, the $(d_1 + \dots + d_\ell)$th coordinate of the $\mathrm{Log}_{E'_{\mathbf{u}}}(\mathbf{v}_{\mathbf{u}})$ equals $(-1)^{r-\ell}\, \mathrm{Li}^{\star}_{(s_r,\dots,s_\ell)}(u_r, \dots, u_\ell)$.

Proof. From the explicit formula for the point $\mathbf{v}_{\mathbf{u}}$ we see that it lies in the domain of convergence of $\mathrm{Log}_{E'_{\mathbf{u}}}$. Hence the Stark logarithmic identity is indeed an actual logarithmic identity, as follows by Remark 2.4. □

5. Star dual t-motives and application to MZV's

We see in §4 that the Anderson-Thakur dual t-motive does not directly give a logarithmic interpretation for MZV's. In [18, 20] Chang, Green and Mishiba found a solution for this problem. Their method consisted of two steps. First, they find a logarithmic interpretation for Carlitz star multiple polylogarithms (see Theorem 4.14, also [19]), then they form a linear combination of these polylogarithms which results in a MZV using the theory of fiber coproducts of t-motives (see [18, 20]). They raised the question whether one could find a more direct way to obtain a logarithmic interpretation for MZV's (see [20, §1.4]).

In this section we give an affirmative answer to the above question of Chang and Mishiba and propose another logarithmic interpretation for MZV's which is much more direct. The key point is to introduce a new dual t-motive called the star dual t-motive so that MZV's are "directly connected" to the associated t-module.

5.1. Star dual t-motives and periods.

We always work with a tuple $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ for $r \geq 1$. In what follows, we will specialize to $\mathfrak{Q} = (Q_1, \dots, Q_r) = (H_{s_1}, \dots, H_{s_r})$ and keep the notation of §4.8.

Remark 5.1. We mention that all the results of this section still hold for any $\mathfrak{Q} = (Q_1, \dots, Q_r) \in \overline{K}[t]^r$ satisfying the condition (4.1). The proofs can be adapted without modification.

We set

$$\Phi^{\star} := \begin{pmatrix} \Phi^{\star}_{1,1} & & \\ \vdots & \ddots & \\ \Phi^{\star}_{r+1,1} & \cdots & \Phi^{\star}_{r+1,r+1} \end{pmatrix} \in \mathrm{Mat}_{r+1}(\overline{K}[t])$$

where for $1 \le \ell \le j \le r+1$,

$$\Phi^{\star}_{j,\ell} = (-1)^{j-\ell} \prod_{\ell \le k < j} Q_k^{(-1)} (t-\theta)^{s_\ell + \cdots + s_r}. \tag{5.1}$$

We also set $\Phi'^{\star} \in \mathrm{Mat}_r(\overline{K}[t])$ to be the $r \times r$ matrix cut out of the top left corner of $\Phi^{\star}$. Let $\mathcal{M}^{\star}$ and $\mathcal{M}'^{\star}$ be the dual t-motives defined by $\Phi^{\star}$ and $\Phi'^{\star}$ respectively. We define

$$\Psi^{\star} := \begin{pmatrix} \Omega^{(s_1+\cdots+s_r)} & 0 & 0 & \dots & 0 \\ -\mathfrak{L}^{\star}(s_1)\Omega^{(s_2+\cdots+s_r)} & \Omega^{(s_2+\cdots+s_r)} & 0 & \dots & 0 \\ \vdots & -\mathfrak{L}^{\star}(s_2)\Omega^{(s_3+\cdots+s_r)} & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ (-1)^{r-1}\mathfrak{L}^{\star}(s_1,\dots,s_{r-1})\Omega^{s_r} & (-1)^{r-2}\mathfrak{L}^{\star}(s_2,\dots,s_{r-1})\Omega^{s_r} & \dots & \Omega^{s_r} & 0 \\ (-1)^{r}\mathfrak{L}^{\star}(s_1,\dots,s_r) & (-1)^{r-1}\mathfrak{L}^{\star}(s_2,\dots,s_r) & \dots & -\mathfrak{L}^{\star}(s_r) & 1 \end{pmatrix}.$$

Then if we set $\Upsilon^{\star} = (\Psi^{\star})^{-1}$, then we use Lemma 4.1 to get

$$\Upsilon^{\star} = \begin{pmatrix} \Omega^{-(s_1+\cdots+s_r)} & 0 & 0 & \dots & 0 \\ \mathfrak{L}(s_1)\Omega^{-(s_1+\cdots+s_r)} & \Omega^{-(s_2+\cdots+s_r)} & 0 & \dots & 0 \\ \vdots & \mathfrak{L}(s_2)\Omega^{-(s_2+\cdots+s_r)} & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \mathfrak{L}(s_{r-1},\dots,s_1)\Omega^{-(s_1+\cdots+s_r)} & \mathfrak{L}(s_{r-1},\dots,s_2)\Omega^{-(s_2+\cdots+s_r)} & \dots & \Omega^{-s_r} & 0 \\ \mathfrak{L}(s_r,\dots,s_1)\Omega^{-(s_1+\cdots+s_r)} & \mathfrak{L}(s_r,\dots,s_2)\Omega^{-(s_2+\cdots+s_r)} & \dots & \mathfrak{L}(s_r)\Omega^{-s_r} & 1 \end{pmatrix}.$$

Note that $\Psi^{\star}$ and $\Upsilon^{\star}$ belongs to $\mathrm{GL}_{r+1}(\mathbb{T})$. Further, by Lemma 4.2 we obtain

$$\Psi^{\star(-1)} = \Phi^{\star}\Psi^{\star}.$$

Lemma 5.2. *The value at $t=\theta$ of the last line of $\Upsilon^{\star}$ is*

$$(\Gamma_{s_1}\dots\Gamma_{s_r}\zeta_A(s_r,\dots,s_1), \Gamma_{s_2}\dots\Gamma_{s_r}\zeta_A(s_r,\dots,s_2), \dots, \Gamma_{s_r}\zeta_A(s_r), 1).$$

Proof. This follows immediately from Equality (4.11). □

Let $\mathbf{m} = \{m_1,\dots,m_r\}$ be the $\overline{K}[t]$-basis of $\mathcal{M}'^{\star}$ with respect to the action of σ represented by $\Phi'^{\star}$. It is not hard to check that $\mathcal{M}'^{\star}$ is a free left $\overline{K}[\sigma]$-module of rank $d = (s_1+\cdots+s_r)+(s_2+\cdots+s_r)+\cdots+s_r$ and

$$\mathbf{w} = \{w_1,\dots,w_d\} := \{(t-\theta)^{s_1+\cdots+s_r-1}m_1,\dots,m_1,\dots,(t-\theta)^{s_r-1}m_r,\dots,m_r\}$$

is a $\overline{K}[\sigma]$-basis of $\mathcal{M}'^{\star}$. We further observe that $(t-\theta)^{\ell}\mathcal{M}'^{\star}/\sigma\mathcal{M}'^{\star} = (0)$ for $\ell \gg 0$.

We denote by $E'^{\star}$ the t-module defined by the dual t-motive $\mathcal{M}'^{\star}$ given by the matrix $\Phi'^{\star}$. We can write down explicitly the maps $\delta_0 : \mathcal{M}'^{\star} \to \mathrm{Mat}_{d\times 1}(\overline{K})$ and $\delta_1 : \mathcal{M}'^{\star} \to \mathrm{Mat}_{d\times 1}(\overline{K})$. For the convenience of the reader we present the former map which is the same as that for the Anderson-Thakur dual t-motives. Let $m \in \mathcal{M}'^{\star} = \overline{K}[t]m_1 + \cdots + \overline{K}[t]m_r$. Then we can write

$$m = \sum_{\ell=1}^{r} (c_{d_\ell-1,\ell}(t-\theta)^{d_\ell-1} + \cdots + c_{0,\ell} + F_\ell(t)(t-\theta)^{d_\ell})m_\ell, \tag{5.2}$$

with $c_{i,\ell} \in \overline{K}$ and $F_\ell(t) \in \overline{K}[t]$. Then

$$\delta_0(m) := (c_{d_1-1,1},\dots,c_{0,1},\dots,c_{d_r-1,r},\dots,c_{0,r})^{\top}. \tag{5.3}$$

5.2. **Integrality properties.** Next, we consider

$$\alpha(\mathcal{M}^\star) = \Phi^\star_{r+1,1} m_1 + \cdots + \Phi^\star_{r+1,r} m_r \in \mathcal{M}'^\star/(\sigma-1)\mathcal{M}'^\star$$

which corresponds to a certain point $\mathbf{v}^\star_{\mathfrak{s}} := \delta_1(\alpha(\mathcal{M}^\star)) \in E'^\star(\overline{K})$.

In this section we prove integrality properties of the Anderson t-module $E'^\star$ and the point $\mathbf{v}^\star_{\mathfrak{s}} \in E'(\overline{K})$ which will be used later to deduce a logarithmic interpretation for ν-adic MZV's from that for MZV's (see Theorem 5.8). Our result is inspired by [23, Theorem 5.3.4] (see Theorem 4.10). Indeed, its proof can be adapted without much modification. For the convenience of the reader we write it down completely below.

Proposition 5.3. *Recall that* $\mathbf{w} = \{w_1, \dots, w_d\}$ *denotes the* $\overline{K}[\sigma]$*-basis*

$$\{(t-\theta)^{s_1+\cdots+s_r-1}m_1, \dots, (t-\theta)m_1, m_1, \dots, (t-\theta)^{s_r-1}m_r, \dots, (t-\theta)m_r, m_r\}$$

of $\mathcal{M}'^\star$*. Let* Ξ *be the set of all the elements of* $\mathcal{M}'^\star$ *of the form* $\sum_{i=1}^d h_i w_i$ *where* $h_i = \sum_{n\geq 0} \sigma^n u_{n,j}$ *with* $u_{n,j} \in A$*.*

Then for $g \in A[t]$ *and* $1 \leq \ell \leq r$*, we have* $gm_\ell \in \Xi$*.*

Proof. Recall that we have put $d_\ell = s_\ell + \cdots + s_r$ for $1 \leq \ell \leq r$. We claim that there exist polynomials $g_{\ell,1}, \dots, g_{\ell,\ell-1} \in A[t]$ such that

$$(t-\theta)^{d_\ell} m_\ell = \sigma(g_{\ell,1}m_1 + \cdots + g_{\ell,\ell-1}m_{\ell-1} + m_\ell).$$

The proof is by induction on ℓ. For $\ell = 1$, we have $(t-\theta)^{d_1}m_1 = \sigma m_1$, and the claim is clear. Suppose that we have proved the claim for $1 \leq i < \ell$. We now show that the claim is true for ℓ. In fact, since $\sigma m_\ell = Q^{\star(-1)}_{\ell,1}(t-\theta)^{d_1}m_1 + \cdots + Q^{\star(-1)}_{\ell,\ell-1}(t-\theta)^{d_{\ell-1}}m_{\ell-1} + (t-\theta)^{d_\ell}m_\ell$ for explicit polynomials $Q^\star_{\ell,j} \in A[t]$ given in (5.1), we get

$$(t-\theta)^{d_\ell} m_\ell = \sigma m_\ell - Q^{\star(-1)}_{\ell,1}(t-\theta)^{d_1}m_1 - \cdots - Q^{\star(-1)}_{\ell,\ell-1}(t-\theta)^{d_{\ell-1}}m_{\ell-1}.$$

By induction it follows that

$$\begin{aligned}
(t-\theta)^{d_\ell} m_\ell &= \sigma m_\ell - Q^{\star(-1)}_{\ell,1}(t-\theta)^{d_1}m_1 - \cdots - Q^{\star(-1)}_{\ell,\ell-1}(t-\theta)^{d_{\ell-1}}m_{\ell-1} \\
&= \sigma m_\ell - \sum_{i=1}^{\ell-1} Q^{\star(-1)}_{\ell,i}\sigma(g_{i,1}m_1 + \cdots + g_{i,i-1}m_{i-1} + m_i) \\
&= \sigma m_\ell - \sum_{i=1}^{\ell-1} \sigma Q^{\star}_{\ell,i}(g_{i,1}m_1 + \cdots + g_{i,i-1}m_{i-1} + m_i) \\
&= \sigma(m_\ell - \sum_{i=1}^{\ell-1} Q^{\star}_{\ell,i}(g_{i,1}m_1 + \cdots + g_{i,i-1}m_{i-1} + m_i)).
\end{aligned}$$

The proof of the claim is now complete.

We are now ready to show by induction on ℓ that for $g \in A[t]$, we have $gm_\ell \in \Xi$. We first assume that $\ell = 1$. We show by induction on the degree of g that $gm_1 \in \Xi$. It is clear that if $\deg g = 0$, then the claim is true. Let $g \in A[t]$ with $\deg g > 0$. We divide g by $(t-\theta)^{d_1}$ and write

$$g = h(t-\theta)^{d_1} + \sum_{j=0}^{d_1-1} a_j(t-\theta)^j$$

with $h \in A[t]$ and $a_0, \dots, a_{d_1-1} \in A$. Since $\sigma m_1 = (t-\theta)^{d_1} m_1$, it follows that

$$\begin{aligned} gm_1 &= h(t-\theta)^{d_1} m_1 + \sum_{j=0}^{d_1-1} a_j (t-\theta)^j m_1 \\ &= \sigma h^{(1)} m_1 + \sum_{j=0}^{d_1-1} a_j (t-\theta)^j m_1. \end{aligned}$$

Since $\deg h < \deg g$, by induction, $h^{(1)} m_1 \in \Xi$. Since $a_0, \dots, a_{d_1-1} \in A$, the sum $\sum_{j=0}^{d_1-1} a_j (t-\theta)^j m_1$ belongs to Ξ. Hence we conclude that $gm_1 \in \Xi$.

Now we consider $1 < \ell \leq r$ and suppose that $gm_i \in \Xi$ for $1 \leq i < \ell$. We show by induction on the degree of g that $gm_\ell \in \Xi$. We divide g by $(t-\theta)^{d_\ell}$ and write

$$g = h(t-\theta)^{d_\ell} + r, \quad \text{with } h, r \in A[t] \text{ and } \deg r < d_\ell.$$

We have seen that there exist polynomials $g_{\ell,1}, \dots, g_{\ell,\ell-1} \in A[t]$ such that

$$(t-\theta)^{d_\ell} m_\ell = \sigma(g_{\ell,1} m_1 + \cdots + g_{\ell,\ell-1} m_{\ell-1} + m_\ell).$$

It follows that

$$\begin{aligned} gm_\ell &= h(t-\theta)^{d_\ell} m_\ell + r m_\ell \\ &= \sigma h^{(1)} (g_{\ell,1} m_1 + \cdots + g_{\ell,\ell-1} m_{\ell-1} + m_\ell) + r m_\ell \\ &= \sigma h^{(1)} m_\ell + \sigma h^{(1)} (g_{\ell,1} m_1 + \cdots + g_{\ell,\ell-1} m_{\ell-1}) + r m_\ell. \end{aligned}$$

The first and second terms belong to Ξ by induction. Since $r \in A[t]$ and $\deg r < d_\ell$, the last term also belongs to Ξ. We conclude that $gm_\ell \in \Xi$ and the proof is finished. □

We prove an analogue version of [23, Theorem 5.3.4] (see Theorem 4.10):

Proposition 5.4. *1) The t-module $E'^\star$ is defined over A.*

2) The point $\mathbf{v}_{\mathfrak{s}}^\star$ is an integral point in $E'^\star(A)$.

Proof. 1) We keep the notation of Proposition 5.3. By Proposition 5.3 we see that for $1 \leq i \leq d$, we have $tw_i \in \Xi$ which means $tw_i = \sum_{j=1}^d h_j w_i$ for some $h_j = \sum_n \sigma^n u_{n,j}$ with $u_{n,j} \in A$. Thus $E'^\star$ is defined over A.

2) We remark that $\delta_1(\Xi) \subset E'^\star(A)$. Since $\mathbf{v}_{\mathfrak{s}}^\star = \delta_1(\Phi_{r+1,1}^\star m_1 + \cdots + \Phi_{r+1,r}^\star m_r)$, it is sufficient to see that all the termes $\Phi_{r+1,1}^\star m_1, \dots, \Phi_{r+1,r}^\star m_r$ belongs to Ξ. In fact, let $1 \leq \ell \leq r$, by the proof of Proposition 5.3 there exist polynomials $g_{\ell,1}, \dots, g_{\ell,\ell-1} \in A[t]$ such that

$$(t-\theta)^{d_\ell} m_\ell = \sigma(g_{\ell,1} m_1 + \cdots + g_{\ell,\ell-1} m_{\ell-1} + m_\ell).$$

Recall that $\Phi_{r+1,\ell}^\star = Q_{r+1,\ell}^{\star(-1)} (t-\theta)^{d_\ell}$ for an explicit polynomial $Q_{r+1,\ell}^\star \in A[t]$ given in (5.1). This implies that

$$\begin{aligned} \Phi_{r+1,\ell}^\star m_\ell &= Q_{r+1,\ell}^{\star(-1)} (t-\theta)^{d_\ell} m_\ell \\ &= \sigma Q_{r+1,\ell}^\star (g_{\ell,1} m_1 + \cdots + g_{\ell,\ell-1} m_{\ell-1} + m_\ell). \end{aligned}$$

We conclude that $\Phi_{r+1,\ell}^\star m_\ell \in \Xi$. The proof is complete. □

5.3. **Logarithm coefficients.** The coefficients of the logarithm series can be calculated following [9]. We set

$$\Theta^\star := ((\Phi^\star)^{-1})^\top = \begin{pmatrix} \Theta^\star_{1,1} & \cdots & \Theta^\star_{1,r+1} \\ & \ddots & \vdots \\ & & \Theta^\star_{r+1,r+1} \end{pmatrix} \in \mathrm{Mat}_{r+1}(\overline{K}(t))$$

where

$$\Theta^\star_{i,i} = \frac{1}{(t-\theta)^{s_i+\cdots+s_r}}$$

and for $1 \leq i < r+1$,

$$\Theta^\star_{i,i+1} = \frac{Q_i^{(-1)}}{(t-\theta)^{s_{i+1}+\cdots+s_r}}.$$

The other coefficients $\Theta^\star_{i,j}$ vanish.

We set

$$\Theta'^\star := ((\Phi'^\star)^{-1})^\top = \begin{pmatrix} \Theta^\star_{1,1} & \cdots & \Theta^\star_{1,r} \\ & \ddots & \vdots \\ & & \Theta^\star_{r,r} \end{pmatrix} \in \mathrm{Mat}_{r}(\overline{K}(t)).$$

If we write

$$\mathrm{Log}_{E'^\star} = \sum_{n\geq 0} P^\star_n \tau^n,$$

then by [9, Proposition 2.2], for $n \geq 0$, the nth coefficient of the logarithm series of $E^\star$ evaluated at $\mathbf{v} \in \overline{K}^d$ is given by

$$P^\star_n \mathbf{v}^{(n)} = \delta_0(\Theta'^{\star(1)} \dots \Theta'^{\star(n)} \iota^{-1}(\mathbf{v})^{(n)}).$$

5.4. **Logarithmic interpretations for MZV's.** Note that

$$\mathbf{f}^\star = ((-1)^r Q_1^{(-1)} \dots Q_r^{(-1)}(t-\theta)^{s_1+\cdots+s_r}, \dots, -Q_r^{(-1)}(t-\theta)^{s_r}),$$
$$\Psi^\star_{\mathbf{f}} = ((-1)^r \mathfrak{L}^\star(s_1,\dots,s_r), (-1)^{r-1}\mathfrak{L}^\star(s_2,\dots,s_r), \dots, -\mathfrak{L}^\star(s_r)).$$

In particular, they verify the hypothesis of Theorem 3.5 by [16, Lemma 5.3.1].

Theorem 3.5 implies the following theorem:

Theorem 5.5. *Recall that for $1 \leq \ell \leq r$, we put $d_\ell := s_\ell + \cdots + s_r$. Then we have*

$$\mathrm{Log}^!_{E'^\star}(\mathbf{v}^\star_{\mathbf{s}}) = \delta_0 \begin{pmatrix} -\mathfrak{L}(s_r,\dots,s_1)\Omega^{-(s_1+\cdots+s_r)} \\ -\mathfrak{L}(s_r,\dots,s_2)\Omega^{-(s_2+\cdots+s_r)} \\ \vdots \\ -\mathfrak{L}(s_r)\Omega^{-s_r} \end{pmatrix}.$$

In particular, for $1 \leq \ell \leq r$, the $(d_1+\cdots+d_\ell)$th coordinate of the right hand side of the above equality equals $-\Gamma_{s_\ell}\dots\Gamma_{s_r}\zeta_A(s_r\dots,s_\ell)$.

Proof. We first estimate the domain of convergence of $\mathrm{Log}_{E'^\star}$. We observe that the (lower triangular) matrix $(\Phi'^\star)^{-1}$ above agrees with Φ'^{-1} from [18, §4.2] along the main diagonal, and that the first subdiagonal agrees up to a factor of (-1), while $(\Phi'^\star)^{-1}_{i,j} = 0$ for all the other (below the subdiagonal) entries. This allows us to use the degree estimates given in [18, Proposition 4.1.3] for the matrix Φ'^{-1} for our

logarithm series $\mathrm{Log}_{E'^\star}$ for the matrix $(\Phi'^\star)^{-1}$. Indeed, following the notation in the proof of [18, Proposition 4.1.3] we fix $\mathbf{u} = (u_1, \dots, u_r) \in \mathrm{Mat}_{1\times r}(\mathbb{C}_\infty[t])$ with

$$u_i = y_{i,1}(t-\theta)^{d_i-1} + y_{i,2}(t-\theta)^{d_i-2} + \cdots + y_{i,d_i}, \quad y_{i,j} \in \mathbb{C}_\infty, \quad 1 \le i \le r.$$

Then, we see that the degree estimates for the entries of $\mathbf{u}^{(n)} \prod_{1\le k\le n}(\Phi'^{-1})^{(n+1-k)}$ coincide with estimates for our $\mathbf{u}^{(n)} \prod_{1\le k\le n}((\Phi'^\star)^{-1})^{(n+1-k)}$ for each term which involves only diagonal or subdiagonal entries of $(\Phi'^\star)^{-1}$. On the other hand, each term which involves any other entry of $(\Phi'^\star)^{-1}$ will be identically zero, since the sub-sub-diagonal coordinates of $(\Phi'^\star)^{-1}$ are all zero. Thus, the formula for the degree of the ℓth component of $\mathbf{u}^{(n)} \prod_{1\le k\le n}((\Phi'^\star)^{-1})^{(n+1-k)}$ will be a subsum of the formula for the degree of the ℓth component of $\mathbf{u}^{(n)} \prod_{1\le k\le n}(\Phi'^{-1})^{(n+1-k)}$. In particular, the degree estimates for $\mathbf{u}^{(n)} \prod_{1\le k\le n}(\Phi'^{-1})^{(n+1-k)}$ will also hold for $\mathbf{u}^{(n)} \prod_{1\le k\le n}((\Phi'^\star)^{-1})^{(n+1-k)}$, since they are bounded above by the maximum of these terms in this sum (see [18, Proposition 4.1.3] for more details). This allows us to conclude using [18, Lemma 4.2.1] that $\mathrm{Log}_{E'^\star}(\mathbf{y})$ converges as long as $\|Q_i\| \le q^{\frac{s_i q}{q-1}}$ and $\mathbf{y} = (y_{1,1}, \dots, y_{1,d_1}, \dots, y_{r,1}, \dots, y_{r,d_r})^\top$ satisfies the condition that $|y_{i,j}|_\infty < q^{j+\frac{d_i}{q-1}}$ for each $1 \le i \le r$ and $1 \le j \le d_i$. To summarize, the radius of convergence of $\mathrm{Log}_{E'^\star}$ is at least as large as that of $\mathrm{Log}_{E'}$.

Next, we turn to analyzing

$$\alpha(\mathcal{M}) = (-1)^r Q_1^{(-1)} \cdots Q_r^{(-1)}(t-\theta)^{s_1+\cdots+s_r} m_1 + \cdots + Q_{r-1}^{(-1)} Q_r^{(-1)}(t-\theta)^{s_{r-1}+s_r} m_{r-1} - Q_r^{(-1)}(t-\theta)^{s_r} m_r.$$

From the defining equation for $\Phi'^\star$ we see that

$$\sigma m_r = (-1)^{r-1} Q_1^{(-1)} \cdots Q_{r-1}^{(-1)}(t-\theta)^{s_1+\cdots+s_r} m_1 + \cdots - Q_{r-1}^{(-1)}(t-\theta)^{s_{r-1}+s_r} m_{r-1} + (t-\theta)^{s_r} m_r.$$

From this we conclude that

$$\alpha(\mathcal{M}) = -Q_r^{(-1)} \sigma(m_r) = -\sigma(Q_r m_r). \tag{5.4}$$

From [5, (3.7.3)] we know that $\|Q_i\| = \|H_{s_i}\| < |\theta|_\infty^{\frac{s_i q}{q-1}}$, so the conditions of Theorem 3.5 (b) are satisfied, which proves the first statement of the theorem.

The second statement of the theorem follows immediately from Lemma 5.2 and the definition of δ_0. □

Remark 5.6. Jing Yu [21] suggested that the logarithmic interpretation for MZV's obtained in Theorem 5.5 could be viewed as a "nice" integral interpretation for the MZV's, thus it may be called a linear form of Anderson logarithms.

5.5. **Logarithmic interpretations for ν-adic MZV's.** Throughout this section we fix a finite place ν of K which corresponds to an irreducible monic polynomial still denoted by ν of A. We let K_ν be the completion of K at ν and let $\mathbb{C}_\nu$ be the completion of an algebraic closure of K_ν. Let $|\cdot|_\nu$ be the normalized ν-adic absolute value on $\mathbb{C}_\nu$. This ν-adic absolute value extends naturally to matrices with entries in $\mathbb{C}_\nu$.

This section aims to present a logarithmic interpretation for ν-adic MZV's. For the depth one case, i.e., for ν-adic zeta values, this was done by Anderson and Thakur (see [5, Theorem 3.8.3]). We mention that Chang and Mishiba in [19] gave another interpretation for these values by combining the Anderson-Thakur dual t-motives and the notion of fiber coproducts. We show that their arguments can

carry over to our setting. Consequently, we deduce from Theorem 5.5 a logarithmic interpretation of ν-adic MZV's (see [20, Theorem 6.2.4]).

In what follows, we always work with a tuple $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ for $r \geq 1$. We work with the t-module $E'^{\star}$ introduced in §5.1 and keep the notation of this section.

Proposition 5.7. *For any $\mathbf{v} \in E'^{\star}(\mathbb{C}_\nu)$ with $|\mathbf{v}|_\nu < 1$, $\mathrm{Log}_{E'^{\star}}(\mathbf{v})$ converges ν-adically in $\mathrm{Lie}_{E'^{\star}}(\mathbb{C}_\nu)$.*

Proof. We write

$$\mathrm{Log}_{E'^{\star}}(\mathbf{v}) = \sum_{i \geq 0} P_i^{\star} \mathbf{v}^{(i)}.$$

Recall that for $i \geq 0$, the ith coefficient of the logarithm series of $E'^{\star}$ evaluated at $\mathbf{v} \in \overline{K}^d$ is given by

$$P_i^{\star} \mathbf{v}^{(i)} = \delta_0(\Theta'^{\star(1)} \dots \Theta'^{\star(i)} \iota^{-1}(\mathbf{v})^{(i)}).$$

Here the matrix $B_i = \Theta'^{\star(1)} \dots \Theta'^{\star(i)}$ is given as follows. We have $B_i[\ell j] = 0$ if $\ell > j$ ($B[\ell j]$ denotes the (ℓ, j)th entry of a matrix B). Further, if $\ell = j$, then

$$B_i[\ell j] = \frac{1}{\mathbb{L}_i^{s_j + \dots + s_r}},$$

where we recall the definition of $\mathbb{L}_i$ from (4.12). For $1 \leq \ell < j \leq r$, we get

$$B_i[\ell j] = \sum_{0 \leq i_\ell < \dots < i_{j-1} < i} \frac{Q_\ell^{(i_\ell)} \dots Q_{j-1}^{(i_{j-1})}}{\mathbb{L}_{i_\ell}^{s_\ell} \dots \mathbb{L}_{i_{j-1}}^{s_{j-1}} \mathbb{L}_i^{s_j + \dots + s_r}}.$$

We consider $w_k = (t - \theta)^s m_j$ (with $1 \leq j \leq r$, $0 \leq s < d_j$) which is an element of the $\overline{K}[\sigma]$-basis

$$\mathbf{w} = \{(t-\theta)^{s_1 + \dots + s_r - 1} m_1, \dots, (t-\theta) m_1, m_1, \dots, (t-\theta)^{s_r - 1} m_r, \dots, (t-\theta) m_r, m_r\}$$

of $\mathcal{M}'^{\star}$. We note that $k = d_1 + \dots + d_j - s$. The vector w_k corresponds to the kth vector in the canonical basis of $\overline{K}^d$.

Letting $P_i^{\star}[k', k]$ the (k', k)th entry of $P_i^{\star}$, we get

$$\begin{aligned}(P_i^{\star}[1, k], \dots, P_i^{\star}[d_1 + \dots + d_r, k])^\top &= P_i^{\star} \iota_{\mathbf{w}}^{-1}(w_k) \\ &= \delta_0(\Theta'^{\star(1)} \dots \Theta'^{\star(i)} \iota_{\mathbf{m}}^{-1}(w_k)^{(i)}) \\ &= \delta_0((B_i[1, j](t - \theta^{q^i})^s, \dots, B_i[r, j](t - \theta^{q^i})^s)^\top).\end{aligned}$$

Recall that the map δ_0 is given explicitly by (5.3). Since we may rewrite the first d_ℓ terms in each coordinate of equation (5.2) in terms of hyperderivatives (see [34, Lemma 2.4.1] or [18, §3.2]), a short calculation using hyperderivatives shows that each $P_i^{\star}[k', k]$ can be written in the following form

$$P_i^{\star}[k', k] = \sum_{\substack{0 \leq i_\ell < \dots < i_{j-1} < i \\ c_\ell, \dots, c_{j-1}, c \in \mathbb{Z}_{\geq 0}}} \frac{Q_{(i_\ell, \dots, i_{j-1}, c_\ell, \dots, c_{j-1}, c)}}{\mathbb{L}_{i_\ell}^{s_\ell + c_\ell} \dots \mathbb{L}_{i_{j-1}}^{s_{j-1} + c_{j-1}} \mathbb{L}_i^{s_j + \dots + s_r + c}} \Bigg|_{t = \theta}$$

where $Q_{(i_\ell, \dots, i_{j-1}, c_\ell, \dots, c_{j-1}, c)} \in \mathbb{F}_q[t, \theta]$ and $c_\ell + \dots + c_{j-1} + c < d_1$.

For $j \in \mathbb{Z}_{\geq 0}$, we use the estimate

$$|L_j|_\nu = |\mathbb{L}_j(\theta)|_\nu \geq |\nu|_\nu^j$$

which implies

$$|P_i^\star|_\nu \leq |\nu|_\nu^{-i(2d_1-1)}$$

and thus

$$|P_i^\star \mathbf{v}^{(i)}|_\nu \leq |\nu|_\nu^{-i(2d_1-1)} |\mathbf{v}|_\nu^{q^i}.$$

Since $|\mathbf{v}|_\nu < 1$, it follows that $|P_i^\star \mathbf{v}^{(i)}|_\nu$ tends to 0 when $i \to +\infty$. This completes the proof. □

Recall that for $1 \leq \ell \leq r$, $d_\ell := s_\ell + \cdots + s_r$. We set

$$a_\nu := (\nu^{d_1} - 1) \dots (\nu^{d_r} - 1).$$

The main result of this section is stated as follows.

Theorem 5.8. *The series* $\mathrm{Log}_{E'^\star}(E'^\star_{a_\nu} \mathbf{v}^\star_{\mathfrak{s}})$ *converges* ν*-adically in* $\mathrm{Lie}_{E'^\star}(\mathbb{C}_\nu)$. *Further, the* d_1*th coordinate of* $\mathrm{Log}_{E'^\star}(E'^\star_{a_\nu} \mathbf{v}^\star_{\mathfrak{s}})$ *equals* $-a_\nu \Gamma_{s_1} \dots \Gamma_{s_r} \zeta_A(s_r, \dots, s_1)_\nu$.

Remark 5.9. Following Chang and Mishiba [20, §6] we define $\zeta_A(s_r, \dots, s_1)_\nu$ to be the value $-\frac{1}{a\Gamma_{s_1} \cdots \Gamma_{s_r}}$ multiplied by the d_1th coordinate of $\mathrm{Log}_{E'^\star}(E'^\star_a \mathbf{v}^\star_{\mathfrak{s}})_\nu$ for some nonzero element $a \in A$ with $|E'^\star_a \mathbf{v}^\star_{\mathfrak{s}}|_\nu < 1$. Note that this value does not depend on the choice of a by [20, Remark 6.2.5]. As motivation for our definition, we note that it generalizes Anderson and Thakur's construction in [5, Thm. 3.8.3(II)] very nicely. As our ν-adic MZV is defined using a logarithm, Theorem 5.8 technically gives a logarithmic interpretation for $\zeta_A(s_r, \dots, s_1)_\nu$, although would be interesting to find the relation of this MZV with the more intrinsic definition of Thakur.

Another interesting question is to see how our definition is related to that given by Chang and Mishiba in [20, Def. 6.1.1]. As pointed out by one of the referees to whom we are grateful for their detailed comments, they should agree but the proof seems highly non-trivial and goes too far from the scope of our paper.

Proof of Theorem 5.8. By Proposition 5.3, $\mathbf{v}^\star_{\mathfrak{s}}$ is a point in $E'^\star(A)$. Then, a short calculation similar to that given in the proof of Theorem 5.5 shows that $E'^\star$ is an iterated extension of tensor powers of the Carlitz module. Thus the arguements of [19, Proposition 4.1.1] carry over without modification. This implies that $|E'^\star_{a_\nu} \mathbf{v}^\star_{\mathfrak{s}}|_\nu < 1$ by [19, Proposition 4.1.1 and Remark 4.2.4]. Thus Theorem 5.8 follows immediately from Proposition 5.7. □

Remark 5.10. In [30] we investigated algebraic relations among Goss's zeta values for function fields of elliptic curves. As one crucial step of our analysis, we had to do some period calculations for some Anderson t-modules (see [30, §3.3]). Note that for Drinfeld modules these calculations follows immediately from basic properties of Anderson generating functions (see for example [22, §4.2]). Our method was based on direct calculations by taking advantage of working with elliptic curves. The motivation of this paper grows from our desire to generalize the aforementioned arguments for general curves. We expect that the method of this paper would provide a general approach to period calculations in our work in progress.

6. Relations with previous works

This section is devoted to comparing the t-modules associated to the star dual t-motives defined in §5 with those arising from the works of Anderson-Thakur [5] and Chang-Mishiba [20]. We start with some examples given by Chang-Mishiba [20] and observe that, in these examples, for the same multiple zeta value, the t-module constructed by the star model has smaller dimension. Next we prove that

indeed this inequality always holds. Finally, we determine integral points in special cases which covers all the examples given in [5].

6.1. **Setup.** In this section, let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ be a tuple with $r \geq 1$. For $1 \leq \ell \leq r$, we put $d_\ell = s_\ell + \cdots + s_r$. In [20] Chang and Mishiba gave a logarithmic interpretation for $\zeta_A(\mathfrak{s})$ (see [20, Theorem 1.4.1]). More precisely, they constructed a t-module $G_\mathfrak{s}$ defined over K, a special point denoted by $\mathbf{v}_\mathfrak{s}^{CM} \in G_\mathfrak{s}(K)$ and a vector $\mathbf{z}_\mathfrak{s}^{CM} \in \operatorname{Lie} G_\mathfrak{s}(\mathbb{C}_\infty)$ such that

1) The d_1th coordinate of $\mathbf{z}_\mathfrak{s}^{CM} \in G_\mathfrak{s}(K)$ equals $\Gamma_{s_1} \dots \Gamma_{s_r} \zeta_A(\mathfrak{s})$.
2) $\operatorname{Exp}_{G_\mathfrak{s}}(\mathbf{z}_\mathfrak{s}^{CM}) = \mathbf{v}_\mathfrak{s}^{CM}$.

We put $\mathfrak{s}' = \operatorname{inv}(\mathfrak{s}) = (s_r, \dots, s_1)$. Then in §5 we constructed a t-module $E'^\star$ defined over A and a special integral point $\mathbf{v}_{\mathfrak{s}'}^\star \in E'^\star(A)$. Theorem 5.5 gives a split-logarithmic identity for $E'^\star$:

$$\operatorname{Log}^!_{E'^\star}(\mathbf{v}_{\mathfrak{s}'}^\star) = \delta_0 \begin{pmatrix} -\mathfrak{L}(s_1, \dots, s_r)\Omega^{-(s_1+\cdots+s_r)} \\ -\mathfrak{L}(s_1, \dots, s_{r-1})\Omega^{-(s_1+\cdots+s_{r-1})} \\ \vdots \\ -\mathfrak{L}(s_1)\Omega^{-s_1} \end{pmatrix}.$$

In particular, the d_1th coordinate of the $\operatorname{Log}^!_{E'^\star}(\mathbf{v}_{\mathfrak{s}'}^\star)$ equals $-\Gamma_{s_1} \dots \Gamma_{s_r} \zeta_A(\mathfrak{s})$.

In the depth one case, i.e., when $r = 1$ and $\mathfrak{s} = (n)$, both constructions coincide with that of Anderson and Thakur given in [5]. The associated t-module is the nth tensor power $C^{\otimes n}$ of the Carlitz module. In [5] they denoted by $\mathbf{Z}_n \in C^{\otimes n}(A)$ the special point and by $\mathbf{z}_n \in \operatorname{Lie} C^{\otimes n}(\mathbb{C}_\infty)$ the associated vector.

6.2. **Relation with the work of Chang-Mishiba.** We first give examples to compare the previous logarithmic interpretations for MZV's given by Chang-Mishiba [20] and by the star dual t-motives (see Theorem 5.5). The two examples are taken from [20]. We observe that, in both cases, the Anderson t-module arising from the star model has smaller dimension and the associated integral point is "simpler".

Example 6.1. The following example is taken from [20, Example 5.4.1]. We consider general q and $\mathfrak{s} = (s_1 = 1, s_2 = 1, s_3 = 2)$. We have $\Gamma_1 = \Gamma_2 = 1$ and $H_1 = H_2 = 1$.

On the one hand, the t-module $G_\mathfrak{s}$ has dimension 10 and is given by

$$(G_\mathfrak{s})_\theta = \left(\begin{array}{cccc|c|cc|cc|c} \theta & 1 & & & & & & & & \\ & \theta & 1 & & & & & & & \\ & & \theta & 1 & & & & & & \\ \tau & & & \theta & -\tau & -\tau & & -\tau & & \tau \\ \hline & & & & \theta+\tau & & & & & \\ \hline & & & & & \theta & 1 & & & \\ & & & & & \tau & \theta & & & \\ \hline & & & & & & & \theta & 1 & \\ & & & & & & & \tau & \theta & -\tau \\ \hline & & & & & & & & & \theta+\tau \end{array}\right).$$

Moreover,

$$\mathbf{v}_\mathfrak{s}^{CM} = (0, 0, 0, 0, 1, 0, 1, 0, -1, 1)^\top,$$

and

$$\mathbf{z}_\mathfrak{s}^{CM} = (*, *, *, \zeta_A(1,1,2), \operatorname{Li}_1^\star(1), *, \operatorname{Li}_2^\star(1), *, -\operatorname{Li}^\star_{(1,1)}(1,1), \operatorname{Li}_1^\star(1))^\top.$$

On the other hand, the t-module $E'^{\star}$ has dimension 7 and is defined by

$$E'^{\star}_{\theta} = \left(\begin{array}{ccccc|cc} \theta & 1 & & & & & \\ & \theta & 1 & & & & \\ & & \theta & 1 & & & \\ \tau & & & \theta & & \tau & \\ \hline & & & & \theta & 1 & \\ & & & & \tau & \theta & \tau \\ \hline & & & & & & \theta+\tau \end{array}\right).$$

The special point is given by

$$-\mathbf{v}^{\star}_{\mathfrak{s}'} = (0,0,0,0,0,0,1)^{\top}.$$

By Theorem 5.5 we have

$$\mathrm{Log}_{E'^{\star}}(-\mathbf{v}^{\star}_{\mathfrak{s}'}) = (*,*,*,\zeta_A(1,1,2),*,\zeta_A(1,1),\zeta_A(1))^{\top}.$$

Note that

$$\dim E'^{\star} = 7 < \dim G_{\mathfrak{s}} = 10.$$

Example 6.2. The following example is taken from [20, Example 5.4.2]. We take $q=2$ and $\mathfrak{s}=(s_1=1,s_2=3)$. We have $\Gamma_1=1,\Gamma_3=\theta^2+\theta$ and $H_1=1,H_3=t+\theta^2$.

On the one hand, the t-module $G_{\mathfrak{s}}$ has dimension 6 and is given by

$$(G_{\mathfrak{s}})_{\theta} = \left(\begin{array}{ccccc|cc} \theta & 1 & & & & & \\ & \theta & 1 & & & & \\ & & \theta & 1 & & & \\ \tau & & & \theta & & -\theta^2\tau & -\tau \\ \hline & & & & & \theta+\tau & \\ \hline & & & & & & \theta+\tau \end{array}\right).$$

Further

$$\mathbf{v}^{CM}_{\mathfrak{s}} = (0,0,0,1,1,\theta+1)^{\top},$$

and

$$\mathbf{z}^{CM}_{\mathfrak{s}} = (*,*,*,(\theta^2+\theta)\zeta_A(1,3),\mathrm{Li}^{\star}_1(1),\theta\mathrm{Li}^{\star}_1(1))^{\top}.$$

On the other hand, the t-module $E'^{\star}$ has dimension 5 and is defined by

$$E'^{\star}_{\theta} = \left(\begin{array}{ccccc|c} \theta & 1 & & & & \\ & \theta & 1 & & & \\ & & \theta & 1 & & \tau \\ \tau & & & \theta & & (\theta^2+\theta)\tau \\ \hline & & & & & \theta+\tau \end{array}\right).$$

The special point is given by

$$-\mathbf{v}^{\star}_{\mathfrak{s}'} = (0,0,0,0,1)^{\top}.$$

By Theorem 5.5 we have

$$\mathrm{Log}_{E'^{\star}}(-\mathbf{v}^{\star}_{\mathfrak{s}'}) = (*,*,*,(\theta^2+\theta)\zeta_A(1,3),(\theta^2+\theta)\zeta_A(1))^{\top}.$$

Note that

$$\dim E'^{\star} = 5 < \dim G_{\mathfrak{s}} = 6.$$

The rest of this section aims to prove that an inequality of dimensions always holds.

Proposition 6.3. *With the above notation, we have* $\dim G_{\mathfrak{s}} \geq \dim E'^{\star}$.

The equality holds if and only if either $r = 1$ *or* $r = 2$ *and* $\mathfrak{s} = (s_1, s_2)$ *with* $1 \leq s_1, s_2 \leq q$.

Proof. We set $w(\mathfrak{s}) := s_1 + \cdots + s_r$ called the weight of $\mathfrak{s}$. By [20, Theorem 5.2.5] there are explicit tuples $\mathfrak{s}_\ell \in \mathbb{N}^{\mathrm{dep}(\mathfrak{s}_\ell)}$ with $w(\mathfrak{s}_\ell) = w(\mathfrak{s})$, $\mathrm{dep}(\mathfrak{s}_\ell) \leq r$, explicit coefficients $b_\ell \in A$ and vectors $\mathbf{u}_\ell \in A^{\mathrm{dep}(\mathfrak{s}_\ell)}$ so that

$$\Gamma_{s_1} \dots \Gamma_{s_r} \zeta_A(\mathfrak{s}) = \sum_\ell b_\ell \cdot (-1)^{\mathrm{dep}(\mathfrak{s}_\ell)-1} \mathrm{Li}^\star_{\mathfrak{s}_\ell}(\mathbf{u}_\ell).$$

Let $\mathfrak{s}_\ell$ be such a tuple. We write $\mathfrak{s}_\ell = (\mathfrak{s}_{\ell,1}, \dots, \mathfrak{s}_{\ell,\mathrm{dep}(\mathfrak{s}_\ell)})$ and set

$$\epsilon(\mathfrak{s}_\ell) := (\mathfrak{s}_{\ell,1} + \cdots + \mathfrak{s}_{\ell,\mathrm{dep}(\mathfrak{s}_\ell)-1}) + \cdots + \mathfrak{s}_{\ell,1}.$$

Note that $\epsilon(\mathfrak{s}_\ell)$ belongs to $\mathbb{Z}_{\geq 0}$. Then it is shown in [20] that

$$\dim G_{\mathfrak{s}} = (s_1 + \cdots + s_r) + \sum_\ell \epsilon(\mathfrak{s}_\ell).$$

By the construction of the star model associated to $\zeta_A(\mathfrak{s})$ we see that

$$\dim E'^{\star} = (s_1 + \cdots + s_r) + \epsilon(\mathfrak{s}).$$

The proposition follows from the fact that there exists ℓ_0 such that $\mathfrak{s}_{\ell_0} = \mathfrak{s}$.

The equality holds if and only if $\mathfrak{s}_\ell = (s_1 + \cdots + s_r)$ for $\ell \neq \ell_0$, which happens only when $r = 1$ or $r = 2$ and $\mathfrak{s} = (s_1, s_2)$ with $1 \leq s_1, s_2 \leq q$. □

6.3. **Relation with the work of Anderson-Thakur.** In this section we extend the previous examples to obtain the following result which explicitly computes integral points and covers all the examples given by Anderson-Thakur when $r = 1$ (see [5, page 187]).

By direct calculations we prove that for $1 \leq n \leq q$, we have $H_n(t) = 1$ and that for $q + 1 \leq n \leq q^2$, we put $k = \lfloor \frac{n-1}{q} \rfloor$ and get

$$H_n(t) = \sum_{j=0}^{k} \binom{n - jq + j - 1}{j} (t^q - t)^{k-j} (t^q - \theta^q)^j.$$

In particular, for $1 \leq n \leq q^2$, we always have

$$\deg H_n(t) \leq kq \leq n - 1.$$

Corollary 6.4. *Let* $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ *be a tuple for* $r \geq 1$ *such that* $1 \leq s_1 \leq q^2$. *We denote by* $\mathfrak{s}' = inv(\mathfrak{s}) = (s_r, \dots, s_1)$. *If we express* $H_{s_1}(t) = \sum_{i=0}^{s_1-1} a_i (t - \theta)^i$, *then*

$$-\mathbf{v}^\star_{\mathfrak{s}'} = (0, \dots, 0, a_{s_1-1}, \dots, a_1, a_0)^\top.$$

Proof. We should keep in mind that we are working with the star model attached to $\mathfrak{s}'$. Since $Q_r := H_{s'_r} = H_{s_1} = \sum_{i=0}^{s_1-1} a_i (t - \theta)^i$, Equation (5.4) implies that

$$\begin{aligned} -\mathbf{v}^\star_{\mathfrak{s}'} &= \delta_1(\sigma Q_r m_r) \\ &= \delta_1(\sigma \sum_{i=0}^{s_1-1} a_i (t - \theta)^i m_r) \\ &= (0, \dots, 0, a_{s_1-1}, \dots, a_1, a_0)^\top. \end{aligned}$$

The proof is finished. □

Remark 6.5. 1) When $r = 1$, we recover the examples given by Anderson-Thakur when $r = 1$ (see [5, page 187]). In this case, we take $r = 1$ and $\mathfrak{s} = (n)$ with $1 \leq n \leq q^2$, hence $\mathfrak{s}' = \mathfrak{s} = (n)$. We see that the point $-\mathbf{v}^\star_{\mathfrak{s}'}$ coincides with the point $\mathbf{Z}_n$ defined by Anderson and Thakur. If we express $H_n(t) = \sum_{i=0}^{n-1} a_i(t-\theta)^i$, then

$$\mathbf{Z}_n = (a_{n-1}, \dots, a_1, a_0)^\top.$$

2) Let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ be a tuple for $r \geq 1$ such that $1 \leq s_1 \leq q$. Thus $\mathfrak{s}' = \mathrm{inv}(\mathfrak{s}) = (s_r, \dots, s_1)$. Since $H_{s_1}(t) = 1$, we get

$$-\mathbf{v}^\star_{\mathfrak{s}'} = (0, \dots, 0, 1)^\top.$$

3) Let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ be a tuple for $r \geq 1$ such that $q+1 \leq s_1 \leq 2q$. Thus $\mathfrak{s}' = \mathrm{inv}(\mathfrak{s}) = (s_r, \dots, s_1)$. It follows that

$$H_{s_1}(t) = (t^q - t) + s_1(t^q - \theta^q) = (s_1 + 1)(t-\theta)^q - (t-\theta) + \theta^q - \theta.$$

Then

$$-\mathbf{v}^\star_{\mathfrak{s}'} = (0, \dots, 0, s_1 + 1, 0, \dots, 0, -1, \theta^q - \theta)^\top$$

where $\theta^q - \theta$ is the dth coordinate and $s_1 + 1$ is the $(d-q)$th coordinate.

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