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Golnaz Badkobeh, Pascal Ochem

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AVOIDING CONJUGACY CLASSES ON THE 5-LETTER ALPHABET

GOLNAZ BADKOBEB* AND PASCAL OCHEM^{†,‡}

Abstract. We construct an infinite word w over the 5-letter alphabet such that for every factor f of w of length at least two, there exists a cyclic permutation of f that is not a factor of w . In other words, w does not contain a non-trivial conjugacy class. This proves the conjecture in Gamard *et al.* [*Theoret. Comput. Sci.* **726** (2018) 1–4].

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1. INTRODUCTION

A *pattern* p is a non-empty finite word over an alphabet $\Delta = \{A, B, C, \dots\}$ of capital letters called *variables*. An *occurrence* of p in a word w is a non-erasing morphism $h : \Delta^* \rightarrow \Sigma^*$ such that $h(p)$ is a factor of w . The *avoidability index* $\lambda(p)$ of a pattern p is the size of the smallest alphabet Σ such that there exists an infinite word over Σ containing no occurrence of p . Bean *et al.* [2] and Zimin [8] characterized unavoidable patterns, *i.e.*, such that $\lambda(p) = \infty$. However, determining the exact avoidability index of an avoidable pattern requires more work. Although patterns with index 4 [2] and 5 [4] have been found, the existence of an avoidable pattern with index at least 6 is an open problem since 2001.

Some techniques in pattern avoidance start by showing that the considered word avoids other structures, such as generalized repetitions [6, 7]. Let us say that a word has property P_i if it does not contain all the conjugates of the same word w with $|w| \geq i$. Recently, in order to study the avoidance of a kind of patterns called circular formulas, Gamard *et al.* [5] obtained that there exists

- a morphic binary word satisfying P_5 ,
- a morphic ternary word satisfying P_3 ,
- a morphic word over the 6-letter alphabet satisfying P_2 .

Recall that a pure morphic word is of the form $m^\omega(0)$ and a morphic word is of the form $h(m^\omega(0))$ for some morphisms m and h . Independently, Bell and Madill [3] obtained a pure morphic word over the 12-letter alphabet that also satisfies P_2 and some other properties.

Keywords and phrases: Combinatorics on words, conjugacy classes.

¹ Goldsmiths, University of London.

² LIRMM, Université de Montpellier, CNRS, Montpellier, France.

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[‡] Corresponding author: ochem@lirmm.fr

It is conjectured that the smallest alphabet allowing an infinite word satisfying P_2 has 5 letters [5], which is best possible. In this paper, we prove this conjecture using a morphic word. This settles the topic of the smallest alphabet needed to satisfy P_i .

2. MAIN RESULT

Let ε denote the empty word. We consider the morphic word $w_5 = G(F^\omega(0))$ defined by the following morphisms.

$$\begin{array}{ll} F(0) = 01, & G(0) = abcd, \\ F(1) = 2, & G(1) = \varepsilon, \\ F(2) = 03, & G(2) = eacd, \\ F(3) = 24, & G(3) = becd, \\ F(4) = 23. & G(4) = be. \end{array}$$

Theorem 2.1. *The morphic word $w_5 \in \Sigma_5^*$ avoids every conjugacy class of length at least 2.*

In order to prove this theorem, it is convenient to express w_5 with the larger morphisms $f = F^3$ and $g = G \circ F^2$ given below. Clearly, $w_5 = g(f^\omega(0))$.

$$\begin{array}{ll} f(0) = 01203, & g(0) = abcdeacd, \\ f(1) = 0124, & g(1) = abcdbecd, \\ f(2) = 0120323, & g(2) = abcdeacdbe, \\ f(3) = 01240324, & g(3) = abcdbecdeacdbeacd, \\ f(4) = 01240323. & g(4) = abcdbecdeacdbe. \end{array}$$

2.1. Avoiding conjugacy classes in $F^\omega(0)$

Here we study the pure morphic word and the conjugacy classes it contains.

Lemma 2.2. *The infinite word $F^\omega(0)$ contains only the conjugacy classes listed in $C = \{F(2), F^2(2), F^d(4), f^d(0)\}$, for all $d \geq 1$.*

Proof. Notice that the factor 01 only occurs as the prefix of the f -image of every letter in $F^\omega(0)$. Moreover, every letter 1 only occurs in $F^\omega(0)$ as the suffix of the factor 01. Let us say that the *index* of a conjugacy class is the number of occurrences of 1 in any of its elements. An easy computation shows that the set of complete conjugacy classes in $F^\omega(0)$ with index at most one is $C_1 = \{F(2), F^2(2), F(4), F^2(4), f(4), f(0)\}$. Let us assume that $F^\omega(0)$ contains a conjugacy class c with index at least two. Let $w \in c$ be such that 01 is a prefix of w . We write $w = ps$ such that the leftmost occurrence of 01 in w is the prefix of s . Then the conjugate sp of w also belongs to c and thus is a factor of $F^\omega(0)$. This implies that the pre-image $v = f^{-1}(w)$ is a factor of $F^\omega(0)$, and so does every conjugate of v . Thus, $F^\omega(0)$ contains a conjugacy class c' such that the elements of c with prefix 01 are the f -images of the elements of c' . Moreover, the index of c' is strictly smaller than the index of c .

Using this argument recursively, we conclude that every complete conjugacy class in $F^\omega(0)$ has a member of the form $f^i(x)$ such that x is an element of a conjugacy class in C_1 .

Now we show that $F(2)$ does not generate larger conjugacy classes in $F^\omega(0)$. We thus have to exhibit a conjugate of $f(F(2)) = F^4(2) = 0120301240324$ that is not a factor of $F^\omega(0)$. A computer check shows that the conjugate 4012030124032 is not a factor of $F^\omega(0)$. Similarly, $F^2(2)$ does not generate larger conjugacy classes in $F^\omega(0)$ since the conjugate 301203012401203230124032 of $f(F^2(2)) = F^5(2) = 012030124012032301240323$ is not a factor of $F^\omega(0)$. \square

2.2. Avoiding conjugacy classes in w_5

We are ready to prove Theorem 2.1. A computer check¹ shows that w_5 avoids every conjugacy class of length at most 1000. Let us assume that w_5 contains a conjugacy class c of length at least 41. Consider a word $w \in c$ with prefix \mathbf{ab} . Notice that \mathbf{ab} only appears in w_5 as the prefix of the g -image of every letter. Since $|w| \geq 41$, w contains at least 2 occurrences of \mathbf{ab} and we write $w = ps$ such that the rightmost occurrence of \mathbf{ab} in w is the prefix of s . Then the conjugate sp of w also belongs to c and thus is a factor of w_5 . This implies that the pre-image $v = g^{-1}(w)$ is a factor of $F^\omega(0)$, and so does every conjugate of v . Thus, $F^\omega(0)$ contains a conjugacy class c' such that the elements of c with prefix \mathbf{ab} are the f -images of the elements of c' .

To finish the proof, it is thus sufficient to show that for every $c' \in C$, there exists a conjugate of $g(c')$ that is not a factor of w_5 . Recall that $C = \{F(2), F^2(2), F^d(4), f^d(0)\}$ for all $d \geq 1$. The computer check mentioned above settles the case of $F(2)$ and $F^2(2)$ since $|g(F(2))| < |g(F^2(2))| = 40 < 1000$. It also settles the case of $f(4)$ and $f(0)$ since $|g(f(0))| < |g(f(4))| = 90 < 1000$.

The next four lemmas handle the remaining cases (with $d \geq 1$):

- $g(f^d(F(4))) = g(f^d(23))$
- $g(f^d(F^2(4))) = g(f^d(0324))$
- $g(f^{d+1}(4)) = g(f^d(01240323))$
- $g(f^{d+1}(0)) = g(f^d(01203))$

Notice that for technical reasons, we do not consider $g(f(4))$ and $g(f(4))$, which are also covered by the computer check.

Lemma 2.3. *Let $p_{23} = e.g(3f(3) \dots f^{d-1}(3).f^d(3))$ and $s_{23} = g(f^{d-1}(01203).f^{d-2}(01203) \dots f(01203)01203).abcdeacdb$. For every $d \geq 0$, the word $T_{23} = p_{23}s_{23}$ is a conjugate of $g(f^d(23))$ that is not a factor of w_5 .*

Proof. It is easy to check that T_{23} is indeed a conjugate of $g(f^d(23))$. Let us assume that T_{23} appears in w_5 .

The letter 3 in $f^\omega(0)$ appears after either 0 or 2. However \mathbf{e} is a suffix of $g(2)$ and not of $g(0)$. Therefore, $e.g(3)$ is a suffix of $g(23)$ only. Since 23 is a suffix of $f(2)$ and not of $f(0)$, then $g(23f(3))$ is a suffix of $g(f(23))$ only. Using this argument recursively, p_{23} is a suffix of $g(f^d(23))$ only.

Now, the letter 3 in $f^\omega(0)$ appears before either 0 or 2, however $\mathbf{abcdeacdb}$ is a prefix of $g(2)$ and not of $g(0)$. Thus $g(01203).abcdeacdb$ is a prefix of $g(012032)$ only. Since 012032 is a prefix of $f(2)$ and not of $f(0)$, then $g(f(01203)012032)$ is a prefix of $g(f(012032))$ only. Using this argument recursively, s_{23} is a prefix of $g(f^{d-1}(012032))$ only. Thus, if T_{23} is a factor of w_5 , then $g(f^d(232))$ is a factor of w_5 . This is a contradiction since 232 is not a factor of $f^\omega(0)$. \square

Lemma 2.4. *Let $p_{0324} = acdbecd.g(24f(24) \dots f^{d-1}(24).f^d(24))$ and $s_{0324} = g(f^{d-1}(01240) \dots f(01240).01240).abcdbecde$. For every $d \geq 0$, the word $T_{0324} = p_{0324}g(f^d(0))s_{0324}$ is a conjugate of $g(f^d(0324))$ that is not a factor of w_5 .*

Proof. Let us assume that T_{0324} appears in w_5 .

The letter 2 in $f^\omega(0)$ appears after either 1 or 3. However $\mathbf{acdbecd}$ is a suffix of $g(3)$ and not of $g(1)$. Therefore $acdbecd.g(24)$ is a suffix of $g(324)$ only. Since 324 is a suffix of $f(3)$ and not of $f(1)$, then $g(324f(24))$ is a suffix of $g(f(324))$ only. Using this argument recursively, p_{0324} is a suffix of $g(f^d(324))$ only.

Now, the letter 0 in $f^\omega(0)$ appears before either 1 or 3. However $\mathbf{abcdbecde}$ is a prefix of $g(3)$ and not of $g(1)$. Thus $g(01240).abcdbecde$ is a prefix of $g(012403)$ only. Since 012403 is a prefix of $f(3)$ and not of $f(1)$, then $g(f(01240)012403)$ is a prefix of $g(f(012403))$ only. Using this argument recursively, s_{0324} is a prefix of $g(f^{d-1}(012403))$ only. Thus, if T_{0324} is a factor of w_5 , then $g(f^d(32403))$ is a factor of w_5 . This is a contradiction since 32403 is not a factor of $f^\omega(0)$. \square

¹See the program at <http://www.lirmm.fr/~ochem/morphisms/conjugacy.htm>

Lemma 2.5. *Let $p_{01240323} = ecdeacdbe.g(0323f(0323)\dots f^{d-1}(0323).f^d(0323))$ and $s_{01240323} = g(f^d(012)f^{d-1}(012)\dots f(012)012).abcdb$. For every $d \geq 0$, the word $T_{01240323} = p_{01240323}s_{01240323}$ is a conjugate of $g(f^d(01240323))$ that is not a factor of w_5 .*

Proof. Let us assume that $T_{01240323}$ appears in w_5 .

The factor 03 in $f^\omega(0)$ appears after either 2 or 4. However $ecdeacdbe$ is a suffix of $g(4)$ and not of $g(2)$. Therefore $ecdeacdbe.g(0323)$ is a suffix of $g(40323)$ only. Since 40323 is a suffix of $f(4)$ and not of $f(2)$, then $g(40323f(0323))$ is a suffix of $g(f(40323))$, using this argument recursively, $p_{01240323}$ is a suffix of $g(f^d(40323))$ only.

Now, the factor 12 in $f^\omega(0)$ appears before either 0 or 4. However $abcdb$ is a prefix of $g(4)$ and not of $g(0)$. Thus $g(012).abcdb$ must only be a prefix of $g(0124)$ and since 0323 is a prefix of $f(4)$ and not of $f(0)$ then $g(f(012)0124)$ is a prefix of $g(f(0124))$ only. Using this argument recursively, $s_{01240323}$ is a prefix of $g(f^d(0124))$ only. Thus, if $T_{01240323}$ is a factor of w_5 , then $g(f^d(403230124))$ is a factor of w_5 . This is a contradiction since 403230124 is not a factor of $f^\omega(0)$. \square

Lemma 2.6. *Let $p_{01203} = d.g(3f(3)\dots f^{d-1}(3).f^d(3))$ and $s_{01203} = g(f^d(012)f^{d-1}(012).f^{d-2}(012)\dots f(012)012).abcdeac$. For every $d \geq 0$, the word $T_{01203} = p_{01203}s_{01203}$ is a conjugate of $g(f^d(01203))$ that is not a factor of w_5 .*

Proof. Let us assume that T_{01203} appears in w_5 .

The letter 3 in $f^\omega(0)$ appears after either 0 or 2. however d is a suffix of $g(0)$ and not of $g(2)$. Therefore $d.g(2)$ is a suffix of $g(12)$ only. Since 12 is a suffix of $f(1)$ and not of $f(3)$, then $g(12f(2))$ is a suffix of $g(f(12))$ only. Using this argument recursively, p_{01203} is a suffix of $g(f^d(12))$ only.

Now, 012 in $f^\omega(0)$ appears before either 1 or 4, however $abcdeac$ is only a prefix of $g(1)$ and not of $g(4)$. Thus $g(012).abcdeac$ is a prefix of $g(0120)$ only. Since 0120 is a prefix of $f(1)$ and not of $f(4)$, then $g(f(012)0120)$ is a prefix of $g(f(0120))$ only. Using this argument recursively, s_{01203} is a prefix of $g(f^d(0120))$. Thus, if T_{01203} is a factor of w_5 , then $g(f^d(030120))$ is a factor of w_5 . This is a contradiction since 030120 is not a factor of $f^\omega(0)$. \square

REFERENCES

- [1] K.A. Baker, G.F. McNulty and W. Taylor, Growth problems for avoidable words. *Theoret. Comput. Sci.* **69** (1989) 19–345.
- [2] D.R. Bean, A. Ehrenfeucht and G.F. McNulty, Avoidable patterns in strings of symbols. *Pacific J. Math.* **85** (1979) 261–294.
- [3] J.P. Bell and B.W. Madill, Iterative Algebras. *Algebr. Represent. Theor.* **18** (2015) 1533–1546.
- [4] R.J. Clark, *Avoidable formulas in combinatorics on words*. Ph.D. thesis, University of California, Los Angeles. Available at http://www.lirmm.fr/~ochem/morphisms/clark_thesis.pdf (2001).
- [5] G. Gamard, P. Ochem, G. Richomme and P. Séebold, Avoidability of circular formulas. *Theoret. Comput. Sci.* **726** (2018) 1–4.
- [6] L. Ilie, P. Ochem and J.O. Shallit, A generalization of repetition threshold. *Theoret. Comput. Sci.* **92** (2004) 71–76.
- [7] P. Ochem, A generator of morphisms for infinite words. *RAIRO: ITA* **40** (2006) 427–441.
- [8] A.I. Zimin, Blocking sets of terms. *Math. USSR Sbornik* **47** (1984) 353–364.