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## AVOIDING CONJUGACY CLASSES ON THE 5-LETTER ALPHABET

GOLNAZ BADKOBEB\* AND PASCAL OCHEM<sup>†,‡</sup>

**Abstract.** We construct an infinite word  $w$  over the 5-letter alphabet such that for every factor  $f$  of  $w$  of length at least two, there exists a cyclic permutation of  $f$  that is not a factor of  $w$ . In other words,  $w$  does not contain a non-trivial conjugacy class. This proves the conjecture in Gamard *et al.* [*Theoret. Comput. Sci.* **726** (2018) 1–4].

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### 1. INTRODUCTION

A *pattern*  $p$  is a non-empty finite word over an alphabet  $\Delta = \{A, B, C, \dots\}$  of capital letters called *variables*. An *occurrence* of  $p$  in a word  $w$  is a non-erasing morphism  $h : \Delta^* \rightarrow \Sigma^*$  such that  $h(p)$  is a factor of  $w$ . The *avoidability index*  $\lambda(p)$  of a pattern  $p$  is the size of the smallest alphabet  $\Sigma$  such that there exists an infinite word over  $\Sigma$  containing no occurrence of  $p$ . Bean *et al.* [2] and Zimin [8] characterized unavoidable patterns, *i.e.*, such that  $\lambda(p) = \infty$ . However, determining the exact avoidability index of an avoidable pattern requires more work. Although patterns with index 4 [2] and 5 [4] have been found, the existence of an avoidable pattern with index at least 6 is an open problem since 2001.

Some techniques in pattern avoidance start by showing that the considered word avoids other structures, such as generalized repetitions [6, 7]. Let us say that a word has property  $P_i$  if it does not contain all the conjugates of the same word  $w$  with  $|w| \geq i$ . Recently, in order to study the avoidance of a kind of patterns called circular formulas, Gamard *et al.* [5] obtained that there exists

- a morphic binary word satisfying  $P_5$ ,
- a morphic ternary word satisfying  $P_3$ ,
- a morphic word over the 6-letter alphabet satisfying  $P_2$ .

Recall that a pure morphic word is of the form  $m^\omega(0)$  and a morphic word is of the form  $h(m^\omega(0))$  for some morphisms  $m$  and  $h$ . Independently, Bell and Madill [3] obtained a pure morphic word over the 12-letter alphabet that also satisfies  $P_2$  and some other properties.

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It is conjectured that the smallest alphabet allowing an infinite word satisfying  $P_2$  has 5 letters [5], which is best possible. In this paper, we prove this conjecture using a morphic word. This settles the topic of the smallest alphabet needed to satisfy  $P_i$ .

## 2. MAIN RESULT

Let  $\varepsilon$  denote the empty word. We consider the morphic word  $w_5 = G(F^\omega(0))$  defined by the following morphisms.

$$\begin{array}{ll} F(0) = 01, & G(0) = abcd, \\ F(1) = 2, & G(1) = \varepsilon, \\ F(2) = 03, & G(2) = eacd, \\ F(3) = 24, & G(3) = becd, \\ F(4) = 23. & G(4) = be. \end{array}$$

**Theorem 2.1.** *The morphic word  $w_5 \in \Sigma_5^*$  avoids every conjugacy class of length at least 2.*

In order to prove this theorem, it is convenient to express  $w_5$  with the larger morphisms  $f = F^3$  and  $g = G \circ F^2$  given below. Clearly,  $w_5 = g(f^\omega(0))$ .

$$\begin{array}{ll} f(0) = 01203, & g(0) = abcdeacd, \\ f(1) = 0124, & g(1) = abcdbecd, \\ f(2) = 0120323, & g(2) = abcdeacdbe, \\ f(3) = 01240324, & g(3) = abcdbecdeacdbeacd, \\ f(4) = 01240323. & g(4) = abcdbecdeacdbe. \end{array}$$

### 2.1. Avoiding conjugacy classes in $F^\omega(0)$

Here we study the pure morphic word and the conjugacy classes it contains.

**Lemma 2.2.** *The infinite word  $F^\omega(0)$  contains only the conjugacy classes listed in  $C = \{F(2), F^2(2), F^d(4), f^d(0)\}$ , for all  $d \geq 1$ .*

*Proof.* Notice that the factor 01 only occurs as the prefix of the  $f$ -image of every letter in  $F^\omega(0)$ . Moreover, every letter 1 only occurs in  $F^\omega(0)$  as the suffix of the factor 01. Let us say that the *index* of a conjugacy class is the number of occurrences of 1 in any of its elements. An easy computation shows that the set of complete conjugacy classes in  $F^\omega(0)$  with index at most one is  $C_1 = \{F(2), F^2(2), F(4), F^2(4), f(4), f(0)\}$ . Let us assume that  $F^\omega(0)$  contains a conjugacy class  $c$  with index at least two. Let  $w \in c$  be such that 01 is a prefix of  $w$ . We write  $w = ps$  such that the leftmost occurrence of 01 in  $w$  is the prefix of  $s$ . Then the conjugate  $sp$  of  $w$  also belongs to  $c$  and thus is a factor of  $F^\omega(0)$ . This implies that the pre-image  $v = f^{-1}(w)$  is a factor of  $F^\omega(0)$ , and so does every conjugate of  $v$ . Thus,  $F^\omega(0)$  contains a conjugacy class  $c'$  such that the elements of  $c$  with prefix 01 are the  $f$ -images of the elements of  $c'$ . Moreover, the index of  $c'$  is strictly smaller than the index of  $c$ .

Using this argument recursively, we conclude that every complete conjugacy class in  $F^\omega(0)$  has a member of the form  $f^i(x)$  such that  $x$  is an element of a conjugacy class in  $C_1$ .

Now we show that  $F(2)$  does not generate larger conjugacy classes in  $F^\omega(0)$ . We thus have to exhibit a conjugate of  $f(F(2)) = F^4(2) = 0120301240324$  that is not a factor of  $F^\omega(0)$ . A computer check shows that the conjugate 4012030124032 is not a factor of  $F^\omega(0)$ . Similarly,  $F^2(2)$  does not generate larger conjugacy classes in  $F^\omega(0)$  since the conjugate 301203012401203230124032 of  $f(F^2(2)) = F^5(2) = 012030124012032301240323$  is not a factor of  $F^\omega(0)$ .  $\square$

## 2.2. Avoiding conjugacy classes in $w_5$

We are ready to prove Theorem 2.1. A computer check<sup>1</sup> shows that  $w_5$  avoids every conjugacy class of length at most 1000. Let us assume that  $w_5$  contains a conjugacy class  $c$  of length at least 41. Consider a word  $w \in c$  with prefix  $\mathbf{ab}$ . Notice that  $\mathbf{ab}$  only appears in  $w_5$  as the prefix of the  $g$ -image of every letter. Since  $|w| \geq 41$ ,  $w$  contains at least 2 occurrences of  $\mathbf{ab}$  and we write  $w = ps$  such that the rightmost occurrence of  $\mathbf{ab}$  in  $w$  is the prefix of  $s$ . Then the conjugate  $sp$  of  $w$  also belongs to  $c$  and thus is a factor of  $w_5$ . This implies that the pre-image  $v = g^{-1}(w)$  is a factor of  $F^\omega(0)$ , and so does every conjugate of  $v$ . Thus,  $F^\omega(0)$  contains a conjugacy class  $c'$  such that the elements of  $c$  with prefix  $\mathbf{ab}$  are the  $f$ -images of the elements of  $c'$ .

To finish the proof, it is thus sufficient to show that for every  $c' \in C$ , there exists a conjugate of  $g(c')$  that is not a factor of  $w_5$ . Recall that  $C = \{F(2), F^2(2), F^d(4), f^d(0)\}$  for all  $d \geq 1$ . The computer check mentioned above settles the case of  $F(2)$  and  $F^2(2)$  since  $|g(F(2))| < |g(F^2(2))| = 40 < 1000$ . It also settles the case of  $f(4)$  and  $f(0)$  since  $|g(f(0))| < |g(f(4))| = 90 < 1000$ .

The next four lemmas handle the remaining cases (with  $d \geq 1$ ):

- $g(f^d(F(4))) = g(f^d(23))$
- $g(f^d(F^2(4))) = g(f^d(0324))$
- $g(f^{d+1}(4)) = g(f^d(01240323))$
- $g(f^{d+1}(0)) = g(f^d(01203))$

Notice that for technical reasons, we do not consider  $g(f(4))$  and  $g(f(4))$ , which are also covered by the computer check.

**Lemma 2.3.** *Let  $p_{23} = e.g(3f(3) \dots f^{d-1}(3).f^d(3))$  and  $s_{23} = g(f^{d-1}(01203).f^{d-2}(01203) \dots f(01203)01203).abcdeacdb$ . For every  $d \geq 0$ , the word  $T_{23} = p_{23}s_{23}$  is a conjugate of  $g(f^d(23))$  that is not a factor of  $w_5$ .*

*Proof.* It is easy to check that  $T_{23}$  is indeed a conjugate of  $g(f^d(23))$ . Let us assume that  $T_{23}$  appears in  $w_5$ .

The letter 3 in  $f^\omega(0)$  appears after either 0 or 2. However  $\mathbf{e}$  is a suffix of  $g(2)$  and not of  $g(0)$ . Therefore,  $e.g(3)$  is a suffix of  $g(23)$  only. Since  $23$  is a suffix of  $f(2)$  and not of  $f(0)$ , then  $g(23f(3))$  is a suffix of  $g(f(23))$  only. Using this argument recursively,  $p_{23}$  is a suffix of  $g(f^d(23))$  only.

Now, the letter 3 in  $f^\omega(0)$  appears before either 0 or 2, however  $\mathbf{abcdeacdb}$  is a prefix of  $g(2)$  and not of  $g(0)$ . Thus  $g(01203).abcdeacdb$  is a prefix of  $g(012032)$  only. Since  $012032$  is a prefix of  $f(2)$  and not of  $f(0)$ , then  $g(f(01203)012032)$  is a prefix of  $g(f(012032))$  only. Using this argument recursively,  $s_{23}$  is a prefix of  $g(f^{d-1}(012032))$  only. Thus, if  $T_{23}$  is a factor of  $w_5$ , then  $g(f^d(232))$  is a factor of  $w_5$ . This is a contradiction since  $232$  is not a factor of  $f^\omega(0)$ .  $\square$

**Lemma 2.4.** *Let  $p_{0324} = acdbecd.g(24f(24) \dots f^{d-1}(24).f^d(24))$  and  $s_{0324} = g(f^{d-1}(01240) \dots f(01240).01240).abcdbecde$ . For every  $d \geq 0$ , the word  $T_{0324} = p_{0324}g(f^d(0))s_{0324}$  is a conjugate of  $g(f^d(0324))$  that is not a factor of  $w_5$ .*

*Proof.* Let us assume that  $T_{0324}$  appears in  $w_5$ .

The letter 2 in  $f^\omega(0)$  appears after either 1 or 3. However  $\mathbf{acdbecd}$  is a suffix of  $g(3)$  and not of  $g(1)$ . Therefore  $acdbecd.g(24)$  is a suffix of  $g(324)$  only. Since  $324$  is a suffix of  $f(3)$  and not of  $f(1)$ , then  $g(324f(24))$  is a suffix of  $g(f(324))$  only. Using this argument recursively,  $p_{0324}$  is a suffix of  $g(f^d(324))$  only.

Now, the letter 0 in  $f^\omega(0)$  appears before either 1 or 3. However  $\mathbf{abcdbecde}$  is a prefix of  $g(3)$  and not of  $g(1)$ . Thus  $g(01240).abcdbecde$  is a prefix of  $g(012403)$  only. Since  $012403$  is a prefix of  $f(3)$  and not of  $f(1)$ , then  $g(f(01240)012403)$  is a prefix of  $g(f(012403))$  only. Using this argument recursively,  $s_{0324}$  is a prefix of  $g(f^{d-1}(012403))$  only. Thus, if  $T_{0324}$  is a factor of  $w_5$ , then  $g(f^d(32403))$  is a factor of  $w_5$ . This is a contradiction since  $32403$  is not a factor of  $f^\omega(0)$ .  $\square$

<sup>1</sup>See the program at <http://www.lirmm.fr/~ochem/morphisms/conjugacy.htm>

**Lemma 2.5.** *Let  $p_{01240323} = ecdeacdbe.g(0323f(0323)\dots f^{d-1}(0323).f^d(0323))$  and  $s_{01240323} = g(f^d(012)f^{d-1}(012)\dots f(012)012).abcdb$ . For every  $d \geq 0$ , the word  $T_{01240323} = p_{01240323}s_{01240323}$  is a conjugate of  $g(f^d(01240323))$  that is not a factor of  $w_5$ .*

*Proof.* Let us assume that  $T_{01240323}$  appears in  $w_5$ .

The factor 03 in  $f^\omega(0)$  appears after either 2 or 4. However  $ecdeacdbe$  is a suffix of  $g(4)$  and not of  $g(2)$ . Therefore  $ecdeacdbe.g(0323)$  is a suffix of  $g(40323)$  only. Since 40323 is a suffix of  $f(4)$  and not of  $f(2)$ , then  $g(40323f(0323))$  is a suffix of  $g(f(40323))$ , using this argument recursively,  $p_{01240323}$  is a suffix of  $g(f^d(40323))$  only.

Now, the factor 12 in  $f^\omega(0)$  appears before either 0 or 4. However  $abcdb$  is a prefix of  $g(4)$  and not of  $g(0)$ . Thus  $g(012).abcdb$  must only be a prefix of  $g(0124)$  and since 0323 is a prefix of  $f(4)$  and not of  $f(0)$  then  $g(f(012)0124)$  is a prefix of  $g(f(0124))$  only. Using this argument recursively,  $s_{01240323}$  is a prefix of  $g(f^d(0124))$  only. Thus, if  $T_{01240323}$  is a factor of  $w_5$ , then  $g(f^d(403230124))$  is a factor of  $w_5$ . This is a contradiction since 403230124 is not a factor of  $f^\omega(0)$ .  $\square$

**Lemma 2.6.** *Let  $p_{01203} = d.g(3f(3)\dots f^{d-1}(3).f^d(3))$  and  $s_{01203} = g(f^d(012)f^{d-1}(012).f^{d-2}(012)\dots f(012)012).abcdeac$ . For every  $d \geq 0$ , the word  $T_{01203} = p_{01203}s_{01203}$  is a conjugate of  $g(f^d(01203))$  that is not a factor of  $w_5$ .*

*Proof.* Let us assume that  $T_{01203}$  appears in  $w_5$ .

The letter 3 in  $f^\omega(0)$  appears after either 0 or 2. however  $d$  is a suffix of  $g(0)$  and not of  $g(2)$ . Therefore  $d.g(2)$  is a suffix of  $g(12)$  only. Since 12 is a suffix of  $f(1)$  and not of  $f(3)$ , then  $g(12f(2))$  is a suffix of  $g(f(12))$  only. Using this argument recursively,  $p_{01203}$  is a suffix of  $g(f^d(12))$  only.

Now, 012 in  $f^\omega(0)$  appears before either 1 or 4, however  $abcdeac$  is only a prefix of  $g(1)$  and not of  $g(4)$ . Thus  $g(012).abcdeac$  is a prefix of  $g(0120)$  only. Since 0120 is a prefix of  $f(1)$  and not of  $f(4)$ , then  $g(f(012)0120)$  is a prefix of  $g(f(0120))$  only. Using this argument recursively,  $s_{01203}$  is a prefix of  $g(f^d(0120))$ . Thus, if  $T_{01203}$  is a factor of  $w_5$ , then  $g(f^d(030120))$  is a factor of  $w_5$ . This is a contradiction since 030120 is not a factor of  $f^\omega(0)$ .  $\square$

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