About the unification type of modal logics between KB and KTB

Philippe Balbiani, Cigdem Gencer

Philippe Balbiani
Çiğdem Gencer

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Abstract. The unification problem in a normal modal logic is to determine, given a formula φ, whether there exists a substitution σ such that $\sigma(\varphi)$ is in that logic. In that case, σ is a unifier of φ. We shall say that a set of unifiers of a unifiable formula φ is minimal complete if for all unifiers σ of φ, there exists a unifier τ of φ in that set such that τ is more general than σ and for all σ, τ in that set, $\sigma \neq \tau$, neither σ is more general than τ, nor τ is more general than σ. When a unifiable formula has no minimal complete set of unifiers, the formula is nullary. We usually distinguish between elementary unification and unification with parameters. In elementary unification, all variables are likely to be replaced by formulas when one applies a substitution. In unification with parameters, some variables—called parameters—remain unchanged. In this paper, we prove that normal modal logics **KB**, **KDB** and **KTB** as well as infinitely many normal modal logics between **KDB** and **KTB** possess nullary formulas for unification with parameters.

Keywords: Normal modal logics **KB**, **KDB** and **KTB**, Unification with parameters, Unification type.

1. Introduction

The unification problem in a normal modal logic[1] is to determine, given a formula φ, whether there exists a substitution σ such that $\sigma(\varphi)$ is in that logic. In that case, σ is a unifier of φ. We shall say that a set of unifiers of a formula φ is minimal complete if for all unifiers σ of φ, there exists a unifier τ of φ in that set such that $\tau \preceq \sigma$, i.e. τ is more general than σ, and for all σ, τ in that set, $\sigma \neq \tau$, neither $\sigma \preceq \tau$, nor $\tau \preceq \sigma$, i.e. neither σ is more general than τ, nor τ is more general than σ. An important question is the following [1,12]: when a formula is unifiable, has it a minimal complete set of unifiers? When the answer is "no", the formula is nullary. When the answer is "yes",

[1] In this paper, all modal logics are normal.

the formula is either unitary, or finitary, or infinitary depending on the cardinality of its minimal complete sets of unifiers.[2] A modal logic is called nullary if it possesses a nullary unifiable formula. Otherwise, it is called either unitary, or finitary, or infinitary depending on the types of its unifiable formulas. We usually distinguish between elementary unification and unification with parameters. In elementary unification, all variables are likely to be replaced by formulas when one applies a substitution. In unification with parameters, some variables—called parameters—remain unchanged.

The problem of checking the unifiability of formulas is a special case of the problem of checking the admissibility of inference rules [20]. Intuitively, for an axiomatically presented modal logic, the admissibility problem asks whether a given inference rule can be added to the axiomatization of the logic without changing the associated set of derivable formulas. Its computability has been studied—for a limited number of transitive modal logics like **K**4 and **S**4—by Jeřábek [15,17] and Rybakov [19]. Aside from these transitive modal logics and for the extensions of **S**5, it is still unknown for numerous modal logics—for example **K**, **KD** and **KT**—whether the problem of checking the admissibility of inference rules is solvable.[3] The significance of the unification type in the research on the problem of checking the unifiability of formulas stems from the fact that if a modal logic is either unitary, or finitary and minimal complete sets of unifiers can be effectively computed then the problem of checking the admissibility of inference rules can be reduced to the problem of checking the unifiability of formulas. See [1,12] for details.

Coming back to unification types in modal logics, it is known that **S**5 is unitary [1, Theorem 3], **KT** is nullary [3, Corollary 8.4], **KD** is nullary [4, Proposition 9], $\mathbf{Alt}_1$ is nullary [7, Proposition 7.7], **KD**45 is unitary [8, Corollary 42], **S**4.3 is unitary [13, Theorem 3.18], **K** is nullary [16, Theorem 3.8] and some transitive modal logics like **K**4 and **S**4 are finitary [14, Theorem 3.5], though **KD**45 has been proved to be unitary only within the context of elementary unification and **KT** and **KD** have been proved to be nullary only within the context of unification with parameters. Taking a look at the literature about unification types in modal logics [1,12], one will quickly note that much remains to be done. For example, the types of simple

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[2]It can be easily proved that if a unifiable formula has several minimal complete sets of unifiers then these sets have the same cardinality.

[3]We follow the same conventions as in [9–11] for talking about modal logics: **S**5 is the least modal logic containing the formulas usually denoted (**T**), (**4**) and (**B**), **KT** is the least modal logic containing the formula usually denoted (**T**), etc.

Church-Rosser modal logics like **KG**, **KDG** and **KTG** are unknown. Even, the type of the least modal logic containing $\Box^a\bot$ is unknown when $a \in \mathbb{N}$ is such that $a \geq 2$. For more on this, see [5].

The argument of Jeřábek [16] proving that **K** is nullary can be explained as follows. Consider the formula $(x \to \Box x)$. It is **K**-unifiable: substitutions like $\sigma_\top(x) = \top$ and $\sigma_d(x) = (\bigwedge\{\Box^l x : l < d\} \wedge \Box^d\bot)$ for each $d \in \mathbb{N}$ are unifiers of it. However, it is nullary. Let us see why. Firstly, for all unifiers τ of $(x \to \Box x)$, either $\tau(x)$ is in **K**, or $(\tau(x) \to \Box^{\mathtt{deg}(\tau(x))}\bot)$ is in **K**—where for all formulas φ, $\mathtt{deg}(\varphi)$ denotes the degree of φ. This is a consequence of the fact that **K** satisfies the following variant of the rule of margins: for all formulas φ, if $(\varphi \to \Box\varphi)$ is in **K** then either φ is in **K**, or $(\varphi \to \Box^{\mathtt{deg}(\varphi)}\bot)$ is in **K**. Secondly, one can prove that for all substitutions τ, $\tau(x)$ is in **K** iff $\sigma_\top \preceq \tau$ and for all unifiers τ of $(x \to \Box x)$, $(\tau(x) \to \Box^{\mathtt{deg}(\tau(x))}\bot)$ is in **K** iff $\sigma_{\mathtt{deg}(\tau(x))} \preceq \tau$. Thirdly, one can prove that for all $d \in \mathbb{N}$, $\sigma_{d+1} \preceq \sigma_d$. As a conclusion, there exists no minimal complete set of unifiers of $(x \to \Box x)$. Indeed, suppose Σ is a minimal complete set of unifiers of $(x \to \Box x)$. Since Σ is minimal complete, therefore let $\tau \in \Sigma$ be such that $\tau \preceq \sigma_0$. Since Σ is a set of unifiers of $(x \to \Box x)$, therefore either $\tau(x)$ is in **K**, or $(\tau(x) \to \Box^{\mathtt{deg}(\tau(x))}\bot)$ is in **K**. In the former case, $\sigma_\top \preceq \tau$. Since $\tau \preceq \sigma_0$, therefore $\sigma_\top \preceq \sigma_0$. Hence, $(\lambda(\sigma_\top(x)) \leftrightarrow \sigma_0(x))$ is in **K** for some substitution λ. Thus, $(\top \to \bot)$ is in **K**: a contradiction. In the latter case, since Σ is a set of unifiers of $(x \to \Box x)$, therefore $\sigma_{\mathtt{deg}(\tau(x))} \preceq \tau$. Since Σ is minimal complete, therefore let $\mu \in \Sigma$ be such that $\mu \preceq \sigma_{\mathtt{deg}(\tau(x))+1}$. Since $\sigma_{\mathtt{deg}(\tau(x))+1} \preceq \sigma_{\mathtt{deg}(\tau(x))}$ and $\sigma_{\mathtt{deg}(\tau(x))} \preceq \tau$, therefore $\mu \preceq \tau$. Since Σ is minimal complete, therefore $\mu = \tau$. Since $\sigma_{\mathtt{deg}(\tau(x))} \preceq \tau$ and $\mu \preceq \sigma_{\mathtt{deg}(\tau(x))+1}$, therefore $\sigma_{\mathtt{deg}(\tau(x))} \preceq \sigma_{\mathtt{deg}(\tau(x))+1}$. Consequently, $(\lambda(\sigma_{\mathtt{deg}(\tau(x))}(x)) \leftrightarrow \sigma_{\mathtt{deg}(\tau(x))+1}(x))$ is in **K** for some substitution λ. Hence, $(\Box^{\mathtt{deg}(\tau(x))+1}\bot \to \Box^{\mathtt{deg}(\tau(x))}\bot)$ is in **K**: a contradiction.

Because of the strong proximity between the modal logics **K**, **KD**, **KT**, **KB**, **KDB** and **KTB** in terms of axiomatization and decidability, the reader may wonder whether Jeřábek's line of reasoning can be used as it is for **KD**, **KT**, **KB**, **KDB** and **KTB**. Obviously, in this line of reasoning, the formulas $\Box^d\bot$ for each $d \in \mathbb{N}$ play an important role—as well as the fact that for all $d \in \mathbb{N}$, $(\Box^{d+1}\bot \to \Box^d\bot)$ is not valid. Unfortunately, when $d \in \mathbb{N}$ is such that $d \geq 1$, $\Box^d\bot$ is equivalent to $\bot$ in **KD**, **KT**, **KDB** and **KTB** and is equivalent to $\Box\bot$ in **KB**. It follows that Jeřábek's line of reasoning has to be seriously adapted if one wants to apply it to **KD**, **KT**, **KB**, **KDB** and **KTB**. Using a parameter p, Balbiani and Gencer [4] have proved that the formula $(x \to (p \wedge \Box(p \to x)))$ is nullary within the context of **KD**. Using distinct parameters p, q, Balbiani [3] has proved that the formula

$((x \to (p \wedge \Box(q \to y))) \wedge (y \to (q \wedge \Box(p \to x))))$ is nullary within the context of **KT**. In this paper, answering to questions put forward by Dzik [12, Chapter 5], we adapt to **KB**, **KDB** and **KTB** as well as to infinitely many modal logics between **KDB** and **KTB** the argument proving that **K** is nullary, though these modal logics will be proved to be nullary only within the context of unification with parameters. We assume the reader is at home with tools and techniques in modal logics. For more on this, see [9–11].

2. Syntax

In this section, we present the syntax of modal logics.

Formulas. Let VAR be a non-empty countable set of *propositional variables* (with typical members denoted x, y, etc) and PAR be a non-empty countable set of *propositional parameters* (with typical members denoted p, q, etc). *Atoms* (denoted α, β, etc) are either variables, or parameters. The set FOR of all *formulas* (with typical members denoted φ, ψ, etc) is inductively defined as follows:

- $\varphi, \psi ::= \alpha \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Box\varphi$.

The Boolean connectives $\top$, $\wedge$, $\to$ and $\leftrightarrow$ are defined by the usual abbreviations. We adopt the standard rules for omission of the parentheses. The modal connective $\Diamond$ is defined as follows:

- $\Diamond\varphi ::= \neg\Box\neg\varphi$.

For all $a \in \mathbb{N}$, the modal connectives $\Box^a$ and $\Diamond^a$ are inductively defined as follows:

- $\Box^0\varphi ::= \varphi$,
- $\Box^{a+1}\varphi ::= \Box\Box^a\varphi$,
- $\Diamond^0\varphi ::= \varphi$,
- $\Diamond^{a+1}\varphi ::= \Diamond\Diamond^a\varphi$.

For all formulas φ, we write "φ^0" to mean "$\neg\varphi$" and we write "φ^1" to mean "φ". For all sets s of formulas, let $\Box s = \{\varphi : \Box\varphi \in s\}$. From now on in this paper,

$$\boxed{\textbf{let } p, q \textbf{ be fixed distinct parameters.}}$$

Let $\boxplus$ and $\boxminus$ be the modal connectives defined as follows:

- $\boxplus\varphi ::= (p^0 \wedge q^0 \rightarrow \Box(p^1 \wedge q^0 \rightarrow \Box(p^0 \wedge q^1 \rightarrow \Box(p^0 \wedge q^0 \rightarrow \varphi))))$,
- $\boxminus\varphi ::= (p^0 \wedge q^0 \rightarrow \Box(p^0 \wedge q^1 \rightarrow \Box(p^1 \wedge q^0 \rightarrow \Box(p^0 \wedge q^0 \rightarrow \varphi))))$.

For all $k \in \mathbb{N}$, the modal connectives $\boxplus^k$ and $\boxminus^k$ are inductively defined as follows:

- $\boxplus^0\varphi ::= \varphi$,
- $\boxplus^{k+1}\varphi ::= \boxplus\boxplus^k\varphi$,
- $\boxminus^0\varphi ::= \varphi$,
- $\boxminus^{k+1}\varphi ::= \boxminus\boxminus^k\varphi$.

For all $k \in \mathbb{N}$, the modal connectives $\boxplus^{<k}$ and $\boxminus^{<k}$ are inductively defined as follows:

- $\boxplus^{<0}\varphi ::= \top$,
- $\boxplus^{<k+1}\varphi ::= (\boxplus^{<k}\varphi \wedge \boxplus^k\varphi)$,
- $\boxminus^{<0}\varphi ::= \top$,
- $\boxminus^{<k+1}\varphi ::= (\boxminus^{<k}\varphi \wedge \boxminus^k\varphi)$.

For all formulas φ, let $\mathtt{var}(\varphi)$ be the set of all variables occurring in φ.

Degrees. The *degree* of a formula φ (in symbols $\mathtt{deg}(\varphi)$) is the nonnegative integer inductively defined as follows:

- $\mathtt{deg}(\alpha) = 0$,
- $\mathtt{deg}(\bot) = 0$,
- $\mathtt{deg}(\neg\varphi) = \mathtt{deg}(\varphi)$,
- $\mathtt{deg}(\varphi \vee \psi) = \max\{\mathtt{deg}(\varphi), \mathtt{deg}(\psi)\}$,
- $\mathtt{deg}(\Box\varphi) = \mathtt{deg}(\varphi) + 1$.

LEMMA 1. *Let φ be a formula.*

1. $\mathtt{deg}(\boxplus(\varphi) = \mathtt{deg}(\varphi) + 3$,
2. $\mathtt{deg}(\boxminus(\varphi) = \mathtt{deg}(\varphi) + 3$,
3. *for all $k \in \mathbb{N}$,* $\mathtt{deg}(\boxplus^k\varphi) = \mathtt{deg}(\varphi) + 3k$,
4. *for all $k \in \mathbb{N}$,* $\mathtt{deg}(\boxminus^k\varphi) = \mathtt{deg}(\varphi) + 3k$,
5. *for all $k \in \mathbb{N}$, if $k = 0$ then* $\mathtt{deg}(\boxplus^{<k}\varphi) = 0$ *else* $\mathtt{deg}(\boxplus^{<k}\varphi) = \mathtt{deg}(\varphi) + 3(k-1)$,

6. *for all* $k \in \mathbb{N}$*, if* $k = 0$ *then* $\mathtt{deg}(\boxminus^{<k}\varphi) = 0$ *else* $\mathtt{deg}(\boxminus^{<k}\varphi) = \mathtt{deg}(\varphi) + 3(k-1)$.

PROOF. (1) and (2): Left to the reader.
(3)–(6): By induction on k. ■

Substitutions. A *substitution* is a function σ associating to each variable x a formula $\sigma(x)$. We shall say that a substitution σ *moves* a variable x if $\sigma(x) \neq x$. Following the standard assumption considered in the literature about the unification problem in modal logics [1,12], we will always assume that substitutions move at most finitely many variables. For all formulas $\varphi(x_1, \ldots, x_m, p_1, \ldots, p_n)$, let $\sigma(\varphi(x_1, \ldots, x_m, p_1, \ldots, p_n))$ be $\varphi(\sigma(x_1), \ldots, \sigma(x_m), p_1, \ldots, p_n)$. The *composition* $\sigma \circ \tau$ of the substitutions σ and τ is the substitution associating to each variable x the formula $\tau(\sigma(x))$.

3. Semantics

In this section, we present the semantics of modal logics.

Frames and Models. A *frame* is a couple $F = (W, R)$ where W is a nonempty set of *states* (with typical members denoted s, t, etc) and R is a binary relation on W. Let $F = (W, R)$ be a frame. For all $a \in \mathbb{N}$, let R^a be the binary relation on W inductively defined as follows:

- sR^0t iff $s = t$,
- $sR^{a+1}t$ iff there exists $u \in W$ such that sRu and uR^at.

We shall say that F is *symmetric* if for all $s, t \in W$, if sRt then tRs. We shall say that F is *serial* if for all $s \in W$, there exists $t \in W$ such that sRt and F is *reflexive* if for all $s \in W$, sRs. Note that if F is reflexive then F is serial. For all $a \in \mathbb{N}$, we shall say that F is *a-reflexive* if for all $s \in W$, there exists $t_0, \ldots, t_a \in W$ such that $t_0 = s$, t_aRt_a and for all $i \in \mathbb{N}$, if $i < a$ then t_iRt_{i+1}. Note that F is 0-reflexive iff F is reflexive. Moreover, for all $a \in \mathbb{N}$, if F is a-reflexive then F is serial and F is $(a+1)$-reflexive. A *model* based on F is a triple $M = (W, R, V)$ where V is a function assigning to each atom α a subset $V(\alpha)$ of W. Given a model $M = (W, R, V)$, the *satisfiability* of a modal formula φ at $s \in W$ (in symbols $M, s \models \varphi$) is inductively defined as follows:

- $M, s \models \alpha$ iff $s \in V(\alpha)$,
- $M, s \not\models \bot$,

- $M, s \models \neg\varphi$ iff $M, s \not\models \varphi$,
- $M, s \models \varphi \vee \psi$ iff either $M, s \models \varphi$, or $M, s \models \psi$,
- $M, s \models \Box\varphi$ iff for all $t \in W$, if sRt then $M, t \models \varphi$.

Truth and Validity. We shall say that a formula φ is *true* in a model $M = (W, R, V)$ if φ is satisfied at all $s \in W$. We shall say that a formula φ is *valid* in a frame F if φ is true in all models based on F. We shall say that a formula φ is *valid* in a class C of frames if φ is valid in all frames of C. From now on in this paper,

$$\boxed{\textbf{we write ``frame'' to mean ``symmetric frame''.}}$$

Let $\mathbf{KB}$ be the set of all formulas valid in the class of all frames. Let $\mathbf{KDB}$ be the set of all formulas valid in the class of all serial frames and $\mathbf{KTB}$ be the set of all formulas valid in the class of all reflexive frames. Obviously, $\mathbf{KTB} \supseteq \mathbf{KDB}$. For all $a \in \mathbb{N}$, let $\mathbf{L}_a$ be the set of all formulas valid in the class of all a-reflexive frames. Note that $\mathbf{L}_0 = \mathbf{KTB}$. Moreover, for all $a \in \mathbb{N}$, $\mathbf{L}_a \supseteq \mathbf{KDB}$ and $\mathbf{L}_a \supseteq \mathbf{L}_{a+1}$. The truth is that

Proposition 1. *1.* $\mathbf{KB}$ *is the least modal logic containing all formulas of the form* $\varphi \to \Box\Diamond\varphi$*,*

2. $\mathbf{KDB}$ *is the least modal logic containing all formulas of the form* $\Box\varphi \to \Diamond\varphi$ *and* $\varphi \to \Box\Diamond\varphi$ *and* $\mathbf{KTB}$ *is the least modal logic containing all formulas of the form* $\Box\varphi \to \varphi$ *and* $\varphi \to \Box\Diamond\varphi$*.*

Proof. This is a standard result. See [11, Theorem 5.14]. ■

In other respect,

Proposition 2. *For all* $a \in \mathbb{N}$*,* $\mathbf{L}_a$ *is the least modal logic containing all formulas of the form* $\Diamond^a((\Box\varphi_1 \to \varphi_1) \wedge \ldots \wedge (\Box\varphi_n \to \varphi_n))$ *(where* n *ranges over* $\mathbb{N}$*) and* $\varphi \to \Box\Diamond\varphi$*.*

Proof. Let $a \in \mathbb{N}$. Let $\mathbf{L}'_a$ be the least modal logic containing all formulas of the form $\Diamond^a((\Box\varphi_1 \to \varphi_1) \wedge \ldots \wedge (\Box\varphi_n \to \varphi_n))$ (where n ranges over $\mathbb{N}$) and $\varphi \to \Box\Diamond\varphi$. Let ψ be a formula.

Suppose $\psi \in \mathbf{L}'_a$. Hence, there exists a proof of ψ in the axiomatical presentation of $\mathbf{L}'_a$ consisting of the standard axioms of $\mathbf{K}$, the standard inference rules of $\mathbf{K}$ and all formulas of the form $\Diamond^a((\Box\varphi_1 \to \varphi_1) \wedge \ldots \wedge (\Box\varphi_n \to \varphi_n))$ (where n ranges over $\mathbb{N}$) and $\varphi \to \Box\Diamond\varphi$. By induction on the length of this proof, the reader may easily verify that $\psi \in \mathbf{L}_a$.

Suppose $\psi \not\in \mathbf{L}'_a$. Hence, by Lindenbaum's Lemma [9, Lemma 4.17], let s_0 be a maximal $\mathbf{L}'_a$-consistent set of formulas such that $\psi \not\in s_0$. Let $F = (W, R)$

be the canonical frame of $\mathbf{L}'_a$ defined by saying that W is the set of all maximal $\mathbf{L}'_a$-consistent sets of formulas and R is the binary relation on W such that sRt iff $\Box s \subseteq t$. Following a standard line of reasoning similar to the one used by Balbiani *et al.* [6] in the proof of their Theorem 3, one may verify that F is a-reflexive; see Annex 1 for details. Let $M = (W, R, V)$ be the model based on F where V is the function assigning to each atom α the subset $V(\alpha) = \{s \in W : \ \alpha \in s\}$. Since $\psi \notin s_0$, therefore by the Truth Lemma [9, Lemma 4.21], $M, s_0 \not\models \psi$. Since $F \models \mathbf{L}_a$, therefore $\psi \notin \mathbf{L}_a$. ■

Proposition 3. *For all $a \in \mathbb{N}$, $\mathbf{L}_a \neq \mathbf{KDB}$ and $\mathbf{L}_a \neq \mathbf{L}_{a+1}$.*

Proof. Let $a \in \mathbb{N}$.

Let $F = (W, R)$ be the serial frame where $W = \{0, 1\}$ and $R = \{(0, 1), (1, 0)\}$. Obviously, $F \models \mathbf{KDB}$. Let $M = (W, R, V)$ be a model based on F such that for some distinct atoms α_0, α_1, $V(\alpha_0) = \{1\}$ and $V(\alpha_1) = \{0\}$. Obviously, $M, 0 \not\models \Diamond^a((\Box\alpha_0 \rightarrow \alpha_0) \wedge (\Box\alpha_1 \rightarrow \alpha_1))$. Since $F \models \mathbf{KDB}$, therefore $\mathbf{L}_a \neq \mathbf{KDB}$.

Let $F = (W, R)$ be the $\mathbf{L}_{a+1}$-frame where $W = \{0, \ldots, a+1\}$ and $R = \{(i, j) : \ |j - i| = 1, \text{ or } i = a+1 \text{ and } j = a+1\}$. Obviously, $F \models \mathbf{L}_{a+1}$. Let $M = (W, R, V)$ be a model based on F such that for some pairwise distinct atoms $\alpha_0, \ldots, \alpha_a$, $V(\alpha_0) = \{1\}$ and for all $n \in \mathbb{N}$, if $n \geq 1$ and $n \leq a$ then $V(\alpha_n) = \{n-1, n+1\}$. Obviously, $M, 0 \not\models \Diamond^a((\Box\alpha_0 \rightarrow \alpha_0) \wedge \ldots \wedge (\Box\alpha_a \rightarrow \alpha_a))$. Since $F \models \mathbf{L}_{a+1}$, therefore $\mathbf{L}_a \neq \mathbf{L}_{a+1}$. ■

Unravelling. In our adaptation to **KB**, **KDB**, **KTB** and $\mathbf{L}_a$ for each $a \in \mathbb{N}$ of the argument proving that **K** is nullary, we will use a way of transforming any model into a tree-like model without affecting satisfiability. The transformation called unravelling will enable us to do this. Seeing that we mainly interest in **KB**, **KDB**, **KTB** and $\mathbf{L}_a$ for each $a \in \mathbb{N}$, we now adapt the standard definition of unravelling [9, Definition 4.51] to the specific properties of these modal logics. Let $F = (W, R)$ be a frame and $s \in W$. The *unravelling* of F around s is the frame $F' = (W', R')$ where W' is the set of all finite sequences of the form $(t_0, \ldots, t_k)$ such that $t_0, \ldots, t_k \in W$, $t_0 = s$ and for all $i \in \mathbb{N}$, if $i < k$ then $t_i R t_{i+1}$ and R' is the binary relation on W' such that $(t_0, \ldots, t_k) R' (u_0, \ldots, u_l)$ iff one of the following conditions holds:

- $l = k - 1$ and $(u_0, \ldots, u_l) = (t_0, \ldots, t_{k-1})$,
- $l = k$, $(u_0, \ldots, u_l) = (t_0, \ldots, t_k)$ and $t_k R u_l$,
- $l = k + 1$ and $(t_0, \ldots, t_k) = (u_0, \ldots, u_{l-1})$.

Note that $(s) \in W'$. Obviously, if F is serial then F' is serial and if F is reflexive then F' is reflexive. Moreover, for all $a \in \mathbb{N}$, if F is a-reflexive then F' is a-reflexive. Let $M = (W, R, V)$ be a model based on F. The *unravelling* of M around s is the model $M' = (W', R', V')$ based on F' where V' is the function assigning to each atom α the subset $V'(\alpha) = \{(t_0, \ldots, t_k) \in W' : t_k \in V(\alpha)\}$ of W'.

LEMMA 2. *Let φ be a formula. For all $(t_0, \ldots, t_k) \in W'$, $M', (t_0, \ldots, t_k) \models \varphi$ iff $M, t_k \models \varphi$.*

PROOF. Let $(t_0, \ldots, t_k) \in W'$. Let $f : W' \longrightarrow W$ be the function defined by $f((u_0, \ldots, u_l)) = u_l$ for each $(u_0, \ldots, u_l) \in W'$. The reader may easily verify that f is a bounded morphism from M' to M. Since $f((t_0, \ldots, t_k)) = t_k$, therefore by [9, Theorem 3.14], $M', (t_0, \ldots, t_k) \models \varphi$ iff $M, t_k \models \varphi$. ■

Let $\mathbf{L}_\omega = \bigcap\{\mathbf{L}_a : a \in \mathbb{N}\}$. The proof of the result stated in Proposition 4 illustrates the use of unravelling.

PROPOSITION 4. $\mathbf{L}_\omega = \mathbf{KDB}$.

PROOF. Suppose $\mathbf{L}_\omega \neq \mathbf{KDB}$. Let φ be a formula such that either $\varphi \in \mathbf{L}_\omega$ and $\varphi \notin \mathbf{KDB}$, or $\varphi \notin \mathbf{L}_\omega$ and $\varphi \in \mathbf{KDB}$. In the former case, let $M = (W, R, V)$ be a serial model and $s \in W$ be such that $M, s \not\models \varphi$. Let $M' = (W', R', V')$ be the unravelling of M around s. Since $M, s \not\models \varphi$, therefore by Lemma 2, $M', (s) \not\models \varphi$. Let $M'' = (W'', R'', V'')$ be the restriction of M' to the set of all finite sequences $(t_0, \ldots, t_k) \in W'$ such that $k \leq \mathtt{deg}(\varphi)$. The reader may easily verify that for all formulas ψ and for all finite sequences $(t_0, \ldots, t_k) \in W'$, if $\mathtt{deg}(\psi) + k \leq \mathtt{deg}(\varphi)$ then $M', (t_0, \ldots, t_k) \models \psi$ iff $M'', (t_0, \ldots, t_k) \models \psi$. Since $M', (s) \not\models \varphi$, therefore $M'', (s) \not\models \varphi$. Let $M''' = (W''', R''', V''')$ be the model such that $W''' = W''$, $R''' = R'' \cup \{((t_0, \ldots, t_k), (t_0, \ldots, t_k))$: the finite sequence $(t_0, \ldots, t_k) \in W'$ is such that $k = \mathtt{deg}(\varphi)\}$ and $V''' = V''$. The reader may easily verify that for all formulas ψ and for all finite sequences $(t_0, \ldots, t_k) \in W'$, if $\mathtt{deg}(\psi) + k \leq \mathtt{deg}(\varphi)$ then $M'', (t_0, \ldots, t_k) \models \psi$ iff $M''', (t_0, \ldots, t_k) \models \psi$. Since $M'', (s) \not\models \varphi$, therefore $M''', (s) \not\models \varphi$. Since M''' is $\mathtt{deg}(\varphi)$-reflexive, therefore $\varphi \notin \mathbf{L}_{\mathtt{deg}(\varphi)}$. Hence, $\varphi \notin \mathbf{L}_\omega$: a contradiction. In the latter case, let $a \in \mathbb{N}$ be such that $\varphi \notin \mathbf{L}_a$. Hence, φ is not valid in the class of all a-reflexive frames. Since every a-reflexive frame is serial, therefore φ is not valid in the class of all serial frames. Thus, $\varphi \notin \mathbf{KDB}$: a contradiction. ■

Properties of the modal connectives $\boxplus$ and $\boxminus$ Within the context of this paper, it is relevant to investigate the properties of the modal connectives $\boxplus$ and $\boxminus$. From now on in this paper,

let L be a fixed modal logic between KB and KTB.

The result stated in Lemma 3 is standard. It follows from the fact that $\mathbf{L}$ contains all formulas of the form $\neg\varphi \rightarrow \Box\neg\Box\varphi$.

Lemma 3. *For all formulas φ, ψ, $(\varphi \rightarrow \Box\psi) \in \mathbf{L}$ iff $(\neg\psi \rightarrow \Box\neg\varphi) \in \mathbf{L}$.*

Proof. See Annex 2 for details. ■

The reader may easily verify that $\mathbf{L}$ contains all formulas of the form $\boxplus(\varphi \rightarrow \psi) \rightarrow (\boxplus\varphi \rightarrow \boxplus\psi)$ and $\boxminus(\varphi \rightarrow \psi) \rightarrow (\boxminus\varphi \rightarrow \boxminus\psi)$ and $\mathbf{L}$ is closed under the generalization rules $\frac{\varphi}{\boxplus\varphi}$ and $\frac{\varphi}{\boxminus\varphi}$. The result stated in Lemma 4 implies that $\mathbf{L}$ is also closed under the tense rules $\frac{\varphi\rightarrow\boxplus\psi}{\neg\psi\rightarrow\boxminus\neg\varphi}$ and $\frac{\varphi\rightarrow\boxminus\psi}{\neg\psi\rightarrow\boxplus\neg\varphi}$.

Lemma 4. *For all formulas φ, ψ, $(\varphi \rightarrow \boxplus\psi) \in \mathbf{L}$ iff $(\neg\psi \rightarrow \boxminus\neg\varphi) \in \mathbf{L}$.*

Proof. See Annex 2 for details. ■

The results stated in Lemmas 5–8 will be used later in our adaptation to $\mathbf{KB}$, $\mathbf{KDB}$, $\mathbf{KTB}$ and $\mathbf{L}_a$ for each $a \in \mathbb{N}$ of the argument proving that $\mathbf{K}$ is nullary.

Lemma 5. *For all $k \in \mathbb{N}$,*

1. $\boxplus^k\top \in \mathbf{L}$,
2. $\boxminus^k\top \in \mathbf{L}$,
3. $\boxplus^{<k}\top \in \mathbf{L}$,
4. $\boxminus^{<k}\top \in \mathbf{L}$.

Proof. See Annex 2 for details. ■

Lemma 6. *Let φ be a formula. For all $k \in \mathbb{N}$,*

1. $(\boxplus^{<k+1}\varphi \leftrightarrow \varphi \wedge \boxplus\boxplus^{<k}\varphi) \in \mathbf{L}$,
2. $(\boxminus^{<k+1}\varphi \leftrightarrow \varphi \wedge \boxminus\boxminus^{<k}\varphi) \in \mathbf{L}$.

Proof. See Annex 2 for details. ■

Lemma 7. *For all $k, l \in \mathbb{N}$,*

1. *if $k > l$ then $(\boxplus^k\bot \rightarrow \boxplus^l\bot) \notin \mathbf{L}$,*
2. *if $k > l$ then $(\boxminus^k\bot \rightarrow \boxminus^l\bot) \notin \mathbf{L}$.*

Proof. Let $k, l \in \mathbb{N}$. Suppose $k > l$. Let $F = (W, R)$ be the frame where $W = \{0, \ldots, 3l\}$ and $R = \{(i, j) : |j - i| \leq 1\}$. Note that F is reflexive. Hence, $F \models \mathbf{KTB}$. Thus, $F \models \mathbf{L}$. Let $M = (W, R, V)$ be a model based on F such that $V(p) = \{i : i = 1 \mod 3\}$ and $V(q) = \{i : i = 2 \mod 3\}$. The reader may easily verify that $M, 0 \not\models (\boxplus^k\bot \rightarrow \boxplus^l\bot)$ and $M, 3l \not\models (\boxminus^k\bot \rightarrow \boxminus^l\bot)$. Consequently, $(\boxplus^k\bot \rightarrow \boxplus^l\bot) \notin \mathbf{L}$ and $(\boxminus^k\bot \rightarrow \boxminus^l\bot) \notin \mathbf{L}$. ■

LEMMA 8. *For all $k \in \mathbb{N}$,*

1. $\boxplus^k \bot \notin \mathbf{L}$*,*

2. $\boxminus^k \bot \notin \mathbf{L}$*.*

PROOF. By Lemma 7. ■

4. Unification

In this section, we present unification in $\mathbf{L}$. For more on the theory of unification, see [2].

Unification Problem. We shall say that a substitution σ is *equivalent* to a substitution τ with respect to a set $\mathcal{X}$ of variables (in symbols $\sigma \simeq_L^{\mathcal{X}} \tau$) if for all variables $x \in \mathcal{X}$, $(\sigma(x) \leftrightarrow \tau(x)) \in \mathbf{L}$. We shall say that a substitution σ is more *general* than a substitution τ with respect to a set $\mathcal{X}$ of variables (in symbols $\sigma \preceq_L^{\mathcal{X}} \tau$) if there exists a substitution υ such that $\sigma \circ \upsilon \simeq_L^{\mathcal{X}} \tau$. Obviously, $\preceq_L^{\mathcal{X}}$ contains $\simeq_L^{\mathcal{X}}$. Moreover, on the set of all substitutions, the binary relation $\simeq_L^{\mathcal{X}}$ is reflexive, symmetric and transitive and the binary relation $\preceq_L^{\mathcal{X}}$ is reflexive and transitive. We shall say that a formula φ is *unifiable* if there exists a substitution σ such that $\sigma(\varphi) \in \mathbf{L}$. In that case, σ is a *unifier* of φ. We shall say that a set Σ of unifiers of a unifiable formula φ is *minimal complete* if

- for all unifiers σ of φ, there exists $\tau \in \Sigma$ such that $\tau \preceq_L^{\mathtt{var}(\varphi)} \sigma$,
- for all $\sigma, \tau \in \Sigma$, $\sigma \neq \tau$, neither $\sigma \preceq_L^{\mathtt{var}(\varphi)} \tau$, nor $\tau \preceq_L^{\mathtt{var}(\varphi)} \sigma$.

Unification Types. An important question is the following: when a formula is unifiable, has it a minimal complete set of unifiers? When the answer is "yes", how large is this set? We shall say that a unifiable formula

- φ is *nullary* if there exists no minimal complete set of unifiers of φ,
- φ is *unitary* if there exists a minimal complete set of unifiers of φ with cardinality 1,
- φ is *finitary* if there exists a finite minimal complete set of unifiers of φ but there exists no with cardinality 1,
- φ is *infinitary* if there exists a minimal complete set of unifiers of φ but there exists no finite one.

We shall say that

- $\mathbf{L}$ is *nullary* if there exists a nullary unifiable formula,
- $\mathbf{L}$ is *unitary* if every unifiable formula is unitary,
- $\mathbf{L}$ is *finitary* if every unifiable formula is either unitary, or finitary and there exists a finitary unifiable formula,
- $\mathbf{L}$ is *infinitary* if every unifiable formula is either unitary, or finitary, or infinitary and there exists an infinitary unifiable formula.

5. Playing with Substitutions

From now on in this paper,

$$\boxed{\textbf{let } x \textbf{ be a fixed variable.}}$$

For all $k \in \mathbb{N}$, let σ_k and τ_k be the substitutions inductively defined as follows:

- $\sigma_0(x) = \bot$,
- for all variables y distinct from x, $\sigma_0(y) = y$,
- $\tau_0(x) = \top$,
- for all variables y distinct from x, $\tau_0(y) = y$,
- $\sigma_{k+1}(x) = (x \wedge \boxplus\sigma_k(x))$,
- for all variables y distinct from x, $\sigma_{k+1}(y) = y$,
- $\tau_{k+1}(x) = \neg(\neg x \wedge \boxminus\neg\tau_k(x))$,
- for all variables y distinct from x, $\tau_{k+1}(y) = y$.

These substitutions will be used in Section 6 to prove that $\mathbf{L}$ possesses nullary formulas. In the meantime, it is relevant to investigate the properties of the substitutions σ_k and τ_k for each $k \in \mathbb{N}$.

LEMMA 9. *For all $k \in \mathbb{N}$,*

1. $(\boxplus^{<k}x \wedge \boxplus^k\bot \rightarrow \sigma_k(x)) \in \mathbf{L}$,

2. $(\boxminus^{<k}\neg x \wedge \boxminus^k\bot \rightarrow \neg\tau_k(x)) \in \mathbf{L}$.

PROOF. See Annex 2 for details. ∎

LEMMA 10. *For all $k \in \mathbb{N}$,*

1. $(\sigma_k(x) \rightarrow x) \in \mathbf{L}$,

2. $(\neg\tau_k(x) \rightarrow \neg x) \in \mathbf{L}$.

PROOF. See Annex 2 for details. ∎

LEMMA 11. *For all* $k \in \mathbb{N}$,

1. $(\sigma_k(x) \to \boxplus\sigma_k(x)) \in \mathbf{L}$,
2. $(\neg\tau_k(x) \to \boxminus\neg\tau_k(x)) \in \mathbf{L}$.

PROOF. See Annex 2 for details. ∎

LEMMA 12. *For all* $k, l \in \mathbb{N}$,

1. *if* $k \leq l$ *then* $(\sigma_k(x) \to \boxplus^l\bot) \in \mathbf{L}$,
2. *if* $k \leq l$ *then* $(\neg\tau_k(x) \to \boxminus^l\bot) \in \mathbf{L}$.

PROOF. See Annex 2 for details. ∎

LEMMA 13. *For all* $k, l \in \mathbb{N}$,

1. *if* $k > l$ *then* $(\sigma_k(x) \to \boxplus^l\bot) \notin \mathbf{L}$,
2. *if* $k > l$ *then* $(\neg\tau_k(x) \to \boxminus^l\bot) \notin \mathbf{L}$.

PROOF. Let $k, l \in \mathbb{N}$.

(1): Suppose $k > l$ and $(\sigma_k(x) \to \boxplus^l\bot) \in \mathbf{L}$. Let υ be the substitution defined as follows:

- $\upsilon(x) = \top$,
- for all variables y distinct from x, $\upsilon(y) = y$.

Since $(\sigma_k(x) \to \boxplus^l\bot) \in \mathbf{L}$, therefore $(\upsilon(\sigma_k(x)) \to \boxplus^l\bot) \in \mathbf{L}$. By Lemma 9, $(\boxplus^{<k}x \wedge \boxplus^k\bot \to \sigma_k(x)) \in \mathbf{L}$. Hence, $(\boxplus^{<k}\upsilon(x) \wedge \boxplus^k\bot \to \upsilon(\sigma_k(x))) \in \mathbf{L}$. Since $\upsilon(x) = \top$, therefore by Lemma 5, $(\boxplus^k\bot \to \upsilon(\sigma_k(x))) \in \mathbf{L}$. Since $(\upsilon(\sigma_k(x)) \to \boxplus^l\bot) \in \mathbf{L}$, therefore $(\boxplus^k\bot \to \boxplus^l\bot) \in \mathbf{L}$. Thus, by Lemma 7, $k \not> l$: a contradiction.

(2): Suppose $k > l$ and $(\neg\tau_k(x) \to \boxminus^l\bot) \in \mathbf{L}$. Let υ be the substitution defined as follows:

- $\upsilon(x) = \bot$,
- for all variables y distinct from x, $\upsilon(y) = y$.

Since $(\neg\tau_k(x) \to \boxminus^l\bot) \in \mathbf{L}$, therefore $(\upsilon(\neg\tau_k(x)) \to \boxminus^l\bot) \in \mathbf{L}$. By Lemma 9, $(\boxminus^{<k}\neg x \wedge \boxminus^k\bot \to \neg\tau_k(x)) \in \mathbf{L}$. Hence, $(\boxminus^{<k}\neg\upsilon(x) \wedge \boxminus^k\bot \to \upsilon(\neg\tau_k(x))) \in \mathbf{L}$. Since $\upsilon(x) = \bot$, therefore by Lemma 5, $(\boxminus^k\bot \to \upsilon(\neg\tau_k(x))) \in \mathbf{L}$. Since $(\upsilon(\neg\tau_k(x)) \to \boxminus^l\bot) \in \mathbf{L}$, therefore $(\boxminus^k\bot \to \boxminus^l\bot) \in \mathbf{L}$. Thus, by Lemma 7, $k \not> l$: a contradiction. ∎

LEMMA 14. *For all* $k, l \in \mathbb{N}$, *1.* $(\boxplus^k\bot\vee\neg\tau_l(x))\not\in\mathbf{L}$*,*

2. $(\boxminus^k\bot\vee\sigma_l(x))\not\in\mathbf{L}$*.*

PROOF. Let $k,l\in\mathbb{N}$.

(1): Suppose $(\boxplus^k\bot\vee\neg\tau_l(x))\in\mathbf{L}$. By Lemma 10, $(\neg\tau_l(x)\rightarrow\neg x)\in\mathbf{L}$. Since $(\boxplus^k\bot\vee\neg\tau_l(x))\in\mathbf{L}$, therefore $(\boxplus^k\bot\vee\neg x)\in\mathbf{L}$. Let υ be the substitution defined as follows:

- $\upsilon(x)=\top$,
- for all variables y distinct from x, $\upsilon(y)=y$.

Since $(\boxplus^k\bot\vee\neg x)\in\mathbf{L}$, therefore $(\boxplus^k\bot\vee\neg\upsilon(x))\in\mathbf{L}$. Since $\upsilon(x)=\top$, therefore $\boxplus^k\bot\in\mathbf{L}$: a contradiction with Lemma 8.

(2): Suppose $(\boxminus^k\bot\vee\sigma_l(x))\in\mathbf{L}$. By Lemma 10, $(\sigma_l(x)\rightarrow x)\in\mathbf{L}$. Since $(\boxminus^k\bot\vee\sigma_l(x))\in\mathbf{L}$, therefore $(\boxminus^k\bot\vee x)\in\mathbf{L}$. Let υ be the substitution defined as follows:

- $\upsilon(x)=\bot$,
- for all variables y distinct from x, $\upsilon(y)=y$.

Since $(\boxminus^k\bot\vee x)\in\mathbf{L}$, therefore $(\boxminus^k\bot\vee\upsilon(x))\in\mathbf{L}$. Since $\upsilon(x)=\bot$, therefore $\boxminus^k\bot\in\mathbf{L}$: a contradiction with Lemma 8. ■

LEMMA 15. *For all* $k,l\in\mathbb{N}$*,*

1. if $k\leq l$ *then* $(\boxplus^k\bot\wedge\sigma_l(x)\leftrightarrow\sigma_k(x))\in\mathbf{L}$*,*

2. if $k\leq l$ *then* $(\boxminus^k\bot\wedge\neg\tau_l(x)\leftrightarrow\neg\tau_k(x))\in\mathbf{L}$*.*

PROOF. See Annex 2 for details. ■

For all $k\in\mathbb{N}$, let λ_k and μ_k be the substitutions defined as follows:

- $\lambda_k(x)=(x\wedge\boxplus^k\bot)$,
- for all variables y distinct from x, $\lambda_k(y)=y$,
- $\mu_k(x)=\neg(\neg x\wedge\boxminus^k\bot)$,
- for all variables y distinct from x, $\mu_k(y)=y$.

LEMMA 16. *For all* $k,l\in\mathbb{N}$*,*

1. if $k\leq l$ *then* $(\lambda_l(\sigma_k(x))\leftrightarrow\sigma_k(x))\in\mathbf{L}$*,*

2. if $k\leq l$ *then* $(\mu_l(\tau_k(x))\leftrightarrow\tau_k(x))\in\mathbf{L}$*.*

PROOF. See Annex 2 for details. ■

LEMMA 17. *For all* $k,l\in\mathbb{N}$*,* *1. if $k \geq l$ then $(\lambda_l(\sigma_k(x)) \leftrightarrow \sigma_l(x)) \in \mathbf{L}$,*

2. if $k \geq l$ then $(\mu_l(\tau_k(x)) \leftrightarrow \tau_l(x)) \in \mathbf{L}$.

PROOF. See Annex 2 for details. ■

LEMMA 18. *For all $k, l \in \mathbb{N}$,*

1. if $k \leq l$ then $\sigma_l \circ \lambda_k \simeq_L^{\{x\}} \sigma_k$,

2. if $k \leq l$ then $\tau_l \circ \mu_k \simeq_L^{\{x\}} \tau_k$.

PROOF. By Lemma 17. ■

LEMMA 19. *For all $k, l \in \mathbb{N}$,*

1. if $k \leq l$ then $\sigma_l \preceq_L^{\{x\}} \sigma_k$,

2. if $k \leq l$ then $\tau_l \preceq_L^{\{x\}} \tau_k$.

PROOF. By Lemma 18. ■

LEMMA 20. *For all $k, l \in \mathbb{N}$,*

1. if $k < l$ then $\sigma_k \not\preceq_L^{\{x\}} \sigma_l$,

2. if $k < l$ then $\tau_k \not\preceq_L^{\{x\}} \tau_l$.

PROOF. Let $k, l \in \mathbb{N}$.

(1): Suppose $k < l$ and $\sigma_k \preceq_L^{\{x\}} \sigma_l$. Let λ be a substitution such that $\sigma_k \circ \lambda \simeq_L^{\{x\}} \sigma_l$. Hence, $(\lambda(\sigma_k(x)) \leftrightarrow \sigma_l(x)) \in \mathbf{L}$. By Lemma 12, $(\sigma_k(x) \rightarrow \boxplus^k \bot) \in \mathbf{L}$. Thus, $(\lambda(\sigma_k(x)) \rightarrow \boxplus^k \bot) \in \mathbf{L}$. Since $(\lambda(\sigma_k(x)) \leftrightarrow \sigma_l(x)) \in \mathbf{L}$, therefore $(\sigma_l(x) \rightarrow \boxplus^k \bot) \in \mathbf{L}$. Consequently, by Lemma 13, $l \not> k$: a contradiction.

(2): Suppose $k < l$ and $\tau_k \preceq_L^{\{x\}} \tau_l$. Let μ be a substitution such that $\tau_k \circ \mu \simeq_L^{\{x\}} \tau_l$. Hence, $(\mu(\tau_k(x)) \leftrightarrow \tau_l(x)) \in \mathbf{L}$. By Lemma 12, $(\neg\tau_k(x) \rightarrow \boxminus^k \bot) \in \mathbf{L}$. Thus, $(\mu(\neg\tau_k(x)) \rightarrow \boxminus^k \bot) \in \mathbf{L}$. Since $(\mu(\tau_k(x)) \leftrightarrow \tau_l(x)) \in \mathbf{L}$, therefore $(\neg\tau_l(x) \rightarrow \boxminus^k \bot) \in \mathbf{L}$. Consequently, by Lemma 13, $l \not> k$: a contradiction. ■

LEMMA 21. *For all $k, l \in \mathbb{N}$,*

1. $\sigma_k \not\preceq_L^{\{x\}} \tau_l$,

2. $\tau_k \not\preceq_L^{\{x\}} \sigma_l$.

PROOF. Let $k, l \in \mathbb{N}$.
(1): Suppose $\sigma_k \preceq_L^{\{x\}} \tau_l$. Let υ be a substitution such that $\sigma_k \circ \upsilon \simeq_L^{\{x\}} \tau_l$. Hence, $(\upsilon(\sigma_k(x)) \leftrightarrow \tau_l(x)) \in \mathbf{L}$. By Lemma 12, $(\sigma_k(x) \rightarrow \boxplus^k \bot) \in \mathbf{L}$.

Thus, $(\upsilon(\sigma_k(x)) \rightarrow \boxplus^k\bot) \in \mathbf{L}$. Since $(\upsilon(\sigma_k(x)) \leftrightarrow \tau_l(x)) \in \mathbf{L}$, therefore $(\boxplus^k\bot \vee \neg\tau_l(x)) \in \mathbf{L}$: a contradiction with Lemma 14.

(2): Suppose $\tau_k \preceq_L^{\{x\}} \sigma_l$. Let υ be a substitution such that $\tau_k \circ \upsilon \simeq_L^{\{x\}} \sigma_l$. Hence, $(\upsilon(\tau_k(x)) \leftrightarrow \sigma_l(x)) \in \mathbf{L}$. By Lemma 12, $(\neg\tau_k(x) \rightarrow \boxminus^k\bot) \in \mathbf{L}$. Thus, $(\upsilon(\neg\tau_k(x)) \rightarrow \boxminus^k\bot) \in \mathbf{L}$. Since $(\upsilon(\tau_k(x)) \leftrightarrow \sigma_l(x)) \in \mathbf{L}$, therefore $(\boxminus^k\bot \vee \sigma_l(x)) \in \mathbf{L}$: a contradiction with Lemma 14. ■

6. Unification Type of a Specific Formula

In this section, we prove that the following specific formula is unifiable and we study its unification type:

$$\boxed{\varphi ::= ((x \rightarrow \boxplus x) \wedge (\neg x \rightarrow \boxminus\neg x)).}$$

By Lemma 4, φ has the same unifiers as the simpler formulas $(x \rightarrow \boxplus x)$ and $(\neg x \rightarrow \boxminus\neg x)$ considered in isolation. Hence, as long as we only consider φ through its unifiers, it does not matter if we are talking about either φ, or $(x \rightarrow \boxplus x)$, or $(\neg x \rightarrow \boxminus\neg x)$.

Lemma 22. *Let σ be a unifier of φ. For all $k \in \mathbb{N}$,*

1. $(\sigma(x) \rightarrow \boxplus^{<k}\sigma(x)) \in \mathbf{L}$,
2. $(\neg\sigma(x) \rightarrow \boxminus^{<k}\neg\sigma(x)) \in \mathbf{L}$.

Proof. See Annex 2 for details. ■

Lemma 23. *For all $k \in \mathbb{N}$,*

1. *σ_k is a unifier of φ,*
2. *τ_k is a unifier of φ.*

Proof. By Lemmas 4 and 11 . ■

Lemma 24. *Let υ be a substitution. If υ is a unifier of φ then*

1. *for all $k \in \mathbb{N}$, the following conditions are equivalent: (a) $\sigma_k \circ \upsilon \simeq_L^{\{x\}} \upsilon$, (b) $\sigma_k \preceq_L^{\{x\}} \upsilon$, (c) $(\upsilon(x) \rightarrow \boxplus^k\bot) \in \mathbf{L}$,*
2. *for all $k \in \mathbb{N}$, the following conditions are equivalent: (d) $\tau_k \circ \upsilon \simeq_L^{\{x\}} \upsilon$, (e) $\tau_k \preceq_L^{\{x\}} \upsilon$, (f) $(\neg\upsilon(x) \rightarrow \boxminus^k\bot) \in \mathbf{L}$.*

Proof. Suppose υ is a unifier of φ.
(1): Let $k \in \mathbb{N}$.
$(a) \Rightarrow (b)$: Suppose $\sigma_k \circ \upsilon \simeq_L^{\{x\}} \upsilon$. Hence, $\sigma_k \preceq_L^{\{x\}} \upsilon$.

$(b) \Rightarrow (c)$: Suppose $\sigma_k \preceq_L^{\{x\}} \upsilon$. Let υ' be a substitution such that $\sigma_k \circ \upsilon' \simeq_L^{\{x\}} \upsilon$. Hence, $(\upsilon'(\sigma_k(x)) \leftrightarrow \upsilon(x)) \in \mathbf{L}$. By Lemma 12, $(\sigma_k(x) \rightarrow \boxplus^k \bot) \in \mathbf{L}$. Thus, $(\upsilon'(\sigma_k(x)) \rightarrow \boxplus^k \bot) \in \mathbf{L}$. Since $(\upsilon'(\sigma_k(x)) \leftrightarrow \upsilon(x)) \in \mathbf{L}$, therefore $(\upsilon(x) \rightarrow \boxplus^k \bot) \in \mathbf{L}$.
$(c) \Rightarrow (a)$: Suppose $(\upsilon(x) \rightarrow \boxplus^k \bot) \in \mathbf{L}$. Since υ is a unifier of φ, therefore by Lemma 22, $(\upsilon(x) \rightarrow \boxplus^{<k} \upsilon(x)) \in \mathbf{L}$. Since $(\upsilon(x) \rightarrow \boxplus^k \bot) \in \mathbf{L}$, therefore $(\upsilon(x) \rightarrow \boxplus^{<k} \upsilon(x) \wedge \boxplus^k \bot) \in \mathbf{L}$. By Lemma 9, $(\boxplus^{<k} x \wedge \boxplus^k \bot \rightarrow \sigma_k(x)) \in \mathbf{L}$. Hence, $(\boxplus^{<k} \upsilon(x) \wedge \boxplus^k \bot \rightarrow \upsilon(\sigma_k(x))) \in \mathbf{L}$. Since $(\upsilon(x) \rightarrow \boxplus^{<k} \upsilon(x) \wedge \boxplus^k \bot) \in \mathbf{L}$, therefore $(\upsilon(x) \rightarrow \upsilon(\sigma_k(x))) \in \mathbf{L}$. By Lemma 10, $(\sigma_k(x) \rightarrow x) \in \mathbf{L}$. Thus, $(\upsilon(\sigma_k(x)) \rightarrow \upsilon(x)) \in \mathbf{L}$. Since $(\upsilon(x) \rightarrow \upsilon(\sigma_k(x))) \in \mathbf{L}$, therefore $(\upsilon(\sigma_k(x)) \leftrightarrow \upsilon(x)) \in \mathbf{L}$. Consequently, $\sigma_k \circ \upsilon \simeq_L^{\{x\}} \upsilon$.
(2): Let $k \in \mathbb{N}$.
$(d) \Rightarrow (e)$: Suppose $\tau_k \circ \upsilon \simeq_L^{\{x\}} \upsilon$. Hence, $\tau_k \preceq_L^{\{x\}} \upsilon$.
$(e) \Rightarrow (f)$: Suppose $\tau_k \preceq_L^{\{x\}} \upsilon$. Let υ' be a substitution such that $\tau_k \circ \upsilon' \simeq_L^{\{x\}} \upsilon$. Hence, $(\upsilon'(\tau_k(x)) \leftrightarrow \upsilon(x)) \in \mathbf{L}$. By Lemma 12, $(\neg\tau_k(x) \rightarrow \boxminus^k \bot) \in \mathbf{L}$. Thus, $(\upsilon'(\neg\tau_k(x)) \rightarrow \boxminus^k \bot) \in \mathbf{L}$. Since $(\upsilon'(\tau_k(x)) \leftrightarrow \upsilon(x)) \in \mathbf{L}$, therefore $(\neg\upsilon(x) \rightarrow \boxminus^k \bot) \in \mathbf{L}$.
$(f) \Rightarrow (d)$: Suppose $(\neg\upsilon(x) \rightarrow \boxminus^k \bot) \in \mathbf{L}$. Since υ is a unifier of φ, therefore by Lemma 22, $(\neg\upsilon(x) \rightarrow \boxminus^{<k} \neg\upsilon(x)) \in \mathbf{L}$. Since $(\neg\upsilon(x) \rightarrow \boxminus^k \bot) \in \mathbf{L}$, therefore $(\neg\upsilon(x) \rightarrow \boxminus^{<k} \neg\upsilon(x) \wedge \boxminus^k \bot) \in \mathbf{L}$. By Lemma 9, $(\boxminus^{<k} \neg x \wedge \boxminus^k \bot \rightarrow \neg\tau_k(x)) \in \mathbf{L}$. Hence, $(\boxminus^{<k} \neg\upsilon(x) \wedge \boxminus^k \bot \rightarrow \upsilon(\neg\tau k(x))) \in \mathbf{L}$. Since $(\neg\upsilon(x) \rightarrow \boxminus^{<k} \neg\upsilon(x) \wedge \boxminus^k \bot) \in \mathbf{L}$, therefore $(\neg\upsilon(x) \rightarrow \upsilon(\neg\tau_k(x))) \in \mathbf{L}$. By Lemma 10, $(\neg\tau_k(x) \rightarrow \neg x) \in \mathbf{L}$. Thus, $(\upsilon(\neg\tau_k(x)) \rightarrow \neg\upsilon(x)) \in \mathbf{L}$. Since $(\neg\upsilon(x) \rightarrow \upsilon(\neg\tau_k(x))) \in \mathbf{L}$, therefore $(\upsilon(\tau_k(x)) \leftrightarrow \upsilon(x)) \in \mathbf{L}$. Consequently, $\tau_k \circ \upsilon \simeq_L^{\{x\}} \upsilon$. ■

LEMMA 25. *Let σ be a substitution. When either $\mathbf{L} = \mathbf{KB}$, or $\mathbf{L} = \mathbf{KDB}$, or $\mathbf{L} = \mathbf{KTB}$, or there exists $a \in \mathbb{N}$ such that $\mathbf{L} = \mathbf{L}_a$, if σ is a unifier of φ then there exists $k \in \mathbb{N}$ such that either $\sigma_k \preceq_L^{\{x\}} \sigma$, or $\tau_k \preceq_L^{\{x\}} \sigma$.*

PROOF. Assume either $\mathbf{L} = \mathbf{KB}$, or $\mathbf{L} = \mathbf{KDB}$, or $\mathbf{L} = \mathbf{KTB}$, or there exists $a \in \mathbb{N}$ such that $\mathbf{L} = \mathbf{L}_a$. Suppose σ is a unifier of φ. Let $k \in \mathbb{N}$ be such that $\mathtt{deg}(\sigma(x)) \leq 3k$. Suppose $\sigma_k \not\preceq_L^{\{x\}} \sigma$ and $\tau_k \not\preceq_L^{\{x\}} \sigma$. Since σ is a unifier of φ, therefore by Lemma 24, $(\sigma(x) \rightarrow \boxplus^k \bot) \notin \mathbf{L}$ and $(\neg\sigma(x) \rightarrow \boxminus^k \bot) \notin \mathbf{L}$. By Propositions 1 and 2 , let $F = (W, R)$ be an L-frame, $M = (W, R, V)$ be a model based on F, $s \in W$, $F' = (W', R')$ be an L-frame, $M' = (W', R', V')$ be a model based on F' and $s' \in W'$ be such that $M, s \not\models (\sigma(x) \rightarrow \boxplus^k \bot)$ and $M', s' \not\models (\neg\sigma(x) \rightarrow \boxminus^k \bot)$. Hence, $M, s \models \sigma(x)$, $M, s \not\models \boxplus^k \bot$, $M', s' \models \neg\sigma(x)$ and $M', s' \not\models \boxminus^k \bot$. Let $v_0, t_1, u_1, v_1, \ldots, t_k, u_k, v_k \in W$

and $v'_0, t'_1, u'_1, v'_1, \ldots, t'_k, u'_k, v'_k \in W'$ be such that $s = v_0$, $s' = v'_0$ and for all $i \in \mathbb{N}$, if $i < k$ then

- $v_i R t_{i+1}$,
- $t_{i+1} R u_{i+1}$,
- $u_{i+1} R v_{i+1}$,
- $v'_i R' t'_{i+1}$,
- $t'_{i+1} R' u'_{i+1}$,
- $u'_{i+1} R' v'_{i+1}$,
- $M, v_i \models p^0 \land q^0$,
- $M, t_{i+1} \models p^1 \land q^0$,
- $M, u_{i+1} \models p^0 \land q^1$,
- $M, v_{i+1} \models p^0 \land q^0$,
- $M', v'_i \models p^0 \land q^0$,
- $M', t'_{i+1} \models p^0 \land q^1$,
- $M', u'_{i+1} \models p^1 \land q^0$,
- $M', v'_{i+1} \models p^0 \land q^0$.

Let $M_s = (W_s, R_s, V_s)$ be the unravelling of M around s and $M'_{s'} = (W'_{s'}, R'_{s'}, V'_{s'})$ be the unravelling of M' around s'. Since $M, s \models \sigma(x)$ and $M', s' \models \neg\sigma(x)$, therefore by Lemma 2, $M_s, (v_0) \models \sigma(x)$ and $M'_{s'}, (v'_0) \models \neg\sigma(x)$. Let $F'' = (W'', R'')$ be the least frame containing the disjoint union of (W_s, R_s) and $(W'_{s'}, R'_{s'})$ and such that for some new states t and u,

- $(v_0, t_1, u_1, v_1, \ldots, t_k, u_k, v_k) R'' t$,
- $t R'' (v_0, t_1, u_1, v_1, \ldots, t_k, u_k, v_k)$,
- $t R'' t$,
- $t R'' u$,
- $u R'' t$,
- $u R'' u$,
- $u R'' (v'_0, t'_1, u'_1, v'_1, \ldots, t'_k, u'_k, v'_k)$,
- $(v'_0, t'_1, u'_1, v'_1, \ldots, t'_k, u'_k, v'_k) R'' u$.

Obviously, F'' is an L-frame. Let $M'' = (W'', R'', V'')$ where

- $V''(p) = V_s(p) \cup V'_{s'}(p) \cup \{t\}$,

- $V''(q) = V_s(q) \cup V'_{s'}(q) \cup \{u\}$,
- for all atoms α, if $\alpha \neq p$ and $\alpha \neq q$ then $V''(\alpha) = V_s(\alpha) \cup V'_{s'}(\alpha)$.

Since $\mathtt{deg}(\sigma(x)) \leq 3k$, $M_s, (v_0) \models \sigma(x)$ and $M'_{s'}, (v'_0) \models \neg\sigma(x)$, therefore $M'', (v_0) \models \sigma(x)$ and $M'', (v'_0) \models \neg\sigma(x)$. Since σ is a unifier of φ, therefore $((\sigma(x) \rightarrow \boxplus\sigma(x)) \wedge (\neg\sigma(x) \rightarrow \boxminus\neg\sigma(x))) \in \mathbf{L}$. Since $M'', (v_0) \models \sigma(x)$ and $M'', (v'_0) \models \neg\sigma(x)$, considering that for all $i \in \mathbb{N}$, if $i < k$ then $M, v_i \models p^0 \wedge q^0$, $M, t_{i+1} \models p^1 \wedge q^0$, $M, u_{i+1} \models p^0 \wedge q^1$, $M, v_{i+1} \models p^0 \wedge q^0$, $M', v'_i \models p^0 \wedge q^0$, $M', t'_{i+1} \models p^0 \wedge q^1$, $M', u'_{i+1} \models p^1 \wedge q^0$ and $M', v'_{i+1} \models p^0 \wedge q^0$, therefore $M'', (v_0, t_1, u_1, v_1, \ldots, t_k, u_k, v_k) \models \sigma(x)$ and $M'', (v'_0, t'_1, u'_1, v'_1, \ldots, t'_k, u'_k, v'_k) \models \neg\sigma(x)$. Since $((\sigma(x) \rightarrow \boxplus\sigma(x)) \wedge (\neg\sigma(x) \rightarrow \boxminus\neg\sigma(x))) \in \mathbf{L}$, considering that $M, v_k \models p^0 \wedge q^0$, $M'', t \models p^1 \wedge q^0$, $M'', u \models p^0 \wedge q^1$ and $M', v'_k \models p^0 \wedge q^0$, therefore $M'', (v_0, t_1, u_1, v_1, \ldots, t_k, u_k, v_k) \models \neg\sigma(x)$ and $M'', (v'_0, t'_1, u'_1, v'_1, \ldots, t'_k, u'_k, v'_k) \models \sigma(x)$. Thus, M'', $(v_0, t_1, u_1, v_1, \ldots, t_k, u_k, v_k) \not\models \sigma(x)$ and $M'', (v'_0, t'_1, u'_1, v'_1, \ldots, t'_k, u'_k, v'_k) \not\models \neg\sigma(x)$: a contradiction. ∎

Proposition 5. *When either* $\mathbf{L} = \mathbf{KB}$*, or* $\mathbf{L} = \mathbf{KDB}$*, or* $\mathbf{L} = \mathbf{KTB}$*, or there exists* $a \in \mathbb{N}$ *such that* $\mathbf{L} = \mathbf{L}_a$*,* φ *is nullary.*

Proof. Assume either $\mathbf{L} = \mathbf{KB}$, or $\mathbf{L} = \mathbf{KDB}$, or $\mathbf{L} = \mathbf{KTB}$, or there exists $a \in \mathbb{N}$ such that $\mathbf{L} = \mathbf{L}_a$. Suppose φ is not nullary. Let Σ be a minimal complete set of unifiers of φ. By Lemma 23, σ_0 is a unifier of φ. Since Σ is a minimal complete set of unifiers of φ, therefore let $\sigma \in \Sigma$ be such that $\sigma \preceq_L^{\{x\}} \sigma_0$. Hence, by Lemma 25, let $k \in \mathbb{N}$ be such that either $\sigma_k \preceq_L^{\{x\}} \sigma$, or $\tau_k \preceq_L^{\{x\}} \sigma$.

Suppose $\sigma_k \preceq_L^{\{x\}} \sigma$. By Lemma 23, σ_{k+1} is a unifier of φ. Since Σ is a minimal complete set of unifiers of φ, therefore let $\sigma' \in \Sigma$ be such that $\sigma' \preceq_L^{\{x\}} \sigma_{k+1}$. Since $\sigma_k \preceq_L^{\{x\}} \sigma$, therefore by Lemma 19, $\sigma' \preceq_L^{\{x\}} \sigma$. Since Σ is a minimal complete set of unifiers of φ, therefore $\sigma' = \sigma$. Since $\sigma_k \preceq_L^{\{x\}} \sigma$ and $\sigma' \preceq_L^{\{x\}} \sigma_{k+1}$, therefore $\sigma_k \preceq_L^{\{x\}} \sigma_{k+1}$: a contradiction with Lemma 20.

Suppose $\tau_k \preceq_L^{\{x\}} \sigma$. Since $\sigma \preceq_L^{\{x\}} \sigma_0$, therefore $\tau_k \preceq_L^{\{x\}} \sigma_0$: a contradiction with Lemma 21. ∎

Now, we are ready for stating the main result.

Theorem 1. *When either* $\mathbf{L} = \mathbf{KB}$*, or* $\mathbf{L} = \mathbf{KDB}$*, or* $\mathbf{L} = \mathbf{KTB}$*, or there exists* $a \in \mathbb{N}$ *such that* $\mathbf{L} = \mathbf{L}_a$*,* $\mathbf{L}$ *is nullary for unification with parameters.*

Proof. By Proposition 5. ∎

7. Conclusion

In this paper, we have adapted to **KB**, **KDB**, **KTB** and $\mathbf{L}_a$ for each $a \in \mathbb{N}$ the argument proving that **K** is nullary, though these modal logics have been proved to be nullary only within the context of unification with parameters—the question of the unification type of **KB**, **KDB**, **KTB** and $\mathbf{L}_a$ for each $a \in \mathbb{N}$ within the context of elementary unification being still to be answered. Much remains to be done. For example, it is not known whether the interval $[\mathbf{K}, \mathbf{KTB}]$ of modal logics contains non-nullary modal logics. If it does then one may wonder about the computability of the problem asking whether a given modal logic in the interval $[\mathbf{K}, \mathbf{KTB}]$ is nullary. In other respect, there are other families of modal logics about which nothing is known for what concerns their unification types: the normal extensions of **K**5; the simple Church-Rosser modal logics like **KG**, **KDG** and **KTG**; the least modal logic containing $\square^a \bot$ when $a \in \mathbb{N}$ is such that $a \geq 2$; the fusions and products of modal logics considered in [18, Sections 2 and 3].

With respect to the normal extensions of **K**5, we believe that all of them are unitary. About the simple Church-Rosser modal logics like **KG**, **KDG** and **KTG**, we believe that they are nullary. If these modal logics are nullary then it is probably non-obvious to prove that they are nullary within the context of elementary unification. Concerning the least modal logic containing $\square^a \bot$ when $a \in \mathbb{N}$ is such that $a \geq 2$, we believe that it is finitary—the question of the unification type of this modal logic was asked by Silvio Ghilardi during the workshop UNIF 2018. As for the fusions and products of modal logics considered in [18, Sections 2 and 3], their rich variety prevents us to have a clear-cut opinion. Nevertheless, we believe, for example, that the fusion of **S**5 with itself is nullary. If the fusion of **S**5 with itself is nullary then it is probably non-obvious to prove that it is nullary within the context of elementary unification.

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Annex 1

In this Annex, we prove that the canonical frame of $\mathbf{L}'_a$ defined in the proof of Proposition 2 is a-reflexive.

Let s be a maximal $\mathbf{L}'_a$-consistent set of formulas. Let $(\varphi_1, \varphi_2, \ldots)$ be an enumeration of FOR. Obviously, for all $n \in \mathbb{N}$, $\Diamond^a((\Box\varphi_1 \to \varphi_1) \wedge \ldots \wedge (\Box\varphi_n \to \varphi_n)) \in s$. Hence, for all $n \in \mathbb{N}$, there exists a n-tuples $(\epsilon^1_1, \ldots, \epsilon^1_n), \ldots, (\epsilon^{a-1}_1, \ldots, \epsilon^{a-1}_n), (\epsilon^a_1, \ldots, \epsilon^a_n)$ of bits such that $\Diamond((\varphi_1^{\epsilon^1_1} \wedge \ldots \wedge \varphi_n^{\epsilon^1_n}) \wedge \ldots \Diamond((\varphi_1^{\epsilon^{a-1}_1} \wedge \ldots \wedge \varphi_n^{\epsilon^{a-1}_n}) \wedge \Diamond((\varphi_1^{\epsilon^a_1} \wedge \ldots \wedge \varphi_n^{\epsilon^a_n}) \wedge (\Box\varphi_1 \to \varphi_1) \wedge \ldots \wedge (\Box\varphi_n \to \varphi_n))) \ldots) \in s$. Thus, there exists a infinite sequences $(\epsilon^1_1, \epsilon^1_2, \ldots), \ldots, (\epsilon^{a-1}_1, \epsilon^{a-1}_2, \ldots), (\epsilon^a_1, \epsilon^a_2, \ldots)$ of bits such that for all $n \in \mathbb{N}$, $\Diamond((\varphi_1^{\epsilon^1_1} \wedge \ldots \wedge \varphi_n^{\epsilon^1_n}) \wedge \ldots \Diamond((\varphi_1^{\epsilon^{a-1}_1} \wedge \ldots \wedge \varphi_n^{\epsilon^{a-1}_n}) \wedge \Diamond((\varphi_1^{\epsilon^a_1} \wedge \ldots \wedge \varphi_n^{\epsilon^a_n}) \wedge (\Box\varphi_1 \to \varphi_1) \wedge \ldots \wedge (\Box\varphi_n \to \varphi_n))) \ldots) \in s$. Let $t_0 = s$. For all $i \in \mathbb{N}$, if $i \geq 1$ and $i \leq a$ then let $t_i = \{\varphi_1^{\epsilon^a_1}, \varphi_2^{\epsilon^a_2}, \ldots\}$. Since for all $n \in \mathbb{N}$, $\Diamond((\varphi_1^{\epsilon^1_1} \wedge \ldots \wedge \varphi_n^{\epsilon^1_n}) \wedge \ldots \Diamond((\varphi_1^{\epsilon^{a-1}_1} \wedge \ldots \wedge \varphi_n^{\epsilon^{a-1}_n}) \wedge \Diamond((\varphi_1^{\epsilon^a_1} \wedge \ldots \wedge \varphi_n^{\epsilon^a_n}) \wedge (\Box\varphi_1 \to \varphi_1) \wedge \ldots \wedge (\Box\varphi_n \to \varphi_n))) \ldots) \in s$, therefore for all $i \in \mathbb{N}$, if $i \geq 1$ and $i \leq a$ then t_i is a maximal $\mathbf{L}'_a$-consistent set of formulas. In other respect, for all $i \in \mathbb{N}$, if $i < a$ then $\Box t_i \subseteq t_{i+1}$. Consequently, for all $i \in \mathbb{N}$, if $i < a$ then $t_i R t_{i+1}$. Moreover, t_a contains all formulas of the form $\Box\varphi \to \varphi$. Hence, $t_a R t_a$.

Annex 2

In this Annex, we prove Lemmas 3, 4, 5, 6, 9, 10, 11, 12, 15, 16, 17 and 22 .

Proof of Lemma 3. Let φ, ψ be formulas. We only consider the "left-to-right" direction of the equivalence, the "right-to-left" direction of the equivalence being proved in a similar way. Suppose $(\varphi \to \Box\psi) \in \mathbf{L}$. Hence, $(\neg\Box\psi \to \neg\varphi) \in \mathbf{L}$. Thus, $(\Box\neg\Box\psi \to \Box\neg\varphi) \in \mathbf{L}$. Since $(\neg\psi \to \Box\neg\Box\psi) \in \mathbf{L}$, therefore $(\neg\psi \to \Box\neg\varphi) \in \mathbf{L}$. ∎

Proof of Lemma 4. Let φ, ψ be formulas. We only consider the "left-to-right" direction of the equivalence, the "right-to-left" direction of the equivalence being proved in a similar way. Suppose $(\varphi \to \boxplus\psi) \in \mathbf{L}$, i.e.

$(\varphi \to (p^0 \wedge q^0 \to \Box(p^1 \wedge q^0 \to \Box(p^0 \wedge q^1 \to \Box(p^0 \wedge q^0 \to \psi))))) \in \mathbf{L}$. Hence, $(p^0 \wedge q^0 \wedge \varphi \to \Box(p^1 \wedge q^0 \to \Box(p^0 \wedge q^1 \to \Box(p^0 \wedge q^0 \to \psi)))) \in \mathbf{L}$. Thus, by Lemma 3, $(\neg(p^1 \wedge q^0 \to \Box(p^0 \wedge q^1 \to \Box(p^0 \wedge q^0 \to \psi))) \to \Box\neg(p^0 \wedge q^0 \wedge \varphi)) \in \mathbf{L}$. Consequently, $(p^1 \wedge q^0 \wedge \neg\Box\neg(p^0 \wedge q^0 \wedge \varphi) \to \Box(p^0 \wedge q^1 \to \Box(p^0 \wedge q^0 \to \psi))) \in \mathbf{L}$. Hence, by Lemma 3, $(\neg(p^0 \wedge q^1 \to \Box(p^0 \wedge q^0 \to \psi)) \to \Box\neg(p^1 \wedge q^0 \wedge \neg\Box\neg(p^0 \wedge q^0 \wedge \varphi))) \in \mathbf{L}$. Thus, $(p^0 \wedge q^1 \wedge \neg\Box\neg(p^1 \wedge q^0 \wedge \neg\Box\neg(p^0 \wedge q^0 \wedge \varphi)) \to \Box(p^0 \wedge q^0 \to \psi)) \in \mathbf{L}$. Consequently, by Lemma 3, $(\neg(p^0 \wedge q^0 \to \psi) \to \Box\neg(p^0 \wedge q^1 \wedge \neg\Box\neg(p^1 \wedge q^0 \wedge \neg\Box\neg(p^0 \wedge q^0 \wedge \varphi)))) \in \mathbf{L}$. Hence, $(\neg\psi \to (p^0 \wedge q^0 \to \Box(p^0 \wedge q^1 \to \Box(p^1 \wedge q^0 \to \Box(p^0 \wedge q^0 \to \neg\varphi))))) \in \mathbf{L}$, i.e. $(\neg\psi \to \boxminus\neg\varphi) \in \mathbf{L}$. ■

Proof of Lemma 5. By induction on k. Suppose for all $k' \in \mathbb{N}$, if $k' < k$ then

- $\boxplus^{k'}\top \in \mathbf{L}$,
- $\boxminus^{k'}\top \in \mathbf{L}$,
- $\boxplus^{<k'}\top \in \mathbf{L}$,
- $\boxminus^{<k'}\top \in \mathbf{L}$.

Case $k = 0$: Hence, $\boxplus^{k}\top = \top$, $\boxminus^{k}\top = \top$, $\boxplus^{<k}\top = \top$ and $\boxminus^{<k}\top = \top$. Thus, $\boxplus^{k}\top \in \mathbf{L}$, $\boxminus^{k}\top \in \mathbf{L}$, $\boxplus^{<k}\top \in \mathbf{L}$ and $\boxminus^{<k}\top \in \mathbf{L}$.
Case $k > 1$: Consequently, by induction hypothesis, $\boxplus^{k-1}\top \in \mathbf{L}$, $\boxminus^{k-1}\top \in \mathbf{L}$, $\boxplus^{<k-1}\top \in \mathbf{L}$ and $\boxminus^{<k-1}\top \in \mathbf{L}$. Hence, $\boxplus\,\boxplus^{k-1}\top \in \mathbf{L}$, $\boxminus\,\boxminus^{k-1}\top \in \mathbf{L}$, $\boxplus^{<k-1}\top \wedge \boxplus^{k-1}\top \in \mathbf{L}$ and $\boxminus^{<k-1}\top \wedge \boxminus^{k-1}\top \in \mathbf{L}$. Thus, $\boxplus^{k}\top \in \mathbf{L}$, $\boxminus^{k}\top \in \mathbf{L}$, $\boxplus^{<k}\top \in \mathbf{L}$ and $\boxminus^{<k}\top \in \mathbf{L}$. ■

Proof of Lemma 6. By induction on k. Suppose for all $k' \in \mathbb{N}$, if $k' < k$ then

- $\boxplus^{<k'+1}\varphi \leftrightarrow \varphi \wedge \boxplus\,\boxplus^{<k'}\varphi \in \mathbf{L}$,
- $\boxminus^{<k'+1}\varphi \leftrightarrow \varphi \wedge \boxminus\,\boxminus^{<k'}\varphi \in \mathbf{L}$.

Case $k = 0$: Hence, $\boxplus^{<k+1}\varphi \leftrightarrow \varphi \wedge \boxplus\,\boxplus^{<k}\varphi = (\top \wedge \varphi) \leftrightarrow (\varphi \wedge \boxplus\top)$ and $\boxminus^{<k+1}\varphi \leftrightarrow \varphi \wedge \boxminus\,\boxminus^{<k}\varphi = (\top \wedge \varphi) \leftrightarrow (\varphi \wedge \boxminus\top)$. Thus, by Lemma 5, $\boxplus^{<k+1}\varphi \leftrightarrow \varphi \wedge \boxplus\,\boxplus^{<k}\varphi \in \mathbf{L}$ and $\boxminus^{<k+1}\varphi \leftrightarrow \varphi \wedge \boxminus\,\boxminus^{<k}\varphi \in \mathbf{L}$.
Case $k > 1$: Consequently, by induction hypothesis, $\boxplus^{<k}\varphi \leftrightarrow \varphi \wedge \boxplus\,\boxplus^{<k-1}\varphi \in \mathbf{L}$ and $\boxminus^{<k}\varphi \leftrightarrow \varphi \wedge \boxminus\,\boxminus^{<k-1}\varphi \in \mathbf{L}$. Hence, $\boxplus^{<k}\varphi \wedge \boxplus^{k}\varphi \leftrightarrow \varphi \wedge \boxplus\,\boxplus^{<k-1}\varphi \wedge \boxplus^{k}\varphi \in \mathbf{L}$ and $\boxminus^{<k}\varphi \wedge \boxminus^{k}\varphi \leftrightarrow \varphi \wedge \boxminus\,\boxminus^{<k-1}\varphi \wedge \boxminus^{k}\varphi \in \mathbf{L}$. Thus, $\boxplus^{<k+1}\varphi \leftrightarrow \varphi \wedge \boxplus\,\boxplus^{<k-1}\varphi \wedge \boxplus\,\boxplus^{k-1}\varphi \in \mathbf{L}$ and $\boxminus^{<k+1}\varphi \leftrightarrow \varphi \wedge \boxminus\,\boxminus^{<k-1}\varphi \wedge \boxminus\,\boxminus^{k-1}\varphi \in \mathbf{L}$. Consequently, $\boxplus^{<k+1}\varphi \leftrightarrow \varphi \wedge \boxplus(\boxplus^{<k-1}\varphi \wedge \boxplus^{k-1}\varphi) \in \mathbf{L}$ and $\boxminus^{<k+1}\varphi \leftrightarrow \varphi \wedge \boxminus(\boxminus^{<k-1}\varphi \wedge \boxminus^{k-1}\varphi) \in \mathbf{L}$. Hence, $\boxplus^{<k+1}\varphi \leftrightarrow \varphi \wedge \boxplus\boxplus^{<k}\varphi \in \mathbf{L}$ and $\boxminus^{<k+1}\varphi \leftrightarrow \varphi \wedge \boxminus\,\boxminus^{<k}\varphi \in \mathbf{L}$. ■

Proof of Lemma 9. By induction on k. Suppose for all $k' \in \mathbb{N}$, if $k' < k$ then

- $\boxplus^{<k'} x \wedge \boxplus^{k'} \bot \rightarrow \sigma_{k'}(x) \in \mathbf{L}$,
- $\boxminus^{<k'} \neg x \wedge \boxminus^{k'} \bot \rightarrow \neg \tau_{k'}(x) \in \mathbf{L}$.

Case $k = 0$: Hence, $\boxplus^{<k} x \wedge \boxplus^{k} \bot \rightarrow \sigma_k(x) = \top \wedge \bot \rightarrow \bot$ and $\boxminus^{<k} \neg x \wedge \boxminus^{k} \bot \rightarrow \neg \tau_k(x) = \top \wedge \bot \rightarrow \neg \top$. Thus, $\boxplus^{<k} x \wedge \boxplus^{k} \bot \rightarrow \sigma_k(x) \in \mathbf{L}$ and $\boxminus^{<k} \neg x \wedge \boxminus^{k} \bot \rightarrow \neg \tau_k(x) \in \mathbf{L}$.
Case $k > 1$: Consequently, by induction hypothesis, $\boxplus^{<k-1} x \wedge \boxplus^{k-1} \bot \rightarrow \sigma_{k-1}(x) \in \mathbf{L}$ and $\boxminus^{<k-1} \neg x \wedge \boxminus^{k-1} \bot \rightarrow \neg \tau_{k-1}(x) \in \mathbf{L}$. Hence, $\boxplus \boxplus^{<k-1} x \wedge \boxplus \boxplus^{k-1} \bot \rightarrow \boxplus \sigma_{k-1}(x) \in \mathbf{L}$ and $\boxminus \boxminus^{<k-1} \neg x \wedge \boxminus \boxminus^{k-1} \bot \rightarrow \boxminus \neg \tau_{k-1}(x) \in \mathbf{L}$. Thus, $x \wedge \boxplus \boxplus^{<k-1} x \wedge \boxplus \boxplus^{k-1} \bot \rightarrow x \wedge \boxplus \sigma_{k-1}(x) \in \mathbf{L}$ and $\neg x \wedge \boxminus \boxminus^{<k-1} \neg x \wedge \boxminus \boxminus^{k-1} \bot \rightarrow \neg x \wedge \boxminus \neg \tau_{k-1}(x) \in \mathbf{L}$. Consequently, by Lemma 6, $\boxplus^{<k} x \wedge \boxplus^{k} \bot \rightarrow \sigma_k(x) \in \mathbf{L}$ and $\boxminus^{<k} \neg x \wedge \boxminus^{k} \bot \rightarrow \neg \tau_k(x) \in \mathbf{L}$. ■

Proof of Lemma 10. We have to consider the following cases.
Case $k = 0$: Hence, $\sigma_k(x) \rightarrow x = \bot \rightarrow x$ and $\neg \tau_k(x) \rightarrow \neg x = \neg \top \rightarrow x$. Thus, $\sigma_k(x) \rightarrow x \in \mathbf{L}$ and $\neg \tau_k(x) \rightarrow \neg x \in \mathbf{L}$.
Case $k > 1$: Consequently, $\sigma_k(x) \rightarrow x = (x \wedge \boxplus \sigma_{k-1}(x)) \rightarrow x$ and $\neg \tau_k(x) \rightarrow \neg x = \neg\neg(\neg x \wedge \boxminus \neg \tau_{k-1}(x)) \rightarrow \neg x$. Hence, $\sigma_k(x) \rightarrow x \in \mathbf{L}$ and $\neg \tau_k(x) \rightarrow \neg x \in \mathbf{L}$. ■

Proof of Lemma 11. By induction on k. Suppose for all $k' \in \mathbb{N}$, if $k' < k$ then

- $\sigma_{k'}(x) \rightarrow \boxplus \sigma_{k'}(x) \in \mathbf{L}$,
- $\neg \tau_{k'}(x) \rightarrow \boxminus \neg \tau_{k'}(x) \in \mathbf{L}$.

Case $k = 0$: Hence, $\sigma_k(x) \rightarrow \boxplus \sigma_k(x) = \bot \rightarrow \boxplus \bot$ and $\neg \tau_k(x) \rightarrow \boxminus \neg \tau_k(x) = \neg \top \rightarrow \boxminus \top$. Thus, $\sigma_k(x) \rightarrow \boxplus \sigma_k(x) \in \mathbf{L}$ and $\neg \tau_k(x) \rightarrow \boxminus \neg \tau_k(x) \in \mathbf{L}$.
Case $k > 1$: Consequently, by induction hypothesis, $\sigma_{k-1}(x) \rightarrow \boxplus \sigma_{k-1}(x) \in \mathbf{L}$ and $\neg \tau_{k-1}(x) \rightarrow \boxminus \neg \tau_{k-1}(x) \in \mathbf{L}$. Hence, by Lemma 10, $\sigma_{k-1}(x) \rightarrow x \wedge \boxplus \sigma_{k-1}(x) \in \mathbf{L}$ and $\neg \tau_{k-1}(x) \rightarrow \neg x \wedge \boxminus \neg \tau_{k-1}(x) \in \mathbf{L}$. Thus, $\sigma_{k-1}(x) \rightarrow \sigma_k(x) \in \mathbf{L}$ and $\neg \tau_{k-1}(x) \rightarrow \neg \tau_k(x) \in \mathbf{L}$. Consequently, $\boxplus \sigma_{k-1}(x) \rightarrow \boxplus \sigma_k(x) \in \mathbf{L}$ and $\boxminus \neg \tau_{k-1}(x) \rightarrow \boxminus \neg \tau_k(x) \in \mathbf{L}$. Since $\sigma_k(x) \rightarrow \boxplus \sigma_{k-1}(x) \in \mathbf{L}$ and $\neg \tau_k(x) \rightarrow \boxminus \neg \tau_{k-1}(x) \in \mathbf{L}$, therefore $\sigma_k(x) \rightarrow \boxplus \sigma_k(x) \in \mathbf{L}$ and $\neg \tau_k(x) \rightarrow \boxminus \neg \tau_k(x) \in \mathbf{L}$. ■

Proof of Lemma 12. By induction on k. Suppose for all $k' \in \mathbb{N}$, if $k' < k$ then for all $l' \in \mathbb{N}$,

- if $k' \leq l'$ then $\sigma_{k'}(x) \rightarrow \boxplus^{l'} \bot \in \mathbf{L}$,

- if $k' \leq l'$ then $\neg\tau_{k'}(x) \to \boxminus^{l'}\bot \in \mathbf{L}$.

Let $l \in \mathbb{N}$ be such that $k \leq l$.
Case $k = 0$**:** Hence, $\sigma_k(x) \to \boxplus^l\bot = \bot \to \boxplus^l\bot$ and $\neg\tau_k(x) \to \boxminus^l\bot = \neg\top \to \boxminus^l\bot$. Thus, $\sigma_k(x) \to \boxplus^l\bot \in \mathbf{L}$ and $\neg\tau_k(x) \to \boxminus^l\bot \in \mathbf{L}$.
Case $k > 1$**:** Since $k \leq l$, therefore $k-1 \leq l-1$ and by induction hypothesis, $\sigma_{k-1}(x) \to \boxplus^{l-1}\bot \in \mathbf{L}$ and $\neg\tau_{k-1}(x) \to \boxminus^{l-1}\bot \in \mathbf{L}$. Consequently, $\boxplus\sigma_{k-1}(x) \to \boxplus\boxplus^{l-1}\bot \in \mathbf{L}$ and $\boxminus\neg\tau_{k-1}(x) \to \boxminus\boxminus^{l-1}\bot \in \mathbf{L}$. Hence, $x \wedge \boxplus\sigma_{k-1}(x) \to \boxplus\boxplus^{l-1}\bot \in \mathbf{L}$ and $\neg x \wedge \boxminus\neg\tau_{k-1}(x) \to \boxminus\boxminus^{l-1}\bot \in \mathbf{L}$. Thus, $\sigma_k(x) \to \boxplus^l\bot \in \mathbf{L}$ and $\neg\tau_k(x) \to \boxminus^l\bot \in \mathbf{L}$. ■

Proof of Lemma 15. By induction on k. Suppose for all $k' \in \mathbb{N}$, if $k' < k$ then for all $l' \in \mathbb{N}$,

- if $k' \leq l'$ then $\boxplus^{k'}\bot \wedge \sigma_{l'}(x) \leftrightarrow \sigma_{k'}(x) \in \mathbf{L}$,
- if $k' \leq l'$ then $\boxminus^{k'}\bot \wedge \neg\tau_{l'}(x) \leftrightarrow \neg\tau_{k'}(x) \in \mathbf{L}$.

Let $l \in \mathbb{N}$ be such that $k \leq l$.
Case $k = 0$**:** Hence, $\boxplus^k\bot \wedge \sigma_l(x) \leftrightarrow \sigma_k(x) = \bot \wedge \sigma_l(x) \leftrightarrow \bot$ and $\boxminus^k\bot \wedge \neg\tau_l(x) \leftrightarrow \neg\tau_k(x) = \bot \wedge \neg\tau_l(x) \leftrightarrow \neg\top$. Thus, $\boxplus^k\bot \wedge \sigma_l(x) \leftrightarrow \sigma_k(x) \in \mathbf{L}$ and $\boxminus^k\bot \wedge \neg\tau_l(x) \leftrightarrow \neg\tau_k(x) \in \mathbf{L}$.
Case $k > 1$**:** Since $k \leq l$, therefore $k-1 \leq l-1$ and by induction hypothesis, $\boxplus^{k-1}\bot \wedge \sigma_{l-1}(x) \leftrightarrow \sigma_{k-1}(x) \in \mathbf{L}$ and $\boxminus^{k-1}\bot \wedge \neg\tau_{l-1}(x) \leftrightarrow \neg\tau_{k-1}(x) \in \mathbf{L}$. Consequently, $\boxplus\boxplus^{k-1}\bot \wedge x \wedge \boxplus\sigma_{l-1}(x) \leftrightarrow x \wedge \boxplus\sigma_{k-1}(x) \in \mathbf{L}$ and $\boxminus\boxminus^{k-1}\bot \wedge \neg x \wedge \boxminus\neg\tau_{l-1}(x) \leftrightarrow \neg x \wedge \boxminus\neg\tau_{k-1}(x) \in \mathbf{L}$. Hence, $\boxplus^k\bot \wedge \sigma_l(x) \leftrightarrow \sigma_k(x) \in \mathbf{L}$ and $\boxminus^k\bot \wedge \neg\tau_l(x) \leftrightarrow \neg\tau_k(x) \in \mathbf{L}$. ■

Proof of Lemma 16. By induction on k. Suppose for all $k' \in \mathbb{N}$, if $k' < k$ then for all $l' \in \mathbb{N}$,

- if $k' \leq l'$ then $\lambda_{l'}(\sigma_{k'}(x)) \leftrightarrow \sigma_{k'}(x) \in \mathbf{L}$,
- if $k' \leq l'$ then $\mu_{l'}(\tau_{k'}(x)) \leftrightarrow \tau_{k'}(x) \in \mathbf{L}$.

Let $l \in \mathbb{N}$ be such that $k \leq l$.
Case $k = 0$**:** Hence, $\lambda_l(\sigma_k(x)) \leftrightarrow \sigma_k(x) = \bot \leftrightarrow \bot$ and $\mu_l(\tau_k(x)) \leftrightarrow \tau_k(x) = \top \leftrightarrow \top$. Thus, $\lambda_l(\sigma_k(x)) \leftrightarrow \sigma_k(x) \in \mathbf{L}$ and $\mu_l(\tau_k(x)) \leftrightarrow \tau_k(x) \in \mathbf{L}$.
Case $k > 1$**:** Since $k \leq l$, therefore $k-1 \leq l$ and by induction hypothesis, $\lambda_l(\sigma_{k-1}(x)) \leftrightarrow \sigma_{k-1}(x) \in \mathbf{L}$ and $\mu_l(\tau_{k-1}(x)) \leftrightarrow \tau_{k-1}(x) \in \mathbf{L}$. Consequently, $x \wedge \boxplus^l\bot \wedge \boxplus\lambda_l(\sigma_{k-1}(x)) \leftrightarrow \boxplus^l\bot \wedge x \wedge \boxplus\sigma_{k-1}(x) \in \mathbf{L}$ and $\neg x \wedge \boxminus^l\bot \wedge \boxminus\neg\mu_l(\tau_{k-1}(x)) \leftrightarrow \boxminus^l\bot \wedge \neg x \wedge \boxminus\neg\tau_{k-1}(x) \in \mathbf{L}$. Hence, $\lambda_l(x \wedge \boxplus\sigma_{k-1}(x)) \leftrightarrow \boxplus^l\bot \wedge \sigma_k(x) \in \mathbf{L}$ and $\mu_l(\neg x \wedge \boxminus\neg\tau_{k-1}(x)) \leftrightarrow \boxminus^l\bot \wedge \neg\tau_k(x) \in \mathbf{L}$. Since $k \leq l$, therefore by Lemma 12, $\lambda_l(\sigma_k(x)) \leftrightarrow \sigma_k(x) \in \mathbf{L}$ and $\mu_l(\tau_k(x)) \leftrightarrow \tau_k(x) \in \mathbf{L}$. ■

Proof of Lemma 17. By induction on k. Suppose for all $k' \in \mathbb{N}$, if $k' < k$ then for all $l' \in \mathbb{N}$,

- if $k' \geq l'$ then $\lambda_{l'}(\sigma_{k'}(x)) \leftrightarrow \sigma_{l'}(x) \in \mathbf{L}$,
- if $k' \geq l'$ then $\mu_{l'}(\tau_{k'}(x)) \leftrightarrow \tau_{l'}(x) \in \mathbf{L}$.

Let $l \in \mathbb{N}$ be such that $k \geq l$.
Case $k = l$: Hence, by Lemma 16, $\lambda_l(\sigma_k(x)) \leftrightarrow \sigma_l(x) \in \mathbf{L}$ and $\mu_l(\tau_k(x)) \leftrightarrow \tau_l(x) \in \mathbf{L}$.
Case $k > l$: Thus, $k-1 \geq l$ and by induction hypothesis, $\lambda_l(\sigma_{k-1}(x)) \leftrightarrow \sigma_l(x) \in \mathbf{L}$ and $\mu_l(\tau_{k-1}(x)) \leftrightarrow \tau_l(x) \in \mathbf{L}$. Consequently, $x \wedge \boxplus^l \bot \wedge \boxplus \lambda_l(\sigma_{k-1}(x)) \leftrightarrow \boxplus^l \bot \wedge x \wedge \boxplus \sigma_l(x) \in \mathbf{L}$ and $\neg x \wedge \boxminus^l \bot \wedge \boxminus \neg \mu_l(\tau_{k-1}(x)) \leftrightarrow \boxminus^l \bot \wedge \neg x \wedge \boxminus \neg \tau_l(x) \in \mathbf{L}$. Hence, $\lambda_l(x \wedge \boxplus \sigma_{k-1}(x)) \leftrightarrow \boxplus^l \bot \wedge \sigma_{l+1}(x) \in \mathbf{L}$ and $\mu_l(\neg x \wedge \boxminus \neg \tau_{k-1}(x)) \leftrightarrow \boxminus^l \bot \wedge \neg \tau_{l+1}(x) \in \mathbf{L}$. Thus, by Lemma 15, $\lambda_l(\sigma_k(x)) \leftrightarrow \sigma_l(x) \in \mathbf{L}$ and $\mu_l(\tau_k(x)) \leftrightarrow \tau_l(x) \in \mathbf{L}$. ■

Proof of Lemma 22. By induction on k. Suppose for all $k' \in \mathbb{N}$, if $k' < k$ then

- $\sigma(x) \rightarrow \boxplus^{<k'} \sigma(x) \in \mathbf{L}$,
- $\neg \sigma(x) \rightarrow \boxminus^{<k'} \neg \sigma(x) \in \mathbf{L}$.

Case $k = 0$: Hence, $\sigma(x) \rightarrow \boxplus^{<k} \sigma(x) = \sigma(x) \rightarrow \top$ and $\neg \sigma(x) \rightarrow \boxminus^{<k} \neg \sigma(x) = \neg \sigma(x) \rightarrow \top$. Thus, $\sigma(x) \rightarrow \boxplus^{<k} \sigma(x) \in \mathbf{L}$ and $\neg \sigma(x) \rightarrow \boxminus^{<k} \neg \sigma(x) \in \mathbf{L}$.
Case $k > 1$: Consequently, by induction hypothesis, $\sigma(x) \rightarrow \boxplus^{<k-1} \sigma(x) \in \mathbf{L}$ and $\neg \sigma(x) \rightarrow \boxminus^{<k-1} \neg \sigma(x) \in \mathbf{L}$. Obviously, by Lemma 11, $\sigma(x) \rightarrow \boxplus^{k-1} \sigma(x) \in \mathbf{L}$ and $\neg \sigma(x) \rightarrow \boxminus^{k-1} \neg \sigma(x) \in \mathbf{L}$. Since $\sigma(x) \rightarrow \boxplus^{<k-1} \sigma(x) \in \mathbf{L}$ and $\neg \sigma(x) \rightarrow \boxminus^{<k-1} \neg \sigma(x) \in \mathbf{L}$, therefore $\sigma(x) \rightarrow \boxplus^{<k-1} \sigma(x) \wedge \boxplus^{k-1} \sigma(x) \in \mathbf{L}$ and $\neg \sigma(x) \rightarrow \boxminus^{<k-1} \neg \sigma(x) \wedge \boxminus^{k-1} \neg \sigma(x) \in \mathbf{L}$. Hence, $\sigma(x) \rightarrow \boxplus^{<k} \sigma(x) \in \mathbf{L}$ and $\neg \sigma(x) \rightarrow \boxminus^{<k} \neg \sigma(x) \in \mathbf{L}$. ■

References

[1] Baader, F., and S. Ghilardi, Unification in modal and description logics, *Logic Journal of the IGPL* 19:705–730, 2011.

[2] Baader, F., and W. Snyder, Unification theory, in *Handbook of Automated Reasoning,* Elsevier, 2001, pp. 439–526.

[3] Balbiani, P., Remarks about the unification type of some non-symmetric non-transitive modal logics, *Logic Journal of the IGPL* 27:639–658, 2019.

[4] Balbiani, P., and Ç. Gencer, *KD* is nullary, *Journal of Applied Non-Classical Logics* 27:196–205, 2017.

[5] Balbiani, P., and Ç. Gencer, Unification in epistemic logics, *Journal of Applied Non-Classical Logics* 27:91–105, 2017.

[6] Balbiani, P., I. Shapirovsky, and V. Shehtman, Every world can see a Sahlqvist world, in *Advances in Modal Logic,* College Publications, 2006, pp. 69–85.

[7] Balbiani, P., and T. Tinchev, Unification in modal logic Alt_1, in *Advances in Modal Logic,* College Publications, 2016, pp. 117–134.

[8] Balbiani, P., and T. Tinchev, Elementary unification in modal logic $KD45$, *Journal of Applied Logic — IFCoLog Journal of Logics and their Applications* 5:301–317, 2018.

[9] Blackburn, P., M. de Rijke, and Y. Venema, *Modal Logic,* Cambridge University Press, 2001.

[10] Chagrov, A., and M. Zakharyaschev, *Modal Logic,* Oxford University Press, 1997.

[11] Chellas, B., *Modal Logic. An Introduction,* Cambridge University Press, 1980.

[12] Dzik, W., *Unification Types in Logic,* Wydawnicto Uniwersytetu Slaskiego, 2007.

[13] Dzik, W., and P. Wojtylak, Projective unification in modal logic, *Logic Journal of the IGPL* 20:121–153, 2012.

[14] Ghilardi, S., Best solving modal equations, *Annals of Pure and Applied Logic* 102:183–198, 2000.

[15] Jeřábek, E., Complexity of admissible rules, *Archive for Mathematical Logic* 46:73–92, 2007.

[16] Jeřábek, E., Blending margins: the modal logic K has nullary unification type, *Journal of Logic and Computation* 25:1231–1240, 2015.

[17] Jeřábek, E., Rules with parameters in modal logic I, *Annals of Pure and Applied Logic* 166:881–933, 2015.

[18] Kurucz, A., Combining modal logics, in *Handbook of Modal Logic,* Elsevier, 2007, pp. 869–924.

[19] Rybakov, V., A criterion for admissibility of rules in the model system **S**4 and the intuitionistic logic, *Algebra and Logic* 23:369–384, 1984.

[20] Rybakov, V., *Admissibility of Logical Inference Rules,* Elsevier Science, 1997.

P. Balbiani, Ç. Gencer
Institut de recherche en informatique de Toulouse
CNRS - Toulouse University
Toulouse
France
Philippe.Balbiani@irit.fr

Ç. Gencer
Faculty of Arts and Sciences
Aydın University
Istanbul
Turkey
cigdemgencer@aydin.edu.tr,cigdem.gencer@irit.fr