

Supplementary Material document for Joint inference on extreme expectiles for multivariate heavy-tailed distributions

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This supplementary material document provides technical lemmas and the proofs of all theoretical results in the main article as well as additional details on the construction of confidence regions and further finite-sample results.

We denote by \xrightarrow{d} and $\xrightarrow{\mathbb{P}}$ weak convergence and convergence in probability of sequences of random variables, respectively, and by $\mathbb{1}\{E\}$ the indicator function of the event E .

A Proofs of main and auxiliary results

We begin this section by stating a couple of facts that will be used multiple times in our proofs. If Z , a random variable having survival function \bar{F} and tail quantile function U , is heavy-tailed with tail index γ , and if $\tau_n \uparrow 1$ and $z_n \rightarrow \infty$ as $n \rightarrow \infty$, then the convergences

$$\frac{\bar{F}(z_n)}{1 - \tau_n} \rightarrow z \in (0, \infty) \quad \text{and} \quad \frac{z_n}{U((1 - \tau_n)^{-1})} \rightarrow z^{-\gamma} \quad \text{as } n \rightarrow \infty$$

are equivalent. Such a local inversion property is, for instance, a consequence of Definition B.1.8 and Proposition B.1.9.10 pp. 366–367 in [de Haan and Ferreira \(2006\)](#). In particular, if $\mathbb{E}|\min(Z, 0)| < \infty$ and $\gamma < 1$, then high expectiles ξ_{τ_n} of Z are well-defined and satisfy

$$\frac{\bar{F}(\xi_{\tau_n})}{1 - \tau_n} \rightarrow \gamma^{-1} - 1 \quad \text{and} \quad \frac{\xi_{\tau_n}}{U((1 - \tau_n)^{-1})} \rightarrow (\gamma^{-1} - 1)^{-\gamma} \quad \text{as } n \rightarrow \infty.$$

See for instance Proposition 1 in [Daouia et al. \(2019\)](#) in the case $p = 2$ (with the notation therein). In what follows, this will be referred to as the asymptotic proportionality of expectiles and quantiles.

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We also recall that, for the heavy-tailed random variable Z with tail index γ , if $a > 0$ is such that $\mathbb{E}|\min(Z, 0)|^a < \infty$ and $\gamma < 1/a$, then $\mathbb{E}|Z|^a < \infty$. Finally, for any $a \geq 1$, if $\gamma < 1/a$ then it is a corollary of Proposition B.1.10 p.369 in [de Haan and Ferreira \(2006\)](#) that

$$\frac{\mathbb{E}([Z - t]^a \mathbb{1}\{Z > t\})}{at^a \bar{F}(t)} = \int_1^\infty (v-1)^{a-1} \frac{\bar{F}(tv)}{\bar{F}(t)} dv \rightarrow \int_1^\infty (v-1)^{a-1} v^{-1/\gamma} dv \text{ as } t \rightarrow \infty. \quad (\text{A.1})$$

We now state and prove some of the crucial auxiliary results we shall use to show our main results. We first state several convergences related to the rescaled bivariate exceedance probability (above high thresholds).

Lemma A.1. *Assume that X and Y are heavy-tailed random variables with continuous distribution functions F_X and F_Y , tail quantile functions U_X and U_Y and tail indices γ_X and γ_Y . Suppose that there is a function R on $[0, \infty]^2 \setminus \{(\infty, \infty)\}$ such that*

$$\forall (x, y) \in [0, \infty]^2 \setminus \{(\infty, \infty)\}, \lim_{s \rightarrow \infty} s \mathbb{P}\left(\bar{F}_X(X) \leq \frac{x}{s}, \bar{F}_Y(Y) \leq \frac{y}{s}\right) = R(x, y).$$

Let $\tau_n \uparrow 1$ and a_n, b_n be two sequences such that $a_n/U_X((1 - \tau_n)^{-1}) \rightarrow a \in (0, \infty)$ and $b_n/U_Y((1 - \tau_n)^{-1}) \rightarrow b \in (0, \infty)$ as $n \rightarrow \infty$.

(i) For any $x, y > 0$, we have, as $n \rightarrow \infty$,

$$\frac{1}{1 - \tau_n} \mathbb{P}(X > a_n x, Y > b_n y) \rightarrow R(a^{-1/\gamma_X} x^{-1/\gamma_X}, b^{-1/\gamma_Y} y^{-1/\gamma_Y}).$$

(ii) For any $x > 0$ and any $\beta \in \mathbb{R}$ such that $1/\gamma_Y > 1 - \beta$, we have, as $n \rightarrow \infty$,

$$\frac{1}{b_n^{1-\beta}} \int_{b_n}^\infty \frac{1}{1 - \tau_n} \mathbb{P}(X > a_n x, Y > t) \frac{dt}{t^\beta} \rightarrow \int_1^\infty R(a^{-1/\gamma_X} x^{-1/\gamma_X}, b^{-1/\gamma_Y} y^{-1/\gamma_Y}) \frac{dy}{y^\beta}.$$

(iii) For any $\alpha, \beta \in \mathbb{R}$ such that $1/\gamma_X > 2(1 - \beta)$ and $1/\gamma_Y > 2(1 - \beta)$, we have, as $n \rightarrow \infty$,

$$\begin{aligned} & \frac{1}{a_n^{1-\alpha}} \frac{1}{b_n^{1-\beta}} \int_{s=a_n}^\infty \int_{t=b_n}^\infty \frac{1}{1 - \tau_n} \mathbb{P}(X > s, Y > t) \frac{ds}{s^\alpha} \frac{dt}{t^\beta} \\ & \rightarrow \iint_{[1, \infty)^2} R(a^{-1/\gamma_X} x^{-1/\gamma_X}, b^{-1/\gamma_Y} y^{-1/\gamma_Y}) \frac{dx}{x^\alpha} \frac{dy}{y^\beta}. \end{aligned}$$

Proof of Lemma A.1. (i) Pick $x, y > 0$. Since F_X and F_Y are continuous,

$$\lim_{s \rightarrow \infty} s \mathbb{P}(X > U_X(s/x), Y > U_Y(s/y)) = R(x, y).$$

Pick an arbitrary $\varepsilon \in (0, 1)$. Using the regular variation property of U_X and U_Y , we find

$$\begin{aligned} & \frac{a_n x}{U_X((1 - \varepsilon)(a_n x)^{1/\gamma_X} (1 - \tau_n)^{-1})} \rightarrow (1 - \varepsilon)^{-\gamma_X} > 1 \\ \text{and } & \frac{b_n y}{U_Y((1 - \varepsilon)(b_n y)^{1/\gamma_Y} (1 - \tau_n)^{-1})} \rightarrow (1 - \varepsilon)^{-\gamma_Y} > 1, \end{aligned}$$

as $n \rightarrow \infty$. By 1-homogeneity of R , this yields

$$\limsup_{n \rightarrow \infty} \frac{1}{1 - \tau_n} \mathbb{P}(X > a_n x, Y > b_n y) \leq (1 - \varepsilon)^{-1} R(a^{-1/\gamma_X} x^{-1/\gamma_X}, b^{-1/\gamma_Y} y^{-1/\gamma_Y}).$$

An analogue lower bound for the limit inferior holds with ε replaced by $-\varepsilon$. Let $\varepsilon \downarrow 0$ to complete the proof.

(ii) A change of variables yields

$$\frac{1}{b_n^{1-\beta}} \int_{b_n}^{\infty} \frac{1}{1 - \tau_n} \mathbb{P}(X > a_n x, Y > t) \frac{dt}{t^\beta} = \int_1^{\infty} \frac{1}{1 - \tau_n} \mathbb{P}(X > a_n x, Y > b_n y) \frac{dy}{y^\beta}.$$

Next, we apply (i) in the present Lemma and we swap limit and integral as follows. From Potter bounds (see Proposition B.1.9.5 in [de Haan and Ferreira, 2006](#)), for any $\delta > 0$, we have for n large enough

$$\frac{1}{1 - \tau_n} \frac{\mathbb{P}(X > a_n x, Y > b_n y)}{y^\beta} \leq \frac{1}{1 - \tau_n} \frac{\mathbb{P}(Y > b_n y)}{y^\beta} \leq D y^{-1/\gamma_Y - \beta + \delta}$$

for all $y \geq 1$, where D is some unimportant positive constant. Since $-1/\gamma_Y - \beta < -1$, the right-hand side defines an integrable function on $[1, \infty)$ for $\delta > 0$ small enough, and thus the dominated convergence theorem yields (ii).

(iii) A change of variables yields

$$\begin{aligned} & \frac{1}{a_n^{1-\alpha}} \frac{1}{b_n^{1-\beta}} \int_{s=a_n}^{\infty} \int_{t=b_n}^{\infty} \frac{1}{1 - \tau_n} \mathbb{P}(X > s, Y > t) \frac{ds}{s^\alpha} \frac{dt}{t^\beta} \\ &= \iint_{[1, \infty)^2} \frac{1}{1 - \tau_n} \mathbb{P}(X > a_n x, Y > b_n y) \frac{dx}{x^\alpha} \frac{dy}{y^\beta}. \end{aligned}$$

We apply again (i) in the present Lemma and we swap limit and integral as follows. Remark that

$$\mathbb{P}(X > a_n x, Y > b_n y) \leq \sqrt{\mathbb{P}(X > a_n x)} \sqrt{\mathbb{P}(Y > b_n y)}.$$

It then follows from Potter bounds again that, for any $\delta > 0$, we have for n large enough

$$\frac{1}{1 - \tau_n} \frac{\mathbb{P}(X > a_n x, Y > b_n y)}{x^\alpha y^\beta} \leq D x^{-1/[2\gamma_X] - \alpha + \delta} y^{-1/[2\gamma_Y] - \beta + \delta}$$

for all $x, y \geq 1$, where D is some unimportant positive constant. Since $-1/[2\gamma_X] - \alpha < -1$ and $-1/[2\gamma_Y] - \beta < -1$, the right-hand side defines an integrable function on $[1, \infty)^2$ for $\delta > 0$ small enough, and we conclude once again using the dominated convergence theorem. \square

Our next auxiliary result is a general weak L^1 law of large numbers for triangular arrays. It is an extension of Theorem 1 p.356 in [Chow and Teicher \(1997\)](#).

Lemma A.2. *Let $(Z_{ni})_{1 \leq i \leq n}$ be an infinite double array of random variables. Assume that*

- For any $n \geq 1$, the Z_{ni} , $1 \leq i \leq n$, are i.i.d. copies of an integrable random variable Z_n ;
- One has $n \mathbb{P}(|Z_n| > 1) \rightarrow 0$ as $n \rightarrow \infty$;
- One has $n \mathbb{E}(|Z_n| \mathbb{1}\{|Z_n| > 1\}) \rightarrow 0$ as $n \rightarrow \infty$;
- One has $n \text{Var}(Z_n \mathbb{1}\{|Z_n| \leq 1\}) \rightarrow 0$ as $n \rightarrow \infty$.

Then $\sum_{i=1}^n [Z_{ni} - \mathbb{E}(Z_n)] \xrightarrow{\mathbb{P}} 0$.

Proof of Lemma A.2. That $\sum_{i=1}^n [Z_{ni} - \mathbb{E}(Z_n \mathbb{1}\{|Z_n| \leq 1\})] \xrightarrow{\mathbb{P}} 0$ is a consequence of Theorem 1 p.356 in [Chow and Teicher \(1997\)](#). Besides

$$n |\mathbb{E}(Z_n) - \mathbb{E}(Z_n \mathbb{1}\{|Z_n| \leq 1\})| \leq n \mathbb{E}(|Z_n| \mathbb{1}\{|Z_n| > 1\}) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

which, combined with the above convergence in probability, concludes the proof. \square

Lemma A.3 is key for the calculation of the limiting covariance structure in Theorem 2.1 of the main paper and its estimation. This lemma is written in terms of the derivative φ_τ of $\eta_\tau/2$, defined by $\varphi_\tau(y) = |\tau - \mathbb{1}\{y \leq 0\}|y$.

Lemma A.3. *Assume that Condition A is satisfied. Assume further that there is $\delta > 0$ such that $\mathbb{E}|\min(X_j, 0)|^{2+\delta} < \infty$ and $0 < \gamma_j < 1/2$ for any $1 \leq j \leq d$. Let $\tau_n \uparrow 1$ as $n \rightarrow \infty$.*

(i) *For any $j, \ell \in \{1, \dots, d\}$ with $j \leq \ell$, we have, as $n \rightarrow \infty$,*

$$\begin{aligned} \frac{\text{Cov}(\varphi_{\tau_n}(X_j - \xi_{\tau_n,j}), \varphi_{\tau_n}(X_\ell - \xi_{\tau_n,\ell}))}{(1 - \tau_n)\xi_{\tau_n,j}\xi_{\tau_n,\ell}} &= \frac{\mathbb{E}(\varphi_{\tau_n}(X_j - \xi_{\tau_n,j})\varphi_{\tau_n}(X_\ell - \xi_{\tau_n,\ell}))}{(1 - \tau_n)\xi_{\tau_n,j}\xi_{\tau_n,\ell}} \\ &\rightarrow \begin{cases} \frac{2\gamma_j}{1 - 2\gamma_j} & \text{if } j = \ell, \\ \iint_{[1,\infty)^2} R_{j,\ell} \left((\gamma_j^{-1} - 1)x_j^{-1/\gamma_j}, (\gamma_\ell^{-1} - 1)x_\ell^{-1/\gamma_\ell} \right) dx_j dx_\ell & \text{if } j < \ell. \end{cases} \end{aligned}$$

(ii) *Suppose further that $n(1 - \tau_n) \rightarrow \infty$ as $n \rightarrow \infty$. Then, for any $j, \ell \in \{1, \dots, d\}$ with $j \leq \ell$, we have*

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n \frac{\varphi_{\tau_n}(X_{i,j} - \xi_{\tau_n,j})\varphi_{\tau_n}(X_{i,\ell} - \xi_{\tau_n,\ell})}{(1 - \tau_n)\xi_{\tau_n,j}\xi_{\tau_n,\ell}} \\ \xrightarrow{\mathbb{P}} \begin{cases} \frac{2\gamma_j}{1 - 2\gamma_j} & \text{if } j = \ell, \\ \iint_{[1,\infty)^2} R_{j,\ell} \left((\gamma_j^{-1} - 1)x_j^{-1/\gamma_j}, (\gamma_\ell^{-1} - 1)x_\ell^{-1/\gamma_\ell} \right) dx_j dx_\ell & \text{if } j < \ell. \end{cases} \end{aligned}$$

Proof of Lemma A.3. We work in the case $j \neq \ell$; the case $j = \ell$ is proved in the same way by noting that (X_j, X_ℓ) is perfectly dependent with (tail) copula $R_{j,j}(s, t) = \min(s, t)$.

We start by showing (i). Since η_τ is continuously differentiable with derivative $2\varphi_\tau$, we have $\mathbb{E}(\varphi_{\tau_n}(X_j - \xi_{\tau_n,j})) = 0$ for any $j \in \{1, \dots, d\}$, by definition of $\xi_{\tau_n,j}$. Write then

$$\begin{aligned} & \text{Cov}(\varphi_{\tau_n}(X_j - \xi_{\tau_n,j}), \varphi_{\tau_n}(X_\ell - \xi_{\tau_n,\ell})) \\ &= (1 - \tau_n)^2 \mathbb{E}((X_j - \xi_{\tau_n,j})(X_\ell - \xi_{\tau_n,\ell}) \mathbb{1}\{X_j \leq \xi_{\tau_n,j}\} \mathbb{1}\{X_\ell \leq \xi_{\tau_n,\ell}\}) \\ &+ \tau_n(1 - \tau_n) \mathbb{E}((X_j - \xi_{\tau_n,j})(X_\ell - \xi_{\tau_n,\ell}) \mathbb{1}\{X_j \leq \xi_{\tau_n,j}\} \mathbb{1}\{X_\ell > \xi_{\tau_n,\ell}\}) \\ &+ \tau_n(1 - \tau_n) \mathbb{E}((X_j - \xi_{\tau_n,j})(X_\ell - \xi_{\tau_n,\ell}) \mathbb{1}\{X_j > \xi_{\tau_n,j}\} \mathbb{1}\{X_\ell \leq \xi_{\tau_n,\ell}\}) \\ &+ \tau_n^2 \mathbb{E}((X_j - \xi_{\tau_n,j})(X_\ell - \xi_{\tau_n,\ell}) \mathbb{1}\{X_j > \xi_{\tau_n,j}\} \mathbb{1}\{X_\ell > \xi_{\tau_n,\ell}\}) \\ &= u_{1,n} + u_{2,n} + u_{3,n} + u_{4,n}. \end{aligned} \quad (\text{A.2})$$

We deal with each term separately. Clearly, by the Cauchy-Schwarz inequality and the square-integrability of the X_j ,

$$|u_{1,n}| \leq (1 - \tau_n)^2 \sqrt{\mathbb{E}((X_j - \xi_{\tau_n,j})^2)} \sqrt{\mathbb{E}((X_\ell - \xi_{\tau_n,\ell})^2)} = O((1 - \tau_n)^2 \xi_{\tau_n,j} \xi_{\tau_n,\ell}). \quad (\text{A.3})$$

Then, by the Cauchy-Schwarz inequality again,

$$\begin{aligned} |u_{2,n}| &\leq (1 - \tau_n) \sqrt{\mathbb{E}((X_j - \xi_{\tau_n,j})^2)} \sqrt{\mathbb{E}((X_\ell - \xi_{\tau_n,\ell})^2 \mathbb{1}\{X_\ell > \xi_{\tau_n,\ell}\})} \\ &= O((1 - \tau_n)^{3/2} \xi_{\tau_n,j} \xi_{\tau_n,\ell}) \end{aligned} \quad (\text{A.4})$$

thanks to (A.1) with $a = 2$ and asymptotic proportionality of expectiles and quantiles. Similarly

$$|u_{3,n}| = O((1 - \tau_n)^{3/2} \xi_{\tau_n,j} \xi_{\tau_n,\ell}). \quad (\text{A.5})$$

We conclude by evaluating the limit of $u_{4,n}$. By an integration by parts,

$$u_{4,n} = \int_{s=\xi_{\tau_n,j}}^{\infty} \int_{t=\xi_{\tau_n,\ell}}^{\infty} \mathbb{P}(X_j > s, X_\ell > t) ds dt (1 + o(1)).$$

Combine the asymptotic proportionality of expectiles and quantiles with Lemma A.1(iii) to get, as $n \rightarrow \infty$,

$$\frac{1}{1 - \tau_n} \frac{1}{\xi_{\tau_n,j} \xi_{\tau_n,\ell}} u_{4,n} \rightarrow \iint_{[1,\infty)^2} R_{j,\ell} \left((\gamma_j^{-1} - 1) x_j^{-1/\gamma_j}, (\gamma_\ell^{-1} - 1) x_\ell^{-1/\gamma_\ell} \right) dx_j dx_\ell. \quad (\text{A.6})$$

Combine (A.2), (A.3), (A.4), (A.5) and (A.6) to conclude the proof of (i).

We now move on to the proof of (ii), for which, according to (i), it is enough to show that

$$\frac{1}{n} \sum_{i=1}^n \frac{\varphi_{\tau_n}(X_{i,j} - \xi_{\tau_n,j}) \varphi_{\tau_n}(X_{i,\ell} - \xi_{\tau_n,\ell}) - \mathbb{E}(\varphi_{\tau_n}(X_j - \xi_{\tau_n,j}) \varphi_{\tau_n}(X_\ell - \xi_{\tau_n,\ell}))}{(1 - \tau_n) \xi_{\tau_n,j} \xi_{\tau_n,\ell}} \xrightarrow{\mathbb{P}} 0. \quad (\text{A.7})$$

Similarly to the early stages of the proof of (i), we break $\varphi_{\tau_n}(X_{i,j} - \xi_{\tau_n,j})\varphi_{\tau_n}(X_{i,\ell} - \xi_{\tau_n,\ell})$ into

$$\begin{aligned} & \varphi_{\tau_n}(X_{i,j} - \xi_{\tau_n,j})\varphi_{\tau_n}(X_{i,\ell} - \xi_{\tau_n,\ell}) \\ &= (1 - \tau_n)^2(X_{i,j} - \xi_{\tau_n,j})(X_{i,\ell} - \xi_{\tau_n,\ell})\mathbb{1}\{X_{i,j} \leq \xi_{\tau_n,j}\}\mathbb{1}\{X_{i,\ell} \leq \xi_{\tau_n,\ell}\} \\ &+ \tau_n(1 - \tau_n)(X_{i,j} - \xi_{\tau_n,j})(X_{i,\ell} - \xi_{\tau_n,\ell})\mathbb{1}\{X_{i,j} \leq \xi_{\tau_n,j}\}\mathbb{1}\{X_{i,\ell} > \xi_{\tau_n,\ell}\} \\ &+ \tau_n(1 - \tau_n)(X_{i,j} - \xi_{\tau_n,j})(X_{i,\ell} - \xi_{\tau_n,\ell})\mathbb{1}\{X_{i,j} > \xi_{\tau_n,j}\}\mathbb{1}\{X_{i,\ell} \leq \xi_{\tau_n,\ell}\} \\ &+ \tau_n^2(X_{i,j} - \xi_{\tau_n,j})(X_{i,\ell} - \xi_{\tau_n,\ell})\mathbb{1}\{X_{i,j} > \xi_{\tau_n,j}\}\mathbb{1}\{X_{i,\ell} > \xi_{\tau_n,\ell}\}. \end{aligned}$$

Now clearly, using (A.2), (A.3), (A.4), (A.5) and (A.6), we have, as $n \rightarrow \infty$,

$$\begin{aligned} & \mathbb{E} \left(\frac{1}{n} \sum_{i=1}^n \frac{(1 - \tau_n)^2(X_{i,j} - \xi_{\tau_n,j})(X_{i,\ell} - \xi_{\tau_n,\ell})\mathbb{1}\{X_{i,j} \leq \xi_{\tau_n,j}\}\mathbb{1}\{X_{i,\ell} \leq \xi_{\tau_n,\ell}\}}{(1 - \tau_n)\xi_{\tau_n,j}\xi_{\tau_n,\ell}} \right) \rightarrow 0, \\ & \mathbb{E} \left(-\frac{1}{n} \sum_{i=1}^n \frac{\tau_n(1 - \tau_n)(X_{i,j} - \xi_{\tau_n,j})(X_{i,\ell} - \xi_{\tau_n,\ell})\mathbb{1}\{X_{i,j} \leq \xi_{\tau_n,j}\}\mathbb{1}\{X_{i,\ell} > \xi_{\tau_n,\ell}\}}{(1 - \tau_n)\xi_{\tau_n,j}\xi_{\tau_n,\ell}} \right) \rightarrow 0, \\ & \mathbb{E} \left(-\frac{1}{n} \sum_{i=1}^n \frac{\tau_n(1 - \tau_n)(X_{i,j} - \xi_{\tau_n,j})(X_{i,\ell} - \xi_{\tau_n,\ell})\mathbb{1}\{X_{i,j} > \xi_{\tau_n,j}\}\mathbb{1}\{X_{i,\ell} \leq \xi_{\tau_n,\ell}\}}{(1 - \tau_n)\xi_{\tau_n,j}\xi_{\tau_n,\ell}} \right) \rightarrow 0, \\ & \text{and } \mathbb{E} \left(\frac{1}{n} \sum_{i=1}^n \frac{(1 - \tau_n^2)(X_{i,j} - \xi_{\tau_n,j})(X_{i,\ell} - \xi_{\tau_n,\ell})\mathbb{1}\{X_{i,j} > \xi_{\tau_n,j}\}\mathbb{1}\{X_{i,\ell} > \xi_{\tau_n,\ell}\}}{(1 - \tau_n)\xi_{\tau_n,j}\xi_{\tau_n,\ell}} \right) \rightarrow 0. \end{aligned}$$

Furthermore, each of the integrands is a sequence of nonnegative random variables, so that the above convergences ensure that the integrands themselves converge in probability to 0. Use then the notation $Z_+ = \max(Z, 0)$ so that to prove (A.7), it is enough to show the convergence

$$\frac{1}{n} \sum_{i=1}^n \frac{(X_{i,j} - \xi_{\tau_n,j})_+(X_{i,\ell} - \xi_{\tau_n,\ell})_+ - \mathbb{E}((X_j - \xi_{\tau_n,j})_+(X_\ell - \xi_{\tau_n,\ell})_+)}{(1 - \tau_n)\xi_{\tau_n,j}\xi_{\tau_n,\ell}} \xrightarrow{\mathbb{P}} 0.$$

For this the key argument is Lemma A.2 applied to the random variables

$$Z_{ni} = \frac{1}{n} \frac{(X_{i,j} - \xi_{\tau_n,j})_+(X_{i,\ell} - \xi_{\tau_n,\ell})_+}{(1 - \tau_n)\xi_{\tau_n,j}\xi_{\tau_n,\ell}} \quad (1 \leq i \leq n).$$

We check each of the assumptions of this Lemma. Clearly, for each n , the Z_{ni} , $1 \leq i \leq n$ are independent copies of the random variable

$$Z_n = \frac{1}{n} \frac{(X_j - \xi_{\tau_n,j})_+(X_\ell - \xi_{\tau_n,\ell})_+}{(1 - \tau_n)\xi_{\tau_n,j}\xi_{\tau_n,\ell}}.$$

The variable Z_n is integrable, by the Cauchy-Schwarz inequality. Note next that

$$\{|Z_n| > 1\} \subset \left\{ \frac{(X_j - \xi_{\tau_n,j})_+}{\xi_{\tau_n,j}\sqrt{n(1 - \tau_n)}} > 1 \right\} \cup \left\{ \frac{(X_\ell - \xi_{\tau_n,\ell})_+}{\xi_{\tau_n,\ell}\sqrt{n(1 - \tau_n)}} > 1 \right\}. \quad (\text{A.8})$$

As a consequence of (A.8),

$$n \mathbb{P}(|Z_n| > 1) \leq n \bar{F}_j \left(\xi_{\tau_n, j} (1 + \sqrt{n(1 - \tau_n)}) \right) + n \bar{F}_\ell \left(\xi_{\tau_n, \ell} (1 + \sqrt{n(1 - \tau_n)}) \right).$$

Let now $\varepsilon > 0$ be such that $(1 + \varepsilon)2\gamma_j < 1$, for any $j \in \{1, \dots, d\}$. By Potter bounds (see Proposition B.1.9.5 in [de Haan and Ferreira, 2006](#)) and asymptotic proportionality of expectiles and quantiles, we obtain that

$$n \mathbb{P}(|Z_n| > 1) = O \left([n(1 - \tau_n)]^{1-1/2\gamma_j+\varepsilon} + [n(1 - \tau_n)]^{1-1/2\gamma_\ell+\varepsilon} \right) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (\text{A.9})$$

Next, we find, by combining (A.8) with the Cauchy-Schwarz inequality,

$$\begin{aligned} & n \mathbb{E}(|Z_n| \mathbb{1}\{|Z_n| > 1\}) \\ & \leq \frac{1}{(1 - \tau_n) \xi_{\tau_n, j} \xi_{\tau_n, \ell}} \sqrt{\mathbb{E}((X_j - \xi_{\tau_n, j})_+^2 \mathbb{1}\{X_j > \xi_{\tau_n, j}(1 + \sqrt{n(1 - \tau_n)})\})} \sqrt{\mathbb{E}((X_\ell - \xi_{\tau_n, \ell})_+^2)} \\ & + \frac{1}{(1 - \tau_n) \xi_{\tau_n, j} \xi_{\tau_n, \ell}} \sqrt{\mathbb{E}((X_\ell - \xi_{\tau_n, \ell})_+^2 \mathbb{1}\{X_\ell > \xi_{\tau_n, \ell}(1 + \sqrt{n(1 - \tau_n)})\})} \sqrt{\mathbb{E}((X_j - \xi_{\tau_n, j})_+^2)}. \end{aligned}$$

Apply first convergence (A.1) with $a = 2$ to get, for n large enough,

$$\begin{aligned} & n \mathbb{E}(|Z_n| \mathbb{1}\{|Z_n| > 1\}) \\ & = O \left(\frac{1}{\xi_{\tau_n, j} \sqrt{1 - \tau_n}} \sqrt{\mathbb{E}((X_j - \xi_{\tau_n, j})_+^2 \mathbb{1}\{X_j > \xi_{\tau_n, j}(1 + \sqrt{n(1 - \tau_n)})\})} \right. \\ & \quad \left. + \frac{1}{\xi_{\tau_n, \ell} \sqrt{1 - \tau_n}} \sqrt{\mathbb{E}((X_\ell - \xi_{\tau_n, \ell})_+^2 \mathbb{1}\{X_\ell > \xi_{\tau_n, \ell}(1 + \sqrt{n(1 - \tau_n)})\})} \right). \end{aligned}$$

We use now the identity

$$\begin{aligned} & \mathbb{E}((Z - t)^2 \mathbb{1}\{Z > t + s\}) \\ & = \mathbb{E}((Z - t - s)^2 \mathbb{1}\{Z > t + s\}) + 2s \mathbb{E}((Z - t - s) \mathbb{1}\{Z > t + s\}) + s^2 \mathbb{P}(Z > t + s) \end{aligned}$$

valid for any square-integrable random variable Z and any $s, t > 0$, along with convergence (A.1) for $a \in \{1, 2\}$, to find, for any $j \in \{1, \dots, d\}$,

$$\frac{\mathbb{E}((X_j - \xi_{\tau_n, j})_+^2 \mathbb{1}\{X_j > \xi_{\tau_n, j}(1 + \sqrt{n(1 - \tau_n)})\})}{(1 - \tau_n) \xi_{\tau_n, j}^2} = O \left(n \bar{F}_j \left(\xi_{\tau_n, j} (1 + \sqrt{n(1 - \tau_n)}) \right) \right).$$

This is part of the upper bound we obtained for the control of $n \mathbb{P}(|Z_n| > 1)$, and therefore

$$\frac{\mathbb{E}((X_j - \xi_{\tau_n, j})_+^2 \mathbb{1}\{X_j > \xi_{\tau_n, j}(1 + \sqrt{n(1 - \tau_n)})\})}{(1 - \tau_n) \xi_{\tau_n, j}^2} = O \left([n(1 - \tau_n)]^{1-1/2\gamma_j+\varepsilon} \right). \quad (\text{A.10})$$

Conclude that, as $n \rightarrow \infty$,

$$n \mathbb{E}(|Z_n| \mathbb{1}\{|Z_n| > 1\}) = O \left([n(1 - \tau_n)]^{1-1/2\gamma_j+\varepsilon} + [n(1 - \tau_n)]^{1-1/2\gamma_\ell+\varepsilon} \right) \rightarrow 0. \quad (\text{A.11})$$

We finish the proof by controlling $n \text{Var}(Z_n \mathbb{1}\{|Z_n| \leq 1\})$ which, since $\text{Var}(Z_n \mathbb{1}\{|Z_n| \leq 1\}) \leq \mathbb{E}(Z_n^2 \mathbb{1}\{|Z_n| \leq 1\})$, can be bounded from above as

$$\begin{aligned}
& n \text{Var}(Z_n \mathbb{1}\{|Z_n| \leq 1\}) \\
& \leq \frac{\mathbb{E}((X_j - \xi_{\tau_n, j})_+^2 (X_\ell - \xi_{\tau_n, \ell})_+^2 \mathbb{1}\{X_j \leq \xi_{\tau_n, j}(1 + \sqrt{n(1 - \tau_n)}), X_\ell > \xi_{\tau_n, \ell}(1 + \sqrt{n(1 - \tau_n)})\})}{n(1 - \tau_n)^2 \xi_{\tau_n, j}^2 \xi_{\tau_n, \ell}^2} \\
& + \frac{\mathbb{E}((X_j - \xi_{\tau_n, j})_+^2 (X_\ell - \xi_{\tau_n, \ell})_+^2 \mathbb{1}\{X_j > \xi_{\tau_n, j}(1 + \sqrt{n(1 - \tau_n)}), X_\ell \leq \xi_{\tau_n, \ell}(1 + \sqrt{n(1 - \tau_n)})\})}{n(1 - \tau_n)^2 \xi_{\tau_n, j}^2 \xi_{\tau_n, \ell}^2} \\
& + \frac{\mathbb{E}((X_j - \xi_{\tau_n, j})_+^2 (X_\ell - \xi_{\tau_n, \ell})_+^2 \mathbb{1}\{X_j \leq \xi_{\tau_n, j}(1 + \sqrt{n(1 - \tau_n)}), X_\ell \leq \xi_{\tau_n, \ell}(1 + \sqrt{n(1 - \tau_n)})\})}{n(1 - \tau_n)^2 \xi_{\tau_n, j}^2 \xi_{\tau_n, \ell}^2} \\
& = v_{1,n} + v_{2,n} + v_{3,n}.
\end{aligned}$$

We control each of these three terms separately. We first note that clearly

$$v_{1,n} \leq \frac{\mathbb{E}((X_\ell - \xi_{\tau_n, \ell})_+^2 \mathbb{1}\{X_\ell > \xi_{\tau_n, \ell}(1 + \sqrt{n(1 - \tau_n)})\})}{(1 - \tau_n) \xi_{\tau_n, \ell}^2}$$

and thus $v_{1,n} \rightarrow 0$ as $n \rightarrow \infty$ by (A.10). Similarly $v_{2,n} \rightarrow 0$ as $n \rightarrow \infty$. To control $v_{3,n}$, we use again the Cauchy-Schwarz inequality to find

$$\begin{aligned}
v_{3,n} & \leq \frac{1}{\sqrt{n(1 - \tau_n)}} \sqrt{\frac{\mathbb{E}((X_j - \xi_{\tau_n, j})_+^4 \mathbb{1}\{X_j \leq \xi_{\tau_n, j}(1 + \sqrt{n(1 - \tau_n)})\})}{(1 - \tau_n) \xi_{\tau_n, j}^4}} \\
& \times \frac{1}{\sqrt{n(1 - \tau_n)}} \sqrt{\frac{\mathbb{E}((X_\ell - \xi_{\tau_n, \ell})_+^4 \mathbb{1}\{X_\ell \leq \xi_{\tau_n, \ell}(1 + \sqrt{n(1 - \tau_n)})\})}{(1 - \tau_n) \xi_{\tau_n, \ell}^4}}.
\end{aligned}$$

The key now is to notice that for any heavy-tailed random variable Z with tail index γ and any $\varepsilon > 0$,

$$\begin{aligned}
\mathbb{E}((Z - t)^4 \mathbb{1}\{Z > t, Z \leq t + s\}) & = 4t^4 \int_1^{1+s/t} (v - 1)^3 \mathbb{P}(Z > tv, Z \leq t + s) dv \\
& \leq Dt^4 \int_1^{1+s/t} v^3 \mathbb{P}(Z > tv) dv \\
& \leq Dt^4 \mathbb{P}(Z > t) \int_1^{1+s/t} v^{3-1/\gamma+\varepsilon} dv \quad (\text{by Potter bounds}) \\
& = Dt^4 \mathbb{P}(Z > t) \times \frac{(1 + s/t)^{4-1/\gamma+\varepsilon} - 1}{4 - 1/\gamma + \varepsilon}
\end{aligned}$$

for t large enough, whatever $s > 0$ is (here D is an unimportant positive constant). This implies, by asymptotic proportionality of expectiles and quantiles and taking into account that $\gamma_j < 1/2$,

$$\sqrt{\frac{\mathbb{E}((X_j - \xi_{\tau_n, j})_+^4 \mathbb{1}\{X_j \leq \xi_{\tau_n, j}(1 + \sqrt{n(1 - \tau_n)})\})}{(1 - \tau_n) \xi_{\tau_n, j}^4}} = o\left(\sqrt{n(1 - \tau_n)}\right).$$

Conclude from the above inequality on $v_{3,n}$ that $v_{3,n} \rightarrow 0$ as $n \rightarrow \infty$. It follows that

$$n \operatorname{Var}(Z_n \mathbb{1}\{|Z_n| \leq 1\}) \leq v_{1,n} + v_{2,n} + v_{3,n} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (\text{A.12})$$

Combine finally (A.9), (A.11), (A.12) with Lemma A.2 to obtain (ii) and conclude the proof. \square

Proof of Theorem 2.1. Note first that for any $j \in \{1, \dots, d\}$, the minimisation criterion defining $\tilde{\xi}_{\tau_n, j}$ can be equivalently rewritten as

$$\begin{aligned} \sqrt{n(1-\tau_n)} \left(\frac{\tilde{\xi}_{\tau_n, j}}{\xi_{\tau_n, j}} - 1 \right) &= \arg \min_{u \in \mathbb{R}} \psi_n^{(j)}(u) \\ \text{with } \psi_n^{(j)}(u) &= \frac{1}{2\xi_{\tau_n, j}^2} \sum_{i=1}^n \left[\eta_{\tau_n} \left(X_{i,j} - \xi_{\tau_n, j} - \frac{u \xi_{\tau_n, j}}{\sqrt{n(1-\tau_n)}} \right) - \eta_{\tau_n}(X_{i,j} - \xi_{\tau_n, j}) \right]. \end{aligned}$$

By (B.12) in the proof of Theorem 2 in Daouia et al. (2018), we obtain

$$\begin{aligned} \psi_n^{(j)}(u) &= -uT_{1,n}^{(j)} + T_{2,n}^{(j)}(u) \\ \text{with } T_{1,n}^{(j)} &= \frac{1}{\sqrt{n(1-\tau_n)}} \sum_{i=1}^n \frac{1}{\xi_{\tau_n, j}} \varphi_{\tau_n}(X_{i,j} - \xi_{\tau_n, j}) \\ \text{and } T_{2,n}^{(j)}(u) &= -\frac{1}{\xi_{\tau_n, j}^2} \sum_{i=1}^n \int_0^{u\xi_{\tau_n, j}/\sqrt{n(1-\tau_n)}} (\varphi_{\tau_n}(X_{i,j} - \xi_{\tau_n, j} - z) - \varphi_{\tau_n}(X_{i,j} - \xi_{\tau_n, j})) dz. \end{aligned}$$

It follows that

$$\sqrt{n(1-\tau_n)} \left(\frac{\tilde{\xi}_{\tau_n, j}}{\xi_{\tau_n, j}} - 1 \right)_{1 \leq j \leq d} = \arg \min_{\mathbf{u} \in \mathbb{R}^d} \left\{ -\sum_{j=1}^d u_j T_{1,n}^{(j)} + \sum_{j=1}^d T_{2,n}^{(j)}(u_j) \right\}.$$

We concentrate on the asymptotic behaviour of the first sum in the right-hand side above. Write

$$\sum_{j=1}^d u_j T_{1,n}^{(j)} = \sum_{i=1}^n \frac{1}{\sqrt{n(1-\tau_n)}} \sum_{j=1}^d \frac{u_j}{\xi_{\tau_n, j}} \varphi_{\tau_n}(X_{i,j} - \xi_{\tau_n, j}) = \sum_{i=1}^n \mathcal{S}_{n,i}.$$

The random variables $\mathcal{S}_{n,i}$ ($1 \leq i \leq n$) thus defined are i.i.d. and have expectation 0. We shall then use Lyapunov's criterion, for which it is sufficient to show that for some $\delta > 0$,

$$\frac{n \mathbb{E}|\mathcal{S}_{n,1}|^{2+\delta}}{[n \operatorname{Var}(\mathcal{S}_{n,1})]^{1+\delta/2}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Choose $\delta > 0$ so small that $\gamma_j < 1/(2+\delta)$ and $\mathbb{E}|\min(X_j, 0)|^{2+\delta} < \infty$ for any j . It is a direct consequence of the Hölder inequality and Lemma 4 in Daouia et al. (2018) that

$$n \mathbb{E}|\mathcal{S}_{n,1}|^{2+\delta} = O([n(1-\tau_n)]^{-\delta/2}) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Besides, by Lemma A.3(i),

$$\begin{aligned}
n \operatorname{Var}(\mathcal{S}_{n,1}) &= \frac{1}{1-\tau_n} \sum_{j=1}^d u_j^2 \frac{1}{\xi_{\tau_n,j}^2} \operatorname{Var}(\varphi_{\tau_n}(X_j - \xi_{\tau_n,j})) \\
&\quad + \frac{1}{1-\tau_n} \sum_{1 \leq j < \ell \leq d} 2u_j u_\ell \frac{1}{\xi_{\tau_n,j} \xi_{\tau_n,\ell}} \operatorname{Cov}(\varphi_{\tau_n}(X_j - \xi_{\tau_n,j}), \varphi_{\tau_n}(X_\ell - \xi_{\tau_n,\ell})) \\
&\rightarrow \operatorname{Var}\left(\sum_{j=1}^d u_j \frac{Z_j}{\gamma_j}\right)
\end{aligned}$$

as $n \rightarrow \infty$, where (Z_1, \dots, Z_d) is a centred Gaussian random vector with covariance matrix $\mathbf{V}^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R})$. If the above limit is 0, then

$$\left(\mathbb{E} \left(\sum_{j=1}^d u_j T_{1,n}^{(j)} \right) = 0 \text{ and } \lim_{n \rightarrow \infty} \operatorname{Var} \left(\sum_{j=1}^d u_j T_{1,n}^{(j)} \right) = 0 \right) \Rightarrow \sum_{j=1}^d u_j T_{1,n}^{(j)} \xrightarrow{\mathbb{P}} 0 = \sum_{j=1}^d u_j \frac{Z_j}{\gamma_j}.$$

Otherwise, Lyapunov's criterion is satisfied, and this entails

$$\sum_{j=1}^d u_j T_{1,n}^{(j)} \xrightarrow{d} \sum_{j=1}^d u_j \frac{Z_j}{\gamma_j}.$$

In other words, $(T_{1,n}^{(1)}, \dots, T_{1,n}^{(d)}) \xrightarrow{d} (Z_1/\gamma_1, \dots, Z_d/\gamma_d)$. We can now repeat the second part of the proof of Theorem 2 in Daouia et al. (2018) to find

$$\psi_n(\mathbf{u}) = \sum_{j=1}^d \psi_n^{(j)}(u_j) = - \sum_{j=1}^d u_j T_{1,n}^{(j)} + \sum_{j=1}^d T_{2,n}^{(j)}(u_j) \xrightarrow{d} - \sum_{j=1}^d u_j \frac{Z_j}{\gamma_j} + \sum_{j=1}^d \frac{u_j^2}{2\gamma_j} = \psi_\infty(\mathbf{u})$$

in the sense of finite-dimensional convergence in distribution (i.e. $(\psi_n(\mathbf{u}_1), \dots, \psi_n(\mathbf{u}_p)) \xrightarrow{d} (\psi_\infty(\mathbf{u}_1), \dots, \psi_\infty(\mathbf{u}_p))$ for all choices of p and $\mathbf{u}_1, \dots, \mathbf{u}_p$). Conclude by applying Theorem 5 in Knight (1999) to the convex and continuous random function $\mathbf{u} \mapsto \psi_n(\mathbf{u})$:

$$\sqrt{n(1-\tau_n)} \left(\frac{\tilde{\xi}_{\tau_n,j}}{\xi_{\tau_n,j}} - 1 \right)_{1 \leq j \leq d} \xrightarrow{d} \arg \min_{\mathbf{u} \in \mathbb{R}^d} \left\{ - \sum_{j=1}^d u_j \frac{Z_j}{\gamma_j} + \sum_{j=1}^d \frac{u_j^2}{2\gamma_j} \right\} = (Z_1, \dots, Z_d).$$

This completes the proof. □

Proof of Theorem 2.2. Let $k = \lfloor n(1-\tau_n) \rfloor$ and set $Q_{\tau_n,j} = U_j(n/k) = q_{1-\lfloor n(1-\tau_n) \rfloor/n,j}$. The quantities $Q_{\tau_n,j}$ and $q_{\tau_n,j}$ are not necessarily equal, but they are asymptotically equivalent, and in fact

$$\sqrt{n(1-\tau_n)} \left(\frac{Q_{\tau_n,j}}{q_{\tau_n,j}} - 1 \right) = o(1)$$

by condition $\sqrt{n(1-\tau_n)}A_j((1-\tau_n)^{-1}) \rightarrow \lambda_j \in \mathbb{R}$ and local uniformity of Condition A(i) (see Theorem 2.3.9 in [de Haan and Ferreira \(2006\)](#)). The key idea is then to apply both Corollary 1 and Lemma 1 in [Stupfler \(2019\)](#), to obtain, for any $j \in \{1, \dots, d\}$,

$$\begin{aligned} \widehat{\gamma}_{\tau_n,j} - \gamma_j &= \frac{\sum_{i=1}^n [\log X_{i,j} - \log Q_{\tau_n,j}] \mathbb{1}\{X_{i,j} > Q_{\tau_n,j}\}}{\sum_{i=1}^n \mathbb{1}\{X_{i,j} > Q_{\tau_n,j}\}} - \gamma_j + o_{\mathbb{P}}\left(\frac{1}{\sqrt{n(1-\tau_n)}}\right) \\ \text{and } \frac{\widehat{q}_{\tau_n,j}}{q_{\tau_n,j}} - 1 &= \gamma_j \left(\frac{1}{\lfloor n(1-\tau_n) \rfloor} \sum_{i=1}^n \mathbb{1}\{X_{i,j} > Q_{\tau_n,j}\} - 1 \right) + o_{\mathbb{P}}\left(\frac{1}{\sqrt{n(1-\tau_n)}}\right). \end{aligned}$$

[The above asymptotic connection between $Q_{\tau_n,j}$ and $q_{\tau_n,j}$ is used for the second equality.] By Lemma 3 in [Stupfler \(2019\)](#) with $f = \log$, we find

$$\frac{\mathbb{E}([\log X_j - \log Q_{\tau_n,j}] \mathbb{1}\{X_j > Q_{\tau_n,j}\})}{\mathbb{P}(X_j > Q_{\tau_n,j})} = \gamma_j + \frac{1}{\sqrt{n(1-\tau_n)}} \frac{\lambda_j}{1-\rho_j} + o\left(\frac{1}{\sqrt{n(1-\tau_n)}}\right).$$

Besides, by Lemma 4 in [Stupfler \(2019\)](#) with $f = \log$, we obtain

$$\begin{aligned} \frac{n^{-1} \sum_{i=1}^n [\log X_{i,j} - \log Q_{\tau_n,j}] \mathbb{1}\{X_{i,j} > Q_{\tau_n,j}\}}{\mathbb{E}([\log X_j - \log Q_{\tau_n,j}] \mathbb{1}\{X_j > Q_{\tau_n,j}\})} &= 1 + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n(1-\tau_n)}}\right) \\ \text{and } \frac{n^{-1} \sum_{i=1}^n \mathbb{1}\{X_{i,j} > Q_{\tau_n,j}\}}{\mathbb{P}(X_j > Q_{\tau_n,j})} &= 1 + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n(1-\tau_n)}}\right). \end{aligned}$$

These three expansions along with the identity $\mathbb{P}(X_j > Q_{\tau_n,j}) = \lfloor n(1-\tau_n) \rfloor / n$ result in

$$\begin{aligned} \widehat{\gamma}_{\tau_n,j} - \gamma_j &= \gamma_j \left(\frac{n^{-1} \sum_{i=1}^n [\log X_{i,j} - \log Q_{\tau_n,j}] \mathbb{1}\{X_{i,j} > Q_{\tau_n,j}\}}{\mathbb{E}([\log X_j - \log Q_{\tau_n,j}] \mathbb{1}\{X_j > Q_{\tau_n,j}\})} - 1 \right) \\ &\quad - \gamma_j \left(\frac{n^{-1} \sum_{i=1}^n \mathbb{1}\{X_{i,j} > Q_{\tau_n,j}\}}{\mathbb{P}(X_j > Q_{\tau_n,j})} - 1 \right) + \frac{1}{\sqrt{n(1-\tau_n)}} \frac{\lambda_j}{1-\rho_j} + o_{\mathbb{P}}\left(\frac{1}{\sqrt{n(1-\tau_n)}}\right), \\ \frac{\widehat{q}_{\tau_n,j}}{q_{\tau_n,j}} - 1 &= \gamma_j \left(\frac{n^{-1} \sum_{i=1}^n \mathbb{1}\{X_{i,j} > Q_{\tau_n,j}\}}{\mathbb{P}(X_j > Q_{\tau_n,j})} - 1 \right) + o_{\mathbb{P}}\left(\frac{1}{\sqrt{n(1-\tau_n)}}\right). \end{aligned}$$

Each of the quantities of interest can therefore be asymptotically viewed as a sum of independent and identically distributed random variables. To obtain the desired result on their joint convergence, we use Lyapunov's CLT combined with the Cramér-Wold device. Let $a_1, b_1, \dots, a_d, b_d$ be real constants and consider

$$\sqrt{n(1-\tau_n)} \sum_{j=1}^d \left[a_j (\widehat{\gamma}_{\tau_n,j} - \gamma_j) + b_j \left(\frac{\widehat{q}_{\tau_n,j}}{q_{\tau_n,j}} - 1 \right) \right] = \sum_{j=1}^d a_j \frac{\lambda_j}{1-\rho_j} + \sum_{i=1}^n S_{n,i} + o_{\mathbb{P}}(1)$$

with

$$S_{n,i} = \sum_{j=1}^d a_j \gamma_j \frac{\sqrt{n(1-\tau_n)}}{n} \left(\frac{[\log X_{i,j} - \log Q_{\tau_n,j}] \mathbb{1}\{X_{i,j} > Q_{\tau_n,j}\}}{\mathbb{E}([\log X_j - \log Q_{\tau_n,j}] \mathbb{1}\{X_j > Q_{\tau_n,j}\})} - 1 \right) \\ + \sum_{j=1}^d (b_j - a_j) \gamma_j \frac{\sqrt{n(1-\tau_n)}}{n} \left(\frac{\mathbb{1}\{X_{i,j} > Q_{\tau_n,j}\}}{\mathbb{P}(X_j > Q_{\tau_n,j})} - 1 \right).$$

The $S_{n,i}$, $1 \leq i \leq n$, are independent, identically distributed and centred random variables. By Lemma 4 in [Stupfler \(2019\)](#) with $f = \log$ combined with Hölder's inequality, there is $\delta > 0$ such that $n \mathbb{E}|S_{n,i}|^{2+\delta} = o(1)$. To complete the proof, it is enough to show that

$$\lim_{n \rightarrow \infty} n \text{Var}(S_{n,1}) = (a_1 \ b_1 \cdots \ a_d \ b_d) \Sigma^Q(\boldsymbol{\gamma}, \mathbf{R}) (a_1 \ b_1 \cdots \ a_d \ b_d)^\top.$$

This last convergence is a straightforward consequence of Lemmas 4 and 6 in [Stupfler \(2019\)](#) combined with the identity

$$\int_0^1 \int_0^1 \frac{R_{j,\ell}(u,v)}{uv} du dv = \int_0^1 \frac{R_{j,\ell}(u,1)}{u} du + \int_0^1 \frac{R_{j,\ell}(1,v)}{v} dv$$

(true because of the 1-homogeneity of $R_{j,\ell}$) and direct but tedious calculations which we omit for the sake of brevity. \square

Proof of Corollary 2.3. Clearly

$$\frac{\widehat{\xi}_{\tau_n,j}}{\xi_{\tau_n,j}} = \frac{(\widehat{\gamma}_{\tau_n,j}^{-1} - 1)^{-\widehat{\gamma}_{\tau_n,j}}}{(\gamma_j^{-1} - 1)^{-\gamma_j}} \times \frac{\widehat{q}_{\tau_n,j}}{q_{\tau_n,j}} \times \frac{q_{\tau_n,j}(\gamma_j^{-1} - 1)^{-\gamma_j}}{\xi_{\tau_n,j}}.$$

All three terms on the right-hand side converge to 1 as $n \rightarrow \infty$, at the rate $1/\sqrt{n(1-\tau_n)}$, by Theorem 2.2 and Proposition 1(i) in [Daouia et al. \(2020\)](#). Linearising thus yields

$$\frac{\widehat{\xi}_{\tau_n,j}}{\xi_{\tau_n,j}} - 1 = \left(\frac{(\widehat{\gamma}_{\tau_n,j}^{-1} - 1)^{-\widehat{\gamma}_{\tau_n,j}}}{(\gamma_j^{-1} - 1)^{-\gamma_j}} - 1 \right) + \left(\frac{\widehat{q}_{\tau_n,j}}{q_{\tau_n,j}} - 1 \right) \\ + \left(\frac{q_{\tau_n,j}(\gamma_j^{-1} - 1)^{-\gamma_j}}{\xi_{\tau_n,j}} - 1 \right) + o_{\mathbb{P}} \left(\frac{1}{\sqrt{n(1-\tau_n)}} \right). \quad (\text{A.13})$$

The delta-method and a straightforward calculation entail

$$\frac{(\widehat{\gamma}_{\tau_n,j}^{-1} - 1)^{-\widehat{\gamma}_{\tau_n,j}}}{(\gamma_j^{-1} - 1)^{-\gamma_j}} - 1 = m(\gamma_j)(\widehat{\gamma}_{\tau_n,j} - \gamma_j) + o_{\mathbb{P}} \left(\frac{1}{\sqrt{n(1-\tau_n)}} \right). \quad (\text{A.14})$$

Besides, using Proposition 1(i) in [Daouia et al. \(2020\)](#) together with $\sqrt{n(1-\tau_n)}q_{\tau_n,j}^{-1} \rightarrow \mu_j$ and $\sqrt{n(1-\tau_n)}A_j((1-\tau_n)^{-1}) \rightarrow \lambda_j$ (as $n \rightarrow \infty$), we get

$$\begin{aligned} & \sqrt{n(1-\tau_n)} \left(\frac{q_{\tau_n,j}(\gamma_j^{-1}-1)^{-\gamma_j}}{\xi_{\tau_n,j}} - 1 \right) \\ &= -\gamma_j(\gamma_j^{-1}-1)^{\gamma_j} \mathbb{E}(X_j)\mu_j - \left(\frac{(\gamma_j^{-1}-1)^{-\rho_j}}{1-\gamma_j-\rho_j} + \frac{(\gamma_j^{-1}-1)^{-\rho_j}-1}{\rho_j} \right) \lambda_j + o(1). \end{aligned} \quad (\text{A.15})$$

Combining (A.13), (A.14) and (A.15) with Theorem 2.2 of the main paper completes the proof. \square

Proof of Theorem 2.4. The idea is to write, for any j ,

$$\log \left(\frac{\bar{\xi}_{\tau'_n,j}^*}{\xi_{\tau'_n,j}} \right) = (\hat{\gamma}_{\tau_n,j} - \gamma_j) \log \left(\frac{1-\tau_n}{1-\tau'_n} \right) + \log \left(\frac{\bar{\xi}_{\tau_n,j}}{\xi_{\tau_n,j}} \right) - \log \left(\left[\frac{1-\tau'_n}{1-\tau_n} \right]^{\gamma_j} \frac{\xi_{\tau'_n,j}}{\xi_{\tau_n,j}} \right)$$

where $\bar{\xi}_{\tau'_n,j}^*$ (resp. $\bar{\xi}_{\tau_n,j}$) can be either $\tilde{\xi}_{\tau'_n,j}^*$ (resp. $\tilde{\xi}_{\tau_n,j}$) or $\hat{\xi}_{\tau'_n,j}^*$ (resp. $\hat{\xi}_{\tau_n,j}$). Then (see *e.g.* the proof of Theorem 5 in [Daouia et al., 2020](#))

$$\log \left(\frac{\bar{\xi}_{\tau'_n,j}^*}{\xi_{\tau'_n,j}} \right) = (\hat{\gamma}_{\tau_n,j} - \gamma_j) \log \left(\frac{1-\tau_n}{1-\tau'_n} \right) + \log \left(\frac{\bar{\xi}_{\tau_n,j}}{\xi_{\tau_n,j}} \right) + O \left(\frac{1}{\sqrt{n(1-\tau_n)}} \right).$$

To show (i) (resp. (ii)), apply now Theorem 2.1 (resp. Corollary 2.3) of the main paper to find

$$\log \left(\frac{\bar{\xi}_{\tau'_n,j}^*}{\xi_{\tau'_n,j}} \right) = (\hat{\gamma}_{\tau_n,j} - \gamma_j) \log \left(\frac{1-\tau_n}{1-\tau'_n} \right) + O_{\mathbb{P}} \left(\frac{1}{\sqrt{n(1-\tau_n)}} \right).$$

The conclusion follows from applying Theorem 2.2 of the main paper. \square

Proposition A.4 is key to showing that the LAWS-based confidence region $\mathcal{E}_{\tau_n,\alpha}$ for an intermediate expectile ξ_{τ_n} has asymptotic coverage probability $1-\alpha$.

Proposition A.4. *Under the conditions of Theorem 2.1, we have*

$$\widehat{\mathbf{V}}_n^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R}) \xrightarrow{\mathbb{P}} \mathbf{V}^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R}).$$

Proof of Proposition A.4. Write

$$\frac{\widehat{F}_{n,j}(\tilde{\xi}_{\tau_n,j})}{1-\tau_n} = \frac{\widehat{F}_{n,j}(\tilde{\xi}_{\tau_n,j})}{\widehat{F}_{n,j}(\xi_{\tau_n,j})} \times \frac{\widehat{F}_{n,j}(\xi_{\tau_n,j})}{\overline{F}_j(\xi_{\tau_n,j})} \times \frac{\overline{F}_j(\xi_{\tau_n,j})}{1-\tau_n}.$$

Clearly $\widehat{F}_{n,j}(\xi_{\tau_n,j})/\overline{F}_j(\xi_{\tau_n,j})$ has expectation 1 and a variance which is a $O(1/[n(1-\tau_n)])$, by straightforward calculations and the asymptotic proportionality of quantiles and expectiles.

Thus $\widehat{F}_{n,j}(\xi_{\tau_{n,j}})/\overline{F}_j(\xi_{\tau_{n,j}}) \xrightarrow{\mathbb{P}} 1$. Using the asymptotic proportionality of quantiles and expectiles again, we find

$$\frac{\widehat{F}_{n,j}(\widetilde{\xi}_{\tau_{n,j}})}{1 - \tau_n} = \frac{\widehat{F}_{n,j}(\widetilde{\xi}_{\tau_{n,j}})}{\widehat{F}_{n,j}(\xi_{\tau_{n,j}})}(\gamma_j^{-1} - 1 + o_{\mathbb{P}}(1)). \quad (\text{A.16})$$

Denote by $\widetilde{\varepsilon}_{n,j} = \widetilde{\xi}_{\tau_{n,j}}/\xi_{\tau_{n,j}} - 1$. By Theorem 2.1, $\widetilde{\varepsilon}_{n,j} \xrightarrow{\mathbb{P}} 0$, and

$$\left| \frac{\widehat{F}_{n,j}(\widetilde{\xi}_{\tau_{n,j}})}{\widehat{F}_{n,j}(\xi_{\tau_{n,j}})} - 1 \right| = O_{\mathbb{P}} \left(\frac{|\widehat{F}_{n,j}(\xi_{\tau_{n,j}}(1 + \widetilde{\varepsilon}_{n,j})) - \widehat{F}_{n,j}(\xi_{\tau_{n,j}})|}{\overline{F}_j(\xi_{\tau_{n,j}})} \right). \quad (\text{A.17})$$

A straightforward calculation shows

$$\frac{|\widehat{F}_{n,j}(\xi_{\tau_{n,j}}(1 + \widetilde{\varepsilon}_{n,j})) - \widehat{F}_{n,j}(\xi_{\tau_{n,j}})|}{\overline{F}_j(\xi_{\tau_{n,j}})} \leq \frac{1}{n\overline{F}_j(\xi_{\tau_{n,j}})} \sum_{i=1}^n \mathbb{1}\{X_{i,j} \in [\xi_{\tau_{n,j}}(1 - |\widetilde{\varepsilon}_{n,j}|), \xi_{\tau_{n,j}}(1 + |\widetilde{\varepsilon}_{n,j}|)]\}.$$

This entails, for any $\varepsilon > 0$, that with arbitrarily large probability as $n \rightarrow \infty$,

$$\frac{|\widehat{F}_{n,j}(\xi_{\tau_{n,j}}(1 + \widetilde{\varepsilon}_{n,j})) - \widehat{F}_{n,j}(\xi_{\tau_{n,j}})|}{\overline{F}_j(\xi_{\tau_{n,j}})} \leq \frac{1}{n\overline{F}_j(\xi_{\tau_{n,j}})} \sum_{i=1}^n \mathbb{1}\{X_{i,j} \in [\xi_{\tau_{n,j}}(1 - \varepsilon), \xi_{\tau_{n,j}}(1 + \varepsilon)]\}.$$

The expectation of the random upper bound is

$$\frac{\overline{F}_j(\xi_{\tau_{n,j}}(1 - \varepsilon)) - \overline{F}_j(\xi_{\tau_{n,j}}(1 + \varepsilon))}{\overline{F}_j(\xi_{\tau_{n,j}})} \rightarrow (1 - \varepsilon)^{-1/\gamma_j} - (1 + \varepsilon)^{-1/\gamma_j} \text{ as } n \rightarrow \infty,$$

and its variance is bounded above by

$$\frac{\overline{F}_j(\xi_{\tau_{n,j}}(1 - \varepsilon)) - \overline{F}_j(\xi_{\tau_{n,j}}(1 + \varepsilon))}{n\overline{F}_j^2(\xi_{\tau_{n,j}})} = O \left(\frac{1}{n\overline{F}_j(\xi_{\tau_{n,j}})} \right) = O \left(\frac{1}{n(1 - \tau_n)} \right) \rightarrow 0.$$

Conclude that, for any $\varepsilon > 0$ and with arbitrarily large probability as $n \rightarrow \infty$,

$$\frac{|\widehat{F}_{n,j}(\xi_{\tau_{n,j}}(1 + \widetilde{\varepsilon}_{n,j})) - \widehat{F}_{n,j}(\xi_{\tau_{n,j}})|}{\overline{F}_j(\xi_{\tau_{n,j}})} \leq 2[(1 - \varepsilon)^{-1/\gamma_j} - (1 + \varepsilon)^{-1/\gamma_j}].$$

Since the upper bound is arbitrarily small as $\varepsilon \downarrow 0$, we find that the left-hand side in (A.17) converges in probability to 0. Note that for any j , $\widehat{\gamma}_{\tau_{n,j}} \xrightarrow{\mathbb{P}} \gamma_j$ (see *e.g.* Theorem 3.2.2 p.70 of [de Haan and Ferreira, 2006](#)); combine this with (A.16) and (A.17) to get, for any j ,

$$\widehat{V}_{n,j,j}^{\text{LAWS}}(\gamma, \mathbf{R}) \xrightarrow{\mathbb{P}} V_{j,j}^{\text{LAWS}}(\gamma, \mathbf{R}).$$

We now concentrate on the convergence of $\widehat{\mathbf{V}}_{n,j,\ell}^{\text{LAWS}}(\gamma, \mathbf{R})$, for $j < \ell$, for which it is enough to focus on the $\overline{m}_{n,j,\ell}$ term. We start by noting that, by Theorem 2.1 of the main paper,

$$\frac{\overline{m}_{n,j,\ell}}{(1 - \tau_n)\widetilde{\xi}_{\tau_n,j}\widetilde{\xi}_{\tau_n,\ell}} = \frac{\overline{m}_{n,j,\ell}}{(1 - \tau_n)\xi_{\tau_n,j}\xi_{\tau_n,\ell}}(1 + o_{\mathbb{P}}(1)).$$

Recall now Lemma 3 in [Daouia et al. \(2018\)](#), which implies that for any $x, h \in \mathbb{R}$ and any $\tau \in (0, 1)$,

$$|\varphi_{\tau}(x - h) - \varphi_{\tau}(x)| \leq |h|(1 - \tau + 2\mathbb{1}\{x > \min(h, 0)\}) \leq 3|h|.$$

Straightforward calculations then yield

$$\begin{aligned} & \left| \varphi_{\tau_n}(X_{i,j} - \widetilde{\xi}_{\tau_n,j})\varphi_{\tau_n}(X_{i,\ell} - \widetilde{\xi}_{\tau_n,\ell}) - \varphi_{\tau_n}(X_{i,j} - \xi_{\tau_n,j})\varphi_{\tau_n}(X_{i,\ell} - \xi_{\tau_n,\ell}) \right| \\ & \leq 3 \left| \widetilde{\xi}_{\tau_n,j} - \xi_{\tau_n,j} \right| \left| \varphi_{\tau_n}(X_{i,\ell} - \widetilde{\xi}_{\tau_n,\ell}) \right| + 3 \left| \widetilde{\xi}_{\tau_n,\ell} - \xi_{\tau_n,\ell} \right| \left| \varphi_{\tau_n}(X_{i,j} - \xi_{\tau_n,j}) \right| \\ & \leq 3 \left| \widetilde{\xi}_{\tau_n,j} - \xi_{\tau_n,j} \right| \left| \varphi_{\tau_n}(X_{i,\ell} - \xi_{\tau_n,\ell}) \right| + 3 \left| \widetilde{\xi}_{\tau_n,\ell} - \xi_{\tau_n,\ell} \right| \left| \varphi_{\tau_n}(X_{i,j} - \xi_{\tau_n,j}) \right| \\ & + 3 \left| \widetilde{\xi}_{\tau_n,j} - \xi_{\tau_n,j} \right| \left| \widetilde{\xi}_{\tau_n,\ell} - \xi_{\tau_n,\ell} \right| (1 - \tau_n + 2\mathbb{1}\{X_{i,\ell} - \xi_{\tau_n,\ell} > \min(\widetilde{\xi}_{\tau_n,\ell} - \xi_{\tau_n,\ell}, 0)\}). \end{aligned}$$

Set $m_{n,j,\ell} = n^{-1} \sum_{i=1}^n \varphi_{\tau_n}(X_{i,j} - \xi_{\tau_n,j})\varphi_{\tau_n}(X_{i,\ell} - \xi_{\tau_n,\ell})$ and combine the above inequality with Theorem 2.1 to get

$$\begin{aligned} \frac{|\overline{m}_{n,j,\ell} - m_{n,j,\ell}|}{(1 - \tau_n)\xi_{\tau_n,j}\xi_{\tau_n,\ell}} &= o_{\mathbb{P}} \left(\frac{1}{n} \sum_{i=1}^n \frac{|\varphi_{\tau_n}(X_{i,\ell} - \xi_{\tau_n,\ell})|}{(1 - \tau_n)\xi_{\tau_n,\ell}} + \frac{1}{n} \sum_{i=1}^n \frac{|\varphi_{\tau_n}(X_{i,j} - \xi_{\tau_n,j})|}{(1 - \tau_n)\xi_{\tau_n,j}} \right) \\ &+ o_{\mathbb{P}} \left(1 + \frac{1}{n(1 - \tau_n)} \sum_{i=1}^n \mathbb{1}\{X_{i,\ell} - \xi_{\tau_n,\ell} > \min(\widetilde{\xi}_{\tau_n,\ell} - \xi_{\tau_n,\ell}, 0)\} \right). \quad (\text{A.18}) \end{aligned}$$

We control each term on the right-hand side of (A.18). Note first that for any $x \in \mathbb{R}$ and $\tau \in (0, 1)$,

$$\frac{|\varphi_{\tau}(x)|}{1 - \tau} \leq |x| \left(1 + \frac{\mathbb{1}\{x > 0\}}{1 - \tau} \right).$$

Hence the bound

$$\frac{1}{n} \sum_{i=1}^n \frac{|\varphi_{\tau_n}(X_{i,j} - \xi_{\tau_n,j})|}{(1 - \tau_n)\xi_{\tau_n,j}} \leq \frac{1}{n} \sum_{i=1}^n \left| \frac{X_{i,j}}{\xi_{\tau_n,j}} - 1 \right| \left(1 + \frac{\mathbb{1}\{X_{i,j} > \xi_{\tau_n,j}\}}{1 - \tau_n} \right).$$

The upper bound has expectation

$$\mathbb{E} \left(\left| \frac{X_j}{\xi_{\tau_n,j}} - 1 \right| \left(1 + \frac{\mathbb{1}\{X_j > \xi_{\tau_n,j}\}}{1 - \tau_n} \right) \right) = 1 + \frac{\gamma_j}{1 - \gamma_j} \frac{\overline{F}_j(\xi_{\tau_n,j})}{1 - \tau_n} + o(1) \rightarrow 2 < \infty$$

by the dominated convergence theorem, asymptotic proportionality of quantiles and expectiles, and (A.1) with $a = 1$. Meanwhile

$$\begin{aligned}
& \text{Var} \left(\frac{1}{n} \sum_{i=1}^n \left| \frac{X_{i,j}}{\xi_{\tau_n,j}} - 1 \right| \left[1 + \frac{\mathbb{1}\{X_{i,j} > \xi_{\tau_n,j}\}}{1 - \tau_n} \right] \right) \\
& \leq \frac{1}{n} \mathbb{E} \left(\left| \frac{X_j}{\xi_{\tau_n,j}} - 1 \right|^2 \left(1 + \frac{\mathbb{1}\{X_j > \xi_{\tau_n,j}\}}{1 - \tau_n} \right)^2 \right) \\
& = O \left(\frac{1}{n} \mathbb{E} \left(\left| \frac{X_j}{\xi_{\tau_n,j}} - 1 \right|^2 \right) + \frac{1}{n(1 - \tau_n)} \mathbb{E} \left(\left| \frac{X_j}{\xi_{\tau_n,j}} - 1 \right|^2 \frac{\mathbb{1}\{X_j > \xi_{\tau_n,j}\}}{1 - \tau_n} \right) \right) \\
& = O \left(\frac{1}{n(1 - \tau_n)} \right) = o(1)
\end{aligned}$$

by the dominated convergence theorem, asymptotic proportionality of quantiles and expectiles, and convergence (A.1) for $a = 2$. Conclude that for any $j \in \{1, \dots, d\}$,

$$\frac{1}{n} \sum_{i=1}^n \frac{|\varphi_{\tau_n}(X_{i,j} - \xi_{\tau_n,j})|}{(1 - \tau_n)\xi_{\tau_n,j}} = O_{\mathbb{P}}(1). \quad (\text{A.19})$$

To control the second term in (A.18), recall that $\tilde{\xi}_{\tau_n,\ell}/\xi_{\tau_n,\ell} \xrightarrow{\mathbb{P}} 1$ by Theorem 2.1. In particular $\tilde{\xi}_{\tau_n,\ell} > \xi_{\tau_n,\ell}/2$ with arbitrarily large probability, and thus

$$\frac{1}{n(1 - \tau_n)} \sum_{i=1}^n \mathbb{1}\{X_{i,\ell} - \xi_{\tau_n,\ell} > \min(\tilde{\xi}_{\tau_n,\ell} - \xi_{\tau_n,\ell}, 0)\} \leq \frac{1}{n(1 - \tau_n)} \sum_{i=1}^n \mathbb{1}\{X_{i,\ell} > \xi_{\tau_n,\ell}/2\}$$

with arbitrarily large probability. By asymptotic proportionality of expectiles and quantiles, the upper bound has expectation $\bar{F}_{\ell}(\xi_{\tau_n,\ell}/2)/(1 - \tau_n) \rightarrow 2^{1/\gamma_{\ell}}(\gamma_{\ell}^{-1} - 1) < \infty$ as $n \rightarrow \infty$, and

$$\text{Var} \left(\frac{1}{n(1 - \tau_n)} \sum_{i=1}^n \mathbb{1}\{X_{i,\ell} > \xi_{\tau_n,\ell}/2\} \right) = \frac{F_{\ell}(\xi_{\tau_n,\ell}/2)\bar{F}_{\ell}(\xi_{\tau_n,\ell}/2)}{n(1 - \tau_n)^2} = O \left(\frac{1}{n(1 - \tau_n)} \right) = o(1).$$

It follows that

$$\frac{1}{n(1 - \tau_n)} \sum_{i=1}^n \mathbb{1}\{X_{i,\ell} - \xi_{\tau_n,\ell} > \min(\tilde{\xi}_{\tau_n,\ell} - \xi_{\tau_n,\ell}, 0)\} = O_{\mathbb{P}}(1). \quad (\text{A.20})$$

Combining (A.18), (A.19) and (A.20), we find

$$\frac{|\bar{m}_{n,j,\ell} - m_{n,j,\ell}|}{(1 - \tau_n)\xi_{\tau_n,j}\xi_{\tau_n,\ell}} \xrightarrow{\mathbb{P}} 0.$$

We conclude the proof by noting that

$$\frac{m_{n,j,\ell}}{(1 - \tau_n)\xi_{\tau_n,j}\xi_{\tau_n,\ell}} \xrightarrow{\mathbb{P}} \iint_{[1,\infty)^2} R_{j,\ell} \left((\gamma_j^{-1} - 1)x_j^{-1/\gamma_j}, (\gamma_{\ell}^{-1} - 1)x_{\ell}^{-1/\gamma_{\ell}} \right) dx_j dx_{\ell}$$

as a consequence of Lemma A.3(ii). The proof of the convergence of $\widehat{\mathbf{V}}_{n,j,\ell}^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R})$ is now complete since $\widehat{\gamma}_{\tau_n,j} \xrightarrow{\mathbb{P}} \gamma_j$ for any j . \square

Proof of Proposition 3.1. Rewrite the conclusion of Theorem 2.1 as

$$\sqrt{n(1-\tau_n)}[\mathbf{V}^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R})]^{-1/2} \begin{pmatrix} \boldsymbol{\xi}_{\tau_n} \\ \widetilde{\boldsymbol{\xi}}_{\tau_n} \end{pmatrix} - \mathbf{1}_d \xrightarrow{d} \mathcal{N}_d(\mathbf{0}_d, \mathbf{I}_d).$$

Use then Proposition A.4 to obtain the consistency of $\widehat{\mathbf{V}}_n^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R})$, apply Slutsky's lemma, and take the d -dimensional Euclidean norm on both sides (which is allowed by the continuous mapping theorem). \square

Proof of Proposition 3.2. The diagonal elements of the matrix $\widehat{\mathbf{V}}_n^{\text{QB}}(\boldsymbol{\gamma}, \mathbf{R})$ are obviously consistent estimators of the corresponding diagonal elements of $\mathbf{V}^{\text{QB}}(\boldsymbol{\gamma}, \mathbf{R})$. The consistency of its off-diagonal elements can be shown as follows. By a straightforward modification of Lemma 7 in Stupfler (2019), $\widehat{R}_{\tau_n,j,\ell}(u, v)$ is a pointwise consistent estimator of $R_{j,\ell}(u, v)$ on $(0, \infty)^2$. It is then sufficient to show the consistency of $\int_0^1 \widehat{R}_{\tau_n,j,\ell}(u, 1) du/u$ and $\int_0^1 \widehat{R}_{\tau_n,j,\ell}(1, u) du/u$ as estimators of $\int_0^1 R_{j,\ell}(u, 1) du/u$ and $\int_0^1 R_{j,\ell}(1, u) du/u$. We only treat the example of $\int_0^1 \widehat{R}_{\tau_n,j,\ell}(u, 1) du/u$, the other being similar. For any $\varepsilon > 0$,

$$\begin{aligned} \left| \int_0^1 \widehat{R}_{\tau_n,j,\ell}(u, 1) \frac{du}{u} - \int_0^1 R_{j,\ell}(u, 1) \frac{du}{u} \right| &\leq \int_{\varepsilon/4}^1 \left| \widehat{R}_{\tau_n,j,\ell}(u, 1) - R_{j,\ell}(u, 1) \right| \frac{du}{u} \\ &\quad + \int_0^{\varepsilon/4} \left(\widehat{R}_{\tau_n,j,\ell}(u, 1) + R_{j,\ell}(u, 1) \right) \frac{du}{u}. \end{aligned}$$

Now $R_{j,\ell}(u, 1) \leq u$ for any $u \in [0, 1]$, so

$$\int_0^{\varepsilon/4} R_{j,\ell}(u, 1) \frac{du}{u} \leq \frac{\varepsilon}{4}.$$

Set $k = k(n) = n(1 - \tau_n)$, so that $k \rightarrow \infty$. It is readily checked that $\widehat{R}_{\tau_n,j,\ell}(u, 1) = 0$ for any $u \leq 1/(2k)$ as soon as $n \geq 2$, and thus, for n large enough,

$$\int_0^{\varepsilon/4} \widehat{R}_{\tau_n,j,\ell}(u, 1) \frac{du}{u} = \int_{1/(2k)}^{\varepsilon/4} \widehat{R}_{\tau_n,j,\ell}(u, 1) \frac{du}{u}.$$

It is also readily checked that, for n large enough, $\widehat{R}_{\tau_n,j,\ell}(u, 1) \leq \lceil ku \rceil / k$ for any $u \in [0, 1]$ with probability 1. Therefore

$$\int_0^{\varepsilon/4} \widehat{R}_{\tau_n,j,\ell}(u, 1) \frac{du}{u} \leq \frac{1}{k} \int_{1/(2k)}^{1/k} \frac{du}{u} + \frac{1}{k} \int_{1/k}^{\varepsilon/4} \lceil ku \rceil \frac{du}{u} \leq \frac{\log 2}{k} + 2 \int_{1/k}^{\varepsilon/4} du \leq \frac{5\varepsilon}{8}$$

for n large enough. Conclude that for n large enough,

$$\left| \int_0^1 \widehat{R}_{\tau_n,j,\ell}(u, 1) \frac{du}{u} - \int_0^1 R_{j,\ell}(u, 1) \frac{du}{u} \right| \leq \frac{7\varepsilon}{8} + \frac{4}{\varepsilon} \int_{\varepsilon/4}^1 \left| \widehat{R}_{\tau_n,j,\ell}(u, 1) - R_{j,\ell}(u, 1) \right| du. \quad (\text{A.21})$$

Construct a finite regular grid of points u_p ($1 \leq p \leq P$) in $[\varepsilon/4, 1]$ such that $u_1 = \varepsilon/4$, $u_P = 1$, and $|u_{p+1} - u_p| \leq h$ for any $1 \leq p \leq P - 1$ (where h will be chosen later). Pick an arbitrary $u \in [\varepsilon/4, 1]$ and let $u_q \leq u \leq u_{q+1}$ be its closest grid points. By monotonicity of $\widehat{R}_{\tau_n, j, \ell}$ and $R_{j, \ell}$ and 1-Lipschitz continuity of $R_{j, \ell}$ (see Theorem 1(iii) in [Schmidt and Stadtmüller \(2006\)](#)), one finds

$$\begin{aligned} & \left| \widehat{R}_{\tau_n, j, \ell}(u, 1) - R_{j, \ell}(u, 1) \right| \\ & \leq \max \left(\widehat{R}_{\tau_n, j, \ell}(u_{q+1}, 1) - R_{j, \ell}(u_{q+1}, 1), R_{j, \ell}(u_q, 1) - \widehat{R}_{\tau_n, j, \ell}(u_q, 1) \right) + h \\ & \leq h + \max_{1 \leq p \leq P} \left| \widehat{R}_{\tau_n, j, \ell}(u_p, 1) - R_{j, \ell}(u_p, 1) \right|. \end{aligned}$$

Conclude, by consistency of $\widehat{R}_{\tau_n, j, \ell}(u, 1)$ at the (finite) set of grid points (u_p) , that whatever $h > 0$ is,

$$\int_{\varepsilon/4}^1 \left| \widehat{R}_{\tau_n, j, \ell}(u, 1) - R_{j, \ell}(u, 1) \right| du \leq 2h \quad (\text{A.22})$$

with arbitrarily high probability as $n \rightarrow \infty$. Choosing $h = \varepsilon^2/64$ and combining (A.21) and (A.22) yields the consistency of $\int_0^1 \widehat{R}_{\tau_n, j, \ell}(u, 1) du/u$. Combine then Corollary 2.3 and Slutsky's lemma. \square

Proof of Theorem 3.3. We examine the joint convergence of the $\widehat{\gamma}_{\tau_n, j}$ and $\widetilde{\xi}_{\tau_n, j}$ by putting them in a common (artificial) minimisation framework:

$$\begin{aligned} & \sqrt{n(1 - \tau_n)} \left(\widehat{\gamma}_{\tau_n, j} - \gamma_j, \frac{\widetilde{\xi}_{\tau_n, j}}{\xi_{\tau_n, j}} - 1 \right)_{1 \leq j \leq d} \\ & = \arg \min_{(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^d \times \mathbb{R}^d} \left\{ \sum_{j=1}^d \frac{1}{2} \left(u_j - \sqrt{n(1 - \tau_n)} (\widehat{\gamma}_{\tau_n, j} - \gamma_j) \right)^2 + \psi_n^{(j)}(v_j) \right\} \\ & = \arg \min_{(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^d \times \mathbb{R}^d} \left\{ \sum_{j=1}^d -u_j \sqrt{n(1 - \tau_n)} (\widehat{\gamma}_{\tau_n, j} - \gamma_j) + \frac{u_j^2}{2} - v_j T_{1, n}^{(j)} + T_{2, n}^{(j)}(v_j) \right\} \end{aligned}$$

with the notation of the proof of Theorem 2.1. The above random cost function is obviously convex and continuous, and we analyse its convergence. It was shown in the proof of Theorem 2.2 (using the notation therein) that

$$\begin{aligned} & \sqrt{n(1 - \tau_n)} (\widehat{\gamma}_{\tau_n, j} - \gamma_j) \\ & = \gamma_j \times \sqrt{n(1 - \tau_n)} \left(\frac{n^{-1} \sum_{i=1}^n [\log X_{i, j} - \log Q_{\tau_n, j}] \mathbb{1}\{X_{i, j} > Q_{\tau_n, j}\}}{\mathbb{E}([\log X_j - \log Q_{\tau_n, j}] \mathbb{1}\{X_j > Q_{\tau_n, j}\})} - 1 \right) \\ & - \gamma_j \times \sqrt{n(1 - \tau_n)} \left(\frac{n^{-1} \sum_{i=1}^n \mathbb{1}\{X_{i, j} > Q_{\tau_n, j}\}}{\mathbb{P}(X_j > Q_{\tau_n, j})} - 1 \right) + \frac{\lambda_j}{1 - \rho_j} + o_{\mathbb{P}}(1) \end{aligned}$$

and that the two random sums on the right-hand side above satisfy Lyapunov's CLT. Besides, as shown in the proof of Theorem 2.1,

$$-\sum_{j=1}^d v_j T_{1,n}^{(j)} + \sum_{j=1}^d T_{2,n}^{(j)}(v_j) = -\sum_{j=1}^d \frac{1}{\sqrt{n(1-\tau_n)}} \sum_{i=1}^n \frac{v_j}{\xi_{\tau_n,j}} \varphi_{\tau_n}(X_{i,j} - \xi_{\tau_n,j}) + \sum_{j=1}^d \frac{v_j^2}{2\gamma_j} + o_{\mathbb{P}}(1)$$

with the random sum on the right-hand side satisfying again Lyapunov's CLT. To show the desired result, by Theorem 5 in [Knight \(1999\)](#), it is enough to show that

$$\left(\frac{\sqrt{n(1-\tau_n)}(\hat{\gamma}_{\tau_n,j} - \gamma_j)}{\gamma_j}, T_{1,n}^{(j)} \right)_{1 \leq j \leq d} \xrightarrow{d} \left(\frac{W_j}{\gamma_j}, \frac{Z_j}{\gamma_j} \right)_{1 \leq j \leq d}$$

where the vector $(W_j, Z_j)_{1 \leq j \leq d}$ has covariance matrix Σ^{LAWS} . Since each of the elements of this vector can be represented as a linear combination of sums of independent and identically distributed random variables having expectation 0 that satisfy Lyapunov's CLT, we only need to evaluate the asymptotic covariance structure of these sums. Taking into account the part of the asymptotic covariance structure already calculated in Theorem 2.1 (between the $\sqrt{n(1-\tau_n)}(\hat{\gamma}_{\tau_n,j} - \gamma_j)$) and Theorem 2.2 (between the $T_{1,n}^{(j)}$) of the main paper, and recalling that $\varphi_{\tau_n}(X_\ell - \xi_{\tau_n,\ell})$ has expectation 0, it is sufficient to show that for any $j, \ell \in \{1, \dots, d\}$ with $j \leq \ell$,

$$\begin{aligned} & \frac{\mathbb{E}(\mathbb{1}\{X_j > Q_{\tau_n,j}\} \varphi_{\tau_n}(X_\ell - \xi_{\tau_n,\ell}))}{\mathbb{P}(X_j > Q_{\tau_n,j}) \xi_{\tau_n,\ell}} \rightarrow \int_1^\infty \bar{R}_{j,\ell} \left(1, (\gamma_\ell^{-1} - 1) x_\ell^{-1/\gamma_\ell} \right) dx_\ell \\ \text{and } & \frac{\mathbb{E}([\log X_j - \log Q_{\tau_n,j}] \mathbb{1}\{X_j > Q_{\tau_n,j}\} \varphi_{\tau_n}(X_\ell - \xi_{\tau_n,\ell}))}{\mathbb{E}([\log X_j - \log Q_{\tau_n,j}] \mathbb{1}\{X_j > Q_{\tau_n,j}\}) \xi_{\tau_n,\ell}} \\ & \rightarrow \frac{1}{\gamma_j} \iint_{[1,\infty)^2} \bar{R}_{j,\ell} \left(x_j^{-1/\gamma_j}, (\gamma_\ell^{-1} - 1) x_\ell^{-1/\gamma_\ell} \right) \frac{dx_j}{x_j} dx_\ell \end{aligned}$$

as $n \rightarrow \infty$, where $\bar{R}_{j,\ell}(s, t) = \min(s, t)$ if $j = \ell$ (recall that (X_j, X_j) is a perfectly dependent pair) and $R_{j,\ell}(s, t)$ otherwise. To show these two convergences we first recall that

$$\varphi_{\tau_n}(X_\ell - \xi_{\tau_n,\ell}) = (1 - \tau_n)(X_\ell - \xi_{\tau_n,\ell}) \mathbb{1}\{X_\ell \leq \xi_{\tau_n,\ell}\} + \tau_n(X_\ell - \xi_{\tau_n,\ell}) \mathbb{1}\{X_\ell > \xi_{\tau_n,\ell}\}.$$

Consequently

$$\begin{aligned} \mathbb{E}(\mathbb{1}\{X_j > Q_{\tau_n,j}\} \varphi_{\tau_n}(X_\ell - \xi_{\tau_n,\ell})) &= (1 - \tau_n) \mathbb{E}((X_\ell - \xi_{\tau_n,\ell}) \mathbb{1}\{X_j > Q_{\tau_n,j}, X_\ell \leq \xi_{\tau_n,\ell}\}) \\ &\quad + \tau_n \mathbb{E}((X_\ell - \xi_{\tau_n,\ell}) \mathbb{1}\{X_j > Q_{\tau_n,j}, X_\ell > \xi_{\tau_n,\ell}\}). \end{aligned}$$

Recalling that $\mathbb{P}(X_j > Q_{\tau_n,j}) = (1 - \tau_n)(1 + o(1))$, and using the Cauchy-Schwarz inequality, the first term on the right-hand side is bounded by

$$(1 - \tau_n) \sqrt{\mathbb{P}(X_j > Q_{\tau_n,j})} \sqrt{\mathbb{E}((X_\ell - \xi_{\tau_n,\ell})^2)} = O((1 - \tau_n)^{3/2} \xi_{\tau_n,\ell}).$$

Then, by an integration by parts,

$$\begin{aligned}
& \frac{\mathbb{E}(\mathbb{1}\{X_j > Q_{\tau_n,j}\} \varphi_{\tau_n}(X_\ell - \xi_{\tau_n,\ell}))}{\mathbb{P}(X_j > Q_{\tau_n,j}) \xi_{\tau_n,\ell}} \\
&= \frac{1}{(1 - \tau_n) \xi_{\tau_n,\ell}} \mathbb{E}((X_\ell - \xi_{\tau_n,\ell}) \mathbb{1}\{X_j > Q_{\tau_n,j}, X_\ell > \xi_{\tau_n,\ell}\}) (1 + o(1)) + o(1) \\
&= \frac{1}{(1 - \tau_n) \xi_{\tau_n,\ell}} \int_{\xi_{\tau_n,\ell}}^{\infty} \mathbb{P}(X_j > Q_{\tau_n,j}, X_\ell > t) dt (1 + o(1)) + o(1).
\end{aligned}$$

Applying Lemma A.1(ii) and using the asymptotic proportionality of quantiles and expectiles, we find

$$\frac{\mathbb{E}(\mathbb{1}\{X_j > Q_{\tau_n,j}\} \varphi_{\tau_n}(X_\ell - \xi_{\tau_n,\ell}))}{\mathbb{P}(X_j > Q_{\tau_n,j}) \xi_{\tau_n,\ell}} \rightarrow \int_1^{\infty} \bar{R}_{j,\ell} \left(1, (\gamma_\ell^{-1} - 1) x_\ell^{-1/\gamma_\ell}\right) dx_\ell \text{ as } n \rightarrow \infty.$$

Similarly

$$\begin{aligned}
& \mathbb{E}([\log X_j - \log Q_{\tau_n,j}] \mathbb{1}\{X_j > Q_{\tau_n,j}\} \varphi_{\tau_n}(X_\ell - \xi_{\tau_n,\ell})) \\
&= (1 - \tau_n) \mathbb{E}([\log X_j - \log Q_{\tau_n,j}](X_\ell - \xi_{\tau_n,\ell}) \mathbb{1}\{X_j > Q_{\tau_n,j}, X_\ell \leq \xi_{\tau_n,\ell}\}) \\
&+ \tau_n \mathbb{E}([\log X_j - \log Q_{\tau_n,j}](X_\ell - \xi_{\tau_n,\ell}) \mathbb{1}\{X_j > Q_{\tau_n,j}, X_\ell > \xi_{\tau_n,\ell}\}).
\end{aligned}$$

Recall the well-known inequality $\log x \leq x - 1$ for any $x \geq 1$ to bound the first term from above by

$$\begin{aligned}
& (1 - \tau_n) \mathbb{E} \left(\left[\frac{X_j}{Q_{\tau_n,j}} - 1 \right] |X_\ell - \xi_{\tau_n,\ell}| \mathbb{1}\{X_j > Q_{\tau_n,j}\} \right) \\
&\leq (1 - \tau_n) \sqrt{\mathbb{E} \left(\left[\frac{X_j}{Q_{\tau_n,j}} - 1 \right]^2 \mathbb{1}\{X_j > Q_{\tau_n,j}\} \right)} \sqrt{\mathbb{E}((X_\ell - \xi_{\tau_n,\ell})^2)} \\
&= O((1 - \tau_n)^{3/2} \xi_{\tau_n,\ell}).
\end{aligned}$$

Here (A.1) with $a = 2$ and the dominated convergence theorem were also used. Recall also Lemma 3 in Stupfler (2019) with $f = \log$ to find, as $n \rightarrow \infty$,

$$\frac{\mathbb{E}([\log X_j - \log Q_{\tau_n,j}] \mathbb{1}\{X_j > Q_{\tau_n,j}\})}{1 - \tau_n} = \frac{\mathbb{E}([\log X_j - \log Q_{\tau_n,j}] \mathbb{1}\{X_j > Q_{\tau_n,j}\})}{\mathbb{P}(X_j > Q_{\tau_n,j})(1 + o(1))} \rightarrow \gamma_j.$$

By an integration by parts then,

$$\begin{aligned}
& \gamma_j \frac{\mathbb{E}([\log X_j - \log Q_{\tau_n,j}] \mathbb{1}\{X_j > Q_{\tau_n,j}\} \varphi_{\tau_n}(X_\ell - \xi_{\tau_n,\ell}))}{\mathbb{E}([\log X_j - \log Q_{\tau_n,j}] \mathbb{1}\{X_j > Q_{\tau_n,j}\}) \xi_{\tau_n,\ell}} \\
&= \frac{1 + o(1)}{(1 - \tau_n) \xi_{\tau_n,\ell}} \mathbb{E}([\log X_j - \log Q_{\tau_n,j}](X_\ell - \xi_{\tau_n,\ell}) \mathbb{1}\{X_j > Q_{\tau_n,j}, X_\ell > \xi_{\tau_n,\ell}\}) + o(1) \\
&= \frac{1 + o(1)}{(1 - \tau_n) \xi_{\tau_n,\ell}} \int_{s=Q_{\tau_n,j}}^{\infty} \int_{t=\xi_{\tau_n,\ell}}^{\infty} \mathbb{P}(X_j > s, X_\ell > t) \frac{ds}{s} dt + o(1).
\end{aligned}$$

Applying Lemma A.1(iii) and using the asymptotic proportionality of quantiles and expectiles, we find

$$\begin{aligned} & \gamma_j \frac{\mathbb{E}([\log X_j - \log Q_{\tau_n, j}] \mathbb{1}\{X_j > Q_{\tau_n, j}\} \varphi_{\tau_n}(X_\ell - \xi_{\tau_n, \ell}))}{\mathbb{E}([\log X_j - \log Q_{\tau_n, j}] \mathbb{1}\{X_j > Q_{\tau_n, j}\}) \xi_{\tau_n, \ell}} \\ & \rightarrow \iint_{[1, \infty)^2} \bar{R}_{j, \ell} \left(x_j^{-1/\gamma_j}, (\gamma_\ell^{-1} - 1) x_\ell^{-1/\gamma_\ell} \right) \frac{dx_j}{x_j} dx_\ell \quad \text{as } n \rightarrow \infty. \end{aligned}$$

The proof is complete. \square

Proof of Theorem 3.4. The quantities $\widehat{\Sigma}_{n, j, \ell}^{\text{LAWS}}(\gamma, \mathbf{R})(1, 1)$ and $\widehat{\mathbf{V}}_{n, j, \ell}^{\text{LAWS}}(\gamma, \mathbf{R})$ are consistent estimators of $\Sigma_{j, \ell}^{\text{LAWS}}(\gamma, \mathbf{R})(1, 1)$ and $\mathbf{V}_{j, \ell}^{\text{LAWS}}(\gamma, \mathbf{R}) = \Sigma_{j, \ell}^{\text{LAWS}}(\gamma, \mathbf{R})(2, 2)$, by Proposition A.4 and the consistency of $\widehat{R}_{\tau_n, j, \ell}$ observed in the proof of Proposition 3.2 of the main paper. A proof similar to that of Proposition A.4 shows that the estimator $\widehat{\Sigma}_{n, j, \ell}^{\text{LAWS}}(\gamma, \mathbf{R})(1, 2)$ is a consistent estimator of $\Sigma_{j, \ell}^{\text{LAWS}}(\gamma, \mathbf{R})(1, 2)$; the idea is to use the consistency of the intermediate quantile $X_{n - \lfloor n(1 - \tau_n) \rfloor, n, j}$ and the intermediate LAWS expectile $\widetilde{\xi}_{\tau_n, \ell}$ to bound them from above and below by quantities arbitrarily close to $q_{\tau_n, j}$ and $\xi_{\tau_n, \ell}$, and then to use a law of large numbers. Conclude that $\widehat{\Sigma}_{n, j, \ell}^{\text{LAWS}}(\gamma, \mathbf{R})$ is a consistent estimator of $\Sigma_{j, \ell}^{\text{LAWS}}(\gamma, \mathbf{R})$, and combine this consistency with the convergence $1/\log d_n \rightarrow 0$ and Theorem 2.4(i) of the main paper to conclude. \square

Proof of Theorem 3.5. We showed in the proof of Proposition 3.2 of the main paper the consistency of $\widehat{R}_{\tau_n, j, \ell}$, $\int_0^1 \widehat{R}_{\tau_n, j, \ell}(u, 1) du/u$ and $\int_0^1 \widehat{R}_{\tau_n, j, \ell}(1, u) du/u$. Using this along with the consistency of the Hill estimators and the convergence $1/\log d_n \rightarrow 0$ (as $n \rightarrow \infty$) shows the consistency of $\widehat{\mathbf{V}}_n^{\star, \text{QB}}(\gamma, \mathbf{R})$ as an estimator of $\mathbf{V}^*(\gamma, \mathbf{R})$. Applying Theorem 2.4(ii) of the main paper concludes the proof. \square

Proposition A.5 and Lemma A.6 below are the key to the proof of Theorem 3.6 of the main paper. We start with Proposition A.5, which is a general result about a likelihood ratio-type test statistic.

Proposition A.5. *Assume that (\mathbf{Z}_n) is a sequence of d -dimensional random vectors such that*

$$v_n(\mathbf{Z}_n - \mathbf{m}_n) \xrightarrow{d} \mathcal{N}_d(\mathbf{0}_d, \Sigma),$$

where $v_n \rightarrow \infty$, $(\mathbf{m}_n = (\mathbf{m}_{n,1}, \dots, \mathbf{m}_{n,d})^\top)$ is a sequence of nonrandom vectors with positive entries that stays bounded away from $\mathbf{0}_d$, and Σ is a positive definite symmetric matrix. Let $(\widehat{\Sigma}_n)$ be a random sequence of positive semidefinite matrices such that $\widehat{\Sigma}_n \xrightarrow{\mathbb{P}} \Sigma$. Set

$$\mathbf{D}_n = v_n \widehat{\Sigma}_n^{-1/2} \left(\mathbf{Z}_n - \frac{\mathbf{Z}_n^\top \widehat{\Sigma}_n^{-1} \mathbf{1}_d}{\mathbf{1}_d^\top \widehat{\Sigma}_n^{-1} \mathbf{1}_d} \mathbf{1}_d \right)$$

which is a sequence of d -dimensional random vectors that is well-defined with arbitrarily high probability as $n \rightarrow \infty$. Let $\|\cdot\|_\infty$ denote the supremum norm.

(i) If $v_n(\mathbf{m}_n - \|\mathbf{m}_n\|_\infty \mathbf{1}_d) \rightarrow \mathbf{0}_d$ then $\mathbf{D}_n^\top \mathbf{D}_n \xrightarrow{d} \chi_{d-1}^2$.

(ii) If $(\|\mathbf{m}_n\|_\infty^{-1} \mathbf{m}_n - \mathbf{1}_d)$ stays bounded away from $\mathbf{0}_d$ then $\mathbf{D}_n^\top \mathbf{D}_n \xrightarrow{\mathbb{P}} \infty$.

Proof of Proposition A.5. Write $\|\cdot\| = \|\cdot\|_\infty$ for notational convenience. Let $\eta > 0$ be such that $\|\mathbf{M} - \Sigma\| \leq \eta \Rightarrow (\mathbf{1}_d^\top \mathbf{M}^{-1} \mathbf{1}_d \in (0, \infty)$ and $\det(\mathbf{M}) > 0)$, and let $A_n = \{\|\hat{\Sigma}_n - \Sigma\| \leq \eta\}$. Then $\mathbb{P}(A_n) \rightarrow 1$ as $n \rightarrow \infty$ and on A_n , the quantity \mathbf{D}_n is well-defined and can be written as

$$\begin{aligned} \mathbf{D}_n &= v_n \hat{\Sigma}_n^{-1/2} \left(\mathbf{Z}_n - \mathbf{m}_n - \frac{(\mathbf{Z}_n - \mathbf{m}_n)^\top \hat{\Sigma}_n^{-1} \mathbf{1}_d}{\mathbf{1}_d^\top \hat{\Sigma}_n^{-1} \mathbf{1}_d} \mathbf{1}_d \right) + v_n \hat{\Sigma}_n^{-1/2} \left(\mathbf{m}_n - \frac{\mathbf{m}_n^\top \hat{\Sigma}_n^{-1} \mathbf{1}_d}{\mathbf{1}_d^\top \hat{\Sigma}_n^{-1} \mathbf{1}_d} \mathbf{1}_d \right) \\ &= v_n \hat{\Sigma}_n^{-1/2} \left(\mathbf{Z}_n - \mathbf{m}_n - \frac{(\mathbf{Z}_n - \mathbf{m}_n)^\top \hat{\Sigma}_n^{-1} \mathbf{1}_d}{\mathbf{1}_d^\top \hat{\Sigma}_n^{-1} \mathbf{1}_d} \mathbf{1}_d \right) \\ &\quad + v_n \hat{\Sigma}_n^{-1/2} \left(\mathbf{m}_n - \|\mathbf{m}_n\| \mathbf{1}_d - \frac{(\mathbf{m}_n - \|\mathbf{m}_n\| \mathbf{1}_d)^\top \hat{\Sigma}_n^{-1} \mathbf{1}_d}{\mathbf{1}_d^\top \hat{\Sigma}_n^{-1} \mathbf{1}_d} \mathbf{1}_d \right). \end{aligned}$$

Let \mathbf{Z} have the $\mathcal{N}_d(\mathbf{0}_d, \Sigma)$ distribution. If $v_n(\mathbf{m}_n - \|\mathbf{m}_n\| \mathbf{1}_d) \rightarrow \mathbf{0}_d$, then clearly

$$\mathbf{D}_n \xrightarrow{d} \Sigma^{-1/2} \left(\mathbf{Z} - \frac{\mathbf{Z}^\top \Sigma^{-1} \mathbf{1}_d}{\mathbf{1}_d^\top \Sigma^{-1} \mathbf{1}_d} \mathbf{1}_d \right) = \mathbf{Y} - \frac{\mathbf{Y}^\top \mathbf{u}}{\mathbf{u}^\top \mathbf{u}} \mathbf{u}$$

where $\mathbf{Y} = \Sigma^{-1/2} \mathbf{Z}$ is standard Gaussian and $\mathbf{u} = \Sigma^{-1/2} \mathbf{1}_d$. The right-hand side corresponds to the Euclidean projection of \mathbf{Y} onto the orthogonal complement of the line spanned by \mathbf{u} . Statement (i) then follows by Cochran's theorem.

To show (ii), let $\mathcal{S}^{d-1} = \{\mathbf{x} \in \mathbb{R}^d \mid \|\mathbf{x}\| = 1\}$ denote the d -dimensional hypersphere, $\mathcal{S}_d^{++}(\mathbb{R})$ be the set of positive definite matrices in dimension d , and consider the mapping

$$\phi : (\mathbf{x}, \mathbf{M}) \in \mathcal{S}^{d-1} \times \mathcal{S}_d^{++}(\mathbb{R}) \mapsto \mathbf{M}^{-1/2} \left(\mathbf{x} - \frac{\mathbf{x}^\top \mathbf{M}^{-1} \mathbf{1}_d}{\mathbf{1}_d^\top \mathbf{M}^{-1} \mathbf{1}_d} \mathbf{1}_d \right) \in \mathbb{R}^d.$$

This mapping is continuous on $\mathcal{S}^{d-1} \times \mathcal{S}_d^{++}(\mathbb{R})$, and on the event A_n ,

$$\mathbf{D}_n = \text{O}_{\mathbb{P}}(1) + v_n \|\mathbf{m}_n\| \phi(\|\mathbf{m}_n\|^{-1} \mathbf{m}_n, \hat{\Sigma}_n).$$

Let $\delta > 0$ be such that $\|\|\mathbf{m}_n\|^{-1} \mathbf{m}_n - \mathbf{1}_d\| \geq \delta$ for any n . On A_n ,

$$\|\phi(\|\mathbf{m}_n\|^{-1} \mathbf{m}_n, \hat{\Sigma}_n)\| \geq \inf\{\|\phi(\mathbf{x}, \mathbf{M})\|, \mathbf{x} \in [0, \infty)^d, \|\mathbf{x}\| = 1, \|\mathbf{x} - \mathbf{1}_d\| \geq \delta, \|\mathbf{M} - \Sigma\| \leq \eta\}.$$

The right-hand side is not zero. Indeed the continuous function $\|\phi\|$ must attain its minimum on the compact set defined by (\mathbf{x}, \mathbf{M}) satisfying the conditions $(\mathbf{x} \in [0, \infty)^d, \|\mathbf{x}\| = 1, \|\mathbf{x} - \mathbf{1}_d\| \geq \delta, \|\mathbf{M} - \Sigma\| \leq \eta)$, and at any such point the function $\|\phi\|$ is nonzero, because

$$\phi(\mathbf{x}, \mathbf{M}) = 0 \Rightarrow \mathbf{x} - \frac{\mathbf{x}^\top \mathbf{M}^{-1} \mathbf{1}_d}{\mathbf{1}_d^\top \mathbf{M}^{-1} \mathbf{1}_d} \mathbf{1}_d = \mathbf{0}_d \Rightarrow \mathbf{x} \in \mathbb{R} \mathbf{1}_d,$$

which cannot happen when $\mathbf{x} \in [0, \infty)^d$, $\|\mathbf{x}\| = 1 = \|\mathbf{1}_d\|$ and $\|\mathbf{x} - \mathbf{1}_d\| \geq \delta$. Conclude that on A_n and for all n , $\phi(\|\mathbf{m}_n\|^{-1}\mathbf{m}_n, \widehat{\Sigma}_n) \geq \kappa$ for a certain constant $\kappa > 0$ and finally

$$\mathbf{D}_n^\top \mathbf{D}_n \geq \|\mathbf{D}_n\|^2 \geq (v_n \|\mathbf{m}_n\| \|\phi(\|\mathbf{m}_n\|^{-1}\mathbf{m}_n, \widehat{\Sigma}_n)\| + O_{\mathbb{P}}(1))^2 \xrightarrow{\mathbb{P}} \infty$$

because $(\|\mathbf{m}_n\|)$ is bounded from below by a nonzero constant. The proof is complete. \square

Lemma A.6 analyses the possible asymptotic behaviour of ratios of extreme expectiles and quantiles pertaining to different marginal distributions. A particular consequence of this result is that the testing problem of Section 3.3 of the main paper is indeed well-defined and is nothing but testing whether extreme quantiles of the marginal distributions of \mathbf{X} are asymptotically equivalent.

Lemma A.6. *Assume that Condition B is satisfied, with $\rho_j < 0$ for any $1 \leq j \leq d$. Assume also that $\mathbb{E}|\min(X_j, 0)| < \infty$ and $0 < \gamma_j < 1$ for any $1 \leq j \leq d$. Pick arbitrary $j, \ell \in \{1, \dots, d\}$ with $j \neq \ell$.*

(i) *The ratios $\xi_{\tau,j}/\xi_{\tau,\ell}$ and $q_{\tau,j}/q_{\tau,\ell}$ have well-defined, possibly infinite, limits as $\tau \uparrow 1$, and these limits are equal.*

(ii) *One has $\lim_{\tau \uparrow 1} \xi_{\tau,j}/\xi_{\tau,\ell} = 1$ if and only if $\lim_{\tau \uparrow 1} q_{\tau,j}/q_{\tau,\ell} = 1$, in which case $\gamma_j = \gamma_\ell$ and*

$$\frac{\xi_{\tau,j}}{\xi_{\tau,\ell}} = 1 + O(q_{\tau,j}^{-1}) + O(|A_j((1-\tau)^{-1})|) + O(|A_\ell((1-\tau)^{-1})|) \text{ as } \tau \uparrow 1.$$

Proof of Lemma A.6. Use the asymptotic proportionality relationship (4) in the main paper to write

$$\frac{\xi_{\tau,j}}{\xi_{\tau,\ell}} = \frac{\xi_{\tau,j}}{q_{\tau,j}} \times \frac{q_{\tau,j}}{q_{\tau,\ell}} \times \frac{q_{\tau,\ell}}{\xi_{\tau,\ell}} = \frac{(\gamma_j^{-1} - 1)^{-\gamma_j}}{(\gamma_\ell^{-1} - 1)^{-\gamma_\ell}} \times \frac{q_{\tau,j}}{q_{\tau,\ell}} (1 + o(1)) \text{ as } \tau \uparrow 1.$$

To show (i), note that if $\gamma_j < \gamma_\ell$ then $q_{\tau,j}/q_{\tau,\ell} = U_j((1-\tau)^{-1})/U_\ell((1-\tau)^{-1}) \rightarrow 0$ as $\tau \uparrow 1$. Similarly, if $\gamma_\ell < \gamma_j$ then $q_{\tau,j}/q_{\tau,\ell} \rightarrow \infty$ as $\tau \uparrow 1$. If now $\gamma_j = \gamma_\ell$, we use the remark below Example 2.3.11 in [de Haan and Ferreira \(2006\)](#) to find that $U_j(t) = c_j t^{\gamma_j} (1 + o(1))$ and $U_\ell(t) = c_\ell t^{\gamma_\ell} (1 + o(1))$ as $t \rightarrow \infty$, where $c_j, c_\ell \in (0, \infty)$ (this follows from the assumption $\rho_j, \rho_\ell < 0$). Therefore

$$\frac{q_{\tau,j}}{q_{\tau,\ell}} = \frac{U_j((1-\tau)^{-1})}{U_\ell((1-\tau)^{-1})} \rightarrow \frac{c_j}{c_\ell} \in (0, \infty) \text{ as } \tau \uparrow 1.$$

Conclude that

$$\lim_{\tau \uparrow 1} \frac{q_{\tau,j}}{q_{\tau,\ell}} = \begin{cases} 0 & \text{if } \gamma_j < \gamma_\ell, \\ \infty & \text{if } \gamma_j > \gamma_\ell, \\ c_j/c_\ell & \text{if } \gamma_j = \gamma_\ell \end{cases} \text{ and thus } \lim_{\tau \uparrow 1} \frac{\xi_{\tau,j}}{\xi_{\tau,\ell}} = \begin{cases} 0 & \text{if } \gamma_j < \gamma_\ell, \\ \infty & \text{if } \gamma_j > \gamma_\ell, \\ c_j/c_\ell & \text{if } \gamma_j = \gamma_\ell. \end{cases}$$

This proves (i). To prove (ii) we note that for either ratio to have a finite limit it is necessary that $\gamma_j = \gamma_\ell = \gamma$, and the limit being 1 imposes that $c_j = c_\ell = c$. It is then a consequence of the equation below Formula (2.3.23) in [de Haan and Ferreira \(2006\)](#) that

$$U_j(t) = ct^\gamma \left(1 + \frac{A_j(t)}{\rho_j} + o(|A_j(t)|) \right) \text{ and } U_\ell(t) = ct^\gamma \left(1 + \frac{A_\ell(t)}{\rho_\ell} + o(|A_\ell(t)|) \right)$$

as $t \rightarrow \infty$. [A similar result, obtained by different means, appears in the proof of Corollary 2 in [Hoga \(2018\)](#).] Conclude that

$$\frac{q_{\tau,j}}{q_{\tau,\ell}} = 1 + O(|A_j((1-\tau)^{-1})|) + O(|A_\ell((1-\tau)^{-1})|)$$

as $\tau \uparrow 1$. Apply then Proposition 1(i) in [Daouia et al. \(2020\)](#) to get

$$\begin{aligned} \frac{\xi_{\tau,j}}{\xi_{\tau,\ell}} &= 1 + O(q_{\tau,j}^{-1}) + O(q_{\tau,\ell}^{-1}) + O(|A_j((1-\tau)^{-1})|) + O(|A_\ell((1-\tau)^{-1})|) \\ &= 1 + O(q_{\tau,j}^{-1}) + O(|A_j((1-\tau)^{-1})|) + O(|A_\ell((1-\tau)^{-1})|) \text{ because } \lim_{\tau \uparrow 1} \frac{q_{\tau,j}}{q_{\tau,\ell}} = 1. \end{aligned}$$

The proof is complete. □

Proof of Theorem 3.6. Define $v_n = \frac{\sqrt{n(1-\tau_n)}}{\log[(1-\tau_n)/(1-\tau'_n)]}$ and $\mathbf{m}_n = \log \boldsymbol{\xi}_{\tau'_n}$. Under H_0 we have, by Lemma [A.6\(ii\)](#),

$$\sup_{1 \leq i, j \leq d} \left| \log \frac{\xi_{\tau,j}}{\xi_{\tau,\ell}} \right| \leq \sum_{1 \leq i, j \leq d} \left| \log \frac{\xi_{\tau,j}}{\xi_{\tau,\ell}} \right| = O \left(q_{\tau,1}^{-1} + \sum_{j=1}^d |A_j((1-\tau)^{-1})| \right).$$

Consequently, if $\|\cdot\|_\infty$ denotes the supremum norm,

$$\begin{aligned} \|v_n(\mathbf{m}_n - \|\mathbf{m}_n\|_\infty \mathbf{1}_d)\|_\infty &= O \left(v_n q_{\tau'_n,1}^{-1} + \sum_{j=1}^d v_n |A_j((1-\tau'_n)^{-1})| \right) \\ &= o \left(v_n q_{\tau_n,1}^{-1} + \sum_{j=1}^d v_n |A_j((1-\tau_n)^{-1})| \right) = o(1). \end{aligned}$$

In other words, under H_0 , $v_n(\mathbf{m}_n - \|\mathbf{m}_n\|_\infty \mathbf{1}_d) \rightarrow \mathbf{0}_d$ as $n \rightarrow \infty$.

Under H_1 , use Lemma [A.6\(i\)](#) to find that there is at least a pair of indices (j, ℓ) for which $\xi_{\tau,j}/\xi_{\tau,\ell} \rightarrow C_{j,\ell} < 1$ as $\tau \uparrow 1$. Therefore

$$\limsup_{n \rightarrow \infty} \|\|\mathbf{m}_n\|_\infty^{-1} \mathbf{m}_{n,j} - 1\| \geq 1 - C_{j,\ell} > 0,$$

meaning that the sequence of vectors $(\|\mathbf{m}_n\|_\infty^{-1} \mathbf{m}_n - \mathbf{1}_d)$ stays bounded away from $\mathbf{0}_d$.

(i) Set $\mathbf{Z}_n = \log \tilde{\boldsymbol{\xi}}_{\tau_n}^* + \hat{\mathbf{b}}/\sqrt{n(1-\tau_n)}$, $\hat{\boldsymbol{\Sigma}}_n = \hat{\mathbf{V}}_n^{*,\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R})$ and $\boldsymbol{\Sigma} = \mathbf{V}^*(\boldsymbol{\gamma}, \mathbf{R})$. The test statistic is

$$\begin{aligned}\Lambda_n^{\text{LAWS}} &= v_n^2 \left(\mathbf{Z}_n - \frac{\mathbf{Z}_n^\top \hat{\boldsymbol{\Sigma}}_n^{-1} \mathbf{1}_d}{\mathbf{1}_d^\top \hat{\boldsymbol{\Sigma}}_n^{-1} \mathbf{1}_d} \mathbf{1}_d \right)^\top \hat{\boldsymbol{\Sigma}}_n^{-1} \left(\mathbf{Z}_n - \frac{\mathbf{Z}_n^\top \hat{\boldsymbol{\Sigma}}_n^{-1} \mathbf{1}_d}{\mathbf{1}_d^\top \hat{\boldsymbol{\Sigma}}_n^{-1} \mathbf{1}_d} \mathbf{1}_d \right) \\ &= \left(v_n \hat{\boldsymbol{\Sigma}}_n^{-1/2} \left(\mathbf{Z}_n - \frac{\mathbf{Z}_n^\top \hat{\boldsymbol{\Sigma}}_n^{-1} \mathbf{1}_d}{\mathbf{1}_d^\top \hat{\boldsymbol{\Sigma}}_n^{-1} \mathbf{1}_d} \mathbf{1}_d \right) \right)^\top \left(v_n \hat{\boldsymbol{\Sigma}}_n^{-1/2} \left(\mathbf{Z}_n - \frac{\mathbf{Z}_n^\top \hat{\boldsymbol{\Sigma}}_n^{-1} \mathbf{1}_d}{\mathbf{1}_d^\top \hat{\boldsymbol{\Sigma}}_n^{-1} \mathbf{1}_d} \mathbf{1}_d \right) \right).\end{aligned}$$

Combine Theorem 2.4 of the main paper, Proposition A.5 and our discussion above to conclude.

(ii) Repeat the proof of (i) with $\mathbf{Z}_n = \log \hat{\boldsymbol{\xi}}_{\tau_n}'^*$ and $\hat{\boldsymbol{\Sigma}}_n = \hat{\mathbf{V}}_n^{*,\text{QB}}(\boldsymbol{\gamma}, \mathbf{R})$. □

B Detailed construction of confidence regions at the intermediate level

We provide here a more detailed discussion of the construction of our joint asymptotic confidence regions for intermediate expectiles. This expands on its corresponding synthesised version in Section 3.1.

Using LAWS estimation Let us first show why constructing confidence regions for intermediate expectiles is a delicate task, even in the univariate case. In Table I, we have reported the (non-)coverage probability of a standard Gaussian 95% asymptotic confidence interval, based on a random sample of size $n = 1,000$, for the intermediate expectile ξ_{τ_n} of level $\tau_n = 1 - 1/\sqrt{n} \approx 0.968$, for the Fréchet, Pareto and Student- t distributions with tail index $\gamma = 1/3$, using the LAWS intermediate expectile estimator. In other words, the asymptotic 95% confidence interval we consider, which is directly deduced from the convergence of a single LAWS estimator in Theorem 2.1, is

$$\tilde{J}_{\tau_n} = \left[\tilde{\xi}_{\tau_n} \left(1 - \sqrt{\frac{2\hat{\gamma}_{\tau_n}^3}{1 - 2\hat{\gamma}_{\tau_n}}} \frac{1.96}{\sqrt{n(1-\tau_n)}} \right), \tilde{\xi}_{\tau_n} \left(1 + \sqrt{\frac{2\hat{\gamma}_{\tau_n}^3}{1 - 2\hat{\gamma}_{\tau_n}}} \frac{1.96}{\sqrt{n(1-\tau_n)}} \right) \right].$$

Here $\hat{\gamma}_{\tau_n}$ denotes the Hill estimator for the X_i based on the top $k = \lfloor n(1-\tau_n) \rfloor$ data points. It can be seen that the coverage probability of this interval is in general in the neighbourhood of twice the nominal level.

In our multivariate case, and at the intermediate level, our main instrument for the construction of asymptotic confidence regions based on LAWS estimators is Theorem 2.1 of the main paper, namely

$$\sqrt{n(1-\tau_n)} \left(\frac{\tilde{\boldsymbol{\xi}}_{\tau_n}}{\boldsymbol{\xi}_{\tau_n}} - \mathbf{1}_d \right) \xrightarrow{d} \mathcal{N}_d(\mathbf{0}_d, \mathbf{V}^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R})). \quad (\text{B.23})$$

Constructing a consistent estimator of the matrix $\mathbf{V}^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R})$ that accurately reflects the uncertainty of LAWS estimators present in finite samples is of course key to the use of this

		MSE	NCP
Estimator	Univariate Fréchet Model ($\eta = 3$)		
LAWS	Standard	9.870	9.29
	Variance-adjusted		5.09
QB	Standard	17.095	56.02
	Bias-adjusted	10.747	4.97
	Univariate Pareto Model ($\zeta = 3$)		
LAWS	Standard	8.520	9.46
	Variance-adjusted		5.01
QB	Standard	18.666	63.22
	Bias-adjusted	10.945	5.11
	Univariate Student- t Model ($\nu = 3$)		
LAWS	Standard	11.449	8.64
	Variance-adjusted		7.27
QB	Standard	13.019	4.84
	Bias-adjusted	13.659	5.82

Table I: Monte Carlo relative Mean Squared Error (MSE, in $\sqrt{\cdot} \times 100$) of the LAWS and QB point estimators, and actual Non-Coverage Probability (NCP, in %) for LAWS and QB confidence interval estimators at the intermediate level $\tau_n = 1 - 1/\sqrt{n}$. For the LAWS estimator, “Standard” refers to the Gaussian confidence interval \tilde{J}_{τ_n} , while “Variance-adjusted” refers to the marginal interval deduced from $\tilde{\mathcal{E}}_{\tau_n, 0.05}$. For the QB estimator, “Standard” refers to the Gaussian confidence interval \hat{J}_{τ_n} , while “Bias-adjusted” refers to its version calculated using the bias correction, namely, the marginal interval deduced from $\hat{\mathcal{E}}_{\tau_n, 0.05}$.

convergence for the design of asymptotic confidence regions. We investigate here an estimator of $\mathbf{V}^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R})$ based on a careful examination of the proof of Theorem 2.1. Recall from this proof that, if $\varphi_\tau(y) = |\tau - \mathbb{1}\{y \leq 0\}|y$ is the derivative of $\eta_\tau/2$, one has

$$\sqrt{n(1-\tau_n)} \begin{pmatrix} \tilde{\xi}_{\tau_n} \\ \xi_{\tau_n} \end{pmatrix} - \mathbf{1}_d = \arg \min_{\mathbf{u} \in \mathbb{R}^d} \left\{ - \sum_{j=1}^d u_j T_{1,n}^{(j)} + \sum_{j=1}^d T_{2,n}^{(j)}(u_j) \right\}$$

where $T_{1,n}^{(j)} = \frac{1}{\sqrt{n(1-\tau_n)}} \sum_{i=1}^n \frac{1}{\xi_{\tau_n, j}} \varphi_{\tau_n}(X_{i,j} - \xi_{\tau_n, j})$

and $T_{2,n}^{(j)}(u) = -\frac{1}{\xi_{\tau_n, j}^2} \sum_{i=1}^n \int_0^{u \xi_{\tau_n, j} / \sqrt{n(1-\tau_n)}} (\varphi_{\tau_n}(X_{i,j} - \xi_{\tau_n, j} - z) - \varphi_{\tau_n}(X_{i,j} - \xi_{\tau_n, j})) dz.$

We find in the proof that $(T_{1,n}^{(1)}, \dots, T_{1,n}^{(d)})$ is asymptotically Gaussian with expectation zero.

Meanwhile, for any j , an application of the law of large numbers and a Taylor expansion of the function φ_{τ_n} heuristically suggest that, as $n \rightarrow \infty$,

$$\begin{aligned} T_{2,n}^{(j)}(u) &\approx -\frac{n}{\xi_{\tau_n,j}^2} \int_0^{u\xi_{\tau_n,j}/\sqrt{n(1-\tau_n)}} (\mathbb{E}(\varphi_{\tau_n}(X_j - \xi_{\tau_n,j} - z)) - \mathbb{E}(\varphi_{\tau_n}(X_j - \xi_{\tau_n,j}))) dz \\ &\approx \frac{n}{\xi_{\tau_n,j}^2} \int_0^{u\xi_{\tau_n,j}/\sqrt{n(1-\tau_n)}} \mathbb{E}(|\tau_n - \mathbb{1}\{X_j \leq \xi_{\tau_n,j}\}|) z dz = \frac{u^2}{2} \left(1 + \frac{2\tau_n - 1}{1 - \tau_n} \bar{F}_j(\xi_{\tau_n,j}) \right). \end{aligned}$$

It therefore follows from classical results on minimisers of convex random functions (see *e.g.* Theorem 5 in [Knight, 1999](#)) that, when n is large, we have the approximation (valid in distribution)

$$\sqrt{n(1-\tau_n)} \left(\frac{\tilde{\xi}_{\tau_n}}{\xi_{\tau_n}} - \mathbf{1}_d \right) \approx \left(\frac{1}{\sqrt{n(1-\tau_n)}} \sum_{i=1}^n \frac{1}{\xi_{\tau_n,j}} \frac{\varphi_{\tau_n}(X_{i,j} - \xi_{\tau_n,j})}{1 + (2\tau_n - 1)\bar{F}_j(\xi_{\tau_n,j})/(1-\tau_n)} \right)_{1 \leq j \leq d}^\top.$$

Since each $\varphi_{\tau_n}(X_j - \xi_{\tau_n,j})$ has expectation 0 (by definition of the expectile $\xi_{\tau_n,j}$), this suggests the following nonparametric approximation of $\mathbf{V}_{j,\ell}^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R})$ for large n :

$$\begin{aligned} \mathbf{V}_{j,\ell}^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R}) &\approx \frac{1}{(1-\tau_n)\xi_{\tau_n,j}\xi_{\tau_n,\ell}} \times \frac{\mathbb{E}(\varphi_{\tau_n}(X_j - \xi_{\tau_n,j})\varphi_{\tau_n}(X_\ell - \xi_{\tau_n,\ell}))}{[1 + (2\tau_n - 1)\bar{F}_j(\xi_{\tau_n,j})/(1-\tau_n)][1 + (2\tau_n - 1)\bar{F}_\ell(\xi_{\tau_n,\ell})/(1-\tau_n)]}. \end{aligned}$$

This approximation is our starting point for the construction of an estimator of $\mathbf{V}_{j,\ell}^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R})$. It is of course consistent with the expression of $\mathbf{V}_{j,\ell}^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R})$ in Theorem 2.1, in the sense that when $n \rightarrow \infty$,

$$\begin{aligned} &\frac{1}{(1-\tau_n)\xi_{\tau_n,j}\xi_{\tau_n,\ell}} \times \frac{\mathbb{E}(\varphi_{\tau_n}(X_j - \xi_{\tau_n,j})\varphi_{\tau_n}(X_\ell - \xi_{\tau_n,\ell}))}{[1 + (2\tau_n - 1)\bar{F}_j(\xi_{\tau_n,j})/(1-\tau_n)][1 + (2\tau_n - 1)\bar{F}_\ell(\xi_{\tau_n,\ell})/(1-\tau_n)]} \\ \rightarrow \mathbf{V}_{j,\ell}^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R}) &= \begin{cases} \frac{2\gamma_j^3}{1 - 2\gamma_j} & \text{if } j = \ell, \\ \gamma_j\gamma_\ell \iint_{[1,\infty)^2} R_{j,\ell} \left((\gamma_j^{-1} - 1)x_j^{-1/\gamma_j}, (\gamma_\ell^{-1} - 1)x_\ell^{-1/\gamma_\ell} \right) dx_j dx_\ell & \text{if } j < \ell. \end{cases} \end{aligned}$$

[See (4) in the main paper and Lemma [A.3](#).] One could estimate each term in this nonparametric approximation of $\mathbf{V}_{j,\ell}^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R})$ directly; this turns out not to be the best-performing solution in practice because it tends to provide an underestimation of the marginal uncertainty on expectiles. Our proposed solution, suggested by the results of extensive Monte-Carlo simulations, is the following. For a diagonal entry $\mathbf{V}_{j,j}^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R})$ of $\mathbf{V}^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R})$, note that Lemma [A.3\(i\)](#) entails

$$\begin{aligned} \mathbf{V}_{j,j}^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R}) &\approx \frac{\mathbb{E}(\varphi_{\tau_n}^2(X_j - \xi_{\tau_n,j}))}{(1-\tau_n)\xi_{\tau_n,j}^2} \times \frac{1}{[1 + (2\tau_n - 1)\bar{F}_j(\xi_{\tau_n,j})/(1-\tau_n)]^2} \\ &\approx \frac{2\gamma_j}{1 - 2\gamma_j} \times \frac{1}{[1 + (2\tau_n - 1)\bar{F}_j(\xi_{\tau_n,j})/(1-\tau_n)]^2} \text{ for large } n. \end{aligned}$$

A further improvement in finite samples is found by noting that $1 + \bar{F}_j(\xi_{\tau_n,j})/(1 - \tau_n) \rightarrow \gamma_j^{-1}$ as $n \rightarrow \infty$, and therefore

$$\mathbf{V}_{j,j}^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R}) \approx \frac{2\gamma_j^2}{1 - 2\gamma_j} \times \frac{1 + \bar{F}_j(\xi_{\tau_n,j})/(1 - \tau_n)}{[1 + (2\tau_n - 1)\bar{F}_j(\xi_{\tau_n,j})/(1 - \tau_n)]^2} \text{ for large } n.$$

For off-diagonal elements, the covariance $\text{Cov}(\varphi_{\tau_n}(X_j - \xi_{\tau_n,j}), \varphi_{\tau_n}(X_\ell - \xi_{\tau_n,\ell}))$ is in practice found to be a good approximation of the direction of dependence within the data; a finite-sample improvement on the estimation of the strength of this dependence is found by writing

$$\mathbf{V}_{j,\ell}^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R}) \approx \gamma_j \gamma_\ell \frac{\mathbb{E}(\varphi_{\tau_n}(X_j - \xi_{\tau_n,j}) \varphi_{\tau_n}(X_\ell - \xi_{\tau_n,\ell}))}{(1 - \tau_n) \xi_{\tau_n,j} \xi_{\tau_n,\ell}} \text{ for large } n.$$

Our estimator of $\mathbf{V}^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R})$ is now constructed by plugging in the LAWS and Hill estimators, the empirical survival functions

$$\widehat{\bar{F}}_{n,j}(x) = 1 - \widehat{F}_{n,j}(x) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{X_{i,j} > x\},$$

and the empirical covariances

$$\bar{m}_{n,j,\ell} = \frac{1}{n} \sum_{i=1}^n \varphi_{\tau_n}(X_{i,j} - \tilde{\xi}_{\tau_n,j}) \varphi_{\tau_n}(X_{i,\ell} - \tilde{\xi}_{\tau_n,\ell}).$$

This results in the estimator $\widehat{\mathbf{V}}_n^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R})$ of $\mathbf{V}^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R})$ given elementwise by

$$\begin{aligned} \widehat{\mathbf{V}}_{n,j,j}^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R}) &= \frac{2\widehat{\gamma}_{\tau_n,j}^2}{1 - 2\widehat{\gamma}_{\tau_n,j}} \times \frac{1 + \widehat{\bar{F}}_{n,j}(\tilde{\xi}_{\tau_n,j})/(1 - \tau_n)}{\left[1 + (2\tau_n - 1)\widehat{\bar{F}}_{n,j}(\tilde{\xi}_{\tau_n,j})/(1 - \tau_n)\right]^2} \text{ for } j = \ell \\ \text{and } \widehat{\mathbf{V}}_{n,j,\ell}^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R}) &= \widehat{\gamma}_{\tau_n,j} \widehat{\gamma}_{\tau_n,\ell} \frac{\bar{m}_{n,j,\ell}}{(1 - \tau_n) \tilde{\xi}_{\tau_n,j} \tilde{\xi}_{\tau_n,\ell}} \text{ for } j \neq \ell. \end{aligned}$$

Under the assumptions of Theorem 2.1 of the main paper, this is indeed a consistent estimator of $\mathbf{V}^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R})$, as Proposition A.4 shows.

When $\mathbf{V}^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R})$ is symmetric positive definite (in particular, no perfect asymptotic dependence between any two components of \mathbf{X} can be present), multiplying the left-hand side in (B.23) by the positive definite inverse square root $[\mathbf{V}^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R})]^{-1/2}$ of $\mathbf{V}^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R})$ and then plugging in our estimator $\widehat{\mathbf{V}}_n^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R})$ produces an asymptotically Gaussian random vector with independent standard Gaussian components. Therefore, if $\|\cdot\|_2$ denotes the Euclidean norm on \mathbb{R}^d and $\chi_{d,1-\alpha}^2$ denotes the $(1 - \alpha)$ -quantile of the chi-square distribution with d degrees of freedom, one has

$$\mathbb{P} \left(\left\| \sqrt{n(1 - \tau_n)} [\widehat{\mathbf{V}}_n^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R})]^{-1/2} \begin{pmatrix} \tilde{\xi}_{\tau_n} \\ \xi_{\tau_n} \end{pmatrix} - \mathbf{1}_d \right\|_2^2 \leq \chi_{d,1-\alpha}^2 \right) \rightarrow 1 - \alpha \text{ as } n \rightarrow \infty.$$

Denoting by $B_d(\mathbf{0}_d, r)$ the closed Euclidean ball in \mathbb{R}^d whose centre is the origin $\mathbf{0}_d$ and radius is r , we find the corresponding $(1 - \alpha)$ -asymptotic LAWS-based confidence region for $\boldsymbol{\xi}_{\tau_n}$ as the random ellipsoid

$$\begin{aligned}\tilde{\mathcal{E}}_{\tau_n, \alpha} &= \tilde{\boldsymbol{\xi}}_{\tau_n} \left[\mathbf{1}_d + [\hat{\mathbf{V}}_n^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R})]^{1/2} B_d \left(\mathbf{0}_d, \sqrt{\chi_{d,1-\alpha}^2/n(1-\tau_n)} \right) \right] \\ &= \left\{ \mathbf{z} \in \mathbb{R}^d \mid \exists \mathbf{u} \in B_d \left(\mathbf{0}_d, \sqrt{\chi_{d,1-\alpha}^2/n(1-\tau_n)} \right), \mathbf{z} = \tilde{\boldsymbol{\xi}}_{\tau_n} \left[\mathbf{1}_d + [\hat{\mathbf{V}}_n^{\text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R})]^{1/2} \mathbf{u} \right] \right\}.\end{aligned}$$

Using QB estimation Like in the LAWS case, the correct use of the QB estimator to construct confidence regions for intermediate expectile estimators is not as straightforward as one could expect. Table I reports the coverage probability of the standard Gaussian 95% asymptotic confidence interval

$$\begin{aligned}\hat{J}_{\tau_n} &= \left[\hat{\xi}_{\tau_n} \left(1 - \hat{\gamma}_{\tau_n} \sqrt{1 + (m(\hat{\gamma}_{\tau_n}))^2} \frac{1.96}{\sqrt{n(1-\tau_n)}} \right), \right. \\ &\quad \left. \hat{\xi}_{\tau_n} \left(1 + \hat{\gamma}_{\tau_n} \sqrt{1 + (m(\hat{\gamma}_{\tau_n}))^2} \frac{1.96}{\sqrt{n(1-\tau_n)}} \right) \right],\end{aligned}$$

constructed by ignoring the bias component and plugging in the Hill estimator to estimate the asymptotic variance. We keep the same setup as in the LAWS case: random samples of size $n = 1,000$ from the Pareto, Fréchet and Student distributions with tail index $\gamma = 1/3$, and the intermediate level $\tau_n = 1 - 1/\sqrt{n} \approx 0.968$. This is often a very poor confidence interval whose coverage probability is very far from the nominal level.

In our multivariate intermediate case, with the QB estimator, our main tool is Corollary 2.3:

$$\sqrt{n(1-\tau_n)} \left(\frac{\hat{\boldsymbol{\xi}}_{\tau_n}}{\boldsymbol{\xi}_{\tau_n}} - \mathbf{1} \right) \xrightarrow{d} \mathcal{N}_d(\mathbf{b}, \mathbf{V}^{\text{QB}}(\boldsymbol{\gamma}, \mathbf{R})).$$

Contrary to the LAWS estimator, the QB estimator is asymptotically biased due to its reliance on the asymptotic proportionality relationship (4) in the main paper. The j th component of this bias is essentially

$$\begin{aligned}\mathbf{b}_j &\approx -\gamma_j(\gamma_j^{-1} - 1)^{\gamma_j} \mathbb{E}(X_j) \frac{\sqrt{n(1-\tau_n)}}{q_{\tau_n, j}} \\ &\quad + \left(\frac{m(\gamma_j)}{1 - \rho_j} - \frac{(\gamma_j^{-1} - 1)^{-\rho_j}}{1 - \gamma_j - \rho_j} - \frac{(\gamma_j^{-1} - 1)^{-\rho_j} - 1}{\rho_j} \right) \sqrt{n(1-\tau_n)} A_j((1-\tau_n)^{-1}).\end{aligned}$$

Two sources of bias therefore arise when using the QB estimator, one proportional to $1/q_{\tau_n, j}$, the other linked to the second-order framework. The correction of the latter source of bias involves estimating accurately the second-order parameter ρ_j , which is a notoriously difficult problem (see *e.g.* the Introduction of Cai et al., 2013), especially from the practical point of view since consistent estimators of ρ_j typically suffer from low rates of convergence, see

e.g. [Goegebeur et al. \(2010, p.2638\)](#) and [Gomes et al. \(2009, p.298\)](#). As such, correcting second-order bias tends to increase finite-sample variability substantially, resulting in confidence regions that may be too conservative. By contrast, the simple expression of the bias component proportional to $1/q_{\tau_n,j}$ makes its correction a straightforward task, with all estimators involved converging at the rate $\sqrt{n(1-\tau_n)}$ or more. This constitutes our rationale for concentrating specifically on the first source of bias with the estimator

$$\hat{\mathbf{b}}_j = -\hat{\gamma}_{\tau_n,j}(\hat{\gamma}_{\tau_n,j}^{-1} - 1)\hat{\gamma}_{\tau_n,j}\bar{X}_{n,j}\frac{\sqrt{n(1-\tau_n)}}{\hat{q}_{\tau_n,j}}, \quad \text{where } \bar{X}_{n,j} = \frac{1}{n}\sum_{i=1}^n X_{i,j}.$$

The covariance matrix $\mathbf{V}^{\text{QB}}(\boldsymbol{\gamma}, \mathbf{R})$, meanwhile, may be estimated as follows:

$$\begin{aligned} \hat{\mathbf{V}}_{n,j,j}^{\text{QB}}(\boldsymbol{\gamma}, \mathbf{R}) &= \hat{\gamma}_{\tau_n,j}^2(1 + (m(\hat{\gamma}_{\tau_n,j}))^2) \quad (\text{with } m(x) = (1-x)^{-1} - \log(x^{-1} - 1)) \\ \text{and } \hat{\mathbf{V}}_{n,j,\ell}^{\text{QB}}(\boldsymbol{\gamma}, \mathbf{R}) &= \hat{\gamma}_{\tau_n,j}\hat{\gamma}_{\tau_n,\ell}\left(\hat{R}_{\tau_n,j,\ell}(1,1)(m(\hat{\gamma}_{\tau_n,j}) - 1)(m(\hat{\gamma}_{\tau_n,\ell}) - 1) \right. \\ &\quad \left. + m(\hat{\gamma}_{\tau_n,j})\int_0^1 \hat{R}_{\tau_n,j,\ell}(u,1)\frac{du}{u} + m(\hat{\gamma}_{\tau_n,\ell})\int_0^1 \hat{R}_{\tau_n,j,\ell}(1,u)\frac{du}{u}\right) \end{aligned}$$

where the estimator of the tail copula $R_{j,\ell}$ is defined as

$$\hat{R}_{\tau_n,j,\ell}(u,v) = \frac{1}{n(1-\tau_n)}\sum_{i=1}^n \mathbb{1}\left\{\frac{n+1-r_{n,i,j}}{(n+1)(1-\tau_n)} \leq u, \frac{n+1-r_{n,i,\ell}}{(n+1)(1-\tau_n)} \leq v\right\}.$$

[Here $r_{n,i,j} = n\hat{F}_{n,j}(X_{i,j})$ denotes the marginal rank of observation $X_{i,j}$ among the observations $X_{1,j}, X_{2,j}, \dots, X_{n,j}$.] This estimator is a slightly modified version of the estimator of the empirical upper tail copula estimator given in Equation (13) in [Schmidt and Stadtmüller \(2006\)](#). Using the quantity $n+1$ in the above formula, rather than the more obvious n , allows one to compute $\hat{\mathbf{V}}_{n,j,\ell}^{\text{QB}}(\boldsymbol{\gamma}, \mathbf{R})$ in a very efficient way by noting that

$$\begin{aligned} &\int_0^1 \hat{R}_{\tau_n,j,\ell}(u,1)\frac{du}{u} \\ &= -\frac{1}{n(1-\tau_n)}\sum_{i=1}^n \log\left(\frac{n+1-r_{n,i,j}}{(n+1)(1-\tau_n)}\right) \mathbb{1}\left\{\frac{n+1-r_{n,i,j}}{(n+1)(1-\tau_n)} \leq 1, \frac{n+1-r_{n,i,\ell}}{(n+1)(1-\tau_n)} \leq 1\right\}. \end{aligned}$$

This estimator $\hat{\mathbf{V}}_n^{\text{QB}}(\boldsymbol{\gamma}, \mathbf{R})$ is a consistent estimator of $\mathbf{V}^{\text{QB}}(\boldsymbol{\gamma}, \mathbf{R})$.

A calculation entirely similar to the one carried out with the LAWS estimator now yields an $(1-\alpha)$ -asymptotic QB confidence region for $\boldsymbol{\xi}_{\tau_n}$ as the random ellipsoid

$$\hat{\mathcal{E}}_{\tau_n,\alpha} = \hat{\boldsymbol{\xi}}_{\tau_n} \left[\mathbf{1}_d - \frac{\hat{\mathbf{b}}}{\sqrt{n(1-\tau_n)}} + [\hat{\mathbf{V}}_n^{\text{QB}}(\boldsymbol{\gamma}, \mathbf{R})]^{1/2} B_d\left(\mathbf{0}_d, \sqrt{\chi_{d,1-\alpha}^2/n(1-\tau_n)}\right) \right].$$

This construction is illustrated in the top panel of Figure [I](#), where it is seen that the QB region tends to have a lower volume than the LAWS region.

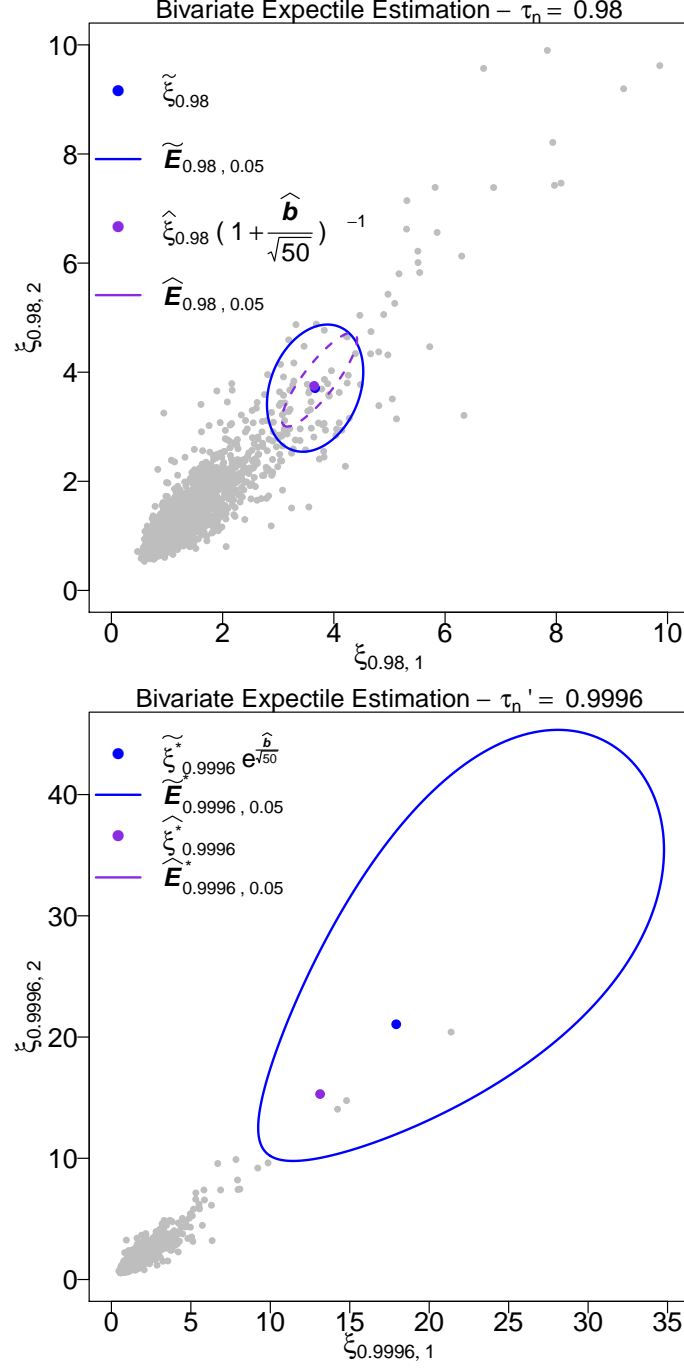


Figure I: Expectile point estimates from the LAWS and QB estimators and associated 95% confidence regions, obtained with one sample of size $n = 2,500$ generated from a Gumbel-Fréchet bivariate distribution with equal marginal tail index $\gamma = 1/3$ and dependence parameter $\vartheta = 3$. Top: Intermediate level, regions $\tilde{\mathcal{E}}_{\tau_n, \alpha}$ and $\hat{\mathcal{E}}_{\tau_n, \alpha}$ with $\tau_n = 1 - 1/\sqrt{n} = 0.98$, bottom: extreme level, regions $\tilde{\mathcal{E}}_{\tau'_n, \alpha}^*$ and $\hat{\mathcal{E}}_{\tau'_n, \alpha}^*$ with $\tau'_n = 1 - 1/n = 0.9996$ and $\tau_n = 1 - 1/\sqrt{n} = 0.98$. The Gumbel-Fréchet model is model (iii) in our simulations Section 4.1.

Let us conclude this section by elaborating why correcting for the bias proportional to $1/q_{\tau_n,j}$ is important regardless of whether second-order bias dominates or not. The key point is that even though the asymptotics in our results will suggest that one component of our vector of expectile estimators is asymptotically biased and the others are not (the biased component will be the one pertaining to the distribution with the lowest γ_j , provided $|\rho_j| > \gamma_j$ for any $1 \leq j \leq d$), this does not mean that these other components should be considered unbiased in finite samples. We illustrate this on the following simple examples. Suppose first that $\mathbf{X} = (X_1, X_2)^\top$ has Fréchet marginal distributions, that is,

$$\forall x > 0, \mathbb{P}(X_1 > x) = 1 - \exp(-x^{-1/\gamma_1}) \text{ and } \mathbb{P}(X_2 > x) = 1 - \exp(-x^{-1/\gamma_2})$$

and assume that $\gamma_1 = 1/3 \approx 0.33$ and $\gamma_2 = 2/5 = 0.4$. This corresponds to random variables X_1 and X_2 having a finite variance but no third moment, which is a fairly typical case when considering heavy-tailed models. For the vector $\mathbf{X} = (X_1, X_2)^\top$, one has $\rho_1 = \rho_2 = -1$. Strictly speaking, if we only take the asymptotics into account, only the source of bias due to $1/q_{\tau_n,1}$ in the first marginal dominates, because $\rho_1 = \rho_2 < -\gamma_2 < -\gamma_1$. One could then think that only the bias due to $1/q_{\tau_n,1}$ should be corrected, and everything else left ignored. However, according to Proposition 1(i) in [Daouia et al. \(2020\)](#), one has, as $n \rightarrow \infty$,

$$\begin{aligned} \frac{\xi_{\tau,j}}{q_{\tau,j}} \approx (\gamma_j^{-1} - 1)^{-\gamma_j} & \left[1 + \frac{\gamma_j(\gamma_j^{-1} - 1)^{\gamma_j} \mathbb{E}(X_j)}{q_{\tau,j}} \right. \\ & \left. + \left(\frac{(\gamma_j^{-1} - 1)^{-\rho_j}}{1 - \gamma_j - \rho_j} + \frac{(\gamma_j^{-1} - 1)^{-\rho_j} - 1}{\rho_j} \right) A_j((1 - \tau)^{-1}) \right] \end{aligned}$$

It is readily checked that $\mathbb{E}(X_1) \approx 1.35$, $\mathbb{E}(X_2) \approx 1.49$ and $A_j(t) = \gamma_j t^{\rho_j}/2$. Consequently, if we fix $\tau = 0.95$, corresponding to $\tau = \tau_n = 1 - k/n$ for $n = 1,000$ and $k = 50$ (close to the setup we consider in our simulation studies), we find $q_{\tau,1} \approx 2.69$ and $q_{\tau,2} \approx 3.28$, and so

$$\frac{\gamma_1(\gamma_1^{-1} - 1)^{\gamma_1} \mathbb{E}(X_1)}{q_{\tau_n,1}} \approx 0.211 \quad \text{and} \quad \frac{\gamma_2(\gamma_2^{-1} - 1)^{\gamma_2} \mathbb{E}(X_2)}{q_{\tau_n,2}} \approx 0.214,$$

while

$$\begin{aligned} & \left(\frac{(\gamma_1^{-1} - 1)^{-\rho_1}}{1 - \gamma_1 - \rho_1} + \frac{(\gamma_1^{-1} - 1)^{-\rho_1} - 1}{\rho_1} \right) A_1((1 - \tau_n)^{-1}) \approx 0.00167 \\ \text{and} \quad & \left(\frac{(\gamma_2^{-1} - 1)^{-\rho_2}}{1 - \gamma_2 - \rho_2} + \frac{(\gamma_2^{-1} - 1)^{-\rho_2} - 1}{\rho_2} \right) A_2((1 - \tau_n)^{-1}) \approx 0.00438. \end{aligned}$$

This calculation means that the error in the approximation $\xi_{\tau_n,j}/q_{\tau_n,j} \approx (\gamma_j^{-1} - 1)^{-\gamma_j}$ due to the $1/q_{\tau_n,j}$ term exceeds 20% in both marginals, while the error due to the second-order source of bias is two degrees of magnitude smaller. This is important when using the QB estimator, of course, but also when using the extrapolating LAWS-based estimator, since (see Formula (11) in the main paper)

$$\left[\frac{1 - \tau'_n}{1 - \tau_n} \right]^{\gamma_j} \frac{\xi_{\tau'_n,j}}{\xi_{\tau_n,j}} = 1 - \frac{\gamma_j(\gamma_j^{-1} - 1)^{\gamma_j} \mathbb{E}(X_j)}{q_{\tau_n,j}} + o(q_{\tau_n,j}^{-1}) + O(A_j((1 - \tau_n)^{-1})).$$

This approximation error translates into bias at the finite-sample level and, if left uncorrected, incorrect coverage for confidence intervals. Here, even though the asymptotics suggest that only the bias due to $1/q_{\tau_n,1}$ should be corrected and everything else left ignored, in a finite sample of this size this would result in a very substantial error.

Interestingly this also tends to happen in cases when the second-order bias, controlled by the value of the parameter ρ_j , is theoretically of the same order as the bias due to the reciprocal of the j th quantile function. Suppose indeed that now the first marginal distribution is a Burr distribution, that is,

$$\forall x > 0, \mathbb{P}(X_1 > x) = (1 + x^{-\rho_1/\gamma_1})^{1/\rho_1}, \text{ but as before } \mathbb{P}(X_2 > x) = 1 - \exp(-x^{-1/\gamma_2}),$$

and assume that $\gamma_1 = -\rho_1 = 1/3$ and $\gamma_2 = 2/5$. In this case, according to the asymptotics, the sources of bias coming from $1/q_{\tau_n,1}$ and $A_1((1 - \tau_n)^{-1})$ in the first marginal should dominate, because $\rho_2 < -\gamma_2 < -\gamma_1 = \rho_1$. It is besides readily checked that $\mathbb{E}(X_1) \approx 0.5$ and $A_1(t) = \gamma_1 t^{\rho_1}$. Consequently, with $\tau = \tau_n = 0.95$, we find $q_{\tau,1} \approx 1.71$, and so

$$\frac{\gamma_1(\gamma_1^{-1} - 1)^{\gamma_1} \mathbb{E}(X_1)}{q_{\tau_n,1}} \approx 0.122 \quad \text{and} \quad \frac{\gamma_2(\gamma_2^{-1} - 1)^{\gamma_2} \mathbb{E}(X_2)}{q_{\tau_n,2}} \approx 0.214 \text{ (the latter as before),}$$

while

$$\begin{aligned} & \left(\frac{(\gamma_1^{-1} - 1)^{-\rho_1}}{1 - \gamma_1 - \rho_1} + \frac{(\gamma_1^{-1} - 1)^{-\rho_1} - 1}{\rho_1} \right) A_1((1 - \tau_n)^{-1}) \approx 0.0590 \\ \text{and } & \left(\frac{(\gamma_2^{-1} - 1)^{-\rho_2}}{1 - \gamma_2 - \rho_2} + \frac{(\gamma_2^{-1} - 1)^{-\rho_2} - 1}{\rho_2} \right) A_2((1 - \tau_n)^{-1}) \approx 0.00438 \text{ (the latter as before).} \end{aligned}$$

In this case the error due to $1/q_{\tau_n,2}$ actually exceeds both errors coming from the first marginal distribution, and the error due to $1/q_{\tau_n,1}$ is twice as high as the error originating from $A_1((1 - \tau_n)^{-1})$. This is despite these latter two sources of bias theoretically having the same importance and dominating the biases coming from the second marginal distribution. Our argument therefore is that in typical cases, one should correct for the bias due to the $1/q_{\tau,j}$ terms in all marginals, as not to do so would negatively affect finite-sample results even if one knew which marginal had the lightest tail (which itself is a seriously unrealistic assumption).

C Additional finite-sample results

This section contains additional results linked to our simulation study and real data analysis in Sections 4 and 5 of the main paper. We start by studying the marginal finite-sample uncertainty about tail expectiles. One can easily deduce from the construction of the LAWS-based confidence region $\hat{\mathcal{E}}_{\tau'_n, \alpha}^*$ an asymptotic $(1 - \alpha)$ -confidence interval for the j th marginal extreme expectile $\xi_{\tau'_n, j}$:

$$\begin{aligned} \tilde{I}_{\tau'_n, j, \alpha}^* = & \left[\tilde{\xi}_{\tau'_n, j}^* \exp \left(\frac{\hat{b}_j}{\sqrt{n(1 - \tau_n)}} - \frac{\log d_n}{\sqrt{n(1 - \tau_n)}} \sqrt{\hat{\mathbf{V}}_{n, j, j}^{\star, \text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R})} z_{1 - \alpha/2} \right), \right. \\ & \left. \tilde{\xi}_{\tau'_n, j}^* \exp \left(\frac{\hat{b}_j}{\sqrt{n(1 - \tau_n)}} + \frac{\log d_n}{\sqrt{n(1 - \tau_n)}} \sqrt{\hat{\mathbf{V}}_{n, j, j}^{\star, \text{LAWS}}(\boldsymbol{\gamma}, \mathbf{R})} z_{1 - \alpha/2} \right) \right] \end{aligned}$$

where $z_{1 - \alpha/2}$ is the quantile of the standard Gaussian distribution at level $1 - \alpha/2$. This can be seen as an adjusted version of the confidence interval based on the LAWS estimator that is considered in [Daouia et al. \(2018\)](#). One may also construct from the QB confidence region $\hat{\mathcal{E}}_{\tau'_n, \alpha}^*$ a QB asymptotic $(1 - \alpha)$ -confidence interval for the j th marginal extreme expectile at level τ'_n :

$$\begin{aligned} \hat{I}_{\tau'_n, j, \alpha}^* = & \left[\hat{\xi}_{\tau'_n, j}^* \exp \left(- \frac{\log d_n}{\sqrt{n(1 - \tau_n)}} \sqrt{\hat{\mathbf{V}}_{n, j, j}^{\star, \text{QB}}(\boldsymbol{\gamma}, \mathbf{R})} z_{1 - \alpha/2} \right), \right. \\ & \left. \hat{\xi}_{\tau'_n, j}^* \exp \left(\frac{\log d_n}{\sqrt{n(1 - \tau_n)}} \sqrt{\hat{\mathbf{V}}_{n, j, j}^{\star, \text{QB}}(\boldsymbol{\gamma}, \mathbf{R})} z_{1 - \alpha/2} \right) \right]. \end{aligned}$$

This is an adjusted version of the confidence interval based on the so-called indirect estimator in [Daouia et al. \(2018\)](#).

C.1 Marginal uncertainty about tail expectiles

Here we simulate $M = 10,000$ samples of $n = 1,000$ independent observations from the Fréchet distribution, having distribution function $F(x) = \exp(-x^{-1/\gamma})$ for $x > 0$; the Pareto distribution, having distribution function $F(x) = 1 - x^{-1/\gamma}$ for $x > 1$; and the Student- t distribution with $1/\gamma$ degrees of freedom. The tail index is chosen to be $\gamma = 1/3$ in each case. For each simulated sample we estimate the (univariate) expectile at the extreme level $\tau'_n = 0.999 = 1 - 1/n$ and we compute the associated confidence intervals $\tilde{I}_{\tau'_n, \alpha}^*$ and $\hat{I}_{\tau'_n, \alpha}^*$ defined in Section 3.2 (there is no dependence on the label of the marginal in this univariate case) with 95% nominal coverage probability. As in the main paper, in the bias correction terms and estimated variances, the Hill estimators $\hat{\gamma}_{\tau_n}$ are replaced by their versions $\hat{\gamma}_{1 - k/n}$ with $k \in [6, 300]$, and the results will be described as a function of this value of k . Then, we compute the actual coverage probability of the corresponding confidence interval estimators. Results are collected in [Figure II](#).

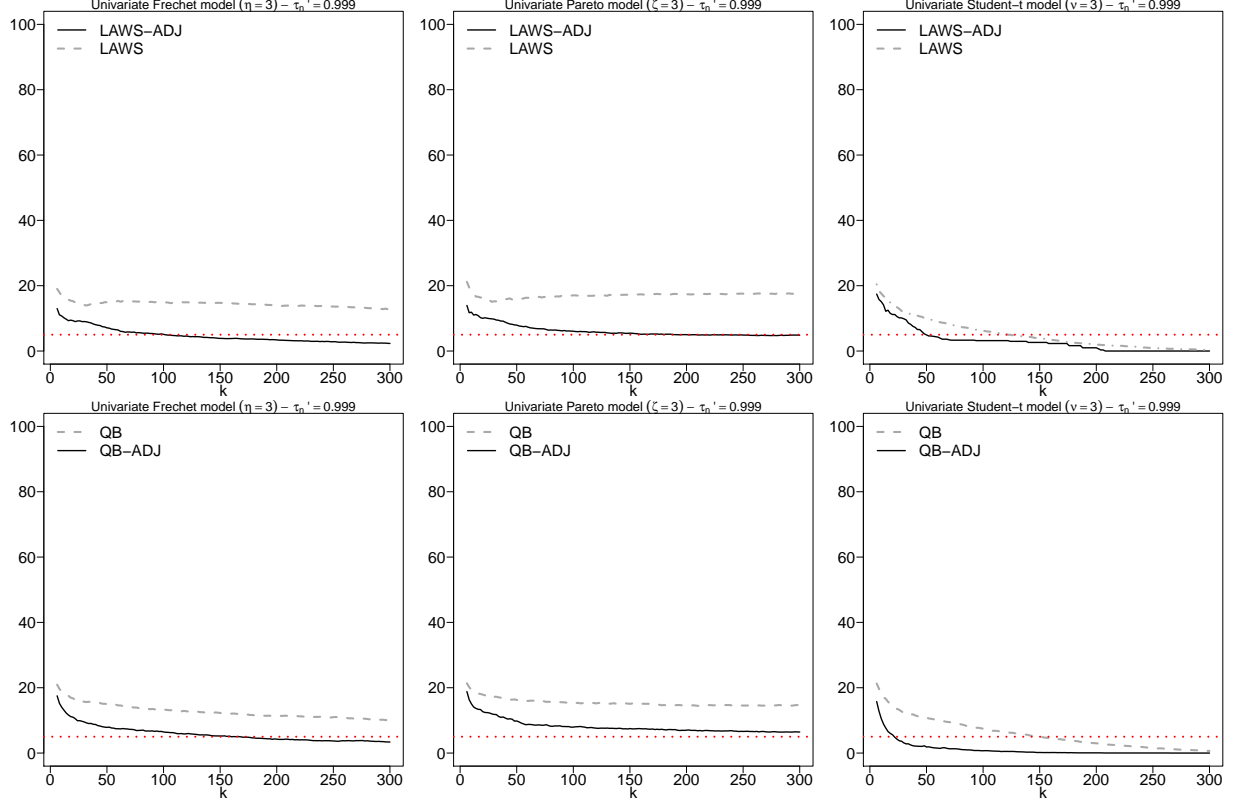


Figure II: Actual non-coverage probabilities (in %) of the confidence intervals of $\xi_{\tau'_n}$, with $n = 1,000$, $\tau'_n = 0.999$ and 95% nominal level (top: LAWS method, bottom: QB method), where in each case the dashed line stands for the naive interval and the solid line for its adjusted counterpart. The horizontal dotted red line represents the 5% nominal non-coverage probability.

The adjusted interval estimators perform comparably in each case, and in fact the LAWS confidence interval has slightly better and more stable coverage, as the top panels show. Our adjusted intervals provide visibly improved results compared to their unadjusted versions for all three distributions, with a remarkable improvement in the LAWS case for the Fréchet and Pareto distributions. By contrast, the actual non-coverage probability of the unadjusted versions is typically in the range of 15-20%. As a conclusion, it appears that in terms of marginal inference at the extreme level, the LAWS and QB extrapolating estimators are comparable, with an advantage for the former once our adjustment to the confidence interval has been applied.

C.2 Joint inference about intermediate and extreme expectiles

We study the finite-sample behaviour of the intermediate LAWS and QB expectile estimators in models (i)-(iv) described in the main paper in Section 4. In each model, we simulate $M = 10^4$ samples of size $n = m \cdot 10^3$, with $m \in \{1, 1.5, 2, 2.5, 5, 10\}$ and dimension $d \in \{2, 3, 4, 5\}$.

We estimate and infer the d -dimensional expectile $\boldsymbol{\xi}_{\tau_n}$, with $\tau_n = 1 - 1/\sqrt{n}$, using the LAWS and QB expectile point estimators and the confidence regions $\tilde{\mathcal{E}}_{\tau_n, \alpha}$ and $\hat{\mathcal{E}}_{\tau_n, \alpha}$, with $\alpha = 0.05$ (95% nominal coverage probability), described in Section 3.1 of the main paper. Then, we compute a Monte Carlo approximation of the relative MSE of the LAWS and QB point estimators across all components and we report the actual (non-)coverage probabilities of the associated confidence regions (see Tables II, III and IV).

With every model except the Gumbel-Fréchet model, the actual coverage probability of the LAWS confidence region estimator is close to the nominal level. With the Gumbel-Fréchet model, permissive confidence regions are generally obtained. This seems to be due to the strong dependence structure of the Gumbel-Fréchet model which is somewhat difficult to estimate accurately. The conclusions for the QB confidence region are similar. By contrast, the naive confidence regions obtained assuming that the margins are independent (and thus ignoring the question of the estimation of the asymptotic dependence between components) provide unsuitable regions whose actual non-coverage probabilities are either substantially higher than desired (for the LAWS estimator) or virtually equal to one (for the QB estimator). Our proposal therefore allows to obtain considerably more accurate confidence regions than existing methods; results do not seem to deteriorate significantly with increasing dimension (at least up to $d = 5$).

C.3 Additional results concerning the real data application

Figures III and IV give additional results on tail index and extreme expectile estimates related to our real data analysis in Section 5, as well as certain bivariate and trivariate confidence regions for extreme expectiles.

m	$d = 2$		$d = 3$		$d = 4$		$d = 5$	
	LAWS	QB	LAWS	QB	LAWS	QB	LAWS	QB
Clayton-Fréchet Model ($\eta = 3$)								
1.0	9.713	10.816	9.244	10.915	10.230	10.864	9.822	10.936
1.5	8.596	9.479	8.739	9.551	8.623	9.488	8.554	9.425
2.0	8.131	8.509	8.075	8.381	7.972	8.460	7.934	8.530
2.5	7.556	7.707	7.578	7.699	7.470	7.760	7.443	7.761
5.0	6.018	6.327	6.410	6.376	6.257	6.263	6.275	6.437
10.0	5.161	4.995	5.098	4.815	5.112	4.729	5.650	4.076
Gaussian-Student- t Model ($\nu = 3$)								
1.0	11.459	13.472	12.313	13.611	11.994	13.611	11.759	13.536
1.5	9.976	11.164	10.115	11.348	10.224	11.272	10.231	11.185
2.0	9.157	10.045	8.949	10.012	9.280	9.991	9.230	9.946
2.5	9.035	8.912	8.679	9.063	8.911	9.018	8.713	9.032
5.0	6.629	6.886	6.702	6.859	6.982	6.902	6.804	6.877
10.0	5.573	5.446	5.998	5.426	5.474	5.416	6.683	5.387
Gumbel-Fréchet Model ($\eta = 3$)								
1.0	9.269	10.922	9.487	10.829	9.888	10.953	9.510	10.783
1.5	8.829	9.582	8.675	9.443	9.883	9.570	8.792	9.520
2.0	7.729	8.520	7.769	8.503	8.012	8.545	7.900	8.428
2.5	7.835	7.732	7.298	7.934	7.499	7.704	7.587	7.879
5.0	6.260	6.230	6.222	6.307	6.057	6.308	6.126	6.276
10.0	5.238	4.953	5.033	5.002	5.558	5.059	5.092	5.049
Multivariate Student- t Model ($\nu = 3$)								
1.0	12.835	13.469	12.438	13.394	12.841	13.365	11.571	13.424
1.5	9.855	11.163	10.206	11.185	10.986	11.270	10.126	11.376
2.0	9.325	9.914	9.227	9.903	9.046	9.913	9.927	9.836
2.5	8.339	9.077	11.012	8.884	8.609	9.111	8.479	9.024
5.0	6.752	6.961	7.523	6.856	6.704	6.895	7.963	6.997
10.0	5.453	5.385	5.497	5.390	8.474	5.414	5.379	5.423

Table II: Monte Carlo relative MSE (averaged across marginals and reported in $\sqrt{\cdot} \times 100$) of the LAWS and QB intermediate estimators at level τ_n , with $n = m \cdot 10^3$ (left column) and $\tau_n = 1 - 1/\sqrt{n}$.

m	$d = 2$	$d = 3$	$d = 4$	$d = 5$
Clayton-Fréchet Model ($\eta = 3$)				
1.0	4.86(11.59)	5.02(13.55)	5.50(15.33)	5.85(17.40)
1.5	4.74(10.66)	4.98(12.87)	5.12(14.12)	5.14(15.22)
2.0	5.01(10.78)	5.03(12.22)	4.95(13.15)	5.12(14.79)
2.5	4.64(9.69)	4.77(11.20)	4.90(12.90)	4.67(13.64)
5.0	4.16(8.78)	4.47(10.65)	4.59(11.07)	4.52(11.72)
10.0	4.01(8.14)	4.75(9.50)	4.34(10.06)	4.46(10.37)
Gaussian-Student- t Model ($\nu = 3$)				
1.0	6.18(8.85)	6.02(9.42)	6.83(11.03)	6.30(11.24)
1.5	4.85(7.20)	5.61(8.52)	6.36(9.37)	5.98(9.50)
2.0	4.86(6.81)	5.28(7.73)	5.70(8.55)	5.95(9.19)
2.5	4.95(6.84)	5.24(7.78)	5.70(8.57)	5.82(9.23)
5.0	4.74(6.47)	5.20(7.11)	5.06(7.06)	5.30(7.74)
10.0	4.58(5.86)	5.45(6.90)	5.49(7.09)	5.17(6.95)
Gumbel-Fréchet Model ($\eta = 3$)				
1.0	5.02(11.47)	4.81(13.64)	4.80(14.22)	4.44(16.24)
1.5	4.30(11.01)	4.44(12.47)	3.90(13.79)	3.73(15.34)
2.0	3.85(9.91)	4.06(12.25)	3.41(13.73)	3.76(15.20)
2.5	3.95(10.73)	3.97(12.60)	3.63(14.13)	3.15(14.81)
5.0	3.69(10.09)	3.34(11.57)	2.87(12.87)	2.51(14.23)
10.0	3.47(9.48)	2.69(11.10)	2.20(12.76)	2.03(13.98)
Multivariate Student- t Model ($\nu = 3$)				
1.0	5.18(8.37)	5.56(9.95)	6.33(11.10)	6.32(11.79)
1.5	4.93(8.11)	5.66(9.42)	5.14(9.75)	5.72(10.91)
2.0	4.47(7.26)	5.16(8.76)	5.33(9.37)	5.79(10.41)
2.5	4.56(7.32)	4.76(7.96)	5.51(9.21)	5.39(9.90)
5.0	4.62(7.16)	4.49(7.51)	4.35(7.81)	4.65(8.66)
10.0	3.99(6.41)	4.28(7.83)	4.67(8.79)	4.76(8.43)

Table III: Monte Carlo actual non-coverage probability (in %) for the LAWS confidence region estimator $\tilde{\mathcal{E}}_{\tau_n, \alpha}$ at the intermediate level, with $n = m \cdot 10^3$ (left column), $\tau_n = 1 - 1/\sqrt{n}$ and 95% nominal level. Between brackets we report the coverage probability obtained assuming independence between the margins.

m	$d = 2$	$d = 3$	$d = 4$	$d = 5$
Clayton-Fréchet Model ($\eta = 3$)				
1.0	6.43(88.81)	7.06(95.11)	7.60(98.06)	7.89(98.95)
1.5	5.80(91.20)	6.60(96.80)	7.18(98.54)	7.19(99.28)
2.0	5.25(92.56)	6.08(97.35)	6.13(99.01)	7.15(99.64)
2.5	5.50(93.25)	5.82(98.03)	6.16(99.30)	6.81(99.83)
5.0	4.84(95.58)	5.11(96.58)	5.88(99.10)	5.79(99.32)
10.0	4.55(97.55)	5.16(97.15)	5.26(99.60)	5.07(99.61)
Gaussian-Student- t Model ($\nu = 3$)				
1.0	6.60(29.32)	6.99(36.66)	7.21(42.65)	7.56(48.00)
1.5	5.54(27.47)	6.44(34.09)	6.47(39.79)	7.28(45.34)
2.0	5.60(27.47)	5.99(32.15)	6.84(37.79)	7.10(42.88)
2.5	5.42(25.02)	5.90(31.79)	6.58(38.16)	6.73(42.78)
5.0	5.27(23.82)	5.06(28.30)	5.99(34.23)	5.83(37.68)
10.0	5.30(22.31)	5.30(26.85)	5.46(30.74)	6.11(35.64)
Gumbel-Fréchet Model ($\eta = 3$)				
1.0	4.21(78.44)	3.82(81.39)	3.35(83.29)	3.47(83.92)
1.5	3.68(81.11)	3.27(82.98)	3.48(84.74)	2.86(86.26)
2.0	3.40(83.02)	3.18(85.36)	3.31(86.99)	3.16(86.94)
2.5	3.54(84.48)	3.22(86.78)	3.55(87.77)	2.86(89.16)
5.0	3.36(87.23)	2.94(89.47)	2.92(91.07)	2.46(91.62)
10.0	3.38(91.34)	2.76(92.97)	2.74(93.17)	2.53(94.42)
Multivariate Student- t Model ($\nu = 3$)				
1.0	5.24(28.66)	5.44(33.16)	6.46(37.84)	6.54(41.79)
1.5	4.75(26.44)	5.34(31.38)	5.66(35.85)	6.37(38.60)
2.0	4.72(25.37)	5.49(30.50)	5.54(34.44)	6.15(37.42)
2.5	4.50(25.26)	5.26(28.73)	5.52(33.72)	5.86(37.96)
5.0	4.73(22.98)	4.36(26.17)	5.05(30.85)	5.24(33.65)
10.0	4.95(22.66)	4.76(25.43)	5.35(29.83)	4.53(32.53)

Table IV: Monte Carlo actual non-coverage probability (in %) for the QB confidence region estimator $\hat{\mathcal{E}}_{\tau_n, \alpha}$ at the intermediate level, with $n = m \cdot 10^3$ (left column), $\tau_n = 1 - 1/\sqrt{n}$ and 95% nominal level. Between brackets we report the coverage probability obtained assuming independence between the margins.

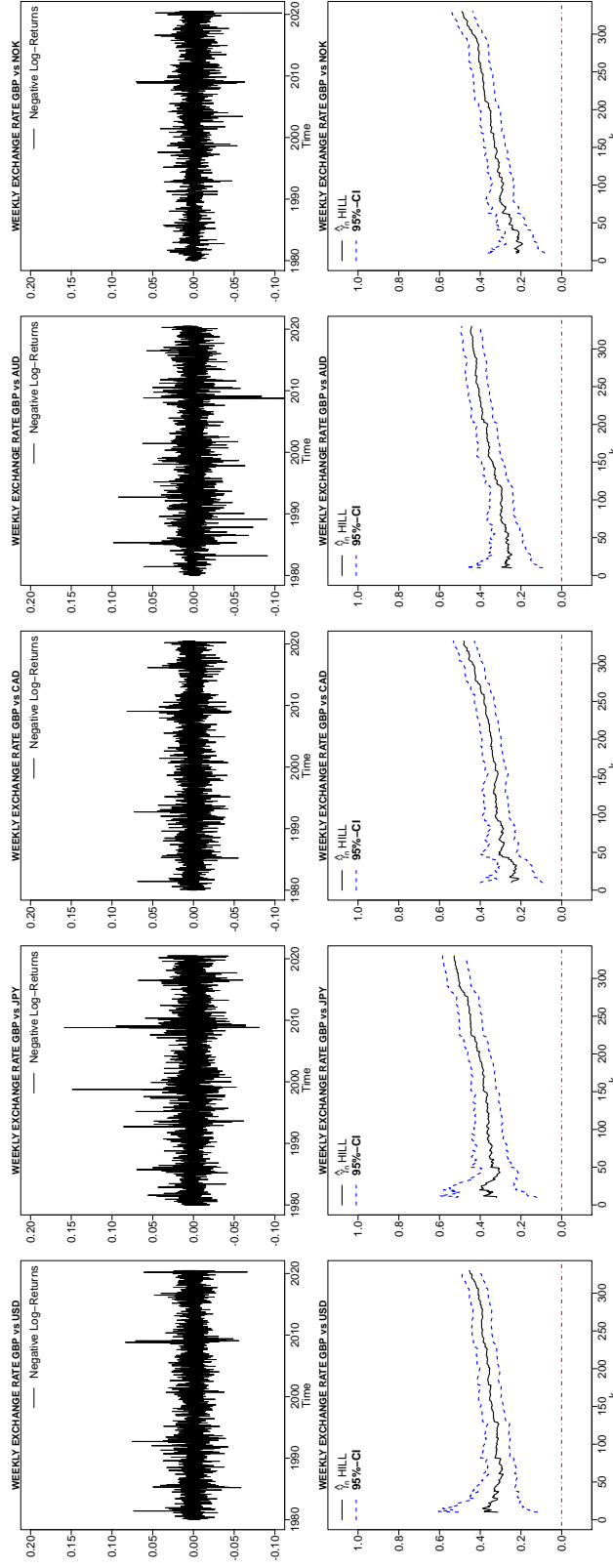


Figure III: Negative weekly log-returns of exchange rates (top row) and tail index estimates and asymptotic 95% confidence intervals (marginal Hill estimators, bottom row).

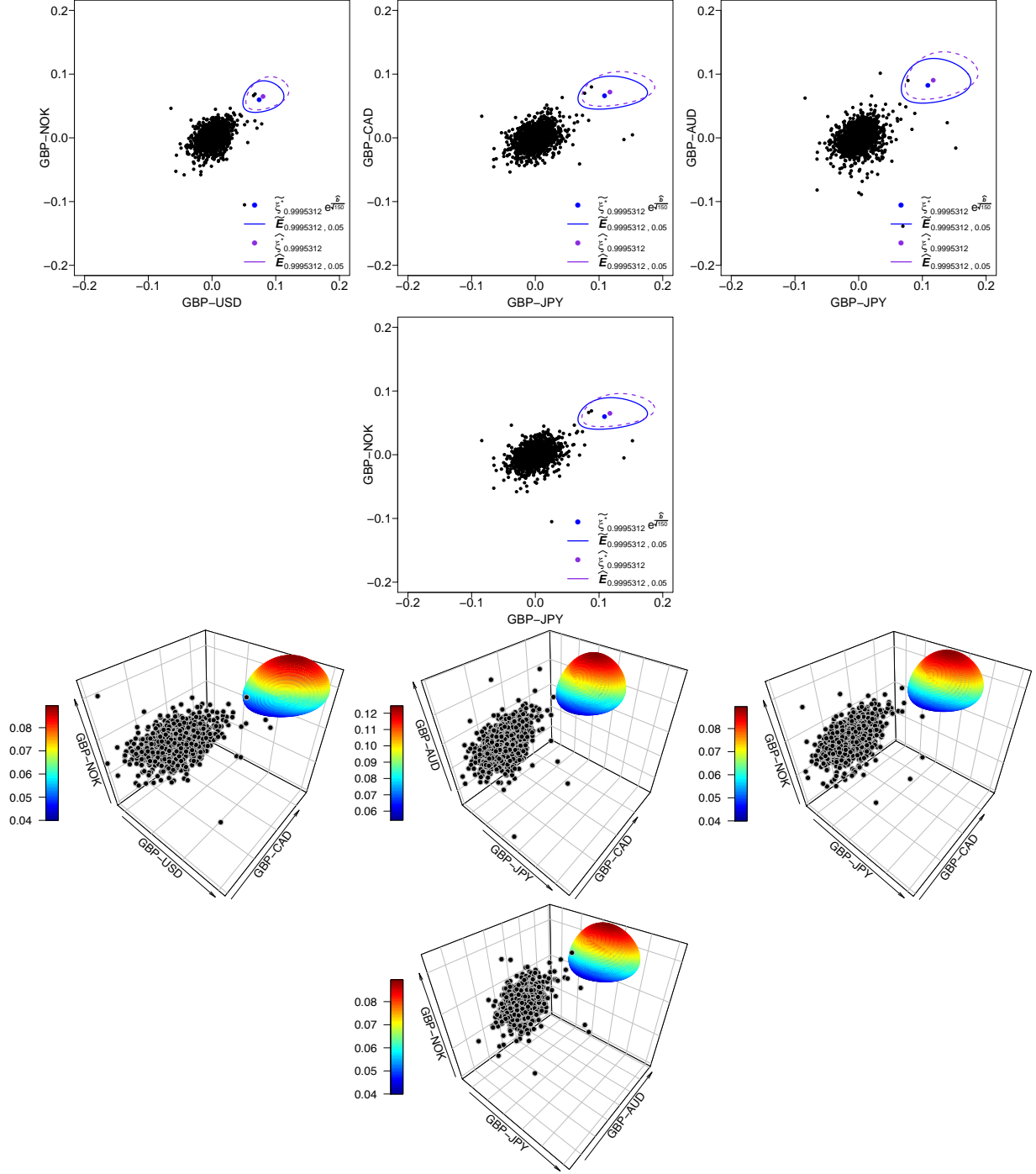


Figure IV: Two- and three-dimensional 95% confidence region estimates for some pairs and triplets of exchange rate returns, obtained with $\tau'_n = 0.9995312$ and $\tau_n = 1 - k/n$ with $k = 150$. In the three-dimensional case we only report LAWS-based confidence regions.

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