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**Abstract**

Usually Fokker-Planck type partial differential equations (PDEs) are well-posed if the initial condition is specified. In this paper, alternatively, we consider the inverse problem which consists in prescribing final data: in particular we give sufficient conditions for existence and uniqueness. In the second part of the paper we provide a probabilistic representation of those PDEs in the form a solution of a McKean type equation corresponding to the time-reversal dynamics of a diffusion process.

**Key words and phrases**. Inverse problem; McKean stochastic differential equation; probabilistic representation of PDEs; time-reversed diffusion; Fokker Planck equation.

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## 1 Introduction

The main objective of the paper consists in studying well-posedness and probabilistic representation of the Fokker-Planck PDE with terminal condition

$$\begin{cases} \partial_t \mathbf{u} &= \frac{1}{2}\sum_{i,j=1}^{d} \partial^2_{ij}\left((\sigma\sigma^\top)_{i,j}(t,x)\mathbf{u}\right) - div\left(b(t,x)\mathbf{u}\right) \\ \mathbf{u}(T) &= \mu, \end{cases} \tag{1.1}$$

EDPTerm0

where $\sigma : [0,T]\times\mathbb{R}^d \to M_{d,m}(\mathbb{R})$, $b : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ and $\mu$ is a prescribed finite Borel measure on $\mathbb{R}^d$. When $\mathbf{u}(t)$ admits a density for some $t \in [0,T]$ we write $\mathbf{u}(t) = u(t,x)dx$. This equation is motivated by applications in various domains of physical sciences and engineering, as heat conduction [3], material science [13] or hydrology [2]. In particular, *hydraulic inversion* is interested in inverting a diffusion process

beck1985inverse

renardy1987mathematical, bal2003marching

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representing the concentration of a pollutant to identify the pollution source location when the final concentration profile is observed. Those models are often formulated by PDE problems which are in general ill-posed because, either the solution is not unique or the solution is not stable. For this issue, the existence is ensured by the fact that the observed contaminant is necessarily originated from some place at a given time (as soon as the model is correct). Several authors have handled the lack of uniqueness problem by introducing regularization methods approaching the problem by well-posed PDEs, see typically [18] and [11]. A second issue, when the problem is well-approximated by a regularized problem, consists in providing a numerical approximating scheme to the backward diffusion process. In particular for (1.2) there are very few results even concerning existence and uniqueness.

Our point of view is that a probabilistic representation of (1.2) can bring new insights to the treatment of the two mentioned issues: well-posedness and numerical approximation. To realize this objective we consider the renormalized PDE

$$\begin{cases} \partial_t \bar{\mathbf{u}} &= \frac{1}{2}\sum_{i,j=1}^{d} \partial_{ij}^2 \left( (\sigma\sigma^\top)_{i,j}(t,x)\bar{\mathbf{u}} \right) - div\left(b(t,x)\bar{\mathbf{u}}\right) \\ \bar{\mathbf{u}}(T) &= \bar{\mu}, \end{cases} \tag{1.2}$$

EDPTerm

where $\bar{\mu} = \frac{\mu}{\mu(\mathbb{R}^d)}$ is a probability measure. We remark that the PDEs (1.2) and (1.1) are equivalent in the sense that a solution (1.2) (resp. (1.1)) provides a solution to the other one. The program consists in considering the McKean type stochastic differential equation (SDE)

$$\begin{cases} Y_t = Y_0 - \int_0^t b\left(T-r, Y_r\right) dr + \int_0^t \left\{ \frac{div_y\left(\Sigma_{i.}\left(T-r, Y_r\right) p_r\left(Y_r\right)\right)}{p_r\left(Y_r\right)} \right\}_{i\in[\![1,d]\!]} dr + \int_0^t \sigma\left(T-r, Y_r\right) d\beta_r, \\ p_t \text{ density law of } \mathbf{p_t} = \text{law of } \mathrm{Y_t}, \mathrm{t} \in ]0, \mathrm{T}[, \\ Y_0 \sim \mathbf{p_T} = \bar{\mu}, \end{cases} \tag{1.3}$$

MKIntro

where $\beta$ is a $m$-dimensional Brownian motion and $\Sigma = \sigma\sigma^\top$, whose solution is the couple $(Y, \mathbf{p})$. Indeed an application of Itô formula (see Proposition 4.3) shows that whenever $(Y, \mathbf{p})$ is a solution of (1.3) then $t \mapsto \mathbf{p}_{T-t}$ is a solution of (1.2).

The idea of considering (1.3) comes from the SDE verified by time-reversal of a diffusion. Time-reversal of Markov processes was explored by several authors: see for instance [8] for the diffusion case in finite dimension, [6] for the diffusion case in infinite dimension and [9] for the jump case.

Consider a *forward* diffusion process $X$ solution of

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s,\ t \in [0,T], \tag{1.4}$$

eq:X

where $\sigma$ and $b$ are Lipschitz coefficients with linear growth and $W$ is a standard Brownian motion on $\mathbb{R}^m$. $\hat{X}_t := X_{T-t}, t \in [0,T]$ will denote the time-reversal process. In [8] the authors gave sufficient general conditions on $\sigma, b$ and the marginal laws $p_t$ of $X_t$ so that $Y := \hat{X}$ is a solution (in law) of the SDE

$$Y_t = X_T - \int_0^t b\left(T-r, Y_r\right) dr + \int_0^t \left\{ \frac{div_y\left(\Sigma_{i.}\left(T-r, Y_r\right) p_{T-r}\left(Y_r\right)\right)}{p_{T-r}\left(Y_r\right)} \right\}_{i\in[\![1,d]\!]} dr + \int_0^t \sigma\left(T-r, Y_r\right) d\beta_r. \tag{1.5}$$

IntroPardoux

The key idea to show well-posedness of the McKean SDE (1.3), is the study of uniqueness of the PDE (1.2) (or (1.1)). For instance, the trivial case of the heat equation with terminal condition produces uniqueness. Suppose indeed that $u : [0,T] \mapsto \mathcal{S}'\left(\mathbb{R}^d\right)$ solves

$$\begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} \\ \mathbf{u}\left(T\right) = \mu. \end{cases} \tag{1.6}$$

HeatPDE

Then, the Fourier transform of $u$, $v(t,\cdot) := \mathcal{F}\mathbf{u}(t,\cdot), t \in [0,T]$ solves the ODE (for fixed $\xi \in \mathbb{R}^d$)

$$\begin{cases} \frac{d}{dt} v(t,\xi) = -|\xi|^2 v(t,\xi), (t,\xi) \in [0,T] \times \mathbb{R}^d \\ v(T,\cdot) = \mathcal{F}\mu. \end{cases} \tag{1.7}$$

HeatODE

This admits at most one solution, since setting $\mathcal{F}\mu = 0$ the unique solution of (1.7) is the null function.

Another relatively simple situation is described below to study uniqueness among the solutions of (1.2) starting in the class of Dirac measures. Suppose for a moment that the PDE in the first line of (1.2), but with initial condition (see (3.2)) is well-posed. Sufficient conditions for this will be provided in Remark 3.3. Let $x \in \mathbb{R}^d$ and $u$ be a solution of (1.2) such that $u(0,\cdot) = \delta_x$. If $X^x$ is the solution of (1.4) with initial condition $x$, it is well-known that the family of laws of $X_t^x, t \in [0,T]$, is a solution of (1.2). So this coincides with $u(t,\cdot)$ and in particular $\mu$ is the law of $X_T^x$. To conclude we only need to determine $x$.

Consider the example when $\sigma$ is continuous bounded non-degenerate and the drift $b$ is affine i.e. $b(s,y) = b_0(s) + b_1(s)y, (s,y) \in [0,T] \times \mathbb{R}^d$, $b_0$ (resp. $b_1$) being mappings from $[0,T]$ to $\mathbb{R}^d$ (resp. to $M_d(R)$). Taking the expectation in the SDE fulfilled by $X^x$, we show that the function $t \mapsto E^x(t) := \mathbb{E}(X_t^x)$ is solution of

$$E^x(t) = \int_{\mathbb{R}^d} y\mu(dy) - \int_t^T (b_0(s) + b_1(s)E^x(s))\, ds.$$

Previous linear ODE has clearly a unique solution. At this point $x = E(0)$ is determined.

Those examples give a flavor of how to tackle the well-posedness issue. However, generalizing those approaches is far more complicated and constitutes the first part of the present work. The contributions of the paper are twofold.

1. We investigate uniqueness for the Fokker-Planck PDE with terminal condition (1.2). This is done in Section 3 in two different situations: the case when the coefficients are bounded and the situation of a PDE associated with an inhomogeneous Ornstein-Uhlenbeck semigroup. In Section 3.3 we show uniqueness when the coefficients are stepwise time-homogeneous. In Theorem 3.13 the coefficients are time-homogeneous, bounded and Hölder, with non-degenerate diffusion. Corollary 3.16 extends previous results to the case of stepwise time-inhomogeneous coefficients. In Section 3.4, Theorem 3.19 treats the Ornstein-Uhlenbeck case. In Section 3.2 we show uniqueness for bounded continuous coefficients for solutions starting in the class $\mathcal{C}$ of multiples of Dirac measures. In Proposition 3.9 we discuss the framework of dimension $d = 1$. Theorem 3.10 is devoted to the case $d \geq 2$. We distinguish the non-degenerate case from the possibly degenerate case but with smooth coefficients: we prove uniqueness for small time horizon $T$.

2. We study existence and uniqueness in law for the McKean SDE (1.3), with some specific remarks concerning strong existence and pathwise uniqueness. We differentiate specifically between existence and uniqueness. After some preliminary considerations in Section 4.1, Sections 4.2 and 4.3 link the well-posedness of the PDE (1.2) to the well-posedness of the McKean SDE (1.3). In particular Proposition 4.6 (resp. Corollary 4.9) links the existence (resp. uniqueness) of (1.2) with (1.3). In Section 4.4, Proposition 4.14 and Theorem 4.16 discuss the case of bounded coefficients. Theorem 4.19 is Section 4.5 is devoted to the case of Ornstein-Uhlenbeck (with not necessarily Gaussian terminal condition), where strong existence and pathwise uniqueness are established.

## 2 Notations and preliminaries

otations

Let us fix $d, m \in \mathbb{N}^*$, $T > 0$. $\mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$ is the linear space of smooth functions with compact support. For a given $p \in \mathbb{N}^*$, $[\![1,p]\!]$ denotes the set of all integers between 1 and $p$ included. $M_{d,m}\left(\mathbb{R}\right)$ stands for the set of $d \times m$ matrices. If $d = m$, we simply use the notation $M_d\left(\mathbb{R}\right)$. For a given $A \in M_d\left(\mathbb{R}\right)$, $Tr\left(A\right)$ (resp. $A^\top$) symbolizes the trace (resp. the transpose) of the matrix $A$. $||A||$ denotes the usual Frobenius norm. $\langle , \rangle$ denotes the usual scalar product on $\mathbb{R}^d$, with associated norm $|.|$. For a given $f : \mathbb{R}^p \to \mathbb{R}^l$, $p, l \in \mathbb{N}^*$, $\partial_j f^i, (i,j) \in [\![1,l]\!] \times [\![1,p]\!]$ denote the partial derivatives of $f$ being defined in the sense of distributions on $\mathbb{R}^p$ whenever they exist. We also introduce the mapping $Jf$ from $\mathbb{R}^p$ to $M_{l,p}\left(\mathbb{R}\right)$ such that $Jf : z \mapsto \left(\partial_j f^i\left(z\right)\right)_{(i,j)\in[\![1,l]\!]\times[\![1,p]\!]}$.

Let $\alpha \in ]0,1[$, $n \in \mathbb{N}$. $\mathcal{C}_b(\mathbb{R}^d)$ (resp. $\mathcal{C}_b^n(\mathbb{R}^d)$) indicates the space of bounded continuous functions (resp. bounded functions of class $\mathcal{C}^n$ such that all the derivatives are bounded). $\mathcal{C}^\alpha(\mathbb{R}^d)$ is the Banach space of bounded $\alpha$-Hölder functions $\mathbb{R}^d \to \mathbb{R}$ equipped with the norm $|.|_\alpha := ||.||_\infty + [.]_\alpha$, where

$$[f]_\alpha := \sup_{x,y\in\mathbb{R}^d, x\neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < \infty.$$

If $n$ is some integer $\mathcal{C}^{\alpha+n}(\mathbb{R}^d)$ is the Banach space of bounded functions $f : \mathbb{R}^d \to \mathbb{R}$ such that all its derivatives up to order $n$ are bounded and such that the derivatives of order $n$ are $\alpha$-Hölder continuous. This is equipped with the norm obtained as the sum of the $C_b^n(\mathbb{R}^d)$-norm plus the sum of the quantities $[g]_\alpha$ where $g$ is an $n$-order derivative of $f$. For more details, see Section 0.2 of [12]. If $E$ is a linear Banach space, we denote by $||.||_E$ the associated operator norm and by $\mathcal{L}\left(E\right)$ the space of linear bounded operators $E \to E$. Often in the sequel we will have $E = \mathcal{C}^{2\alpha}(\mathbb{R}^d)$.

lunardi_1995

$\mathcal{P}\left(\mathbb{R}^d\right)$ (resp. $\mathcal{M}_+\left(\mathbb{R}^d\right), \mathcal{M}_f\left(\mathbb{R}^d\right)$) denotes the set of probability measures (resp. non-negative finite valued measure, finite signed measures) on $\left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$. We also denote by $\mathcal{S}\left(\mathbb{R}^d\right)$ the space of Schwartz functions and by $\mathcal{S}'\left(\mathbb{R}^d\right)$ the space of tempered distributions. For all $\phi \in \mathcal{S}\left(\mathbb{R}^d\right)$ and $\mu \in \mathcal{M}_f\left(\mathbb{R}^d\right)$, we set the notations

$$\mathcal{F}\phi : \xi \mapsto \int_{\mathbb{R}^d} e^{-i\langle\xi,x\rangle}\phi\left(x\right)dx,\ \mathcal{F}\mu : \xi \mapsto \int_{\mathbb{R}^d} e^{-i\langle\xi,x\rangle}\mu\left(dx\right).$$

Given a mapping $\mathbf{u} : [0,T] \to \mathcal{M}_f\left(\mathbb{R}^d\right)$, we convene that when for $t \in [0,T]$, $\mathbf{u}\left(t\right)$ has a density, this is denoted by $u\left(t,\cdot\right)$. We also introduce, for a given $t$ in $[0,T]$, the differential operator,

$$L_t f := \frac{1}{2}\sum_{i,j=1}^d \Sigma_{ij}(t,\cdot)\partial_{ij} f + \sum_{i=1}^d b_i\left(t,\cdot\right)\partial_i f, \tag{2.1}$$

EqOpL

$f \in C^2(\mathbb{R}^d)$ and denote by $L_t^*$ its formal adjoint, which means that for a given signed measure $\eta$

$$L_t^*\eta := \frac{1}{2}\sum_{i,j=1}^d \partial_{ij}^2\left(\Sigma_{i,j}(t,x)\eta\right) - div\left(b(t,x)\eta\right)\ . \tag{2.2}$$

EqOpL*

With this notation, equation (1.1) rewrites

EDPTerm0

$$\begin{cases} \partial_t \mathbf{u} = L_t^*\mathbf{u} \\ \mathbf{u}\left(T\right) = \mu. \end{cases} \tag{2.3}$$

BackwardFokke

In the sequel we will often make use of the following assumptions.

Lip1d **Assumption 1.** *$b, \sigma$ are Lipschitz in space uniformly in time, with linear growth.*

Zvon1 **Assumption 2.** *$b$ and $\sigma$ are bounded and $\Sigma$ is continuous.*

Zvon3 **Assumption 3.** *There exists $\epsilon > 0$ such that for all $t \in [0,T]$, $\xi \in \mathbb{R}^d$, $x \in \mathbb{R}^d$*

$$\langle \Sigma(t,x)\xi, \xi \rangle \geq \epsilon \left|\xi\right|^2. \tag{2.4}$$

For a given random variable $X$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{L}_{\mathbb{P}}(X)$ denotes its law under $\mathbb{P}$ and $\mathbb{E}_{\mathbb{P}}(X)$ its expectation under $\mathbb{P}$. When self-explanatory, the subscript will be omitted in the sequel.

# 3 A Fokker-Planck PDE with terminal condition

S3

## 3.1 Preliminary results on uniqueness

In this section, we consider a Fokker-Planck type PDE with terminal condition for which the notion of solution is clarified in the following definition.

Def **Definition 3.1.** *Fix $\mu \in \mathcal{M}_f(\mathbb{R}^d)$. We say that a mapping $\mathbf{u}$ from $[0,T]$ to $\mathcal{M}_f(\mathbb{R}^d)$ solves the PDE (1.1), if for all $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and all $t \in [0,T]$*

$$\int_{\mathbb{R}^d} \phi(y)\,\mathbf{u}(t)(dy) = \int_{\mathbb{R}^d} \phi(y)\,\mu(dy) - \int_t^T \int_{\mathbb{R}^d} L_s\phi(y)\,\mathbf{u}(s)(dy)\,ds. \tag{3.1}$$

weak

We consider the following assumption related to a given class $\mathcal{C} \subseteq \mathcal{M}_+(\mathbb{R}^d)$.

GH1 **Assumption 4.** *For all $\nu \in \mathcal{C}$, the PDE*

$$\begin{cases} \partial_t \mathbf{u} = L_t^* \mathbf{u} \\ \mathbf{u}(0) = \nu \end{cases} \tag{3.2}$$

Fokker

*admits at most one solution $\mathbf{u} : [0,T] \to \mathcal{M}_+(\mathbb{R}^d)$.*

We recall that, for a given $\nu \in \mathcal{M}_f(\mathbb{R}^d)$, $\mathbf{u} : [0,T] \to \mathcal{M}_f(\mathbb{R}^d)$ is a solution of the PDE (3.2) if for all $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and all $t \in [0,T]$,

$$\int_{\mathbb{R}^d} \phi(y)\,\mathbf{u}(t)(dy) = \int_{\mathbb{R}^d} \phi(y)\,\nu(dy) + \int_0^t \int_{\mathbb{R}^d} L_s\phi(y)\,\mathbf{u}(s)(dy)\,ds. \tag{3.3}$$

weakbis

Suppose there is an $\mathcal{M}_+(\mathbb{R}^d)$-valued solution of (3.2) $\mathbf{u}$ and Assumption 4 with respect to some class $\mathcal{C}$ holds and such that $\mathbf{u}(0) \in \mathcal{C}$. Then this unique solution will be denoted by $\mathbf{u}^\nu$ in the sequel. We remark that, whenever Assumption 4 holds with respect to a given $\mathcal{C} \subseteq \mathcal{P}(\mathbb{R}^d)$, then (3.2) admits at most one $\mathcal{M}_+(\mathbb{R}^d)$-valued solution with any initial value belonging to $\mathbb{R}_+^* \mathcal{C} := (\alpha\nu)_{\alpha>0, \nu\in\mathcal{C}}$.

We start with a simple but fundamental observation.

PFundam **Proposition 3.2.** *Let us suppose $\sigma, b$ to be locally bounded, $\nu$ be a Borel probability on $\mathbb{R}^d$, $\alpha > 0$, $\xi$ be a r.v. distributed according to $\nu$. Suppose that there is a solution $X$ of SDE*

$$X_t = \xi + \int_0^t b(r, X_r)\,dr + \int_0^t \sigma(r, X_r)\,dW_r,\ t \in [0,T],\ \mathbb{P}-\text{a.s.}, \tag{3.4}$$

EqLin

*where $W$ is an $m$-dimensional standard Brownian motion. Then the $\mathcal{M}_+(\mathbb{R}^d)$-valued function $t \mapsto \alpha\mathcal{L}(X_t)$ is a solution of (3.2) with initial value $\alpha\nu$.*

*Proof.* One first applies Itô formula to $\varphi(X_t)$, where $\varphi$ is a smooth function with compact support and then one takes the expectation. □

R1 **Remark 3.3.** *1. Suppose that the coefficients $b, \Sigma$ are bounded. Assumption 4 holds with respect to $\mathcal{C} := \mathcal{M}_+\left(\mathbb{R}^d\right)$ as soon as the martingale problem associated with $b, \Sigma$ admits uniqueness for all initial condition of the type $\delta_x, x \in \mathbb{R}^d$. Indeed, this is a consequence of Lemma 2.3 in [5].*

*2. Suppose $b$ and $\sigma$ with linear growth. Let $\nu \in \mathcal{M}_+\left(\mathbb{R}^d\right)$ not vanishing (resp. $\nu \in \mathcal{P}\left(\mathbb{R}^d\right)$). The existence of a $\mathcal{M}_+\left(\mathbb{R}^d\right)$-valued (resp. $\mathcal{P}\left(\mathbb{R}^d\right)$-valued) solution for (3.2) (even on $t \geq 0$) is ensured when the martingale problem associated to $b, \Sigma$ admits existence (and consequently when the SDE (3.4) admits weak existence) with initial condition $\nu$ (resp. $\frac{\nu}{\|\nu\|}$). This follows by Proposition 3.2. We remark that, for example, this happens when the coefficients $b, \sigma$ are continuous with linear growth: see Theorem 12.2.3 in [17] for the case of bounded coefficients, the general case can be easily obtained by truncation.*

*3. The martingale problem associated to $b, \Sigma$ is well-posed for all deterministic initial condition, for instance in the following cases.*

- *When $\Sigma, b$ have linear growth and $\Sigma$ is continuous and non-degenerate, i.e. Assumption 3, see [17] Corollary 7.1.7 and Theorem 10.2.2.*
- *Suppose $d = 1$ and $\sigma$ is bounded. When $\sigma$ is lower bounded by a positive constant on each compact set, see [17], Exercise 7.3.3.*
- *When $d = 2$, $\Sigma$ is non-degenerate and $\sigma$ and $b$ are time-homogeneous and bounded, see [17], Exercise 7.3.4.*
- *When $\sigma, b$ are Lipschitz with linear growth (with respect to the space variable), in which case we have even strong solutions of the corresponding stochastic differential equation.*

LC313 **Lemma 3.4.** *Let $T > 0$ be arbitrary and $\nu \in \mathcal{P}\left(\mathbb{R}^d\right)$. We suppose the validity of Assumptions 2 and 3. Then there is a unique $\mathcal{M}_+\left(\mathbb{R}^d\right)$-valued solution $\mathbf{u}$ to (3.2) with $\mathbf{u}(0) = \nu$. Moreover $\mathbf{u}^\nu$ takes values in $\mathcal{P}(\mathbb{R}^d)$.*

*Proof.* Existence follows by items 2. and 3. of Remark 3.3. Uniqueness is a consequence of items 1. and 3. of the same Remark. □

Below we give two uniqueness results for the PDE (1.2).

P1 **Proposition 3.5.** *Suppose Assumption 4 holds with respect to a given $\mathcal{C} \subseteq \mathcal{M}_+(\mathbb{R}^d)$. Suppose that for all $\nu \in \mathcal{C}$ there exists an $\mathcal{M}_+(\mathbb{R}^d)$-valued solution of (3.2) with initial value $\nu$. Then, the following properties are equivalent.*

1. *The mapping from $\mathcal{C}$ to $\mathcal{M}_+(\mathbb{R}^d)$ $\nu \mapsto \mathbf{u}^\nu(T)$ is injective.*
2. *For all $\mu \in \mathcal{M}_+(\mathbb{R}^d)$, the PDE (2.3) with terminal value $\mu$ admits at most a solution in the sense of Definition 3.1 among all $\mathcal{M}_+\left(\mathbb{R}^d\right)$-valued solutions starting in the class $\mathcal{C}$.*

*Proof.* Concerning the converse implication, consider $(\nu, \nu') \in \mathcal{C}^2$ such that $\mathbf{u}^\nu(T) = \mathbf{u}^{\nu'}(T)$ and suppose that uniqueness holds for equation (2.3) for all terminal values in $\mathcal{M}_+\left(\mathbb{R}^d\right)$ in the sense of Definition 3.1 among non-negative measure-valued solutions starting in the class $\mathcal{C}$. We remark that $\mathbf{u}^\nu, \mathbf{u}^{\nu'}$ are such solutions and are associated to the same terminal value. Uniqueness gives $\mathbf{u}^\nu = \mathbf{u}^{\nu'}$ and in particular $\nu = \nu'$.

Concerning the direct implication, consider $\mathbf{u}^1, \mathbf{u}^2$ two non-negative measure-valued solutions of equation (1.2) in the sense of Definition 3.1, with the same terminal value in $\mathcal{M}_+\left(\mathbb{R}^d\right)$, such that $\mathbf{u}^i\left(0\right), i \in \{1,2\}$, belong to $\mathcal{C}$ and suppose that $\nu \mapsto \mathbf{u}^\nu\left(T\right)$ is injective from $\mathcal{C}$ to $\mathcal{M}_+\left(\mathbb{R}^d\right)$. Setting $\nu^i := \mathbf{u}^i\left(0\right)$, we remark that for a given $i \in \{1,2\}$

$$\begin{cases} \partial_t \mathbf{u}^i = L_t^* \mathbf{u}^i \\ \mathbf{u}^i\left(0\right) = \nu_i, \end{cases} \tag{3.5}$$

FPBis

in the sense of identity (3.3). Then, the fact $\mathbf{u}^1\left(T\right) = \mathbf{u}^2\left(T\right)$ gives $\mathbf{u}^{\nu_1}\left(T\right) = \mathbf{u}^{\nu_2}\left(T\right)$. By injectivity $\nu_1 = \nu_2$ and the result follows by Assumption 4. □

Proceeding in the same way as for the proof of Proposition 3.5 we obtain the following.

P2 **Proposition 3.6.** *Suppose that for all $\nu \in \mathcal{M}_f\left(\mathbb{R}^d\right)$, there exists a unique solution $\mathbf{u}^\nu$ of (3.2) with initial value $\nu$. Then, the following properties are equivalent.*

1. *The mapping $\nu \mapsto \mathbf{u}^\nu(T)$ is injective.*
2. *For all $\mu \in \mathcal{M}_f(\mathbb{R}^d)$, the PDE (1.1) with terminal value $\mu$ admits at most a solution in the sense of Definition 3.1.*

RP1 **Remark 3.7.** 1. *Suppose that the coefficients $\Sigma, b$ are bounded. Then, any measure-valued solution $\mathbf{u} : [0,T] \to \mathcal{M}_+(\mathbb{R}^d)$ of (3.2) such that $\mathbf{u}(0) \in \mathcal{P}(\mathbb{R}^d)$ takes values in $\mathcal{P}(\mathbb{R}^d)$. Indeed, this can be shown approaching the function $\varphi \equiv 1$ from below by smooth functions with compact support.*

2. *Replacing $\mathcal{M}_+(\mathbb{R}^d)$ with $\mathcal{P}(\mathbb{R}^d)$ in Assumption 4, item 2. in Proposition 3.5 can be stated also replacing $\mathcal{M}_+(\mathbb{R}^d)$ with $\mathcal{P}(\mathbb{R}^d)$.*

## 3.2 Uniqueness: the case of Dirac initial conditions

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In this section we give examples of functions $b, \sigma$ for which uniqueness of (2.3) among $\mathcal{M}_+(\mathbb{R}^d)$-valued solutions is ensured, supposing Assumption 4 is in force with respect to $\mathcal{C} := (\alpha\delta_x)_{\alpha>0, x\in\mathbb{R}^d}$.

Ralpha **Remark 3.8.** *Let $\alpha \geq 0$. Let $x \in \mathbb{R}^d$. Suppose that there is a solution $X^x$ of SDE (3.4) with $\xi = x$.*

1. *By Proposition 3.2, the $\mathcal{M}_+\left(\mathbb{R}^d\right)$-valued mapping $t \mapsto \alpha\mathcal{L}\left(X_t^x\right)$ is a solution of (3.2) with initial value $\alpha\delta_x$.*
2. *$t \mapsto \alpha\mathcal{L}\left(X_t^x\right)$ can be identified with $\mathbf{u}^{\alpha\delta_x}$ and in particular $\int_{\mathbb{R}^d} \mathbf{u}^{\alpha\delta_x}\left(t\right)\left(dy\right) = \alpha,\ \forall t \in [0,T]$.*

If Assumption 1 holds, $X^x$ denotes the unique solution of equation (3.4) with initial value $x \in \mathbb{R}^d$. We start with the case of dimension $d = m = 1$.

propLip1 **Proposition 3.9.** *Suppose the validity of Assumption 4 with $\mathcal{C} = (\alpha\delta_x)_{\alpha>0, x\in\mathbb{R}}$ and 1 with $d = m = 1$. Then, for all $\mu \in \mathcal{M}_+\left(\mathbb{R}\right)$, equation (1.2) with terminal value $\mu$ admits at most one solution in the sense of Definition 3.1 among the $\mathcal{M}_+\left(\mathbb{R}\right)$-valued solutions starting in $\mathcal{C}$.*

*Proof.* Fix $(x,y) \in \mathbb{R}^2$ and $\alpha, \beta \geq 0$ such that

$$\mathbf{u}^{\alpha\delta_x}\left(T\right) = \mathbf{u}^{\beta\delta_y}\left(T\right). \tag{3.6}$$

identity

It suffices to show that $\alpha = \beta$ and $x = y$ to conclude, thanks to Proposition 3.5. By item 2. of Remark 3.8, we have $\alpha = \beta$ and consequently $\mathcal{L}_{\mathbb{P}}\left(X_T^x\right) = \mathcal{L}_{\mathbb{P}}\left(X_T^y\right)$. In particular $\mathbb{E}\left(X_T^x\right) = \mathbb{E}\left(X_T^y\right)$. Since $b, \sigma$ are Lipschitz in space, they have bounded derivatives in the sense of distributions that we denote by $\partial_x b$ and $\partial_x \sigma$.

Set $Z^{x,y} := X^y - X^x$. We have

$$Z_t^{x,y} = (y-x) + \int_0^t b_s^{x,y} Z_s^{x,y} ds + \int_0^t \sigma_s^{x,y} Z_s^{x,y} dW_s, \forall t \in [0,T], \tag{3.7}$$ EDol

where for a given $s \in [0,T]$

$$b_s^{x,y} = \int_0^1 \partial_x b\left(s, aX_s^y + (1-a)X_s^x\right) da,\ \sigma_s^{x,y} = \int_0^1 \partial_x \sigma\left(s, aX_s^y + (1-a)X_s^x\right) da.$$

The unique solution of (3.7) is well-known to be

$$Z^{x,y} = \exp\left(\int_0^{\cdot} b_s^{x,y} ds\right) \mathcal{E}\left(\int_0^{\cdot} \sigma_s^{x,y} dW_s\right)(y-x),$$

where $\mathcal{E}(\cdot)$ denotes the Doléans exponential. Finally, we have

$$\mathbb{E}\left(\exp\left(\int_0^T b_s^{x,y} ds\right) \mathcal{E}\left(\int_0^{\cdot} \sigma_s^{x,y} dW_s\right)_T\right)(y-x) = 0.$$

Since the quantity appearing in the expectation is strictly positive, we conclude $x = y$. □

We continue now with a discussion concerning the multidimensional case $d \geq 2$. The uniqueness result below only holds when the time-horizon is small enough. Later, in Section 3.3 we will present in a framework of piecewise time-homogeneous coefficients results which are valid for any time-horizon. Theorem 3.10 distinguishes two cases: the first one with regular possibly degenerate coefficients, the second one with non-degenerate possibly irregular coefficients.

propLipd **Theorem 3.10.** *We suppose Assumption 4 with $\mathcal{C} = (\alpha\delta_x)_{\alpha>0, x\in\mathbb{R}^d}$ and the validity of either item (a) or (b) below.*

*(a) Assumption 1.*

*(b) Assumptions 2 and 3.*

*There is $T > 0$ small enough such that the following holds. For all $\mu \in \mathcal{M}_+\left(\mathbb{R}^d\right)$, equation (1.2) admits at most one solution in the sense of Definition 3.1 among the $\mathcal{M}_+\left(\mathbb{R}^d\right)$-valued solutions starting in $\mathcal{C}$.*

The proof of item (a) of Theorem 3.10 relies on a basic lemma of moments estimation.

Lemma **Lemma 3.11.** *We suppose Assumption 1. Let $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$. Then, $\sup_{t\in[0,T]} \mathbb{E}\left(|X_t^x - X_t^y|^2\right) \leq |y-x|^2 e^{KT}$, with $K := 2K^b + \sum_{j=1}^m \left(K^{\sigma,j}\right)^2$, where*

$$K^b := \sup_{s\in[0,T]} || \, ||Jb(s,\cdot)|| \, ||_\infty$$

*and for all $j \in [\![1,m]\!]$*

$$K^{\sigma,j} := \sup_{s\in[0,T]} || \, ||J\sigma_{\cdot j}(s,\cdot)|| \, ||_\infty .$$

**Proof** (of Lemma 3.11).
For a given $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$ we set

$$Z_t^{x,y} := X_t^y - X_t^x, t \in [0,T].$$

We have

$$Z_t^{x,y} = y - x + \int_0^t B_r^{x,y} Z_r^{x,y} dr + \sum_{j=1}^m \int_0^t C_r^{x,y,j} Z_r^{x,y} dW_r^j,\ t \in [0,T], \tag{3.8}$$ EZxy

with, for all $r \in [0,T]$

$$B_r^{x,y} := \int_0^1 Jb\left(r, aX_r^y + (1-a)X_r^x\right) da, \quad C_r^{x,y,j} := \int_0^1 J\sigma_{\cdot j}\left(r, aX_r^y + (1-a)X_r^x\right) da, \forall\, j \in [\![1,m]\!].$$

By the classical existence and uniqueness theorem for SDEs with Lipschitz coefficients we know that

$$\mathbb{E}(\sup_{s\leq T} |X_s^z|^2) < \infty, \tag{3.9}$$ SQI

for all $z \in \mathbb{R}^d$. This implies

$$\mathbb{E}(\sup_{t\in[0,T]} |Z_t^{x,y}|^2) < \infty. \tag{3.10}$$ sup

Now, Itô's formula gives, for all $t \in [0,T]$

$$|Z_t^{x,y}|^2 = |y-x|^2 + 2\int_0^t \langle B_r^{x,y} Z_r^{x,y}, Z_r^{x,y}\rangle\, dr + \sum_{j=1}^m \int_0^t \left|C_r^{x,y,j} Z_r^{x,y}\right|^2 dr + 2\sum_{i=1}^d M_t^{x,y,i}, \tag{3.11}$$ ItoSquareNorm

where, for a given $i \in [\![1,d]\!]$, $M^{x,y,i}$ denotes the local martingale $\int_0^{\cdot} Z_s^{x,y,i} \sum_{j=1}^m \left(C_s^{x,y,j} Z_s^{x,y}\right)_i dW_s^j$.
Consequently, for all $i \in [\![1,d]\!]$, we have

$$\begin{aligned}
\sqrt{[M^{x,y,i}]_T} &= \sqrt{\sum_{j=1}^m \int_0^T \left(Z_r^{x,y,i}\right)^2 \left(C_r^{x,y,j} Z_r^{x,y}\right)_i^2 dr}, \\
&\leq \sqrt{\sum_{j=1}^m \int_0^T \left|C_r^{x,y,j} Z_r^{x,y}\right|^2 |Z_r^{x,y}|^2\, dr}, \\
&\leq \sqrt{T\sum_{j=1}^m \left(K^{\sigma,j}\right)^2} \sup_{r\in[0,T]} |Z_r^{x,y}|^2.
\end{aligned} \tag{3.12}$$ EMForm

By the latter inequality and (3.10), we know that $\mathbb{E}\left([M^{x,y,i}]_T^{\frac{1}{2}}\right) < \infty$, so for all $i \in [\![1,d]\!]$, $M^{x,y,i}$ is a true martingale. Taking expectation in identity (3.11), we obtain

$$\mathbb{E}\left(|Z_t^{x,y}|^2\right) = |y-x|^2 + \int_0^t \mathbb{E}\left(2\left\langle B_r^{x,y} Z_r^{x,y}, Z_r^{x,y}\right\rangle + \sum_{k=1}^m \left|C_r^{x,y,k} Z_r^{x,y}\right|^2\right) dr.$$

Hence, thanks to Cauchy-Schwarz inequality and to the definition of $K^b$ and $K^{\sigma,j}$ for all $j \in [\![1,m]\!]$

$$\mathbb{E}\left(|Z_t^{x,y}|^2\right) \leq |y-x|^2 + K\int_0^t \mathbb{E}\left(|Z_r^{x,y}|^2\right) dr$$

and we conclude via Gronwall's Lemma. □

**Proof** (of Theorem 3.10).

Fix $(x_1,x_2) \in \mathbb{R}^d \times \mathbb{R}^d, \alpha, \beta \geq 0$ such that

$$\mathbf{u}^{\alpha\delta_{x_1}}(T) = \mathbf{u}^{\beta\delta_{x_2}}(T). \tag{3.13}$$

To conclude, it suffices to show $\alpha = \beta$ and $x_1 = x_2$ thanks to Proposition 3.5.

1. We suppose first Assumption Lip1d. Once again, item 2. of Remark Ralpha gives $\alpha = \beta$ and

$$\mathbb{E}\left(X_T^{x_1}\right) = \mathbb{E}\left(X_T^{x_2}\right). \tag{3.14}$$ Eequal

Adopting the same notations as in the proof of Lemma Lemma, a similar argument as in (EMForm), together with (sup) allow to show that the local martingale part of $Z^{x_1,x_2} = X^{x_2} - X^{x_1}$ defined in (EZxy) is a true martingale. So, taking the expectation in (EMForm) with $x = x_1, y = x_2$, by Lemma Lemma we obtain

$$\begin{aligned}\left|\mathbb{E}\left(X_T^{x_2} - X_T^{x_1}\right) - (x_2 - x_1)\right| &\leq K_b \int_0^T \mathbb{E}|X_r^{x_2} - X_r^{x_1}|dr \\ &\leq K_b \int_0^T \sqrt{\mathbb{E}\left(|X_r^{x_2} - X_r^{x_1}|\right)^2} dr \\ &\leq \frac{K}{2} T e^{\frac{K}{2}T} \left|x_2 - x_1\right|.\end{aligned}$$

Remembering (Eequal), this implies

$$\left(1 - \frac{K}{2} T e^{\frac{K}{2}T}\right) |x_2 - x_1| \leq 0.$$

Taking $T$ such that $\frac{K}{2}T < M$ with $Me^M < 1$, we have $1 - \frac{K}{2}Te^{\frac{K}{2}T} > 0$, which implies $|x_2 - x_1| = 0$.

2. We suppose here Assumptions Zvon1 and Zvon3. Firstly, point 1. of Theorem 1. in [Z19] ensures the existence of probability spaces $\left(\Omega^i, \mathcal{F}^i, \mathbb{P}^i\right),\ i \in \{1,2\}$ on which are defined respectively two $m$-dimensional Brownian motions $W^1, W^2$ and two processes $X^1, X^2$ such that

$$X_t^i = x_i + \int_0^t b\left(s, X_s^i\right) ds + \int_0^t \sigma\left(s, X_s^i\right) dW_s^i,\ \mathbb{P}^i\text{−a.s., t} \in [0, \mathrm{T}].$$

Once again, item 2. of Remark Ralpha implies $\alpha_1 = \alpha_2$ and

$$\mathcal{L}_{\mathbb{P}^1}\left(X_T^1\right) = \mathcal{L}_{\mathbb{P}^2}\left(X_T^2\right). \tag{3.15}$$ TermLaw

Secondly, point b. of Theorem 3 in [Z19] shows that for every given bounded $D \subset \mathbb{R}^d$, for all $\phi : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ belonging to $W_p^{1,2}\left([0,T] \times D\right)$ (see Definition of that space in [Z19]) for a given $p > d+2$, we have for all $t \in [0,T], i \in \{1,2\}$,

$$\phi\left(t, X_t^i\right) = \phi\left(0, x_i\right) + \int_0^t \left(\partial_t + L_s\right)\phi\left(s, X_s^i\right) ds + \int_0^t J\phi\left(s, X_s^i\right)\sigma\left(s, X_s^i\right) dW_s^i,\ \mathbb{P}^i\text{−a.s.} \tag{3.16}$$ TSDE

where the application of $\partial_t + L_t, t \in [0,T]$ has to be understood componentwise.

Thirdly, Theorem 2. in [Z19] shows that if $T$ is sufficiently small, then the system of $d$ PDEs

$$\forall\,(t,x) \in [0,T] \times \mathbb{R}^d,\ \begin{cases}\partial_t \phi\,(t,x) + L_t \phi\,(t,x) = 0, \\ \phi\,(T,x) = x,\end{cases} \tag{3.17}$$ E317

admits a solution $\phi$ in $W_p^{1,2}\left([0,T] \times D\right)$ for all $p > 1$ and all bounded $D \subset \mathbb{R}^d$. Moreover the partial derivatives in space of $\phi$ are bounded (in particular $J\phi$ is bounded) and $\phi\,(t,\cdot)$ is injective for all $t \in [0,T]$.

Combining now (3.17) with identity (3.16), we observe that $\phi\left(.,X^i\right), i\in\{1,2\}$, are local martingales. Using additionally the fact that $J\phi$ and $\sigma$ are bounded, it is easy to show that they are true martingales. Taking the expectation in (3.16) with respect to $\mathbb{P}^i, i=1,2$, gives

$$\phi(0,x_i)=\mathbb{E}_{\mathbb{P}^i}\left(\phi\left(T,X_T^i\right)\right), i\in\{1,2\}.$$

In parallel, identity (3.15) gives

$$\mathbb{E}_{\mathbb{P}^1}\left(\phi\left(T,X_T^1\right)\right)=\mathbb{E}_{\mathbb{P}^2}\left(\phi\left(T,X_T^2\right)\right).$$

So, $\phi(0,x_1)=\phi(0,x_2)$. We conclude that $x_1=x_2$ since $\phi(0,\cdot)$ is injective.

□

## 3.3 Uniqueness: the case of bounded, non-degenerate coefficients

SGP

In this section we consider the case of time-homogeneous, bounded and Hölder coefficients in dimension $d\geq 1$. We suppose that Assumption 3 holds and consider the following one.

Lun1 **Assumption 5.** *1. $b,\sigma$ are time-homogeneous and bounded.*

*2. For all $(i,j)\in[\![1,d]\!]^2$, $b_i,\Sigma_{ij}\in\mathcal{C}^{2\alpha}\left(\mathbb{R}^d\right)$, for a given $\alpha\in]0,\frac{1}{2}[$.*

We refer to the differential operator (2.1) $L_t$ and we simply set here $L\equiv L_t$.

liminary **Remark 3.12.** *Suppose the validity of Assumptions 3, 5.*

1. *Let $T>0$. Proposition 4.2 in [5] implies that for every $\nu\in\mathcal{M}_f\left(\mathbb{R}^d\right)$, there exists a unique $\mathcal{M}_f\left(\mathbb{R}^d\right)$-valued solution of equation (3.2) with initial value $\nu$. This unique solution will be denoted by $\mathbf{u}^\nu$. In the sequel $T$ will be omitted.*

2. *We remark that the uniqueness result mentioned in item 1. is unknown in the case of general bounded coefficients. In the general framework, only a uniqueness result for non-negative solutions is available, see Remark 3.3 1.*

3. *Since $L$ is time-homogeneous, taking into account Assumptions 3, 5, operating a shift, uniqueness of (3.2) also holds replacing the initial time 0 by any other initial time, for every initial value in $\mathcal{M}_f\left(\mathbb{R}^d\right)$, with any other maturity $T$.*

P315 **Theorem 3.13.** *Suppose the validity of Assumptions 3 and 5. Then, for all $\mu\in\mathcal{M}_f\left(\mathbb{R}^d\right)$, equation (1.2) with terminal value $\mu$ admits at most one $\mathcal{M}_f\left(\mathbb{R}^d\right)$-valued solution in the sense of Definition 3.1.*

By Theorems 3.1.12, 3.1.14 and Corollary 3.1.16 in [12] the differential operator $L$ suitably extends as a map $\mathcal{D}(L)=\mathcal{C}^{2\alpha+2}(\mathbb{R}^d)\subset\mathcal{C}^{2\alpha}(\mathbb{R}^d)\mapsto\mathcal{C}^{2\alpha}\left(\mathbb{R}^d\right)$ and that extension is sectorial, see Definition 2.0.1 in [12]. We set $E:=\mathcal{C}^{2\alpha}\left(\mathbb{R}^d\right)$. By the considerations below that Definition, in (2.0.2) and (2.0.3) therein, one defines $P_t:=e^{tL}, P_t:E\to E, t\geq 0$. By Proposition 2.1.1 in [12], $(P_t)_{t\geq 0}$ is a semigroup and $t\mapsto P_t$ is analytical on $]0,+\infty[$ with values in $\mathcal{L}(E)$, with respect to $||.||_E$.

Before proving the theorem, we provide two lemmata.

key_1 **Lemma 3.14.** *Suppose the validity of Assumptions 3 and 5. Then, for all $\phi \in E$ and all $\nu \in \mathcal{M}_f\left(\mathbb{R}^d\right)$, the function from $\mathbb{R}_+^*$ to $\mathbb{R}$*

$$t \mapsto \int_{\mathbb{R}^d} P_t\phi\left(x\right)\nu\left(dx\right)$$

*is analytic.*

*Proof.* The result can be easily established using the fact that $\phi \mapsto P_t\phi$ with values in $\mathcal{L}(E)$ is analytic and the fact that the map $\psi \mapsto \int_{\mathbb{R}^d}\psi(x)\nu(dx)$ is linear and bounded.

□

key_2 **Lemma 3.15.** *Suppose the validity of Assumptions 3 and 5. Let $T > 0$. Then for all $\nu \in \mathcal{M}_f\left(\mathbb{R}^d\right)$, $t \in [0,T]$ and $\phi \in E$ we have the identity*

$$\int_{\mathbb{R}^d} P_t\phi\left(x\right)\nu\left(dx\right) = \int_{\mathbb{R}^d} \phi\left(x\right)\mathbf{u}^\nu\left(t\right)\left(dx\right), \tag{3.18}$$ EL310

*where $\mathbf{u}^\nu$ was defined in point 1. of Remark 3.12.* RPreliminary

*Proof.* Let $\nu \in \mathcal{M}_f\left(\mathbb{R}^d\right)$. We denote by $\mathbf{v}^\nu$ the mapping from $[0,T]$ to $\mathcal{M}_f\left(\mathbb{R}^d\right)$ such that $\forall t \in [0,T]$, $\forall \phi \in E$

$$\int_{\mathbb{R}^d} \phi(x)\mathbf{v}^\nu\left(t\right)\left(dx\right) = \int_{\mathbb{R}^d} P_t\phi(x)\nu(dx). \tag{3.19}$$ ERiesz

Previous expression defines the measure $\mathbf{v}^\nu(t,\cdot)$ since $\phi \mapsto \int_{\mathbb{R}^d} P_t\phi(x)\nu(dx)$ is continuous with respect to the sup-norm, using $\|P_t\phi\|_\infty \leq \|\phi\|_\infty$, and Lebesgue dominated convergence theorem.
By approximating elements of $E$ with elements of $\mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$, it will be enough to prove (3.18) for $\phi \in \mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$.
Our goal is to show that $\mathbf{v}^\nu$ is a $\mathcal{M}_f\left(\mathbb{R}^d\right)$-valued solution of (3.2) with initial value $\nu$ to conclude $\mathbf{v}^\nu = \mathbf{u}^\nu$ via point 1. of Remark 3.12 and so to prove (3.18) for $\phi \in \mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$.
Let $t \in [0,T]$ and $\phi \in \mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$. On the one hand, point (i) of Proposition 2.1.1 in [12] gives

$$LP_t\phi = P_tL\phi, \tag{3.20}$$ LP

since $\mathcal{C}_c^\infty\left(\mathbb{R}^d\right) \subset \mathcal{D}\left(L\right) = \mathcal{C}^{2\alpha+2}\left(\mathbb{R}^d,\mathbb{R}\right)$. On the other hand, for all $s \in [0,t]$, we have

$$\begin{aligned}
\left|LP_s\phi\right|_E &= \left|P_sL\phi\right|_{2\alpha} \\
&\leq \left|\left|P_s\right|\right|_E \left|L\phi\right|_E \\
&\leq M_0e^{\omega s}\left|L\phi\right|_E,
\end{aligned}$$

with $M_0, \omega$ the real parameters appearing in Definition 2.0.1 in [12] and using point (iii) of Proposition 2.1.1 in the same reference. Then the mapping $s \mapsto LP_s\phi$ belongs obviously to $L^1([0,t];E)$ and point (ii) of Proposition 2.1.4 in [12] combined with identity (3.20) gives

$$P_t\phi = \phi + \int_0^t P_sL\phi ds.$$

Back to our main goal, using in particular Fubini's theorem, we have

$$\begin{aligned}
\int_{\mathbb{R}^d} P_t\phi\left(x\right)\nu\left(dx\right) &= \int_{\mathbb{R}^d} \phi\left(x\right)\nu\left(dx\right) + \int_{\mathbb{R}^d}\int_0^t P_sL\phi\left(x\right)ds\nu\left(dx\right) \\
&= \int_{\mathbb{R}^d} \phi\left(x\right)\nu\left(dx\right) + \int_0^t\int_{\mathbb{R}^d} P_sL\phi\left(x\right)\nu\left(dx\right)ds \\
&= \int_{\mathbb{R}^d} \phi\left(x\right)\nu\left(dx\right) + \int_0^t\int_{\mathbb{R}^d} L\phi\left(x\right)\mathbf{v}^\nu\left(s\right)\left(dx\right)ds.
\end{aligned}$$

This shows that $\mathbf{v}^\nu$ is a solution of equation (3.2). Fokker □

**Proof** (of Theorem 3.13).
Let $\nu, \nu' \in \mathcal{M}_f\left(\mathbb{R}^d\right)$ such that

$$\mu_T := \mathbf{u}^{\nu}(T) = \mathbf{u}^{\nu'}(T).$$

Thanks to Proposition 3.6, it suffices to show that $\nu = \nu'$ i.e.

$$\forall \phi \in \mathcal{C}_c^{\infty}\left(\mathbb{R}^d\right), \int_{\mathbb{R}^d} \phi(x)\,\nu(dx) = \int_{\mathbb{R}^d} \phi(x)\,\nu'(dx).$$

Since $T > 0$ is arbitrary, by Remark 3.12 we can consider $\mathbf{u}^{\nu,2T}$ and $\mathbf{u}^{\nu',2T}$, defined as the corresponding $\mathbf{u}^{\nu}$ and $\mathbf{u}^{\nu'}$ functions obtained replacing the horizon $T$ with $2T$. They are defined on $[0, 2T]$ and by Remark 3.12 1. (uniqueness on $[0, T]$), they constitute extensions of the initial $\mathbf{u}^{\nu}$ and $\mathbf{u}^{\nu'}$.
By Remark 3.12 3., the uniqueness of an $\mathcal{M}_f\left(\mathbb{R}^d\right)$-valued solution of (3.2) (for $t \in [T, 2T]$, with $T$ as initial time) holds for

$$\begin{cases} \partial_t \mathbf{u}(\tau) = L^* \mathbf{u}(\tau),\ T \le \tau \le 2T \\ \mathbf{u}(T) = \mu_T. \end{cases} \tag{3.21}$$

FPShift

Now, the functions $\mathbf{u}^{\nu,2T}$ and $\mathbf{u}^{\nu',2T}$ solve (3.21) on $[T, 2T]$. This gives in particular

$$\forall \tau \ge T,\ \forall \phi \in \mathcal{C}_c^{\infty}\left(\mathbb{R}^d\right), \int_{\mathbb{R}^d} \phi(x)\,\mathbf{u}^{\nu,2T}(\tau)(dx) = \int_{\mathbb{R}^d} \phi(x)\,\mathbf{u}^{\nu',2T}(\tau)(dx). \tag{3.22}$$

IdLawBis

Fix $\phi \in \mathcal{C}_c^{\infty}\left(\mathbb{R}^d\right)$. Combining now the results of Lemmata 3.14 and 3.15, we obtain that the function

$$\tau \mapsto \int_{\mathbb{R}^d} \phi(x)\,\mathbf{u}^{\nu,2T}(\tau)(dx) - \int_{\mathbb{R}^d} \phi(x)\,\mathbf{u}^{\nu',2T}(\tau)(dx) \tag{3.23}$$

ETau

defined on $[0, 2T]$, is zero on $[T, 2T]$ and analytic on $]0, 2T]$. Hence it is zero on $]0, 2T]$. By (3.18) we obtain

$$\int_{\mathbb{R}^d} P_\tau \phi(x)\,(\nu - \nu')(dx) = 0,\ \forall t \in ]0, 2T]. \tag{3.24}$$

ETaubis

Separating $\nu$ and $\nu'$ in positive and negative components, we can finally apply dominated convergence theorem in (3.23) to send $\tau$ to $0^+$. This is possible thanks to points (i) of Proposition 2.1.4 and (iii) of Proposition 2.1.1 in [12] together with the representation (3.18). Indeed $P_\tau \phi(x) \to \phi(x)$ for every $\phi \in E, x \in \mathbb{R}^d$ when $\tau \to 0^+$. This shows $\nu = \nu'$ and ends the proof.

□

For the sake of applications it is useful to formulate a piecewise time-homogeneous version of Theorem 3.13.

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**Corollary 3.16.** *Let $n \in \mathbb{N}^*$. Let $0 = t_0 < \ldots < t_n = T$ be a partition. For $k \in [\![2, n]\!]$ (resp. $k = 1$) we denote $I_k = ]t_{k-1}, t_k]$ (resp. $[t_0, t_1]$). Suppose that the following holds.*

1. *For all $k \in [\![1, n]\!]$, the restriction of $\sigma$ (resp. $b$) to $I_k \times \mathbb{R}^d$ is a time-homogeneous function $\sigma^k : \mathbb{R}^d \to M_d(\mathbb{R})$ (resp. $b^k : \mathbb{R}^d \to \mathbb{R}^d$).*
2. *Assumption 3.*
3. *Assumption 5 is verified for each $\sigma^k, b^k$ and $\Sigma^k$, where we have set $\Sigma^k := \sigma^k {\sigma^k}^\top$.*

*Then, for all $\mu \in \mathcal{M}_f\left(\mathbb{R}^d\right)$, equation (1.2) with terminal value $\mu$ admits at most one $\mathcal{M}_f\left(\mathbb{R}^d\right)$-valued solution in the sense of Definition 3.1.*

*Proof.* For each given $k \in [\![1,n]\!]$, we introduce the PDE operator $L^k$ defined by

$$L^k := \frac{1}{2}\sum_{i,j=1}^{d} \Sigma_{ij}^k \partial_{ij} + \sum_{i=1}^{d} b_i^k \partial_i. \tag{3.25}$$ OpLk

Let now $\mathbf{u}^1, \mathbf{u}^2$ be two solutions of (II.2) with same terminal value $\mu$.

The measure-valued functions $\mathbf{v}^i := \mathbf{u}^i(\cdot + t_{n-1}), i \in \{1,2\}$ defined on $[0, T - t_{n-1}]$ are solutions of

$$\begin{cases} \partial_t \mathbf{v} = (L^n)^* \mathbf{v} \\ \mathbf{v}(T - t_{n-1}, \cdot) = \mu, \end{cases} \tag{3.26}$$ BackwardFokke

in the sense of Definition 3.1 replacing $T$ by $T - t_{n-1}$ and $L$ by $L^n$. Then, Theorem 3.13 gives $\mathbf{v}^1 = \mathbf{v}^2$ and consequently $\mathbf{u}^1 = \mathbf{u}^2$ on $[t_{n-1}, T]$. To conclude, we proceed by backward induction.

□

## 3.4 Uniqueness: the case of Ornstein-Uhlenbeck semigroup

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In this section, we consider the case $b := (s,x) \mapsto C(s)x$ with $C$ continuous from $[0,T]$ to $M_d(\mathbb{R})$ and $\sigma$ continuous from $[0,T]$ to $M_{d,m}(\mathbb{R})$. We set $\Sigma := \sigma\sigma^\top$. We also denote by $\mathcal{D}(t),\ t \in [0,T]$, the unique solution of

$$\mathcal{D}(t) = I - \int_0^t C(s)^\top \mathcal{D}(s) ds,\ t \in [0,T].$$

We recall that for every $t \in [0,T]$, $\mathcal{D}(t)$ is invertible and

$$\mathcal{D}^{-1}(t) = I + \int_0^t C(s)^\top \mathcal{D}^{-1}(s) ds,\ t \in [0,T].$$

For previous and similar properties, see Chapter 8 of [4]. pronson

In that setting, the classical Fokker-Planck PDE for finite measures reads

$$\begin{cases} \partial_t \mathbf{u}(t) = \sum_{i,j=1}^{d} \Sigma(t)_{ij} \partial_{ij} \mathbf{u}(t) - \sum_{i=1}^{d} \partial_i \left( (C(t)x)_i \mathbf{u}(t) \right) \\ \mathbf{u}(0) = \nu \in \mathcal{M}_f(\mathbb{R}^d). \end{cases} \tag{3.27}$$ FP_OU

dOU_Uniq **Proposition 3.17.** *For all $\nu \in \mathcal{M}_f(\mathbb{R}^d)$, equation (3.27) with initial value $\nu$ admits at most one $\mathcal{M}_f(\mathbb{R}^d)$-valued solution.*

*Proof.*

1. Let $\nu \in \mathcal{M}_f(\mathbb{R}^d)$ and $\mathbf{u}$ be a solution of (3.2) with initial value $\nu$. Identity (3.3) can be extended to $\mathcal{S}(\mathbb{R}^d)$ since for all $t \in [0,T]$, $\mathbf{u}(t)$ belongs to $\mathcal{M}_f(\mathbb{R}^d)$. Then, $t \mapsto \mathcal{F}\mathbf{u}(t)$ verifies

$$\mathcal{F}\mathbf{u}(t)(\xi) = \mathcal{F}\nu(\xi) + \int_0^t \left\langle C(s)^\top \xi, \nabla \mathcal{F}\mathbf{u}(s) \right\rangle ds - \frac{1}{2}\int_0^t \langle \Sigma(s)\xi, \xi \rangle \mathcal{F}\mathbf{u}(s)\, ds,\ (t,\xi) \in [0,T] \times \mathbb{R}^d. \tag{3.28}$$ WeakOUPDE

In fact, the integrand inside the first integral has to be understood as a Schwartz distribution: in particular the symbol $\nabla$ is understood in the sense of distributions and for each given $s \in [0,T]$, $\left\langle C(s)^\top \xi, \nabla \mathcal{F}\mathbf{u}(s) \right\rangle$ denotes the tempered distribution

$$\varphi \mapsto \sum_{i=1}^{d} \partial_i \mathcal{F}\mathbf{u}(s) \left( \xi \mapsto \left( C(s)^\top \xi \right)_i \varphi(\xi) \right).$$

Indeed, even though for any $t$, $\mathcal{F}\mathbf{u}(t)$ is a function, the equation (3.28) has to be understood in $\mathcal{S}'(\mathbb{R}^d)$. Hence, for all $\phi \in \mathcal{S}(\mathbb{R}^d)$, this gives

$$\begin{aligned}
&\int_{\mathbb{R}^d} \phi(\xi)\,\mathcal{F}\mathbf{u}(t)(\xi)\,d\xi - \int_{\mathbb{R}^d} \phi(\xi)\,\mathcal{F}\nu(\xi)\,\phi(\xi)d\xi \qquad (3.29)\\
&\quad = -i\sum_{k,l=1}^{d}\int_0^t C(s)_{kl}\int_{\mathbb{R}^d}\xi_l\mathcal{F}\phi_k(\xi)\,\mathbf{u}(s)(d\xi)\,ds - \frac{1}{2}\int_0^t\int_{\mathbb{R}^d}\langle\Sigma(s)\xi,\xi\rangle\,\mathcal{F}\mathbf{u}(s)(\xi)\,\phi(\xi)d\xi ds\\
&\quad = -\sum_{k,l=1}^{d}\int_0^t C(s)_{kl}\int_{\mathbb{R}^d}\mathcal{F}(\partial_l\phi_k)(\xi)\,\mathbf{u}(s)(d\xi)\,ds - \frac{1}{2}\int_0^t\int_{\mathbb{R}^d}\langle\Sigma(s)\xi,\xi\rangle\,\mathcal{F}\mathbf{u}(s)(\xi)\,d\xi ds\\
&\quad = -\int_0^t\int_{\mathbb{R}^d}\left(div_\xi\left(C(s)^\top\xi\phi(\xi)\right) + \frac{1}{2}\langle\Sigma(s)\xi,\xi\rangle\,\phi(\xi)\right)\mathcal{F}\mathbf{u}(s)(\xi)d\xi ds,
\end{aligned}$$

where $\phi_k : \xi \mapsto \xi_k\phi(\xi)$ for a given $k \in [\![1,d]\!]$.

2. Let now $\mathbf{v} : [0,T] \to \mathcal{M}_f(\mathbb{R}^d)$ defined by

$$\int_{\mathbb{R}^d}\phi(x)\,\mathbf{v}(t)(dx) = \int_{\mathbb{R}^d}\phi\left(\mathcal{D}(t)^\top x\right)\mathbf{u}(t)(dx), \qquad (3.30)$$ MeasChange

$t \in [0,T], \phi \in \mathcal{C}_b(\mathbb{R}^d)$. For every $\xi \in \mathbb{R}^d$, we set $\phi(x) = \exp(-i\langle\xi,x\rangle)$ in (3.30) to obtain

$$\mathcal{F}\mathbf{v}(t)(\xi) = \mathcal{F}\mathbf{u}(t)(\mathcal{D}(t)\xi), \qquad (3.31)$$ EAcont

for all $\xi \in \mathbb{R}^d$, for all $t \in [0,T]$.

3. We want now to show that, for each $\xi$, $t \mapsto \mathcal{F}\mathbf{v}(t)$ fulfills an ODE. To achieve this, suppose for a moment that $(t,\xi) \mapsto \mathcal{F}\mathbf{u}(t)(\xi)$ is differentiable with respect to the variable $\xi$. Then, on the one hand, we have for all $(t,\xi) \in [0,T]\times\mathbb{R}^d$,

$$\mathcal{F}\mathbf{u}(t)(\xi) = \mathcal{F}\nu(\xi) + \int_0^t\left\langle C(s)^\top\xi, \nabla_\xi\mathcal{F}\mathbf{u}(s)(\xi)\right\rangle ds - \frac{1}{2}\int_0^t\langle\Sigma(s)\xi,\xi\rangle\,\mathcal{F}\mathbf{u}(s)(\xi)\,ds, \qquad (3.32)$$ strongPDE

thanks to identity (3.28). This means in particular that, for each given $\xi \in \mathbb{R}^d$, $t \mapsto \mathcal{F}\mathbf{u}(t)(\xi)$ is differentiable almost everywhere on $[0,T]$.

On the other hand, for almost every $t \in [0,T]$ and all $\xi \in \mathbb{R}^d$, we have

$$\begin{aligned}
\partial_t\mathcal{F}\mathbf{v}(t)(\xi) &= \partial_t\mathcal{F}\mathbf{u}(t)(\mathcal{D}(t)\xi) + \sum_{i=1}^{d}\left(\frac{d}{dt}(\mathcal{D}(t)\xi)\right)_i\partial_i\mathcal{F}\mathbf{u}(t)(\mathcal{D}(t)\xi),\\
&= \partial_t\mathcal{F}\mathbf{u}(t)(\mathcal{D}(t)\xi) - \sum_{i=1}^{d}\left(C(t)^\top\mathcal{D}(t)\xi\right)_i\partial_i\mathcal{F}\mathbf{u}(t)(\mathcal{D}(t)\xi),\\
&= -\frac{1}{2}\langle\Sigma(t)\mathcal{D}(t)\xi,\mathcal{D}(t)\xi\rangle\,\mathcal{F}\mathbf{v}(t)(\xi), \qquad (3.33)
\end{aligned}$$ ETechnical

where from line 1 to line 2, we have used the fact $\frac{d}{dt}(\mathcal{D}(t)\xi) = -C(t)^\top\mathcal{D}(t)\xi$ for all $(t,\xi) \in [0,T]\times\mathbb{R}^d$ and from line 2 to line 3, the identity (3.32). Since $t \mapsto \mathcal{F}\mathbf{v}(t)(\xi)$ is absolutely continuous by (3.31), (3.33) implies

$$\mathcal{F}\mathbf{v}(t)(\xi) = \mathcal{F}\nu(\xi) - \frac{1}{2}\int_0^t\langle\Sigma(s)\mathcal{D}(s)\xi,\mathcal{D}(s)\xi\rangle\,\mathcal{F}\mathbf{v}(s)(\xi)\,ds, \xi \in \mathbb{R}^d, \qquad (3.34)$$ EDOFourierFwd

for all $t \in [0,T]$.

4. Now, if $(t,\xi) \mapsto \mathcal{F}\mathbf{u}(t)(\xi)$ is not necessarily differentiable in the variable $\xi$, we will be able to prove (3.34) still holds by making use of calculus in the sense of distributions.

5. Suppose that (3.34) holds. This gives

$$\mathcal{F}\mathbf{u}(t)(\xi) = e^{-\int_0^t \frac{\left|\sigma(s)^\top \xi\right|^2}{2} ds} \mathcal{F}\nu\left(\mathcal{D}^{-1}(t)\,\xi\right). \tag{3.35}$$

6. The proof is now concluded after we have established the (3.34). Since both sides of it are continuous in $(t,\xi)$, it will be enough to show the equality as $\mathcal{S}'(\mathbb{R}^d)$-valued. This can be done differentiating (3.28), considered as an equality in $\mathcal{S}'(\mathbb{R}^d)$. For this we will apply Lemma 3.18 setting $\Phi := \mathcal{F}\mathbf{u}(t)$ for every fixed $t \in [0,T]$ and differentiating in time. We set $\Phi_t(\xi) = \mathcal{F}\mathbf{v}(t)(\xi),\ \xi \in \mathbb{R}^d$ and $\Phi_t(\varphi) = \int_{\mathbb{R}^d} \varphi(\xi)\Phi_t(\xi)\mathrm{d}\xi, \varphi \in \mathcal{S}(\mathbb{R}^d)$. We remark that $\Phi_t$ is compatible with the one defined in (3.36). (3.34) will the directly follow from Lemma 3.18.

□

**Lemma 3.18.** *Let $\Phi \in \mathcal{S}'\left(\mathbb{R}^d\right), t \in [0,T]$. We denote by $\Phi_t$ the element of $\mathcal{S}'\left(\mathbb{R}^d\right)$ such that for all $\varphi \in \mathcal{S}\left(\mathbb{R}^d\right)$*

$$\Phi_t(\varphi) := \det\left(\mathcal{D}^{-1}(t)\right) \Phi\left(\varphi\left(\mathcal{D}^{-1}(t)\,\cdot\right)\right). \tag{3.36}$$

*Then, for all $t \in [0,T]$*

$$\Phi_t(\varphi) = \Phi(\varphi) - \sum_{i=1}^d \int_0^t \left(\partial_i \Phi\right)_s \left(x \mapsto \left(C(s)^\top \mathcal{D}(s)\,x\right)_i \varphi(x)\right) ds. \tag{3.37}$$

*Proof.* We begin with the case $\Phi \in \mathcal{S}\left(\mathbb{R}^d\right)$ (or only $\mathcal{C}^\infty\left(\mathbb{R}^d\right)$). In this case,

$$\Phi_t(x) = \Phi\left(\mathcal{D}(t)\,x\right),\ x \in \mathbb{R}^d, t \in [0,T].$$

Hence, for every $t \in [0,T]$

$$\begin{aligned}
\frac{d}{dt}\Phi_t(x) &= \left\langle \frac{d}{dt}\left(\mathcal{D}(t)\,x\right), \nabla\Phi\left(\mathcal{D}(t)\,x\right)\right\rangle \\
&= -\left\langle C(t)^\top \mathcal{D}(t)\,x, \nabla\Phi\left(\mathcal{D}(t)\,x\right)\right\rangle \\
&= -\sum_{i=1}^d \left(C(t)^\top \mathcal{D}(t)\,x\right)_i \left(\partial_i\Phi\right)_t(x),
\end{aligned}$$

Now, coming back to the general case, let $\Phi \in \mathcal{S}'\left(\mathbb{R}^d\right)$ and $(\phi_\epsilon)_{\epsilon>0}$ a sequence of mollifiers in $\mathcal{S}\left(\mathbb{R}^d\right)$, converging to the Dirac measure. Then for all $\epsilon > 0$, the function $\Phi * \phi_\epsilon : x \mapsto \Phi\left(\phi_\epsilon(x - \cdot)\right)$ belongs to $\mathcal{S}'\left(\mathbb{R}^d\right) \cap \mathcal{C}^\infty\left(\mathbb{R}^d\right)$. By the first part of the proof, (3.37) holds replacing $\Phi = \Phi \star \varphi_\varepsilon$. Now, this converges to $\Phi$ in $\mathcal{S}'\left(\mathbb{R}^d\right)$ when $\epsilon$ tends to $0^+$. (3.37) follows sending $\epsilon$ to $0^+$. Indeed, for all $\varphi \in \mathcal{S}\left(\mathbb{R}^d\right)$, $t \in [0,T]$, setting

$\check{\phi}_\epsilon : y \mapsto \phi_\epsilon(-y)$, we have

$$\begin{aligned}
\Phi_t(\varphi) &= \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^d} \varphi(x) \left(\Phi * \phi_\epsilon\right)_t (x)\, dx \\
&= \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^d} \varphi(x) \Phi * \phi_\epsilon (x)\, dx - \lim_{\epsilon\to 0^+} \sum_{i=1}^d \int_0^t \det\left(\mathcal{D}^{-1}(s)\right) \int_{\mathbb{R}^d} \left(C(s)^\top x\right)_i \varphi\left(\mathcal{D}^{-1}(s)\, x\right) \partial_i \Phi * \phi_\epsilon(x) dx ds \\
&= \Phi(\varphi) - \lim_{\epsilon\to 0^+} \sum_{i=1}^d \int_0^t \det\left(\mathcal{D}^{-1}(s)\right) \partial_i \Phi \left(\left(\left(C(s)^\top \cdot\right)_i \varphi\left(\mathcal{D}^{-1}(s)\cdot\right)\right) * \check{\phi}_\epsilon\right) ds \\
&= \Phi(\varphi) - \sum_{i=1}^d \int_0^t \det\left(\mathcal{D}^{-1}(s)\right) \partial_i \Phi \left(\left(C(s)^\top \cdot\right)_i \varphi\left(\mathcal{D}^{-1}(s)\cdot\right)\right) ds \\
&= \Phi(\varphi) - \sum_{i=1}^d \int_0^t \left(\partial_i \Phi\right)_s \left(x \mapsto \left(C(s)^\top \mathcal{D}(s)\, x\right)_i \varphi(x)\right) ds.
\end{aligned}$$

To conclude, it remains to justify the commutation between the limit in $\epsilon$ and the integral in time from line 3 to line 4 using Lebesgue dominated convergence theorem. On the one hand, for a given $i \in [\![1, d]\!]$, the fact $\partial_i \Phi$ belongs to $\mathcal{S}'\left(\mathbb{R}^d\right)$ implies that there exists $C > 0$, $N \in \mathbb{N}$ such that for all $\varphi \in \mathcal{S}\left(\mathbb{R}^d\right)$

$$\left|\partial_i \Phi(\varphi)\right| \leq C \sup_{|\alpha|\leq N} \sup_{x\in\mathbb{R}^d} \left(1 + |x|^2\right)^N \left|\partial_x^\alpha \varphi(x)\right|,$$

see Chapter 1, Exercise 8 in [16]. On the other hand, the quantities

$$\sup_{x\in\mathbb{R}^d} \left(1 + |x|^2\right)^N \left|\partial_x^\alpha \left(x_j \varphi(\mathcal{D}^{-1}(s)\cdot)\right) * \check{\phi}_\epsilon\right|$$

are bounded uniformly in the couple $(s, \epsilon)$, for all $j \in [\![1, d]\!]$, $\alpha \in \mathbb{N}^d$, taking also into account that the function $s \mapsto \mathcal{D}^{-1}(s)$ is continuous and therefore bounded. Since $C$ is also continuous on $[0, T]$, we are justified to use Lebesgue's dominated convergence theorem.

□

dOU_Uniq **Theorem 3.19.** *For all $\mu \in \mathcal{M}_f\left(\mathbb{R}^d\right)$, equation (1.2) with terminal value $\mu$ admits at most one $\mathcal{M}_f\left(\mathbb{R}^d\right)$-valued solution in the sense of Definition 3.1.*

*Proof.* Let $\mu \in \mathcal{M}_f\left(\mathbb{R}^d\right)$ and $\mathbf{u}$ a solution of (2.3) with terminal value $\mu$. Then, $\mathbf{u}$ solves equation (3.2) with initial value $\mathbf{u}(0)$. As a consequence, by I (3.35) appearing at the end of the proof of Proposition 3.17, for all $\xi \in \mathbb{R}^d$,

$$\mathcal{F}\mu(\xi) = e^{-\int_0^T \frac{\left|\sigma(s)^\top \xi\right|^2}{2} ds} \mathcal{F}\mathbf{u}(0)\left(\mathcal{D}^{-1}(T)\,\xi\right),$$

so that

$$\mathcal{F}\mathbf{u}(0)(\xi) = e^{\int_0^T \frac{\left|\sigma(s)^\top \xi\right|^2}{2} ds} \mathcal{F}\mu\left(\mathcal{D}(T)\,\xi\right).$$

Hence, $\mathbf{u}(0)$ is entirely determined by $\mu$ and Proposition 3.17 gives the result. □

# 4 McKean SDEs related to time-reversal of diffusions

S4

## 4.1 Preliminary considerations

Prelim In this last section we concentrate on the analysis of the well-posedness of the McKean SDE (1.3).

Regarding $b:[0,T]\times\mathbb{R}^d\mapsto\mathbb{R}^d$, $\sigma:[0,T]\times\mathbb{R}^d\mapsto M_{d,m}(\mathbb{R})$, we set $\widehat{b}:=b(T-.,\cdot)$, $\widehat{\sigma}:=\sigma(T-.,\cdot)$, $\widehat{\Sigma}:=\widehat{\sigma}^\top\widehat{\sigma}$. Given a probability-valued function $\mathbf{p}:[0,T]\to\mathcal{P}(\mathbb{R}^d)$, we denote by $p_t$ the density of $\mathbf{p}(t)$, for $t\in[0,T]$, whenever it exists. For the McKean type SDE (1.3), we consider the following notion of solution.

MKSol **Definition 4.1.** *On a given filtered probability space $\left(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},\mathbb{P}\right)$ equipped with an $m$-dimensional $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion $\beta$, a* **solution** *of equation (1.3) is a couple $(Y,\mathbf{p})$ fulfilling (1.3) with Brownian motion $\beta$, such that $Y$ is $(\mathcal{F}_t)_{t\in[0,T]}$-adapted and such that for all $i\in[\![1,d]\!]$, all compact $K\subset\mathbb{R}^d$, all $\tau<T$*

$$\int_0^\tau\int_K\left|div_y\left(\widehat{\Sigma}_{i.}(r,y)\,p_r(y)\right)\right|dydr<\infty. \tag{4.1}$$ IdInt

RDefMK **Remark 4.2.** *For a given solution $(Y,\mathbf{p})$ of equation (1.3), identity (4.1) appearing in Definition 4.1 implies in particular that, for all $i\in[\![1,d]\!]$, all $\tau<T$*

$$\int_0^\tau\left|\frac{div_y\left(\widehat{\Sigma}_{i.}(r,Y_r)\,p_r(Y_r)\right)}{p_r(Y_r)}\right|dr<\infty,\ \mathbb{P}-\text{a.s.}$$

The terminology stating that (1.3) constitutes a probabilistic representation of (1.2) because is justified by the result below.

PProbRep **Proposition 4.3.** *Suppose $b,\sigma$ locally bounded. If $(Y,\mathbf{p})$ is a solution of (1.3) in the sense of Definition 4.1, then $\mathbf{p}(T-\cdot)$ is a solution of (1.1), with $\mu=\mathbf{p}(0)$ in the sense of Definition 3.1.*

*Proof.* Let $(Y,\mathbf{p})$ be a solution of (1.3) in the sense of Definition 4.1 with a Brownian motion symbolized by $\beta$. Let $\phi\in\mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$ and $t\in]0,T]$. Itô's formula gives

$$\phi(Y_{T-t})=\phi(Y_0)+\int_0^{T-t}\left\langle\tilde{b}(s,Y_s;p_s),\nabla\phi(Y_s)\right\rangle+\frac{1}{2}Tr\left(\widehat{\Sigma}(s,Y_s)\,\nabla^2\phi(Y_s)\right)ds+\int_0^{T-t}\nabla\phi(Y_s)^\top\sigma(s,Y_s)\,d\beta_s, \tag{4.2}$$ Ito

with

$$\tilde{b}(s,y;p_s):=\left\{\frac{div_y\left(\widehat{\Sigma}_{j.}(s,y)\,p_s(y)\right)}{p_s(y)}\right\}_{j\in[\![1,d]\!]}-\widehat{b}(s,y),\quad(s,y)\in]0,T[\times\mathbb{R}^d.$$

We now want to take the expectation in identity (4.2). On the one hand, Remark 4.2, implies that for all $i\in[\![1,d]\!]$ and $s\in]0,T[$

$$\int_0^T ds\mathbb{E}\left|\frac{div_y\left(\widehat{\Sigma}_{i.}(s,Y_s)\,p_s(Y_s)\right)}{p_s(Y_s)}\partial_i\phi(Y_s)\right|<\infty.$$

On the other hand

$$\int_0^T\mathbb{E}\left\{Tr\left(\widehat{\Sigma}(s,Y_s)\,\nabla^2\phi(Y_s)\right)\right\}ds=\sum_{i,j=1}^d\int_0^T\int_{\mathbb{R}^d}\widehat{\Sigma}_{ij}(s,y)\,\partial_{ij}\phi(y)\,p_s(y)\,dyds\ \text{p.s.}$$

Previous expression is finite since $\sigma$ is bounded on compact sets and the partial derivatives of $\phi$ have compact supports. With similar arguments we prove that $\int_0^T ds\mathbb{E}\left|\left\langle\widehat{b}(s,Y_s),\nabla\phi(Y_s)\right\rangle\right|<\infty,\ s\in]0,T[$. Moreover, fixing $s\in]0,T[$, integrating by parts we have

$$\begin{aligned}\mathbb{E}\left\{\left\langle\tilde{b}(s,Y_s;p_s),\nabla\phi(Y_s)\right\rangle\right\}&=\sum_{k,j=1}^d\int_{\mathbb{R}^d}\partial_k\left(\widehat{\Sigma}_{jk}(s,y)\,p_s(y)\right)\partial_j\phi(y)\,dy-\int_{\mathbb{R}^d}\left\langle\widehat{b}(s,y),\nabla\phi(y)\right\rangle p_s(y)\,dy \\ &=-\int_{\mathbb{R}^d}Tr\left(\widehat{\Sigma}(s,y)\,\nabla^2\phi(y)\right)p_s(y)\,dy-\int_{\mathbb{R}^d}\left\langle\widehat{b}(s,y),\nabla\phi(y)\right\rangle p_s(y)\,dy.\end{aligned} \tag{4.3}$$ E42quater

Now, the quadratic variation of the local martingale $M^Y := \int_0^\cdot \nabla\phi(Y_s)^\top \sigma(s,Y_s)\, d\beta_s$ yields

$$\left[M^Y\right] = \int_0^\cdot \nabla\phi(Y_s)^\top \Sigma(s,Y_s)\, \nabla\phi(Y_s)\, ds.$$

We remark in particular that $\mathbb{E}\left(\left[M^Y\right]_T\right) < \infty$ since $\sigma$ is bounded on compact sets and $\phi$ has compact support. This shows $M^Y$ is a true (even square integrable) martingale and all terms involved in (4.2) are integrable.

At this point we evaluate the expectation in (4.2) taking into account the considerations above together with (4.1) and (4.3). We obtain

$$\mathbb{E}\left(\phi\left(Y_{T-t}\right)\right) = \int_{\mathbb{R}^d} \phi(y)\, \mu(dy) - \int_0^{T-t}\int_{\mathbb{R}^d} L_{T-s}\phi(y)\, p_s(y)\, dyds.$$

Applying the change of variable $t \mapsto T-t$, we finally obtain the identity

$$\int_{\mathbb{R}^d} \phi(y)\, p_{T-t}(y)\, dy = \int_{\mathbb{R}^d} \phi(y)\, \mu(dy) - \int_t^T \int_{\mathbb{R}^d} L_s\phi(y)\, p_{T-s}(y)\, dyds,$$

which means that $\mathbf{p}(T-\cdot)$ solves (1.2) in the sense of Definition 3.1 with terminal value $\mu$. □

We also provide the different notions of existence and uniqueness for (1.3) we will use in the sequel.

MKDSol **Definition 4.4.** *Let $\mathcal{A}$ be a class of measure-valued functions from $[0,T]$ to $\mathcal{P}\left(\mathbb{R}^d\right)$.*

1. *We say that (1.3) admits* **existence in law** *in $\mathcal{A}$, if there exists a complete filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$ equipped with an $m$-dimensional $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion $\beta$ and a couple $(Y, \mathbf{p})$ solution of (1.3) in the sense of Definition 4.1 such that $\mathbf{p}$ belongs to $\mathcal{A}$.*
2. *Let $\left(Y^1, \mathbf{p^1}\right)$, $\left(Y^2, \mathbf{p^2}\right)$ be two solutions of (1.3) in the sense of Definition 4.1 associated to some complete filtered probability spaces $\left(\Omega^1, \mathcal{F}^1, \left(\mathcal{F}^1_t\right)_{t\in[0,T]}, \mathbb{P}^1\right)$, $\left(\Omega^2, \mathcal{F}^2, \left(\mathcal{F}^2_t\right)_{t\in[0,T]}, \mathbb{P}^2\right)$ respectively, equipped with Brownian motions $\beta^1, \beta^2$ respectively and such that $\mathbf{p^1}, \mathbf{p^2}$ belong to $\mathcal{A}$. We say that (1.3) admits* **uniqueness in law** *in $\mathcal{A}$, if $Y^1_0, Y^2_0$ have the same law implies that $Y^1, Y^2$ have the same law.*
3. *We say that (1.3) admits* **strong existence** *in $\mathcal{A}$ if for any complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ equipped with an $m$-dimensional $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion $\beta$, there exists a solution $(Y, \mathbf{p})$ of equation (1.3) in the sense of Definition 4.1 such that $\mathbf{p}$ belongs to $\mathcal{A}$.*
4. *We say that (1.3) admits* **pathwise uniqueness** *in $\mathcal{A}$ of if for any complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ equipped with an $m$-dimensional $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion $\beta$, for any solutions $\left(Y^1, \mathbf{p^1}\right)$, $\left(Y^2, \mathbf{p^2}\right)$ of (1.3) in the sense of Definition 4.1 such that $Y^1_0 = Y^2_0$, $\mathbb{P}-$a.s. and $\mathbf{p^1}, \mathbf{p^2}$ belong to $\mathcal{A}$, we have $Y^1 = Y^2$, $\mathbb{P}-$a.s.*

We finally define the sets in which we will formulate existence and uniqueness results in the sequel.

NAC1_2 **Notation 1.** *1. For a given $\mathcal{C} \subseteq \mathcal{P}\left(\mathbb{R}^d\right)$, $\mathcal{A}_\mathcal{C}$ denotes the set of measure-valued functions from $[0,T]$ to $\mathcal{P}\left(\mathbb{R}^d\right)$ $\mathbf{p}$ such that $\mathbf{p}(T)$ belongs to $\mathcal{C}$. Furthermore, for a given measure-valued function $\mathbf{p} : [0,T] \mapsto \mathcal{P}\left(\mathbb{R}^d\right)$, we will denote*

$$b(t,\cdot;\mathbf{p}_t) := \left\{ \frac{div_y\left(\widehat{\Sigma}_{i.}\, p_t\right)}{p_t} \right\}_{i\in[\![1,d]\!]}, \tag{4.4}$$ EBP

*for almost all $t \in [0,T]$ whenever $p_t$ exists and the right-hand side quantity is well-defined. The function $(t,x) \mapsto b(t,x;\mathbf{p}_t)$ is defined on $[0,T]\times\mathbb{R}^d$ with values in $\mathbb{R}^d$.*

2. *Let $\mathcal{A}_1$ (resp. $\mathcal{A}_2$) denote the set of measure-valued functions from $[0,T]$ to $\mathcal{P}\left(\mathbb{R}^d\right)$ $\mathbf{p}$ such that, for all $t \in [0,T[$, $\mathbf{p}(t)$ admits a density $p_t$ with respect to the Lebesgue measure on $\mathbb{R}^d$ and such that $(t,x) \mapsto b(t,x;\mathbf{p}_t)$ is locally bounded (resp. is locally Lipschitz in space with linear growth) on $[0,T[\times\mathbb{R}^d$.*

We state now existence and uniqueness results for equation (1.3) in different settings.

## 4.2 PDE with terminal condition and existence for the McKean SDE

MKEX

The existence result for equation (1.3) will be based on two pillars: the reachability condition constituted by the existence of a solution of the Fokker-Planck PDE with terminal condition and the time-reversal techniques of [8]. More precisely, we suppose that Assumption 4 is in force for a fixed $\mathcal{C} \subseteq \mathcal{P}\left(\mathbb{R}^d\right)$ and consider the following extra assumptions, i.e. Assumptions 6, 7 and 8, still with respect to $(\mathcal{C},\mu)$.

MKEx_1 **Assumption 6.** *The backward PDE (1.1) with terminal condition $\mu$ admits at least an $\mathcal{M}_+\left(\mathbb{R}^d\right)$-valued solution $\mathbf{u}$ in the sense of Definition 3.1 verifying the following.*

1. *$\mathbf{u}(0)$ belongs to $\mathcal{C}$.*
2. *$\forall t \in ]0,T[$, $\mathbf{u}(t)$ admits a density with respect to the Lebesgue measure on $\mathbb{R}^d$ (denoted by $u(t,\cdot)$) and for all $t_0 > 0$ and all compact $K \subset \mathbb{R}^d$*

$$\int_{t_0}^{T}\int_K |u(t,x)|^2 + \sum_{i=1}^{d}\sum_{j=1}^{m} |\sigma_{ij}(t,x)\,\partial_i u(t,x)|^2\,dxdt < \infty. \tag{4.5}$$ HP

R45 **Remark 4.5.** *Suppose Assumption 1 holds and let $\mathbf{u}$ be the measure-valued function appearing in Assumption 6. Then (4.5) implies that the family of densities $u(T-t,\cdot), t \in ]0,T[$ verifies condition (4.1) appearing in Definition 4.1. To show this, it suffices to check that for all $t_0 > 0$, all compact $K \subset \mathbb{R}^d$ and all $(i,j,k) \in [\![1,d]\!]^2 \times [\![1,m]\!]$*

$$\int_{t_0}^{T}\int_K |\partial_j(\sigma_{ik}(s,y)\,\sigma_{jk}(s,y)\,u(s,y))|\,dyds < \infty. \tag{4.6}$$ Integr

*The integrand appearing in (4.6) is well-defined. Indeed, in the sense of distributions we have*

$$\partial_j(\sigma_{ik}\sigma_{jk}u) = \sigma_{ik}\sigma_{jk}\partial_j u + u(\sigma_{ik}\partial_j\sigma_{jk} + \sigma_{jk}\partial_j\sigma_{ik}); \tag{4.7}$$ Deriv

*moreover the components of $\sigma$ are Lipschitz, so they are (together with their space derivatives) locally bounded. Also $u$ and $\sigma_{jk}\partial_j$ are square integrable by (4.5). This implies (4.6).*

MKEx_2 **Assumption 7.** *Let $\mathbf{u}$ be the measure-valued mapping appearing in Assumption 6. We suppose that $\mu$ admits a density and $\mathbf{u}(T-\cdot)|_{[0,T[\times\mathbb{R}^d}$ belongs to $\mathcal{A}_1$.*

We introduce two new assumptions.

MKEx_3 **Assumption 8.** *Let $\mathbf{u}$ be the measure-valued mapping appearing in Assumption 6. We suppose that $\mu$ admits a density and $\mathbf{u}(T-\cdot)|_{[0,T[\times\mathbb{R}^d}$ belongs to $\mathcal{A}_2$.*

We remark that Assumption 8 implies 7.

KEx_Prop **Proposition 4.6.** *Suppose the validity of Assumptions 1, Assumption 4 with respect to $\mathcal{C}$ and Assumption 6 with respect to $(\mathcal{C},\mu)$. Then (1.3) admits existence in law in $\mathcal{A}_{\mathcal{C}}$.*
*In particular if, moreover, Assumption 7 (resp. 8) holds, then (1.3) admits existence in law in $\mathcal{A}_{\mathcal{C}} \cap \mathcal{A}_1$ (resp. strong existence in $\mathcal{A}_{\mathcal{C}} \cap \mathcal{A}_2$).*

*Proof.* By Assumption 6, there is an $\mathcal{M}_+\left(\mathbb{R}^d\right)$-valued solution $\mathbf{u}$ of equation (1.1) in the sense of Definition 3.1 such that $\mathbf{u}(T) = \mu$ and $\mathbf{u}(0)$ belongs to $\mathcal{C}$. We consider now a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$ equipped with an $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion $W$. Let $X_0$ be a r.v. distributed according to $\mathbf{u}(0)$. Under Assumption 1, it is well-known that there is a solution $X$ to

$$X_t = X_0 + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dW_s,\ t \in [0,T]. \tag{4.8}$$ SDE

Now, by Proposition 3.2, $t \mapsto \mathcal{L}(X_t)$ is a $\mathcal{P}\left(\mathbb{R}^d\right)$-valued solution of equation (3.2) in the sense of (3.3) with initial value $\mathbf{u}(0) \in \mathcal{C}$. Then Assumption 4 gives

$$\mathcal{L}(X_t) = \mathbf{u}(t),\ t \in [0,T], \tag{4.9}$$ MKIdLaw

since $\mathbf{u}$ solves also (3.2) with initial value $\mathbf{u}(0) \in \mathcal{C}$. This implies in particular that $\mathbf{u}$ is probability valued and that for all $t \in ]0,T[$, $X_t$ has $u(t,\cdot)$ as a density fulfilling condition (4.5) in Assumption 6.
Combining this observation with Assumption 1, Theorem 2.1 in [8] states that there exists a filtered probability space $\left(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{Q}\right)$ equipped with the Brownian motion $\beta$ and a copy of $\hat{X}$ (still denoted by the same letter) such that $\widehat{X}$ fulfills the first lign of (1.3) with $\beta$ and

$$\mathbf{p}(t) = \mathbf{u}(T-t),\ t \in ]0,T[. \tag{4.10}$$ Eup

Finally, existence in law for (1.3) in the sense of Definition 4.1 holds since $(\widehat{X}, \mathbf{u}(T-\cdot))$ is a solution of (1.3) on the same filtered probability space and the same Brownian motion above. This occurs in $\mathcal{A}_{\mathcal{C}}$ since $\mathcal{L}\left(\widehat{X}_T\right) \in \mathcal{C}$ thanks to equality (4.9) for $t = T$.
We discuss rapidly the *in particular* point.

- Suppose that Assumption 7; then $\mathbf{u}(T-\cdot)$ belongs to $\mathcal{A}_{\mathcal{C}} \cap \mathcal{A}_1$ and we also have existence in law in $\mathcal{A}_{\mathcal{C}} \cap \mathcal{A}_1$.
- Suppose the validity of Assumption 8. Then, (4.10), strong existence and pathwise uniqueness for the first line of (1.3) holds by classical arguments since the coefficients are locally Lipschitz with linear growth, see [14] Exercise (2.10), and Chapter IX.2 and [14], Th. 12. section V.12. of [15]. By Yamada-Watanabe theorem this implies uniqueness in law, which shows that $\mathbf{u}(T-\cdot)$ constitutes the marginal laws of the considered strong solutions. This concludes the proof of strong existence in $\mathcal{A}_{\mathcal{C}} \cap \mathcal{A}_2$ since $\mathbf{u}(T-\cdot)$ belongs to $\mathcal{A}_{\mathcal{C}} \cap \mathcal{A}_2$, by Assumption 8.

□

istence2 **Remark 4.7.** *By (4.10), the second component* $\mathbf{p}$ *of the solution of (1.3) is given by* $\mathbf{u}(T-\cdot)$.

## 4.3 PDE with terminal condition and uniqueness for the McKean SDE

MKUNIQ

In this subsection we discuss some questions related to uniqueness for equation (1.3). We state the following hypothesis related to $(\mu, \mathcal{C})$ where $\mathcal{C}$ is a given subset of $\mathcal{P}\left(\mathbb{R}^d\right)$.

APDETerm **Assumption 9.** *The equation (1.1) with terminal condition* $\mu$ *admits at most a* $\mathcal{P}\left(\mathbb{R}^d\right)$*-valued solution* $\mathbf{u}$ *in the sense of Definition 3.1 such that* $\mathbf{u}(0)$ *belongs to* $\mathcal{C}$.

We recall that Section 3.2 provides various classes of examples where Assumption 9 holds.

MKProp **Proposition 4.8.** *Suppose the validity of Assumption 9 with respect to $(\mu, \mathcal{C})$ and suppose $b, \sigma$ to be locally bounded. Let $\left(Y^i, \mathbf{p}^i\right),\ i \in \{1,2\}$ be two solutions of equation (II.3) in the sense of Definition 4.1 such that $\mathbf{p}^1(T), \mathbf{p}^2(T)$ belong to $\mathcal{C}$. Then,*

$$\mathbf{p}^1 = \mathbf{p}^2.$$

*Proof.* Proposition 4.3 shows that $\mathbf{p}^1(T-\cdot), \mathbf{p}^2(T-\cdot)$ are $\mathcal{P}\left(\mathbb{R}^d\right)$-valued solutions of equation (II.2) in the sense of Definition 3.1 with terminal value $\mu$. Assumption 9 gives the result since $\mathbf{p}^1(T), \mathbf{p}^2(T)$ belong to $\mathcal{C}$. □

As a corollary, we establish some consequences about uniqueness in law and pathwise uniqueness results for equation (II.3) in the classes $\mathcal{A}_1$ and $\mathcal{A}_2$.

Coro **Corollary 4.9.** *Suppose the validity of Assumption 9 with respect to $(\mu, \mathcal{C})$. Then, the following results hold.*

1. *If $b$ is locally bounded, $\sigma$ is continuous and if the non-degeneracy Assumption 3 holds then (II.3) admits uniqueness in law in $\mathcal{A}_{\mathcal{C}} \cap \mathcal{A}_1$.*
2. *If $(b, \sigma)$ are locally Lipschitz with linear growth in space, then (II.3) admits pathwise uniqueness in $\mathcal{A}_{\mathcal{C}} \cap \mathcal{A}_2$.*

*Proof.* If $(Y, \mathbf{p})$ is a solution of (II.3) and is such that $\mathbf{p}(T)$ belongs to $\mathcal{C}$, then by Proposition 4.8 $\mathbf{p}$ is determined by $\mu = \mathcal{L}(Y_0)$.
To show that item 1. (resp. 2.) holds, it suffices to show that the classical SDE

$$dX_t = b\left(t, X_t; \mathbf{p}_t\right) - \widehat{b}\left(t, X_t\right) dt + \widehat{\sigma}\left(t, X_t\right) dW_t,\ t \in [0, T[, \tag{4.11}$$ FrozenSDE

where $b$ was defined in (4.4) and $W$ an $m$-dimensional Brownian motion, admits uniqueness in law (resp. pathwise uniqueness). The mentioned uniqueness in law is a consequence of Theorem 10.1.3 in [17] and pathwise uniqueness holds by [14] Exercise (2.10), and Chapter IX.2 and [15] Th. 12. Section V.12. □

## 4.4 Well-posedness for the McKean SDE: the bounded coefficients case

amples44

In this section, we state a significant result related to existence and uniqueness in law together with pathwise uniqueness for equation (II.3). In particular we obtain existence and uniqueness in law for (II.3) in the class $\mathcal{A}_1$

We formulate the following hypotheses.

oothness **Assumption 10.** *1. Assumption 3 holds.*

2. *The functions $\sigma$ is Lipschitz (in space).*
3. *The functions $\sigma$, $b$, $\left(\nabla_x b_i\right)_{i \in [\![1,d]\!]}$, $\left(\nabla_x \Sigma_{ij}\right)_{i,j \in [\![1,d]\!]}$ are continuous bounded and $\nabla_x^2 \Sigma$ is Hölder continuous with exponent $\alpha \in ]0,1[$ in space uniformly in time.*

othness1 **Assumption 11.** *$\Sigma$ is supposed to be Hölder continuous in time*

Runu **Remark 4.10.** *Under Assumption 10, for every $\nu \in \mathcal{P}(\mathbb{R}^d)$ there exists a unique $\mathcal{P}\left(\mathbb{R}^d\right)$-valued solution $\mathbf{u}^\nu$ of (3.2). Indeed the assumptions of Lemma 3.4 are fulfilled.*

We continue with a fundamental lemma whose proof will appear in the Appendix.

FriedAr **Lemma 4.11.** *Suppose the validity of Assumptions 10 and 11. Then, for all $\nu \in \mathcal{P}\left(\mathbb{R}^d\right)$, $\mathbf{u}^\nu(t)$ admits a density $u^\nu(t,\cdot) \in \mathcal{C}^1\left(\mathbb{R}^d\right)$ for all $t \in ]0,T]$. Furthermore, for each compact $K$ of $]0,T]\times\mathbb{R}^d$, there are strictly positive constants $C_1^K, C_2^K, C_3^K$, also depending on $\nu$ such that*

$$C_1^K \leq u^\nu(t,x) \leq C_2^K \tag{4.12}$$

$$|\partial_i u^\nu(t,x)| \leq C_3^K,\ i \in [\![1,d]\!], \tag{4.13}$$

*for all $(t,x) \in K$.*

P49 **Lemma 4.12.** *Suppose that the initial condition $\mu$ equals $\mathbf{u}^\nu(T)$ for some $\nu \in \mathcal{P}\left(\mathbb{R}^d\right)$. We suppose the following.*

1. *Assumptions 10.*
2. *$\mathbf{u}^\nu(t)$ admits a density $u^\nu(t,\cdot) \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$, for all $t \in ]0,T]$.*
3. *For each compact $K$ of $]0,T]\times\mathbb{R}^d$, there are strictly positive constants $C_1^K, C_2^K, C_3^K$, also depending on $\nu$ such that (4.12) and (4.13) hold $\forall (t,x) \in K$.*

*Then equation (1.3) admits existence in law in $\mathcal{A}_1$.*

CP49 **Corollary 4.13.** *We suppose the validity of Assumptions 10 and and 11.*

1. *Suppose the existence of $\nu \in \mathcal{P}(\mathbb{R}^d)$ such that $\mathbf{u}^\nu(T) = \mu$. Then, equation (1.3) admits existence in law in $\mathcal{A}_1$. Moreover, if $\nu$ is a Dirac mass, existence in law occurs in $\mathcal{A}_{(\delta_x)_{x\in\mathbb{R}^d}} \cap \mathcal{A}_1$.*
2. *Otherwise (1.3) does not admit existence in law.*

*Proof.*

1. The first part is a direct consequence of Lemma 4.11, Lemma 4.12 and expression (4.4). If in addition, $\nu$ is a Dirac mass, then $\mathbf{u}^\nu(0)$ belongs to $\mathcal{C} := (\delta_x)_{x\in\mathbb{R}^d}$, hence existence in law occurs in $\mathcal{A}_{\mathcal{C}} \cap \mathcal{A}_1$ again by Proposition 4.6.
2. Otherwise suppose ab absurdo that $(Y,\mathbf{p})$ is a solution of (1.3). By Proposition 4.3 $\mathbf{p}(T-\cdot)$ is a solution of (2.3). We set $\nu_0 = \mathbf{p}(T)$ so that $\mathbf{p}(T-\cdot)$ verifies also (3.2) with initial value $\nu_0$. Since, by Lemma 3.4 uniqueness holds for (3.2), it follows that $\mathbf{p}(T-\cdot) = \mathbf{u}^{\nu_0}$ which concludes the proof of item 2.

□

**Proof** (of Lemma 4.12). Suppose $\mu = \mathbf{u}^\nu(T)$ for some $\nu \in \mathcal{P}\left(\mathbb{R}^d\right)$.
We recall that Assumption 4 holds with respect to $\mathcal{C} := \mathcal{P}\left(\mathbb{R}^d\right)$ by Remark 3.3 1.
In view of applying Proposition 4.6, we need to check that Assumptions 6 and 7 hold with respect to $(\mu,\mathcal{C})$. Assumption 6 is verified by $\mathbf{u} = \mathbf{u}^\nu$. Indeed the function $\mathbf{u}^\nu$ is a $\mathcal{P}\left(\mathbb{R}^d\right)$-valued solution of (1.2) with terminal value $\mu$ and such that $\mathbf{u}^\nu(0)$ belongs to $\mathcal{C}$. Condition (4.5) appearing in Assumption 6 is satisfied with $\mathbf{u} = \mathbf{u}^\nu$ thanks to the right-hand side of inequalities (4.12) and (4.13) and the fact that $\sigma$ is bounded. Hence Assumption 6 holds with respect to $(\mu,\mathcal{C})$.
It remains to show Assumption 7 holds i.e. that

$$(t,x) \mapsto \frac{div_x\left(\widehat{\Sigma}_{i\cdot}(t,x)u^\nu(T-t,x)\right)}{u^\nu(T-t,x)}$$

is locally bounded on $[0,T[\times\mathbb{R}^d$. To achieve this, we fix $i\in[\![1,d]\!]$ and a bounded open subset $\mathcal{O}$ of $[0,T[\times\mathbb{R}^d$. For $(t,x)\in\mathcal{O}$ we have

$$\left|\frac{div_x\left(\widehat{\Sigma}_{i.}\left(t,x\right)u^{\nu}\left(T-t,x\right)\right)}{u^{\nu}\left(T-t,x\right)}\right|\leq\left|div_x\left(\widehat{\Sigma}_{i.}\left(t,x\right)\right)\right|+\left|\widehat{\Sigma}_{i.}\left(t,x\right)\right|\frac{\left|\nabla_x u^{\nu}\left(T-t,x\right)\right|}{u^{\nu}\left(T-t,x\right)}.$$

The latter quantity is locally bounded in $t,x$ thanks to the boundedness of $\Sigma, div_x\left(\widehat{\Sigma}_{i.}\right)$ and inequalities (4.12) and (4.13). Hence, Assumption 7 holds. This ends the proof. □

TExUniq **Proposition 4.14.** *Suppose the validity of Assumption 10 and 11. The following results hold.*

1. *Let us suppose $d=1$. Suppose $\mu$ equals $\mathbf{u}^{\delta_{x_0}}(T)$ for some $x_0\in\mathbb{R}^d$. Then (11.3) admits existence and uniqueness in law in $\mathcal{A}_{(\delta_x)_{x\in\mathbb{R}^d}}\cap\mathcal{A}_1$, pathwise uniqueness in $\mathcal{A}_{(\delta_x)_{x\in\mathbb{R}^d}}\cap\mathcal{A}_2$.*
2. *Let $d\geq 2$. There is a maturity $T$ sufficiently small (only depending on the Lipschitz constant of $b,\sigma$) such that the following result holds. Suppose $\mu$ equals $\mathbf{u}^{\delta_{x_0}}(T)$ for some $x_0\in\mathbb{R}^d$. Then (11.3) admits existence and uniqueness in law in $\mathcal{A}_{(\delta_x)_{x\in\mathbb{R}^d}}\cap\mathcal{A}_1$, pathwise uniqueness in $\mathcal{A}_{(\delta_x)_{x\in\mathbb{R}^d}}\cap\mathcal{A}_2$.*

*Proof.* By Assumptions 10 and 11, Corollary 4.13 implies that (11.3) admits existence in law in the two cases in the specific classes. To check the uniqueness in law and pathwise uniqueness results, we wish to apply Corollary 4.9. It suffices to check Assumption 9 because the other hypotheses are included in Assumption 10. Below we verify Assumption 9 with respect to $(\mu,(\delta_x)_{x\in\mathbb{R}})$, for the separate two cases.

1. Fix $x_0\in\mathbb{R}^d$. This will follow from Proposition 3.9 that holds under Assumption 1 which is a consequence of Assumption 10.
2. We proceed as for previous case but applying Theorem 3.10 instead of Proposition 3.9.

□

We state now the most important results of the section.

xUniqBis **Theorem 4.15.** *Suppose $b,\sigma$ are time-homogeneous, Assumption 10 and suppose there is $\nu\in\mathcal{P}\left(\mathbb{R}^d\right)$ (a priori not known) such that $\mu=\mathbf{u}^{\nu}(T)$.*

1. *(11.3) admits existence and uniqueness in law. Moreover existence in law holds in $\mathcal{A}_1$.*
2. *(11.3) admits pathwise uniqueness in $\mathcal{A}_2$.*

*Proof.* 1. (a) First, Assumption 11 trivially holds since $b,\sigma$ are time-homogeneous. Then, point 1 of Corollary 4.13 implies that (11.3) admits existence in law (in $\mathcal{A}_1$) since Assumption 10 holds.

(b) Let $(Y,\mathbf{p})$ be a solution of (11.3). Proceeding as in the proof of item 2. of Corollary 4.13, we obtain that $\mathbf{p}(T-\cdot)=\mathbf{u}^{\nu_0}$ with $\nu_0=\mathbf{p}(T)$. Then, Lemma 4.11 and the fact that $\sigma$ is bounded allow to show that $\mathbf{p}$ belongs to $\mathcal{A}_1$, see (4.4) in Notation 1.

(c) To conclude it remains to show uniqueness in law in $\mathcal{A}_1$. For this we wish to apply point 1. of Corollary 4.9. To achieve this, we check Assumption 9 with respect to $\left(\mu,\mathcal{P}\left(\mathbb{R}^d\right)\right)$. This is a consequence of Assumptions 3 and 5 and Theorem 3.13 This concludes the proof of item 1.

2. Concerning pathwise uniqueness in $\mathcal{A}_2$, we proceed as for uniqueness in law but applying point 2 of Corollary 4.9. This is valid since Assumption 10 implies that $b,\sigma$ are bounded and Lipschitz.

□

In the result below we extend Theorem 4.15 to the case when the coefficients $b, \sigma$ are piecewise time-homogeneous.

TC313 **Theorem 4.16.** *Let $n \in \mathbb{N}^*$. Let $0 = t_0 < \ldots < t_n = T$ be a partition. For $k \in [\![2, n]\!]$ (resp. $k = 1$) we denote $I_k = ]t_{k-1}, t_k]$ (resp. $[t_0, t_1]$). Suppose that the following holds.*

1. *For all $k \in [\![1, n]\!]$ the restriction of $\sigma$ (resp. $b$) to $I_k \times \mathbb{R}^d$ is a time-homogeneous function $\sigma^k : \mathbb{R}^d \to M_d(\mathbb{R})$ (resp. $b^k : \mathbb{R}^d \to \mathbb{R}^d$).*
2. *Assumption 3.*
3. *$\sigma$ is Lipschitz in space uniformly in time.*
4. *The functions $\sigma^k$, $b^k$, $\left(\nabla_x b_i^k\right)_{i \in [\![1,d]\!]}$, $\left(\nabla_x \Sigma_{ij}^k\right)_{i,j \in [\![1,d]\!]}$ are continuous bounded and $\nabla_x^2 \Sigma^k$ is Hölder continuous with exponent $\alpha \in ]0, 1[$.*

*Suppose $\mu$ equals $\mathbf{u}^\nu(T)$ for some $\nu \in \mathcal{P}\left(\mathbb{R}^d\right)$. Then equation (1.3) admits existence and uniqueness in law. Existence in law holds in $\mathcal{A}_1$.*

RC313 **Remark 4.17.** *A similar remark as in Corollary 4.13 holds for the Theorems 4.15 and 4.16. If there is no $\nu \in \mathcal{P}(\mathbb{R}^d)$ such that $\mathbf{u}^\nu(T) = \mu$, then (1.3) does not admit existence in law.*

**Proof** of Theorem 4.16. We recall that by Lemma 3.4, $\mathbf{u}^{\nu_0}$ is well-defined for all $\nu_0 \in \mathcal{P}\left(\mathbb{R}^d\right)$.

1. We first show that $\mathbf{u}^{\nu_0}$ verifies (4.12) and (4.13). Indeed, fix $k \in [\![1, n]\!]$. The restriction $\mathbf{u_k}$ of $\mathbf{u}^{\nu_0}$ to $\bar{I}_k$ is a solution $\mathbf{v}$ of the first line (3.2) replacing $[0, T]$ with $\bar{I}_k$, $L$ by $L^k$ defined in (3.25), with initial condition $\mathbf{v}(t_{k-1}) = \mathbf{u}^{\nu_0}(t_{k-1})$. That restriction is even the unique solution, using Lemma 3.4 replacing $[0, T]$ with $\bar{I}_k$. We apply Lemma 4.11 replacing $[0, T]$ with $\bar{I}_k$, taking into account Assumptions 1.0 and 1.1, which holds trivially replacing $\sigma, b, \Sigma$ with $\sigma^k, b^k, \Sigma^k$ This implies that $\mathbf{u}^{\nu_0}$ verifies (4.12) and (4.13) replacing $[0, T]$ with $\bar{I}_k$, and therefore on the whole $[0, T]$.
2. Existence in law in $\mathcal{A}_1$, follows now by Lemma 4.12.
3. It remains to show uniqueness in law. Let $(Y, \mathbf{p})$ be a solution of (1.3). We set $\nu_0 := \mathbf{p}(T)$. Since $\mathbf{u}^{\nu_0}$ and $\mathbf{p}(T - \cdot)$ solve (3.2), Lemma 3.4 implies that $\mathbf{p}$ is uniquely determined. Similarly as in item 1.(b) of the proof of Theorem 4.15, item 1. of the present proof and Lemma 4.11 allow to show that $\mathbf{p}$ belongs to $\mathcal{A}_1$.
4. It remains to show uniqueness in law in $\mathcal{A}_1$. For this, Corollary 3.16 implies Assumption 9 with $\mathcal{C} = \mathcal{P}(\mathbb{R}^d)$. Uniqueness of (1.3) in the class $\mathcal{A}_1$ follows now by Corollary 4.9, which ends the proof.

□

## 4.5 Well-posedness for the McKean SDE: the Ornstein-Uhlenbeck semigroup

Sex In this section we consider the case $b : (s, x) \mapsto C(s)\, x$ with $C$ continuous from $[0, T]$ to $\mathbb{R}^d$ and $\sigma$ continuous from $[0, T]$ to $M_{d,m}(\mathbb{R})$. We also suppose that for all $t \in [0, T]$, $\sigma(t)$ is invertible. We denote by $\mathcal{C}(t), t \in [0, T]$, the unique solution of the matrix-valued ODE

$$\mathcal{C}(t) = I + \int_0^t C(s)\mathcal{C}(s)ds.$$

For a given $x_0 \in \mathbb{R}^d$ and a given t $\in ]0,T]$, we denote by $p_t^{x_0}$ the density of a Gaussian random vector with mean $m_t^{x_0} = \mathcal{C}(t)x_0$ and covariance matrix $Q_t = \mathcal{C}(t)\int_0^t \mathcal{C}^{-1}(s)\Sigma(s)\mathcal{C}^{-1}(s)^\top ds\mathcal{C}(t)^\top$. Note that for all $t \in ]0,T]$, $Q_t$ is strictly positive definite, in particular it is invertible. Indeed, for every $t \in [0,T]$, $\Sigma(t)$ is strictly positive definite. By continuity in $t$, $\int_0^t \mathcal{C}^{-1}(s)\Sigma(s)\mathcal{C}^{-1}(s)^\top ds$ is also strictly positive definite and finally the same holds for $Q_t$. For a given $\nu \in \mathcal{P}\left(\mathbb{R}^d\right)$, $t \in ]0,T]$, we set the notation

$$p_t^\nu : x \mapsto \int_{\mathbb{R}^d} p_t^{x_0}(x)\,\nu(dx_0). \tag{4.14}$$ Epnu

At this level, we need a lemma.

OU_lemma **Lemma 4.18.** *Let $\nu \in \mathcal{P}\left(\mathbb{R}^d\right)$. The measure-valued function $t \mapsto p_t^\nu(x)dx$ is the unique solution of (3.2) with initial value $\nu$ and we denote it by $\mathbf{u}^\nu$. Furthermore, $\mathbf{u}^\nu(T-\cdot)$ belongs to $\mathcal{A}_2$.*

*Proof.* 1. We denote immediately $\mathbf{u}^\nu(t)(dx) := p_t^\nu(x)dx,\ t \in ]0,T]$. By Chapter 5, Section 5.6 in [10], for every $t \in ]0,T]$, $p_t^{x_0}$ is the density of the random variable $X_t^{x_0}$, where $X^{x_0}$ is the unique strong solution of (3.4) with initial value $x_0$. The mapping $t \mapsto p_t^{x_0}(x)dx$ is a solution of (3.2) by Proposition 3.2, with initial condition $\delta_{x_0}$. Consequently, by superposition, $\mathbf{u}^\nu$ is a solution of (3.2) with initial value $\nu$.

2. $\mathbf{u}^\nu$ is the unique solution of (3.2) because of Proposition 3.17.

3. It remains to show that $\mathbf{u}^\nu(T-\cdot)$ belongs to $\mathcal{A}_2$, namely that for all $i \in [\![1,d]\!]$

$$(t,x) \mapsto \frac{div_x\left(\Sigma(T-t)_{i\cdot}\, p_{T-t}^\nu(x)\right)}{p_{T-t}^\nu(x)},$$

is locally Lipschitz with linear growth in space on $[0,T[\times\mathbb{R}^d$.

Fix $i \in [\![1,d]\!]$, $t \in [0,T[$ and $x \in \mathbb{R}^d$. Remembering the fact, $p_{T-t}^{x_0}$ is a Gaussian law with mean $m_{T-t}^{x_0}$ and covariance matrix $Q_{T-t}$ for a given $x_0 \in \mathbb{R}^d$, we have

$$\frac{div_x\left(\Sigma(T-t)_{i\cdot}\, p_{T-t}^\nu(x)\right)}{p_{T-t}^\nu(x)} = -\frac{1}{p_{T-t}^\nu(x)}\int_{\mathbb{R}^d}\left\langle \Sigma(T-t)_{i\cdot}, Q_{T-t}^{-1}\left(x - m_{T-t}^{x_0}\right)\right\rangle p_{T-t}^{x_0}(x)\,\nu(dx_0). \tag{4.15}$$ div_OU

Let $K$ be a compact subset of $]0,T]\times\mathbb{R}^d$; then there is $M_K > 0$ such that for all $(t,x) \in K$, $x_0 \in \mathbb{R}^d$,

$$\left|\left\langle \Sigma(T-t)_{i\cdot}, Q_{T-t}^{-1}\left(x - m_{T-t}^{x_0}\right)\right\rangle p_{T-t}^{x_0}(x)\right| \le |\Sigma(T-t)_{i\cdot}|\,\left|\left|Q_{T-t}^{-1}\right|\right|\left|x - m_{T-t}^{x_0}\right| p_{T-t}^{x_0}(x) \le M_K.$$

This follows because $t \mapsto \Sigma(T-t)$ and $t \mapsto Q_{T-t}^{-1}$ are continuous on $[0,T[$ and, setting

$$c_K := \inf\{t | (t,x) \in K\}, \quad m_K := \sup_{a\in\mathbb{R}} |a| \exp\left(-c_K\frac{a^2}{2}\right),$$

we have

$$|x - m_{T-t}^{x_0}| p_{T-t}^{x_0}(x) \le m_K,\ \forall (t,x) \in K.$$

To show that left-hand side of (4.15) is locally bounded on $[0,T[\times\mathbb{R}^d$ it remains to show that $(t,x) \mapsto \int_{\mathbb{R}^d} p_{T-t}^{x_0}(x)\nu(dx_0)$ is lower bounded on $K$. Indeed, let $I$ be a compact of $\mathbb{R}^d$. Since $(t,x,x_0) \mapsto p_{T-t}^{x_0}(x)$ is strictly positive and continuous is lower bounded by a constant $c(K,I)$. The result follows choosing $I$ such that $\nu(I) > 0$.

To conclude, it remains to show that the functions $(t,x) \mapsto \frac{div_x\left(\Sigma(T-t)_i . p^\nu_{T-t}(x)\right)}{p^\nu_{T-t}(x)}$, $i \in [\![1,d]\!]$ defined on $[0,T[\times\mathbb{R}^d$ has locally bounded spatial derivatives, which implies that they are Lipschitz with linear growth on each compact of $[0,T[\times\mathbb{R}^d$. By technical but easy computations, the result follows using the fact the real functions $a \mapsto |a|^m \exp\left(-\frac{a^2}{2}\right)$, $m=1,2$, are bounded.

□

We give now a global well-posedness result for equation (1.3).

OU_WellP **Theorem 4.19.** *1. Suppose the initial condition $\mu$ equals $\mathbf{u}^\nu(T)$ for some $\nu \in \mathcal{P}\left(\mathbb{R}^d\right)$. Then, equation (1.3) admits existence in law, strong existence, uniqueness in law and pathwise uniqueness.*

*2. Otherwise (1.3) does not admit any solution.*

*Proof.* Item 2. can be proved using similar arguments as for the proof of Corollary 4.13. Let $(Y,\mathbf{p})$ be a solution of (1.3) and set $\nu_0 = \mathbf{p}(T)$. By Proposition 4.3, $\mathbf{p}(T-\cdot)$ is a solution of (2.3), so that $\mathbf{p}(T-\cdot)$ verifies also (3.2) with initial value $\nu_0$. Since, by Proposition 3.17, uniqueness holds for (3.2), it follows that $\mathbf{p}(T-\cdot) = \mathbf{u}^{\nu_0}$ which concludes the proof of item 2.
We prove now item 1. For this, taking into account Proposition 4.8 and Yamada-Watanabe theorem and related results for classical SDEs, it suffices to show strong existence and pathwise uniqueness. We set $\mathcal{C} := \mathcal{P}\left(\mathbb{R}^d\right)$

1. Concerning the strong existence statement, we want to apply Proposition 4.6. For this we have to check the validity of Assumption 1, Assumption 4 with respect to $\mathcal{C}$ and Assumptions 6, 8 hold with respect to $(\mu,\mathcal{C})$.

   Since $b,\sigma$ are affine, Assumption 1 trivially holds. Furthermore, Assumption 4 holds with respect to $\mathcal{C}$ thanks to Proposition 3.17.

   Now, $\mathbf{u}^\nu$ is a probability valued solution of (1.1) with terminal value $\mu$. Furthermore, Lemma 4.18 shows that $\mathbf{u}^\nu$, being the unique solution of solution of (3.2), is such that, for all $t \in ]0,T]$, $\mathbf{u}^\nu(t)$ admits $p^\nu_t$ (see (4.14)) for density. Then, relation (4.5) holds since, by the considerations above (4.14) $(t,x) \mapsto p^\nu_t(x)$ is locally bounded with locally bounded spatial derivatives. Hence, Assumption 6 holds with respect to $(\mu,\mathcal{C})$. Finally, Lemma 4.18 implies that $\mathbf{u}^\nu(T-\cdot)$ belongs to $\mathcal{A}_2$. Hence, Assumption 8 holds with respect to $(\mu,\mathcal{C})$. At this point Proposition 4.6 implies existence in law.

2. Let $(Y,\mathbf{p})$ be a solution of equation (1.3). Proposition 4.3 implies that $\mathbf{p}(T-\cdot)$ solves (1.2). Then, Proposition 3.17 gives $\mathbf{p}(T-\cdot) = \mathbf{u}^{\nu_0}$ with $\nu_0 = \mathbf{p}(T)$. Lemma 4.18 implies $\mathbf{p}$ belongs to $\mathcal{A}_2$.

3. It remains to show pathwise uniqueness in $\mathcal{A}_2$. Assumption 9 holds with respect to $(\mu,\mathcal{C})$ thanks to Theorem 3.19. Now, point 2 of Corollary 4.9 implies pathwise uniqueness in $\mathcal{A}_2$ since $b,\sigma$ are locally Lipschitz with linear growth in space.

□

# Appendix

## 4.6 Proof of Lemma 4.11 FriedAr

Let $\nu \in \mathcal{P}\left(\mathbb{R}^d\right)$. For each given $t \in [0,T]$, we denote by $G_t$ the differential operator such that for all $f \in \mathcal{C}^2\left(\mathbb{R}^d\right)$

$$G_t f = \frac{1}{2}\sum_{i,j=1}^{d} \partial_{ij}\left(\Sigma_{ij}\left(t,\cdot\right) f\right) - \sum_{i=1}^{d} \partial_i\left(b_i\left(t,\cdot\right) f\right).$$

Assumption 10 smoothness implies that for a given $f \in \mathcal{C}^2\left(\mathbb{R}^d\right)$, $G_t f$ can be rewritten in the two following ways:

$$G_t f = \frac{1}{2}\sum_{i,j=1}^{d} \Sigma_{ij}(t,\cdot)\partial_{ij} f + \sum_{i=1}^{d}(\sum_{j=1}^{d} \partial_i \Sigma_{ij}(t,\cdot) - b_i(t,\cdot))\partial_i f + c^1(t,\cdot) f, \tag{4.16}$$ Friedman

with

$$c^1 : (t,x) \mapsto \frac{1}{2}\sum_{i,j=1}^{d} \partial_{ij}\Sigma_{ij}(t,x) - \sum_{i=1}^{d} \partial_i b_i(t,x).$$

$$G_t f = \frac{1}{2}\sum_{i,j=1}^{d} \partial_j(\partial_i \Sigma_{ij}(t,\cdot) f + \Sigma_{ij}(t,\cdot)\partial_i f - \sum_{i=1}^{d} b_i(t,\cdot)\partial_i f) - \sum_{i=1}^{d} \partial_i b_i(t,\cdot) f. \tag{4.17}$$ Aronson

On the one hand, combining identity (4.16) Friedman with Assumption 10 smoothness, there exists a fundamental solution $\Gamma$ (in the sense of Definition stated in Section 1. p.3 of [7] friedman 1964) of $\partial_t u = G_t u$, thanks to Theorem 10. Section 6 Chap. 1. in the same reference. Furthermore, there exists $C_1, C_2 > 0$ such that for all $i \in [\![1,d]\!]$, $x,\xi \in \mathbb{R}^d$, $\tau \in [0,T]$, $t > \tau$,

$$\left|\Gamma\left(x,t,\xi,\tau\right)\right| \leq C_1\left(t-\tau\right)^{-\frac{d}{2}} \exp\left(-\frac{C_2\left|x-\xi\right|^2}{4\left(t-\tau\right)}\right), \tag{4.18}$$ PropFriedman_1

$$\left|\partial_{x_i}\Gamma\left(x,t,\xi,\tau\right)\right| \leq C_1\left(t-\tau\right)^{-\frac{d+1}{2}} \exp\left(-\frac{C_2\left|x-\xi\right|^2}{4\left(t-\tau\right)}\right), \tag{4.19}$$ PropFriedman_2

thanks to identities (6.12), (6.13) in Section 6 Chap. 1 in [7] friedman 1964.

On the other hand, combining Identity (4.17) Aronson with Assumption 10 smoothness, there exists a weak fundamental solution $\Theta$ of $\partial_t u = G_t u$ thanks to Theorem 5 in [1] AronsonGeneral. In addition, there exists $K_1, K_2, K_3 > 0$ such that for almost every $x,\xi \in \mathbb{R}^d$ , $\tau \in [0,T]$, $t \geq \tau$

$$\frac{1}{K_1}\left(t-\tau\right)^{-\frac{d}{2}} \exp\left(-\frac{K_2\left|x-\xi\right|^2}{4\left(t-\tau\right)}\right) \leq \Theta\left(x,t,\xi,\tau\right) \leq K_1\left(t-\tau\right)^{-\frac{d}{2}} \exp\left(-\frac{K_3\left|x-\xi\right|^2}{4\left(t-\tau\right)}\right), \tag{4.20}$$ PropAronson

thanks to point (ii) of Theorem 10 in [1] AronsonGeneral.

Our goal is now to show that $\Gamma$ and $\Theta$ coincide. To this end, we adapt the argument developed at the beginning of Section 7 in [1] AronsonGeneral. Fix a function $H$ from $[0,T]\times\mathbb{R}^d$ belonging to $\mathcal{C}_c^\infty\left([0,T]\times\mathbb{R}^d\right)$. Identity (7.6) in Theorem 12 Chap 1. Section 1. of [7] friedman 1964 implies in particular that the function

$$u : (t,x) \mapsto \int_0^t \int_{\mathbb{R}^d} \Gamma\left(x,t,\xi,\tau\right) H\left(\tau,\xi\right) d\xi d\tau,$$

is continuously differentiable in time, two times continuously differentiable in space and is a solution of the Cauchy problem

$$\begin{cases} \partial_t u\left(t,x\right) = G_t u\left(t,x\right) + H\left(t,x\right), \ (t,x) \in ]0,T]\times\mathbb{R}^d, \\ u\left(0,\cdot\right) = 0. \end{cases} \tag{4.21}$$ CauchyPb

It is consequently also a weak (i.e. distributional) solution of (4.21), which belongs to $\mathcal{E}^2(]0,T]\times\mathbb{R}^d)$ (see definition of that space in [1]) since $u$ is bounded thanks to inequality (4.18) and the fact that $H$ is bounded. Then, point (ii) of Theorem 5 in [1] says that

$$(t,x)\mapsto\int_0^t\int_{\mathbb{R}^d}\Theta\left(x,t,\xi,\tau\right)H\left(\tau,\xi\right)d\xi d\tau$$

is the unique weak solution in $\mathcal{E}^2(]0,T]\times\mathbb{R}^d)$ of (4.21). This implies that for every $(t,x)\in]0,T]\times\mathbb{R}^d$ we have

$$\int_0^t\int_{\mathbb{R}^d}\left(\Gamma-\Theta\right)\left(x,t,\xi,\tau\right)H\left(\tau,\xi\right)d\xi d\tau=0.$$

Point (i) of Theorem 5 in [1] (resp inequality (4.18)) implies that $\Theta$ (resp. $\Gamma$) belongs to $L^p\left(]0,T]\times\mathbb{R}^d\right)$ as a function of $(\xi,\tau)$, for an arbitrary $p\geq d+2$. Then, we conclude that for all $(t,x)\in]0,T]\times\mathbb{R}^d$,

$$\Theta\left(x,t,\xi,\tau\right)=\Gamma\left(x,t,\xi,\tau\right),\ d\xi d\tau a.e. \tag{4.22}$$ coincide

for all $(\tau,\xi)\in[0,t[\times\mathbb{R}^d$. This happens by density of $\mathcal{C}_c^\infty\left([0,T]\times\mathbb{R}^d\right)$ in $L^q\left(]0,T]\times\mathbb{R}^d\right)$, $q$ being the conjugate of $p$.

This, together with (4.20) and the fact that $\Gamma$ is continuous in $(\tau,\xi)$ implies that (4.20) holds for all $(\tau,\xi)\in[0,t[\times\mathbb{R}^d$ and therefore

$$\frac{1}{K_1}\left(t-\tau\right)^{-\frac{d}{2}}\exp\left(-\frac{K_2\left|x-\xi\right|^2}{4\left(t-\tau\right)}\right)\leq\Gamma\left(x,t,\xi,\tau\right)\leq K_1\left(t-\tau\right)^{-\frac{d}{2}}\exp\left(-\frac{K_3\left|x-\xi\right|^2}{4\left(t-\tau\right)}\right). \tag{4.23}$$ PropAronsonBis

We introduce

$$q_t:=x\mapsto\int_{\mathbb{R}^d}\Gamma\left(x,t,\xi,0\right)\nu\left(d\xi\right).$$

By (4.23), with $\tau=0$ we get

$$q_t\left(x\right)\geq\frac{1}{K_1}t^{-\frac{d}{2}}\int_{\mathbb{R}^d}\exp\left(-\frac{K_2\left|x-\xi\right|^2}{4t}\right)\nu\left(d\xi\right). \tag{4.24}$$ PropFriedman_

We denote now by $\mathbf{v}^\nu$ the measure-valued mapping such that $\mathbf{v}^\nu\left(0,\cdot\right)=\nu$ and for all $t\in]0,T]$, $\mathbf{v}^\nu\left(t\right)$ has density $q_t$ with respect to the Lebesgue measure on $\mathbb{R}^d$. We want to show that $\mathbf{v}^\nu$ is a solution of (3.2) with initial value $\nu$ to conclude $\mathbf{u}^\nu=\mathbf{v}^\nu$ thanks to the validity of Assumption 4 because of Remark 3.3 1. and 3. To this end, we remark that the definition of a fundamental solution for $\partial_t u=G_t u$ says that $u$ is a $C^{1,2}$ solution and consequently also a solution in the sense of distributions. In particular for all $\phi\in\mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$, for all $t\geq\epsilon>0$

$$\int_{\mathbb{R}^d}\phi\left(x\right)\mathbf{v}^\nu\left(t\right)\left(dx\right)=\int_{\mathbb{R}^d}\phi\left(x\right)\mathbf{v}^\nu\left(\epsilon\right)\left(dx\right)+\int_\epsilon^t\int_{\mathbb{R}^d}L_s\phi\left(x\right)\mathbf{v}^\nu\left(s\right)\left(dx\right)ds. \tag{4.25}$$ NearFP

To conclude, it remains to send $\epsilon$ to $0^+$. Theorem 15 section 8. Chap 1. and point (ii) of the definition stated p. 27 in [7] imply in particular that for all $\phi\in\mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$, $\xi\in\mathbb{R}^d$,

$$\int_{\mathbb{R}^d}\Gamma\left(x,\epsilon,\xi,0\right)\phi\left(x\right)dx\underset{\epsilon\to0^+}{\longrightarrow}\phi\left(\xi\right).$$

Fix now $\phi\in\mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$. In particular thanks to Fubini's theorem, (4.20) and Lebesgue's dominated conver-

gence theorem we have

$$\begin{aligned}\int_{\mathbb{R}^d} \phi(x)\, \mathbf{v}^\nu(\epsilon)\,(dx) &= \int_{\mathbb{R}^d} \phi(x) \int_{\mathbb{R}^d} \Gamma(x,\epsilon,\xi,0)\,\nu(d\xi)\,dx \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Gamma(x,\epsilon,\xi,0)\,\phi(x)\,dx\nu(d\xi) \\ &\underset{\epsilon\to 0^+}{\longrightarrow} \int_{\mathbb{R}^d} \phi(\xi)\,\nu(d\xi).\end{aligned}$$

By (4.25) $\mathbf{v}^\nu$ is a solution of (3.2) and consequently $\mathbf{u}^\nu = \mathbf{v}^\nu$, so that, for every $t \in ]0,T]$, $\mathbf{u}^\nu(t)$ admits $u^\nu(t,\cdot) = q_t$ for density with respect to the Lebesgue measure on $\mathbb{R}^d$. Now, integrating the inequalities (4.18), (4.19) with respect to $\nu$ and combining this with inequality (4.24), we obtain the existence of $K_1, K_2, C_1, C_2 > 0$ such that for all $t \in ]0,T]$, for all $x \in \mathbb{R}^d$, for all $i \in [\![1,d]\!]$

$$\frac{1}{K_1} t^{-\frac{d}{2}} \int_{\mathbb{R}^d} \exp\left(-\frac{K_2\,|x-\xi|^2}{4t}\right)\nu(d\xi) \le u^\nu(t,x) \le K_1 t^{-\frac{d}{2}},$$

$$|\partial_i u^\nu(t,x)| \le C_1 t^{-\frac{d+1}{2}}.$$

Consequently, the upper bounds in (4.12) and (4.13) hold. Concerning the lower bound in (4.12), let $I$ be a compact subset of $\mathbb{R}^d$ such that $\nu(I) > 0$, the result follows since $(t,x,\xi) \mapsto \exp\left(-\frac{K_2|x-\xi|^2}{4t}\right)$ is strictly positive, continuous and therefore lower bounded by a strictly positive constant on $K \times I$ for each compact $K$ of $]0,T] \times \mathbb{R}^d$.

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## References

[1] D. G. Aronson. Non-negative solutions of linear parabolic equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 22:607–694, 1968.

[2] A. C. Bagtzoglou and J. Atmadja. Marching-jury backward beam equation and quasi-reversibility methods for hydrologic inversion: Application to contaminant plume spatial distribution recovery. *Water Resources Research*, 39(2), 2003.

[3] J. V. Beck, B. Blackwell, and Ch. R. St. Clair Jr. *Inverse heat conduction: Ill-posed problems*. James Beck, 1985.

[4] R. Bronson and G. B. Costa. *Matrix Methods: Applied Linear Algebra*. Academic Press, 2008.

[5] A. Figalli. Existence and uniqueness of martingale solutions for SDEs with rough or degenerate coefficients. *J. Funct. Anal.*, 254(1):109–153, 2008.

[6] H. Föllmer and A. Wakolbinger. Time reversal of infinite-dimensional diffusions. *Stochastic Process. Appl.*, 22(1):59–77, 1986.

[7] A. Friedman. *Partial differential equations of parabolic type*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964.

[8] U. G. Haussmann and É. Pardoux. Time reversal of diffusions. *Ann. Probab.*, 14(4):1188–1205, 1986.

[9] J. Jacod and Ph. Protter. Time reversal on Lévy processes. *Ann. Probab.*, 16(2):620–641, 1988.

[10] I. Karatzas and S. E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[11] R. Lattès and J.-L. Lions. *The method of quasi-reversibility. Applications to partial differential equations*. Translated from the French edition and edited by Richard Bellman. Modern Analytic and Computational Methods in Science and Mathematics, No. 18. American Elsevier Publishing Co., Inc., New York, 1969.

[12] A. Lunardi. *Analytic semigroups and optimal regularity in parabolic problems*, volume 16 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Verlag, Basel, 1995.

[13] M. Renardy, W. J. Hrusa, and J. A. Nohel. *Mathematical problems in viscoelasticity*, volume 35 of *Pitman Monographs and Surveys in Pure and Applied Mathematics*. Longman Scientific & Technical, Harlow; John Wiley & Sons, Inc., New York, 1987.

[14] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

[15] L. C. G. Rogers and D. Williams. *Diffusions, Markov processes, and martingales. Vol. 2*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2000. Itô calculus, Reprint of the second (1994) edition.

[16] W. Rudin. *Functional analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, second edition, 1991.

[17] D. W. Stroock and S. R. S. Varadhan. *Multidimensional diffusion processes*. Classics in Mathematics. Springer-Verlag, Berlin, 2006. Reprint of the 1997 edition.

[18] A. N. Tikhonov and V. Y. Arsenin. *Solutions of ill-posed problems*. V. H. Winston & Sons, Washington, D.C.: John Wiley & Sons, New York-Toronto, Ont.-London, 1977. Translated from the Russian, Preface by translation editor Fritz John, Scripta Series in Mathematics.

[19] A. K. Zvonkin. A transformation of the phase space of a diffusion process that will remove the drift. *Mat. Sb. (N.S.)*, 93(135):129–149, 152, 1974.