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COMMUTATION OF SHINTANI DESCENT AND JORDAN DECOMPOSITION

FRANÇOIS DIGNE AND JEAN MICHEL

Dedicated to the memory of T.A. Springer

ABSTRACT. Let \mathbf{G}^F be a finite group of Lie type, where \mathbf{G} is a reductive group defined over $\overline{\mathbb{F}}_q$ and F is a Frobenius root. Lusztig's Jordan decomposition parametrises the irreducible characters in a rational series $\mathcal{E}(\mathbf{G}^F, (s)_{\mathbf{G}^*F^*})$ where $s \in \mathbf{G}^{*F^*}$ by the series $\mathcal{E}(C_{\mathbf{G}^*}(s)^{F^*}, 1)$. We conjecture that the Shintani twisting preserves the space of class functions generated by the union of the $\mathcal{E}(\mathbf{G}^F, (s')_{\mathbf{G}^*F^*})$ where $(s')_{\mathbf{G}^*F^*}$ runs over the semi-simple classes of \mathbf{G}^{*F^*} geometrically conjugate to s ; further, extending the Jordan decomposition by linearity to this space, we conjecture that there is a way to fix Jordan decomposition such that it maps the Shintani twisting to the Shintani twisting on disconnected groups defined by Deshpande, which acts on the linear span of $\coprod_{s'} \mathcal{E}(C_{\mathbf{G}^*}(s')^{F^*}, 1)$. We show a non-trivial case of this conjecture, the case where \mathbf{G} is of type A_{n-1} with n prime.

1. DESHPANDE'S APPROACH TO SHINTANI DESCENT

We follow [3]. Let \mathbf{H} be an algebraic group over an algebraically closed field k ; we identify \mathbf{H} to its points $\mathbf{H}(k)$ over k . Let γ_1, γ_2 be two commuting bijective isogenies on \mathbf{H} . We define the following subset of $(\mathbf{H} \rtimes \langle \gamma_1 \rangle) \times (\mathbf{H} \rtimes \langle \gamma_2 \rangle)$:

$$R_{\gamma_1, \gamma_2} = \{(x\gamma_1, y\gamma_2) \mid x, y \in \mathbf{H}, [x\gamma_1, y\gamma_2] = 1\}$$

where $[u, v]$ is the commutator $uvu^{-1}v^{-1}$. A matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$ defines a map

$$R_{\gamma_1, \gamma_2} \xrightarrow{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} R_{\gamma_1^a \gamma_2^c, \gamma_1^b \gamma_2^d} : (x\gamma_1, y\gamma_2) \mapsto ((x\gamma_1)^a (y\gamma_2)^c, (x\gamma_1)^b (y\gamma_2)^d)$$

There is an action of \mathbf{H} on R_{γ_1, γ_2} by simultaneous conjugation: $(x\gamma_1, y\gamma_2) \xrightarrow{\mathrm{ad} g} (gx\gamma_1g^{-1}, gy\gamma_2g^{-1})$, which commutes with the maps $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and we denote by $R_{\gamma_1, \gamma_2} / \sim \mathbf{H}$ the space of orbits under this action.

Assume now that \mathbf{H} is connected and γ_1 is a Frobenius root (an isogeny such that some finite power is a Frobenius morphism). Then, by the Lang-Steinberg theorem any $x\gamma_1 \in \mathbf{H}\gamma_1$ is \mathbf{H} -conjugate to γ_1 , and we can take as representatives of the \mathbf{H} -orbits pairs of the form $(\gamma_1, y\gamma_2)$; on these pairs there is only an action of the fixator of γ_1 , that is \mathbf{H}^{γ_1} . Further, the condition $[\gamma_1, y\gamma_2] = 1$ is equivalent to $y \in \mathbf{H}^{\gamma_1}$ (since γ_1 and γ_2 commute). We can thus interpret R_{γ_1, γ_2} as the \mathbf{H}^{γ_1} -conjugacy classes on the coset $\mathbf{H}^{\gamma_1}\gamma_2$, which we denote by $\mathbf{H}^{\gamma_1}\gamma_2 / \sim \mathbf{H}^{\gamma_1}$. If a, c and γ_2 are such that $\gamma_1^a \gamma_2^c$ is still a Frobenius root, we can thus interpret the map

given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as a map $\mathbf{H}^{\gamma_1 \gamma_2} / \sim \mathbf{H}^{\gamma_1} \rightarrow \mathbf{H}^{\gamma_1^a \gamma_2^c \gamma_1^b \gamma_2^d} / \sim \mathbf{H}^{\gamma_1^a \gamma_2^c}$, a “generalised Shintani descent”.

We are interested here in the case of $R_{F, \text{Id}}$ where F is a Frobenius root. The matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ defines a map, that we call the Shintani twisting and will just denote by $\text{Sh} : R_{F, \text{Id}} \rightarrow R_{F, \text{Id}} : (x^F, y) \mapsto (x^F y, y)$. With the identification above of $R_{F, \text{Id}} / \sim \mathbf{H}$ with the pairs (F, y) which identifies it with $\mathbf{H}^F / \sim \mathbf{H}^F$, if we write $y = \lambda \cdot {}^F \lambda^{-1}$ using the Lang-Steinberg theorem, Sh maps (F, y) to $(Fy, y) = (F\lambda \cdot {}^F \lambda^{-1}, \lambda \cdot {}^F \lambda^{-1})$ which is conjugate by ${}^F \lambda^{-1}$ to $(F, {}^F \lambda^{-1} \lambda)$, thus we recover the usual definition of the Shintani twisting Sh as being induced by the correspondence $\lambda \cdot {}^F \lambda^{-1} \mapsto {}^F \lambda^{-1} \lambda$.

The advantage of Deshpande’s approach is that the map $\text{Sh} : R_{F, \text{Id}} \rightarrow R_{F, \text{Id}}$ still makes sense when \mathbf{H} is disconnected; this time the interpretation of $R_{F, \text{Id}} / \sim \mathbf{H}$ is different: the \mathbf{H} -orbits on $\mathbf{H}F$ are parametrised by $H^1(F, \mathbf{H}/\mathbf{H}^0)$. If $\sigma \in \mathbf{H}$ is a representative of such an \mathbf{H} -orbit, the \mathbf{H} -orbits of pairs $(\sigma F, y)$ are in bijection with $\mathbf{H}^{\sigma F} / \sim \mathbf{H}^{\sigma F}$, so we see that we must consider all rational forms of \mathbf{H} corresponding to the various representatives $\sigma F \in H^1(F, \mathbf{H}/\mathbf{H}^0)$ together, and Sh will act on the disjoint union of the conjugacy classes of each of these forms. Let σF be such a representative, let \mathbf{H}_1 be a σF -stable coset of \mathbf{H}^0 in \mathbf{H} , let $\sigma' \in \mathbf{H}_1^{\sigma F}$ and let us compute the image by Sh of a commuting pair $(\sigma F, \sigma' y)$ where $y \in \mathbf{H}^0$ (thus $y \in (\mathbf{H}^0)^{\sigma F}$). Let us write using the Lang-Steinberg theorem $y = \lambda \cdot {}^{\sigma' \sigma F} \lambda^{-1}$. Then Sh maps $(\sigma F, \sigma' y)$ to $(\sigma F \sigma' y, \sigma' y) = (\sigma' \sigma F \lambda \cdot {}^{\sigma' \sigma F} \lambda^{-1}, \sigma' \lambda \cdot {}^{\sigma' \sigma F} \lambda^{-1})$ (using $[\sigma F, \sigma'] = 1$) which is conjugate by ${}^{\sigma' \sigma F} \lambda^{-1}$ to $(\sigma' \sigma F, \sigma' \cdot {}^{\sigma' \sigma F} \lambda^{-1} \lambda)$. Thus

Proposition 1. *Sh is induced on $\sigma'(\mathbf{H}^0)^{\sigma F}$, where σ and σ' are two representatives of \mathbf{H}/\mathbf{H}^0 such that σF and σ' commute, by the correspondence between the element $\sigma' \lambda \cdot {}^{\sigma' \sigma F} \lambda^{-1}$ of $\sigma'(\mathbf{H}^0)^{\sigma F}$ and the element $\sigma' \cdot {}^{\sigma' \sigma F} \lambda^{-1} \lambda$ of $\sigma'(\mathbf{H}^0)^{\sigma' \sigma F}$.*

Assume now that \mathbf{H}/\mathbf{H}^0 is commutative and that we can lift all elements of \mathbf{H}/\mathbf{H}^0 to commuting representatives, such that F -stable elements lift to F -stable representatives; then we can lift all pairs $(\sigma F, \sigma')$ as in Proposition 1 which commute in \mathbf{H}/\mathbf{H}^0 to commuting pairs, thus we can see Sh as a linear map on the space $\mathcal{C}(\mathbf{H}, F) := \bigoplus_{\sigma \in H^1(F, \mathbf{H}/\mathbf{H}^0)} \mathcal{C}(\mathbf{H}^{\sigma F})$, the direct sum of the spaces of class functions on the various rational forms $\mathbf{H}^{\sigma F}$.

Definition 2. *Let \mathbf{H} be an algebraic group with a Frobenius root F , then we say that \mathbf{H} satisfies condition (*) if we can lift elements of \mathbf{H}/\mathbf{H}^0 to commuting elements of \mathbf{H} , such that F -stable elements of \mathbf{H}/\mathbf{H}^0 lift to F -stable elements.*

Beware that Sh on class functions is defined by $\text{Sh}(f)(x) = f(\text{Sh}(x))$ thus it maps class functions on $\sigma'(\mathbf{H}^0)^{\sigma' \sigma F}$ to class functions on $\sigma'(\mathbf{H}^0)^{\sigma F}$.

Note finally that when \mathbf{H}^0 is trivial and F acts trivially on \mathbf{H}/\mathbf{H}^0 , the computations of this section recover the well-known action of $\text{GL}_2(\mathbb{Z})$ on the Drinfeld double of the finite group $\mathbf{H}/\mathbf{H}^0 = \mathbf{H}$. See, for example [2, 8.4.2].

2. CONJECTURES

Definition 3. *We extend the definition of Deligne-Lusztig characters to a disconnected reductive group \mathbf{H} with a Frobenius root F by $R_{\mathbf{T}}^{\mathbf{H}}(s) := \text{Ind}_{\mathbf{H}^0 F}^{\mathbf{H}^F} R_{\mathbf{T}}^{\mathbf{H}^0}(s)$ where \mathbf{T} runs over F -stable maximal tori of \mathbf{H} and s over \mathbf{T}^{*F} . We call unipotent*

characters the irreducible components of the $R_{\mathbf{T}}^{\mathbf{H}}(1)$ and we denote by $\mathcal{E}(\mathbf{H}^F, 1)$ the set of unipotent characters of \mathbf{H}^F .

From now on, \mathbf{G} will be a connected reductive group with a Frobenius root F and s an F^* -stable semi-simple element of \mathbf{G}^* , the group dual to \mathbf{G} . We denote by $\mathcal{E}(\mathbf{G}^F, (s)_{\mathbf{G}^*F^*})$ the \mathbf{G}^{*F^*} -Lusztig series associated with s ; this series consists of the irreducible components of the $R_{\mathbf{T}^*}^{\mathbf{G}}(s)$ where \mathbf{T}^* runs over F -stable maximal tori of $C_{\mathbf{G}^*}(s)$. We denote by $\mathcal{E}(\mathbf{G}^F, (s))$ the geometric Lusztig series associated with s , that is the union of the series $\mathcal{E}(\mathbf{G}^F, (s')_{\mathbf{G}^*F^*})$ where $(s')_{\mathbf{G}^*F^*}$ runs over the \mathbf{G}^{*F^*} -classes in the geometric class of s .

Let $\varepsilon_{\mathbf{G}} := (-1)^{F-\text{rank } \mathbf{G}}$. Lusztig's Jordan decomposition of characters in a reductive group with a not necessarily connected centre is given by the following theorem (see for example [5, Theorem 11.5.1]).

Theorem 4. *Let \mathbf{G} be a connected reductive group with Frobenius root F and (\mathbf{G}^*, F^*) be dual to \mathbf{G} ; for any semi-simple element $s \in \mathbf{G}^{*F^*}$, there is a bijection from $\mathcal{E}(\mathbf{G}^F, (s)_{\mathbf{G}^*F^*})$ to $\mathcal{E}(C_{\mathbf{G}^*}(s)^{F^*}, 1)$. This bijection may be chosen such that, extended by linearity to virtual characters, it sends $\varepsilon_{\mathbf{G}} R_{\mathbf{T}^*}^{\mathbf{G}}(s)$ to $\varepsilon_{C_{\mathbf{G}^*}(s)^0} R_{\mathbf{T}^*}^{C_{\mathbf{G}^*}(s)}(1)$ for any F^* -stable maximal torus \mathbf{T}^* of $C_{\mathbf{G}^*}(s)$.*

Conjecture 5. *Let \mathbf{G} be a connected reductive group with Frobenius root F . Then Sh , viewed as a linear operator on $\mathcal{C}(\mathbf{G}^F)$, preserves the subspace spanned by $\mathcal{E}(\mathbf{G}^F, (s))$ for each geometric class (s) .*

Proposition 6. *Assume that F is the Frobenius morphism corresponding to an \mathbb{F}_q -structure on \mathbf{G} . Then Conjecture 5 holds when the characteristic is almost good for \mathbf{G} in the sense of [7, 1.12] and $Z\mathbf{G}$ is connected or when \mathbf{G} is of type A and either q is large or F acts trivially on $Z\mathbf{G}/(Z\mathbf{G})^0$.*

Proof. The case of type A when F acts trivially on $Z\mathbf{G}/(Z\mathbf{G})^0$ results from [1, Théorème 5.5.4]. Let us prove the other cases. In [9, Section 3.2] it is shown that the characteristic functions of character sheaves are eigenvectors of Sh ; then in [9, Theorem 5.7] and [10, Theorem 3.2 and Theorem 4.1] (resp. [1, Corollaire 24.11]) it is shown that when $Z\mathbf{G}$ is connected and the characteristic is almost good (resp. in type A with q large) the characteristic functions of character sheaves coincide up to scalars with “almost characters”. Since for each s , the space $\mathbb{C}\mathcal{E}(\mathbf{G}^F, (s))$ is spanned by a subset of almost characters, this gives the result. \square

It is probable that the above proof applies to more reductive groups (see [11], [12], and also [1, Introduction]), but it is difficult to find appropriate statements in the literature.

Assuming conjecture 5, a choice of a Jordan decomposition as specified by Theorem 4 will map the operator Sh on $\mathbb{C}\mathcal{E}(\mathbf{G}^F, (s))$ to a linear operator on the space

$$\mathcal{C}(C_{\mathbf{G}^*}(s)^{F^*}, F^*, 1) := \bigoplus_{\sigma \in H^1(F^*, C_{\mathbf{G}^*}(s)/C_{\mathbf{G}^*}^0(s))} \mathbb{C}\mathcal{E}(C_{\mathbf{G}^*}(s)^{\sigma F^*}, 1).$$

Conjecture 7. *Let $s \in \mathbf{G}^{*F^*}$. Then $C_{\mathbf{G}^*}(s)$ satisfies (*) and Sh on $\mathcal{C}(C_{\mathbf{G}^*}(s), F^*)$ preserves the subspace $\mathcal{C}(C_{\mathbf{G}^*}(s)^{F^*}, F^*, 1)$, and the choice of a Jordan decomposition in Theorem 4 may be refined so that it maps Sh on $\mathbb{C}\mathcal{E}(\mathbf{G}^F, (s))$ to Sh on $\mathcal{C}(C_{\mathbf{G}^*}(s)^{F^*}, F^*, 1)$.*

3. THE CASE OF A GROUP OF TYPE A_{n-1} WHEN n IS PRIME.

Proposition 8. *Conjectures 5 and 7 hold for any reductive group of type A_{n-1} , with n prime, if they hold for SL_n .*

Proof. We start with two lemmas.

Lemma 9. *Let $\tilde{\mathbf{G}}$ be a connected reductive group with a Frobenius root F . Let \mathbf{G} be a closed connected F -stable subgroup of $\tilde{\mathbf{G}}$ with same derived group. Let F^* be a Frobenius root dual to F and let $s \in \mathbf{G}^{*F^*}$ be such that $C_{\mathbf{G}^*}(s)^{F^*} = C_{\mathbf{G}^*}^0(s)^{F^*}$. Then the characters in $\mathcal{E}(\mathbf{G}^F, (s)_{\mathbf{G}^*F^*})$ are the restrictions from $\tilde{\mathbf{G}}^F$ to \mathbf{G}^F of the characters in $\mathcal{E}(\tilde{\mathbf{G}}^F, (\tilde{s})_{\tilde{\mathbf{G}}^*F^*})$ where $\tilde{s} \in \tilde{\mathbf{G}}^{*F^*}$ lifts $s \in \mathbf{G}^{*F^*}$.*

Proof. By definition the characters in $\mathcal{E}(\tilde{\mathbf{G}}^F, (\tilde{s})_{\tilde{\mathbf{G}}^*F^*})$ are those which occur in some $R_{\tilde{\mathbf{T}}}^{\tilde{\mathbf{G}}}(\tilde{s})$, where $\tilde{\mathbf{T}}$ is a maximal torus such that $\tilde{\mathbf{T}}^* \ni \tilde{s}$, and by for instance [5, 11.3.10] we have $\mathrm{Res}_{\tilde{\mathbf{G}}^F}^{\tilde{\mathbf{G}}} R_{\tilde{\mathbf{T}}}^{\tilde{\mathbf{G}}}(\tilde{s}) = R_{\tilde{\mathbf{T}} \cap \mathbf{G}}^{\mathbf{G}}(s)$. It is thus sufficient for proving the lemma to prove that for any $\tilde{\chi} \in \mathcal{E}(\tilde{\mathbf{G}}^F, (\tilde{s})_{\tilde{\mathbf{G}}^*F^*})$ its restriction χ to \mathbf{G}^F is irreducible.

By, for instance [5, 11.3.9] we have $\langle \chi, \chi \rangle_{\mathbf{G}^F} = |\{\theta \in \mathrm{Irr}(\tilde{\mathbf{G}}^F/\mathbf{G}^F) \mid \tilde{\chi}\theta = \tilde{\chi}\}|$. It is thus sufficient to prove that for any non-trivial such θ we have $\tilde{\chi}\theta \neq \tilde{\chi}$. But a non-trivial θ corresponds to a non-trivial $z \in Z(\tilde{\mathbf{G}}^*)^{F^*}$ with trivial image in \mathbf{G}^* , and since $\tilde{\chi}\theta \in \mathcal{E}(\tilde{\mathbf{G}}^F, (\tilde{s}z)_{\tilde{\mathbf{G}}^*F^*})$ it is sufficient to prove that \tilde{s} and $\tilde{s}z$ are not $\tilde{\mathbf{G}}^{*F^*}$ -conjugate. But, indeed if ${}^g(\tilde{s}z) = \tilde{s}$ then ${}^{\bar{g}}s = s$ where \bar{g} is the image of g in \mathbf{G}^* , thus $\bar{g} \in C_{\mathbf{G}^*}(s)^{F^*} = C_{\mathbf{G}^*}^0(s)^{F^*}$, whence $g \in C_{\tilde{\mathbf{G}}^*}(\tilde{s})$ (a contradiction) since the preimage of $C_{\mathbf{G}^*}^0(s)$ is $C_{\tilde{\mathbf{G}}^*}^0(\tilde{s})$ (by for example [5, 3.5.1 (i)]). \square

Lemma 10. *Consider a quotient $1 \rightarrow Z \rightarrow \mathbf{G}_1 \xrightarrow{\pi} \mathbf{G} \rightarrow 1$, where Z is a connected subgroup of \mathbf{G}_1 . Assume that \mathbf{G} and \mathbf{G}_1 have Frobenius roots both denoted by F and that π commutes with F ; then π is surjective from the conjugacy classes of \mathbf{G}_1^F to that of \mathbf{G}^F , and commutes with Sh .*

Proof. The group Z is F -stable since π commutes with F . Let $x \in \mathbf{G}^F$. Since Z is connected, by the Lang-Steinberg theorem $\pi^{-1}(x)^F$ is non-empty whence the surjectivity of π on elements, hence on conjugacy classes. If $y \in \pi^{-1}(x)^F$ the image by π of $\mathrm{Sh}(y)$ is $\mathrm{Sh}(x)$, whence the commutation of Sh with π . \square

Let now \mathbf{G} be an arbitrary reductive group of type A_{n-1} . Since n is prime, the only semi-simple connected reductive groups of type A_{n-1} are SL_n and PGL_n ; thus any connected reductive group \mathbf{G} of type A_{n-1} is the almost direct product of the derived \mathbf{G}' of \mathbf{G} , equal to SL_n or PGL_n , by a torus \mathbf{S} .

In the case $\mathbf{G}' = \mathrm{PGL}_n$ the almost direct product is direct since PGL_n has a trivial centre. If \mathbf{T} is an F -stable maximal torus of \mathbf{G} , then it has a decomposition $\mathbf{T}_1 \times \mathbf{S}$ where $\mathbf{T}_1 = \mathbf{T} \cap \mathbf{G}'$ is F -stable since \mathbf{G}' is F -stable. It is possible to find an F -stable complement \mathbf{S}' of \mathbf{T}_1 (see for example the proof of [6, 2.2]) and then \mathbf{G} has an F -stable product decomposition $\mathbf{G}' \times \mathbf{S}'$. Since Sh is trivial on a torus the conjectures are reduced to the case of $\mathbf{G}' = \mathrm{PGL}_n$.

The same argument reduces the case of a direct product $\mathbf{G} = \mathbf{G}' \times \mathbf{S}$ where $\mathbf{G}' = \mathrm{SL}_n$ to the case of SL_n . The other possibility when $\mathbf{G}' = \mathrm{SL}_n$ is an almost direct product $\mathrm{SL}_n \times \mathbf{S}$ amalgamated by $Z\mathrm{SL}_n$; this is isomorphic to a product of the form $\mathrm{GL}_n \times \mathbf{S}'$; in such a group all centralisers are connected, as well as in

the dual group (isomorphic to \mathbf{G}) thus both conjectures are trivial since Sh is the identity (by for example [4, IV, 1.1]).

Finally, in PGL_n the action of Sh is trivial by Lemma 10 applied to the quotient $\text{GL}_n \xrightarrow{\pi} \text{PGL}_n$, and in the dual SL_n semisimple elements have connected centralisers which are Levi subgroups of SL_n ; considering the embedding of these centralisers in the corresponding Levi of GL_n , on which Sh is trivial since it is isomorphic to a product of GL_{n_i} we get that Sh is trivial on the unipotent characters of these centralisers by Lemma 9. \square

We have thus shown that we only need to consider SL_n . Up to isomorphism there are two possible \mathbb{F}_q -structures on SL_n (only one if $n = 2$) thus F will be one of the Frobenius endomorphisms F_+ or F_- where $\text{SL}_n^{F_+} = \text{SL}_n(\mathbb{F}_q)$ and $\text{SL}_n^{F_-} = \text{SU}_n(\mathbb{F}_q)$. When we want to consider both cases simultaneously but keep track whether $F = F_+$ or $F = F_-$, we will denote the Frobenius by F_ε with $\varepsilon \in \{-1, 1\}$, where we always take $\varepsilon = 1$ if $n = 2$. We will use the dual group PGL_n , the inclusion $\text{SL}_n \subset \text{GL}_n$ and the quotient $\text{GL}_n \rightarrow \text{PGL}_n$. Since (GL_n, F) is its own dual, we will write F (instead of F^*) for the Frobenius map on the dual of GL_n and on PGL_n . We choose for F_+ the standard Frobenius which raises all matrix entries to the q -th power, and choose for F_- the map given by $x \mapsto F_+({}^t(x^{-1}))$. This choice is such that on the torus of diagonal matrices \mathbf{T}^* of \mathbf{G}^* , F_- acts by raising all the eigenvalues to the power $-q$, and acts trivially on $W_{\mathbf{G}^*}(\mathbf{T}^*)$. The torus \mathbf{T}^* is split for F_+ and of type w_0 (the longest element of the Weyl group) with respect to a quasi-split torus for F_- .

Proposition 11. *For the group SL_n , n prime, with Frobenius $F = F_\varepsilon$ conjecture 5 holds. Further, conjecture 7 holds if it holds when $q \equiv \varepsilon \pmod{n}$ for the series $\mathcal{E}(\text{SL}_n^F, (s))$ with $s \in \text{PGL}_n^F$ geometrically conjugate to $\text{diag}(1, \zeta, \zeta^2, \dots, \zeta^{n-1})$ where $\zeta \in \overline{\mathbb{F}}_q$ is a non-trivial n -th root of 1.*

Proof. Lemma 9 applied with $\mathbf{G} = \text{SL}_n$ and $\tilde{\mathbf{G}} = \text{GL}_n$ shows that conjectures 5 and 7 hold when $C_{\text{PGL}_n}(s)^F = C_{\text{PGL}_n}^0(s)^F$. Indeed, in this case, since Sh is trivial in GL_n and commutes with the restriction to SL_n by Lemma 10, Lemma 9 shows that Sh is trivial on $\mathcal{E}(\text{SL}_n^F, (s))$; on the other hand, since $C_{\text{PGL}_n}^0(s)$ is the quotient with central kernel of a product of GL_{n_i} , the unipotent characters of $C_{\text{PGL}_n}(s)^F = C_{\text{PGL}_n}^0(s)^F$ can be lifted to this product (see [5, Proposition 11.3.8]). Since Sh is trivial in GL_{n_i} and commutes which such a quotient with connected central kernel by Lemma 10, it is trivial on the unipotent characters of $C_{\text{PGL}_n}(s)^F$.

We have thus reduced the study of conjectures 5 and 7 to the case of semi-simple elements $s \in \text{PGL}_n$ with $C_{\text{PGL}_n}(s)^F \neq C_{\text{PGL}_n}^0(s)^F$.

By [5, Lemma 11.2.1(iii)], since n is prime a semi-simple element $s \in \text{PGL}_n$ has a non-connected centraliser if and only if $|C_{\text{PGL}_n}(s)/C_{\text{PGL}_n}^0(s)| = n$. Since this group of components is a subquotient of the Weyl group \mathfrak{S}_n , it has order n if and only if it is a subgroup of \mathfrak{S}_n generated by an n -cycle. An easy computation shows that such an s is as in the statement of Proposition 11. Now the centraliser of s is the semidirect product of \mathbf{T}^* with the cyclic group generated by the n -cycle $c = (1, \dots, n)$. Assume that the geometric class of s has an F -stable representative s' in a torus of type $w \in \mathfrak{S}_n$ with respect to \mathbf{T}^* (or equivalently that $w^F s = s$). Then $C_{\text{PGL}_n}(s')^F \simeq C_{\text{PGL}_n}(s)^{w^F}$ hence is equal to $C_{\text{PGL}_n}^0(s')^F$ unless w commutes with c , that is $w = c^i$ for some i . Since c centralises s we get that s is F -stable,

which means for $F = F_\varepsilon$ that n divides $q - \varepsilon$, in particular Conjecture 7 holds when $q \not\equiv \varepsilon \pmod{n}$; we recall that we always take $\varepsilon = 1$ when $n = 2$.

Finally, conjecture 5 holds in any case, since Sh preserves all geometric series except possibly that of s and preserves orthogonality of characters, thus preserves also the series $\mathcal{E}(\mathrm{SL}_n^F, (s))$. \square

Henceforth we assume that n is prime and divides $q - \varepsilon$, and study conjecture 7 for the particular s of Proposition 11.

Assumption 12. *We choose ζ , a primitive n -th root of unity in $\overline{\mathbb{F}}_q$. We choose an isomorphism $\overline{\mathbb{F}}_q^\times \simeq (\mathbb{Q}/\mathbb{Z})_{p'}$ which maps ζ to $1/n$ and a group embedding $\mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{C}^\times$ which maps $1/n$ to $\zeta_{\mathbb{C}} := \exp(2i\pi/n)$.*

We denote by \mathbf{T} the diagonal torus of SL_n and by $\tilde{\mathbf{T}}$ the diagonal torus of GL_n ; we choose the dual torus \mathbf{T}^* to be the diagonal torus of PGL_n . We let $s \in \mathbf{T}^*$ be the image of $\mathrm{diag}(1, \zeta, \zeta^2, \dots, \zeta^{n-1}) \in \tilde{\mathbf{T}}$.

Sh on $C_{\mathrm{PGL}_n}(s)$.

We first study Sh on $\mathcal{C}(C_{\mathrm{PGL}_n}(s)^F, F, 1)$. We have $C_{\mathrm{PGL}_n}(s) = \mathbf{T}^* \rtimes \langle c \rangle$ where c is the permutation matrix representing the cycle $(1, 2, \dots, n) \in \mathfrak{S}_n$, which acts on \mathbf{T}^* by sending $\mathrm{diag}(t_1, \dots, t_n)$ to $\mathrm{diag}(t_n, t_1, \dots, t_{n-1})$. On \mathbf{T}^* the Frobenius F_ε acts by $t \mapsto t^{\varepsilon q}$, and acts trivially on $W_{\mathbf{G}^*}(\mathbf{T}^*)$. Note that $C_{\mathbf{G}^*}(s)$ satisfies condition (*) since c is F -stable. Since $C_{\mathrm{PGL}_n}(s)/C_{\mathrm{PGL}_n}^0(s) = \langle c \rangle$, and F acts trivially on c , we have $H^1(F, C_{\mathrm{PGL}_n}(s)/C_{\mathrm{PGL}_n}^0(s)) = \langle c \rangle$ and the geometric class of s splits into n rational classes parametrised by the powers of c . A representative of the class parametrised by c^j is x^s where x is such that $x^{-1}F x = c^j$. This representative lies in a maximal torus $\mathbf{T}_{c^j}^* = {}^x\mathbf{T}^*$ of type c^j with respect to \mathbf{T}^* . Choosing the PGL_n -conjugacy by x^{-1} to identify $(\mathbf{T}_{c^j}^*, F)$ with $(\mathbf{T}^*, c^j F)$, we identify back the representative of the class parametrised by c^j with the element s in $\mathbf{T}^{*c^j F}$. We have

$$\mathcal{C}(C_{\mathrm{PGL}_n}(s)^F, F, 1) = \bigoplus_{j=0}^{n-1} \mathbb{C}\mathcal{E}(\mathbf{T}^{*c^j F} \rtimes \langle c \rangle, 1).$$

Since \mathbf{T}^* is a torus, the unipotent characters of $\mathbf{T}^{*c^j F} \rtimes \langle c \rangle$ are the n extensions of the trivial character of $\mathbf{T}^{*c^j F}$. We parametrise these by $Z\mathrm{SL}_n \times \mathbb{Z}/n\mathbb{Z}$ in the following way: if $z_0 = \mathrm{diag}(\zeta, \dots, \zeta)$, we call $\theta_{z_0^i, k}$ the character of $\mathbf{T}^{*c^k F} \rtimes \langle c \rangle$ which is trivial on $\mathbf{T}^{*c^k F}$ and equal to $\zeta_{\mathbb{C}}^i$ on c . This allows us to define another basis of $\mathcal{C}(C_{\mathrm{PGL}_n}(s)^F, F, 1)$, the ‘‘Mellin transforms’’, defined for $j, k \in \mathbb{Z}/n\mathbb{Z}$ by $\theta_{j, k} := \sum_{z \in Z\mathrm{SL}_n} \hat{s}_j(z) \theta_{z, k}$, where \hat{s}_j is the character of $\mathbf{T}^{*c^j F}$ corresponding to the element $s \in \mathbf{T}^{*c^j F}$ through duality. The point is that it is more convenient to compute the action of Sh on the Mellin transforms:

Proposition 13. *We have $\mathrm{Sh} \theta_{j, i} = \theta_{j, i+j}$ unless $n = 2$ (thus $\varepsilon = 1$) and $q \equiv -1 \pmod{4}$. In this last case we have $\mathrm{Sh} \theta_{j, i} = \theta_{j, i+j-1}$*

Proof.

Lemma 14. *With the conventions of Assumption 12, when n is odd, we have*

$$\hat{s}_j(z_0^i) = \zeta_{\mathbb{C}}^{ij}; \text{ when } n = 2 \text{ (thus } \varepsilon = 1) \text{ we have } \hat{s}_j(z_0) = \begin{cases} (-1)^{(q-1)/2} & \text{if } j = 0 \\ (-1)^{(q+1)/2} & \text{if } j = 1 \end{cases}.$$

Proof. The correspondence between $\mathbf{T}^{*c^j F}$ and $\text{Irr}(\mathbf{T}^{c^j F})$ comes from the diagram of exact sequences

$$\begin{array}{ccccccc} X(\mathbf{T}) & \xrightarrow{c^j F-1} & X(\mathbf{T}) & \xrightarrow{\text{Res}} & \text{Irr}(\mathbf{T}^{c^j F}) & \rightarrow & 1 \\ \parallel & & \parallel & & \downarrow \sim & & \\ Y(\mathbf{T}^*) & \xrightarrow{c^j F-1} & Y(\mathbf{T}^*) & \xrightarrow{\pi} & \mathbf{T}^{*c^j F} & \rightarrow & 1 \end{array}$$

where π maps $y \in Y(\mathbf{T}^*)$ to $N_{F^m/c^j F}(y(1/(q^m-1)))$ for any m such that $(c^j F)^m = F^m$ (see [5, Proposition 11.1.7]); here $y(1/(q^m-1))$ is defined using the choices of Assumption 12 and $N_{F^m/c^j F}$ is the norm map on \mathbf{T}^* given by $x \mapsto x \cdot c^j F x \dots (c^j F)^{m-1} x$.

When $j = 0$ we can take $m = 2$ in the above diagram. We have $s = N_{F^2/F}(y(1/(q^2-1))) = y(1/(\varepsilon q - 1))$ where y is the cocharacter mapping $\lambda \in \overline{\mathbb{F}}_q^\times$ to the image in \mathbf{T}^* of

$$\text{diag}(1, \lambda^{(\varepsilon q-1)/n}, \lambda^{2(\varepsilon q-1)/n}, \dots, \lambda^{(n-1)(\varepsilon q-1)/n}) \in \tilde{\mathbf{T}}.$$

The element y interpreted as a character of \mathbf{T}^F maps $z_0 = \text{diag}(\zeta, \dots, \zeta) \in Z \text{SL}_n$ to $\zeta_{\mathbb{C}}^{\frac{\varepsilon q-1}{n} \frac{n(n-1)}{2}} = \zeta_{\mathbb{C}}^{(n-1)(\varepsilon q-1)/2}$ which is equal to 1 when n is odd—thus in this case \hat{s}_0 is trivial on $Z \text{SL}_n$. If $n = 2$ we have $\varepsilon = 1$ and we get $\hat{s}_0(\text{diag}(-1, -1)) = (-1)^{(q-1)/2}$.

When $j \neq 0$, we can take $m = 2n$. We want to find $y \in Y(\mathbf{T}^*)$ such that $s = N_{F^{2n}/c^j F}(y(1/(q^{2n}-1)))$. Since $(c^j F)^n = F^n$ we have

$$N_{F^{2n}/c^j F}(x) = N_{F^n/c^j F}(x) F^n(N_{F^n/c^j F}(x)).$$

Let y_0 be a cocharacter mapping $\lambda \in \mathbb{F}_{q^{2n}}^\times$ to the image in \mathbf{T}^* of $\text{diag}(\lambda, 1, \dots, 1)$.

We have $(c^j F)^k(\lambda, 1, \dots, 1) = (1, \dots, 1, \lambda^{(\varepsilon q)^k}, 1, \dots)$ where $\lambda^{(\varepsilon q)^k}$ is at the place indexed by $kj + 1$, hence at the place indexed by i in $N_{F^n/c^j F}(y_0(\lambda))$ the exponent of λ is such that $i \equiv kj + 1 \pmod{n}$, which since $\lambda \in \mathbb{F}_{q^n}^\times$ is equivalent to $k = (i-1)j'$ where $jj' \equiv 1 \pmod{n}$. Hence $N_{F^n/c^j F}(y_0(\lambda))$ is the image in \mathbf{T}^* of $\text{diag}(\lambda, \lambda^{(\varepsilon q)^{j'}}, \lambda^{(\varepsilon q)^{2j'}}, \dots, \lambda^{(\varepsilon q)^{(n-1)j'})}$ and $N_{F^{2n}/c^j F}(y_0(\lambda))$ is the image in \mathbf{T}^* of $\text{diag}(\lambda^{1+(\varepsilon q)^n}, \lambda^{(1+(\varepsilon q)^n)(\varepsilon q)^{j'}}, \lambda^{(1+(\varepsilon q)^n)(\varepsilon q)^{2j'}}, \dots, \lambda^{(1+(\varepsilon q)^n)(\varepsilon q)^{(n-1)j'})}$. If we set $\mu = \lambda^{1+(\varepsilon q)^n}$, the element $\text{diag}(1, \mu^{(\varepsilon q)^{j'}-1}, \mu^{(\varepsilon q)^{2j'}-1}, \dots, \mu^{(\varepsilon q)^{(n-1)j'}-1})$ has same image in \mathbf{T}^* . For $\mu^{\varepsilon q-1} = \zeta^j$ we have $\mu^{(\varepsilon q)^{kj'}-1} = \zeta^{j(1+\varepsilon q+\dots+(\varepsilon q)^{kj'-1})} = \zeta^{jkj'} = \zeta^k$, the second equality since $q \equiv \varepsilon \pmod{n}$. Hence $N_{F^{2n}/c^j F}(y_0(\frac{j}{n(\varepsilon q-1)((\varepsilon q)^n+1)})) = s$, thus we can take $y = \frac{j(q^{2n}-1)}{n(\varepsilon q-1)((\varepsilon q)^n+1)} y_0 = \frac{j((\varepsilon q)^n-1)}{n(\varepsilon q-1)} y_0$ (note that $\frac{(\varepsilon q)^n-1}{n(\varepsilon q-1)}$ is an integer). As a character of $\mathbf{T}^{c^j F}$ it maps the element $z_0 = \text{diag}(\zeta, \dots, \zeta) \in Z \text{SL}_n$ to $\zeta_{\mathbb{C}}^{\frac{j((\varepsilon q)^n-1)}{n(\varepsilon q-1)}}$. We now use:

Lemma 15. *If n is odd, $k > 0$ and n^k divides $\varepsilon q - 1$, we have $\frac{(\varepsilon q)^n-1}{n(\varepsilon q-1)} \equiv 1 \pmod{n^k}$.*

Proof. Let us write $\varepsilon q = 1 + an^k$. We have $\frac{(\varepsilon q)^n-1}{\varepsilon q-1} = 1 + \varepsilon q + \dots + (\varepsilon q)^{n-1} \equiv n + a \frac{n(n-1)}{2} n^k \pmod{n^{k+1}}$. Since n is odd it divides $n(n-1)/2$, so that $\frac{(\varepsilon q)^n-1}{\varepsilon q-1} \equiv n \pmod{n^{k+1}}$ which, dividing by n , gives the result. \square

This lemma applied with $k = 1$ shows that \hat{s}_j maps z_0^i to $\zeta_{\mathbb{C}}^{ij}$ when n is odd. If $n = 2$ (thus $\varepsilon = 1$) we get $\hat{s}_1(\text{diag}(-1, -1)) = (-1)^{(q+1)/2}$. \square

We now compute the Mellin transforms $\theta_{j,k}$. If n is odd, we have by Lemma 14

$$\theta_{j,i}(c^k) = \sum_{l \in \mathbb{Z}/n\mathbb{Z}} \hat{s}_j(z_0^l) \theta_{z_0^l, i}(c^k) = \sum_{l \in \mathbb{Z}/n\mathbb{Z}} \zeta_{\mathbb{C}}^{jl} \zeta_{\mathbb{C}}^{lk} = \begin{cases} 0 & \text{if } j+k \neq 0, \\ n & \text{if } j = -k. \end{cases}$$

We see that $\theta_{j,i}$ is a function supported by the coset $\mathbf{T}^{c^i F} \cdot c^{-j} \subset \mathbf{T}^{c^i F} \rtimes \langle c \rangle$ and is constant equal to n on this coset. By proposition 1, Sh maps a constant function on this coset to the constant function on $\mathbf{T}^{c^{i+j} F} \cdot c^{-j}$ with same value, hence $\text{Sh } \theta_{j,i} = \theta_{j,i+j}$.

If $n = 2$ we have

$$\theta_{j,i}(c^k) = (-1)^k \hat{s}_j(z_0) + \hat{s}_j(1) = 1 + \begin{cases} (-1)^{k+(q-1)/2} & \text{if } j = 0, \\ (-1)^{k+(q+1)/2} & \text{if } j = 1. \end{cases}$$

Thus $\theta_{j,i}$ is supported by $\mathbf{T}^{c^i F} \cdot c^j$ if $q \equiv 1 \pmod{4}$ and by $\mathbf{T}^{c^i F} \cdot c^{1-j}$ if $q \equiv -1 \pmod{4}$ and is constant equal to 2 on its support. We get the same result as in the odd case for the action of Sh when $q \equiv 1 \pmod{4}$. If $q \equiv -1 \pmod{4}$, since Sh maps functions on $\mathbf{T}^{c^i} \cdot c^{1-j}$ to functions on $\mathbf{T}^{c^{i+j-1}} \cdot c^{1-j}$ we get $\text{Sh } \theta_{j,i} = \theta_{j,i+j-1}$. \square

Proposition 13 shows that Sh preserves the space $\mathcal{C}(C_{\text{PGL}_n}(s)^F, F, 1)$. Note that the computation made in the proof of Lemma 14 and Definition 3 show that $R_{\mathbf{T}^{c^i} \rtimes \langle c \rangle}^{\mathbf{T}^{c^i}} \text{Id} = \theta_{0,i}$ unless $n = 2$ and $q \equiv -1 \pmod{4}$, in which case $R_{\mathbf{T}^{c^i} \rtimes \langle c \rangle}^{\mathbf{T}^{c^i}} \text{Id} = \theta_{1,i}$.

Sh on SL_n^F .

For computing the other side of conjecture 7, that is Sh on SL_n^F , we first parametrise the characters in $\cup_j \mathcal{E}(\text{SL}_n^F, (s_j)_{\text{PGL}_n^F})$, where $s_j \in (\mathbf{T}_{c^j}^*)^F$ is an F -stable representative of the rational class that we parametrised above by $s \in \mathbf{T}^{*c^j F}$. We use the following notation: for $z \in Z \text{SL}_n$ we denote by Γ_z the Gelfand-Graev character indexed by z as in [5, Definition 12.3.3]; that is $\Gamma_z = \text{Ind}_{\mathbf{U}^F}^{\text{SL}_n^F} \psi_z$ where \mathbf{U} is the unipotent radical of the Borel subgroup consisting of upper triangular matrices and ψ_z is a regular character of \mathbf{U}^F indexed by z . This labelling depends on the choice of a regular character ψ_1 : we have $\psi_z = {}^t \psi_1$ where $t \in \mathbf{T}$ satisfies $t^{-1} F t = z$.

Let $\tilde{s}_j \in (\mathbf{T}_{c^j}^*)^F$ be a lifting of s_j . By [5, Proposition 11.3.10] $R_{\mathbf{T}_{c^j} \cap \text{SL}_n}^{\text{SL}_n}(s_j)$ is the restriction of $R_{\mathbf{T}_{c^j}}^{\text{GL}_n}(\tilde{s}_j)$, hence the series $\mathcal{E}(\text{SL}_n^F, (s_j)_{\text{PGL}_n^F})$ is the set of irreducible components of the restrictions of the elements of $\mathcal{E}(\text{GL}_n^F, (\tilde{s}_j))$. Moreover since $C_{\text{GL}_n}(\tilde{s}_j)$ is a torus, the character $R_{\mathbf{T}_{c^j}}^{\text{GL}_n}(\tilde{s}_j)$ is irreducible, hence is the only character in $\mathcal{E}(\text{GL}_n^F, (\tilde{s}_j))$; thus this character must be a component of the (unique) Gelfand-Graev character of GL_n^F by [5, Theorem 12.4.12]. The restriction of this character to SL_n^F is equal to $\sum_{z \in Z \text{SL}_n} \chi_{z,j}$ where $\chi_{z,j}$ is the unique irreducible component of the Gelfand-Graev character Γ_z in the series $\mathcal{E}(\text{SL}_n^F, (s_j))$ (see [5, Corollary 12.4.10]). Thus we have $\mathcal{E}(\text{SL}_n^F, (s_j)) = \{\chi_{z,j} \mid z \in Z \text{SL}_n\}$. It is again more convenient to compute Sh on the basis formed by the Mellin transforms $\chi_{j,k} := \sum_{z \in Z \text{SL}_n} \hat{s}_j(z) \chi_{z,k}$.

Proposition 16. *We have $\text{Sh } \chi_{j,i} = \chi_{j,i+j}$ unless $n = 2$ (thus $\varepsilon = 1$) and $q \equiv -1 \pmod{4}$. In this last case we have $\text{Sh } \chi_{j,i} = \chi_{j,i+j-1}$.*

Proof. We note first that $\chi_{z,j}(g)$ is independent of z if $C_{\text{SL}_n}(g)$ is connected. Indeed, $\chi_{z,j}$ and $\chi_{z',j}$ are conjugate by an element $x \in \text{GL}_n^F$ since they are two components of the restriction of an irreducible character of GL_n^F . We have thus $\chi_{z',j}(g) = \chi_{z,j}(xg)$. We can multiply x by a central element to obtain an element $y \in \text{SL}_n$ and, since ${}^y g = xg \in \text{SL}_n^F$, we have $y^{-1} \cdot {}^F y \in C_{\text{SL}_n}(g)$. If this centraliser is connected, using the Lang-Steinberg theorem we can multiply y by an element of $C_{\text{SL}_n}(g)$ to get a rational element y' , whence $\chi_{z',j}(g) = \chi_{z,j}(y'g) = \chi_{z,j}(g)$. For such an element g we thus have $\chi_{j,i}(g) = 0$ if $j \neq 0$.

Since $C_{\text{SL}_n}(g)$ is connected we have $\text{Sh}(g) = g$ thus $\text{Sh } \chi_{j,i}(g) = \chi_{j,i}(g)$, in particular $\text{Sh } \chi_{0,i} = \chi_{0,i}$ and if $j \neq 0$ we have $(\text{Sh } \chi_{j,i})(g) = \chi_{j,i+j}(g) = 0$, whence $(\text{Sh } \chi_{j,i})(g) = \chi_{j,i+j}(g)$ for all j .

It remains to consider the conjugacy classes of SL_n which have a non-connected centraliser.

Lemma 17. *When n is prime the only elements of SL_n which have a non-connected centraliser are the zu with $z \in Z\text{SL}_n$ and u regular unipotent.*

Proof. Let su be the Jordan decomposition of an element of SL_n with s semi-simple and u unipotent. We have $C_{\text{SL}_n}(su) = C_{C_{\text{SL}_n}(s)}(u)$. The group $C_{\text{SL}_n}(s)$ is a Levi subgroup of SL_n , that is the subgroup of elements of determinant 1 in a product $\prod_{i=1}^k \text{GL}_{n_i}$. Thus $C_{\text{SL}_n}(su)$ is the subgroup of elements with determinant 1 in a product $\prod_{i=1}^k \mathbf{H}_i$ where \mathbf{H}_i is the centraliser of u in GL_{n_i} , which is connected. We claim that if $k > 1$ the group $C_{\text{SL}_n}(su)$ is connected. Indeed, since n is prime, if $k > 1$ the n_i are coprime so that there exist integers a_i satisfying $\sum_{i=1}^k a_i n_i = -1$. Then the map $(h_1, \dots, h_k) \mapsto (h_1 \lambda^{a_1}, \dots, h_k \lambda^{a_k}, \lambda)$ where $\lambda = \det(h_1 h_2 \dots h_k)$ is an isomorphism from $\mathbf{H}_1 \times \dots \times \mathbf{H}_k$ to $C_{\text{SL}_n}(su) \times \overline{\mathbb{F}}_q$. Hence this last group is connected, thus its projection $C_{\text{SL}_n}(su)$ is also connected.

It remains to look at the centralisers of elements zu with u unipotent and $z \in Z\text{SL}_n$, that is the centralisers of unipotent elements. By [4, IV, Proposition 4.1], since n is prime, $C_{\text{SL}_n}(u)$ is connected unless u is regular and when u is regular we have $C_{\text{SL}_n}(u) = R_u(C_{\text{SL}_n}(u)).Z\text{SL}_n$ which is not connected. \square

Thus to prove the proposition we have only to consider the classes of the elements zu with u regular unipotent and $z \in Z\text{SL}_n$. Fix a rational regular unipotent element u_1 . The conjugacy classes of rational regular unipotent elements are parametrised by $H^1(F, C_{\text{SL}_n}(u_1)/C_{\text{SL}_n}^0(u_1)) = H^1(F, Z\text{SL}_n) = Z\text{SL}_n$ (the last equality since $q \equiv \varepsilon \pmod{n}$): a representative u_z of the class parametrised by the F -class of $z \in Z\text{SL}_n$ is ${}^t u_1$ where $t \in \mathbf{T}$ satisfies $t^{-1} {}^F t = z$. By [4, IV, Proposition 1.2] Sh maps the SL_n^F -class of $zu_{z'}$ to that of $zu_{zz'}$. Now $\chi_{z,k}$ being a component of the restriction of the irreducible character $R_{\mathbf{T}, c^k}^{\text{GL}_n}(\tilde{s}_k)$ has central character equal to \hat{s}_k , independently of z , whence $\chi_{z,k}(z' u_{z''}) = \hat{s}_k(z') \chi_{z,k}(u_{z''})$. By [5, Corollary 12.3.13], there is a family of Gauss sums σ_z indexed by $Z\text{SL}_n$ such that

$$\chi_{z,k}(u_{z''}) = \sum_{z' \in H^1(F, Z\text{SL}_n)} \sigma_{z'' z'^{-1}} \langle (-1)^{F\text{-semi-simple rank}(\text{SL}_n)} D(\chi_{z,k}), \Gamma_{z'} \rangle_{\text{SL}_n^F},$$

where D is the Curtis-Alvis duality. If $k \neq 0$ and $F = F_+$ or if $F = F_-$ we claim that $(-1)^{F\text{-semi-simple rank}(\text{SL}_n)} D(\chi_{z,k}) = \chi_{z,k}$: indeed, in both cases the character

$R_{\tilde{\mathbf{T}}_{c^k}}^{\mathrm{GL}_n}(\tilde{s}_k)$ is cuspidal since the torus $\tilde{\mathbf{T}}_{c^k}$ is not contained in a proper rational Levi subgroup. Indeed when $F = F_+$ and $k \neq 0$ (resp. $F = F_-$), the type c^k (resp. the type $w_0 c^k$) of \mathbf{T}_{c^k} with respect to a quasi-split torus is not contained in a standard parabolic subgroup of \mathfrak{S}_n . Thus $\chi_{z,k}$ is cuspidal as a component of the restriction to SL_n^F of $R_{\tilde{\mathbf{T}}_{c^k}}^{\mathrm{GL}_n}(\tilde{s}_k)$, whence our claim since the duality on cuspidal characters is the multiplication by $(-1)^{F\text{-semi-simple rank}(\mathrm{SL}_n)}$ (take $\mathbf{L} = \mathrm{SL}_n$ in [5, 7.2.9]). If $k = 0$ and $F = F_+$, the characters $\chi_{z,0}$ are the irreducible components of $R_{\mathbf{T}}^{\mathrm{SL}_n}(\hat{s}_0) = \mathrm{Ind}_{\mathbf{B}^F}^{\mathrm{SL}_n^F} \hat{s}_0$, where \mathbf{B} is the Borel subgroup of upper triangular matrices and \hat{s}_0 has been lifted to \mathbf{B}^F . The endomorphism algebra of this induced character is isomorphic to the group algebra of $\mathbb{Z}/n\mathbb{Z}$, hence the components of $\mathrm{Ind}_{\mathbf{B}^F}^{\mathrm{SL}_n^F} \hat{s}_0$ are parametrised by the characters of $\mathbb{Z}/n\mathbb{Z}$ and by [8, Theorem B] the effect of the duality is to multiply the parameters by the sign character of the endomorphism algebra which is trivial if n is odd and is -1 if $n = 2$.

If n is odd or if $n = 2$ and $k \neq 0$, we thus have $\chi_{z,k}(u_{z''}) = \sigma_{z''z^{-1}}$. If $n = 2$ and $k = 0$ we have $\chi_{z,0}(u_{z''}) = -\sigma_{z''z'}$ where $\{z, z'\} = \{1, \mathrm{diag}(-1, -1)\}$.

We consider first the case n odd. By Lemma 14 we have $\hat{s}_k(z_0^i) = \zeta_{\mathbb{C}}^{ki}$. Thus the values of the Mellin transforms are $\chi_{j,k}(z' u_{z''}) = \sum_i \hat{s}_j(z_0^i) \chi_{z_0^i, k}(z' u_{z''}) = \sum_i \hat{s}_j(z_0^i) \hat{s}_k(z') \sigma_{z''z_0^{-i}} = \hat{s}_k(z') \sum_i \zeta_{\mathbb{C}}^{ij} \sigma_{z''z_0^{-i}}$. Let us put $z' = z_0^l$; we get $\chi_{j,k}(z_0^l u_{z''}) = \zeta_{\mathbb{C}}^{lk} \sum_i \zeta_{\mathbb{C}}^{ij} \sigma_{z''z_0^{-i}}$. Now as recalled above we have $\mathrm{Sh}(z' u_{z''}) = z' u_{z''z'}$, whence $(\mathrm{Sh} \chi_{j,k})(z_0^l u_{z''}) = \chi_{j,k}(z_0^l u_{z_0^l z''}) = \zeta_{\mathbb{C}}^{lk} \sum_i \zeta_{\mathbb{C}}^{ij} \sigma_{z''z_0^{l-i}}$. Taking $i-l$ as new variable in the summation we get $(\mathrm{Sh} \chi_{j,k})(z_0^l u_{z''}) = \zeta_{\mathbb{C}}^{lk} \sum_i \zeta_{\mathbb{C}}^{(i+l)j} \sigma_{z''z_0^{-i}} = \zeta_{\mathbb{C}}^{l(j+k)} \sum_i \zeta_{\mathbb{C}}^{ij} \sigma_{z''z_0^{-i}} = \chi_{j,j+k}(z_0^l u_{z''})$, which gives the proposition for n odd.

If $n = 2$ we have $z_0 = \mathrm{diag}(-1, -1)$. We have $\chi_{j,k}(z u_{z''}) = \hat{s}_k(z)(\hat{s}_j(1)\chi_{z_0^0, k}(u_{z''}) + \hat{s}_j(z_0)\chi_{z_0, k}(u_{z''}))$. We get $\chi_{j,0}(z u_{z''}) = \hat{s}_0(z)(-\sigma_{z_0 z''} - \hat{s}_j(z_0)\sigma_{z''})$ and $\chi_{j,1}(z u_{z''}) = \hat{s}_1(z)(\sigma_{z''} + \hat{s}_j(z_0)\sigma_{z_0 z''})$. If \hat{s}_j is the identity character then $\chi_{j,0}$ and $\chi_{j,1}$ are invariant by Sh since $\mathrm{Sh}(u_{z''}) = u_{z''}$ and $\mathrm{Sh}(z_0 u_{z''}) = z_0 u_{z_0 z''}$. If \hat{s}_j is not trivial then, $\chi_{j,0}(z u_{z''})$ and $\chi_{j,1}(z u_{z''})$ are equal if $z = 1$ and opposite if $z \neq 1$, thus, using again $\mathrm{Sh}(u_{z''}) = u_{z''}$ and $\mathrm{Sh}(z_0 u_{z''}) = z_0 u_{z_0 z''}$ we see that $\chi_{j,0}$ and $\chi_{j,1}$ are exchanged by Sh . By Lemma 14, if $q \equiv 1 \pmod{4}$ the character \hat{s}_0 is trivial and we get the same result as in the odd case. If $q \equiv -1 \pmod{4}$, the character \hat{s}_1 is trivial and Sh exchanges $\chi_{0,0}$ and $\chi_{0,1}$ and fixes $\chi_{1,0}$ and $\chi_{1,1}$, which is the announced result. \square

We can now state:

Proposition 18. *For $s = \mathrm{diag}(1, \zeta, \dots, \zeta^{n-1})$, the bijection $J : \chi_{j,i} \mapsto \theta_{j,i}$ from $\mathcal{CE}(\mathrm{SL}_n^F, (s))$ to $\mathcal{CE}(C_{\mathrm{PGL}_n}(s)^F, F, 1)$ restricts on characters to a refinement of the Jordan decomposition which satisfies Conjecture 7.*

Proof. Propositions 13 and 16 give the commutation of J with Sh . It remains to show that $J(R_{\mathbf{T}_{c^j}}^{\mathrm{SL}_n}(s_j)) = R_{\mathbf{T}_{c^j}^*}^{\mathbf{T}_{c^j}^* \rtimes \langle c \rangle}(1)$. As we have noticed after the proof of

Proposition 13 we have $R_{\mathbf{T}_{c^j}}^{\mathbf{T}_{c^j}^* \rtimes \langle c \rangle} \mathrm{Id} = \begin{cases} \theta_{0,j} & \text{for } n \text{ odd or } q \equiv 1 \pmod{4} \\ \theta_{1,j} & \text{for } n = 2, q \equiv -1 \pmod{4} \end{cases}$. On the

other hand for n odd or $q \equiv 1 \pmod{4}$, we have $\chi_{0,j} = \sum_z \hat{s}_0(z) \chi_{z,j} = \sum_z \chi_{z,j}$ since in that case \hat{s}_0 is the trivial character. For $n = 2, q \equiv -1 \pmod{4}$, we have $\chi_{1,j} = \sum_z \hat{s}_1(z) \chi_{z,j} = \sum_z \chi_{z,j}$ since in that case \hat{s}_1 is the trivial character. By definition of $\chi_{z,j}$ we have $\sum_z \chi_{z,j} = R_{\mathbf{T}_{c^j}}^{\mathrm{SL}_n}(s_j)$, whence the proposition. \square

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