



Commutation of Shintani descent and Jordan decomposition

François Digne, Jean Michel

► To cite this version:

François Digne, Jean Michel. Commutation of Shintani descent and Jordan decomposition. 2020. hal-02901155v2

HAL Id: hal-02901155

<https://hal.science/hal-02901155v2>

Preprint submitted on 20 Jul 2020 (v2), last revised 12 Oct 2020 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

COMMUTATION OF SHINTANI DESCENT AND JORDAN DECOMPOSITION

FRANÇOIS DIGNE AND JEAN MICHEL

ABSTRACT. Let \mathbf{G}^F be a finite group of Lie type, where \mathbf{G} is a reductive group defined over $\overline{\mathbb{F}}_q$ and F is a Frobenius root. Lusztig's Jordan decomposition parametrises the irreducible characters in a rational series $\mathcal{E}(\mathbf{G}^F, (s)_{\mathbf{G}^{*F^*}})$ where $s \in \mathbf{G}^{*F^*}$ by the series $\mathcal{E}(C_{\mathbf{G}^*}(s)^{F^*}, 1)$. We conjecture that the Shintani twisting preserves the space of class functions generated by the union of the $\mathcal{E}(\mathbf{G}^F, (s')_{\mathbf{G}^{*F^*}})$ where $(s')_{\mathbf{G}^{*F^*}}$ runs over the semi-simple classes of \mathbf{G}^{*F^*} geometrically conjugate to s ; further, extending the Jordan decomposition by linearity to this space, we conjecture that there is a way to fix Jordan decomposition such that it maps the Shintani twisting to the Shintani twisting on disconnected groups defined by Deshpande, which acts on the linear span of $\coprod_{s'} \mathcal{E}(C_{\mathbf{G}^*}(s')^{F^*}, 1)$. We show a non-trivial case of this conjecture, the case where \mathbf{G} is of type A_{n-1} with n prime.

1. DESHPANDE'S APPROACH TO SHINTANI DESCENT

We follow [3]. Let \mathbf{H} be an algebraic group over an algebraically closed field k ; we identify \mathbf{H} to its points $\mathbf{H}(k)$ over k . Let γ_1, γ_2 be two commuting bijective isogenies on \mathbf{H} . We define the following subset of $(\mathbf{H} \rtimes \langle \gamma_1 \rangle) \times (\mathbf{H} \rtimes \langle \gamma_2 \rangle)$:

$$R_{\gamma_1, \gamma_2} = \{(x\gamma_1, y\gamma_2) \mid x, y \in \mathbf{H}, [x\gamma_1, y\gamma_2] = 1\}$$

where $[u, v]$ is the commutator $uvu^{-1}v^{-1}$. A matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$ defines a map

$$R_{\gamma_1, \gamma_2} \xrightarrow{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} R_{\gamma_1^a \gamma_2^c, \gamma_1^b \gamma_2^d} : (x\gamma_1, y\gamma_2) \mapsto ((x\gamma_1)^a (y\gamma_2)^c, (x\gamma_1)^b (y\gamma_2)^d)$$

There is an action of \mathbf{H} on R_{γ_1, γ_2} by simultaneous conjugation: $(x\gamma_1, y\gamma_2) \xrightarrow{\mathrm{ad} g} (gx\gamma_1g^{-1}, gy\gamma_2g^{-1})$, which commutes with the maps $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and we denote by $R_{\gamma_1, \gamma_2} / \sim \mathbf{H}$ the space of orbits under this action.

Assume now that \mathbf{H} is connected and γ_1 is a Frobenius root (an isogeny such that some finite power is a Frobenius morphism). Then, by the Lang-Steinberg theorem any $x\gamma_1 \in \mathbf{H}\gamma_1$ is \mathbf{H} -conjugate to γ_1 , and we can take as representatives of the \mathbf{H} -orbits pairs of the form $(\gamma_1, y\gamma_2)$; on these pairs there is only an action of the fixator of γ_1 , that is \mathbf{H}^{γ_1} . Further, the condition $[\gamma_1, y\gamma_2] = 1$ is equivalent to $y \in \mathbf{H}^{\gamma_1}$ (since γ_1 and γ_2 commute). We can thus interpret R_{γ_1, γ_2} as the \mathbf{H}^{γ_1} -conjugacy classes on the coset $\mathbf{H}^{\gamma_1}\gamma_2$, which we denote by $\mathbf{H}^{\gamma_1}\gamma_2 / \sim \mathbf{H}^{\gamma_1}$. If a, b and γ_2 are such that $\gamma_1^a \gamma_2^c$ is still a Frobenius root, we can thus interpret the map

given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as a map $\mathbf{H}^{\gamma_1}\gamma_2/\sim \mathbf{H}^{\gamma_1} \rightarrow \mathbf{H}^{\gamma_1^a\gamma_2^c}\gamma_1^b\gamma_2^d/\sim \mathbf{H}^{\gamma_1^a\gamma_2^c}$, a “generalised Shintani descent”.

We are interested here by the case of $R_{F,\text{Id}}$ where F is a Frobenius root. The matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ defines a map, that we call the Shintani twisting and will just denote by $\text{Sh} : R_{F,\text{Id}} \rightarrow R_{F,\text{Id}} : (xF, y) \mapsto (xFy, y)$. With the identification above of $R_{F,\text{Id}}/\sim \mathbf{H}$ with the pairs (F, y) which identifies it with $\mathbf{H}^F/\sim \mathbf{H}^F$, if we write $y = \lambda \cdot {}^F\lambda^{-1}$ using the Lang-Steinberg theorem, Sh maps (F, y) to $(Fy, y) = (F\lambda \cdot {}^F\lambda^{-1}, \lambda \cdot {}^F\lambda^{-1})$ which is conjugate by ${}^F\lambda^{-1}$ to $(F, {}^F\lambda^{-1}\lambda)$, thus we recover the usual definition of the Shintani twisting Sh as being induced by the correspondence $\lambda \cdot {}^F\lambda^{-1} \mapsto {}^F\lambda^{-1}\lambda$.

The advantage of Deshpande’s approach is that the map $\text{Sh} : R_{F,\text{Id}} \rightarrow R_{F,\text{Id}}$ still makes sense when \mathbf{H} is disconnected; this time the interpretation of $R_{F,\text{Id}}/\sim \mathbf{H}$ is different: the \mathbf{H} -orbits on \mathbf{H}^F are parametrised by $H^1(F, \mathbf{H}/\mathbf{H}^0)$. If $\sigma \in \mathbf{H}$ is a representative of such a \mathbf{H} -orbit, the \mathbf{H} -orbits of pairs $(\sigma F, y)$ are in bijection with $\mathbf{H}^{\sigma F}/\sim \mathbf{H}^{\sigma F}$, so we see that we must consider all rational forms of \mathbf{H} corresponding to the various representatives $\sigma F \in H^1(F, \mathbf{H}/\mathbf{H}^0)$ together, and Sh will act on the disjoint union of the conjugacy classes of each of these forms. Let σF be such a representative, let \mathbf{H}_1 be a σF -stable coset of \mathbf{H}^0 in \mathbf{H} , let $\sigma' \in \mathbf{H}_1^{\sigma F}$ and let us compute the image by Sh of a commuting pair $(\sigma F, \sigma' y)$ where $y \in \mathbf{H}^0$ (thus $y \in (\mathbf{H}^0)^{\sigma F}$). Let us write using the Lang-Steinberg theorem $y = \lambda \cdot {}^{\sigma' \sigma F}\lambda^{-1}$. Then Sh maps $(\sigma F, \sigma' y)$ to $(\sigma F \sigma' y, \sigma' y) = (\sigma' \sigma F \lambda \cdot {}^{\sigma' \sigma F}\lambda^{-1}, \sigma' \lambda \cdot {}^{\sigma' \sigma F}\lambda^{-1})$ (using $[\sigma F, \sigma'] = 1$) which is conjugate by ${}^{\sigma' \sigma F}\lambda^{-1}$ to $(\sigma' \sigma F, \sigma' \cdot {}^{\sigma' \sigma F}\lambda^{-1}\lambda)$. Thus

Proposition 1. *Sh is induced on $\sigma'(\mathbf{H}^0)^{\sigma F}$, where σ and σ' are two representatives of \mathbf{H}/\mathbf{H}^0 such that σF and σ' commute, by the correspondence between the element $\sigma' \lambda \cdot {}^{\sigma' \sigma F}\lambda^{-1}$ of $\sigma'(\mathbf{H}^0)^{\sigma F}$ and the element $\sigma' \cdot {}^{\sigma' \sigma F}\lambda^{-1}\lambda$ of $\sigma'(\mathbf{H}^0)^{\sigma' \sigma F}$.*

Assume now that \mathbf{H}/\mathbf{H}^0 is commutative and that we can lift all elements of \mathbf{H}/\mathbf{H}^0 to commuting representatives, such that F -stable elements lift to F -stable representatives; then we can lift all pairs $(\sigma F, \sigma')$ as in Proposition 1 which commute in \mathbf{H}/\mathbf{H}^0 to commuting pairs, thus we can see Sh as a linear map on the space $\mathcal{C}(\mathbf{H}, F) := \bigoplus_{\sigma \in H^1(F, \mathbf{H}/\mathbf{H}^0)} \mathcal{C}(\mathbf{H}^{\sigma F})$, the direct sum of the spaces of class functions on the various rational forms $\mathbf{H}^{\sigma F}$.

Beware that Sh on class functions is defined by $\text{Sh}(f)(x) = f(\text{Sh}(x))$ thus it maps class functions on $\sigma'(\mathbf{H}^0)^{\sigma' \sigma F}$ to class functions on $\sigma'(\mathbf{H}^0)^{\sigma F}$.

Note finally that when \mathbf{H}^0 is trivial and F acts trivially on \mathbf{H}/\mathbf{H}^0 , the computations of this section recover the well-known action of $\text{GL}_2(\mathbb{Z})$ on the Drinfeld double of the finite group $\mathbf{H}/\mathbf{H}^0 = \mathbf{H}$. See, for example [2, 8.4.2].

2. CONJECTURES

Definition 2. *We extend the definition of Deligne-Lusztig characters to a disconnected reductive group \mathbf{H} with a Frobenius root F by $R_{\mathbf{T}}^{\mathbf{H}}(s) := \text{Ind}_{\mathbf{H}^0 F}^{\mathbf{H}^F} R_{\mathbf{T}}^{\mathbf{H}^0}(s)$ where \mathbf{T} runs over F -stable maximal tori of \mathbf{H} and s over \mathbf{T}^{*F*} . We call unipotent characters the irreducible components of the $R_{\mathbf{T}}^{\mathbf{H}}(1)$ and we denote by $\mathcal{E}(\mathbf{H}^F, 1)$ the set of unipotent characters of \mathbf{H}^F .*

From now on, \mathbf{G} will be a connected reductive group with a Frobenius root F and s an F^* -stable semi-simple element of \mathbf{G}^* , the group dual to \mathbf{G} . We denote

by $\mathcal{E}(\mathbf{G}^F, (s)_{\mathbf{G}^{*F^*}})$ the \mathbf{G}^{*F^*} -Lusztig series associated with s ; this series consists of the irreducible components of the $R_{\mathbf{T}^*}^{\mathbf{G}}(s)$ where \mathbf{T}^* runs over F -stable maximal tori of $C_{\mathbf{G}^*}(s)$. We denote by $\mathcal{E}(\mathbf{G}^F, (s))$ the geometric Lusztig series associated with s , that is the union of the series $\mathcal{E}(\mathbf{G}^F, (s')_{\mathbf{G}^{*F^*}})$ where $(s')_{\mathbf{G}^{*F^*}}$ runs over the \mathbf{G}^{*F^*} -classes in the geometric class of s .

Let $\varepsilon_{\mathbf{G}} := (-1)^{F - \text{rank } \mathbf{G}}$. Lusztig's Jordan decomposition of characters in a reductive group with a non necessarily connected centre is given by the following theorem (see for example [5, Theorem 11.5.1]).

Theorem 3. *Let \mathbf{G} be a connected reductive group with Frobenius root F and (\mathbf{G}^*, F^*) be dual to \mathbf{G} ; for any semi-simple element $s \in \mathbf{G}^{*F^*}$, there is a bijection from $\mathcal{E}(\mathbf{G}^F, (s)_{\mathbf{G}^{*F^*}})$ to $\mathcal{E}(C_{\mathbf{G}^*}(s)^{F^*}, 1)$. This bijection may be chosen such that, extended by linearity to virtual characters, it sends $\varepsilon_{\mathbf{G}} R_{\mathbf{T}^*}^{\mathbf{G}}(s)$ to $\varepsilon_{C_{\mathbf{G}^*}(s)^0} R_{\mathbf{T}^*}^{C_{\mathbf{G}^*}(s)}(1)$ for any F^* -stable maximal torus \mathbf{T}^* of $C_{\mathbf{G}^*}(s)$.*

Conjecture 4. *Let \mathbf{G} be a connected reductive group with Frobenius root F . Then Sh , viewed as a linear operator on $\mathcal{C}(\mathbf{G}^F)$, preserves the subspace spanned by $\mathcal{E}(\mathbf{G}^F, (s))$ for each geometric class (s) .*

Proposition 5. *Assume that F is the Frobenius morphism corresponding to an \mathbb{F}_q -structure on \mathbf{G} . Then Conjecture 4 holds when the characteristic is almost good for \mathbf{G} and $Z\mathbf{G}$ is connected or when \mathbf{G} is of type A and q is large or F acts trivially on $Z\mathbf{G}/(Z\mathbf{G})^0$.*

Proof. The case of type A when F acts trivially on $Z\mathbf{G}/(Z\mathbf{G})^0$ results from [1, Théorème 5.5.4]. Let us prove the other cases. In [8, Section 3.2] it is shown that the characteristic functions of character sheaves are eigenvectors of Sh ; then in [8, Theorem 5.7] and [9, Theorem 3.2 and Theorem 4.1] (resp. [1, Corollaire 24.11]) it is shown that when $Z\mathbf{G}$ is connected and the characteristic is almost good (resp. in type A with q large) the characteristic functions of character sheaves coincide up to scalars with “almost characters”. Since for each s , the space $\mathbb{C}\mathcal{E}(\mathbf{G}^F, (s))$ is spanned by a subset of almost characters, this gives the result. \square

It is probable that the above proof applies to more reductive groups (see [10], [11], and also [1, Introduction]), but it is difficult to find appropriate statements in the literature.

Before stating the next conjecture, let us state a condition on centralizers of semi-simple elements

Definition 6. *Let \mathbf{G} be a connected reductive group with a Frobenius root F , let s be a semi-simple element of \mathbf{G}^F , and let $\mathbf{H} = C_{\mathbf{G}}(s)$. Then we say that s satisfies condition $(*)$ if we can lift elements of \mathbf{H}/\mathbf{H}^0 to commuting elements of \mathbf{H} , such that F -stable elements of \mathbf{H}/\mathbf{H}^0 lift to F -stable elements.*

We conjecture that condition $(*)$ is automatically satisfied.

Assuming conjecture 4, and that $s \in \mathbf{G}^{*F^*}$ satisfies $(*)$, a choice of a Jordan decomposition as specified by Theorem 3 will map Sh to a linear operator on the space

$$\mathcal{C}(C_{\mathbf{G}^*}(s)^{F^*}, F^*, 1) := \oplus_{\sigma \in H^1(F^*, C_{\mathbf{G}^*}(s)/C_{\mathbf{G}^*}^0(s))} \mathbb{C}\mathcal{E}(C_{\mathbf{G}^*}(s)^{\sigma F^*}, 1).$$

Conjecture 7. *Let $s \in \mathbf{G}^{*F^*}$. Then s satisfies $(*)$ and Sh on $\mathcal{C}(C_{\mathbf{G}^*}(s), F^*)$ preserves the subspace $\mathcal{C}(C_{\mathbf{G}^*}(s)^{F^*}, F^*, 1)$, and the choice of a Jordan decomposition in Theorem 3 may be refined so that it maps Sh on $\mathcal{CE}(\mathbf{G}^F, (s))$ to Sh on $\mathcal{C}(C_{\mathbf{G}^*}(s)^{F^*}, F^*, 1)$.*

3. THE CASE OF A GROUP OF TYPE A_{n-1} WHEN n IS PRIME.

Lemma 8. *Let $\mathbf{G} \subset \tilde{\mathbf{G}}$ be an embedding of connected reductive groups which induces an isomorphism on derived groups, let F and F^* be dual Frobenius roots, and let $s \in \mathbf{G}^{*F^*}$ be such that $C_{\mathbf{G}^*}(s)^{F^*} = C_{\mathbf{G}^*}^0(s)^{F^*}$. Then the characters in $\mathcal{E}(\mathbf{G}^F, (s)_{\mathbf{G}^{*F^*}})$ are the restrictions from $\tilde{\mathbf{G}}^F$ to \mathbf{G}^F of the characters in $\mathcal{E}(\tilde{\mathbf{G}}^F, (\tilde{s})_{\tilde{\mathbf{G}}^{*F^*}})$ where $\tilde{s} \in \tilde{\mathbf{G}}^{*F^*}$ lifts $s \in \mathbf{G}^{*F^*}$.*

Proof. By definition the characters in $\mathcal{E}(\tilde{\mathbf{G}}^F, (\tilde{s})_{\tilde{\mathbf{G}}^{*F^*}})$ are those which occur in some $R_{\tilde{\mathbf{T}}}^{\tilde{\mathbf{G}}}(\tilde{s})$, where $\tilde{\mathbf{T}}$ is a maximal torus such that $\tilde{\mathbf{T}}^* \ni \tilde{s}$, and by for instance [5, 11.3.10] we have $\text{Res}_{\tilde{\mathbf{T}} \cap \mathbf{G}}^{\tilde{\mathbf{G}}} R_{\tilde{\mathbf{T}}}^{\tilde{\mathbf{G}}}(\tilde{s}) = R_{\tilde{\mathbf{T}} \cap \mathbf{G}}^{\mathbf{G}}(s)$. It is thus sufficient for proving the lemma to prove that for any $\tilde{\chi} \in \mathcal{E}(\tilde{\mathbf{G}}^F, (\tilde{s})_{\tilde{\mathbf{G}}^{*F^*}})$ its restriction χ to \mathbf{G}^F is irreducible.

By, for instance [5, 11.3.9] we have $\langle \chi, \chi \rangle_{\mathbf{G}^F} = |\{\theta \in \text{Irr}(\tilde{\mathbf{G}}^F/\mathbf{G}^F) \mid \tilde{\chi}\theta = \tilde{\chi}\}|$. It is thus sufficient to prove that for any non-trivial such θ we have $\tilde{\chi}\theta \neq \tilde{\chi}$. But a non-trivial θ corresponds to a non-trivial $z \in Z(\tilde{\mathbf{G}}^*)^{F^*}$ with trivial image in \mathbf{G}^* , and since $\tilde{\chi}\theta \in \mathcal{E}(\tilde{\mathbf{G}}^F, (\tilde{s}z)_{\tilde{\mathbf{G}}^{*F^*}})$ it is sufficient to prove that \tilde{s} and $\tilde{s}z$ are not $\tilde{\mathbf{G}}^{*F^*}$ -conjugate. But, indeed if ${}^g(\tilde{s}z) = \tilde{s}$ then ${}^{\bar{g}}s = s$ where \bar{g} is the image of g in \mathbf{G}^* , thus $\bar{g} \in C_{\mathbf{G}^*}(s)^{F^*} = C_{\mathbf{G}^*}^0(s)^{F^*}$, whence $g \in C_{\tilde{\mathbf{G}}^*}(\tilde{s})$ (a contradiction) since the preimage of $C_{\mathbf{G}^*}^0(s)$ is $C_{\tilde{\mathbf{G}}^*}^0(\tilde{s})$ (by for example [5, 3.5.1 (i)]). \square

Lemma 9. *Consider a quotient $1 \rightarrow Z \rightarrow \mathbf{G}_1 \xrightarrow{\pi} \mathbf{G} \rightarrow 1$ where Z is a connected F -stable subgroup of $Z\mathbf{G}_1$. Then π is surjective between conjugacy classes of \mathbf{G}_1^F and of \mathbf{G}^F , and commutes with Sh .*

Proof. Let $x \in \mathbf{G}^F$. Since Z is connected, by the Lang-Steinberg theorem $\pi^{-1}(x)^F$ is non-empty whence the surjectivity of π on conjugacy classes. If $y \in \pi^{-1}(x)^F$ the image by π of $\text{Sh}(y)$ is $\text{Sh}(x)$, whence the commutation of Sh with π . \square

Let us show now that if \mathbf{G} is of type A_{n-1} , then conjectures 4 and 7 hold if they hold for SL_n . Since n is prime, the only semi-simple connected reductive groups of type A_{n-1} are SL_n and PGL_n ; thus any connected reductive group \mathbf{G} of type A_{n-1} is the almost direct product of the derived \mathbf{G}' of \mathbf{G} , equal to SL_n or PGL_n , by a torus \mathbf{S} .

In the case $\mathbf{G}' = \text{PGL}_n$ the almost direct product is direct since PGL_n has a trivial center. If \mathbf{T} is an F -stable maximal torus of \mathbf{G} , then it has a decomposition $\mathbf{T}_1 \times \mathbf{S}$ where $\mathbf{T}_1 = \mathbf{T} \cap \mathbf{G}'$ is F -stable since \mathbf{G}' is F -stable. It is possible to find an F -stable complement \mathbf{S}' of \mathbf{T}_1 (see for example the proof of [6, 2.2]) and then \mathbf{G} has an F -stable product decomposition $\mathbf{G}' \times \mathbf{S}'$. Since Sh is trivial on a torus the conjectures are reduced to the case of $\mathbf{G}' = \text{PGL}_n$.

The same argument reduces the case of a direct product $\mathbf{G} = \mathbf{G}' \times \mathbf{S}$ where $\mathbf{G}' = \text{SL}_n$ to the case of SL_n . The other possibility when $\mathbf{G}' = \text{SL}_n$ is an almost direct product $\text{SL}_n \rtimes \mathbf{S}$ amalgamated by $Z\text{SL}_n$; this is isomorphic to a product of the form $\text{GL}_n \rtimes \mathbf{S}'$; in such a group all centralizers are connected, as well as in

the dual group (isomorphic to \mathbf{G}) thus both conjectures are trivial since Sh is the identity (by for example [4, IV, 1.1]).

Finally, in PGL_n the action of Sh is trivial by Lemma 9 applied to the quotient $\mathrm{GL}_n \xrightarrow{\pi} \mathrm{PGL}_n$, and in the dual SL_n semisimple elements have connected centralizers which are Levi subgroups of SL_n ; considering the embedding of these centralizers in the corresponding Levi of GL_n , on which Sh is trivial since it is isomorphic to a product of GL_{n_i} we get that Sh is trivial on the unipotent characters of these centralizers by Lemma 8.

We have thus shown that we only need to consider SL_n . Up to isomorphism there are two possible \mathbb{F}_q -structures on SL_n (only one if $n = 2$) thus F will be one of the Frobenius endomorphisms F_+ or F_- where $\mathrm{SL}_n^{F_+} = \mathrm{SL}_n(\mathbb{F}_q)$ and $\mathrm{SL}_n^{F_-} = \mathrm{SU}_n(\mathbb{F}_q)$. We will use the dual group PGL_n , the inclusion $\mathrm{SL}_n \subset \mathrm{GL}_n$ and the quotient $\mathrm{GL}_n \rightarrow \mathrm{PGL}_n$. Since (GL_n, F_{\pm}) is its own dual, we will write F_{\pm} (instead of F_{\pm}^*) for the Frobenius map on the dual of GL_n and on PGL_n . We choose for F_+ the standard Frobenius which raises all matrix entries to the q -th power, and choose for F_- the map given by $x \mapsto F_+({}^t(x^{-1}))$. This choice is such that on the torus of diagonal matrices \mathbf{T}^* of \mathbf{G}^* , F_- acts by raising all the eigenvalues to the power $-q$, and acts trivially on $W_{\mathbf{G}^*}(\mathbf{T}^*)$. The torus \mathbf{T}^* is split for F_+ and of type w_0 (the longest element of the Weyl group) with respect to a quasi-split torus for F_- .

Lemma 8 applied with $\mathbf{G} = \mathrm{SL}_n$ and $\tilde{\mathbf{G}} = \mathrm{GL}_n$ shows that conjectures 4 and 7 hold when $C_{\mathrm{PGL}_n}(s)^F = C_{\mathrm{PGL}_n}^0(s)^F$. Indeed, in this case, since Sh is trivial in GL_n and commutes with the restriction to SL_n by Lemma 9, Lemma 8 shows that Sh is trivial on $\mathcal{E}(\mathrm{SL}_n^F, (s))$; on the other hand, since $C_{\mathrm{PGL}_n}^0(s)$ is the quotient with central kernel of a product of GL_{n_i} , the unipotent characters of $C_{\mathrm{PGL}_n}(s)^F = C_{\mathrm{PGL}_n}^0(s)^F$ can be lifted to this product (see [5, Proposition 11.3.8]). Since Sh is trivial in GL_{n_i} and commutes with such a quotient with connected central kernel by Lemma 9, it is trivial on the unipotent characters of $C_{\mathrm{PGL}_n}(s)^F$.

We have thus reduced the study of conjectures 4 and 7 to the case of semi-simple elements $s \in \mathrm{PGL}_n$ with $C_{\mathrm{PGL}_n}(s)^F \neq C_{\mathrm{PGL}_n}^0(s)^F$.

By [5, Lemma 11.2.1(iii)], since n is prime a semi-simple element $s \in \mathrm{PGL}_n$ has a non-connected centraliser if and only if $|C_{\mathrm{PGL}_n}(s)/C_{\mathrm{PGL}_n}^0(s)| = n$. Since this group of components is a subquotient of the Weyl group \mathfrak{S}_n , it has order n if and only if it is a subgroup of \mathfrak{S}_n generated by an n -cycle. An easy computation shows that up to conjugacy such an s is the image in PGL_n of the matrix $\mathrm{diag}(1, \zeta, \zeta^2, \dots, \zeta^{n-1}) \in \mathbf{T}^*$ where ζ is a primitive (that is, non-trivial, since n is prime) n -th root of 1. Now the centralizer of s is the semidirect product of \mathbf{T}^* with the cyclic group generated by the n -cycle $c = (1, \dots, n)$. Assume that the geometric class of $s = \mathrm{diag}(1, \zeta, \zeta^2, \dots, \zeta^{n-1})$ has an F -stable representative s' in a torus of type $w \in \mathfrak{S}_n$ with respect to \mathbf{T}^* (or equivalently that ${}^w F s = s$). Then $C_{\mathrm{PGL}_n}(s')^F \simeq C_{\mathrm{PGL}_n}(s)^{wF}$ hence is equal to $C_{\mathrm{PGL}_n}^0(s')^F$ unless w commutes with c , that is $w = c^i$ for some i . Since c centralizes s we get that s is F -stable, which means that n divides $q - \varepsilon$ when $F = F_{\varepsilon}$ where we write F_{ε} for F_+ , (resp. F_-) for $\varepsilon = +1$, (resp. -1); we recall that we always take $\varepsilon = 1$ when $n = 2$.

So the proof of the conjectures 4 and 7 is reduced to the series $\mathcal{E}(\mathrm{SL}_n^F, (s))$ with $s \in \mathrm{PGL}_n^F$ geometrically conjugate to $\mathrm{diag}(1, \zeta, \zeta^2, \dots, \zeta^{n-1})$, and both conjectures are true if $q \not\equiv \varepsilon \pmod{n}$. Further, conjecture 4 holds in any case, since Sh preserves all geometric series except that of s and preserves orthogonality of characters, thus preserves also the series $\mathcal{E}(\mathrm{SL}_n^F, (s))$.

Henceforth we assume that n is prime and divides $q - \varepsilon$, and study conjecture 7 for this particular s .

Assumption 10. *We choose ζ , an n -th root of unity in \mathbb{F}_q . We choose an isomorphism $\overline{\mathbb{F}}_q^\times \simeq (\mathbb{Q}/\mathbb{Z})_{p'}$ which maps ζ to $1/n$ and a group embedding $\mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{C}^\times$ which maps $1/n$ to $\zeta_{\mathbb{C}} := \exp(2i\pi/n)$.*

We denote by \mathbf{T} the diagonal torus of SL_n and $\tilde{\mathbf{T}}$ the diagonal torus of GL_n ; we choose the dual torus \mathbf{T}^* to be the diagonal torus of PGL_n . Let $s \in \mathbf{T}^*$ be the image of $\mathrm{diag}(1, \zeta, \zeta^2, \dots, \zeta^{n-1}) \in \tilde{\mathbf{T}}$.

Sh on $C_{\mathrm{PGL}_n}(s)$.

We first study Sh on $\mathcal{C}(C_{\mathrm{PGL}_n}(s)^F, F, 1)$. We have $C_{\mathrm{PGL}_n}(s) = \mathbf{T}^* \rtimes \langle c \rangle$ where c is the permutation matrix representing the cycle $(1, 2, \dots, n) \in \mathfrak{S}_n$, which acts on \mathbf{T}^* by sending $\mathrm{diag}(t_1, \dots, t_n)$ to $\mathrm{diag}(t_n, t_1, \dots, t_{n-1})$. On \mathbf{T}^* the Frobenius F_ε acts by $t \mapsto t^{\varepsilon q}$, and acts trivially on $W_{\mathbf{G}^*}(\mathbf{T}^*)$. Note that s satisfies condition (*) since c is F -stable. Since $C_{\mathrm{PGL}_n}(s)/C_{\mathrm{PGL}_n}^0(s) = \langle c \rangle$, and F acts trivially on c , we have $H^1(F, C_{\mathrm{PGL}_n}(s)/C_{\mathrm{PGL}_n}^0(s)) = \langle c \rangle$ and the geometric class of s splits into n rational classes parametrised by the powers of c . A representative of the class parametrised by c^j is ${}^x s$ where x is such that $x^{-1} F x = c^j$. This representative is in a maximal torus $\mathbf{T}_{c^j}^* = {}^x \mathbf{T}^*$ of type c^j with respect to \mathbf{T}^* . Choosing the PGL_n -conjugacy by x^{-1} to identify $(\mathbf{T}_{c^j}^*, F)$ with $(\mathbf{T}^*, c^j F)$, we identify back the representative of the class parametrised by c^j with the element s of $\mathbf{T}^{*c^j F}$. We have

$$\mathcal{C}(C_{\mathrm{PGL}_n}(s)^F, F, 1) = \bigoplus_{j=0}^{n-1} \mathbb{C}\mathcal{E}(\mathbf{T}^{*c^j F} \rtimes \langle c \rangle, 1).$$

Since \mathbf{T}^* is a torus, the unipotent characters of $\mathbf{T}^{*c^j F} \rtimes \langle c \rangle$ are the n extensions of the trivial character of $\mathbf{T}^{*c^j F}$. We parametrise these by $Z\mathrm{SL}_n \times \mathbb{Z}/n\mathbb{Z}$ in the following way: if $z_0 = \mathrm{diag}(\zeta, \dots, \zeta)$, we call $\theta_{z_0, k}$ the character of $\mathbf{T}^{*c^k F} \rtimes \langle c \rangle$ which is trivial on $\mathbf{T}^{*c^k F}$ and equal to $\zeta_{\mathbb{C}}^i$ on c . This allows to define another basis of $\mathcal{C}(C_{\mathrm{PGL}_n}(s)^F, F, 1)$, the ‘‘Mellin transforms’’, defined for $j, k \in \mathbb{Z}/n\mathbb{Z}$ by $\theta_{j, k} := \sum_{z \in Z\mathrm{SL}_n} \hat{s}_j(z) \theta_{z, k}$, where \hat{s}_j is the character of $\mathbf{T}^{c^j F}$ corresponding to the element $s \in \mathbf{T}^{*c^j F}$ through duality. The point is that it is more convenient to compute the action of Sh on the Mellin transforms:

Proposition 11. *We have $\mathrm{Sh} \theta_{j, i} = \theta_{j, i+j}$ unless $n = 2$ (thus $\varepsilon = 1$) and $q \equiv -1 \pmod{4}$. In this last case we have $\mathrm{Sh} \theta_{j, i} = \theta_{j, i+j-1}$*

Proof.

Lemma 12. *With the conventions of Assumption 10, when n is odd, we have*

$$\hat{s}_j(z_0^i) = \zeta_{\mathbb{C}}^{ij}; \text{ when } n = 2 \text{ (thus } \varepsilon = 1) \text{ we have } \hat{s}_j(z_0) = \begin{cases} (-1)^{(q-1)/2} & \text{if } j = 0 \\ (-1)^{(q+1)/2} & \text{if } j = 1 \end{cases}.$$

Proof. The correspondence between $\mathbf{T}^{*c^j F}$ and $\text{Irr}(\mathbf{T}^{c^j F})$ comes from the diagram of exact sequences

$$\begin{array}{ccccccc} X(\mathbf{T}) & \xrightarrow{c^j F-1} & X(\mathbf{T}) & \xrightarrow{\text{Res}} & \text{Irr}(\mathbf{T}^{c^j F}) & \rightarrow & 1 \\ \parallel & & \parallel & & \downarrow \sim & & \\ Y(\mathbf{T}^*) & \xrightarrow{c^j F-1} & Y(\mathbf{T}^*) & \xrightarrow{\pi} & \mathbf{T}^{*c^j F} & \rightarrow & 1 \end{array}$$

where π maps $y \in Y(\mathbf{T}^*)$ to $N_{F^m/c^j F}(y(1/(q^m-1)))$ for any m such that $(c^j F)^m = F^m$ (see [5, Proposition 11.1.7]); here $y(1/(q^m-1))$ is defined using the choices of Assumption 10 and $N_{F^m/c^j F}$ is the norm map on \mathbf{T}^* given by $x \mapsto x \cdot c^j F x \dots (c^j F)^{m-1} x$.

When $j = 0$ we can take $m = 2$ in the above diagram. We have $s = N_{F^2/F}(y(1/(q^2-1))) = y(1/(\varepsilon q - 1))$ where y is the cocharacter mapping $\lambda \in \overline{\mathbb{F}}_q$ to the image in \mathbf{T}^* of

$$\text{diag}(1, \lambda^{(\varepsilon q-1)/n}, \lambda^{2(\varepsilon q-1)/n}, \dots, \lambda^{(n-1)(\varepsilon q-1)/n}) \in \tilde{\mathbf{T}}.$$

The element y interpreted as a character of \mathbf{T}^F maps $z_0 = \text{diag}(\zeta, \dots, \zeta) \in Z \text{SL}_n$ to $\zeta_{\mathbb{C}}^{\frac{\varepsilon q-1}{n} \frac{n(n-1)}{2}} = \zeta_{\mathbb{C}}^{(n-1)(\varepsilon q-1)/2}$ which is equal to 1 when n is odd—thus in this case \hat{s}_0 is trivial on $Z \text{SL}_n$. If $n = 2$ we have $\varepsilon = 1$ and we get $\hat{s}_0(\text{diag}(-1, -1)) = (-1)^{(q-1)/2}$.

When $j \neq 0$, we can take $m = 2n$. We want to find $y \in Y(\mathbf{T}^*)$ such that $s = N_{F^{2n}/c^j F}(y(1/(q^{2n}-1)))$. Since $(c^j F)^n = F^n$ we have $N_{F^{2n}/c^j F}(x) = N_{F^n/c^j F}(x) F^n(N_{F^n/c^j F}(x))$. Let y_0 be a cocharacter mapping $\lambda \in \mathbb{F}_{q^{2n}}^\times$ to the image in \mathbf{T}^* of $\text{diag}(\lambda, 1, \dots, 1)$. We have $(c^j F)^k(\lambda, 1, \dots, 1) = (1, \dots, 1, \lambda^{(\varepsilon q)^k}, 1, \dots)$ where $\lambda^{(\varepsilon q)^k}$ is at the place indexed by $kj + 1$, hence at the place indexed by i in $N_{F^n/c^j F}(y_0(\lambda))$ the exponent of λ is such that $i \equiv kj + 1 \pmod{n}$, which since $\lambda \in \mathbb{F}_{q^n}$ is equivalent to $k = (i-1)j'$ where $jj' \equiv 1 \pmod{n}$. Hence $N_{F^n/c^j F}(y_0(\lambda))$ is the image in \mathbf{T}^* of $\text{diag}(\lambda, \lambda^{(\varepsilon q)^{j'}}, \lambda^{(\varepsilon q)^{2j'}}, \dots, \lambda^{(\varepsilon q)^{(n-1)j'}})$ and $N_{F^{2n}/c^j F}(y_0(\lambda))$ is the image in \mathbf{T}^* of $\text{diag}(\lambda^{1+(\varepsilon q)^n}, \lambda^{(1+(\varepsilon q)^n)(\varepsilon q)^{j'}}, \lambda^{(1+(\varepsilon q)^n)(\varepsilon q)^{2j'}}, \dots, \lambda^{(1+(\varepsilon q)^n)(\varepsilon q)^{(n-1)j'}})$. If we set $\mu = \lambda^{1+(\varepsilon q)^n}$, the element $\text{diag}(1, \mu^{(\varepsilon q)^{j'}-1}, \mu^{(\varepsilon q)^{2j'}-1}, \dots, \mu^{(\varepsilon q)^{(n-1)j'}-1})$ has same image in \mathbf{T}^* . For $\mu^{\varepsilon q-1} = \zeta^j$ we have $\mu^{(\varepsilon q)^{kj'}-1} = \zeta^{j(1+\varepsilon q+\dots+(\varepsilon q)^{kj'-1})} = \zeta^{jkj'} = \zeta^k$, the second equality since $q \equiv \varepsilon \pmod{n}$. Hence $N_{F^{2n}/c^j F}(y_0(\frac{j}{n(\varepsilon q-1)((\varepsilon q)^n+1)})) = s$, thus we can take $y = \frac{j(q^{2n}-1)}{n(\varepsilon q-1)((\varepsilon q)^n+1)} y_0 = \frac{j((\varepsilon q)^n-1)}{n(\varepsilon q-1)} y_0$ (note that $\frac{(\varepsilon q)^n-1}{n(\varepsilon q-1)}$ is an integer). As a character of $\mathbf{T}^{c^j F}$ it maps the element $z_0 = \text{diag}(\zeta, \dots, \zeta) \in Z \text{SL}_n$ to $\zeta_{\mathbb{C}}^{\frac{j((\varepsilon q)^n-1)}{n(\varepsilon q-1)}}$. We now use:

Lemma 13. *If n is odd, $k > 0$ and n^k divides $\varepsilon q - 1$, we have $\frac{(\varepsilon q)^n-1}{n(\varepsilon q-1)} \equiv 1 \pmod{n^k}$.*

Proof. Let us write $\varepsilon q = 1 + an^k$. We have $\frac{(\varepsilon q)^n-1}{\varepsilon q-1} = 1 + \varepsilon q + \dots + (\varepsilon q)^{n-1} \equiv n + a \frac{n(n-1)}{2} n^k \pmod{n^{k+1}}$. Since n is odd it divides $n(n-1)/2$, so that $n + a \frac{n(n-1)}{2} n^k \equiv n \pmod{n^{k+1}}$ which, dividing by n , gives the result. \square

This lemma applied with $k = 1$ shows that \hat{s}_j maps z_0^i to $\zeta_{\mathbb{C}}^{ij}$ when n is odd. If $n = 2$ (thus $\varepsilon = 1$) we get $\hat{s}_1(\text{diag}(-1, -1)) = (-1)^{(q+1)/2}$. \square

We now compute the Mellin transforms $\theta_{j,k}$. If n is odd, we have by Lemma 12

$$\theta_{j,i}(c^k) = \sum_{l \in \mathbb{Z}/n\mathbb{Z}} \hat{s}_j(z_0^l) \theta_{z_0^l, i}(c^k) = \sum_{l \in \mathbb{Z}/n\mathbb{Z}} \zeta_{\mathbb{C}}^{jl} \zeta_{\mathbb{C}}^{lk} = \begin{cases} 0 & \text{if } j+k \neq 0, \\ n & \text{if } j = -k. \end{cases}$$

We see that $\theta_{j,i}$ is a function supported by the coset $\mathbf{T}^{c^i F} \cdot c^{-j} \subset \mathbf{T}^{c^i F} \rtimes \langle c \rangle$ and is constant equal to n on this coset. By proposition 1, Sh maps a constant function on this coset to the constant function on $\mathbf{T}^{c^{i+j} F} \cdot c^{-j}$ with same value, hence $\text{Sh } \theta_{j,i} = \theta_{j,i+j}$.

If $n = 2$ we have

$$\theta_{j,i}(c^k) = (-1)^k \hat{s}_j(z_0) + \hat{s}_j(1) = 1 + \begin{cases} (-1)^{k+(q-1)/2} & \text{if } j = 0, \\ (-1)^{k+(q+1)/2} & \text{if } j = 1. \end{cases}$$

Thus $\theta_{j,i}$ is supported by $\mathbf{T}^{c^i F} \cdot c^j$ if $q \equiv 1 \pmod{4}$ and by $\mathbf{T}^{c^i F} \cdot c^{1-j}$ if $q \equiv -1 \pmod{4}$ and is constant equal to 2 on its support. We get the same result as in the odd case for the action of Sh when $q \equiv 1 \pmod{4}$. If $q \equiv -1 \pmod{4}$, since Sh maps functions on $\mathbf{T}^{c^i} c^{1-j}$ to functions on $\mathbf{T}^{c^{i+j-1}} c^{1-j}$ we get $\text{Sh } \theta_{j,i} = \theta_{j,i+j-1}$. \square

Proposition 11 shows that Sh preserves the space $\mathcal{C}(C_{\text{PGL}_n}(s)^F, F, 1)$. Note that the computation made in the proof of Lemma 12 and Definition 2 show that $R_{\mathbf{T}_{c^i}}^{\mathbf{T}_{c^i} \rtimes \langle c \rangle} \text{Id} = \theta_{0,i}$ unless $n = 2$ and $q \equiv -1 \pmod{4}$, in which case $R_{\mathbf{T}_{c^i}}^{\mathbf{T}_{c^i} \rtimes \langle c \rangle} \text{Id} = \theta_{1,i}$.

Sh on SL_n^F .

For computing the other the side of conjecture 7, that is Sh on SL_n^F , we first parametrise the characters in $\cup_j \mathcal{E}(\text{SL}_n^F, (s_j)_{\text{PGL}_n^F})$, where $s_j \in (\mathbf{T}_{c^j}^*)^F$ is an F -stable representative of the rational class that we parametrised above by $s \in \mathbf{T}^{*c^j F}$. We use the following notation: for $z \in Z \text{SL}_n$ we denote by Γ_z the Gelfand-Graev character indexed by z as in [5, Definition 12.3.3]; that is $\Gamma_z = \text{Ind}_{\mathbf{U}^F}^{\text{SL}_n^F} \psi_z$ where \mathbf{U} is the unipotent radical of the Borel subgroup consisting of upper triangular matrices and ψ_z is a regular character of \mathbf{U}^F indexed by z . This labelling depends on the choice of a regular character ψ_1 : we have $\psi_z = {}^t \psi_1$ where $t \in \mathbf{T}$ satisfies $t^{-1} F t = z$.

Let $\tilde{s}_j \in (\tilde{\mathbf{T}}_{c^j}^*)^F$ be a lifting of s_j . By [5, Proposition 11.3.10] $R_{\mathbf{T}_{c^j} \cap \text{SL}_n}^{\text{SL}_n} (s_j)$ is the restriction of $R_{\mathbf{T}_{c^j}}^{\text{GL}_n} (\tilde{s}_j)$, hence the series $\mathcal{E}(\text{SL}_n^F, (s_j)_{\text{PGL}_n^F})$ is the set of irreducible components of the restrictions of the elements of $\mathcal{E}(\text{GL}_n^F, (\tilde{s}_j))$. Moreover since $C_{\text{GL}_n}(\tilde{s}_j)$ is a torus, the character $R_{\mathbf{T}_{c^j}}^{\text{GL}_n} (\tilde{s}_j)$ is irreducible, hence is the only character in $\mathcal{E}(\text{GL}_n^F, (\tilde{s}_j))$; thus this character must be a component of the (unique) Gelfand-Graev character of GL_n^F by [5, Theorem 12.4.12]. The restriction of this character to SL_n^F is equal to $\sum_{z \in Z \text{SL}_n} \chi_{z,j}$ where $\chi_{z,j}$ is the unique irreducible component of the Gelfand-Graev character Γ_z in the series $\mathcal{E}(\text{SL}_n^F, (s_j))$ (see [5, Corollary 12.4.10]). Thus we have $\mathcal{E}(\text{SL}_n^F, (s_j)) = \{\chi_{z,j} \mid z \in Z \text{SL}_n\}$. It is again more convenient to compute Sh on the basis formed by the Mellin transforms $\chi_{j,k} := \sum_{z \in Z \text{SL}_n} \hat{s}_j(z) \chi_{z,k}$.

Proposition 14. *We have $\text{Sh } \chi_{j,i} = \chi_{j,i+j}$ unless $n = 2$ (thus $\varepsilon = 1$) and $q \equiv -1 \pmod{4}$. In this last case we have $\text{Sh } \chi_{j,i} = \chi_{j,i+j-1}$.*

Proof. We note first that $\chi_{z,j}(g)$ is independent of z if $C_{\text{SL}_n}(g)$ is connected. Indeed, $\chi_{z,j}$ and $\chi_{z',j}$ are conjugate by an element $x \in \text{GL}_n^F$ since they are two components of the restriction of an irreducible character of GL_n^F . We have thus $\chi_{z',j}(g) = \chi_{z,j}(xg)$. We can multiply x by a central element to obtain an element $y \in \text{SL}_n$ and, since $y^q g = xg \in \text{SL}_n^F$, we have $y^{-1} \cdot y^q g \in C_{\text{SL}_n}(g)$. If this centraliser is connected, using the Lang-Steinberg theorem we can multiply y by an element of $C_{\text{SL}_n}(g)$ to get a rational element y' , whence $\chi_{z',j}(g) = \chi_{z,j}(y'g) = \chi_{z,j}(g)$. For such an element g we thus have $\chi_{j,i}(g) = 0$ if $j \neq 0$.

Since $C_{\text{SL}_n}(g)$ is connected we have $\text{Sh}(g) = g$ thus $\text{Sh } \chi_{j,i}(g) = \chi_{j,i}(g)$, in particular $\text{Sh } \chi_{0,i} = \chi_{0,i}$ and if $j \neq 0$ we have $(\text{Sh } \chi_{j,i})(g) = \chi_{j,i+j}(g) = 0$, whence $(\text{Sh } \chi_{j,i})(g) = \chi_{j,i+j}(g)$ for all j .

It remains to consider the conjugacy classes of SL_n which have a non-connected centraliser.

Lemma 15. *When n is prime the only elements of SL_n which have a non-connected centraliser are the zu with $z \in Z \text{SL}_n$ and u regular unipotent.*

Proof. Let su be the Jordan decomposition of an element of SL_n with s semi-simple and u unipotent. We have $C_{\text{SL}_n}(su) = C_{C_{\text{SL}_n}(s)}(u)$. The group $C_{\text{SL}_n}(s)$ is a Levi subgroup of SL_n , that is the subgroup of elements of determinant 1 in a product $\prod_{i=1}^k \text{GL}_{n_i}$. Thus $C_{\text{SL}_n}(su)$ is the subgroup of elements with determinant 1 in a product $\prod_{i=1}^k \mathbf{H}_i$ where \mathbf{H}_i is the centraliser of u in GL_{n_i} , which is connected. We claim that if $k > 1$ the group $C_{\text{SL}_n}(su)$ is connected. Indeed, since n is prime, if $k > 1$ the n_i are coprime so that there exist integers a_i satisfying $\sum_{i=1}^k a_i n_i = -1$. Then the map $(h_1, \dots, h_k) \mapsto (h_1 \lambda^{a_1}, \dots, h_k \lambda^{a_k}, \lambda)$ where $\lambda = \det(h_1 h_2 \dots h_k)$ is an isomorphism from $\mathbf{H}_1 \times \dots \times \mathbf{H}_k$ to $C_{\text{SL}_n}(su) \times \overline{\mathbb{F}}_q$. Hence this last group is connected, thus its projection $C_{\text{SL}_n}(su)$ is also connected.

It remains to look at the centralisers of elements zu with u unipotent and $z \in Z \text{SL}_n$, that is the centralisers of unipotent elements. By [4, IV, Proposition 4.1], since n is prime, $C_{\text{SL}_n}(u)$ is connected unless u is regular and when u is regular we have $C_{\text{SL}_n}(u) = R_u(C_{\text{SL}_n}(u)).Z \text{SL}_n$ which is not connected. \square

Thus to prove the proposition we have only to consider the classes of the elements zu with u regular unipotent and $z \in Z \text{SL}_n$. Fix a rational regular unipotent element u_1 . The conjugacy classes of rational regular unipotent elements are parametrised by $H^1(F, C_{\text{SL}_n}(u_1)/C_{\text{SL}_n}^0(u_1)) = H^1(F, Z \text{SL}_n) = Z \text{SL}_n$ (the last equality since $q \equiv \varepsilon \pmod{n}$): a representative u_z of the class parametrised by the F -class of $z \in Z \text{SL}_n$ is ${}^t u_1$ where $t \in \mathbf{T}$ satisfies $t^{-1} F t = z$. By [4, IV, Proposition 1.2] Sh maps the SL_n^F -class of $zu_{z'}$ to that of $zu_{zz'}$. Now $\chi_{z,k}$ being a component of the restriction of the irreducible character $R_{\mathbf{T}_{c^k}}^{\text{GL}_n}(\tilde{s}_k)$ has central character equal to \hat{s}_k , independently of z , whence $\chi_{z,k}(z' u_{z''}) = \hat{s}_k(z') \chi_{z,k}(u_{z''})$. By [5, Corollary 12.3.13], there is a family of Gauss sums σ_z indexed by $Z \text{SL}_n$ such that

$$\chi_{z,k}(u_{z''}) = \sum_{z' \in H^1(F, Z \text{SL}_n)} \sigma_{z'' z'^{-1}} \langle (-1)^{F\text{-semi-simple rank}(\text{SL}_n)} D(\chi_{z,k}), \Gamma_{z'} \rangle_{\text{SL}_n^F},$$

where D is the Curtis-Alvis duality. If $k \neq 0$ and $F = F_+$ or if $F = F_-$ we claim that $(-1)^{F\text{-semi-simple rank}(\text{SL}_n)} D(\chi_{z,k}) = \chi_{z,k}$: indeed, in both cases the character

$R_{\tilde{\mathbf{T}}_{c^k}}^{\mathrm{GL}_n}(\tilde{s}_k)$ is cuspidal since the torus $\tilde{\mathbf{T}}_{c^k}$ is not contained in a proper rational Levi subgroup. Indeed when $F = F_+$ and $k \neq 0$ (resp. $F = F_-$), the type c^k (resp. the type $w_0 c^k$) of \mathbf{T}_{c^k} with respect to a quasi-split torus is not contained in a standard parabolic subgroup of \mathfrak{S}_n . Thus $\chi_{z,k}$ is cuspidal as a component of the restriction to SL_n^F of $R_{\tilde{\mathbf{T}}_{c^k}}^{\mathrm{GL}_n}(\tilde{s}_k)$, whence our claim since the duality on cuspidal characters is the multiplication by $(-1)^{F\text{-semi-simple rank}(\mathrm{SL}_n)}$ (take $\mathbf{L} = \mathrm{SL}_n$ in [5, 7.2.9]). If $k = 0$ and $F = F_+$, the characters $\chi_{z,0}$ are the irreducible components of $R_{\mathbf{T}}^{\mathrm{SL}_n}(\hat{s}_0) = \mathrm{Ind}_{\mathbf{B}^F}^{\mathrm{SL}_n^F} \hat{s}_0$, where \mathbf{B} is the Borel subgroup of upper triangular matrices and \hat{s}_0 has been lifted to \mathbf{B}^F . The endomorphism algebra of this induced character is isomorphic to the algebra of $\mathbb{Z}/n\mathbb{Z}$, hence the components of $\mathrm{Ind}_{\mathbf{B}^F}^{\mathrm{SL}_n^F} \hat{s}_0$ are parametrised by the characters of $\mathbb{Z}/n\mathbb{Z}$ and by [7] the effect of the duality is to multiply the parameters by the sign character of the endomorphism algebra which is trivial if n is odd and is -1 if $n = 2$.

If n is odd or if $n = 2$ and $k \neq 0$, we thus have $\chi_{z,k}(u_{z''}) = \sigma_{z''z^{-1}}$. If $n = 2$ and $k = 0$ we have $\chi_{z,0}(u_{z''}) = -\sigma_{z''z'}$ where $\{z, z'\} = \{1, \mathrm{diag}(-1, -1)\}$.

We consider first the case n odd. By Lemma 12 we have $\hat{s}_k(z_0^i) = \zeta_{\mathbb{C}}^{ki}$. Thus the values of the Mellin transforms are $\chi_{j,k}(z' u_{z''}) = \sum_i \hat{s}_j(z_0^i) \chi_{z_0^i, k}(z' u_{z''}) = \sum_i \hat{s}_j(z_0^i) \hat{s}_k(z') \sigma_{z''z_0^{-i}} = \hat{s}_k(z') \sum_i \zeta_{\mathbb{C}}^{ij} \sigma_{z''z_0^{-i}}$. Let us put $z' = z_0^l$; we get $\chi_{j,k}(z_0^l u_{z''}) = \zeta_{\mathbb{C}}^{lk} \sum_i \zeta_{\mathbb{C}}^{ij} \sigma_{z''z_0^{-i}}$. Now as recalled above we have $\mathrm{Sh}(z' u_{z''}) = z' u_{z'z''}$, whence $(\mathrm{Sh} \chi_{j,k})(z_0^l u_{z''}) = \chi_{j,k}(z_0^l u_{z'z''}) = \zeta_{\mathbb{C}}^{lk} \sum_i \zeta_{\mathbb{C}}^{ij} \sigma_{z''z_0^{l-i}}$. Taking $i - l$ as new variable in the summation we get $(\mathrm{Sh} \chi_{j,k})(z_0^l u_{z''}) = \zeta_{\mathbb{C}}^{lk} \sum_i \zeta_{\mathbb{C}}^{(i+l)j} \sigma_{z''z_0^{-i}} = \zeta_{\mathbb{C}}^{l(j+k)} \sum_i \zeta_{\mathbb{C}}^{ij} \sigma_{z''z_0^{-i}} = \chi_{j,j+k}(z_0^l u_{z''})$, which gives the proposition for n odd.

If $n = 2$ we have $z_0 = \mathrm{diag}(-1, -1)$. We have $\chi_{j,k}(z u_{z''}) = \hat{s}_k(z)(\hat{s}_j(1)\chi_{z_0^0, k}(u_{z''}) + \hat{s}_j(z_0)\chi_{z_0, k}(u_{z''}))$. We get $\chi_{j,0}(z u_{z''}) = \hat{s}_0(z)(-\sigma_{z_0 z''} - \hat{s}_j(z_0)\sigma_{z''})$ and $\chi_{j,1}(z u_{z''}) = \hat{s}_1(z)(\sigma_{z''} + \hat{s}_j(z_0)\sigma_{z_0 z''})$. If \hat{s}_j is the identity character then $\chi_{j,0}$ and $\chi_{j,1}$ are invariant by Sh since $\mathrm{Sh}(u_{z''}) = u_{z''}$ and $\mathrm{Sh}(z_0 u_{z''}) = z_0 u_{z_0 z''}$. If \hat{s}_j is not trivial then, $\chi_{j,0}(z u_{z''})$ and $\chi_{j,1}(z u_{z''})$ are equal if $z = 1$ and opposite if $z \neq 1$, thus, using again $\mathrm{Sh}(u_{z''}) = u_{z''}$ and $\mathrm{Sh}(z_0 u_{z''}) = z_0 u_{z_0 z''}$ we see that $\chi_{j,0}$ and $\chi_{j,1}$ are exchanged by Sh . By Lemma 12, if $q \equiv 1 \pmod{4}$ the character \hat{s}_0 is trivial and we get the same result as in the odd case. If $q \equiv -1 \pmod{4}$, the character \hat{s}_1 is trivial and Sh exchanges $\chi_{0,0}$ and $\chi_{0,1}$ and fixes $\chi_{1,0}$ and $\chi_{1,1}$, which is the announced result. \square

We can now state:

Proposition 16. *For $s = \mathrm{diag}(1, \zeta, \dots, \zeta^{n-1})$, the bijection $J : \chi_{j,i} \rightarrow \theta_{j,i}$ from $\mathbb{CE}(\mathrm{SL}_n^F, (s))$ to $\mathbb{CE}(C_{\mathrm{PGL}_n}(s)^F, F, 1)$ restricts on characters to a refinement of the Jordan decomposition which satisfies Conjecture 7.*

Proof. Propositions 11 and 14 give the commutation of J with Sh . It remains to show that $J(R_{\mathbf{T}_{c^j}}^{\mathrm{SL}_n}(s_j)) = R_{\mathbf{T}_{c^j}^*}^{\mathbf{T}_{c^j}^* \rtimes \langle c \rangle}(1)$. As we have noticed after the proof of

Proposition 11 we have $R_{\mathbf{T}_{c^j}}^{\mathbf{T}_{c^j} \rtimes \langle c \rangle} \mathrm{Id} = \begin{cases} \theta_{0,j} & \text{for } n \text{ odd or } q \equiv 1 \pmod{4} \\ \theta_{1,j} & \text{for } n = 2, q \equiv -1 \pmod{4} \end{cases}$. On the

other hand for n odd or $q \equiv 1 \pmod{4}$, we have $\chi_{0,j} = \sum_z \hat{s}_0(z) \chi_{z,j} = \sum_z \chi_{z,j}$ since in that case \hat{s}_0 is the trivial character. For $n = 2, q \equiv -1 \pmod{4}$, we have $\chi_{1,j} = \sum_z \hat{s}_1(z) \chi_{z,j} = \sum_z \chi_{z,j}$ since in that case \hat{s}_1 is the trivial character. By definition of $\chi_{z,j}$ we have $\sum_z \chi_{z,j} = R_{\mathbf{T}_{c^j}}^{\mathrm{SL}_n}(s_j)$, whence the proposition. \square

We thank Cédric Bonnafé for having suggested improvements to a previous version of this text.

REFERENCES

- [1] C. Bonnafé, Sur les caractères des groupes réductifs finis à centre non connexe: applications aux groupes spéciaux linéaires et unitaires, *Astérisque* **306** (2006) 1–174
- [2] M. Broué, On characters of finite groups, *Mathematical Lectures from Peking University* (2017) Springer.
- [3] T. Deshpande, Shintani descent for algebraic groups and almost characters of unipotent groups, *Compositio Math.* **152** (2016) 1697–1724
- [4] F. Digne and J. Michel, Fonctions L des variétés de Deligne-Lusztig et descente de Shintani *Mémoires SMF* **20** (1985) 1–144
- [5] F. Digne and J. Michel, Representations of finite groups of Lie type *LMS Student Texts* **95** (2020) CUP
- [6] M. Enguehard and J. Michel, The Sylow subgroups of a finite reductive group, *Bull. Acad. Sinica* **13** (2018) 227–247.
- [7] K. McGovern, Multiplicities of principal series representations of finite groups with split (B, N) -pairs, *J. Algebra* **77** (1982) 419–442
- [8] T. Shoji, Character sheaves and almost characters of reductive groups, *Adv. Math.* **111** (1995) 244–313
- [9] T. Shoji, Character sheaves and almost characters of reductive groups II, *Adv. Math.* **111** (1995) 314–354
- [10] T. Shoji, Lusztig’s conjecture for finite special linear groups, *Representation Theory* **10** (2006) 164–222
- [11] J-L. Waldspurger, Une conjecture de Lusztig pour les groupes classiques, *Mémoires SMF* **96** (2004), 1–166