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# COMMUTATION OF SHINTANI DESCENT AND JORDAN DECOMPOSITION

FRANÇOIS DIGNE AND JEAN MICHEL

ABSTRACT. Let  $\mathbf{G}^F$  be a finite group of Lie type, where  $\mathbf{G}$  is a reductive group defined over  $\overline{\mathbb{F}}_q$  and  $F$  is a Frobenius root. Lusztig's Jordan decomposition parametrises the irreducible characters in a rational series  $\mathcal{E}(\mathbf{G}^F, (s)_{\mathbf{G}^*F^*})$  where  $s \in \mathbf{G}^{*F^*}$  by the series  $\mathcal{E}(C_{\mathbf{G}^*}(s)^{F^*}, 1)$ . We conjecture that the Shintani twisting preserves the space of class functions generated by the union of the  $\mathcal{E}(\mathbf{G}^F, (s')_{\mathbf{G}^*F^*})$  where  $(s')_{\mathbf{G}^*F^*}$  runs over the semi-simple classes of  $\mathbf{G}^{*F^*}$  geometrically conjugate to  $s$ ; further, extending the Jordan decomposition by linearity to this space, we conjecture that there is a way to fix Jordan decomposition such that it maps the Shintani twisting to the Shintani twisting on disconnected groups defined by Deshpande, which acts on the linear span of  $\coprod_{s'} \mathcal{E}(C_{\mathbf{G}^*}(s')^{F^*}, 1)$ . We show a non-trivial case of this conjecture, the case where  $\mathbf{G}$  is of type  $A_{n-1}$  with  $n$  prime.

Deshpande

## 1. DESHPANDE'S APPROACH TO SHINTANI DESCENT

We follow [3]. Let  $\mathbf{H}$  be an algebraic group over an algebraically closed field  $k$ ; we identify  $\mathbf{H}$  to its points  $\mathbf{H}(k)$  over  $k$ . Let  $\gamma_1, \gamma_2$  be two commuting bijective isogenies on  $\mathbf{H}$ . We define the following subset of  $(\mathbf{H} \rtimes \langle \gamma_1 \rangle) \times (\mathbf{H} \rtimes \langle \gamma_2 \rangle)$ :

$$R_{\gamma_1, \gamma_2} = \{(x\gamma_1, y\gamma_2) \mid x, y \in \mathbf{H}, [x\gamma_1, y\gamma_2] = 1\}$$

where  $[u, v]$  is the commutator  $uvu^{-1}v^{-1}$ . A matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$  defines a map

$$R_{\gamma_1, \gamma_2} \xrightarrow{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} R_{\gamma_1^a \gamma_2^c, \gamma_1^b \gamma_2^d} : (x\gamma_1, y\gamma_2) \mapsto ((x\gamma_1)^a (y\gamma_2)^c, (x\gamma_1)^b (y\gamma_2)^d)$$

There is an action of  $\mathbf{H}$  on  $R_{\gamma_1, \gamma_2}$  by simultaneous conjugation:  $(x\gamma_1, y\gamma_2) \xrightarrow{\mathrm{ad} g} (gx\gamma_1g^{-1}, gy\gamma_2g^{-1})$ , which commutes with the maps  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , and we denote by  $R_{\gamma_1, \gamma_2} / \sim \mathbf{H}$  the space of orbits under this action.

Assume now that  $\mathbf{H}$  is connected and  $\gamma_1$  is a Frobenius root (an isogeny such that some finite power is a Frobenius morphism). Then, by the Lang-Steinberg theorem any  $x\gamma_1 \in \mathbf{H}\gamma_1$  is  $\mathbf{H}$ -conjugate to  $\gamma_1$ , and we can take as representatives of the  $\mathbf{H}$ -orbits pairs of the form  $(\gamma_1, y\gamma_2)$ ; on these pairs there is only an action of the fixator of  $\gamma_1$ , that is  $\mathbf{H}^{\gamma_1}$ . Further, the condition  $[\gamma_1, y\gamma_2] = 1$  is equivalent to  $y \in \mathbf{H}^{\gamma_1}$  (since  $\gamma_1$  and  $\gamma_2$  commute). We can thus interpret  $R_{\gamma_1, \gamma_2}$  as the  $\mathbf{H}^{\gamma_1}$ -conjugacy classes on the coset  $\mathbf{H}^{\gamma_1}\gamma_2$ , which we denote by  $\mathbf{H}^{\gamma_1}\gamma_2 / \sim \mathbf{H}^{\gamma_1}$ . If  $a, b$  and  $\gamma_2$  are such that  $\gamma_1^a \gamma_2^c$  is still a Frobenius root, we can thus interpret the map

given by  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  as a map  $\mathbf{H}^{\gamma_1 \gamma_2} / \sim \mathbf{H}^{\gamma_1} \rightarrow \mathbf{H}^{\gamma_1^a \gamma_2^c \gamma_1^b \gamma_2^d} / \sim \mathbf{H}^{\gamma_1^a \gamma_2^c}$ , a “generalised Shintani descent”.

We are interested here by the case of  $R_{F, \text{Id}}$  where  $F$  is a Frobenius root. The matrix  $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$  defines a map, that we call the Shintani twisting and will just denote by  $\text{Sh} : R_{F, \text{Id}} \rightarrow R_{F, \text{Id}} : (x^F, y) \mapsto (x^F y, y)$ . With the identification above of  $R_{F, \text{Id}} / \sim \mathbf{H}$  with the pairs  $(F, y)$  which identifies it with  $\mathbf{H}^F / \sim \mathbf{H}^F$ , if we write  $y = \lambda \cdot {}^F \lambda^{-1}$  using the Lang-Steinberg theorem,  $\text{Sh}$  maps  $(F, y)$  to  $(Fy, y) = (F\lambda \cdot {}^F \lambda^{-1}, \lambda \cdot {}^F \lambda^{-1})$  which is conjugate by  ${}^F \lambda^{-1}$  to  $(F, {}^F \lambda^{-1} \lambda)$ , thus we recover the usual definition of the Shintani twisting  $\text{Sh}$  as being induced by the correspondence  $\lambda \cdot {}^F \lambda^{-1} \mapsto {}^F \lambda^{-1} \lambda$ .

The advantage of Deshpande’s approach is that the map  $\text{Sh} : R_{F, \text{Id}} \rightarrow R_{F, \text{Id}}$  still makes sense when  $\mathbf{H}$  is disconnected; this time the interpretation of  $R_{F, \text{Id}} / \sim \mathbf{H}$  is different: the  $\mathbf{H}$ -orbits on  $\mathbf{H}F$  are parametrised by  $H^1(F, \mathbf{H}/\mathbf{H}^0)$ . If  $\sigma \in \mathbf{H}$  is a representative of such a  $\mathbf{H}$ -orbit, the  $\mathbf{H}$ -orbits of pairs  $(\sigma F, y)$  are in bijection with  $\mathbf{H}^{\sigma F} / \sim \mathbf{H}^{\sigma F}$ , so we see that we must consider all rational forms of  $\mathbf{H}$  corresponding to the various representatives  $\sigma F \in H^1(F, \mathbf{H}/\mathbf{H}^0)$  together, and  $\text{Sh}$  will act on the disjoint union of the conjugacy classes of each of these forms. Let  $\sigma F$  be such a representative, let  $\mathbf{H}_1$  be a  $\sigma F$ -stable coset of  $\mathbf{H}^0$  in  $\mathbf{H}$ , let  $\sigma' \in \mathbf{H}_1^{\sigma F}$  and let us compute the image by  $\text{Sh}$  of a commuting pair  $(\sigma F, \sigma' y)$  where  $y \in \mathbf{H}^0$  (thus  $y \in (\mathbf{H}^0)^{\sigma F}$ ). Let us write using the Lang-Steinberg theorem  $y = \lambda \cdot {}^{\sigma' \sigma F} \lambda^{-1}$ . Then  $\text{Sh}$  maps  $(\sigma F, \sigma' y)$  to  $(\sigma F \sigma' y, \sigma' y) = (\sigma' \sigma F \lambda \cdot {}^{\sigma' \sigma F} \lambda^{-1}, \sigma' \lambda \cdot {}^{\sigma' \sigma F} \lambda^{-1})$  (using  $[\sigma F, \sigma'] = 1$ ) which is conjugate by  ${}^{\sigma' \sigma F} \lambda^{-1}$  to  $(\sigma' \sigma F, \sigma' \cdot {}^{\sigma' \sigma F} \lambda^{-1} \lambda)$ . Thus

**Sh** **Proposition 1.** *Sh is induced on  $\sigma'(\mathbf{H}^0)^{\sigma F}$ , where  $\sigma$  and  $\sigma'$  are two representatives of  $\mathbf{H}/\mathbf{H}^0$  such that  $\sigma F$  and  $\sigma'$  commute, by the correspondence between the element  $\sigma' \lambda \cdot {}^{\sigma' \sigma F} \lambda^{-1}$  of  $\sigma'(\mathbf{H}^0)^{\sigma F}$  and the element  $\sigma' \cdot {}^{\sigma' \sigma F} \lambda^{-1} \lambda$  of  $\sigma'(\mathbf{H}^0)^{\sigma' \sigma F}$ .*

Assume now that  $\mathbf{H}/\mathbf{H}^0$  is commutative and that we can lift all elements of  $\mathbf{H}/\mathbf{H}^0$  to commuting representatives, such that  $F$ -stable elements lift to  $F$ -stable representatives; then we can lift all pairs  $(\sigma F, \sigma')$  as in Proposition 1 which commute in  $\mathbf{H}/\mathbf{H}^0$  to commuting pairs, thus we can see  $\text{Sh}$  as a linear map on the space  $\mathcal{C}(\mathbf{H}, F) := \bigoplus_{\sigma \in H^1(F, \mathbf{H}/\mathbf{H}^0)} \mathcal{C}(\mathbf{H}^{\sigma F})$ , the direct sum of the spaces of class functions on the various rational forms  $\mathbf{H}^{\sigma F}$ .

Beware that  $\text{Sh}$  on class functions is defined by  $\text{Sh}(f)(x) = f(\text{Sh}(x))$  thus it maps class functions on  $\sigma'(\mathbf{H}^0)^{\sigma' \sigma F}$  to class functions on  $\sigma'(\mathbf{H}^0)^{\sigma F}$ .

Note finally that when  $\mathbf{H}^0$  is trivial and  $F$  acts trivially on  $\mathbf{H}/\mathbf{H}^0$ , the computations of this section recover the well-known action of  $\text{GL}_2(\mathbb{Z})$  on the Drinfeld double of the finite group  $\mathbf{H}/\mathbf{H}^0 = \mathbf{H}$ . See, for example [2, 8.4.2].

## 2. CONJECTURES

**RTG non connected**

**Definition 2.** *We extend the definition of Deligne-Lusztig characters to a disconnected reductive group  $\mathbf{H}$  with a Frobenius root  $F$  by  $R_{\mathbf{T}}^{\mathbf{H}}(s) := \text{Ind}_{\mathbf{H}^0 F}^{\mathbf{H}^F} R_{\mathbf{T}}^{\mathbf{H}^0}(s)$  where  $\mathbf{T}$  runs over  $F$ -stable maximal tori of  $\mathbf{H}$  and  $s$  over  $\mathbf{T}^{*F^*}$ . We call unipotent characters the irreducible components of the  $R_{\mathbf{T}}^{\mathbf{H}}(1)$  and we denote by  $\mathcal{E}(\mathbf{H}^F, 1)$  the set of unipotent characters of  $\mathbf{H}^F$ .*

From now on,  $\mathbf{G}$  will be a connected reductive group with a Frobenius root  $F$  and  $s$  an  $F^*$ -stable semi-simple element of  $\mathbf{G}^*$ , the group dual to  $\mathbf{G}$ . We denote

by  $\mathcal{E}(\mathbf{G}^F, (s)_{\mathbf{G}^{*F^*}})$  the  $\mathbf{G}^{*F^*}$ -Lusztig series associated with  $s$ ; this series consists of the irreducible components of the  $R_{\mathbf{T}^*}^{\mathbf{G}}(s)$  where  $\mathbf{T}^*$  runs over  $F$ -stable maximal tori of  $C_{\mathbf{G}^*}(s)$ . We denote by  $\mathcal{E}(\mathbf{G}^F, (s))$  the geometric Lusztig series associated with  $s$ , that is the union of the series  $\mathcal{E}(\mathbf{G}^F, (s')_{\mathbf{G}^{*F^*}})$  where  $(s')_{\mathbf{G}^{*F^*}}$  runs over the  $\mathbf{G}^{*F^*}$ -classes in the geometric class of  $s$ .

Let  $\varepsilon_{\mathbf{G}} := (-1)^{F-\text{rank } \mathbf{G}}$ . Lusztig's Jordan decomposition of characters in a reductive group with a non necessarily connected centre is given by the following theorem (see for example [5, Theorem 11.5.1]).

Jordan decomposition

**Theorem 3.** *Let  $\mathbf{G}$  be a connected reductive group with Frobenius root  $F$  and  $(\mathbf{G}^*, F^*)$  be dual to  $\mathbf{G}$ ; for any semi-simple element  $s \in \mathbf{G}^{*F^*}$ , there is a bijection from  $\mathcal{E}(\mathbf{G}^F, (s)_{\mathbf{G}^{*F^*}})$  to  $\mathcal{E}(C_{\mathbf{G}^*}(s)^{F^*}, 1)$ . This bijection may be chosen such that, extended by linearity to virtual characters, it sends  $\varepsilon_{\mathbf{G}} R_{\mathbf{T}^*}^{\mathbf{G}}(s)$  to  $\varepsilon_{C_{\mathbf{G}^*}(s)^0} R_{\mathbf{T}^*}^{C_{\mathbf{G}^*}(s)}(1)$  for any  $F^*$ -stable maximal torus  $\mathbf{T}^*$  of  $C_{\mathbf{G}^*}(s)$ .*

Sh preserve series

**Conjecture 4.** *Let  $\mathbf{G}$  be a connected reductive group with Frobenius root  $F$ . Then Sh, viewed as a linear operator on  $\mathcal{C}(\mathbf{G}^F)$ , preserves the subspace spanned by  $\mathcal{E}(\mathbf{G}^F, (s))$  for each geometric class  $(s)$ .*

**Proposition 5.** *Assume that  $F$  is the Frobenius morphism corresponding to an  $\mathbb{F}_q$ -structure on  $\mathbf{G}$ . Then Conjecture 4 holds when the characteristic is almost good for  $\mathbf{G}$  and  $Z\mathbf{G}$  is connected or when  $\mathbf{G}$  is of type A and  $q$  is large or  $F$  acts trivially on  $Z\mathbf{G}/(Z\mathbf{G})^0$ .*

*Proof.* The case of type A when  $F$  acts trivially on  $Z\mathbf{G}/(Z\mathbf{G})^0$  results from [1, Théorème 5.5.4]. Let us prove the other cases. In [8, Section 3.2] it is shown that the characteristic functions of character sheaves are eigenvectors of Sh; then in [8, Theorem 5.7] and [9, Theorem 3.2 and Theorem 4.1] (resp. [1, Corollaire 24.11]) it is shown that when  $Z\mathbf{G}$  is connected and the characteristic is almost good (resp. in type A with  $q$  large) the characteristic functions of character sheaves coincide up to scalars with “almost characters”. Since for each  $s$ , the space  $\mathbb{C}\mathcal{E}(\mathbf{G}^F, (s))$  is spanned by a subset of almost characters, this gives the result.  $\square$

It is probable that the above proof applies to more reductive groups (see [10], [11], and also [1, Introduction]), but it is difficult to find appropriate statements in the literature.

Before stating the next conjecture, let us state a condition on centralizers of semi-simple elements

**Definition 6.** *Let  $\mathbf{G}$  be a connected reductive group with a Frobenius root  $F$ , let  $s$  be a semi-simple element of  $\mathbf{G}^F$ , and let  $\mathbf{H} = C_{\mathbf{G}}(s)$ . Then we say that  $s$  satisfies condition (\*) if we can lift elements of  $\mathbf{H}/\mathbf{H}^0$  to commuting elements of  $\mathbf{H}$ , such that  $F$ -stable elements of  $\mathbf{H}/\mathbf{H}^0$  lift to  $F$ -stable elements.*

We conjecture that condition (\*) is automatically satisfied.

Assuming conjecture 4, and that  $s \in \mathbf{G}^{*F^*}$  satisfies (\*), a choice of a Jordan decomposition as specified by Theorem 3 will map Sh to a linear operator on the space

$$\mathcal{C}(C_{\mathbf{G}^*}(s)^{F^*}, F^*, 1) := \bigoplus_{\sigma \in H^1(F^*, C_{\mathbf{G}^*}(s)/C_{\mathbf{G}^*}^0(s))} \mathbb{C}\mathcal{E}(C_{\mathbf{G}^*}(s)^{\sigma F^*}, 1).$$

Sh commute Jordan

**Conjecture 7.** *Let  $s \in \mathbf{G}^{*F^*}$ . Then  $s$  satisfies (\*) and Sh on  $\mathcal{C}(C_{\mathbf{G}^*}(s), F^*)$  preserves the subspace  $\mathcal{C}(C_{\mathbf{G}^*}(s)^{F^*}, F^*, 1)$ , and the choice of a Jordan decomposition in Theorem 3 may be refined so that it maps Sh on  $\mathcal{CE}(\mathbf{G}^F, (s))$  to Sh on  $\mathcal{C}(C_{\mathbf{G}^*}(s)^{F^*}, F^*, 1)$ .*

### 3. THE CASE OF A GROUP OF TYPE $A_{n-1}$ WHEN $n$ IS PRIME.

restriction

**Lemma 8.** *Let  $\mathbf{G} \subset \tilde{\mathbf{G}}$  be an embedding of connected reductive groups which induces an isomorphism on derived groups, let  $F$  and  $F^*$  be dual Frobenius roots, and let  $s \in \mathbf{G}^{*F^*}$  be such that  $C_{\mathbf{G}^*}(s)^{F^*} = C_{\mathbf{G}^*}^0(s)^{F^*}$ . Then the characters in  $\mathcal{E}(\mathbf{G}^F, (s)_{\mathbf{G}^{*F^*}})$  are the restrictions from  $\tilde{\mathbf{G}}^F$  to  $\mathbf{G}^F$  of the characters in  $\mathcal{E}(\tilde{\mathbf{G}}^F, (\tilde{s})_{\tilde{\mathbf{G}}^{*F^*}})$  where  $\tilde{s} \in \tilde{\mathbf{G}}^{*F^*}$  lifts  $s \in \mathbf{G}^{*F^*}$ .*

*Proof.* By definition the characters in  $\mathcal{E}(\tilde{\mathbf{G}}^F, (\tilde{s})_{\tilde{\mathbf{G}}^{*F^*}})$  are those which occur in some  $R_{\tilde{\mathbf{T}}}^{\tilde{\mathbf{G}}}(\tilde{s})$ , where  $\tilde{\mathbf{T}}$  is a maximal torus such that  $\tilde{\mathbf{T}}^* \ni \tilde{s}$ , and by for instance [5, 11.3.10] we have  $\text{Res}_{\tilde{\mathbf{G}}^F}^{\tilde{\mathbf{G}}^F} R_{\tilde{\mathbf{T}}}^{\tilde{\mathbf{G}}}(\tilde{s}) = R_{\tilde{\mathbf{T}} \cap \mathbf{G}}^{\tilde{\mathbf{G}}}(s)$ . It is thus sufficient for proving the lemma to prove that for any  $\tilde{\chi} \in \mathcal{E}(\tilde{\mathbf{G}}^F, (\tilde{s})_{\tilde{\mathbf{G}}^{*F^*}})$  its restriction  $\chi$  to  $\mathbf{G}^F$  is irreducible.

By, for instance [5, 11.3.9] we have  $\langle \chi, \chi \rangle_{\mathbf{G}^F} = |\{\theta \in \text{Irr}(\tilde{\mathbf{G}}^F/\mathbf{G}^F) \mid \tilde{\chi}\theta = \tilde{\chi}\}|$ . It is thus sufficient to prove that for any non-trivial such  $\theta$  we have  $\tilde{\chi}\theta \neq \tilde{\chi}$ . But a non-trivial  $\theta$  corresponds to a non-trivial  $z \in Z(\tilde{\mathbf{G}}^*)^{F^*}$  with trivial image in  $\mathbf{G}^*$ , and since  $\tilde{\chi}\theta \in \mathcal{E}(\tilde{\mathbf{G}}^F, (\tilde{s}z)_{\tilde{\mathbf{G}}^{*F^*}})$  it is sufficient to prove that  $\tilde{s}$  and  $\tilde{s}z$  are not  $\tilde{\mathbf{G}}^{*F^*}$ -conjugate. But, indeed if  ${}^g(\tilde{s}z) = \tilde{s}$  then  ${}^{\bar{g}}s = s$  where  $\bar{g}$  is the image of  $g$  in  $\mathbf{G}^*$ , thus  $\bar{g} \in C_{\mathbf{G}^*}(s)^{F^*} = C_{\mathbf{G}^*}^0(s)^{F^*}$ , whence  $g \in C_{\tilde{\mathbf{G}}^*}(\tilde{s})$  (a contradiction) since the preimage of  $C_{\mathbf{G}^*}^0(s)$  is  $C_{\tilde{\mathbf{G}}^*}^0(\tilde{s})$  (by for example [5, 3.5.1 (i)]).  $\square$

quotient

**Lemma 9.** *Consider a quotient  $1 \rightarrow Z \rightarrow \mathbf{G}_1 \xrightarrow{\pi} \mathbf{G} \rightarrow 1$  where  $Z$  is a connected  $F$ -stable subgroup of  $Z\mathbf{G}_1$ . Then  $\pi$  is surjective between conjugacy classes of  $\mathbf{G}_1^F$  and of  $\mathbf{G}^F$ , and commutes with Sh.*

*Proof.* Let  $x \in \mathbf{G}^F$ . Since  $Z$  is connected, by the Lang-Steinberg theorem  $\pi^{-1}(x)^F$  is non-empty whence the surjectivity of  $\pi$  on conjugacy classes. If  $y \in \pi^{-1}(x)^F$  the image by  $\pi$  of  $\text{Sh}(y)$  is  $\text{Sh}(x)$ , whence the commutation of Sh with  $\pi$ .  $\square$

Let us show now that if  $\mathbf{G}$  is of type  $A_{n-1}$ , then conjectures 4 and 7 hold if they hold for  $\text{SL}_n$ . Since  $n$  is prime, the only semi-simple connected reductive groups of type  $A_{n-1}$  are  $\text{SL}_n$  and  $\text{PGL}_n$ ; thus any connected reductive group  $\mathbf{G}$  of type  $A_{n-1}$  is the almost direct product of the derived  $\mathbf{G}'$  of  $\mathbf{G}$ , equal to  $\text{SL}_n$  or  $\text{PGL}_n$ , by a torus  $\mathbf{S}$ .

In the case  $\mathbf{G}' = \text{PGL}_n$  the almost direct product is direct since  $\text{PGL}_n$  has a trivial center. If  $\mathbf{T}$  is an  $F$ -stable maximal torus of  $\mathbf{G}$ , then it has a decomposition  $\mathbf{T}_1 \times \mathbf{S}$  where  $\mathbf{T}_1 = \mathbf{T} \cap \mathbf{G}'$  is  $F$ -stable since  $\mathbf{G}'$  is  $F$ -stable. It is possible to find an  $F$ -stable complement  $\mathbf{S}'$  of  $\mathbf{T}_1$  (see for example the proof of [6, 2.2]) and then  $\mathbf{G}$  has an  $F$ -stable product decomposition  $\mathbf{G}' \times \mathbf{S}'$ . Since Sh is trivial on a torus the conjectures are reduced to the case of  $\mathbf{G}' = \text{PGL}_n$ .

The same argument reduces the case of a direct product  $\mathbf{G} = \mathbf{G}' \times \mathbf{S}$  where  $\mathbf{G}' = \text{SL}_n$  to the case of  $\text{SL}_n$ . The other possibility when  $\mathbf{G}' = \text{SL}_n$  is an almost direct product  $\text{SL}_n \times \mathbf{S}$  amalgamated by  $Z\text{SL}_n$ ; this is isomorphic to a product of the form  $\text{GL}_n \times \mathbf{S}'$ ; in such a group all centralizers are connected, as well as in

the dual group (isomorphic to  $\mathbf{G}$ ) thus both conjectures are trivial since  $\text{Sh}$  is the identity (by for example [4, IV, 1.1]).

Finally, in  $\text{PGL}_n$  the action of  $\text{Sh}$  is trivial by Lemma 9 applied to the quotient  $\text{GL}_n \xrightarrow{\pi} \text{PGL}_n$ , and in the dual  $\text{SL}_n$  semisimple elements have connected centralizers which are Levi subgroups of  $\text{SL}_n$ ; considering the embedding of these centralizers in the corresponding Levi of  $\text{GL}_n$ , on which  $\text{Sh}$  is trivial since it is isomorphic to a product of  $\text{GL}_{n_i}$  we get that  $\text{Sh}$  is trivial on the unipotent characters of these centralizers by Lemma 8.

We have thus shown that we only need to consider  $\text{SL}_n$ . Up to isomorphism there are two possible  $\mathbb{F}_q$ -structures on  $\text{SL}_n$  (only one if  $n = 2$ ) thus  $F$  will be one of the Frobenius endomorphisms  $F_+$  or  $F_-$  where  $\text{SL}_n^{F_+} = \text{SL}_n(\mathbb{F}_q)$  and  $\text{SL}_n^{F_-} = \text{SU}_n(\mathbb{F}_q)$ . We will use the dual group  $\text{PGL}_n$ , the inclusion  $\text{SL}_n \subset \text{GL}_n$  and the quotient  $\text{GL}_n \rightarrow \text{PGL}_n$ . Since  $(\text{GL}_n, F_{\pm})$  is its own dual, we will write  $F_{\pm}$  (instead of  $F_{\pm}^*$ ) for the Frobenius map on the dual of  $\text{GL}_n$  and on  $\text{PGL}_n$ . We choose for  $F_+$  the standard Frobenius which raises all matrix entries to the  $q$ -th power, and choose for  $F_-$  the map given by  $x \mapsto F_+({}^t(x^{-1}))$ . This choice is such that on the torus of diagonal matrices  $\mathbf{T}^*$  of  $\mathbf{G}^*$ ,  $F_-$  acts by raising all the eigenvalues to the power  $-q$ , and acts trivially on  $W_{\mathbf{G}^*}(\mathbf{T}^*)$ . The torus  $\mathbf{T}^*$  is split for  $F_+$  and of type  $w_0$  (the longest element of the Weyl group) with respect to a quasi-split torus for  $F_-$ .

Lemma 8 applied with  $\mathbf{G} = \text{SL}_n$  and  $\tilde{\mathbf{G}} = \text{GL}_n$  shows that conjectures 4 and 7 hold when  $C_{\text{PGL}_n}(s)^F = C_{\text{PGL}_n}^0(s)^F$ . Indeed, in this case, since  $\text{Sh}$  is trivial in  $\text{GL}_n$  and commutes with the restriction to  $\text{SL}_n$  by Lemma 9, Lemma 8 shows that  $\text{Sh}$  is trivial on  $\mathcal{E}(\text{SL}_n^F, (s))$ ; on the other hand, since  $C_{\text{PGL}_n}^0(s)$  is the quotient with central kernel of a product of  $\text{GL}_{n_i}$ , the unipotent characters of  $C_{\text{PGL}_n}(s)^F = C_{\text{PGL}_n}^0(s)^F$  can be lifted to this product (see [5, Proposition 11.3.8]). Since  $\text{Sh}$  is trivial in  $\text{GL}_{n_i}$  and commutes with such a quotient with connected central kernel by Lemma 9, it is trivial on the unipotent characters of  $C_{\text{PGL}_n}(s)^F$ .

We have thus reduced the study of conjectures 4 and 7 to the case of semi-simple elements  $s \in \text{PGL}_n$  with  $C_{\text{PGL}_n}(s)^F \neq C_{\text{PGL}_n}^0(s)^F$ .

By [5, Lemma 11.2.1(iii)], since  $n$  is prime a semi-simple element  $s \in \text{PGL}_n$  has a non-connected centraliser if and only if  $|C_{\text{PGL}_n}(s)/C_{\text{PGL}_n}^0(s)| = n$ . Since this group of components is a subquotient of the Weyl group  $\mathfrak{S}_n$ , it has order  $n$  if and only if it is a subgroup of  $\mathfrak{S}_n$  generated by an  $n$ -cycle. An easy computation shows that up to conjugacy such an  $s$  is the image in  $\text{PGL}_n$  of the matrix  $\text{diag}(1, \zeta, \zeta^2, \dots, \zeta^{n-1}) \in \mathbf{T}^*$  where  $\zeta$  is a primitive (that is, non-trivial, since  $n$  is prime)  $n$ -th root of 1. Now the centralizer of  $s$  is the semidirect product of  $\mathbf{T}^*$  with the cyclic group generated by the  $n$ -cycle  $c = (1, \dots, n)$ . Assume that the geometric class of  $s = \text{diag}(1, \zeta, \zeta^2, \dots, \zeta^{n-1})$  has an  $F$ -stable representative  $s'$  in a torus of type  $w \in \mathfrak{S}_n$  with respect to  $\mathbf{T}^*$  (or equivalently that  ${}^w F s = s$ ). Then  $C_{\text{PGL}_n}(s')^F \simeq C_{\text{PGL}_n}(s)^{wF}$  hence is equal to  $C_{\text{PGL}_n}^0(s')^F$  unless  $w$  commutes with  $c$ , that is  $w = c^i$  for some  $i$ . Since  $c$  centralizes  $s$  we get that  $s$  is  $F$ -stable, which means that  $n$  divides  $q - \varepsilon$  when  $F = F_{\varepsilon}$  where we write  $F_{\varepsilon}$  for  $F_+$ , (resp.  $F_-$ ) for  $\varepsilon = +1$ , (resp.  $-1$ ); we recall that we always take  $\varepsilon = 1$  when  $n = 2$ .

So the proof of the conjectures 4 and 7 is reduced to the series  $\mathcal{E}(\text{SL}_n^F, (s))$  with  $s \in \text{PGL}_n^F$  geometrically conjugate to  $\text{diag}(1, \zeta, \zeta^2, \dots, \zeta^{n-1})$ , and both conjectures are true if  $q \not\equiv \varepsilon \pmod{n}$ . Further, conjecture 4 holds in any case, since  $\text{Sh}$  preserves all geometric series except that of  $s$  and preserves orthogonality of characters, thus preserves also the series  $\mathcal{E}(\text{SL}_n^F, (s))$ .

Henceforth we assume that  $n$  is prime and divides  $q - \varepsilon$ , and study conjecture 7 for this particular  $s$ .

assumptions

**Assumption 10.** We choose  $\zeta$ , an  $n$ -th root of unity in  $\mathbb{F}_q$ . We choose an isomorphism  $\overline{\mathbb{F}}_q^\times \simeq (\mathbb{Q}/\mathbb{Z})_p'$  which maps  $\zeta$  to  $1/n$  and a group embedding  $\mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{C}^\times$  which maps  $1/n$  to  $\zeta_{\mathbb{C}} := \exp(2i\pi/n)$ .

We denote by  $\mathbf{T}$  the diagonal torus of  $\mathrm{SL}_n$  and  $\tilde{\mathbf{T}}$  the diagonal torus of  $\mathrm{GL}_n$ ; we choose the dual torus  $\mathbf{T}^*$  to be the diagonal torus of  $\mathrm{PGL}_n$ . Let  $s \in \mathbf{T}^*$  be the image of  $\mathrm{diag}(1, \zeta, \zeta^2, \dots, \zeta^{n-1}) \in \tilde{\mathbf{T}}$ .

Sh on  $C_{\mathrm{PGL}_n}(s)$ .

We first study Sh on  $\mathcal{C}(C_{\mathrm{PGL}_n}(s)^F, F, 1)$ . We have  $C_{\mathrm{PGL}_n}(s) = \mathbf{T}^* \rtimes \langle c \rangle$  where  $c$  is the permutation matrix representing the cycle  $(1, 2, \dots, n) \in \mathfrak{S}_n$ , which acts on  $\mathbf{T}^*$  by sending  $\mathrm{diag}(t_1, \dots, t_n)$  to  $\mathrm{diag}(t_n, t_1, \dots, t_{n-1})$ . On  $\mathbf{T}^*$  the Frobenius  $F_\varepsilon$  acts by  $t \mapsto t^{\varepsilon q}$ , and acts trivially on  $W_{\mathbf{G}^*}(\mathbf{T}^*)$ . Note that  $s$  satisfies condition (\*) since  $c$  is  $F$ -stable. Since  $C_{\mathrm{PGL}_n}(s)/C_{\mathrm{PGL}_n}^0(s) = \langle c \rangle$ , and  $F$  acts trivially on  $c$ , we have  $H^1(F, C_{\mathrm{PGL}_n}(s)/C_{\mathrm{PGL}_n}^0(s)) = \langle c \rangle$  and the geometric class of  $s$  splits into  $n$  rational classes parametrised by the powers of  $c$ . A representative of the class parametrised by  $c^j$  is  ${}^x s$  where  $x$  is such that  $x^{-1} F x = c^j$ . This representative is in a maximal torus  $\mathbf{T}_{c^j}^* = {}^x \mathbf{T}^*$  of type  $c^j$  with respect to  $\mathbf{T}^*$ . Choosing the  $\mathrm{PGL}_n$ -conjugacy by  $x^{-1}$  to identify  $(\mathbf{T}_{c^j}^*, F)$  with  $(\mathbf{T}^*, c^j F)$ , we identify back the representative of the class parametrised by  $c^j$  with the element  $s$  of  $\mathbf{T}^{*c^j F}$ . We have

$$\mathcal{C}(C_{\mathrm{PGL}_n}(s)^F, F, 1) = \bigoplus_{j=0}^{n-1} \mathbb{C}\mathcal{E}(\mathbf{T}^{*c^j F} \rtimes \langle c \rangle, 1).$$

Since  $\mathbf{T}^*$  is a torus, the unipotent characters of  $\mathbf{T}^{*c^j F} \rtimes \langle c \rangle$  are the  $n$  extensions of the trivial character of  $\mathbf{T}^{*c^j F}$ . We parametrise these by  $Z\mathrm{SL}_n \times \mathbb{Z}/n\mathbb{Z}$  in the following way: if  $z_0 = \mathrm{diag}(\zeta, \dots, \zeta)$ , we call  $\theta_{z_0, k}$  the character of  $\mathbf{T}^{*c^k F} \rtimes \langle c \rangle$  which is trivial on  $\mathbf{T}^{*c^k F}$  and equal to  $\zeta_{\mathbb{C}}^i$  on  $c$ . This allows to define another basis of  $\mathcal{C}(C_{\mathrm{PGL}_n}(s)^F, F, 1)$ , the ‘‘Mellin transforms’’, defined for  $j, k \in \mathbb{Z}/n\mathbb{Z}$  by  $\theta_{j, k} := \sum_{z \in Z\mathrm{SL}_n} \hat{s}_j(z) \theta_{z, k}$ , where  $\hat{s}_j$  is the character of  $\mathbf{T}^{*c^j F}$  corresponding to the element  $s \in \mathbf{T}^{*c^j F}$  through duality. The point is that it is more convenient to compute the action of Sh on the Mellin transforms:

Sh theta

**Proposition 11.** We have  $\mathrm{Sh} \theta_{j, i} = \theta_{j, i+j}$  unless  $n = 2$  (thus  $\varepsilon = 1$ ) and  $q \equiv -1 \pmod{4}$ . In this last case we have  $\mathrm{Sh} \theta_{j, i} = \theta_{j, i+j-1}$

*Proof.*

value of hat s

**Lemma 12.** With the conventions of Assumption 10, when  $n$  is odd, we have

$$\hat{s}_j(z_0^i) = \zeta_{\mathbb{C}}^{ij}; \text{ when } n = 2 \text{ (thus } \varepsilon = 1) \text{ we have } \hat{s}_j(z_0) = \begin{cases} (-1)^{(q-1)/2} & \text{if } j = 0 \\ (-1)^{(q+1)/2} & \text{if } j = 1 \end{cases}.$$

*Proof.* The correspondence between  $\mathbf{T}^{*c^j F}$  and  $\text{Irr}(\mathbf{T}^{c^j F})$  comes from the diagram of exact sequences

$$\begin{array}{ccccccc} X(\mathbf{T}) & \xrightarrow{c^j F-1} & X(\mathbf{T}) & \xrightarrow{\text{Res}} & \text{Irr}(\mathbf{T}^{c^j F}) & \rightarrow & 1 \\ & & \parallel & & \downarrow \sim & & \\ Y(\mathbf{T}^*) & \xrightarrow{c^j F-1} & Y(\mathbf{T}^*) & \xrightarrow{\pi} & \mathbf{T}^{*c^j F} & \rightarrow & 1 \end{array}$$

where  $\pi$  maps  $y \in Y(\mathbf{T}^*)$  to  $N_{F^m/c^j F}(y(1/(q^m-1)))$  for any  $m$  such that  $(c^j F)^m = F^m$  (see [5, Proposition 11.1.7]); here  $y(1/(q^m-1))$  is defined using the choices of Assumption 10 and  $N_{F^m/c^j F}$  is the norm map on  $\mathbf{T}^*$  given by  $x \mapsto x \cdot c^j F x \cdots (c^j F)^{m-1} x$ .

When  $j = 0$  we can take  $m = 2$  in the above diagram. We have  $s = N_{F^2/F}(y(1/(q^2-1))) = y(1/(\varepsilon q - 1))$  where  $y$  is the cocharacter mapping  $\lambda \in \overline{\mathbb{F}}_q$  to the image in  $\mathbf{T}^*$  of

$$\text{diag}(1, \lambda^{(\varepsilon q-1)/n}, \lambda^{2(\varepsilon q-1)/n}, \dots, \lambda^{(n-1)(\varepsilon q-1)/n}) \in \tilde{\mathbf{T}}.$$

The element  $y$  interpreted as a character of  $\mathbf{T}^F$  maps  $z_0 = \text{diag}(\zeta, \dots, \zeta) \in Z \text{SL}_n$  to  $\zeta_{\mathbb{C}}^{\frac{\varepsilon q-1}{n} \frac{n(n-1)}{2}} = \zeta_{\mathbb{C}}^{(n-1)(\varepsilon q-1)/2}$  which is equal to 1 when  $n$  is odd—thus in this case  $\hat{s}_0$  is trivial on  $Z \text{SL}_n$ . If  $n = 2$  we have  $\varepsilon = 1$  and we get  $\hat{s}_0(\text{diag}(-1, -1)) = (-1)^{(q-1)/2}$ .

When  $j \neq 0$ , we can take  $m = 2n$ . We want to find  $y \in Y(\mathbf{T}^*)$  such that  $s = N_{F^{2n}/c^j F}(y(1/(q^{2n}-1)))$ . Since  $(c^j F)^n = F^n$  we have  $N_{F^{2n}/c^j F}(x) = N_{F^n/c^j F}(x)F^n(N_{F^n/c^j F}(x))$ . Let  $y_0$  be a cocharacter mapping  $\lambda \in \mathbb{F}_{q^{2n}}^\times$  to the image in  $\mathbf{T}^*$  of  $\text{diag}(\lambda, 1, \dots, 1)$ . We have  $(c^j F)^k(\lambda, 1, \dots, 1) = (1, \dots, 1, \lambda^{(\varepsilon q)^k}, 1, \dots)$  where  $\lambda^{(\varepsilon q)^k}$  is at the place indexed by  $kj + 1$ , hence at the place indexed by  $i$  in  $N_{F^n/c^j F}(y_0(\lambda))$  the exponent of  $\lambda$  is such that  $i \equiv kj + 1 \pmod{n}$ , which since  $\lambda \in \mathbb{F}_{q^n}$  is equivalent to  $k = (i-1)j'$  where  $jj' \equiv 1 \pmod{n}$ . Hence  $N_{F^n/c^j F}(y_0(\lambda))$  is the image in  $\mathbf{T}^*$  of  $\text{diag}(\lambda, \lambda^{(\varepsilon q)^{j'}}, \lambda^{(\varepsilon q)^{2j'}}, \dots, \lambda^{(\varepsilon q)^{(n-1)j'})}$  and  $N_{F^{2n}/c^j F}(y_0(\lambda))$  is the image in  $\mathbf{T}^*$  of  $\text{diag}(\lambda^{1+(\varepsilon q)^n}, \lambda^{(1+(\varepsilon q)^n)(\varepsilon q)^{j'}}, \lambda^{(1+(\varepsilon q)^n)(\varepsilon q)^{2j'}}, \dots, \lambda^{(1+(\varepsilon q)^n)(\varepsilon q)^{(n-1)j'})}$ . If we set  $\mu = \lambda^{1+(\varepsilon q)^n}$ , the element  $\text{diag}(1, \mu^{(\varepsilon q)^{j'}-1}, \mu^{(\varepsilon q)^{2j'}-1}, \dots, \mu^{(\varepsilon q)^{(n-1)j'}-1})$  has same image in  $\mathbf{T}^*$ . For  $\mu^{\varepsilon q-1} = \zeta^j$  we have  $\mu^{(\varepsilon q)^{kj'}-1} = \zeta^{j(1+\varepsilon q+\dots+(\varepsilon q)^{kj'-1})} = \zeta^{jkj'} = \zeta^k$ , the second equality since  $q \equiv \varepsilon \pmod{n}$ . Hence  $N_{F^{2n}/c^j F}(y_0(\frac{j}{n(\varepsilon q-1)((\varepsilon q)^n+1)})) = s$ , thus we can take  $y = \frac{j(q^{2n}-1)}{n(\varepsilon q-1)((\varepsilon q)^n+1)} y_0 = \frac{j((\varepsilon q)^n-1)}{n(\varepsilon q-1)} y_0$  (note that  $\frac{(\varepsilon q)^n-1}{n(\varepsilon q-1)}$  is an integer). As a character of  $\mathbf{T}^{c^j F}$  it maps the element  $z_0 = \text{diag}(\zeta, \dots, \zeta) \in Z \text{SL}_n$  to  $\zeta_{\mathbb{C}}^{\frac{j((\varepsilon q)^n-1)}{n(\varepsilon q-1)}}$ . We now use:

**Lemma 13.** *If  $n$  is odd,  $k > 0$  and  $n^k$  divides  $\varepsilon q - 1$ , we have  $\frac{(\varepsilon q)^n-1}{n(\varepsilon q-1)} \equiv 1 \pmod{n^k}$ .*

*Proof.* Let us write  $\varepsilon q = 1 + an^k$ . We have  $\frac{(\varepsilon q)^n-1}{\varepsilon q-1} = 1 + \varepsilon q + \dots + (\varepsilon q)^{n-1} \equiv n + a \frac{n(n-1)}{2} n^k \pmod{n^{k+1}}$ . Since  $n$  is odd it divides  $n(n-1)/2$ , so that  $n + a \frac{n(n-1)}{2} n^k \equiv n \pmod{n^{k+1}}$  which, dividing by  $n$ , gives the result.  $\square$

This lemma applied with  $k = 1$  shows that  $\hat{s}_j$  maps  $z_0^i$  to  $\zeta_{\mathbb{C}}^{ij}$  when  $n$  is odd. If  $n = 2$  (thus  $\varepsilon = 1$ ) we get  $\hat{s}_1(\text{diag}(-1, -1)) = (-1)^{(q+1)/2}$ .  $\square$



We now compute the Mellin transforms  $\theta_{j,k}$ . If  $n$  is odd, we have by Lemma 12

$$\theta_{j,i}(c^k) = \sum_{l \in \mathbb{Z}/n\mathbb{Z}} \hat{s}_j(z_0^l) \theta_{z_0^l, i}(c^k) = \sum_{l \in \mathbb{Z}/n\mathbb{Z}} \zeta_{\mathbb{C}}^{jl} \zeta_{\mathbb{C}}^{lk} = \begin{cases} 0 & \text{if } j+k \neq 0, \\ n & \text{if } j = -k. \end{cases}$$

We see that  $\theta_{j,i}$  is a function supported by the coset  $\mathbf{T}^{c^i F} \cdot c^{-j} \subset \mathbf{T}^{c^i F} \rtimes \langle c \rangle$  and is constant equal to  $n$  on this coset. By proposition 1, Sh maps a constant function on this coset to the constant function on  $\mathbf{T}^{c^{i+j} F} \cdot c^{-j}$  with same value, hence  $\text{Sh } \theta_{j,i} = \theta_{j,i+j}$ .

If  $n = 2$  we have

$$\theta_{j,i}(c^k) = (-1)^k \hat{s}_j(z_0) + \hat{s}_j(1) = 1 + \begin{cases} (-1)^{k+(q-1)/2} & \text{if } j = 0, \\ (-1)^{k+(q+1)/2} & \text{if } j = 1. \end{cases}$$

Thus  $\theta_{j,i}$  is supported by  $\mathbf{T}^{c^i F} \cdot c^j$  if  $q \equiv 1 \pmod{4}$  and by  $\mathbf{T}^{c^i F} \cdot c^{1-j}$  if  $q \equiv -1 \pmod{4}$  and is constant equal to 2 on its support. We get the same result as in the odd case for the action of Sh when  $q \equiv 1 \pmod{4}$ . If  $q \equiv -1 \pmod{4}$ , since Sh maps functions on  $\mathbf{T}^{c^i} c^{1-j}$  to functions on  $\mathbf{T}^{c^{i+j-1}} c^{1-j}$  we get  $\text{Sh } \theta_{j,i} = \theta_{j,i+j-1}$ .  $\square$

Proposition 11 shows that Sh preserves the space  $\mathcal{C}(C_{\text{PGL}_n}(s)^F, F, 1)$ . Note that the computation made in the proof of Lemma 12 and Definition 2 show that  $R_{\mathbf{T}^{c^i} \rtimes \langle c \rangle}^{\mathbf{T}^{c^i} \rtimes \langle c \rangle} \text{Id} = \theta_{0,i}$  unless  $n = 2$  and  $q \equiv -1 \pmod{4}$ , in which case  $R_{\mathbf{T}^{c^i} \rtimes \langle c \rangle}^{\mathbf{T}^{c^i} \rtimes \langle c \rangle} \text{Id} = \theta_{1,i}$ .

### Sh on $\text{SL}_n^F$ .

For computing the other the side of conjecture 7, that is Sh on  $\text{SL}_n^F$ , we first parametrise the characters in  $\cup_j \mathcal{E}(\text{SL}_n^F, (s_j)_{\text{PGL}_n^F})$ , where  $s_j \in (\mathbf{T}_{c^j}^*)^F$  is an  $F$ -stable representative of the rational class that we parametrised above by  $s \in \mathbf{T}^{*c^j F}$ . We use the following notation: for  $z \in Z \text{SL}_n$  we denote by  $\Gamma_z$  the Gelfand-Graev character indexed by  $z$  as in [5, Definition 12.3.3]; that is  $\Gamma_z = \text{Ind}_{\mathbf{U}^F}^{\text{SL}_n^F} \psi_z$  where  $\mathbf{U}$  is the unipotent radical of the Borel subgroup consisting of upper triangular matrices and  $\psi_z$  is a regular character of  $\mathbf{U}^F$  indexed by  $z$ . This labelling depends on the choice of a regular character  $\psi_1$ : we have  $\psi_z = {}^t \psi_1$  where  $t \in \mathbf{T}$  satisfies  $t^{-1} F t = z$ .

Let  $\tilde{s}_j \in (\tilde{\mathbf{T}}_{c^j}^*)^F$  be a lifting of  $s_j$ . By [5, Proposition 11.3.10]  $R_{\mathbf{T}_{c^j} \cap \text{SL}_n}^{\text{SL}_n}(s_j)$  is the restriction of  $R_{\mathbf{T}_{c^j}}^{\text{GL}_n}(\tilde{s}_j)$ , hence the series  $\mathcal{E}(\text{SL}_n^F, (s_j)_{\text{PGL}_n^F})$  is the set of irreducible components of the restrictions of the elements of  $\mathcal{E}(\text{GL}_n^F, (\tilde{s}_j))$ . Moreover since  $C_{\text{GL}_n}(\tilde{s}_j)$  is a torus, the character  $R_{\mathbf{T}_{c^j}}^{\text{GL}_n}(\tilde{s}_j)$  is irreducible, hence is the only character in  $\mathcal{E}(\text{GL}_n^F, (\tilde{s}_j))$ ; thus this character must be a component of the (unique) Gelfand-Graev character of  $\text{GL}_n^F$  by [5, Theorem 12.4.12]. The restriction of this character to  $\text{SL}_n^F$  is equal to  $\sum_{z \in Z \text{SL}_n} \chi_{z,j}$  where  $\chi_{z,j}$  is the unique irreducible component of the Gelfand-Graev character  $\Gamma_z$  in the series  $\mathcal{E}(\text{SL}_n^F, (s_j))$  (see [5, Corollary 12.4.10]). Thus we have  $\mathcal{E}(\text{SL}_n^F, (s_j)) = \{\chi_{z,j} \mid z \in Z \text{SL}_n\}$ . It is again more convenient to compute Sh on the basis formed by the Mellin transforms  $\chi_{j,k} := \sum_{z \in Z \text{SL}_n} \hat{s}_j(z) \chi_{z,k}$ .

Sh chi

**Proposition 14.** *We have  $\text{Sh } \chi_{j,i} = \chi_{j,i+j}$  unless  $n = 2$  (thus  $\varepsilon = 1$ ) and  $q \equiv -1 \pmod{4}$ . In this last case we have  $\text{Sh } \chi_{j,i} = \chi_{j,i+j-1}$ .*

*Proof.* We note first that  $\chi_{z,j}(g)$  is independent of  $z$  if  $C_{\text{SL}_n}(g)$  is connected. Indeed,  $\chi_{z,j}$  and  $\chi_{z',j}$  are conjugate by an element  $x \in \text{GL}_n^F$  since they are two components of the restriction of an irreducible character of  $\text{GL}_n^F$ . We have thus  $\chi_{z',j}(g) = \chi_{z,j}(xg)$ . We can multiply  $x$  by a central element to obtain an element  $y \in \text{SL}_n$  and, since  ${}^y g = xg \in \text{SL}_n^F$ , we have  $y^{-1} \cdot {}^F y \in C_{\text{SL}_n}(g)$ . If this centraliser is connected, using the Lang-Steinberg theorem we can multiply  $y$  by an element of  $C_{\text{SL}_n}(g)$  to get a rational element  $y'$ , whence  $\chi_{z',j}(g) = \chi_{z,j}(y'g) = \chi_{z,j}(g)$ . For such an element  $g$  we thus have  $\chi_{j,i}(g) = 0$  if  $j \neq 0$ .

Since  $C_{\text{SL}_n}(g)$  is connected we have  $\text{Sh}(g) = g$  thus  $\text{Sh } \chi_{j,i}(g) = \chi_{j,i}(g)$ , in particular  $\text{Sh } \chi_{0,i} = \chi_{0,i}$  and if  $j \neq 0$  we have  $(\text{Sh } \chi_{j,i})(g) = \chi_{j,i+j}(g) = 0$ , whence  $(\text{Sh } \chi_{j,i})(g) = \chi_{j,i+j}(g)$  for all  $j$ .

It remains to consider the conjugacy classes of  $\text{SL}_n$  which have a non-connected centraliser.

**Lemma 15.** *When  $n$  is prime the only elements of  $\text{SL}_n$  which have a non-connected centraliser are the  $zu$  with  $z \in Z \text{SL}_n$  and  $u$  regular unipotent.*

*Proof.* Let  $su$  be the Jordan decomposition of an element of  $\text{SL}_n$  with  $s$  semi-simple and  $u$  unipotent. We have  $C_{\text{SL}_n}(su) = C_{C_{\text{SL}_n}(s)}(u)$ . The group  $C_{\text{SL}_n}(s)$  is a Levi subgroup of  $\text{SL}_n$ , that is the subgroup of elements of determinant 1 in a product  $\prod_{i=1}^k \text{GL}_{n_i}$ . Thus  $C_{\text{SL}_n}(su)$  is the subgroup of elements with determinant 1 in a product  $\prod_{i=1}^k \mathbf{H}_i$  where  $\mathbf{H}_i$  is the centraliser of  $u$  in  $\text{GL}_{n_i}$ , which is connected. We claim that if  $k > 1$  the group  $C_{\text{SL}_n}(su)$  is connected. Indeed, since  $n$  is prime, if  $k > 1$  the  $n_i$  are coprime so that there exist integers  $a_i$  satisfying  $\sum_{i=1}^k a_i n_i = -1$ . Then the map  $(h_1, \dots, h_k) \mapsto (h_1 \lambda^{a_1}, \dots, h_k \lambda^{a_k}, \lambda)$  where  $\lambda = \det(h_1 h_2 \dots h_k)$  is an isomorphism from  $\mathbf{H}_1 \times \dots \times \mathbf{H}_k$  to  $C_{\text{SL}_n}(su) \times \overline{\mathbb{F}}_q$ . Hence this last group is connected, thus its projection  $C_{\text{SL}_n}(su)$  is also connected.

It remains to look at the centralisers of elements  $zu$  with  $u$  unipotent and  $z \in Z \text{SL}_n$ , that is the centralisers of unipotent elements. By [4, IV, Proposition 4.1], since  $n$  is prime,  $C_{\text{SL}_n}(u)$  is connected unless  $u$  is regular and when  $u$  is regular we have  $C_{\text{SL}_n}(u) = R_u(C_{\text{SL}_n}(u)).Z \text{SL}_n$  which is not connected.  $\square$

Thus to prove the proposition we have only to consider the classes of the elements  $zu$  with  $u$  regular unipotent and  $z \in Z \text{SL}_n$ . Fix a rational regular unipotent element  $u_1$ . The conjugacy classes of rational regular unipotent elements are parametrised by  $H^1(F, C_{\text{SL}_n}(u_1)/C_{\text{SL}_n}^0(u_1)) = H^1(F, Z \text{SL}_n) = Z \text{SL}_n$  (the last equality since  $q \equiv \varepsilon \pmod{n}$ ): a representative  $u_z$  of the class parametrised by the  $F$ -class of  $z \in Z \text{SL}_n$  is  ${}^t u_1$  where  $t \in \mathbf{T}$  satisfies  $t^{-1} {}^F t = z$ . By [4, IV, Proposition 1.2]  $\text{Sh}$  maps the  $\text{SL}_n^F$ -class of  $zu_{z'}$  to that of  $zu_{zz'}$ . Now  $\chi_{z,k}$  being a component of the restriction of the irreducible character  $R_{\mathbf{T}, c^k}^{\text{GL}_n}(\tilde{s}_k)$  has central character equal to  $\hat{s}_k$ , independently of  $z$ , whence  $\chi_{z,k}(z' u_{z''}) = \hat{s}_k(z') \chi_{z,k}(u_{z''})$ . By [5, Corollary 12.3.13], there is a family of Gauss sums  $\sigma_z$  indexed by  $Z \text{SL}_n$  such that

$$\chi_{z,k}(u_{z''}) = \sum_{z' \in H^1(F, Z \text{SL}_n)} \sigma_{z'' z'^{-1}} \langle (-1)^{F\text{-semi-simple rank}(\text{SL}_n)} D(\chi_{z,k}), \Gamma_{z'} \rangle_{\text{SL}_n^F},$$

where  $D$  is the Curtis-Alvis duality. If  $k \neq 0$  and  $F = F_+$  or if  $F = F_-$  we claim that  $(-1)^{F\text{-semi-simple rank}(\text{SL}_n)} D(\chi_{z,k}) = \chi_{z,k}$ : indeed, in both cases the character

$R_{\tilde{\mathbf{T}}_{c^k}}^{\mathrm{GL}_n}(\tilde{s}_k)$  is cuspidal since the torus  $\tilde{\mathbf{T}}_{c^k}$  is not contained in a proper rational Levi subgroup. Indeed when  $F = F_+$  and  $k \neq 0$  (resp.  $F = F_-$ ), the type  $c^k$  (resp. the type  $w_0 c^k$ ) of  $\mathbf{T}_{c^k}$  with respect to a quasi-split torus is not contained in a standard parabolic subgroup of  $\mathfrak{S}_n$ . Thus  $\chi_{z,k}$  is cuspidal as a component of the restriction to  $\mathrm{SL}_n^F$  of  $R_{\tilde{\mathbf{T}}_{c^k}}^{\mathrm{GL}_n}(\tilde{s}_k)$ , whence our claim since the duality on cuspidal characters is the multiplication by  $(-1)^{F\text{-semi-simple rank}(\mathrm{SL}_n)}$  (take  $\mathbf{L} = \mathrm{SL}_n$  in [5, 7.2.9]). If  $k = 0$  and  $F = F_+$ , the characters  $\chi_{z,0}$  are the irreducible components of  $R_{\mathbf{T}}^{\mathrm{SL}_n}(\hat{s}_0) = \mathrm{Ind}_{\mathbf{B}^F}^{\mathrm{SL}_n^F} \hat{s}_0$ , where  $\mathbf{B}$  is the Borel subgroup of upper triangular matrices and  $\hat{s}_0$  has been lifted to  $\mathbf{B}^F$ . The endomorphism algebra of this induced character is isomorphic to the algebra of  $\mathbb{Z}/n\mathbb{Z}$ , hence the components of  $\mathrm{Ind}_{\mathbf{B}^F}^{\mathrm{SL}_n^F} \hat{s}_0$  are parametrised by the characters of  $\mathbb{Z}/n\mathbb{Z}$  and by [7] the effect of the duality is to multiply the parameters by the sign character of the endomorphism algebra which is trivial if  $n$  is odd and is  $-1$  if  $n = 2$ .

If  $n$  is odd or if  $n = 2$  and  $k \neq 0$ , we thus have  $\chi_{z,k}(u_{z''}) = \sigma_{z''z^{-1}}$ . If  $n = 2$  and  $k = 0$  we have  $\chi_{z,0}(u_{z''}) = -\sigma_{z''z'}$  where  $\{z, z'\} = \{1, \mathrm{diag}(-1, -1)\}$ .

We consider first the case  $n$  odd. By Lemma 12 we have  $\hat{s}_k(z_0^i) = \zeta_{\mathbb{C}}^{ki}$ . Thus the values of the Mellin transforms are  $\chi_{j,k}(z' u_{z''}) = \sum_i \hat{s}_j(z_0^i) \chi_{z_0^i, k}(z' u_{z''}) = \sum_i \hat{s}_j(z_0^i) \hat{s}_k(z') \sigma_{z''z_0^{-i}} = \hat{s}_k(z') \sum_i \zeta_{\mathbb{C}}^{ij} \sigma_{z''z_0^{-i}}$ . Let us put  $z' = z_0^l$ ; we get  $\chi_{j,k}(z_0^l u_{z''}) = \zeta_{\mathbb{C}}^{lk} \sum_i \zeta_{\mathbb{C}}^{ij} \sigma_{z''z_0^{-i}}$ . Now as recalled above we have  $\mathrm{Sh}(z' u_{z''}) = z' u_{z''z'}$ , whence  $(\mathrm{Sh} \chi_{j,k})(z_0^l u_{z''}) = \chi_{j,k}(z_0^l u_{z_0^l z''}) = \zeta_{\mathbb{C}}^{lk} \sum_i \zeta_{\mathbb{C}}^{ij} \sigma_{z''z_0^{l-i}}$ . Taking  $i-l$  as new variable in the summation we get  $(\mathrm{Sh} \chi_{j,k})(z_0^l u_{z''}) = \zeta_{\mathbb{C}}^{lk} \sum_i \zeta_{\mathbb{C}}^{(i+l)j} \sigma_{z''z_0^{-i}} = \zeta_{\mathbb{C}}^{l(j+k)} \sum_i \zeta_{\mathbb{C}}^{ij} \sigma_{z''z_0^{-i}} = \chi_{j,j+k}(z_0^l u_{z''})$ , which gives the proposition for  $n$  odd.

If  $n = 2$  we have  $z_0 = \mathrm{diag}(-1, -1)$ . We have  $\chi_{j,k}(z u_{z''}) = \hat{s}_k(z)(\hat{s}_j(1)\chi_{z_0^0, k}(u_{z''}) + \hat{s}_j(z)\chi_{z_0, k}(u_{z''}))$ . We get  $\chi_{j,0}(z u_{z''}) = \hat{s}_0(z)(-\sigma_{z_0 z''} - \hat{s}_j(z_0)\sigma_{z''})$  and  $\chi_{j,1}(z u_{z''}) = \hat{s}_1(z)(\sigma_{z''} + \hat{s}_j(z_0)\sigma_{z_0 z''})$ . If  $\hat{s}_j$  is the identity character then  $\chi_{j,0}$  and  $\chi_{j,1}$  are invariant by  $\mathrm{Sh}$  since  $\mathrm{Sh}(u_{z''}) = u_{z''}$  and  $\mathrm{Sh}(z_0 u_{z''}) = z_0 u_{z_0 z''}$ . If  $\hat{s}_j$  is not trivial then,  $\chi_{j,0}(z u_{z''})$  and  $\chi_{j,1}(z u_{z''})$  are equal if  $z = 1$  and opposite if  $z \neq 1$ , thus, using again  $\mathrm{Sh}(u_{z''}) = u_{z''}$  and  $\mathrm{Sh}(z_0 u_{z''}) = z_0 u_{z_0 z''}$  we see that  $\chi_{j,0}$  and  $\chi_{j,1}$  are exchanged by  $\mathrm{Sh}$ . By Lemma 12, if  $q \equiv 1 \pmod{4}$  the character  $\hat{s}_0$  is trivial and we get the same result as in the odd case. If  $q \equiv -1 \pmod{4}$ , the character  $\hat{s}_1$  is trivial and  $\mathrm{Sh}$  exchanges  $\chi_{0,0}$  and  $\chi_{0,1}$  and fixes  $\chi_{1,0}$  and  $\chi_{1,1}$ , which is the announced result.  $\square$

We can now state:

**Proposition 16.** *For  $s = \mathrm{diag}(1, \zeta, \dots, \zeta^{n-1})$ , the bijection  $J : \chi_{j,i} \rightarrow \theta_{j,i}$  from  $\mathbb{C}\mathcal{E}(\mathrm{SL}_n^F, (s))$  to  $\mathbb{C}\mathcal{E}(\mathrm{CPGL}_n(s)^F, F, 1)$  restricts on characters to a refinement of the Jordan decomposition which satisfies Conjecture 7.*

*Proof.* Propositions 11 and 14 give the commutation of  $J$  with  $\mathrm{Sh}$ . It remains to show that  $J(R_{\mathbf{T}_{c^j}}^{\mathrm{SL}_n}(s_j)) = R_{\mathbf{T}_{c^j}^* \rtimes \langle c \rangle}^{\mathbf{T}_{c^j}^* \rtimes \langle c \rangle}(1)$ . As we have noticed after the proof of

Proposition 11 we have  $R_{\mathbf{T}_{c^j}}^{\mathbf{T}_{c^j}^* \rtimes \langle c \rangle} \mathrm{Id} = \begin{cases} \theta_{0,j} & \text{for } n \text{ odd or } q \equiv 1 \pmod{4} \\ \theta_{1,j} & \text{for } n = 2, q \equiv -1 \pmod{4} \end{cases}$ . On the

other hand for  $n$  odd or  $q \equiv 1 \pmod{4}$ , we have  $\chi_{0,j} = \sum_z \hat{s}_0(z)\chi_{z,j} = \sum_z \chi_{z,j}$  since in that case  $\hat{s}_0$  is the trivial character. For  $n = 2, q \equiv -1 \pmod{4}$ , we have  $\chi_{1,j} = \sum_z \hat{s}_1(z)\chi_{z,j} = \sum_z \chi_{z,j}$  since in that case  $\hat{s}_1$  is the trivial character. By definition of  $\chi_{z,j}$  we have  $\sum_z \chi_{z,j} = R_{\mathbf{T}_{c^j}}^{\mathrm{SL}_n}(s_j)$ , whence the proposition.  $\square$

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