

THE CONTINUOUS SUBSOLUTION PROBLEM FOR COMPLEX HESSIAN EQUATIONS

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ABSTRACT. Let $\Omega \subset \mathbb{C}^n$ be a bounded strongly m -pseudoconvex domain ($1 \leq m \leq n$) and μ a positive Borel measure on Ω . We study the Dirichlet problem for the complex Hessian equation $(dd^c u)^m \wedge \beta^{n-m} = \mu$ on Ω .

First we give a sufficient condition on the measure μ in terms of its domination by the m -Hessian capacity which guarantees the existence of a continuous solution to the associated Dirichlet problem with a continuous boundary datum.

As an application, we prove that if the equation has a continuous m -subharmonic subsolution whose modulus of continuity satisfies a Dini type condition, then the equation has a continuous solution with an arbitrary continuous boundary datum. Moreover when the measure has a finite mass, we give a precise quantitative estimate on the modulus of continuity of the solution.

One of the main steps in the proofs is to establish a new capacity estimate showing that the m -Hessian measure of a continuous m -subharmonic function on Ω with zero boundary values is dominated by an explicit function of the m -Hessian capacity with respect to Ω , involving the modulus of continuity of φ . Another important ingredient is a new weak stability estimate on the Hessian measure of a continuous m -subharmonic function.

INTRODUCTION

Complex Hessian equations are important examples of fully non-linear PDE's of second order on complex manifolds. They interpolate between (linear) complex Poisson equations ($m = 1$) and (non linear) complex Monge-Ampère equations ($m = n$). They arise in many geometric problems, including the J -flow [SW] and quaternionic geometry [AV]. They have attracted the attention of many researchers these last years. An account of the most relevant papers connected to this problem have been mentioned in [BZ20]. We will not repeat them here and refer to this paper and the references therein.

0.1. Statement of the problem. Let $\Omega \Subset \mathbb{C}^n$ be a bounded domain and m a fixed integer such that $1 \leq m \leq n$. We consider the following general Dirichlet problem for the complex m -Hessian equation :

The Dirichlet problem: Let $g \in C^0(\partial\Omega)$ be a continuous function (the boundary datum) and μ a positive Borel measure on Ω (the right hand side). The Dirichlet problem with boundary datum g and right hand side μ consists in finding a function $U \in \mathcal{SH}_m(\Omega) \cap C^0(\Omega)$ satisfying the following properties :

$$(0.1) \quad \begin{cases} (dd^c U)^m \wedge \beta^{n-m} = \mu, & \text{on } \Omega, \quad (\dagger) \\ U|_{\partial\Omega} = g, & \text{on } \partial\Omega, \quad (\dagger\dagger) \end{cases}$$

The equation (\dagger) must be understood in the sense of currents on Ω (see section 2).

The equality $(\dagger\dagger)$ means that $\lim_{z \rightarrow \zeta} U(z) = g(\zeta)$ for any $\zeta \in \partial\Omega$.

Observe that the comparison principle implies the uniqueness of the solution to the Dirichlet problem (0.1) when it exists. We will denote it by $U_{g,\mu} = U_{g,\mu}^\Omega$.

Recall the usual notations $d = \partial + \bar{\partial}$ and $d^c := (i/2)(\bar{\partial} - \partial)$ so that $dd^c = i\partial\bar{\partial}$. Given a real function $u \in \mathcal{C}^2(\Omega)$, for each integer $1 \leq k \leq n$, we denote by $\sigma_k(u)$ the continuous function defined at each point $z \in \Omega$ as the k -th symmetric polynomial of the eigenvalues $\lambda_1(z) \leq \dots \leq \lambda_n(z)$ of the complex Hessian matrix $\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z)\right)$ of u i.e.

$$\sigma_k(u)(z) := \sum_{1 \leq j_1 < \dots < j_k \leq n} \lambda_{j_1}(z) \cdots \lambda_{j_k}(z), \quad z \in \Omega.$$

A simple computation shows that

$$(dd^c u)^k \wedge \beta^{n-k} = \frac{(n-k)! k!}{n!} \sigma_k(u) \beta^n,$$

pointwise on Ω for $1 \leq k \leq m$, where $\beta := dd^c|z|^2$ is the usual Kähler form on \mathbb{C}^n .

We say that a real function $u \in \mathcal{C}^2(\Omega)$ is m -subharmonic on Ω if for any $1 \leq k \leq m$, we have $\sigma_k(u) \geq 0$ pointwise on Ω .

Observe that the function u is 1-subharmonic on Ω ($m = 1$) if it is subharmonic on Ω and $\sigma_1(u) = (1/4)\Delta u$, while u is n -subharmonic on Ω ($m = n$) if u is plurisubharmonic on Ω and $\sigma_n(u) = \det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z)\right)$.

It was shown by Z. Błocki in [Bl05], that it is possible to define a general notion of m -subharmonic function using the concept of m -positive currents (see section 2). Moreover, identifying positive (n, n) -currents with positive Radon measures, it is possible to define the k -Hessian measure $(dd^c u)^k \wedge \beta^{n-k}$ when $1 \leq k \leq m$ for any (locally) bounded m -subharmonic function u on Ω (see section 2).

Several questions related to the Dirichlet problem (0.1) will be addressed.

1. The first problem is to find a necessary and sufficient condition on μ which guarantees the existence of a solution to the Dirichlet problem (0.1).

2. The second problem is to study the regularity of the solution $U_{g,\mu}^\Omega$ in terms of the regularity of the data (g, μ) .

When $\mu = 0$, the Dirichlet problem (0.1) can be solved using the Perron method as for the complex Monge-Ampère equation (see [Bl05], [Ch16a]).

When $g = 0$ and μ is a positive Borel measure on Ω , the Dirichlet problem is much more difficult. A necessary condition for the existence of a solution to (0.1) is the existence of a subsolution.

For the complex Monge-Ampère equation, S. Kołodziej proved that if the Dirichlet problem (0.1) has a bounded subsolution, then it has a bounded solution ([Kol95]). The same result for the Hessian equation was proved by N. C. Nguyen in [N13].

The particular case of the Dirichlet problem (0.1) we are interested in can be formulated as follows.

The continuous subsolution problem : Let μ be a positive Borel measure on Ω . Assume that there exists a continuous function $\varphi \in \mathcal{SH}_m(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$ satisfying the following conditions :

$$(0.2) \quad \mu \leq (dd^c \varphi)^m \wedge \beta^{n-m}, \text{ on } \Omega, \text{ and } \varphi|_{\partial\Omega} \equiv 0.$$

(i) Does the Dirichlet problem (0.1) admit a continuous solution $U_{\mu,g}$ for any continuous boundary datum g ?

(ii) In this case, is it possible to estimate the modulus of continuity of the solution $U_{\mu,g}$ in terms of the modulus of continuity of φ and g and some characteristic function related to μ ?

The continuous subsolution problem stated above has attracted a lot of attention these last years. It was formulated by Kołodziej for the complex Monge-Ampère equation and addressed in [DGZ16] in the case of the existence of Hölder continuous subsolution.

There has been many articles on the subject. The Hölder continuous subsolution problem was solved very recently for positive Borel measures with finite mass in [BZ20]. For more details on the previous results on this problem, we refer to [BZ20] and the references therein.

Recently S. Kołodziej and N.C. Nguyen gave a Dini type sufficient condition on the modulus of continuity of the subsolution which guarantees the existence of a continuous plurisubharmonic solution for the complex Monge-Ampère equation ([KN20a]).

0.2. Main results. Our main goal in this paper is to give a partial answer to the "Continuous Subsolution Problem". Namely, we will give sufficient condition of Dini type on the modulus of continuity of the subsolution φ which guarantees the existence of a continuous solution to

the Dirichlet problem (0.1). Moreover we will give a precise estimate of the modulus of continuity of the solution $U_{\mu,g}$ when μ has finite mass.

We will improve and extend the result of [KN20a] to the Hessian equation using an original idea from [KN20b], some new ideas from [BZ20] and an idea from a former preliminary draft of this project which has not been completed. Moreover our main result improves the Hölder continuous subsolution theorem obtained in [BZ20]. The terminology will be defined in the next section.

Our first main result gives a sufficient condition on the Borel measure μ in terms of its diffusion with respect to the m -Hessian capacity which guarantees the existence of a continuous solution to the Dirichlet problem (0.1).

Theorem 1. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded strongly m -pseudoconvex domain and μ be a positive Borel measure on Ω with finite mass. Assume that μ is diffuse with respect to the m -Hessian capacity i.e. there exists a constant $A > 0$ such that for any compact set $K \subset \Omega$,*

$$\mu(K) \leq A c_m(K) \gamma(c_m(K)),$$

where $\gamma : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous increasing function on \mathbb{R}^+ which satisfies the following Dini type condition

$$(0.3) \quad \int_{0^+} \frac{\gamma(t)^{1/m}}{t} dt < +\infty.$$

Then for any continuous boundary datum $g \in \mathcal{C}^0(\partial\Omega)$, the Dirichlet problem (0.1) admits a unique solution $U = U_{\mu,g} \in \mathcal{SH}_m(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$.

The capacity $c_m(K) = c_m(K, \Omega)$ will be defined in the next section.

Our second main result gives a new comparison inequality which will be applied to positive Borel measures without restriction on their support nor on their mass.

Let us fix $0 < r < m/(n-m)$ and $0 < b < 2n$ and define the following functions for $t \in \mathbb{R}^+$:

$$(0.4) \quad \ell_m(t) := \begin{cases} t^r, & \text{if } 1 \leq m < n, \\ \exp(-bt^{-1/n}), & \text{if } m = n. \end{cases}$$

Theorem 2. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded m -hyperconvex domain and $\varphi \in \mathcal{SH}_m(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$ with $\varphi = 0$ on $\partial\Omega$.*

Then there exists a constant $B = B(m, n, \varphi, \Omega) > 0$ such that for any compact set $K \subset \Omega$,

$$\int_K (dd^c \varphi)^m \wedge \beta^{n-m} \leq B \{ \vartheta_m(c_m(K)) + [\vartheta_m(c_m(K))]^m \} c_m(K),$$

where $\vartheta_m(t) := \kappa_\varphi \circ \theta_m \circ \ell_m(t)$, κ_φ is the modulus of continuity of φ and θ_m is an inverse of the function $t \mapsto t^{2m} \kappa_\varphi(t)^{1-m}$.

The constant B in the theorem is explicit (see (4.18)).

Theorem 2 generalizes the estimate proved in [BZ20] in the Hölder continuous case.

As a consequence of Theorem 1 and Theorem 2, we will deduce the following two results which solves the continuous subsolution problem under a Dini type condition on the modulus of continuity of the subsolution.

Since the two results are different for complex Monge-Ampère equations and Hessian equations, we will state them separately.

Theorem 3. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded strongly m -pseudoconvex domain with $1 \leq m < n$ and μ a positive Borel measure on Ω .*

Assume that there exists $\varphi \in \mathcal{SH}_m(\Omega) \cap C^0(\overline{\Omega})$ such that

$$(0.5) \quad \mu \leq (dd^c \varphi)^m \wedge \beta^{n-m}, \text{ weakly on } \Omega \text{ and } \varphi|_{\partial\Omega} \equiv 0,$$

and the modulus of continuity κ_φ of φ satisfies the following Dini type condition:

$$(0.6) \quad \int_{0^+} \frac{[\kappa_\varphi(t)]^{1/m}}{t} dt < +\infty,$$

Then for any continuous function $g \in C^0(\partial\Omega)$, there exists a unique function $U = U_{g,\mu} \in \mathcal{SH}_m(\Omega) \cap C^0(\Omega)$ such that

$$(dd^c U)^m \wedge \beta^{n-m} = \mu, \text{ weakly on } \Omega \text{ and } U|_{\partial\Omega} = g.$$

Moreover if $\mu(\Omega) < +\infty$, the $\widehat{\kappa}$ -modulus of continuity of U satisfies the following estimate

$$\widehat{\kappa}_U(\delta) \leq C \widehat{\kappa}_m(\delta),$$

where $\widehat{\kappa}_m(\delta)$ is given by the equation (5.7) and $C = C(m, n, \mu, \varphi, \Omega) > 0$ is a uniform constant.

For Complex Monge-Ampère equations (the case $m = n$) we obtain a much better result.

Theorem 4. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain and μ a positive Borel measure on Ω .*

Assume that there exists $\varphi \in \mathcal{PSH}(\Omega) \cap C^0(\overline{\Omega})$ such that

$$(0.7) \quad \mu \leq (dd^c \varphi)^n \text{ weakly on } \Omega \text{ and } \varphi|_{\partial\Omega} \equiv 0.$$

and the modulus of continuity κ_φ of φ satisfies the following Dini type condition:

$$(0.8) \quad \int_{0^+} \frac{[\kappa_\varphi(t)]^{1/n}}{t|\log t|} dt < +\infty.$$

Then for any continuous function $g \in \mathcal{C}^0(\partial\Omega)$, there exists a unique function $U = U_{g,\mu} \in \mathcal{PSH}(\Omega) \cap \mathcal{C}^0(\Omega)$ such that

$$(dd^c U)^n = \mu, \text{ weakly on } \Omega, \text{ and } U|_{\partial\Omega} = g.$$

Moreover if $\mu(\Omega) < +\infty$, the $\widehat{\kappa}$ -modulus of continuity of U satisfies the following estimate

$$\widehat{\kappa}_U(\delta) \leq C \widehat{\kappa}_n(\delta),$$

where $\widehat{\kappa}_n(\delta)$ is given by the equation (5.7) with $m = n$ and $C = C(n, \mu, \varphi, \Omega) > 0$ is a uniform constant.

Here the $\widehat{\kappa}$ -modulus of continuity of a given function $\phi : \Omega \rightarrow \mathbb{R}$ is defined for $0 < \delta < \delta_0$ by

$$(0.9) \quad \widehat{\kappa}_\phi(\delta) := \sup_{z \in \Omega_\delta} (\widehat{\phi}_\delta(z) - \phi(z)),$$

where

$$(0.10) \quad \widehat{\phi}_\delta(z) := \int_{\mathbb{B}} \phi(z + \delta\xi) d\lambda_{\mathbb{B}}(\xi),$$

for $z \in \Omega_\delta := \{z \in \Omega; \text{dist}(z, \partial\Omega) > \delta\}$ and $0 < \delta < \delta_0$.

The existence of a continuous solution under the Dini type condition in Theorem 4 was proved recently in ([KN20a]) by a slightly different method. However our estimate on the modulus of continuity of the solution improves the result of [KN20a] where the measure μ is supposed to have a compact support.

0.3. Organization of the paper. In section 1, we give the necessary definitions and preliminaries that will be needed in the sequel.

Section 2 contains some news results which will play a crucial role in the proofs of our main results. We first give a new estimate on the behaviour near the boundary of a domain of the m -Hessian measure of a continuous m -potential in terms of its modulus of continuity and the m -Hessian capacity on the domain. Then we prove continuity properties of these measures acting on normalized potentials.

Section 3 contains the proof of Theorem 1. We first give a priori uniform estimates and then prove continuity of the Hessian potentials of Borel measures which are diffuse with respect to the corresponding capacity (see definition 3.1). Then we establish a new stability estimate that improves the one obtained in [BZ20] under a weaker domination

condition on the measure. This estimate is inspired by an estimate proved in [BGZ08] in the spirit of [EGZ09] in the case of compact Kähler manifolds.

Section 4 contains the proof of Theorem 2 as well as some consequences. The proof of this theorem consists in extending a similar result proved in [BZ20] in the Hölder continuous case.

Section 5 contains the proofs of Theorem 3 and Theorem 4. These proofs are done at the same time in several steps following the same scheme. We first use the weak stability result Theorem 3.11 to reduce the estimation of the modulus of continuity of the solution to the Dirichlet problem (0.1) to the estimate of the L^m -norm of the difference of two normalized potentials with respect to the measure μ using its domination by the Hessian measure of the subsolution. Then we use results from Section 3 to estimate the L^m -norm with respect to μ in terms of the L^m -norm with respect to the Lebesgue measure following a scheme which has become standard and which was initiated in [EGZ09] and completed in [GKZ08] (see also [DDGKPZ15] and [GZ17]).

Section 6 contains an extension of the main results by dropping the assumption on boundary values of the subsolution as well as an example of a singular measure that satisfies the hypothesis of Theorem 3 and Theorem 4.

1. PRELIMINARY RESULTS

In this section, we recall the basic properties of m -subharmonic functions and some known results we will use throughout the paper.

1.1. Hessian potentials. For a hermitian $n \times n$ matrix $a = (a_{j,\bar{k}})$ with complex coefficients, we denote by $\lambda_1, \dots, \lambda_n$ the eigenvalues of the matrix a . For any $1 \leq k \leq n$ we define the k -th trace of a by the formula

$$S_k(a) := \sum_{1 \leq j_1 < \dots < j_k \leq n} \lambda_{j_1} \cdots \lambda_{j_k},$$

which is the k -th elementary symmetric polynomial of the eigenvalues $(\lambda_1, \dots, \lambda_n)$ of a .

Recall that $d = \partial + \bar{\partial}$ and define $d^c := i(\bar{\partial} - \partial)$ so that $dd^c = 2i\partial\bar{\partial}$ and denote by

$$\beta := dd^c|z|^2$$

the standard Kähler form on \mathbb{C}^n .

Let $\mathbb{C}_{(1,1)}^n$ be the space of real $(1, 1)$ -forms on \mathbb{C}^n with constant coefficients, and define the cone of m -positive $(1, 1)$ -forms on \mathbb{C}^n by

$$\Theta_m := \{\omega \in \mathbb{C}_{(1,1)}^n; \omega \wedge \beta^{n-1} \geq 0, \dots, \omega^m \wedge \beta^{n-m} \geq 0\}.$$

Definition 1.1. 1) A smooth $(1, 1)$ -form ω on Ω is said to be m -postive on Ω if for any $z \in \Omega$, $\omega(z) \in \Theta_m$.

2) A function $u : \Omega \rightarrow \mathbb{R} \cup \{-\infty\}$ is said to be m -subharmonic on Ω if it is subharmonic on Ω (not identically $-\infty$ on any component) and for any collection of smooth m -positive $(1, 1)$ -forms $\omega_1, \dots, \omega_{m-1}$ on Ω , the following inequality holds in the sense of currents

$$dd^c u \wedge \omega_1 \wedge \dots \wedge \omega_{m-1} \wedge \beta^{n-m} \geq 0,$$

in the sense of currents on Ω .

We denote by $\mathcal{SH}_m(\Omega)$ the positive convex cone of m -subharmonic functions on Ω which are not identically $-\infty$ on any component of Ω . These are the m -Hessian potentials.

We give below the most basic properties of m -subharmonic functions that will be used in the sequel (see [Bl05], [Lu12]).

Proposition 1.2. 1. If $u \in \mathcal{C}^2(\Omega)$, then u is m -subharmonic on Ω if and only if $(dd^c u)^k \wedge \beta^{n-k} \geq 0$ pointwise on Ω for $k = 1, \dots, m$.

2. $\mathcal{PSH}(\Omega) = \mathcal{SH}_n(\Omega) \subsetneq \mathcal{SH}_{n-1}(\Omega) \subsetneq \dots \subsetneq \mathcal{SH}_1(\Omega) = \mathcal{SH}(\Omega)$.

3. $\mathcal{SH}_m(\Omega) \subset L_{loc}^1(\Omega)$ is a positive convex cone.

4. If u is m -subharmonic on Ω and $f : I \rightarrow \mathbb{R}$ is a convex, increasing function on some interval containing the image of u , then $f \circ u$ is m -subharmonic on Ω .

5. The limit of a decreasing sequence of functions in $\mathcal{SH}_m(\Omega)$ is m -subharmonic on Ω when it is not identically $-\infty$ on any component.

6. Let u be an m -subharmonic function on Ω . Let v be an m -subharmonic function on a domain $\Omega' \subset \mathbb{C}^n$ with $\Omega \cap \Omega' \neq \emptyset$. If $u \geq v$ on $\Omega \cap \partial\Omega'$, then the function

$$z \mapsto w(z) := \begin{cases} \max(u(z), v(z)) & \text{if } z \in \Omega \cap \Omega' \\ u(z) & \text{if } z \in \Omega \setminus \Omega' \end{cases}$$

is m -subharmonic on Ω .

1.2. Approximation of Hessian potentials. Another ingredient which will be important is the regularization process. Let χ be a fixed positive radial Borel function with compact support in the unit ball $\mathbb{B} \subset \mathbb{C}^n$ and $\int_{\mathbb{C}^n} \chi(\zeta) d\lambda_{2n}(\zeta) = 1$. For any $0 < \delta < \delta_0 := \text{diam}(\Omega)$, we set $\chi_\delta(\zeta) = \frac{1}{\delta^{2n}} \chi(\frac{\zeta}{\delta})$ and $\Omega_\delta = \{z \in \Omega; \text{dist}(z, \partial\Omega) > \delta\}$.

Let $u \in \mathcal{SH}_m(\Omega) \subset L_{loc}^1(\Omega)$ and define its standard δ -regularization by the formula

$$(1.1) \quad u \star \chi_\delta(z) := \int_{\Omega} u(z - \zeta) \chi_\delta(\zeta) d\lambda_{2n}(\zeta), z \in \Omega_\delta.$$

Then it is easy to see that u_δ is m -subharmonic and smooth on Ω_δ and decreases to u in Ω as δ decreases to 0.

Observe that when $\chi = \chi^{\mathbb{B}} := (1/\tau_n)\mathbf{1}_{\mathbb{B}}$ is the normalized characteristic function of the unit ball, then $u \star \chi_\delta = \widehat{u}_\delta$ is the mean-value function of u defined on Ω_δ by

$$\widehat{u}_\delta(z) := (1/\tau_n) \int_{\mathbb{B}} u(z + \delta\zeta) d\lambda_{2n}(\zeta), z \in \Omega_\delta,$$

where $\tau_n := \lambda_{2n}(\mathbb{B})$.

Lemma 1.3. *Let $u \in \mathcal{SH}_m(\Omega) \cap L^1(\Omega)$. Then for $0 < \delta < \delta_0$, its δ -regularization extends to \mathbb{C}^n by the formula*

$$(1.2) \quad u \star \chi_\delta u(z) := \int_{\Omega} u(\zeta) \chi_\delta(z - \zeta) d\lambda_{2n}(\zeta), z \in \mathbb{C}^n,$$

and have the following properties :

- 1) the function u_δ is m -subharmonic on Ω_δ , smooth on \mathbb{C}^n provided that χ is smooth;
- 2) (u_δ) decreases to u in Ω as δ decreases to 0;
- 3) the mean-value function \widehat{u}_δ satisfies the estimate

$$\int_{\Omega_\delta} (\widehat{u}_\delta(z) - u(z)) d\lambda_{2n}(z) \leq a_n \delta^2 \int_{\Omega_\delta} dd^c u \wedge \beta^{n-1},$$

where $a_n > 0$ is a constant which does not depend on u nor on δ .

Proof. The first and second property are clear. The third one follows from Poisson-Jensen formula for subharmonic functions (see [GKZ08], [Ze20]). \square

Let us introduce the notions of m -pseudoconvexity that will be used in the sequel.

Definition 1.4. 1. We say that the open set $\Omega \Subset \mathbb{C}^n$ is m -hyperconvex if it admits a defining function $\rho : \Omega \rightarrow]-\infty, 0[$ which is a bounded continuous m -subharmonic on Ω (see [Lu12, Lu15]).

2. We say that the open set $\Omega \Subset \mathbb{C}^n$ is strongly m -pseudoconvex if Ω admits a smooth defining function ρ which is strictly m -subharmonic in a neighbourhood of $\bar{\Omega}$ and satisfies $|\nabla\rho(z)| > 0$ pointwise on $\partial\Omega = \{\rho = 0\}$. In this case we can choose ρ so that

$$(1.3) \quad (dd^c \rho)^k \wedge \beta^{n-k} \geq \beta^n \text{ for } 1 \leq k \leq m,$$

pointwise on Ω .

Example 1.5. 1. Any euclidean ball in \mathbb{C}^n is strongly m -pseudoconvex and any polydisc in \mathbb{C}^n ($n \geq 2$) is m -hyperconvex but not strongly m -pseudoconvex.

2. The domain $\{z \in \mathbb{C}^n; \sum_{1 \leq j \leq n} |z_j| < 1\}$ is a bounded m -hyperconvex domain with Lipschitz but not smooth boundary, hence it is not strongly m -pseudoconvex.

The following lemma will be also needed. For a function $g \in C^0(\partial\Omega)$, we denote by $\mathcal{SH}_m^g(\Omega)$ the set of functions $w \in \mathcal{SH}_m(\Omega) \cap L^\infty(\Omega)$ such that $w = g$ on $\partial\Omega$ i.e. for any $\zeta \in \Omega$, $\lim_{z \rightarrow \zeta} w(z) = g(\zeta)$.

Lemma 1.6. *Let $g \in C^0(\partial\Omega)$ and $w \in \mathcal{SH}_m^g(\Omega)$. Then there exists a decreasing sequence (w_j) of functions in $\mathcal{SH}_m^g(\Omega) \cap C^0(\bar{\Omega})$ which converges to w pointwise on Ω .*

Proof. First take any decreasing sequence of continuous functions (h_j) on $\bar{\Omega}$ which converges to w on Ω . We can arrange so that $h_j = g$ on $\partial\Omega$. Indeed take the harmonic extension G of g to Ω and then the sequence $\min\{h_j, G\}$ satisfies the requirement.

Now set

$$w_j := \sup\{v \in \mathcal{SH}_m(\Omega); v \leq h_j\}.$$

By [BZ20], we know that the sequence (w_j) satisfies all the requirements of the lemma. \square

1.3. Remarks on the modulus of continuity. Let $\phi : \Omega \rightarrow \mathbb{R}$ be a continuous function. We fix $\delta_0 > 0$ so that $\Omega_{\delta_0} \neq \emptyset$ and recall the following definition for $0 < \delta < \delta_0$ and $z \in \Omega_\delta$,

$$(1.4) \quad \widehat{\phi}_\delta(z) := \int_{\mathbb{B}} \phi(z + \delta\zeta) d\lambda_{\mathbb{B}}(\zeta),$$

where $\lambda_{\mathbb{B}}$ is the normalized Lebesgue measure on \mathbb{B} .

We introduce the modulus of (uniform) continuity of ϕ on Ω defined for $\varepsilon > 0$ by the formula

$$(1.5) \quad \kappa_\phi(\varepsilon) := \sup\{|\phi(z) - \phi(z')|; z, z' \in \Omega, |z - z'| \leq \varepsilon\}.$$

Then ϕ extends to a uniformly continuous function on $\bar{\Omega}$ if and only if $\lim_{\varepsilon \rightarrow 0^+} \kappa_\phi(\varepsilon) = 0$.

We introduce another modulus of continuity defined for $0 < \delta < \delta_0$ by the formula

$$(1.6) \quad \widehat{\kappa}_\phi(\delta) := \sup_{\Omega_\delta} \left(\widehat{\phi}_\delta(z) - \phi(z) \right).$$

We see immediately that $\widehat{\kappa}_\phi(\delta) \leq \kappa_\phi(\delta)$ for any $0 < \delta < \delta_0$.

These moduli quantify the continuity of ϕ on Ω . While the (full) modulus of continuity κ_ϕ characterizes uniform continuity of ϕ on $\bar{\Omega}$, the (relative) modulus of continuity $\widehat{\kappa}_\phi$ only characterizes the continuity of ϕ on Ω . Indeed the condition $\lim_{\delta \rightarrow 0^+} \widehat{\kappa}_\phi(\delta) = 0$ implies that the function ϕ is continuous on Ω , but it does not imply the extension of the function ϕ by continuity to $\bar{\Omega}$ as the example of a harmonic function on Ω shows.

We will state a result from [Ze20] which clarifies the relations between these notions of continuity in some cases.

We need some definitions.

Definition 1.7. 1. A continuous function $\kappa : [0, l] \rightarrow \mathbb{R}^+$ is a modulus of continuity if it is increasing, subadditive and satisfies $\kappa(0) = 0$. It's always possible to extend such a function to the whole \mathbb{R}^+ with the same properties.

2. A function $\phi : \Omega \rightarrow \mathbb{R}$ is said to be κ -continuous near the boundary $\partial\Omega$ if there exists $0 < \delta_1 < \delta_0$ small enough and a constant $M_1 > 0$ such that for any $\zeta \in \partial\Omega$ and any $z \in \Omega$ with $|z - \zeta| \leq \delta \leq \delta_1$, we have $|u(z) - u(\zeta)| \leq M_1 \kappa(\delta)$.

Uniform continuity on $\bar{\Omega}$ implies uniform continuity near the boundary $\partial\Omega$. However as observed above, the condition $\lim_{\delta \rightarrow 0} \widehat{\kappa}_\phi(\delta) = 0$ implies the continuity of ϕ on Ω but it does not imply continuity near the boundary $\partial\Omega$.

We first introduce the following condition on κ .

$$(1.7) \quad \exists A > 1, \limsup_{t \rightarrow 0^+} \frac{\kappa(At)}{A\kappa(t)} < \frac{1}{2n}.$$

Observe that the condition (1.7) is satisfied by any logarithmic Hölder modulus of continuity $\kappa_{\alpha, \nu}(t) := t^\alpha (-\log t)^\nu$ for $0 < t \ll 1$ with $0 \leq \alpha < 1$ and $\nu \in \mathbb{R}$, with $\nu < 0$ when $\alpha = 0$.

The following lemma is proved in [Ze20].

Lemma 1.8. *Let κ be a modulus of continuity satisfying (1.7). Let $\Omega \Subset \mathbb{C}^n$ be a bounded domain and $u \in \mathcal{SH}(\Omega) \cap L^\infty(\bar{\Omega})$. Assume that u is κ -continuous near $\partial\Omega$. Then the following properties are equivalent:*

(i) $\exists c_1 > 0, \exists \delta_1$ with $0 < \delta_1 < \delta_0$ such that for any $0 < \delta < \delta_1$

$$\widehat{u}_\delta(z) \leq u(z) + c_1 \kappa(\delta), \text{ for any } z \in \Omega_\delta,$$

(ii) $\exists c_2 > 0, \exists \delta_2$ with $0 < \delta_2 < \delta_0$ such that for any $0 < \delta < \delta_2$,

$$\sup_{\bar{B}(z, \delta)} u \leq u(z) + c_2 \kappa(\delta), \text{ for any } z \in \Omega_\delta.$$

Moreover if one of these conditions is satisfied then u is κ -continuous on $\bar{\Omega}$ i.e. $\kappa_u \leq C \kappa$, where $C > 0$ is a uniform constant.

1.4. Complex Hessian operators. Following [Bl05], we can define the Hessian operators acting on (locally) bounded m -subharmonic functions as follows. Given $u_1, \dots, u_k \in \mathcal{SH}_m(\Omega) \cap L^\infty(\Omega)$ ($1 \leq k \leq m$), one can define inductively the following positive $(m-k, m-k)$ -current on Ω

$$dd^c u_1 \wedge \dots \wedge dd^c u_k \wedge \beta^{n-m} := dd^c(u_1 dd^c u_2 \wedge \dots \wedge dd^c u_k \wedge \beta^{n-m}).$$

In particular, if $u \in \mathcal{SH}_m(\Omega) \cap L_{loc}^\infty(\Omega)$, the positive current $(dd^c u)^m \wedge \beta^{n-m}$ can be identified to a positive Borel measure on Ω , the so called m -Hessian measure of u defined by :

$$(dd^c u)^m \wedge \beta^{n-m} = \frac{m!(n-m)!}{n!} \sigma_m(u) \beta^n.$$

Observe that when $m = 1$, $\sigma_1(u) = dd^c u \wedge \beta^{n-1} / \beta^n$ is the Riesz measure of u (up to a positive constant), while $\sigma_n(u) = (dd^c u)^n / \beta^n$ is the complex Monge-Ampère measure of u on Ω .

It is then possible to extend Bedford-Taylor theory to this context. In particular, Chern-Levine Nirenberg inequalities hold and the Hessian operators are continuous under local uniform convergence and monotone convergence pointwise a.e. on Ω of sequences of functions in $\mathcal{SH}_m(\Omega) \cap L_{loc}^\infty(\Omega)$ (see [Bl05], [Lu12]).

We define $\mathcal{E}_m^0(\Omega)$ to be the positive convex cone of negative functions $\phi \in \mathcal{SH}_m(\Omega) \cap L^\infty(\Omega)$ such that

$$\int_{\Omega} (dd^c \phi)^m \wedge \beta^{n-m} < +\infty, \quad \phi|_{\partial\Omega} \equiv 0.$$

These are the "test functions" in m -Hessian Potential Theory in the sense that Stokes theorem is valid for these functions (see [Lu12]).

More precisely it follows from [Lu12, Lu15] that the following property holds: if $\phi \in \mathcal{E}_m^0(\Omega)$ and $u, v \in \mathcal{SH}_m(\Omega) \cap L^\infty(\Omega)$ with $u \leq 0$, then for $0 \leq k \leq m-1$,

$$(1.8) \quad \int_{\Omega} (-\phi) dd^c u \wedge (dd^c v)^k \wedge \beta^{n-k-1} \leq \int_{\Omega} (-u) dd^c \phi \wedge (dd^c v)^k \wedge \beta^{n-k-1}.$$

An important tool in the corresponding Potential Theory is the Comparison Principle.

Proposition 1.9. *Assume that $u, v \in \mathcal{SH}_m(\Omega) \cap L^\infty(\Omega)$ and for any $\zeta \in \partial\Omega$, $\liminf_{z \rightarrow \zeta} (u(z) - v(z)) \geq 0$. Then*

$$\int_{\{u < v\}} (dd^c v)^m \wedge \beta^{n-m} \leq \int_{\{u < v\}} (dd^c u)^m \wedge \beta^{n-m}.$$

Consequently, if $(dd^c u)^m \wedge \beta^{n-m} \leq (dd^c v)^m \wedge \beta^{n-m}$ weakly on Ω , then $u \geq v$ in Ω .

It follows from the comparison principle that if the Dirichlet problem (0.1) admits a solution, then it is unique.

Let us recall the following estimates due to Cegrell ([Ceg04]) for complex Monge-Ampère operators and extended by Charabati to complex Hessian operators ([Ch16a]).

Lemma 1.10. *Let $u, v, w \in \mathcal{E}_m^0(\Omega)$. Then for any $1 \leq k \leq m - 1$*

$$\int_{\Omega} dd^c u \wedge (dd^c v)^k \wedge (dd^c w)^{m-k-1} \wedge \beta^{n-m} \leq I_m(u)^{\frac{1}{m}} I_m(v)^{\frac{k}{m}} I_m(w)^{\frac{m-k-1}{m}},$$

where $I_m(u) := \int_{\Omega} (dd^c u)^m \wedge \beta^{n-m}$.

In particular, if Ω is strongly m -hyperconvex, then

$$\int_{\Omega} dd^c u \wedge (dd^c w)^k \wedge \beta^{n-k-1} \leq c_{m,n} (I_m(u))^{\frac{1}{m}} (I_m(w))^{\frac{k}{m}},$$

and

$$\int_{\Omega} dd^c u \wedge \beta^{n-1} \leq c_{m,n} (I_m(u))^{\frac{1}{m}}$$

where $c_{m,n} > 0$ is a uniform constant.

The following consequence will be useful in the sequel. This result is usually stated for plurisubharmonic functions on a bounded domain with boundary values 0. Let us give a more general version using Cegrell inequalities (see [BZ20]).

Corollary 1.11. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded strongly m -pseudoconvex domain. Assume that $u, v \in \mathcal{SH}_m(\Omega) \cap L^\infty(\Omega)$ satisfy $u \leq v$ on Ω and for any $\zeta \in \partial\Omega$, $\lim_{z \rightarrow \zeta} (u(z) - v(z)) = 0$. Then*

$$\int_{\Omega} (dd^c v)^m \wedge \beta^{n-m} \leq \int_{\Omega} (dd^c u)^m \wedge \beta^{n-m}.$$

We will need the following result which was proved by Blocki for the complex Monge-Ampère operator ([Bl93])

Lemma 1.12. *Let $\psi, v, w \in \mathcal{SH}_m(\Omega) \cap L^\infty(\Omega)$ such that $\psi \leq 0$, $v \leq w$ and $\lim_{z \rightarrow \zeta} (w(z) - v(z)) = 0$. Then*

$$\int_{\Omega} (w - v)^m (dd^c \psi)^m \wedge \beta^{n-m} \leq m! \|\psi\|_{\infty}^m \int_{\Omega} (dd^c v)^m \wedge \beta^{n-m}$$

The proof in this case is the same as in [Bl93] since it essentially only uses the integration by parts formula

2. HESSIAN MEASURES OF CONTINUOUS POTENTIALS

2.1. Hessian capacities. An important tool in dealing with our problems is the notion of capacity. This was introduced by Bedford and Taylor in their pioneer work for the complex Monge-Ampère operator (see [BT82]). Let us recall the corresponding notion of capacity we will use here (see [Lu12], [SA13]). Let $\Omega \Subset \mathbb{C}^n$ be a m -hyperconvex domain. The m -Hessian capacity is defined as follows. For any compact set $K \subset \Omega$,

$$c_m(K, \Omega) := \sup \left\{ \int_K (dd^c u)^m \wedge \beta^{n-m}; u \in \mathcal{SH}_m(\Omega), -1 \leq u \leq 0 \right\}.$$

We can extend this capacity as an outer capacity on Ω . Given a set $S \subset \Omega$, we define the inner capacity of S by the formula

$$c_m(S, \Omega) := \sup \{c_m(K, \Omega); K \text{ compact } K \subset S\}.$$

The outer capacity of S is defined by the formula

$$c_m^*(S, \Omega) := \inf \{c_m(U, \Omega); U \text{ is open } U \supset S\},$$

One can show that $c_m^*(\cdot, \Omega)$ is a Choquet capacity and then any Borel set $B \subset \Omega$ is capacitable and for any compact set $K \subset \Omega$,

$$(2.1) \quad c_m(K, \Omega) = \int_{\Omega} (dd^c u_K^*)^m \wedge \beta^{n-m},$$

where u_K is the relative equilibrium potential of (K, Ω) defined by the formula :

$$u_K := \sup \{u \in \mathcal{SH}_m(\Omega); u \leq 0 \text{ in } \Omega, u \leq -\mathbf{1}_K \text{ on } \Omega\},$$

and u_K^* is its upper semi-continuous regularization on Ω (see [Lu12]).

It is well known that u_K^* is m -subharmonic on Ω , $-1 \leq u_K^* \leq 0$, $u_K^* = -1$ quasi-everywhere (with respect to c_m) on Ω and $u_K^* \rightarrow 0$ as $z \rightarrow \partial\Omega$ (see [Lu12]).

2.2. Hessian mass estimates near the boundary. Here we prove a comparison inequality which seems to be new even in the case of a complex Monge-Ampère measure. This will play a crucial role in the proof of Theorem 2 and may have an interest in its own. It is a generalization of an estimate proved in [BZ20] for Hessian measures of Hölder continuous potentials.

Lemma 2.1. *Let $\Omega \Subset \mathbb{C}^n$ be a m -hyperconvex domain and $\varphi \in \text{SH}_m(\Omega) \cap L^\infty(\Omega)$. Then for any compact set $K \subset \Omega$ we have*

$$\int_K (dd^c \varphi)^m \wedge \beta^{n-m} \leq (\text{osc}_K \varphi)^m c_m(K, \Omega).$$

If moreover φ is continuous in $\bar{\Omega}$ and $\varphi = 0$ on $\partial\Omega$, then for any compact subset $K \subset \Omega$, we have

$$\int_K (dd^c \varphi)^m \wedge \beta^{n-m} \leq [\kappa(\delta_K(\partial\Omega))]^m c_m(K, \Omega),$$

where $\kappa = \kappa_\varphi$ is the modulus of continuity of φ on $\bar{\Omega}$ and

$$\delta_K(\partial\Omega) := \sup_{z \in K} \text{dist}(z, \partial\Omega)$$

is the Hausdorff distance of K to the boundary $\partial\Omega$.

Proof. 1) We can assume that $\max_K \varphi = 0$ and $\varphi \not\equiv 0$. Then $a := \text{osc}_K \varphi = -\inf_K \varphi > 0$ and then the function $v := a^{-1}\varphi$ is m -subharmonic on Ω , and satisfies the inequalities $v \leq 0$ on Ω and $v \geq -1$ on K .

Fix $\varepsilon > 0$ and let u_K be the relative extremal m -subharmonic function of (K, Ω) . Then $K \subset \{(1 + \varepsilon)u_K^* < v\} \cup \{u_K < u_K^*\}$. Since the set $\{u_K < u_K^*\}$ has zero m -capacity (see [Lu12]), it follows from the comparison principle that

$$\begin{aligned} \int_K (dd^c v)^m \wedge \beta^{n-m} &\leq \int_{\{(1+\varepsilon)u_K^* < v\}} (dd^c v)^m \wedge \beta^{n-m} \\ &\leq (1 + \varepsilon)^m \int_{\{(1+\varepsilon)u_K^* < v\}} (dd^c u_K^*)^m \wedge \beta^{n-m} \\ &\leq (1 + \varepsilon)^m \int_\Omega (dd^c u_K^*)^m \wedge \beta^{n-m} = (1 + \varepsilon)^m c_m(K, \Omega). \end{aligned}$$

The last identity follows from [Lu12]. Letting $\varepsilon \rightarrow 0$, we obtain the first statement.

2) Fix a compact set $K \subset \Omega$. Since κ is the modulus of continuity of φ , we have for any $\zeta \in \partial\Omega$ and $z \in K$ $\varphi(\zeta) - \varphi(z) \leq \kappa(|\zeta - z|)$. Since $\varphi = 0$ in $\partial\Omega$, we obtain that for any $z \in K$,

$$-\varphi(z) \leq \kappa(\delta_K(\partial\Omega)).$$

Hence $\text{osc}_K \varphi \leq \kappa(\delta_K(\partial\Omega))$. Applying the first statement to φ we obtain the required inequality. \square

2.3. Hessian measures acting on Hessian potentials. We will study continuity properties of the functional associated to the Hessian measure of a function $\varphi \in \mathcal{SH}_m(\Omega) \cap C^0(\bar{\Omega})$, acting on the space $\mathcal{SH}_m(\Omega) \cap L^\infty(\Omega)$.

Let $g \in C^0(\partial\Omega)$ be a continuous function on $\partial\Omega$ and $R > 0$ a positive constant. We denote by $\mathcal{SH}_m^g(\Omega, R)$ the set of functions $v \in \mathcal{SH}_m(\Omega) \cap$

$L^\infty(\Omega)$ such that

$$\int_{\Omega} (dd^c v)^m \wedge \beta^{n-m} \leq R, \quad \text{and} \quad \lim_{z \rightarrow \zeta} v(z) = g(\zeta), \quad \forall \zeta \in \partial\Omega.$$

The following result improves previous estimates given in [N14] and [BZ20].

Theorem 2.2. *Let $\varphi \in \mathcal{SH}_m(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$ and $g \in \mathcal{C}^0(\partial\Omega)$ be given functions. Then there exists $C_m = C(m, \Omega, g, R) > 0$ such that for every $u, v \in \mathcal{SH}_m^g(\Omega, R)$, we have*

$$(2.2) \quad \int_{\Omega} |u - v|^m (dd^c \varphi)^m \wedge \beta^{n-m} \leq C_m \kappa_\varphi \circ \theta_m (\|u - v\|_m^m),$$

where $\|u - v\|_m := (\int_{\Omega} |u - v|^m d\lambda_{2n})^{1/m}$, θ_m is the reciprocal of the function $t \mapsto t^{2m} \kappa_\varphi^{1-m}(t)$.

Proof. Observe that for any $\varepsilon > 0$, $u_\varepsilon := \max\{u - \varepsilon, v\} \in \mathcal{SH}_m^g(\Omega)$, $u_\varepsilon \geq v$ and $u_\varepsilon = v$ near the boundary $\partial\Omega$. By the comparison principle, this implies that $u_\varepsilon \in \mathcal{SH}_m^g(\Omega, R)$. Therefore, replacing u by u_ε , we can assume that $u \geq v$ on Ω and $u = v$ near the boundary, for the inequality (2.2) will follow since $|u - v| = (\max\{u, v\} - u) + (\max\{u, v\} - v)$.

On the other hand by approximation on the support S of $u - v$ which is compact, we can assume that u and v are smooth on a neighbourhood of S .

Then it remains to estimate the following integral

$$I_m := \int_{\Omega} (u - v)^m (dd^c \varphi)^m \wedge \beta^{n-m}.$$

Observe first that we can extend φ by continuity to \mathbb{C}^n with the same modulus of continuity. Indeed, it is easy to see that the function defined for $z \in \mathbb{C}^n$ by the following formula

$$\tilde{\varphi}(z) := \sup\{\varphi(\zeta) - \kappa_\varphi(|z - \zeta|); \zeta \in \bar{\Omega}\}.$$

is the required extension. For simplicity, it will be denote by φ .

Then we denote by φ_δ the smooth approximants of φ on \mathbb{C}^n , defined by (1.2).

We know that for $0 < \delta < \delta_0$, $\varphi_\delta \in \mathcal{SH}_m(\Omega_\delta) \cap \mathcal{C}^\infty(\mathbb{C}^n)$.

Since φ_δ is not m -subharmonic on the whole Ω , we will consider its m -subharmonic envelope defined by the formula :

$$(2.3) \quad \psi_\delta(z) := \sup\{\psi(z); \psi \in \mathcal{SH}_m(\Omega), \psi \leq \varphi_\delta \text{ on } \Omega\}, \quad z \in \Omega.$$

We know by [BZ20, Theorem 3.3] that $\psi_\delta \in \mathcal{SH}_m(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$, $\psi_\delta \leq \varphi_\delta$ on Ω and

$$(2.4) \quad (dd^c \psi_\delta)^m \wedge \beta^{n-m} \leq (\sigma_m(\varphi_\delta))_+,$$

weakly on Ω , where $(\sigma_m(\varphi_\delta))_+$ is defined pointwise on Ω by $(\sigma_m(\varphi_\delta))_+(z) = \sigma_m(\varphi_\delta)(z)$ for $z \in \Omega$ such that $dd^c \varphi_\delta(z) \in \Theta_m$ and $(\sigma_m(\varphi_\delta))_+(z) = 0$ otherwise.

To prove the required estimate, we write for $0 < \delta < \delta_0$

$$I_m = A_m(\delta) + B_m(\delta),$$

where

$$A_m(\delta) := \int_{\Omega} (u - v)^m [(dd^c \varphi)^m - (dd^c \psi_\delta)^m] \wedge \beta^{n-m}.$$

and

$$B_m(\delta) := \int_{\Omega} (u - v)^m (dd^c \psi_\delta)^m \wedge \beta^{n-m}.$$

We estimate each term separately for fixed $0 < \delta < \delta_0$.

To estimate the first term, observe that

$$((dd^c \varphi)^m - (dd^c \psi_\delta)^m) \wedge \beta^{n-m} = dd^c(\varphi - \psi_\delta) \wedge T,$$

where $T := \sum_{j=0}^{m-1} (dd^c \varphi)^j \wedge (dd^c \psi_\delta)^{m-j-1} \wedge \beta^{n-m}$.

Then

$$A_m(\delta) = \int_{\Omega} (u - v)^m dd^c(\varphi - \psi_\delta - \kappa_\varphi(\delta)) \wedge T.$$

Integration by parts yields

$$A_m(\delta) = \int_{\Omega} (\psi_\delta - \varphi + \kappa_\varphi(\delta)) [-dd^c(u - v)^m] \wedge T.$$

An easy computation shows that

$$(2.5) \quad \begin{aligned} -dd^c(u - v)^m \wedge T &\leq m(u - v)^{m-1} dd^c(v - u) \wedge T \\ &\leq m(u - v)^{m-1} dd^c v \wedge T, \end{aligned}$$

in the sense of currents on Ω .

Observe that from the definition we have $\psi_\delta \leq \varphi_\delta \leq \varphi + \kappa_\varphi(\delta)$ on Ω . On the other hand, since $\varphi - \kappa_\varphi(\delta) \leq \varphi_\delta$ on Ω , it follows that $\psi_\delta - \varphi + \kappa_\varphi(\delta) \geq 0$ on Ω . Combining the two estimates we conclude that $0 \leq \psi_\delta - \varphi + \kappa_\varphi(\delta) \leq 2\kappa_\varphi(\delta)$, and then

$$A_m(\delta) \leq 2m \kappa_\varphi(\delta) \int_{\Omega} (u - v)^{m-1} dd^c v \wedge T.$$

By definition of T we have

$$\begin{aligned} & \int_{\Omega} (u - v)^{m-1} dd^c v \wedge T \\ &= \sum_{j=0}^{m-1} \int_{\Omega} (u - v)^{m-1} dd^c v \wedge (dd^c \varphi)^j \wedge (dd^c \psi_{\delta})^{m-j-1} \wedge \beta^{n-m} \end{aligned}$$

Observe that if we write

$$(dd^c \varphi)^j \wedge (dd^c \psi_{\delta})^{m-j-1} \wedge \beta^{n-m} = dd^c w \wedge S_j,$$

where $w = \varphi$ or $w = \psi_{\delta}$, then as before by integration by parts using an inequality analogous to (2.5) with k instead of m and $dd^c v \wedge S_j$ instead of T , we obtain that for $1 \leq k \leq m$,

$$\int_{\Omega} (u - v)^k dd^c v \wedge dd^c w \wedge S_j \leq k \|w\|_{\Omega} \int_{\Omega} (u - v)^{k-1} (dd^c v)^2 \wedge S_j.$$

Repeating the integration by parts we finally get

$$(2.6) \quad A_m(\delta) \leq 2m! \|\varphi\|_{\Omega}^{m-1} \kappa_{\varphi}(\delta) \int_{\Omega} (dd^c v)^m \wedge \beta^{n-m} \leq C_1 \kappa_{\varphi}(\delta),$$

where $C_1 := mm! R \|\varphi\|_{\Omega}^{m-1}$.

To estimate the second term, we need to establish the following estimate $0 < \delta < \delta_0$,

$$(2.7) \quad dd^c \varphi_{\delta} \leq M_2 \frac{\kappa_{\varphi}(\delta)}{\delta^2} \beta, \text{ pointwise on } \Omega,$$

where $M_2 > 0$ is a uniform constant.

Indeed, by differentiating the integral formula $\varphi_{\delta}(z) = \varphi \star \chi_{\delta}(z)$ and by making an obvious change of variables, we obtain for $j, k = 1, \dots, n$

$$\begin{aligned} \partial_j \partial_{\bar{k}} \varphi_{\delta}(z) &= \delta^{-2} \int_{\mathbb{C}^n} \varphi(z - \delta \eta) \partial_j \partial_{\bar{k}} \chi(\eta) d\lambda_{2n}(\eta) \\ &= \delta^{-2} \int_{\mathbb{C}^n} [\varphi(z - \delta \eta) - \varphi(z)] \partial_j \partial_{\bar{k}} \chi(\eta) d\lambda_{2n}(\eta), \end{aligned}$$

where the last equation follows from the fact that by Stokes formula $\int_{\mathbb{C}^n} \partial_j \partial_{\bar{k}} \chi(\eta) d\lambda_{2n}(\eta) = 0$ since χ is a smooth test function with compact support. Thus the estimate (2.7) follows from the last equation since the support of χ is contained in the unit ball.

Now from the inequalities (2.4) and (2.7), it follows that

$$(dd^c \psi_{\delta})^m \wedge \beta^{n-m} \leq M_2^m \frac{\kappa_{\varphi}^m(\delta)}{\delta^{2m}} \beta^n, \text{ weakly on } \Omega.$$

Therefore we have

$$(2.8) \quad B_m(\delta) \leq C_2 \frac{\kappa_\varphi^m(\delta)}{\delta^{2m}} \int_\Omega (u-v)^m \beta^n,$$

where $C_2 := M_2^m$.

From (2.6) and (2.8) we conclude that

$$\int_\Omega (u-v)^m (dd^c \varphi)^m \wedge \beta^{n-m} \leq C_1 \kappa_\varphi(\delta) + C_2 \frac{\kappa_\varphi^m(\delta)}{\delta^{2m}} \int_\Omega (u-v)^m \beta^n.$$

We want to optimize the right hand side by taking $\delta > 0$ so that

$$\delta^{2m} \kappa_\varphi^{1-m}(\delta) = \int_\Omega (u-v)^m \beta^n = \|u-v\|_m^m,$$

i.e. $\delta = \theta_m(\|u-v\|_m^m)$, where θ_m is the reciprocal of the function $t \mapsto t^{2m} \kappa_\varphi^{1-m}(t)$. This is possible if $\|u-v\|_m^m \leq \theta_m^{-1}(\delta_0)$ so that $\delta < \delta_0$. Then applying the previous estimate we obtain the estimate of the Lemma in this case.

Now assume that $\|u-v\|_m^m > \theta_m^{-1}(\delta_0)$. By Lemma 1.12, we have

$$\int_\Omega (u-v)^m (dd^c \varphi)^m \wedge \beta^{n-m} \leq m! \|\varphi\|_\infty^m \int_\Omega (dd^c v)^m \wedge \beta^{n-m} \leq m! R \|\varphi\|_\infty^m.$$

We see that we obtain the inequality of the Lemma 2.2 by increasing the constant C consequently. \square

Corollary 2.3. *Under the same assumptions as the Theorem 2.2, we have*

$$\int_\Omega |u-v|^m \sigma_m(\varphi) \leq C(m) \kappa_\varphi \circ \theta_m(M \|u-v\|_1),$$

where $M := [\|u-v\|_\infty]^{m-1}$.

Proof. Apply Theorem 2.2 and observe that

$$\|u-v\|_m^m \leq \|u-v\|_\infty^{m-1} \|u-v\|_1.$$

The required inequality follows immediately, since κ_φ is non decreasing. \square

2.4. Global approximants to the solution. Let $u \in \mathcal{SH}_m(\Omega) \cap \mathcal{C}(\bar{\Omega})$. We define the volume mean-values of u as follows:

$$(2.9) \quad \hat{u}_\delta(z) := \frac{1}{\tau_{2n} \delta^{2n}} \int_{|\zeta-z| \leq \delta} u(\zeta) dV_{2n}(\zeta), z \in \Omega_\delta,$$

where τ_{2n} is the volume of the unit ball in \mathbb{C}^n .

We need the following lemma which was proved in [Ch16b] in the Hölder continuous case.

Lemma 2.4. *Let $u \in \mathcal{SH}_m(\Omega) \cap L^\infty(\Omega)$ such that there exists two functions $v, w : \bar{\Omega} \rightarrow \mathbb{R}$ continuous on $\bar{\Omega}$ such that $v \leq u \leq w$ on Ω and $v = w$ on $\partial\Omega$.*

Then there exist $\delta_0 > 0$ small enough, depending on Ω , such that for any $0 < \delta < \delta_0$ the function defined by

$$(2.10) \quad \tilde{u}_\delta = \begin{cases} \max\{\hat{u}_\delta - \hat{\kappa}(\delta), u\} & \text{on } \Omega_\delta, \\ u & \text{on } \Omega \setminus \Omega_\delta, \end{cases}$$

is a bounded m -subharmonic function on Ω which satisfies the inequalities

$$0 \leq \tilde{u}_\delta - u \leq \hat{u}_\delta - u \leq \tilde{u}_\delta - u + \hat{\kappa}(\delta), \quad \text{on } \Omega_\delta,$$

where $\hat{\kappa}(\delta) := \hat{\kappa}_v(\delta) + \hat{\kappa}_w(\delta) + \delta$ for $0 < \delta < \delta_0$.

Moreover $\tilde{u}_\delta = u$ in a neighbourhood of $\partial\Omega_\delta$ in Ω .

Proof. By the gluing property (see Proposition 1.2), it is enough to prove that for $\delta > 0$ small enough, $\hat{u}_\delta - \hat{\kappa}(\delta) \leq u$ on $\partial\Omega_\delta$.

Indeed fix $0 < \delta < \delta_0 < 1$ and fix $z \in \partial\Omega_\delta$. Then there exists $\zeta \in \partial\Omega$ such that $|z - \zeta| = \delta$. Hence

$$\begin{aligned} \hat{u}_\delta(z) \leq \hat{w}_\delta(z) &\leq w(\zeta) + \hat{\kappa}_w(\delta) \\ &\leq v(\zeta) + \hat{\kappa}_w(\delta) \\ &\leq v(z) + \hat{\kappa}_v(\delta) + \hat{\kappa}_w(\delta) \\ &< v(z) + \hat{\kappa}(\delta) \leq u(z) + \hat{\kappa}(\delta) \end{aligned}$$

which proves the required condition. Observe that, since v is continuous, the set $\{\hat{u}_\delta - \hat{\kappa}(\delta) < v(z)\}$ is a neighbourhood of $\partial\Omega_\delta$. Hence $\hat{u}_\delta - \hat{\kappa}(\delta) \leq u$ is a neighbourhood of $\partial\Omega_\delta$ and then $\tilde{u}_\delta = u$ in a neighbourhood of $\partial\Omega_\delta$. \square

The following estimate will play a crucial role in the proof of Theorem 3 and Theorem 4.

Corollary 2.5. *Let Ω be a bounded strongly m -pseudoconvex domain and μ a positive Borel measure on Ω with finite mass. Assume there exists $\varphi \in \mathcal{SH}_m(\Omega) \cap \mathcal{C}(\bar{\Omega})$ such that $\varphi = 0$ on $\partial\Omega$ and $\mu \leq \sigma_m(\varphi)$ weakly on Ω . Let $g \in \mathcal{C}(\partial\Omega)$ and $u \in \mathcal{SH}_m(\Omega) \cap L^\infty(\Omega)$ satisfying $\sigma_m(u) \leq \mu$ weakly on Ω and $u = g$ on $\partial\Omega$.*

Then there exists two continuous m -subharmonic functions v and w on Ω satisfying the requirements of Lemma 2.4 so that the corresponding functions $(\tilde{u}_\delta)_{0 < \delta < \delta_0}$ defined by the formula (2.10) satisfy the following estimates:

$$\int_{\Omega_\delta} (\tilde{u}_\delta - u)^m d\mu \leq C(m, \mu) \kappa \circ \theta_m(D\delta^2), \quad 0 < \delta < \delta_0,$$

where $C(m, \mu) = C(m, \Omega, g, \mu) > 0$ and $D = D(m, n, \varphi, g) > 0$ are uniform constants, $\kappa(\delta) := \kappa_\varphi(\delta) + \kappa_g(\sqrt{\delta}) + \delta$.

Proof. We want to apply Lemma 2.4 and Corollary 2.3. To this end, we need to construct two functions v and w satisfying the requirement of the Lemma 2.4. Let w be the maximal m -subharmonic function on Ω with boundary values g . By [Ch16b], we have $\kappa_w(\delta) \leq \kappa_g(\sqrt{\delta})$ and by the comparison principle we have $u \leq w$ on Ω .

Moreover, the function $v := \varphi + w$ is m -subharmonic on Ω , continuous on $\bar{\Omega}$ with $\kappa_v(\delta) \leq \kappa_\varphi(\delta) + \kappa_g(\sqrt{\delta})$ and $v = g$ on $\partial\Omega$. Since $\sigma_m(v) \geq \sigma_m(\varphi)$ and $\sigma_m(u) \leq \mu \leq \sigma_m(\varphi)$ weakly on Ω , it follows from the comparison principle that $v \leq u$ on Ω .

Therefore we can apply Lemma 2.4 to construct global approximants $(\tilde{u}_\delta)_{0 < \delta < \delta_0}$ given by the formula (2.10). Since $\tilde{u}_\delta \geq u$ on Ω_δ , and $\tilde{u}_\delta = u$ in a neighbourhood of $\Omega \setminus \Omega_\delta$, it follows from Corollary 1.11 that

$$\int_{\Omega} (dd^c \tilde{u}_\delta)^m \wedge \beta^{n-m} \leq \int_{\Omega} (dd^c u)^m \beta^{n-m} \leq \mu(\Omega).$$

By Corollary 2.3, we have for $0 < \delta < \delta_0$

$$(2.11) \quad \int_{\Omega} (\tilde{u}_\delta - u)^m d\mu \leq C_m \kappa_\varphi \circ \theta_m(M \|\tilde{u}_\delta - u\|_1),$$

where $M := (\text{osc}_{\Omega} u)^{m-1}$ and $C_m = C(m, \Omega, g, \mu) > 0$ is a uniform constant.

Now observe that $\tilde{u}_\delta - u = 0$ on $\Omega \setminus \Omega_\delta$ and $\tilde{u}_\delta - u \leq \hat{u}_\delta - u$ on Ω_δ . This yields

$$(2.12) \quad \|\tilde{u}_\delta - u\|_1 \leq \int_{\Omega_\delta} (\hat{u}_\delta - u) d\lambda_{2n}.$$

Since $\mu(\Omega) < \infty$, by Lemma 1.3, we have $\int_{\Omega_\delta} (\hat{u}_\delta - u) d\lambda_{2n} \leq a_n \|\Delta u\|_{\Omega} \delta^2$, where $a_n > 0$ is a positive uniform constant. By Lemma 1.10, we have $\|\Delta u\|_{\Omega} \leq c_{m,n} \mu(\Omega)^{1/m} < +\infty$. Hence from (2.12) we conclude that

$$(2.13) \quad \|\tilde{u}_\delta - u\|_1 \leq D' \delta^2,$$

where $D' = D'(m, n, \mu) > 0$ is a uniform constant.

Moreover since $\varphi + w \leq u \leq w$, we have

$$(2.14) \quad M := (\text{osc}_{\Omega} u)^{m-1} \leq (\text{osc}_{\Omega} w)^{m-1} + \|\varphi\|_{\bar{\Omega}}.$$

The conclusion follows from (2.11), (2.13) and (2.14). \square

3. CONTINUITY OF THE PLURIPOTENTIAL OF A DIFFUSE MEASURE

3.1. Diffuse Borel measures. We will use the following terminology from Potential Theory (see [Po16]).

Definition 3.1. Let μ be a positive Borel measure on Ω .

1. We say that μ is diffuse with respect to the capacity $c_m = c_m(\cdot, \Omega)$ if $\mu(K) = 0$ whenever $K \subset \Omega$ is a compact set with $c_m(K, \Omega) = 0$.

2. We associate to μ a natural one variable function as follows :

(3.1)

$$\Gamma_\mu(t) = \Gamma_{\mu,m}(t) := \sup\{\mu(K); K \subset \Omega \text{ is compact, } c_m(K, \Omega) \leq t\}.$$

It follows from the definition that Γ_μ is non decreasing right continuous function on \mathbb{R}^+ which satisfies the following property: for any compact set $K \subset \Omega$, we have

$$(3.2) \quad \mu(K) \leq \Gamma_\mu(c_m(K)),$$

where $c_m(K) = c_m(K, \Omega)$.

Observe that by inner regularity of the measure μ , this inequality is satisfied for any Borel set $K \subset \Omega$.

3. If Γ is a non-decreasing right continuous function on \mathbb{R}^+ , we say that μ is Γ -diffuse (with respect to the m -Hessian capacity) if for any compact subset $K \subset \Omega$, with $c_m(K, \Omega) \leq 1$,

$$(3.3) \quad \mu(K) \leq \Gamma(c_m(K)).$$

This means that $\Gamma_\mu(t) \leq \Gamma(t)$, for any $t \in [0, 1]$.

Let us mention that S. Kołodziej was the first to relate the domination of the measure μ by the Monge-Ampère capacity to the regularity of the solution to complex Monge-Ampère equations (see [Kol96]).

The following lemma is easy to prove (see [Po16]).

Lemma 3.2. *A positive Borel measure μ on Ω is diffuse (with respect to the m -Hessian capacity) if and only if $\lim_{t \rightarrow 0^+} \Gamma_\mu(t) = 0$.*

Let us give a simple example.

Example 3.3. Let $\phi \in \mathcal{SH}_m(\Omega) \cap L^\infty(\Omega)$ and $\sigma_\phi := (dd^c \phi)^m \wedge \beta^{n-m}$ be its m -Hessian measure. Set $M := \text{osc}_\Omega \phi$. Then from the definition of the m -Hessian capacity, we have for any compact subset $K \subset \Omega$,

$$\sigma_\phi(K) \leq A c_m(K), \quad \text{where } A := M^m.$$

This implies that the measure σ_ϕ is diffuse with respect to the m -Hessian capacity on Ω and $\Gamma_{\sigma_\phi}(t) \leq At$ for any $t \in \mathbb{R}^+$.

An example of Kołodziej shows that there exists a Borel measure μ such that $\mu \leq c_m$, but μ is not the Monge-Ampère of a bounded plurisubharmonic function (see [Kol96]).

The following examples due to Dinew and Kołodziej are more involved.

Example 3.4. 1. Assume that $1 \leq m < n$. Then Dinew and Kołodziej proved in [DK14] that the volume measure λ_{2n} is diffuse with respect to capacity. Namely for any $1 < r < \frac{n}{n-m}$, there exists a constant $N(r) > 0$ such that for any compact subset $K \subset \Omega$,

$$(3.4) \quad \lambda_{2n}(K) \leq N(r)c_m(K)^r.$$

Observe that this estimate is sharp in terms of the exponent when $m < n$. This can be seen by taking $\Omega = \mathbb{B}$ the unit ball and $K := \bar{\mathbb{B}}_r \subset \mathbb{B}$ the closed ball of radius $r \in]0, 1[$, since $c_m(\bar{\mathbb{B}}_r, \mathbb{B}) \approx r^{2(n-m)}$ as $r \rightarrow 0$ (see [Lu12]).

Let $0 \leq f \in L^p(\Omega)$ with $p > n/m$. Then $\frac{n(p-1)}{p(n-m)} > 1$. By Hölder inequality and inequality (3.4) we obtain: for any $1 < \tau < \frac{n(p-1)}{p(n-m)}$ there exists a constant $M(\tau) > 0$ such that for any compact set $K \subset \Omega$,

$$\int_K f d\lambda_{2n} \leq M(\tau) \|f\|_p c_m(K)^\tau.$$

2. When $m = n$ the domination is much more precise. It was proved in [ACKPZ09] that for any $0 < b < 2n$, there exists a constant $B > 0$ such that for any compact subset $K \subset \Omega$,

$$(3.5) \quad \lambda_{2n}(K) \leq B c_n(K) \exp\left(-b [c_n(K)]^{-1/n}\right).$$

Let $0 \leq f \in L^p(\Omega)$ with $p > 1$, then by Hölder inequality and inequality (3.5), for any $0 < b < 2n(p-1)/p$, there exists a constant $B' > 0$ such that for any compact set $K \subset \Omega$,

$$\int_K f d\lambda_{2n} \leq B' \|f\|_p c_n(K) \exp\left(-b [c_n(K)]^{-1/n}\right).$$

Theorem 2 will provide us with new examples.

The condition (3.3) plays an important role in the following stability result which will be a crucial point in the proof of our theorems (see [EGZ09, GKZ08, Ch16a]).

3.2. Uniform a priori estimates. The following lemma is elementary, but it turns out to play a crucial role.

Lemma 3.5. *Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a decreasing right continuous function such that $\lim_{s \rightarrow +\infty} f(s) = 0$ and let $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a non-decreasing function which satisfies the following Dini condition*

$$(3.6) \quad \int_{0^+} \frac{\eta(t)}{t} dt < +\infty.$$

Assume that for any $t \in [0, 1]$ and any $s > 0$, we have

$$(3.7) \quad t f(s+t) \leq f(s) \cdot \eta(f(s)).$$

Then $f(s) = 0$ for all $s \geq S_\infty$, where

$$S_\infty := s_0 + \int_0^{ef(s_0)} \frac{\eta(t)}{t} dt,$$

and $s_0 \geq 0$ satisfies the condition

$$\eta(f(s_0)) \leq 1/e < 1.$$

Observe that the Dini condition implies that $\lim_{t \rightarrow 0^+} \eta(t) = 0$. This Lemma is a reformulation of a Lemma of Kolodziej [Kol05] in the spirit of [EGZ09] and [BGZ08]). Its proof is a variant of the proof of [EGZ09, Lemma 2.4]. We will give it here for convenience for the reader.

Proof. Since $\lim_{s \rightarrow +\infty} f(s) = 0$, there exists $s_0 > 0$ such that $f(s_0) < 1/e$. If $f(s_0) = 0$ we are done. If $f(s_0) > 0$, there exists $s > s_0$ such $f(s) < f(s_0)/e$ since $\lim_{s \rightarrow +\infty} f(s) = 0$. Therefore we can set

$$s_1 := \inf\{s > s_0 ; f(s) < f(s_0)/e\}.$$

By (3.7) we have

$$f(s_0 + 1) \leq f(s_0)\eta(f(s_0)) < f(s_0)/e,$$

which implies that $s_0 < s_1 \leq s_0 + 1$.

By definition of s_1 , there exists a sequence s'_k decreasing to s_1 such that $f(s'_k) < f(s_0)/e$ for any $k > 0$. Since f is right continuous, it follows that $f(s_1) = \lim_{k \rightarrow +\infty} f(s'_k) \leq f(s_0)/e$.

Thus we have proved that $s_0 < s_1 \leq s_0 + 1$ and $f(s_1) \leq f(s_0)/e$.

We will construct by induction an increasing sequence $(s_j)_{j \geq 0}$ of positive numbers such that for any $j \in \mathbb{N}$

$$s_j < s_{j+1} \leq s_j + 1 \text{ and } f(s_{j+1}) \leq f(s_j)/e$$

Indeed assume by induction that for a fixed $j \geq 1$, s_1, \dots, s_j are constructed with the required properties. Then the number

$$s_{j+1} := \inf\{s > s_j ; f(s) < f(s_j)/e\}.$$

is well defined and by the same reasoning for s_1 we see that it satisfies the required properties.

On the other hand, since $f(s_{j+1}) \leq f(s_j)/e$, from (3.7), it follows that for any $s \in]s_j, s_{j+1}[$ we have

$$(s - s_j)f(s) \leq f(s_j)\eta(f(s_j)) \leq ef(s)\eta(f(s_j)),$$

since $s_j < s < s_{j+1}$ and then $f(s) \geq f(s_j)/e$.

Therefore for any $j \in \mathbb{N}$ and $s \in]s_j, s_{j+1}[$, $s - s_j \leq e\eta(f(s_j))$, hence for any $j \in \mathbb{N}$,

$$s_{j+1} - s_j \leq e\eta(f(s_j))$$

Moreover since $f(s_j) \leq f(s_{j-1})/e$ for any $j \geq 1$, it follows that $f(s_j) \leq f(s_0)/e^j$ and then

$$s_{j+1} - s_j \leq e\eta(f(s_0)e^{-j}),$$

for all $j \in \mathbb{N}$ since η is non decreasing.

Therefore

$$\begin{aligned} s_\infty := \lim_{j \rightarrow +\infty} s_j &= s_0 + \sum_{j \geq 0} (s_{j+1} - s_j) \leq s_0 + e \sum_{j=0}^{+\infty} \eta(f(s_0)e^{-j}) \\ &\leq s_0 + e \int_0^{+\infty} \eta(f(s_0)e^{-x+1}) dx. \end{aligned}$$

A simple change of variables yields

$$s_\infty \leq s_0 + e \int_0^{ef(s_0)} \frac{\eta(t)}{t} dt.$$

Recall that by construction $s_j \leq s_\infty$ and $f(s_{j+1}) \leq f(s_0)/e^j$ for any $j \in \mathbb{N}$. Therefore for any $j \in \mathbb{N}$

$$0 \leq f(s_\infty) \leq f(s_{j+1}) \leq f(s_0)/e^j,$$

which implies that that $f(s_\infty) = 0$.

Therefore if we define

$$S_\infty := s_0 + e \int_0^{ef(s_0)} \frac{\eta(t)}{t} dt,$$

we conclude that $f(s) = 0$ for any $s \geq S_\infty$.

Finally observe that, since $\eta(f(s_0)) \leq 1/e < 1$, we have $f(s_0) \leq \eta^{-1}(1/e)$. \square

Remark 3.6. 1. Observe that if $\eta(f(0)) \leq 1/e$ then $f(s) = 0$ for any $s \geq S_\infty$, where

$$S_\infty := e \int_0^{ef(0)} \frac{\eta(t)}{t} dt.$$

2. If η does not have the monotonicity property, we can replace in the statement of the lemma η by the least non decreasing majorant function of η define by

$$\bar{\eta}(t) := \sup\{\eta(s) \geq 0; s \leq t\}, t \geq 0.$$

We now deduce a uniform a priori estimate on solutions to complex Hessian equations.

Lemma 3.7. *Let $u, v \in \mathcal{SH}_m(\Omega) \cap L^\infty(\Omega)$ be such that*

$$\liminf_{z \rightarrow \partial\Omega} (u - v)(z) \geq 0.$$

Then for any $t > 0, s > 0$, we have

$$(3.8) \quad t^m \text{Cap}_m(\{u < v - s - t\}, \Omega) \leq \int_{\{u < v - s\}} (dd^c u)^m \wedge \beta^{n-m}.$$

The lemma is well known. It follows from the comparison principle (see [Kol96], [EGZ09], [GKZ08], [Ch16a], [KN20b]).

Corollary 3.8. *Let μ be a positive Borel measure on Ω with finite mass. Assume that μ is Γ -diffuse with respect to the m -Hessian capacity and the function $\gamma(t) := \Gamma(t)/t$ is non-decreasing and satisfies the following Dini type condition*

$$(3.9) \quad \int_{0^+} \frac{\gamma^{1/m}(t)}{t} dt < +\infty.$$

Assume that $u \in \mathcal{SH}_m(\Omega) \cap L^\infty(\Omega)$ is the solution to the Dirichlet problem (0.1) with boundary datum $g \in C^0(\partial\Omega)$ and right hand side μ . Then we have the following uniform estimate on Ω

$$(3.10) \quad \min_{\partial\Omega} g - 2e (\mu(\Omega)/a)^{1/m} - \int_0^a \frac{\gamma^{1/m}(t)}{t} dt \leq u \leq \max_{\partial\Omega} g,$$

where

$$a := e^m \gamma^{-1}(1/e^m).$$

In particular

$$(3.11) \quad \text{osc}_{\bar{\Omega}} u \leq \text{osc}_{\partial\Omega} g + 2e (\mu(\Omega)/a)^{1/m} + \int_0^a \frac{\gamma^{1/m}(t)}{t} dt.$$

Proof. Set $\alpha := \min_{\partial\Omega} g$. Then $\liminf_{z \rightarrow \partial\Omega} (u - \alpha) \geq 0$. Then we can apply Lemma 3.7 with u and $v = \alpha$ and get the following estimate for

any $s, t > 0$

$$(3.12) \quad \begin{aligned} t^m c_m(\{u - \alpha < -s - t\}, \Omega) &\leq \int_{\{u - \alpha < -s\}} (dd^c u)^m \wedge \beta^{n-m} \\ &\leq \mu(\{u - \alpha < -s\}). \end{aligned}$$

By definition of Γ , we deduce that for any $s, t > 0$

$$t^m c_m(\{u - \alpha < -s - t\}, \Omega) \leq \Gamma(c_m(\{u - \alpha < -s\}, \Omega)).$$

Define $f(s) := c_m(\{u - \alpha < -s\}, \Omega)^{1/m}$ for $s > 0$. Then we see that the condition of the Lemma 3.5 is satisfied with $\eta(t) := \gamma^{1/m}(t^m)$.

Applying Lemma 3.5 we conclude that $f(s) = 0$ for $s \geq S_\infty$. This means that $u \geq \alpha - S_\infty$ outside a set of zero capacity. Since such set is of Lebesgue measure zero, it follows that $u \geq \alpha - S_\infty$.

On the other hand by the classical maximum principle we also have $u \leq M = M_g := \max_{\partial\Omega} g$ in Ω .

Therefore we have the following uniform bound on u in Ω

$$(3.13) \quad \min_{\partial\Omega} g - S_\infty \leq u \leq \max_{\partial\Omega} g,$$

where

$$S_\infty \leq s_0 + \int_0^{e\eta^{-1}(1/e)} \frac{\eta(t)}{t} dt = s_0 + (1/m) \int_0^a \frac{\gamma^{1/m}(t)}{t} dt,$$

and

$$a := [e\eta^{-1}(1/e)]^m = e^m \gamma^{-1}(1/e^m).$$

We need to establish a uniform estimate on the initial time s_0 . Recall that s_0 satisfies $\eta(f(s_0)) \leq 1/e$. This condition is equivalent to the following one

$$c_m(\{u < \alpha - s_0\}, \Omega) \leq ae^{-m}.$$

Observe that by (3.12) we have for $t > 0$

$$c_m(\{u - \alpha < -2t\}) \leq \mu(\Omega)/t^m$$

Then choosing $t_0 = e(\mu(\Omega)/a)^{1/m}$ and $s_0 := 2t_0$, we obtain the estimate $f(s_0) \leq a$ and then

$$S_\infty \leq 2e(\mu(\Omega)/a)^{1/m} + \int_0^a \frac{\gamma^{1/m}(t)}{t} dt.$$

Therefore from this upper bound and (3.13), we obtain the uniform estimate (3.10). \square

3.3. Existence of a continuous solution : Proof of Theorem 1.

We first prove a weak stability theorem in terms of capacity in the spirit of a similar result of [EGZ09].

Lemma 3.9. *Let μ be a positive Borel measure with finite mass on Ω and u, v be two bounded m -subharmonic functions on Ω such that*

$$\liminf_{z \rightarrow \partial\Omega} (u - v)(z) \geq 0 \text{ and } (dd^c u)^m \wedge \beta^{n-m} \leq \mu,$$

in the sense of currents on Ω . Assume that μ is Γ -diffuse with respect to the m -Hessian capacity and the function $\gamma(t) := \Gamma(t)/t$ is non-decreasing and satisfies the Dini type condition (3.9).

Then there exists a uniform constant $B > 0$ such that for any $\varepsilon > 0$, we have

$$\sup_{\Omega} (v - u)_+ \leq \varepsilon + B \int_0^{\varsigma(\varepsilon)} \frac{\gamma^{1/m}(t)}{t} dt,$$

where

$$\varsigma(\varepsilon) := e^m [c_m(\{v - u > \varepsilon\}, \Omega)].$$

Proof. Fix $\varepsilon > 0$ and apply Lemma 3.7 with $t \in [0, 1]$ and $s + \varepsilon$. Then we obtain

$$(3.14) \quad t^m c_m(\{u - v < -\varepsilon - t - s\}, \Omega) \leq \mu(\{u - v < -\varepsilon - s\}).$$

Set $f(s) := [c_m(\{u - v < -\varepsilon - s\}, \Omega)]^{1/m}$ for $s \in \mathbb{R}^+$. Then by the domination condition we deduce that for $t \in [0, 1]$ and $s \in \mathbb{R}^+$

$$tf(s + t) \leq f(s)\gamma^{1/m}(f(s)^m).$$

Now we can apply Lemma 3.5 with $\eta(t) := \gamma^{1/m}(t^m)$.

There are two cases to be considered:

1) If $\eta(f(0)) \leq 1/e$, then by Lemma 3.5 we conclude that $c_m(\{u - v < -\varepsilon - s\}, \Omega) = 0$ if $s \geq S_\infty$ i.e.

$$\sup_{\Omega} (v - u) \leq \varepsilon + S_\infty = \varepsilon + \int_0^{ef(0)} \frac{\eta(t)}{t} dt.$$

A simple change of variables leads to the estimates.

$$(3.15) \quad \sup_{\Omega} (v - u) \leq \varepsilon + (1/m) \int_0^{\varsigma(\varepsilon)} \frac{\gamma^{1/m}(t)}{t} dt,$$

where $\varsigma(\varepsilon) = e^m f(0)^m = e^m [c_m(\{v - u > \varepsilon\}, \Omega)]$.

2) If $\eta(f(0)) > 1/e$, then

$$\varsigma(\varepsilon) := e^m [c_m(\{v - u > \varepsilon\}, \Omega)] = e^m f(0)^m \geq e^m [\eta^{-1}(1/e)]^m =: a.$$

Hence

$$\int_0^{\varsigma(\varepsilon)} \frac{\gamma^{1/m}(t)}{t} dt \geq A := \int_0^a \frac{\gamma^{1/m}(t)}{t} dt.$$

On the other hand since $u \geq v$ in $\partial\Omega$, by the uniform estimate, we have $v - u \leq \max_{\partial\Omega} u - u \leq \text{osc}_\Omega u \leq M = M(\eta, \cdot, \Omega)$, hence if we let $B_0 := MA^{-1}$ we get

$$(3.16) \quad \sup_{\Omega} (v - u) \leq B_0 \int_0^{\varsigma(\varepsilon)} \frac{\gamma^{1/m}(t)}{t} dt$$

Comparing the estimates (3.15) and (3.16) we obtain the estimate required in the theorem with $B := \max\{B_0, 1/m\}$. \square

We prove Theorem 1 on the existence of a continuous solution to the Dirichlet problem for diffuse measures satisfying the Dini condition (3.9).

Proof. The proof will be done in three steps.

1) Existence of a bounded solution. Indeed, since μ is diffuse with respect to c_m , it follows from the generalized Radon-Nikodym theorem that there exists a function $v \in \mathcal{SH}_m(\Omega) \cap L^\infty(\Omega)$ and $F \in L^1(\Omega, \sigma_m(v))$ such that $\mu = F\sigma_m(v)$ on Ω (see [Ceg98], [Lu12]).

Set $F_j := \min\{F, j^m\}$ for $j \in \mathbb{N}$. Then we have $\mu_j := F_j\sigma_m(v) \leq j^m\sigma_m(v) = \sigma_m(jv)$ with $jv \in \mathcal{SH}_m(\Omega) \cap L^\infty(\Omega)$. By the bounded subsolution theorem, there exists $u_j \in \mathcal{SH}_m(\Omega) \cap L^\infty(\Omega)$ such that $u_j = g$ and $\sigma_m(u_j) = \mu_j = F_j\sigma_m(v)$. By the comparison principle, the sequence $(u_j)_{j \in \mathbb{N}}$ is decreasing and by Corollary 3.8, the sequence (u_j) is uniformly bounded on Ω . Therefore it converges to $u \in \mathcal{SH}_m(\Omega) \cap L^\infty(\Omega)$. By the continuity of the Hessian operator with respect to decreasing sequences, it follows that $\sigma_m(u) = \mu$ weakly on Ω .

2) Boundary values of the solution. Since $\sigma_m(u_j) \leq \sigma_m(u_k) \leq \mu$ for $k \geq j$, it follows that the measures μ_j are uniformly Γ -diffuse with respect to c_m and then by Lemma 3.9, there is a uniform constant B such that for any $k \geq j$,

$$\sup_{\Omega} (u_j - u_k) \leq \varepsilon + B \int_0^{\varsigma_{j,k}(\varepsilon)} \frac{\gamma^{1/m}(t)}{t} dt,$$

where

$$\varsigma_{j,k}(\varepsilon) := e^m c_m(\{u_j - u_k > \varepsilon\}).$$

Since the sequence (u_j) is decreasing, it follows that it converges in capacity with respect to c_m on each compact set in Ω (see [Lu12]). Observe that since $\lim_{j,k \rightarrow +\infty} \varsigma_{j,k}(\varepsilon) = 0$, for j, k large enough, we have $u_j - u_k \leq \varepsilon/2$ near the boundary of Ω . This implies that for j, k large enough, the sets $\{u_j - u_k > \varepsilon\}$ are contained in a fixed compact set in Ω . Therefore for any $\varepsilon > 0$, $\lim_{j,k \rightarrow +\infty} \varsigma_{j,k}(\varepsilon) = 0$.

It follows that the sequence (u_j) converges uniformly to u on Ω . Hence $u = g$ on $\partial\Omega$.

3) Continuity of the solution u : By Lemma 1.6, there exists a decreasing sequence (w_j) of continuous m -subharmonic functions in $\bar{\Omega}$ which converges to u pointwise in Ω and such that $w_j = g$ on $\partial\Omega$.

Fix $\varepsilon > 0$ and $j_0 \gg 1$ large enough so that $w_j \geq g - \varepsilon$ on $\partial\Omega$ for $j \geq j_0$. Then $\liminf_{z \rightarrow \partial\Omega} (u - w_j + 2\varepsilon) > 0$ for $j \geq j_0$. Then applying Lemma 3.9 with u and $v = w_j - 2\varepsilon$ we obtain for $j \geq j_0$

$$w_j - 2\varepsilon - u \leq \varepsilon + B \int_0^{\varsigma_j(\varepsilon)} \frac{\gamma^{1/m}(t)}{t} dt,$$

where

$$\varsigma_j(\varepsilon) = e^m c_m(\{w_j > u + 3\varepsilon\}).$$

Since (w_j) decreases to u pointwise in Ω , it follows as before that it converges in capacity over any compact subset of Ω . Observe that for $j \geq j_0$, we have $\{u < w_j - 3\varepsilon\} \subset \{u < w_{j_0} - 3\varepsilon\} \Subset \Omega$. Hence $\lim_{j \rightarrow +\infty} \varsigma_j(\varepsilon) = 0$, and then

$$\lim_{j \rightarrow +\infty} \max_{\bar{\Omega}} (w_j - u) \leq 4\varepsilon.$$

As $\varepsilon > 0$ is arbitrary, it follows that the sequence (w_j) converges uniformly in $\bar{\Omega}$ to u , hence u is continuous in $\bar{\Omega}$. \square

In the case of infinite mass, we can prove the following result using Theorem 1.

Corollary 3.10. *Let μ be a positive Borel measure on Ω with $\mu(\Omega) = +\infty$. Assume that the following two conditions are satisfied*

(i) μ is Γ -diffuse with respect to the m -Hessian capacity with Γ satisfying the Dini type condition (3.9),

(ii) the Dirichlet problem (0.1) admits a bounded subsolution $\psi \in \mathcal{SH}_m(\Omega) \cap L^\infty(\Omega)$ i.e. $(dd^c\psi)^m \wedge \beta^{n-m} \geq \mu$ weakly on Ω and $\psi = 0$ on $\partial\Omega$.

Then for any continuous boundary datum $g \in \mathcal{C}^0(\partial\Omega)$, the Dirichlet problem (0.1) admits a unique solution $U = U_{\mu,g} \in \mathcal{SH}_m(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$.

Proof. Let $(K_j)_{j \in \mathbb{N}}$ be an increasing sequence of relatively compact Borel subsets of Ω such that $\Omega = \cup_j K_j$. Set $\mu_j := \mathbf{1}_{K_j}$ for $j \in \mathbb{N}$. By Theorem 1, for each $j \in \mathbb{N}$ there exists $u_j \in \mathcal{SH}_m(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$ such that $(dd^c u_j)^m \wedge \beta^{n-m} = \mu_j$ and $u_j = g$ on $\partial\Omega$. By the comparison principle, (u_j) is a decreasing sequence.

On the other hand let w_g the maximal m -subharmonic function on Ω with $w_g = g$ on $\partial\Omega$. Then $\psi_g := \psi + w_g \in \mathcal{SH}_m(\Omega) \cap L^\infty(\Omega)$ and $(dd^c \psi_g)^m \wedge \beta^{n-m} \geq \mu \geq \mu_j$. By the comparison principle it

follows that $w_g \geq u_j \geq \psi_g$ on Ω . Therefore (u_j) decreases to a function $u \in \mathcal{SH}_m(\Omega) \cap L^\infty(\Omega)$ such that $u = g$ on $\partial\Omega$ and $(dd^c u)^m \wedge \beta^{n-m} = \mu$ weakly on Ω .

We need to prove that u is continuous on $\bar{\Omega}$. Indeed choose $K_j := \{\psi < -\varepsilon_j\}$, for $j \in \mathbb{N}$, where $(\varepsilon_j)_{j \in \mathbb{N}}$ is decreasing sequence of positive numbers converging to 0 so that $\mu(\{\psi = -\varepsilon_j\}) = 0$ for any $j \in \mathbb{N}$. Then $\psi_j := u_j + \max\{\psi, -\varepsilon_j\} \in \mathcal{SH}_m(\Omega) \cap L^\infty(\Omega)$, $\psi_j = g$ on $\partial\Omega$ and we have

$$(dd^c \psi_j)^m \wedge \beta^{n-m} \geq (dd^c u_j)^m \wedge \beta^{n-m} + (dd^c \max\{\psi, -\varepsilon_j\})^m \wedge \beta^{n-m} = \mu,$$

weakly on Ω .

By the comparison principle it follows that

$$u_j + \max\{\psi, -\varepsilon_j\} \leq u \leq u_j \text{ on } \bar{\Omega}.$$

This proves that (u_j) converges to u uniformly on $\bar{\Omega}$, hence u is continuous on $\bar{\Omega}$. \square

3.4. Weak uniform stability theorem. The role of the weak stability theorem L^1 - L^∞ was discovered in [EGZ09], where the Hölder continuity of the solution to the Dirichlet problem for the complex Monge-Ampère equation on compact homogenous manifolds was proved. Since then, this result became the main tool in deriving estimates on the modulus of continuity of solutions to the complex Monge-Ampère and Hessian equations.

In order to estimate the modulus of continuity of the solution in this general context, we need to prove a similar result.

Denote by

$$(3.17) \quad J_\Gamma(\tau) := \int_0^\tau \frac{\gamma^{1/m}(t)}{t} dt, \quad \tau \in \mathbb{R}^+.$$

Theorem 3.11. *Let μ be a positive Borel measure on Ω with finite mass. Assume that μ is Γ -diffuse with respect to the m -Hessian capacity and satisfies the Dini type condition (3.9).*

Let $u, v \in \mathcal{SH}_m(\Omega) \cap L^\infty(\Omega)$ be such that $\liminf_{z \in \partial\Omega} (u - v)(z) \geq 0$ and

$$(dd^c u)^m \wedge \beta^{n-m} \leq \mu,$$

in the sense of currents on Ω .

Then

$$\sup_\Omega (v - u)_+ \leq B h_\Gamma(e^m \|v - u\|_{m,\mu}^m)$$

where $B > 0$ is a uniform constant, $\|(v - u)_+\|_{m,\mu}^m := \int_\Omega (v - u)_+^m d\mu$ and h_Γ is the reciprocal of the function $s \mapsto s^{2m} J_\Gamma^{-1}(s)$ on \mathbb{R}^+ .

Observe that h is a continuous increasing function on \mathbb{R}^+ such that $h(0) = 0$.

Proof. We fix $\varepsilon > 0$ and apply Lemma 3.9 to obtain the estimate

$$\sup_{\Omega} (v - u)_+ \leq \varepsilon + B \int_0^{\varsigma(\varepsilon)} \frac{\gamma^{1/m}(t)}{t} dt,$$

where

$$\varsigma(\varepsilon) := e^m c_m(\{v - u > \varepsilon\}, \Omega).$$

To estimate $\varsigma(\varepsilon)$ we apply Lemma 3.7 with $s = t = \varepsilon/2$ which yields

$$\begin{aligned} c_m(\{v - u > \varepsilon\}, \Omega) &\leq 2^m \varepsilon^{-m} \int_{v-u \geq \varepsilon/2} (dd^c u)^m \wedge \beta^{n-m} \\ (3.18) \qquad \qquad \qquad &\leq 2^{2m} \varepsilon^{-2m} \int_{\Omega} (v - u)_+^m (dd^c u)^m \wedge \beta^{n-m} \end{aligned}$$

Hence

$$\varsigma(\varepsilon) \leq 2^{2m} e^m \varepsilon^{-2m} \|(v - u)_+\|_{m,\mu}^m.$$

Then by Lemma 3.9

$$\sup_{\Omega} (v - u)_+ \leq \varepsilon + B J_{\Gamma} \left(2^m e^m \varepsilon^{-2m} \|(v - u)_+\|_{m,\mu}^m \right).$$

Therefore if we choose $\varepsilon := h_{\Gamma}(2^m e^m \|(v - u)_+\|_{m,\mu}^m)$ we obtain the required estimate. \square

Corollary 3.12. *Let μ be a positive Borel measure on Ω with finite mass. Assume that μ is Γ -diffuse with respect to the m -Hessian capacity with $\Gamma(t) := t^{1+a}$, where $a > 0$. Let $u, v \in \mathcal{SH}_m(\Omega) \cap L^{\infty}(\Omega)$ be such that $\liminf_{z \in \partial\Omega} (u - v)(z) \geq 0$ and*

$$(dd^c u)^m \wedge \beta^{n-m} \leq \mu,$$

in the sense of currents on Ω .

Then there exists a uniform constant $A = A(a, m) > 0$ such that

$$\sup_{\Omega} (v - u)_+ \leq A (\|(v - u)_+\|_{m,\mu})^{\nu}$$

where $\|(v - u)_+\|_{m,\mu} := [\int_{\Omega} (v - u)_+^m d\mu]^{1/m}$ and $\nu := \frac{a}{2a+1}$.

Proof. It is a straight forwards consequence of Theorem 3.11. Indeed it is enough to compute h_{Γ} in this case. A simple computation shows that $J_{\Gamma}(\tau) = \frac{m}{a} \tau^{a/m}$. Hence $h_{\Gamma}(t) = C(a, m) t^{\frac{a}{2am+m}}$ for $t > 0$. \square

4. MASS ESTIMATES FOR HESSIAN MEASURES

For the proof of Theorem 2, we will use the same method as in [BZ20] which was inspired by an idea in [KN20b]. However, since our measure has not a compact support nor a bounded mass, we need to use the control on the behaviour of the mass of the m -Hessian of the subsolution close to the boundary, given by Lemma 2.1.

4.1. Proof of Theorem 2. Recall the volume estimate stated before.

Let us fix $0 < r < m/(n - m)$ and $0 < b < 2n$ and define the following function on \mathbb{R}^+ :

$$(4.1) \quad \ell_m(t) := \begin{cases} t^r, & \text{if } 1 \leq m < n, \\ \exp(-bt^{-1/n}), & \text{if } m = n. \end{cases}$$

Then the estimates (3.4) and (3.5) can be written as follows: there exists a constant $B_m > 0$ such that for any Borel set $S \subset \Omega$,

$$(4.2) \quad \lambda_{2n}(S) \leq B_m \ell_m(c_m(S, \Omega)) c_m(S, \Omega),$$

where B_m depends on m, r and Ω when $m < n$ and B_n depends on n, b and Ω

Recall that $\varphi \in \mathcal{SH}_m(\Omega) \cap C^0(\bar{\Omega})$ with $\varphi = 0$ on $\partial\Omega$ and we want to estimate the mass of the Hessian measure $\sigma_m(\varphi)$ on compact sets in Ω .

Proof. We extend φ as a continuous function in the whole of \mathbb{C}^n with the same modulus of continuity and denote by φ the extension. Then denote by φ_δ ($0 < \delta < \delta_0$) the smooth approximants of φ in Ω , defined as usual, for $z \in \bar{\Omega}$,

$$\varphi_\delta(z) = \int_{\Omega} \varphi(\xi) \chi_\delta(z - \xi) d\lambda(\xi).$$

Observe that for $0 < \delta < \delta_0$ and $z \in \Omega_\delta := \{z \in \Omega; \text{dist}(z, \partial\Omega) > \delta\}$,

$$\varphi_\delta(z) = \int_{\Omega} \varphi(z - \zeta) \chi_\delta(\zeta) d\lambda(\zeta),$$

and then $\varphi_\delta \in \mathcal{SH}_m(\Omega_\delta) \cap C^\infty(\mathbb{C}^n)$.

Since $\varphi \in C^0(\bar{\Omega})$, we have $\varphi_\delta \leq \varphi + \kappa_\varphi(\delta)$ on Ω .

We consider the m -subharmonic envelope of φ_δ on Ω defined by the formula

$$\psi_\delta := \sup\{\psi \in \mathcal{SH}_m(\Omega); \psi \leq \varphi_\delta \text{ on } \Omega\}.$$

It follows from [BZ20, Theorem 3.3] that $\psi_\delta \in \mathcal{SH}_m(\Omega)$ and $\psi_\delta \leq \varphi_\delta$ on Ω .

Fix $0 < \delta < \delta_0$ and a compact set $K \subset \Omega_\delta$ and consider the following set

$$E := \{3\kappa(\delta)u_K^* + \psi_\delta < \varphi - 2\kappa(\delta)\} \subset \Omega.$$

Since κ is the modulus of continuity of φ on $\bar{\Omega}$, we have $\varphi - \kappa(\delta) \leq \varphi_\delta \leq \varphi + \kappa(\delta)$ on Ω and then $\varphi - \kappa(\delta) \leq \psi_\delta \leq \varphi_\delta \leq \varphi(z) + \kappa(\delta)$ on Ω . Therefore $\liminf_{z \rightarrow \partial\Omega} (\psi_\delta - \varphi + \kappa(\delta)) \geq 0$, and then $E \Subset \Omega$. By the comparison principle, we conclude that

$$(4.3) \quad \begin{aligned} \int_E (dd^c \varphi)^m \wedge \beta^{n-m} &\leq \int_E (dd^c (3\kappa(\delta)u_K^* + \psi_\delta))^m \wedge \beta^{n-m} \\ &\leq 3\kappa(\delta)L \int_E (dd^c (u_K^* + \psi_\delta))^m \wedge \beta^{n-m} \\ &\quad + \int_E (dd^c \psi_\delta)^m \wedge \beta^{n-m}, \end{aligned}$$

where $L := \max_{0 \leq j \leq m-1} (3\kappa(\delta_0))^j$.

Observe that $-1 + \varphi - \kappa(\delta) \leq u_K^* + \psi_\delta \leq \varphi + \kappa(\delta)$ on Ω , hence $|u_K^* + \psi_\delta| \leq \sup_\Omega |\varphi| + 1 + \kappa(\delta_0) =: M_0$ on Ω .

Therefore from inequality (4.3), it follows that

$$(4.4) \quad \int_E (dd^c \varphi)^m \wedge \beta^{n-m} \leq 3\kappa(\delta)LM_0^m c_m(E, \Omega) + \int_E (dd^c \psi_\delta)^m \wedge \beta^{n-m}.$$

Moreover we have

$$(4.5) \quad dd^c \varphi_\delta \leq \frac{M_1 \kappa(\delta)}{\delta^2} \beta, \quad \text{pointwise on } \Omega,$$

where $M_1 > 0$ is a uniform constant depending only on Ω .

By [BZ20, Theorem 3.3], we have

$$(4.6) \quad (dd^c \psi_\delta)^m \wedge \beta^{n-m} \leq (\sigma_m(\varphi_\delta))_+ \leq \frac{M_1^m \kappa(\delta)^m}{\delta^{2m}} \beta^n,$$

in the sense of currents on Ω .

Therefore

$$(4.7) \quad \int_E (dd^c \psi_\delta)^m \wedge \beta^{n-m} \leq M_1^m \kappa(\delta)^m \delta^{-2m} \lambda_{2n}(E).$$

Let us denote for simplicity $c_m(\cdot) := c_m(\cdot, \Omega)$. Then from (4.4) and (4.5), we deduce that

$$(4.8) \quad \int_E (dd^c \varphi)^m \wedge \beta^{n-m} \leq 3\kappa(\delta)LM_0^m c_m(E) + M_1^m \kappa(\delta)^m \delta^{-2m} \lambda_{2n}(E)$$

From (4.8) and (4.2), it follows that

$$(4.9) \quad \int_E (dd^c \varphi)^m \wedge \beta^{n-m} \leq 3\kappa(\delta) LM_0^m c_m(E) + B_m M_1^m \kappa(\delta)^m \delta^{-2m} \ell_m(c_m(E)) c_m(E).$$

Since $\varphi - \kappa(\delta) \leq \psi_\delta \leq \varphi_\delta \leq \varphi + \kappa(\delta)$ on Ω , it follows that $E \subset \{u_K^* < -1/3\}$.

The comparison principle yields the following estimate

$$(4.10) \quad c_m(E, \Omega) \leq 3^m c_m(K, \Omega).$$

Indeed fix $v \in SH_m(\Omega)$ with $-1 \leq v \leq 0$. Then $E \subset \{3u_K^* < -1\} \subset \{3u_K^* < v\} \Subset \Omega$ and the comparison principle implies that

$$\int_E (dd^c v)^m \wedge \beta^{n-m} \leq \int_{\{3u_K^* < v\}} 3^m (dd^c u_K^*)^m \wedge \beta^{n-m} \leq 3^m c_m(K, \Omega).$$

Taking the supremum over v we obtain the estimate (4.10).

Since $K \setminus \{u_K < u_K^*\} \subset \{u_K^* = -1\} \subset E$ and $K \cap \{u_K < u_K^*\}$ has zero capacity, we see that $\int_K (dd^c \varphi)^m \wedge \beta^{n-m} \leq \int_E (dd^c \varphi)^m \wedge \beta^{n-m}$.

Therefore we finally deduce from (4.8), (4.2) and (4.6) that for a fixed $0 < \delta < \delta_0$ and any compact set $K \subset \Omega_\delta$, we have

$$\int_K (dd^c \varphi)^m \wedge \beta^{n-m} \leq A_0 \kappa(\delta) c_m(K) + A_1 \kappa(\delta)^m \delta^{-2m} \ell_m(c_m(K)) c_m(K).$$

where $A_0 := 3^{m+1} L_0 M_0^m$ and $A_1 := M_1^m 3^{mr}$.

By inner regularity of the capacity, we deduce that the previous estimate holds for any Borel subset $S \subset \Omega_\delta$ i.e.

$$(4.11) \quad \int_S (dd^c \varphi)^m \wedge \beta^{n-m} \leq A_0 \kappa(\delta) c_m(S) + A_1 \kappa(\delta)^m \delta^{-2m} \ell_m(c_m(S)) c_m(S).$$

Let $K \subset \Omega$ be any fixed compact set and $0 < \delta < \delta_0$. Then

$$\int_K (dd^c \varphi)^m \wedge \beta^{n-m} = \int_{K \cap \Omega_\delta} (dd^c \varphi)^m \wedge \beta^{n-m} + \int_{K \setminus \Omega_\delta} (dd^c \varphi)^m \wedge \beta^{n-m}.$$

We will estimate each term separately. By (4.11) the first term is estimated easily: for $0 < \delta < \delta_0$, we have

$$(4.12) \quad \int_{K \cap \Omega_\delta} (dd^c \varphi)^m \wedge \beta^{n-m} \leq A_0 \kappa(\delta) c_m(K) + A_1 \kappa(\delta)^m \delta^{-2m} \ell_m(c_m(K)) c_m(K).$$

To estimate the second term we apply Lemma 2.1 for the Borel set $S := K \setminus \Omega_\delta$. Since $\delta_B(\partial\Omega) \leq \delta$ we get

$$\int_{K \setminus \Omega_\delta} (dd^c \varphi)^m \wedge \beta^{n-m} \leq \kappa(\delta)^m c_m(K).$$

Finally we obtain for $0 < \delta < \delta_0$,

$$(4.13) \quad \int_K (dd^c \varphi)^m \wedge \beta^{n-m} \leq 3\kappa_\varphi(\delta) LM_0^m c_m(E, \Omega) \\ + M_1^m \kappa_\varphi(\delta)^m \delta^{-2m} \lambda_{2n}(E).$$

Therefore we finally deduce from (4.8), (4.13), (3.4) and (4.6) that for a fixed $0 < \delta < \delta_0$ and any compact set $K \subset \Omega$, we have

$$(4.14) \quad \int_K (dd^c \varphi)^m \wedge \beta^{n-m} \leq A_0 \kappa(\delta) c_m(K) \\ + A_2 \kappa(\delta)^m \delta^{-2m} \ell_m(c_m(K)),$$

where $\kappa = \kappa_\varphi$, $A_0 := 2^{m+1} L_0 M_0^m$ and $A_2 := 3^m N(r) M_1^m$.

We want to optimize the right hand side of (4.14) by choosing δ so that $\kappa(\delta) = \kappa(\delta)^m \delta^{-2m} \ell_m(c_m(K))$ i.e.

$$\kappa(\delta)^{1-m} \delta^{2m} = \ell_m(c_m(K)).$$

Observe that the function $x \mapsto \kappa(x)^{1-m} x^{2m}$ is continuous on \mathbb{R}^+ and takes the values 0 at $t = 0$ and $+\infty$ at $t = +\infty$. Therefore for any $y > 0$, the equation $y = \kappa(x)^{1-m} x^{2m}$ has at least one solution $x > 0$. Let us define the lower inverse function of the function $x \mapsto \kappa(x)^{1-m} x^{2m}$ by the following formula:

$$(4.15) \quad \theta_m(y) := \inf\{x > 0; \kappa(x)^{1-m} x^{2m} = y\}, y > 0.$$

Choose $\delta > 0$ to be $\delta = \theta_m[\ell_m(c_m(K))]$.

Set $\vartheta_m(t) := \kappa \circ \theta_m(\ell_m(t))$ and observe that if $\delta_K(\partial\Omega) \leq \vartheta_m(c_m(K))$, then by Lemma 2.1 we get

$$(4.16) \quad \int_K (dd^c \varphi)^m \wedge \beta^{n-m} \leq [\vartheta(c_m(K))]^m c_m(K).$$

Now assume that $\vartheta_m(c_m(K)) < \delta_K(\partial\Omega) \leq \delta_0$. Then we can take $\delta := \vartheta_m(c_m(K))$ in the inequality (4.14) and get

$$(4.17) \quad \int_K (dd^c \varphi)^m \wedge \beta^{n-m} \leq B \vartheta(c_m(K)) c_m(K) + [\vartheta(c_m(K))]^m c_m(K).$$

Combining inequalities (4.16) and (4.17), we obtain the estimate of the theorem with the constant B given by the following formula:

$$(4.18) \quad B := A_0 + A_1 + 1.$$

□

4.2. Some consequences. For $m < n$ we recover the result of ([BZ20]).

Corollary 4.1. *Let $\Omega \Subset \mathbb{C}^n$ be a m -hyperconvex domain and $\varphi \in \mathcal{SH}_m(\Omega) \cap C^\alpha(\overline{\Omega})$ such that $\varphi = 0$ in $\partial\Omega$.*

Then for any $0 < \epsilon < \alpha m / [(n - m)(2m + \alpha(1 - m))]$, there exists a constant $A = A(m, n, \alpha, \epsilon, \Omega) > 0$ such that for every compact $K \subset \Omega$, we have

$$\int_K (dd^c \varphi)^m \wedge \beta^{n-m} \leq A [c_m(K)]^{1+\epsilon}.$$

Proof. Since $\kappa_\varphi(t) = \kappa_0 t^\alpha$, by Theorem 2, we have for any compact $K \subset \Omega$,

$$\int_K (dd^c \varphi)^m \wedge \beta^{n-m} \leq A \{ \vartheta_m(c_m(K)) + [\vartheta_m(c_m(K))]^m \} c_m(K),$$

where $\vartheta_m(t) := \kappa_\varphi \circ \theta_m \circ \ell_m(t)$ and $\theta_m^{-1}(t) := t^{2m+\alpha(1-m)}$. On the other hand $\ell_m(t) = t^r$ with $0 < r < m/(n-m)$, hence $\vartheta_m(t) = t^{\alpha r / [2m+\alpha(1-m)]}$. □

For $m = n$ we obtain a much more precise result.

Corollary 4.2. *Let $\Omega \Subset \mathbb{C}^n$ be a hyperconvex domain. Let $\varphi \in \mathcal{PSH}(\Omega) \cap C^\alpha(\overline{\Omega})$ such that $\varphi = 0$ in $\partial\Omega$. Then for any $0 < q < \frac{2n\alpha}{2n+(1-n)\alpha}$, there exists a constant $Q = Q(n, q, \alpha, \Omega) > 0$ such that for every compact $K \subset \Omega$,*

$$\int_K (dd^c \varphi)^n \leq Q c_n(K, \Omega) \exp\left(-q [c_n(K, \Omega)]^{-1/n}\right).$$

Proof. Since $\kappa_\varphi(t) = \kappa_0 t^\alpha$, by Theorem 2, we have for any compact $K \subset \Omega$,

$$\int_K (dd^c \varphi)^m \wedge \beta^{n-m} \leq A \{ \vartheta_m(c_m(K)) + [\vartheta_m(c_m(K))]^m \} c_m(K),$$

where $\vartheta_m(t) := \kappa \circ \theta_m \circ \ell_m(t)$ and $\theta_m^{-1}(t) := t^{2m+\alpha(1-m)}$.

When $m = n$ we have $\ell_n(t) = \exp(-bt^{-1/n})$ with $b < 2n$, hence $\vartheta_n(t) = \exp(-qt^{-1/n})$, where $q = \frac{\alpha b}{2n+(1-n)\alpha}$. □

From this result, we deduce a global exponential integrability theorem for plurisubharmonic functions in the Cegrell class with respect to Borel measures with Hölder continuous Monge-Ampère potentials.

Let $\mathcal{F}(\Omega)$ defined as the class of negative plurisubharmonic functions ψ on Ω such that there exists a decreasing sequence of plurisubharmonic test functions (ψ_j) in $\mathcal{E}^0(\Omega)$ such that $\sup_j \int_\Omega (dd^c \psi_j)^n < +\infty$ (see [Ceg04]).

We denote by $\dot{\mathcal{F}}(\Omega)$ the set of functions $\psi \in \mathcal{F}(\Omega)$ normalized by the condition $\int_{\Omega} (dd^c \psi)^n \leq 1$.

Corollary 4.3. *Let $\Omega \Subset \mathbb{C}^n$ be a hyperconvex domain and $\varphi \in \mathcal{PSH}(\Omega) \cap \mathcal{C}^{\alpha}(\bar{\Omega})$ such that $\varphi = 0$ in $\partial\Omega$. Let $k > n$ and $0 < q < q_n(\alpha) := \frac{2n\alpha}{2n+(1-n)\alpha}$. Then there exists a constant $\tilde{Q} = \tilde{Q}(k, n, q) > 0$ such that for any $\psi \in \dot{\mathcal{F}}(\Omega)$,*

$$\int_{\Omega} (-\psi)^k e^{-q\psi} (dd^c \varphi)^n \leq \tilde{Q}.$$

In particular, for any compact subset $E \Subset \Omega$, there exists a constant $R = R(E, k, n, q) > 0$ such that for any $\psi \in \dot{\mathcal{F}}(\Omega)$,

$$\int_E e^{-q\psi} (dd^c \varphi)^n \leq R.$$

Proof. Indeed fix $k > n$, and $0 < q < q_n(\alpha)$. Then we have

$$\int_{\Omega} (-\psi)^k e^{-q\psi} (dd^c \varphi)^n = \int_0^{+\infty} (kt^{k-1} + qt^k) e^{qt} dt \int_{\{\psi < -t\}} (dd^c \varphi)^n.$$

On the other hand, by [CKZ05] there exists a constant $p_n > 0$ such that for any $\psi \in \dot{\mathcal{F}}(\Omega)$ and any $t > 0$, we have

$$c_n(\{\psi < -t\}) \leq p_n t^{-n} \int_{\Omega} (dd^c \psi)^n \leq p_n t^{-n}.$$

Choosing q' such that $q < q' < q_n(\alpha)$ and applying Corollary 4.2 with the exponent q' instead of q , we obtain

$$\begin{aligned} \int_{\Omega} (-\psi)^k e^{-q\psi} (dd^c \varphi)^n &\leq Q \int_0^{+\infty} (kt^{k-n-1} + \varepsilon t^{k-n}) e^{(q-q')t} dt \\ &=: \tilde{Q}(k, n, q) < +\infty, \end{aligned}$$

since $k > n$ and $q' > q$.

To prove the second statement, observe first that if $E \Subset \Omega$ is a compact subset, for any $\psi \in \dot{\mathcal{F}}(\Omega)$,

$$\int_{\Omega} (-\psi)^k e^{-q\psi} (dd^c \varphi)^n \geq (-\max_E \psi)^k \int_E e^{-q\psi} (dd^c \varphi)^n.$$

To conclude we need the following two well known facts about plus-subharmonic functions:

- the map $\psi \mapsto \max_E \psi$ is continuous on $\mathcal{PSH}(\Omega)$ for the $L^1_{loc}(\Omega)$ -topology by Hartog's lemma;
- the set $\dot{\mathcal{F}}(\Omega) \subset \mathcal{PSH}(\Omega)$ is compact for the $L^1_{loc}(\Omega)$ -topology and $\max_E \psi < 0$ for any $\dot{\mathcal{F}}(\Omega)$ (see [Ze09]).

Therefore there exists a constant $m(E, \Omega) > 0$ such that $(-\max_E \psi)^k \geq m(E, \Omega)^k$ for any $\psi \in \dot{\mathcal{F}}(\Omega)$. \square

Local exponential integrability of plurisubharmonic functions with respect to Borel measures with Hölder continuous Monge-Ampère potentials were first obtained in [DNS10].

5. MODULUS OF CONTINUITY OF THE SOLUTION

Now we are ready to prove Theorem 3 and Theorem 4 from the introduction using Theorem 2, Corollary 3.10 and Theorem 2.2.

5.1. Proofs of Theorem 3 and Theorem 4. Recall that we are given a Borel measure μ on Ω such that there exists $\varphi \in \mathcal{SH}_m(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$ with $\varphi|_{\partial\Omega} \equiv 0$ and satisfying the inequality $(dd^c\varphi)^m \wedge \beta^{n-m} \geq \mu$ weakly on Ω .

The goal is first to prove that if modulus of continuity κ_φ of φ satisfies the Dini type condition (0.6) for $1 \leq m < n$ or (0.8) for $m = n$ respectively, then for any boundary value datum $g \in \mathcal{C}^0(\partial\Omega)$, the Dirichlet problem (0.1) has a unique continuous solution. Moreover when $\mu(\Omega) < +\infty$, we will give an estimate on the modulus of continuity of the solution.

Recall that the function h_m is defined by its reciprocal as follows $\tau = h_m(t)$ is the unique solution to the following equation:

$$(5.1) \quad h_m^{-1}(\tau) := \tau^{2m} J_m^{-1}(\tau), \quad J_m(\tau) := \int_0^\tau [\kappa_\varphi \circ \theta_m \circ \ell_m(t)]^{1/m} \frac{dt}{t},$$

where $\ell_m(s)$ is defined by (4.1) and θ_m is the inverse of the function $t \mapsto t^{2m} \kappa_\varphi(t)^{1-m}$.

We are going to prove the two theorems at the same time since the proofs only differ in the last step.

Proof. There are two steps in the proof.

1. *Existence of a continuous solution.* Since $\mu \leq (dd^c\varphi)^m \wedge \beta^{n-m}$ and $\varphi = 0$ on $\partial\Omega$, it follows from Theorem 2, that μ is Γ -diffuse with $\Gamma(t) = t\gamma_m(t)$ and $\gamma_m(t) := \kappa_\varphi \circ \theta_m \circ \ell_m(t)$.

We claim that that the conditions (0.6) and (0.8) of Theorem 3 and Theorem 4 respectively imply that the Dini condition (3.9) holds for γ_m in both cases.

Indeed assume first that $1 \leq m < n$. Then $\ell_m(t) = t^r$, where $1 < r < m/(n-m)$. By the change of variable $s = \ell_m(t) = t^r$ we obtain

$$\int_{0^+}^1 \gamma(t)^{1/m} \frac{dt}{t} = \frac{1}{r} \int_{0^+}^1 \kappa_\varphi^{1/m}(\theta_m(s)) \frac{ds}{s}.$$

Then the change of variables $x = \theta_m(s)$ allows to write $s = \theta_m^{-1}(x) = x^{2m} \kappa_\varphi^{1-m}(x)$ which implies $\frac{ds}{s} = 2m \frac{dx}{x} + (1-m)d\kappa_\varphi(x)$. Then an easy computation shows that

$$\begin{aligned} \int_{0+}^1 \gamma(t)^{1/m} \frac{dt}{t} &= \frac{2m}{r} \int_{0+}^{\theta_m(1)} \kappa_\varphi^{1/m}(x) \frac{dx}{x} \\ &+ \frac{m(1-m)}{r} \kappa_\varphi^{1/m}(\theta_m(1)). \end{aligned}$$

This proves that the condition (0.6) of Theorem 3 is equivalent to the Dini condition (3.9) for the function $\gamma = \gamma_m$.

Now assume that $m = n$. Then $\ell_n(t) = e^{-b/t^{1/n}}$. We set $s = \ell_n(t)$. Then $\frac{dt}{t} = n \frac{ds}{s(-\log s)}$. Hence

$$\int_{0+}^1 \gamma(t)^{1/n} \frac{dt}{t} = n \int_{0+}^1 \kappa_\varphi^{1/n}(\theta_n(s)) \frac{ds}{s(-\log s)}.$$

Now observe that $x = \theta_n(s)$ satisfies $s = \theta^{-1}(x) = x^{2n} \kappa_\varphi^{1-n}(x)$.

Since κ_φ is increasing, it follows that $s \geq c_1 x^{2n}$ and then $x = \theta_n(s) \leq (s/c_1)^{1/2n}$ for $s \in]0, 1]$. Therefore

$$\int_{0+}^1 \gamma(t)^{1/n} \frac{dt}{t} \leq n \int_0^a \kappa_\varphi^{1/n}((s/c_1)^{1/2n}) \frac{ds}{s(-\log s)},$$

Now the change of variable $x = (s/c_1)^{1/2n}$ leads to the inequality

$$\int_{0+}^1 \gamma(t)^{1/n} \frac{dt}{t} \leq 2n^2 \int_{0+}^a \kappa_\varphi^{1/n}(x) \frac{dx}{x(-\log(c_1 x))},$$

This shows that the condition (0.8) in Theorem 4 implies that the Dini condition (3.9) holds for the function $\gamma = \gamma_n$. This proves our claim about γ_n . A more careful computation shows that actually the two conditions (0.8) and (3.9) are equivalent, but we don't need that here.

By Corollary 3.10, it follows that there is a unique function $u \in \mathcal{SH}_m(\Omega) \cap \mathcal{C}^0(\Omega)$ such that

$$(dd^c u)^m \wedge \beta^{n-m} = \mu,$$

in the weak sense on Ω and $u = g$ on $\partial\Omega$.

Step 2 : Estimation of the partial $\widehat{\kappa}$ -modulus of continuity. Assume that $\mu(\Omega) < +\infty$. We want to estimate the modulus of continuity of the solution u .

For $0 < \delta < \delta_0$ and denote as before by $\widehat{u}_\delta(z)$ the mean value of u on the ball $B(z, \delta) \subset \Omega$. By Lemma 2.4, the global approximants defined

for $0 < \delta < \delta_0$ by

$$\tilde{u}_\delta := \begin{cases} \max\{\hat{u}_\delta - \kappa(\delta), u\} & \text{on } \Omega_\delta, \\ u & \text{on } \Omega \setminus \Omega_\delta \end{cases}$$

with $\kappa(\delta) := \kappa_\varphi(\delta) + \kappa_g(\sqrt{\delta}) + \delta$ satisfy the following properties:

- \tilde{u}_δ is m -subharmonic and bounded on Ω ,
- $\tilde{u}_\delta(z) = u(z)$ on $\Omega \setminus \Omega_\delta$
- $0 \leq \tilde{u}_\delta(z) - u(z) \leq \hat{u}_\delta(z) - u(z) \leq \tilde{u}_\delta(z) - u(z) + \kappa(\delta)$ for $z \in \Omega_\delta$.

Therefore we can apply Corollary 2.5 and get for $0 < \delta < \delta_0$,

$$(5.2) \quad \begin{aligned} \int_{\Omega} (\tilde{u}_\delta - u)^m d\mu &= \int_{\Omega_\delta} (\tilde{u}_\delta - u)^m d\mu \\ &\leq C_m \kappa_\varphi \circ \theta_m (D\delta^2). \end{aligned}$$

By Theorem 3.11 it follows that

$$(5.3) \quad \sup_{\Omega} (\tilde{u}_\delta - u) \leq B h_m (2^m e^m \|\tilde{u}_\delta - u\|_{m,\mu}^m),$$

where $D > 0$ is a uniform constant, and $h_m = h_{\Gamma_m}$ is defined by the formula (5.1).

Therefore from equation (5.2) and (5.3) we deduce that

$$\sup_{\Omega} (\tilde{u}_\delta - u) \leq B h_m [C_m \kappa_\varphi \circ \theta_m (D\delta^2)],$$

which implies

$$(5.4) \quad \sup_{\Omega_\delta} (\tilde{u}_\delta - u) \leq B h_m [C_m \kappa_\varphi \circ \theta_m (D\delta^2)].$$

Recall that $\hat{u}_\delta - u \leq \tilde{u}_\delta - u + \kappa(\delta)$ on Ω_δ . Hence for $0 < \delta < \delta_0$

$$(5.5) \quad \sup_{\Omega_\delta} (\hat{u}_\delta - u) \leq \sup_{\Omega} (\tilde{u}_\delta - u) + \kappa(\delta).$$

Using (5.4) and (5.5) we finally get for $0 < \delta < \delta_0$

$$(5.6) \quad \sup_{\Omega_\delta} (\hat{u}_\delta - u) \leq B \tilde{\kappa}(\delta),$$

where

$$(5.7) \quad \tilde{\kappa}(\delta) := h_m [C_m \kappa_\varphi \circ \theta_m (D\delta^2)] + \kappa_\varphi(\delta) + \kappa_g(\sqrt{\delta}) + \delta,$$

and h_m is defined by (5.1). □

5.2. Some consequences. Let us state corollaries of Theorem 3 and Theorem 4 to show how the estimates obtained so far are more precise compared to previous ones (see [KN20b], [BZ20]).

Corollary 5.1. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded strongly m -pseudoconvex domain with $1 \leq m \leq n$ and μ a positive Borel measure on Ω . Assume that there exists $\varphi \in \mathcal{SH}_m(\Omega) \cap C^\alpha(\bar{\Omega})$ such that*

$$(5.8) \quad \mu \leq (dd^c \varphi)^m \wedge \beta^{n-m}, \text{ weakly on } \Omega \text{ and } \varphi|_{\partial\Omega} \equiv 0.$$

Then for any continuous function $g \in C^{2\alpha}(\partial\Omega)$, there exists a unique function $U = U_{g,\mu} \in \mathcal{SH}_m(\Omega) \cap C^0(\bar{\Omega})$ such that

$$(dd^c U)^m \wedge \beta^{n-m} = \mu, \text{ and } U = g \text{ on } \partial\Omega.$$

Moreover if $\mu(\Omega) < +\infty$, $U \in C^{\tilde{\alpha}}(\bar{\Omega})$ for any $\tilde{\alpha} < \tilde{\alpha}_m$, where

$$(5.9) \quad \tilde{\alpha}_m := \frac{2\tilde{r}\alpha^2}{m\tilde{m}[\tilde{m} + 2\alpha\tilde{r}]},$$

and $\tilde{r} := \frac{m}{n-m}$ and $\tilde{m} := 2m + \alpha(1-m)$.

Proof. We want to apply Theorem 3. Here we have $\kappa_\varphi(t) = \kappa_0 t^\alpha$, $\ell_m(t) = t^r$ with $0 < r < m/(n-m)$. By Theorem 2, μ is Γ -diffuse with $\Gamma(t) = At\kappa_\varphi \circ \theta_m(\ell_m(t))$ and θ_m is the inverse of the function $t \mapsto t^{2m}\kappa_\varphi(t)^{1-m} = t^{2m+\alpha(1-m)}$. Thus $\Gamma(t) = At^{1+r\alpha/\tilde{m}}$.

Then $J_m(\tau) = A^{1/m} \frac{m\tilde{m}}{r\alpha} \tau^{\alpha r/m\tilde{m}}$ and $h_m^{-1}(t) = A'(m, \alpha) t^{2m+m\tilde{m}/\alpha r}$.

Finally by (5.7) the $\widehat{\kappa}$ -modulus of continuity of the solution is dominated as follows

$$\widehat{\kappa}_U(\delta) \leq C'(\alpha, m, n, \Omega) \delta^{\frac{2r\alpha^2}{m\tilde{m}[\tilde{m}+2\alpha r]}}.$$

As before we apply Lemma 1.8 to conclude. \square

Corollary 5.2. *Under the assumption of Theorem 4, with $\varphi \in C^\alpha(\bar{\Omega})$ with $0 < \alpha \leq 1$ and $g \in C^{2\alpha}(\partial\Omega)$, the solution $U := U_{g,\mu}$ to the Dirichlet problem is Hölder continuous and its modulus of continuity satisfies the following estimate*

$$\kappa_U(\delta) \leq C\delta^{\alpha/n\tilde{n}}(-\log \delta)^{1/2},$$

where $\tilde{n} := (2-\alpha)n + \alpha$ and $C > 0$ is a positive uniform constant.

Here κ_U is the usual modulus of continuity of U on $\bar{\Omega}$ defined as follows:

$$(5.10) \quad \kappa_U(\delta) := \sup\{|U(z) - U(z')|; z, z' \in \bar{\Omega}, |z - z'| \leq \delta\}$$

The precise relationship between κ_U and $\widehat{\kappa}_U$ will be discussed in section 2.3 (see [Ze20] for more details).

Proof. We want to apply Theorem 4. Here we have $\kappa_\varphi(t) = \kappa_0 t^\alpha$, $\ell_n(t) = e^{-bt^{-1/n}}$ with $0 < b < 2n$. By Theorem 2, μ is Γ -diffuse with $\Gamma(t) = A_0 t \kappa_\varphi \circ \theta_n(\ell_n(t))$ and θ_n is the inverse of the function $t \mapsto t^{2n} \kappa_\varphi(t)^{2-n} = t^{2n+\alpha(1-n)}$. Thus

$$\Gamma(t) = A_0 t e^{-b_1 t^{-1/n}},$$

where $b_1 := \alpha b / [2n + \alpha(1 - n)]$.

Then for $\tau > 0$, we have

$$J_n(\tau) = A_0^{1/n} \int_0^\tau e^{-b_2 t^{-1/n}} \frac{dt}{t}$$

where $b_2 := b_1/n$.

By the change of variable $s = t^{-1/n}$ we get

$$J_n(\tau) = n A_0^{1/n} \int_{\tau^{-1/n}}^{+\infty} e^{-b_2 s} \frac{ds}{s}.$$

Fix $\tau_0 > 0$. Then for $0 < \tau < \tau_0$ we have

$$J_n(\tau) \leq A_1 \int_{\tau^{-1/n}}^{+\infty} e^{-b_2 s} ds = A_2 e^{-b_2 \tau^{-1/n}},$$

where $A_1 := n A_0^{1/n} \tau_0^{1/n}$ and $A_2 := A_1/b_2$.

Therefore given $\varepsilon > 0$, there exists $y_0 > 0$ and a constant $A_3 > 0$ such that for $0 < y < y_0$ we have

$$h_n^{-1}(y) := y^{2n} J_n^{-1}(y) \geq \frac{b_2 y^{2n}}{(-\log(y/A_2))^n}.$$

An easy computation shows that there exists $x_0 > 0$ small enough such that for $0 < x < x_0$, $h_n(x) \leq A_3 x^{1/2n} (-\log x)^{1/2}$

By Theorem 4, the $\widehat{\kappa}$ -modulus of continuity of the solution U to the Dirichlet problem (0.1) in this case satisfies $\widehat{\kappa}_U(\delta) \leq \widehat{\kappa}(\delta)$, where

$$\widehat{\kappa}(\delta) := A_4 \delta^{\alpha/n\bar{n}} (-\log \delta)^{1/2}.$$

Now we need to apply Lemma 1.8 to conclude. Indeed it's clear that the modulus of continuity $\widehat{\kappa}$ obtained above satisfies the condition (1.5). Moreover the function U is $\widehat{\kappa}$ -continuous near the boundary by Corollary 2.5. \square

6. APPLICATIONS

Let $\Omega \Subset \mathbb{C}^n$ be a bounded strongly m -pseudoconvex domain, μ a positive Borel measure on Ω and m be an integer such that $1 \leq m \leq n$. Here we will prove that all previous results still hold without assuming

that the subsolution has boundary values 0, but with a lost on the control of the modulus of continuity of the solution.

Theorem 6.1. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded strongly m -pseudoconvex domain with $1 \leq m \leq n$ and μ a positive Borel measure on Ω . Assume that there exists $\psi \in \mathcal{SH}_m(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$ such that*

$$(6.1) \quad \mu \leq (dd^c\psi)^m \wedge \beta^{n-m}, \text{ weakly on } \Omega.$$

Assume that the modulus of continuity κ_ψ of ψ satisfies the following Dini type condition:

$$(6.2) \quad \int_{0^+} \frac{[\kappa_\psi(t)]^{1/m}}{t |\log t|^{\epsilon_m}} dt < +\infty,$$

where $\epsilon_n = 1$ and $\epsilon_m = 0$ if $1 \leq m < n$. Then for any continuous function $g \in \mathcal{C}^0(\partial\Omega)$, there exists a unique function $U = U_{g,\mu} \in \mathcal{SH}_m(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$ such that

$$(dd^cU)^m \wedge \beta^{n-m} = \mu, \text{ and } U = g \text{ on } \partial\Omega.$$

Moreover if $\mu(\Omega) < +\infty$, the $\widehat{\kappa}$ -modulus of continuity of U satisfies the following estimate:

$$\widehat{\kappa}_U(\delta) \leq C \widehat{\kappa}_m(\delta),$$

where $\widehat{\kappa}_m(\delta)$ is given by the following formula.

$$(6.3) \quad \widehat{\kappa}_m(\delta) := h_m [C_m \tilde{\kappa}_\psi \circ \theta_m (D\delta^2)] + \tilde{\kappa}_\psi(\delta) + \kappa_g(\sqrt{\delta}) + \delta,$$

where $\tilde{\kappa}_\psi(\delta) = \kappa_\psi(\sqrt{\delta})$ and h_m is defined by (5.1).

Proof. We want to apply Theorem 3 and Theorem 4. We claim that there exists a function $\varphi \in \mathcal{SH}_m(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$ such that $\varphi = 0$ on $\partial\Omega$ with $\kappa_\varphi(\delta) \leq \kappa_\psi(\sqrt{\delta})$ and $\mu \leq (dd^c\varphi)^m \wedge \beta^{n-m}$, weakly on Ω .

Indeed, consider the maximal m -subharmonic function on Ω with boundary values $-\psi$ i.e.

$$\tilde{\varphi}(z) := \sup\{v(z); v \in \mathcal{SH}_m(\Omega), v \leq -\psi, \text{ on } \partial\Omega\}.$$

We know by [Ch16b] that $\tilde{\varphi} \in \mathcal{SH}_m(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$, $\tilde{\varphi} = -\psi$ on $\partial\Omega$ and $\kappa_{\tilde{\varphi}}(\delta) \leq \kappa_\psi(\sqrt{\delta})$.

It follows that the function $\varphi = \tilde{\varphi} + \psi \in \mathcal{SH}_m(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$ and satisfies the inequality

$$(dd^c\varphi)^m \wedge \beta^{n-m} \geq (dd^c\tilde{\varphi})^m \wedge \beta^{n-m} + (dd^c\psi)^m \wedge \beta^{n-m} \geq \mu,$$

weakly on Ω . Moreover $\varphi = 0$ on $\partial\Omega$ and $\kappa_\varphi(\delta) \leq \kappa_\psi(\sqrt{\delta})$. It is clear that the modulus of continuity of φ satisfies the Dini condition (6.2) so that we can apply Theorem 3 in the case $m < n$ and Theorem 4 in the case $m = n$, which proves the theorem. \square

Let us give an example of application of this result.

Example 6.2. Let $1 \leq m \leq n$. Let us embed $\mathbb{R}^m \hookrightarrow \mathbb{C}^m$ as the real part of \mathbb{C}^m and consider the following Borel measure $\mu := \lambda_m \otimes \lambda_{2n-2m} \sim H_{2n-m}$ on the unit ball $\mathbb{B} \subset \mathbb{C}^n \simeq \mathbb{C}^m \times \mathbb{C}^{n-m}$. This is a singular positive measure on \mathbb{B}

We consider the following function :

$$\psi(z) := \sum_{1 \leq j \leq m} (\Re z_j)_+ + \sum_{m < j \leq n} |z_j|^2 - 2.$$

It is easy to check that ψ is plurisubharmonic and Lipschitz on any bounded domain in \mathbb{C}^n which satisfies the following inequality

$$(6.4) \quad \mu \leq A(dd^c \psi)^m \wedge \beta^{n-m}$$

in the sense of currents on \mathbb{C}^n , where $A > 0$ is a positive constant.

Moreover $\int_{\mathbb{B}} (dd^c \psi)^m \wedge \beta^{n-m} < +\infty$. Therefore by Theorem 6.1, there exists $u \in \mathcal{SH}_m(\mathbb{B}) \cap C^0(\bar{\mathbb{B}})$ such that $(dd^c u)^m \wedge \beta^{n-m} = \mu$ on \mathbb{B} and $u = 0$ on $\partial \mathbb{B}$. Moreover $u \in C^{\alpha'}(\bar{\mathbb{B}})$ for any where $\alpha' < \hat{\alpha}_m$, where $\tilde{\alpha}_m$ is given by the formula

$$\hat{\alpha}_m := \frac{\tilde{r}}{2\tilde{m}[m\tilde{m} + \tilde{r}]}, \text{ with } \tilde{m} := 2m + (1 - m)/2.$$

When $m = n$ we can improve this exponent since by Corollary 5.2, the modulus of continuity satisfies $\kappa_u(\delta) \leq Const \delta^{1/2n\tilde{m}} (-\log \delta)^{1/2}$.

Observe that the unit ball \mathbb{B} can be replaced by any bounded strongly m -pseudoconvex domain $\Omega \Subset \mathbb{C}^n$.

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