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Obukhov Length as a Scale for the Nondimensionalization of a Turbulence Equation System

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1 **Obukhov Length as a Scale for the**
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4 **Jun-Ichi Yano · Marta Waclwczyk**

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24 **Keywords** Nondimensionalization scale · Obukhov length · Stably-stratified
25 turbulence · Vertical scale ·

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1 Introduction

Theories of atmospheric boundary-layer turbulence have been developed by heavily relying on the so-called dimensional analyses (Barenblatt 1996). This methodology is alternatively called the scaling approach in atmospheric boundary-layer studies, as reviewed by *e.g.*, Holtslag and Nieuwstadt (1986), Foken (2006). Some key variables controlling a given regime of turbulence are first identified, then various characteristic scales of the system (length, velocity, temperature, etc) are determined from these key controlling variables by a dimensional consistency. For example, the Obukhov length (Obukhov 1948) is defined from the frictional velocity and the buoyancy flux from a dimensional analysis.

For performing a dimensional analysis correctly, a certain ingenuity is required for choosing proper controlling variables of a given system. No systematic methodology exists for choosing them, but the choice is solely based on physical intuitions. A wrong choice of controlling variables can lead to totally meaningless results (*cf.*, Batchelor 1954). With absence of an analytical solution to turbulent flows as well as difficulties in observations and numerical modelling, usefulness of those proposed scales is often hard to judge. As a result, the Obukhov length is hardly a unique choice. There are various efforts to introduce alternative scales as reviewed throughout the text, but the most notably one is a theory based on gradient-based scales (Sorbjan 2006, 2010, 2016).

However, the dimensional analysis is not a sole possibility of defining characteristic scales of a system. In atmospheric large-scale dynamics, they are typically derived by a nondimensionalization. Arguably, this procedure is more straightforward and formal: we simply introduce characteristic scales for all the variables for nondimensionalizing them. These scales cannot be arbitrary, because we expect that terms in the equation balance each other, thus their orders of magnitudes must match each other. These conditions, in turn, constrain these characteristic scales in a natural manner. Advantage of the nondimensionalization approach is that these scales are defined not only by dimensional consistencies, but also by requirements of balance between the terms in a system. The latter are stronger constraints than the former. The Rossby radius of deformation is a classical example of a characteristic scale identified by a nondimensionalization. This scale characterizes the quasi-geostrophic system (Sec. 3.12, Pedlosky 1987). The depth of the Ekman layer is another such example (Sec. 4.3, Pedlosky 1987).

The present paper attempts to derive the Obukhov length by following this principle of nondimensionalization. For this goal, the principle is applied to a turbulent system. This scale is a core of the celebrated Monin–Obukhov similarity theory (Monin and Obukhov 1954). In spite of its importance in describing boundary-layer turbulence, the basis of this Obukhov scaling is often questioned. Especially, various studies suggest that this similarity theory breaks down in a strongly-stratified limit (*cf.*, King 1990, Howell and Sun 1999, Mahrt 1999). Various efforts for generalization of the Monin–Obukhov

71 theory already exist (*e.g.*, Zilitinkevich and Calanca 2000, Zilitinkevich 2002,
72 Zilitinkevich and Easu 2007). However, as far as the authors are aware of, all
73 these efforts are under frameworks of the dimensional analysis. The present pa-
74 per is going to suggest a procedure beyond those efforts by analysing governing
75 equations of turbulence more directly.

76 A similar analysis is already performed by Mahrt (1982) more specifically
77 focusing on gravity-wave currents in the boundary layer. Unfortunately, he
78 has only considered a first of step of the nondimensionalization, called “scale
79 analysis” but without actually writing down a nondimensional set of equations.
80 Advantage of his study is, in turn, in actually solving a closed set of equations.
81 In contrast, the present study performs the nodimensionalization in a more
82 systematic manner by taking equations for turbulence statistics. However, the
83 chosen set of equations chosen is hardly self-contained. A full nondimensional-
84 ization of a turbulence model is performed by Nieuwstadt (1984). However, an
85 adopted model already contains a closure, thus the final nondimensionalization
86 result also depends on the closure. In the present study, we consider turbu-
87 lence equations without closure for this reason. The present analysis is close
88 in the spirit of George *et al.* (2000) in seeking to identify characteristic scales
89 of a given system by directly examining a balance condition in a governing
90 equation, although the latter study does not go through a path of nondimen-
91 sionalization. Link of the present study to theirs further suggests possibilities
92 of applying various methodologies of multiscale asymptotic expansions to the
93 atmospheric boundary-layer problems, though such attempts are left for future
94 studies.

95 The paper proceeds as follows. In the next section, the Richardson num-
96 ber is introduced by a nondimensionalization of a linear perturbation problem
97 under a presence of a background vertical wind shear. This example serves
98 for double purposes. First, since the Richardson number plays a key role in
99 the Monin–Obukhov theory, its significance must be better identified in a self-
100 contained manner (*cf.*, Loboeki 2013). Second, it provides a simple demonstra-
101 tion of how characteristic scales can be identified by a nondimensionalization
102 procedure. Here, the analysis can also be performed for a fully nonlinear case
103 as well, but without any major changes in conclusions, except for a conse-
104 quence of absolute magnitudes of dependent variables can be estimated as a
105 result. This is considered a standard problem in instabilities theories of fluid
106 mechanics, originating from Miles (1961) and Howard (1961). In standard
107 analyses, the Richardson number is introduced in retrospect only after solving
108 the instability problem in dimensional form, with an exception of Sec. 8.1 of
109 Townsend (1976), which outlines a nondimensionalization procedure for this
110 problem. A nondimensionalization is performed originally in this section, but
111 merely intended to serve as a pedagogical purpose for a demonstration.

112 The main analyses are presented in Secs. 3 and 4, where the turbulent-
113 kinetic energy (TKE) and the eddy buoyancy equations are examined, respec-
114 tively. In these two sections, the nondimensionalization procedure is tailored
115 towards a style of the similarity theory for the turbulence. A full nondimension-
116 alization of this system under a standard procedure is provided in Appendix

117 separately as a reference. Link to the similarity theory is discussed in Sec. 5,
118 and the paper is concluded by final remarks in Sec. 6.

119 Throughout the paper, Boussinesq approximation is adopted for the anal-
120 ysis for a simplicity. Though an extension of the analysis under an anelastic
121 approximation is relatively straightforward, there is no much advantage in this
122 generalization, but only with a consequence of making the analysis more in-
123 volved. For the same reason, a two-dimensional system is considered in Sec. 2
124 with x and z taken as horizontal and vertical coordinates, respectively.

125 The bar, $\bar{}$, and the prime, \prime , signs are used throughout for designating the
126 mean and the deviation from the former. The former is assumed to depend
127 only on height, z . The latter corresponds to the turbulence fluctuation when a
128 fully nonlinear problem is considered, as in Secs. 3 and 4. On the other hand,
129 under a linear stability analysis in Sec. 2, the primed quantities designate the
130 perturbation variables. Due to the linearization, the perturbation variables
131 may grow to infinity under an unstable situation, whereas the turbulent fluctu-
132 ations are bounded by full nonlinearity. Nevertheless, the formal definition
133 of the prime sign itself does not change over these sections.

134 2 A Linear Perturbation Problem and the Richardson Number

135 As already stated, only a perturbation problem is considered in this section by
136 neglecting the nonlinearity by following a standard shear instability problem
137 (Miles 1961, Howard 1961). Also by following it, the viscosity is neglected in
138 this section. Thus, the governing equations of the problem are given by:

$$\frac{\partial u'}{\partial t} + w' \frac{d\bar{u}}{dz} + \bar{u} \frac{\partial}{\partial x} u' = -\frac{\partial \phi'}{\partial x}, \quad (1a)$$

$$\frac{\partial w'}{\partial t} + \bar{u} \frac{\partial}{\partial x} w' = -\frac{\partial \phi'}{\partial z} + b', \quad (1b)$$

$$\frac{\partial b'}{\partial t} + w' \frac{d\bar{b}}{dz} = 0, \quad (1c)$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0. \quad (1d)$$

139 Here, u and w are the horizontal and the vertical components of the velocity,
140 ϕ the pressure divided by a reference density, b the buoyancy, which may
141 be evaluated from the potential temperature, θ , as $b = g\theta/\theta_0$, where g is the
142 acceleration of the gravity, θ_0 is a reference value of the potential temperature.
143 For the moist atmosphere, the potential temperature must be furthermore
144 replaced by the virtual potential temperature for accounting for a contribution
145 of moisture to the buoyancy.

146 Nondimensionalizations are performed on the variables by designating the
147 nondimensional variables by the dagger, \dagger . The nondimensionalization scales
148 for the dependent variables are designated by the subscript $*$. Thus,

$$u' = u_* u^\dagger, \quad w' = u_* w^\dagger, \quad \phi' = \phi_* \phi^\dagger, \quad b' = b_* b^\dagger,$$

149

$$\frac{\partial}{\partial t} = \frac{1}{t_*} \frac{\partial}{\partial t^\dagger}, \quad \frac{\partial}{\partial x} = \frac{1}{z_*} \frac{\partial}{\partial x^\dagger}, \quad \frac{\partial}{\partial z} = \frac{1}{z_*} \frac{\partial}{\partial z^\dagger},$$

150 and also

$$\bar{u} = \bar{u}_* \bar{u}^\dagger, \quad \frac{d\bar{u}}{dz} = \left(\frac{d\bar{u}}{dz} \right)_* \frac{d\bar{u}^\dagger}{dz^\dagger}, \quad \frac{d\bar{b}}{dz} = \left(\frac{d\bar{b}}{dz} \right)_* \frac{d\bar{b}^\dagger}{dz^\dagger}.$$

151 Here, the velocity and the spatial scales are not differentiated in horizontal
 152 and vertical directions for maintaining the order of magnitude of the pressure
 153 gradient the same in both directions, thus $x_* = z_*$ and $w_* = u_*$. Also note that
 154 the background-state gradients, $d\bar{u}/dz$ and $d\bar{b}/dz$, are nondimensionalized by
 155 directly using the scales for these gradients, rather than by \bar{u}/z_* and \bar{b}/z_* ,
 156 respectively, for the purpose deriving a standard definition of the Richardson
 157 number below.

158 By substitutions of those expressions into the equations, we find the fol-
 159 lowing relations for the nondimensionalization scales:

$$t_* = \frac{z_*}{\bar{u}_*}, \quad \phi_* = \bar{u}_* u_*, \quad b_* = \frac{\bar{u}_* u_*}{z_*}. \quad (2a, b, c)$$

160 Here, the last relation can be interpreted as a definition of the length scale by
 161 re-writing it as:

$$z_* = \frac{\bar{u}_* u_*}{b_*}, \quad (3)$$

162 if all the variables in the right hand side can be considered to be prescribed.

163 As a result, a nondimensionalized set of equations is given by

$$\frac{\partial u^\dagger}{\partial t^\dagger} + \left[\frac{z_*}{\bar{u}_*} \left(\frac{d\bar{u}}{dz} \right)_* \right] \frac{d\bar{u}^\dagger}{dz^\dagger} w^\dagger + \bar{u}^\dagger \frac{\partial u^\dagger}{\partial x^\dagger} = - \frac{\partial \phi^\dagger}{\partial x^\dagger} \quad (4a)$$

$$\frac{\partial w^\dagger}{\partial t^\dagger} + \bar{u}^\dagger \frac{\partial w^\dagger}{\partial x^\dagger} = - \frac{\partial \phi^\dagger}{\partial z^\dagger} + b^\dagger \quad (4b)$$

$$\frac{\partial b^\dagger}{\partial t^\dagger} + \left(\frac{z_*}{\bar{u}_*} \right)^2 \left(\frac{d\bar{b}}{dz} \right)_* \frac{d\bar{b}^\dagger}{dz^\dagger} w^\dagger = 0 \quad (4c)$$

$$\frac{\partial u^\dagger}{\partial x^\dagger} + \frac{\partial w^\dagger}{\partial z^\dagger} = 0 \quad (4d)$$

164 Note that by choosing the nondimensionalization scales by Eqs. (2a, b, c),
 165 there is no constant factor in front of almost all the terms in the equations,
 166 except for the two in Eqs. (4a) and (4c). Here, by choosing a length scale, z_* ,
 167 in an appropriate manner, we can remove a constant factor from one of these
 168 two terms, but not from both of them.

169 There are two options for choosing the length scale, z_* : (i) by setting a
 170 factor in front of the second term of Eq. (4a) unity, or (ii) by setting a factor
 171 in front of the second term in Eq. (4c) unity. The option (i) amounts to set
 172 the spatial scale (shear scale) as that of the background wind shear, and we
 173 obtain

$$z_* = z_{*u} \equiv \bar{u}_* / (d\bar{u}/dz)_*. \quad (5a)$$

174 The option (ii) leads to

$$z_* = z_{*b} \equiv \bar{u}_*/(d\bar{b}/dz)_*^{1/2}. \quad (5b)$$

175 We may call the latter the buoyancy–gradient scale.

176 As a result, the set of equations also reduces to with the option (i):

$$\begin{aligned} \frac{\partial u^\dagger}{\partial t^\dagger} + \frac{d\bar{u}^\dagger}{dz^\dagger} w^\dagger + \bar{u}^\dagger \frac{\partial u^\dagger}{\partial x^\dagger} &= -\frac{\partial \phi^\dagger}{\partial x^\dagger}, \\ \frac{\partial w^\dagger}{\partial t^\dagger} + \bar{u}^\dagger \frac{\partial w^\dagger}{\partial x^\dagger} &= -\frac{\partial \phi^\dagger}{\partial z^\dagger} + b^\dagger, \\ \frac{\partial b^\dagger}{\partial t^\dagger} + Ri \left(\frac{d\bar{b}^\dagger}{dz^\dagger} \right) w^\dagger &= 0, \\ \frac{\partial u^\dagger}{\partial x^\dagger} + \frac{\partial w^\dagger}{\partial z^\dagger} &= 0, \end{aligned}$$

177 and with the option (ii):

$$\begin{aligned} \frac{\partial u^\dagger}{\partial t^\dagger} + Ri^{-1/2} \frac{d\bar{u}^\dagger}{dz^\dagger} w^\dagger + \bar{u}^\dagger \frac{\partial u^\dagger}{\partial x^\dagger} &= -\frac{\partial \phi^\dagger}{\partial x^\dagger}, \\ \frac{\partial w^\dagger}{\partial t^\dagger} + \bar{u}^\dagger \frac{\partial w^\dagger}{\partial x^\dagger} &= -\frac{\partial \phi^\dagger}{\partial z^\dagger} + b^\dagger, \\ \frac{\partial b^\dagger}{\partial t^\dagger} + \frac{d\bar{b}^\dagger}{dz^\dagger} w^\dagger &= 0, \\ \frac{\partial u^\dagger}{\partial x^\dagger} + \frac{\partial w^\dagger}{\partial z^\dagger} &= 0. \end{aligned}$$

178 Here, Ri is the Richardson number defined by a ratio of the two characteristic
179 scales:

$$Ri = \left(\frac{z_{*u}}{z_{*b}} \right)^2 = \frac{(d\bar{b}/dz)_*}{(d\bar{u}/dz)_*^2}. \quad (6)$$

180 We find that when the shear is more dominant than the buoyancy gradi-
181 ent (stratification), *i.e.*, $Ri < 1$, the scaling based on the shear scale, z_{*u} ,
182 (Eq. 5a) is relevant, and when the stratification is more dominant than the
183 shear, *i.e.*, $Ri > 1$, the scaling based on the buoyancy–gradient scale, z_{*b} ,
184 (Eq. 5b) becomes relevant. We expect that these Ri -dependent characteristics
185 of the system are still valid also for fully turbulent regimes, that are going to
186 be addressed in the next two sections. An equivalent nondimensionalization,
187 as in this section, is performed, separately, in the Appendix for a full turbu-
188 lence system considered in Secs. 3 and 4, and essentially the same conclusion
189 is drawn.

190 In this manner, we have demonstrated how naturally characteristic scales
191 (not only the length scale) of a system can be determined by a nondimension-
192 alization. A question that we are going to address in the next two sections
193 is whether the Obukhov length can be derived in a similar manner for fully
194 turbulent flows.

195 Furthermore, the result obtained in this section already has implications
 196 in the boundary–layer turbulence, because the identified characteristic scales,
 197 z_{u*} and z_{b*} , (Eqs. 5a, b) are expected to characterize the typical size of eddies
 198 of given regimes, and thus, also characterize the resulting mixing lengths, l_* .
 199 In this respect, it may be worthwhile to note that, for example, Grisogono
 200 (2010) propose to use two different mixing lengths,

$$l_* = \frac{v'_*}{(d\bar{u}/dz)_*}, \quad (7a)$$

$$l_* = \frac{v'_*}{(d\bar{b}/dz)_*^{1/2}}, \quad (7b)$$

201 depending on the Richardson number, Ri . Here, v'_* is a scale for the velocity.
 202 A similar scale (buoyancy scale), defined by

$$l_* = \frac{w'_*}{(d\bar{b}/dz)_*^{1/2}}, \quad (8)$$

203 is introduced by Stull (1973), Zeman and Tennekes (1977), Brost and Wyn-
 204 gaard (1978). Hunt *et al.* (1985), in turn, suggest by field data analysis that
 205 the buoyancy scale (Eq. 8) characterizes both the vertical heat transport and
 206 the temperature-variance production in the stably-stratified boundary layer.
 207 These definitions reduces to z_{*u} and z_{*b} , respectively, with small and large
 208 Richardson numbers by re–setting as $v'_* = \bar{u}_*$ and $w'_* = \bar{u}_*$. This condition is
 209 expected to be satisfied when a system is fully turbulent.

210 3 Turbulent System: Turbulent Kinetic Energy Equation

211 3.1 Obukhov Length

212 In the following two sections, nondimensionalization of stably-stratified tur-
 213 bulence system is considered. The goal is to derive the Obukhov length as a
 214 natural consequence of nondimensionalization, in a similar manner as demon-
 215 strated in the last section how a characteristic length of a system is identified.
 216 Thus, we focus on the equations containing the vertical buoyancy flux, $\overline{w'b'}$,
 217 and the vertical momentum stress, $\overline{u'w'}$. The Obukhov length is defined by a
 218 ratio of fractional powers of those two quantities:

$$L = \frac{(\overline{u'w'})_*^{3/2}}{(\overline{w'b'})_*}. \quad (9)$$

219 Here, $(\overline{u'w'})_*$ and $(\overline{w'b'})_*$ are the characteristic scales for the vertical momen-
 220 tum stress and the vertical buoyancy flux, respectively. Note that unlike in the
 221 dimensional analysis, under a nondimensionalization procedure, these scales do
 222 not necessarily refer to actual values at a particular vertical level (say, the sur-
 223 face). Even when such a choice to be made, the chosen values must also be
 224 representative for the whole vertical stretch of the boundary layer.

This definition is often further simplified into

$$L = \frac{u^{*3}}{(\overline{w'b'})_*}$$

by introducing a frictional velocity, u^* , defined by

$$u^{*2} = (\overline{u'w'})_*. \quad (10)$$

Here, note that in a standard nondimensionalization procedure, the orders of magnitudes of those flux terms are estimated by

$$\begin{aligned} (\overline{u'w'})_* &= u_* w_*, \\ (\overline{w'b'})_* &= w_* b_*. \end{aligned}$$

By substituting these two expressions into (9),

$$L = \frac{u_*^{3/2} w_*^{1/2}}{b_*}.$$

If an isotropic scaling (*i.e.*, $w_* = u_*$) can further be assumed as in the last section, and also $\bar{u}_* = u_*$, the last expression reduces to Eq. (3) with $L = z_*$. In this manner, taking the Obukhov length as an example here, we see that the turbulence length scales assume more than what are typically assumed for the nondimensionalization scales. See Secs. 3.7 and 5 for the further discussions.

3.2 Turbulent Kinetic Energy Equation

In this section, we consider the turbulent kinetic energy (TKE) equation, because it contains both the vertical buoyancy flux, $\overline{w'b'}$, and the vertical momentum stress, $\overline{u'w'}$. Based on a result from the last section, we expect that a closed scaling for the nondimensionalization is obtained by considering this equation when the shear effect is strong (*i.e.*, $Ri \leq 1$). On the other hand, when the stratification becomes strong enough (*i.e.*, $Ri \gg 1$), we expect that we also have to consider the eddy–buoyancy equation, which also contains a vertical buoyancy flux term. The latter will be considered in the next section.

By following a standard formulation of the atmospheric boundary-layer turbulence, only the vertical flux terms are considered assuming horizontal homogeneity. This is solely for simplifying the analysis focusing on the goal of deriving the Obukhov length. Horizontal heterogeneity is expected to be important for some stable-stratified atmospheric turbulent flows, but this extension is left for a future study.

A standard TKE equation (*e.g.*, Deardorff 1983) is given by:

$$\frac{\partial \overline{\mathbf{v}'^2}}{\partial t} \frac{1}{2} = \overline{w'b'} - \overline{u'w'} \frac{\partial \bar{u}}{\partial z} - \frac{\partial}{\partial z} \overline{w'(\mathbf{v}'^2 + \phi')} - \varepsilon \quad (11)$$

Here, the overbar designates a horizontal average, \mathbf{v} is a velocity vector, and ε the dissipation rate.

253 3.3 The Balance: $\overline{w'b'} \sim (\overline{u'w'})d\bar{u}/dz$

254 We first consider a balance between two terms that involve the vertical buoy-
 255 ancy flux, $\overline{w'b'}$, and the vertical momentum stress, $\overline{u'w'}$, respectively, because
 256 this balance is likely to lead to a derivation of the Obukhov length. This
 257 amounts to consider a balance between the first and the second terms. This
 258 balance requirement leads to:

$$(\overline{w'b'})_* = (\overline{u'w'})_* \left(\frac{d\bar{u}}{dz} \right)_*. \quad (12)$$

259 This condition may be used to estimate a vertical scale, z_* , by introducing a
 260 typical change, \bar{u}_* , of the background wind over this scale so that

$$\left(\frac{d\bar{u}}{dz} \right)_* = \frac{\bar{u}_*}{z_*}. \quad (13)$$

261 By substituting this relation back to the balance condition (12), we obtain an
 262 estimate of the length scale given by

$$z_* = \frac{\bar{u}_* (\overline{u'w'})_*}{(\overline{w'b'})_*}. \quad (14)$$

263 Our next goal is to try to show the equivalence of this length scale with that
 264 of Obukhov.

265 3.4 The Balance: $(\overline{u'w'})d\bar{u}/dz \sim \partial(\overline{w'\mathbf{v}'^2})/\partial z$

266 This length scale (14) reduces to the Obukhov length (9) if we can set $\bar{u}_* =$
 267 $(\overline{u'w'})_*^{1/2}$. To get an answer to this question, we need an estimate of the wind-
 268 shear strength, \bar{u}_* , relative to the eddy. Thus, this amounts to consider a
 269 balance between the second and the third terms.

270 This balance condition is given by

$$(\overline{u'w'})_* \left(\frac{d\bar{u}}{dz} \right)_* = \frac{(\overline{w'\mathbf{v}'^2})_*}{z_*}.$$

271 To obtain a more explicit expression for $(d\bar{u}/dz)_*$ in terms of the turbulence
 272 fluctuations, we further note

$$(\overline{u'w'})_* = u_* w_*, \quad (15a)$$

$$(\overline{w'\mathbf{v}'^2})_* = u_*^2 w_*. \quad (15b)$$

273 By substituting Eqs. (15a, b) into the balance condition above, it reduces to

$$\left(\frac{d\bar{u}}{dz} \right)_* = \frac{u_*}{z_*}, \quad (16)$$

274 and $\bar{u}_* = u_*$. Thus,

$$z_* = \frac{u_* (\overline{u'w'})_*}{(\overline{w'b'})_*}. \quad (17)$$

275 The stress term can further be re-written as

$$(\overline{u'w'}) = \hat{\epsilon} u_*^2$$

276 by introducing an aspect ratio, $\hat{\epsilon}$, of the system. As a result,

$$u_* = \hat{\epsilon}^{-1/2} (\overline{u'w'})^{1/2}. \quad (18)$$

277 By further substituting (18) into (17), we finally obtain

$$z_* = \frac{\hat{\epsilon}^{-1/2} (\overline{u'w'})_*^{3/2}}{(\overline{w'b'})_*}. \quad (19)$$

278 If the turbulence is isotropic, the length scale (19) reduces to the Obukhov
279 length (9). However, keep in mind that stably-stratified turbulence is often
280 observed to be quasi two dimensional (*i.e.*, $\hat{\epsilon} \ll 1$), thus the Obukhov length
281 may underestimate the vertical scale.

282 Most importantly, keep in mind that the balance (16) is possible only if the
283 wind-shear is strong enough, and the Richardson number, Ri , is the order of
284 unity or less. When the Richardson number, Ri , is very small, this balance is
285 no longer valid (the wind-shear term drops as $O(Ri)$), and the perturbation-
286 buoyancy equation must be considered instead, as in Sec. 4, in defining the
287 length scale of the system.

288 3.5 The Balance: $\overline{w'b'} \sim \partial(\overline{w'v'^2})/\partial z$

289 In completing the analysis, we also consider the balance between the first and
290 the third terms, because this term provides an alternative estimate of the
291 vertical scale, z_* . It is given by

$$z_* = \frac{(\overline{w'u'^2})_*}{(\overline{w'b'})_*}. \quad (20)$$

292 Note that the estimate of the length scale by Eq. (20) is valid regardless of the
293 magnitude of the Richardson number, Ri , of the system, unlike Eq. (19).

294 3.6 Deardorff Velocity Scale

295 Eq. (20) can further be re-written by noting that

$$(\overline{w'u'^2})_* = u_*^2 w_* = \hat{\epsilon} u_*^3 = \hat{\epsilon}^{-2} w_*^3$$

296 Rearrangement after the substitution leads to a velocity scale

$$w_* = [\hat{\epsilon}^2 z_* (\overline{w'b'})_*]^{1/3} \quad (21)$$

297 This velocity scale reduces to one introduced by Deardorff (1970) and Tennekes
298 (1970), when we set $\hat{\epsilon} = 1$ and z_* the boundary-layer depth.

3.7 Link to Eddy–Diffusion Formulation

Those readers who are with a strong background in boundary–layer meteorology, Eqs. (15a, b) may not be immediately clear. Here, the right hand sides are the expressions used in the standard nondimensionalization procedure, as considered in Sec. 2. As already remarked at the end of Sec. 3.1, the dimensional analysis in the boundary–layer meteorology, more specific scale variables, such as fluxes, are considered for deriving the characteristic scales of a system. On the other hand, under a nondimensionalization procedure, only the orders of magnitudes of the variables in concern, thus if $u \sim u_*$, $w \sim w_*$, etc, we can immediately write as Eqs. (15a, b). Keep in mind that, by definition, the values in the left–hand sides are also only the orders of magnitude estimates. They do not correspond at all to any actual flux values at any vertical levels, as expected in standard boundary–layer similarity theories.

However, it may be useful to realize that the same expressions can also be derived by adopting eddy–diffusion formulations. Under the latter, the left hand sides in Eqs. (15a, b) may be represented as

$$\overline{(u'w')}_* = -\nu_t \frac{d\bar{u}}{dz} \quad (22a)$$

$$\overline{(w'\mathbf{v}'^2)}_* = -\nu_t \frac{d\bar{u}^2}{dz} \quad (22b)$$

Here, the eddy–diffusion coefficient, ν_t , may further be written as

$$\nu_t = u_* l_*$$

in terms of a velocity scale, u_* , and a mixing length, l_* . By substituting this expression into Eqs. (22a, b), we finally obtain

$$\overline{(u'w')}_* = u_* l_* \frac{\bar{u}_*}{z_*} f^\dagger(z^\dagger) \quad (23a)$$

$$\overline{(w'\mathbf{v}'^2)}_* = u_* l_* \frac{\bar{u}_*^2}{z_*} g^\dagger(z^\dagger) \quad (23b)$$

Here, $f^\dagger(z^\dagger)$ and $g^\dagger(z^\dagger)$ are universal functions (*cf.*, Sec. 5 for further discussions). By normalizing them as, say, $f^\dagger(1) = 1$ and $g^\dagger(1) = 1$, and also assuming $l_* = z_*$, $\bar{u}_* = u_*$, $\hat{\epsilon} = 1$, Eqs. (23a, b) reduce to Eqs. (15a, b). However, here, more steps are required to reach the same conclusion.

4 Turbulent System: Buoyancy Perturbation Equation

The analysis in Sec. 2 suggests that as the Richardson number, Ri , becomes larger than unity, the shear term in the TKE equation (or the momentum equation) becomes less important, so that it loses a basis of using this term for deriving the Obukhov length from Eq. (14). As the Richardson number increases, in turn, the buoyancy stratification, $d\bar{b}/dz$, plays a more important

328 role. For considering the contribution of this term, in this section, we consider
 329 the buoyancy perturbation equation:

$$\frac{\partial b'}{\partial t} + w' \frac{d\bar{b}}{dz} + \frac{\partial w' b'}{\partial z} = Q'. \quad (24)$$

330 Here, Q' is a perturbation diabatic heating.

331 By considering a balance between the second and the third terms in the
 332 left hand side, we obtain as a length scale:

$$z_* = \frac{\overline{(w' b')}_*}{w_* (d\bar{b}/dz)_*}. \quad (25)$$

333 This is expected to be a characteristic scale of a turbulent flow when the per-
 334 turbation buoyancy becomes a dominant under a strong stratification, being
 335 consistent with a more general analysis in the Appendix.

336 This scale is somehow akin to the external static-stability scale

$$L_N = \frac{\overline{(u' w')}_*^{1/2}}{(d\bar{b}/dz)_*^{1/2}}.$$

337 as introduced by Kitaigorodskii (1988), and considered, especially, by Zilitinke-
 338 vich and Esau (2005). More specifically, Zilitinkevich and Calanca (2000) sug-
 339 gests a nondimensional parameter, L/L_N , to define a transition from a regime
 340 dominated by the Obukhov length, L , to L_N . A link to the buoyancy-gradient
 341 scale introduced by Eq. (5b) is also noted. Recall that the latter is further
 342 linked to the buoyancy scale introduced by Eqs. (7b, c). Sorbjan (2006, 2010,
 343 2016), in turn, develops his gradient-based similarity theory based on the
 344 buoyancy scale.

345 Zilitinkevich and Esau (2005) argue that the scale, L_N , becomes relevant
 346 when the vertical eddy heat flux is small. In turn, we argue that the scale
 347 defined by Eq. (25) becomes relevant when the Richardson number, Ri , is
 348 large enough. The scale, L_N , can be obtained by assuming that Ozmidov
 349 scale, $L_o = \varepsilon^{1/2} (d\bar{b}/dz)_*^{3/2}$, is equal to another length scale, $z_* = \overline{(u' w')}_*^{3/2} / \varepsilon$,
 350 which defines the characteristic turbulence length (Grisogono 2010).

351 It transpires that the scale, L_N , is purely based on a dimensional analysis.
 352 In contrast, the scale (25) is derived from an actual balance in an equation.
 353 These two scales become equivalent when all the flux scales are re-written
 354 in terms of the scales of more basic variables as in Eqs. (15a, b), and also
 355 the relation (3) is invoked. We expect that values of these two scales are also
 356 similar numerically in practice. Note further that under this equivalence, the
 357 nondimensional parameter, L/L_N , also reduces to the Richardson number, Ri .
 358 However, as emphasized in the next section, in employing a length scale in a
 359 context of a dimensional analysis leading to a similarity theory, it must be
 360 exact numerical value rather than just an order-of-magnitude estimate, as in
 361 a standard nondimensionalization.

362 5 Discussions: Link to the Similarity Theory

363 A basis for the similarity theory may be provided from the nondimension-
 364 alization of the system considered in the last two sections in the following
 365 manner. As an example, let us consider the TKE equation (11). Note that the
 366 TKE equation is hardly self-contained. However, for the sake of a heuristic
 367 argument, let us suppose that every variable in Eq. (11) can be determined
 368 self-consistently by solving it. Alternatively, we may just suppose, though not
 369 feasible in reality, that all the necessary higher-order moments are also nondi-
 370 mensionalized in a similar manner, and all the necessary equations are solved
 371 to obtain all the variables in Eq. (11).

372 Steady solutions for these nondimensionalized variables are given in terms
 373 of nondimensional functions, say, $f^\dagger(z^\dagger)$, $g^\dagger(z^\dagger)$, $q^\dagger(z^\dagger)$, etc. as

$$\overline{w^\dagger b^\dagger} = f^\dagger(z^\dagger) \quad (26a)$$

$$\overline{u^\dagger w^\dagger} = g^\dagger(z^\dagger) \quad (26b)$$

$$\frac{\partial \bar{u}^\dagger}{\partial z^\dagger} = q^\dagger(z^\dagger), \quad (26c)$$

374 etc. After dimensionalizations,

$$\overline{w' b'} = (\overline{w' b'})_* f^\dagger\left(\frac{z}{z_*}\right) \quad (27a)$$

$$\overline{u' w'} = (\overline{u' w'})_* g^\dagger\left(\frac{z}{z_*}\right) \quad (27b)$$

$$\frac{\partial \bar{u}}{\partial z} = \left(\frac{\partial \bar{u}}{\partial z}\right)_* q^\dagger\left(\frac{z}{z_*}\right) \quad (27c)$$

375 Furthermore, let us suppose that the only necessary boundary conditions re-
 376 quired for determining these variables are their surface values. In that case,
 377 we set $(\overline{w' b'})_* = (\overline{w' b'})_0$, $(\overline{u' w'})_* = (\overline{u' w'})_0$, and $(\partial \bar{u} / \partial z)_* = (\partial \bar{u} / \partial z)_0$ with
 378 the subscript, 0, designating the surface values. Here, those eddy flux values
 379 are more precisely defined at the top of the viscous boundary layer, but these
 380 can also be equated with the actual values of the fluxes from the surface.

381 As a result, general solutions to the system are given by

$$\overline{w' b'} = (\overline{w' b'})_0 f^\dagger\left(\frac{z}{z_*}\right) \quad (28a)$$

$$\overline{u' w'} = (\overline{u' w'})_0 g^\dagger\left(\frac{z}{z_*}\right) \quad (28b)$$

$$\frac{\partial \bar{u}}{\partial z} = \left(\frac{\partial \bar{u}}{\partial z}\right)_0 q^\dagger\left(\frac{z}{z_*}\right) \quad (28c)$$

382 Based on the arguments so far, we may conclude that the nondimensional
 383 functions, $f^\dagger(z/z_*)$, $g^\dagger(z/z_*)$, $q^\dagger(z/z_*)$, are universal only depending on the
 384 nondimensionalization scale, z_* . This is an essence of the similarity theory
 385 (*cf.*, Sorbjan 1989).

Here, for this last statement to be valid in strict manner, it is not sufficient that the nondimensionalization scale, z_* , simply measures a characteristic scale of a system, merely as an estimate of an order of magnitude, as the nondimensionalization intends to do. Instead, this length scale, z_* , must be re-scaled in a precise manner as the system environment changes so that the universality of the functions f , g^\dagger , and q^\dagger is maintained.

Being consistent with this requirement for developing a similarity theory, in the last two sections, attempt is made to define the length scale, z_* , of the system in a more strict manner based on the flux scales, $(\overline{u'w'})_*$, $(\overline{w'b'})_*$ etc. rather than simply in terms of the variable scales, u_* , w_* , etc. For applying a similarity theory in a strict manner, these flux values must also be defined at a specific vertical level, probably, most conveniently at the surface.

In this respect, under a systematic nondimensionalization procedure, the present study has failed to derive the Obukhov length, L , (Eq. 9) in any strict manner as a nondimensionalization scale, z_* . The closest we have obtained is Eq. (14), which defines the length scale in terms of both the wind stress magnitude, $(\overline{u'w'})_*$, and the wind-shear scale, \bar{u}_* . It reduces to the Obukhov length only under an extra scale argument (15a, b) for isotropic turbulence (*i.e.*, $\hat{\epsilon} = 1$). We should realize that this reduction is hardly exact from a point of view of the similarity theory, either, thus in applying this scale to the universal functions, f^\dagger , g^\dagger , and q^\dagger , some minor adjustment factors must be multiplied on the nondimensionalized vertical coordinate, z^\dagger .

Most importantly, this scaling is valid only when the Richardson number is $Ri \leq 1$. When $Ri \gg 1$, with strong stratification, the buoyancy-gradient scale defined by Eq. (25) becomes more relevant rather than the shear-based scale (14).

However, the TKE equation identifies an alternative vertical scale defined by Eq. (20), which is expected to be valid independent of the value of the Richardson number, thus also independent of the turbulence regimes. This new length scale is not remotely different from the Obukhov length, but obtained by replacing the frictional velocity, u^* , by the scale, $(\overline{w'u'^2})_*^{1/3}$, for the vertical flux of the turbulent kinetic energy.

6 Summary and Further Remarks

The present study has derived the Obukhov length as a result of a balance between the vertical buoyancy flux and the shear-generation terms in the TKE equation. However, this balance is valid only with the Richardson number, Ri , of the order of unity or less. When the Richardson number is much larger than unity, the system is characterized by another length scale defined by a ratio between the vertical buoyancy eddy flux and a vertical advection rate of buoyancy controlled by background stratification (Eq. 25). This scale is akin to the buoyancy scale introduced by Stull (1973), Zeman and Tennekes (1977), Brost and Wyngaard (1978), and the external static-stability scale considered by Zilitinkevich and Esau (2005).

429 The present nondimensionalization analysis has also identified a new length
430 scale defined as a ratio between the the TKE vertical flux, $(w'u'^2)_*$, and the
431 vertical buoyancy flux, $(w'b')_*$ (Eq. 20). This scale may be adopted as an alter-
432 native length scale for developing a boundary-layer turbulence similarity the-
433 ory. A next step would be to examine this possibility based on field-campaign
434 data sets.

435 An important aim behind the present study has been to demonstrate how
436 a characteristic scale of a turbulent system can be identified directly by nondi-
437 mensionalization of a partial differential-equation system describing turbu-
438 lence. For this demonstrative purpose, the analysis has been performed with
439 the simplest possible turbulence system, assuming a horizontal homogeneity.
440 Also only a limited set of equations is examined, namely, the TKE and the
441 buoyancy-perturbation equations. The choice is made specifically with a goal
442 in mind of identifying the Obukhov length so that equations contain the di-
443 mensional parameters used in its definition.

444 An equivalent nondimensionalization analysis with a full turbulence equa-
445 tion system is still to be performed. Nevertheless, the present preliminary
446 analysis already suggests a fruitfulness of such an investigation. It is expected
447 that different turbulent regimes are identified by changing orders of magni-
448 tudes of the Richardson number, as the present study has already suggested.
449 Such an analysis is expected to provide a more solid theoretical basis for inter-
450 preting the various different turbulent regimes phenomenologically identified
451 for the stably-stratified turbulence (*cf.*, Mahrt 1999). Various further gener-
452 alizations are equally feasible. The present analysis has been performed under
453 an assumption of quasi-stationarity of the system. However, some of the tur-
454 bulence regimes under stable stratification may be fundamentally transient
455 (*cf.*, Caughey *et al.* 1979). A role of horizontal heterogeneity is another aspect
456 to be investigated, especially in a context of stably-stratified turbulence. For
457 example, under certain situations, the horizontal heat transport, a term that
458 is often neglected in theoretical studies, becomes a key process in heat budget
459 (*e.g.*, Wittch 1991). A role of anisotropy of the flow with $\hat{\epsilon} \ll 1$ is still to be
460 carefully examined as well.

461 Another aspect, that is not discussed herein, is a role of boundary condi-
462 tions in solving the turbulence problems. When our focus is on a layer close
463 enough to the surface (*e.g.*, surface layer), a contribution from a top of the
464 planetary layer may be neglected, as a basic premise of the Monin-Obukhov
465 theory as well as in subsequent generalizations. However, when a problem con-
466 cerns a whole depth of the boundary layer, the depth of the boundary layer
467 becomes another parameter to be considered. As pointed out by *e.g.*, Holtslag
468 and Nieuwstadt (1986), the problem must be solved by explicitly taking into
469 account of a condition at the top of the boundary layer.

470 A critical difference must also be recognized between a typical dimensional
471 analysis performed in the turbulence studies and the nondimensionalization
472 procedure considered in the present study. It is common in turbulence studies
473 to take characteristic scales (not only the length scale, but more generally) to
474 be functions of height, as manifested as a *local similarity theory* (Nieuwstadt

1984, Sec. 4.4.1, Sorbjan 1989; Mahrt 1999). For example, a frictional velocity can be evaluated as a function of height, under a dimensional analysis, and the given result is used for furthermore evaluating the Obukhov length. On the other hand, a single fixed scale must be adopted for each variable in nondimensionalization. If the velocity is ever nondimensionalized by a scale that depends on height, a height–dependence of the dimensional velocity must somehow be partitioned between a nondimensional velocity and a velocity scale for a nondimensionalization. Such a partitioning is hardly trivial, thus usually not considered.

A major exception to this rule is when variables are defined purely locally, without influence of their spatial gradients (*e.g.*, with absence of transport processes). In this case, nondimensionalization may be performed independently at every spatial point. Conversely, it suggests that a local similarity theory is valid in a strict sense, only if a system is locally in equilibrium at each vertical level without significant effects of vertical transport, as theoretically expected as a z -less stratification state (Neustadt 1984). In other words, a local similarity theory is strictly valid only for the flows with strong two dimensionality. The last remark further suggests how a validity of a given similarity theory can be verified by a more direct nondimensionalization of a given system, more generally speaking. Further investigations along this line are much anticipated.

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Appendix: Full Nondimensionalization of the Turbulent System

The appendix presents a full nondimensionalization of the system consisting of Eqs. (11) and (24). By following the same procedure as in Sec. 2, we find the following relationships between the nondimensionalization scales:

$$\phi_* = u_*^2, \varepsilon_* = \frac{\hat{\varepsilon}}{z_*} u_*^3, t_* = \frac{z_*}{\hat{\varepsilon} u_*}, b_* = \frac{u_*^2}{z_*}, Q_* = \frac{\hat{\varepsilon} u_*^3}{z_*^2} \quad (29)$$

In analogous manner as in Sec. 2, we obtain a nondimensionalized system:

$$\frac{\partial}{\partial t^\dagger} \frac{\overline{\mathbf{v}^{\dagger'2}}}{2} = \overline{w^\dagger' b^\dagger'} - \left[\frac{z_*}{u_*} \left(\frac{\partial u}{\partial z} \right)_* \right] \overline{u^\dagger' w^\dagger'} \frac{\partial \bar{u}^\dagger}{\partial z^\dagger} - \frac{\partial}{\partial z^\dagger} \overline{w^\dagger' (\mathbf{v}^{\dagger'2} + \phi^{\dagger'})} - \varepsilon^\dagger \quad (30a)$$

$$\frac{\partial b^\dagger'}{\partial t^\dagger} + \left(\frac{z_*}{u_*} \right)^2 \left(\frac{db}{dz} \right)_* \overline{w^\dagger' d\bar{b}^\dagger} + \frac{\partial w^\dagger' b^\dagger'}{\partial z^\dagger} = Q^\dagger \quad (30b)$$

Here, the vertical scale, z_* , is left unspecified. As in Sec. 2, we have two options to choose: (i) $(z_*/u_*) \overline{(\partial u / \partial z)}_* = 1$, or (ii) $(z_*/u_*)^2 \overline{(db/dz)}_* = 1$. Each leads to an option of (i) the wind–shear scale, $z_* = u_*/\overline{(\partial u / \partial z)}_*$ or (ii) the buoyancy–gradient scale, $z_* = u_*/\overline{(db/dz)}_*^{1/2}$. In respective cases, we obtain (i) $(z_*/u_*)^2 \overline{(db/dz)}_* = Ri$, and (ii) $(z_*/u_*) \overline{(\partial u / \partial z)}_* = Ri^{-1/2}$. Recall that the Richardson number, Ri , is defined by Eq. (6).

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