HAL
open science

# Abstraction of Monotone Systems Based on Feedback Controllers

Vladimir Sinyakov, Antoine Girard

## To cite this version:

Vladimir Sinyakov, Antoine Girard. Abstraction of Monotone Systems Based on Feedback Controllers. IFAC 2020 - 21st IFAC World Congress, Jul 2020, Berlin, Germany. 10.1016/j.ifacol.2020.12.2342 . hal-02900533

**HAL Id: hal-02900533**

**https://hal.science/hal-02900533v1**

Submitted on 16 Jul 2020

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Abstraction of Monotone Systems Based on Feedback Controllers ⋆

**Vladimir Sinyakov** * **Antoine Girard** *

* *Université Paris-Saclay, CNRS, CentraleSupélec, Laboratoire des signaux et systèmes, 91190, Gif-sur-Yvette, France (e-mail:* `firstname.lastname@l2s.centralesupelec.fr`*)*

**Abstract:** In this paper, we consider the problem of computation of efficient symbolic abstractions for a certain subclass of continuous-time monotone control systems. The new abstraction algorithm utilizes the properties of such systems to build symbolic models with the same number of states but fewer transitions in comparison to the one produced by the standard algorithm. At the same time, the new abstract system is at least as controllable as the standard one. The proposed algorithm is based on the solution of a region-to-region control synthesis problem. This solution is formally obtained using the theory of viscosity solutions of the dynamic programming equation and the theory of differential equations with discontinuous right-hand side. In the new abstraction algorithm the symbolic controls are essentially the feedback controllers which solve this control synthesis problem. The improvement in the number of transitions is achieved by reducing the number of successors for each symbolic control. The approach is illustrated by an example which compares the two abstraction algorithms.

## 1. INTRODUCTION

Synthesis of feedback controllers for nonlinear dynamical systems is one of the key problems in control theory. Formal methods approach suggests splitting this problem into several subproblems with the first one being the contruction of a symbolic abstract system (or abstraction) which is usually a system with finite number of states and transitions (see Tabuada (2008); Belta et al. (2017)). These abstractions capture the behavior of the original system in such a way that a controller built to solve the control problem for an abstract system can be refined to a respective controller for the original system. The notions of an alternating simulation relation, an approximate alternating simulation relation and a feedback refinement relation are used to formalize such properties.

There are several known methods of abstraction. Some of those methods require the control system to satisfy certain sets of conditions to be applicable. One of the more general methods is based on partitioning of the state space and on discretizing the control space. This abstraction method utilizes the notion of alternating simulation relation and can be applied to a very wide class of systems but is especially efficient when the reachable sets originated from partition elements can be efficiently computed or approximated (see Scott and Barton (2013); Kurzhanski and Varaiya (2014); Kostousova (2014); Sinyakov (2015); Meyer et al. (2019)). One of such types of control systems is monotone systems or, more generally, mixed-monotone systems. Due to its generality and popularity we will refer to this method as "standard" throughout the paper. In this paper we specify a subclass of monotone systems for which there is a more efficient abstraction algorithm. This new algorithm and the formal proof of its correctness constitute the main contribution of the paper.

The method we present here also utilizes the partitioning of the state space. Unlike in the standard algorithm, each symbolic control in this method corresponds to a certain feedback controller for the original system as opposed to an open-loop control function (see e.g. Caines and Wei (1998)). Intuitively, we use a feedback controller such that the interval approximation of the reachable set (of the closed-loop system) from a partition element is the smallest in size or, more precisely, that it is minimal with respect to inclusion in a certain class $\mathscr{A}$ of interval sets for which we are able to construct the respective controllers. That way we expect to have fewer transitions correponding to a single symbolic control. The considered class $\mathscr{A}$ of interval sets has a description in terms of viscosity solutions of the related Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation (see Crandall and Lions (1983); Fleming and Soner (1995)). These intervals can be also described by certain differential equations with discontinuous right-hand side (see Filippov (1988)). We utilize both frameworks to establish the existence and uniqueness of the minimal element as well as the method of its practical construction.

The problem of polytope-to-polytope control for nonlinear control systems in relation with symbolic control has been considered extensively in the literature (see Belta and Habets (2006); Girard and Martin (2012); Ben Sassi and Girard (2013); Sloth and Wisniewski (2014); Meyer et al. (2016)). It has been shown (see e.g. Saoud et al. (2018)) that for controllability reasons it is sometimes important to consider "flat" partition elements. Moreover, depending on the system and the partition element, a minimal reachable set may be also flat. These considerations pose the main technical difficulty in the proof of correctness of our construction.

⋆ This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 725144).

The paper is structured as follows. In Section 2 we define the problem of calculating the minimal (in a certain class $\mathscr{A}$) target set to which we can control the system from a given initial set (Problem 1). The main result of Section 3 suggests that every target set in the considered class $\mathscr{A}$ corresponds to a viscosity supersolution (upper solution) of the related backward HJBI equation. Once we have a supersolution, the feedback controller can be constructed (or verified) using the idea of extremal aiming (see, e.g. Subbotin (1995)). In Section 4 we first obtain the description of $\mathscr{A}$ in terms of differential equations with discontinuous right-hand side. Then we prove the existence and uniqueness of the minimal element of class $\mathscr{A}$. Finally, we define the controller and prove that it solves Problem 1.

In Section 5 we utilize the controllers obtained in Section 4 to define the new abstraction. Each symbolic control input $v$ is associated with a particular controller $u(t,x)$ (instead of an open-loop or a constant control as in the standard algorithm). The transitions from a state $q$ with a control $v$ in the abstract system are enabled for every partition element which intersects the respective reachable set overapproximation. In Section 6 we compare the standard and the new abstraction algorithms on a 3-dimensional example of a temperature regulation problem.

*Notations*: For $x \in \mathbb{R}^n$, $\|x\|_\infty = \max_i |x_i|$ is the infinity norm. Let $d(x,X)$ denote the distance $\inf_{z\in X} \|x-z\|_\infty$ between $x \in \mathbb{R}^n$ and $X \subseteq \mathbb{R}^n$. Given vectors $x, x' \in \mathbb{R}^n$, $x \preceq x'$ stands for $x_i \le x'_i$ for all $i = 1,\dots,n$. Using this partial order, we define multi-dimensional interval sets as follows: for $\underline{x}, \overline{x} \in \mathbb{R}^n$, $[\underline{x}, \overline{x}] = \{x \mid x \succeq \underline{x},\ x \preceq \overline{x}\}$. For a set-valued map $W\colon [0,T] \rightrightarrows \mathbb{R}^{n_w}$, the space of all Lebesgue measurable functions $w(\cdot)$ on $[0,T]$ such that $w(t) \in W(t)$ a.e. is denoted by $L^\infty([0,T], W(\cdot))$.

## 2. CONTROLLER SYNTHESIS PROBLEM FOR MONOTONE SYSTEMS

Consider a nonlinear system of the following type ($i = 1,\dots,n_x$):

$$\dot{x}_i = f_i(t,x,u_i,w), \quad t \in [0,T]. \tag{1}$$

Here $x \in \mathbb{R}^{n_x}$ is the state, $u \in U = [\underline{u}, \overline{u}] \subset \mathbb{R}^{n_x}$ is the control and $w \in W = [\underline{w}, \overline{w}] \subset \mathbb{R}^{n_w}$ is the disturbance. Here we allow the case when $\underline{u}_j = \overline{u}_j$ for some indices $j$. In this case, there are essentially less control parameters than $n_x$.

The set of admissible open-loop controls is $\mathcal{U}(t,\tau) = L^\infty([t,\tau], U)$. The set of admissible realizations of the disturbance is $\mathcal{W}(t,\tau) = L^\infty([t,\tau], W)$. Let $\mathbf{x}(t;\tau,x,u(\cdot),w(\cdot))$ denote a trajectory of the system satisfying the initial condition $x(\tau) = x$ and corresponding to the control $u(\cdot)$ and disturbance $w(\cdot)$. Finally, let $X^{u(\cdot)}(t;t_0,X^0)$ denote the reachable set

$$\{x \in \mathbb{R}^{n_x} \mid \exists x^0 \in X^0,\ \exists w(\cdot) \in \mathcal{W}(t_0,t) : \\ \mathbf{x}(t;t_0,x^0,u(\cdot),w(\cdot)) = x\}.$$

The conditions on the considered class of systems are summarized in the following.

*Assumption 1.* The right-hand side of system (1) is continuous in $(t,x,u,w)$, globally Lipschitz in $(x,u)$ uniformly in $(t,w)$ and satisfies the following monotonicity condition: $f_i$ is nondecreasing in $x_j$, $u_i$ and $w_k$ for all $i,j = 1,\dots,n_x$, $i \ne j$ and all $k = 1,\dots,n_w$. Such systems are called *monotone* with respect to state $x$ and input $(u,w)$.

Let us consider a class $\mathscr{A}$ of target interval sets $X^1$ which we will define below. For a controller $u\colon [0,T] \times \mathbb{R}^{n_x} \to U$ and a disturbance realization $w(\cdot)$, we will consider the closed-loop system:

$$\dot{x}_i = f_i(t,x,u_i(t,x),w(t)), \quad t \in [0,T]. \tag{2}$$

*Problem 1.* Given a system (1) satisfying Assumption 1, an initial interval set $X^0 = [\underline{x}^0, \overline{x}^0] \subset \mathbb{R}^{n_x}$ and a time horizon $T > 0$, find a minimal by inclusion set $X^1$ in a class $\mathscr{A}$ and a controller $u(t,x)$ such that

- the closed-loop system has a solution for all initial data and all admissible disturbances and every solution exists on the whole interval $[0,T]$;
- all trajectories of the closed-loop system originated from $X^0$ at $t = 0$ reach $X^1$ at $t = T$.

Since the inclusion relation $\subseteq$ induces only a partial order on subsets of $\mathbb{R}^{n_x}$, a minimal by inclusion set $X^1$ may be not unique in general. However, it will be unique in the case discussed in this paper.

Let us now introduce the type of classes $\mathscr{A}$ of target sets under consideration. Fix a trajectory $\hat{x}(\cdot) = \mathbf{x}(\cdot;0,x^0,\hat{u}(\cdot),\hat{w}(\cdot))$ of system (1) such that $x^0 \in X^0$. Consider a class $\mathscr{A}^{\hat{x}(\cdot)}$ consisting of all interval sets $X^1$ for which there exists a Lipschitz continuous interval-valued map $X(t)$ satisfying the following properties:

(a) $X(0) = X^0$, $X(T) = X^1$;
(b) for all $t \in [0,T]$, $x \in X(t)$, and all $w(\cdot) \in \mathcal{W}(t,T)$ there exists $u(\cdot) \in \mathcal{U}(t,T)$ such that $\mathbf{x}(\tau;t,x,u(\cdot),w(\cdot)) \in X(\tau)$ for all $\tau \in [t,T]$;
(c) $\hat{x}(t) \in X(t)$ for all $t \in [0,T]$.

*Remark 1.* Minimal interval over-approximation $X^+(t) = [\mathbf{x}(t;0,\underline{x}_0,\hat{u}(.),\underline{w}), \mathbf{x}(t;0,\overline{x}_0,\hat{u}(.),\overline{w})]$ of the reachable set $X^{\hat{u}(\cdot)}(t;0,X^0)$ gives an example of such interval-valued map.

*Remark 2.* Property (b) is sometimes called weak invariance of $X(t)$ with respect to differential inclusion $\dot{x}_i \in f_i(t,x,U,w(t))$. Property (c) implies the following: consider $X(t) = [\underline{x}(t), \overline{x}(t)]$ and let $\overline{x}_j(\tau) = \hat{x}_j(\tau)$ for some $j$ and $\tau \in [0,T)$. If $\overline{x}_j(\cdot)$ and $\hat{x}_j(\cdot)$ are differentiable at $\tau$ then $\dot{\hat{x}}_j(\tau) \le \dot{\overline{x}}_j(\tau)$. Similarly, one may prove that if $\underline{x}_j(\tau) = \hat{x}_j(\tau)$ then $\dot{\underline{x}}_j(\tau) \le \dot{\hat{x}}_j(\tau)$ if both derivatives exist at $\tau$.

It is known that the problem of controller synthesis for a reachability specification can be solved by considering the corresponding problem of dynamic optimization (see Subbotin (1995); Kurzhanski and Varaiya (2001)). Namely, given a supersolution of the backward Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation, a reachability controller can be obtained, for example, by utilizing the idea of extremal aiming. With this in mind, let us formally translate our description of the problem into the Hamilton-Jacobi setting.

## 3. PRELIMINARIES ON THE HAMILTONIAN FORMALISM

Discussion in this section is applicable to a general nonlinear system with uncertainty

$$\dot{x} = f(t, x, u, w), \quad t \in [0, T]. \tag{3}$$

Here $x \in \mathbb{R}^{n_x}$ is the state, $u \in U \subset \mathbb{R}^{n_u}$ is the control and $w \in W \subset \mathbb{R}^{n_w}$ is the disturbance. We assume that sets $U$ and $W$ are convex and compact. The respective sets of admissible open-loop controls and disturbance realizations are $L^\infty([t, \tau], U)$ and $L^\infty([t, \tau], W)$ as before.

Consider an initial set $X^0$ and let us represent it as a sublevel set of some function $\sigma(\cdot)$:

$$X^0 = \{x \in \mathbb{R}^{n_x} \mid \sigma(x) \le 0\}.$$

Similarly, given a target set $X^1$, let us represent it as a sublevel set of some other function $\psi(\cdot)$:

$$X^1 = \{x \in \mathbb{R}^{n_x} \mid \psi(x) \le 0\}.$$

Consider now the HJBI equation

$$V_t + \min_{u \in U} \max_{w \in W} \langle V_x, f(t, x, u, w) \rangle = 0. \tag{4}$$

*Assumption 2.* We impose the following assumptions.

(1) The right-hand side of (3) is continuous in $(t, x, u, w)$, globally Lipschitz in $(x, u)$ uniformly in $(t, w)$;
(2) For system (3), Isaacs minimax condition is satisfied: for all $p \in \mathbb{R}^{n_x}$

$$\min_{u \in U} \max_{w \in W} \langle p, f(t, x, u, w) \rangle = \max_{w \in W} \min_{u \in U} \langle p, f(t, x, u, w) \rangle. \tag{5}$$

In the following let $H(t, x, p)$ denote the expression

$$\min_{u \in U} \max_{w \in W} \langle p, f(t, x, u, w) \rangle.$$

As mentioned above, we may obtain a controller, which steers system (3) to $X^1$ at $t = T$, by computing a supersolution (or the actual solution) of equation (4) with the terminal condition

$$V(T, x) = \psi(x) \tag{6}$$

backwards in time. To guarantee that every point of $X^0$ is controllable, condition $V(0, x) \le \sigma(x)$ for all $x \in \mathbb{R}^{n_x}$ must be satisfied.

However, since $X^1$ is an unknown part of the solution of Problem 1, we have to employ another approach. Intuitively, one may try to consider equation (4) forward in time with the initial condition

$$V(0, x) = \sigma(x) \tag{7}$$

and put

$$X^1 = \{x \in \mathbb{R}^{n_x} \mid V(T, x) \equiv \psi(x) \le 0\}.$$

In general, the forward solution $V$ of (4), (7) is not even a supersolution of (4), (6). However, for system (1) the forward subsolutions, which we construct below, turn out to be backward supersolutions indeed.

Let us now remind precisely the definitions of viscosity solutions in the considered cases (see Crandall and Lions (1983); Fleming and Soner (1995)). For equation (4) considered in forward time we have

- A function $V$ is a forward viscosity subsolution of (4) if and only if for all $(t, x) \in (0, T] \times X$
$$q + H(t, x, p) \le 0 \quad \forall (q, p) \in D^+V(t, x); \tag{8}$$
- A function $V$ is a forward viscosity supersolution of (4) if and only if for all $(t, x) \in (0, T] \times X$
$$q + H(t, x, p) \ge 0 \quad \forall (q, p) \in D^-V(t, x); \tag{9}$$
- $V$ is a forward viscosity solution if it is both a sub- and a supersolution.

Here $D^+V(t, x)$ denotes the superdifferential of $V$ at $(t, x)$ and $D^-V(t, x)$ denotes the subdifferential of $V$ at $(t, x)$.

For equation (4) considered in backward time we have

- A function $V$ is a backward viscosity subsolution of (4) if and only if for all $(t, x) \in [0, T) \times X$
$$q + H(t, x, p) \ge 0 \quad \forall (q, p) \in D^+V(t, x); \tag{10}$$
- A function $V$ is a backward viscosity supersolution of (4) if and only if for all $(t, x) \in [0, T) \times X$
$$q + H(t, x, p) \le 0 \quad \forall (q, p) \in D^-V(t, x); \tag{11}$$
- $V$ is a backward viscosity solution if it is both a sub- and a supersolution.

The next lemma and the following corollary show the connection between Problem 1 and the HJBI equation (4).

*Lemma 1.* Consider a continuous set-valued map $X(t)$, $t \in [0, T]$ with closed values and let $L > 0$ be the Lipschitz constant of the right-hand side of (3). Under Assumption 2, $X(t)$ satisfies property (b) of the definition of class $\mathscr{A}^{\hat{x}(\cdot)}$ if and only if the function

$$V(t, x) = e^{-Lt} d(x, X(t))$$

is a backward supersolution of equation (4).

*Corollary 1.* Let the assumptions of Lemma 1 hold. If $X(t)$ satisfies property (b) of the definition of class $\mathscr{A}^{\hat{x}(\cdot)}$ and $X(\cdot)$ is Lipschitz continuous and convex-valued then the function $V(t, x) = e^{-Lt} d(x, X(t))$ is a forward subsolution of equation (4).

In the next section we utilize this corollary to obtain a description of $\mathscr{A}^{\hat{x}(\cdot)}$ in terms of equations with discontinuous right-hand side (Corollary 2).

## 4. SOLUTION OF THE SYNTHESIS PROBLEM

In this section we provide the solution to Problem 1. From this point onward we consider Assumptions 1 and 2 being satisfied.

### *4.1 Minimal reachable sets*

In this subsection we find equations which define the minimal target set $X^1$ in Problem 1.

Given an arbitrary interval $X^0 = [\underline{x}^0, \overline{x}^0]$, let us consider a Lipschitz continuous interval-valued map $X(t) = [\underline{x}(t), \overline{x}(t)]$ such that $X(0) = X^0$. We introduce the function $\sigma(\cdot)$:

$$\sigma(x) \equiv d(x, X^0) = \max_i \max\{x_i - \overline{x}_i^0,\ \underline{x}_i^0 - x_i, 0\}. \tag{12}$$

Now let us define the function

$$V(t, x) = e^{-Lt} \max_i \max\{x_i - \overline{x}_i(t),\ \underline{x}_i(t) - x_i, 0\} \tag{13}$$

where $L > 0$ is the Lipschitz constant of the right-hand side of (1):

$$|f_i(t,x,u_i,w) - f_i(t,y,u_i,w)| \le L\|x-y\|_\infty.$$

As mentioned in the previous section, to obtain a controller which solves the reachability problem for a target set $X^1 = [\underline{x}(T), \overline{x}(T)]$, we need a backward supersolution of (4), (6). Under the assumptions of Corollary 1, a backward supersolution of the form (13) is also a forward subsolution of (4), (7). Therefore, let us now give a criterion for (13) to be a forward subsolution.

*Lemma 2.* Function $V$ is a viscosity subsolution of (4), (7) in forward time if and only if

$$\begin{aligned} \dot{\overline{x}}_i(t) &\ge f_i(t,\overline{x}(t),\underline{u}_i,\overline{w}), \\ \dot{\underline{x}}_i(t) &\le f_i(t,\underline{x}(t),\overline{u}_i,\underline{w}) \end{aligned} \tag{14}$$

a.e. on $[0,T]$.

Thus, for every interval-valued map $X(t)$ in the definition of class $\mathscr{A}^{\hat{x}(\cdot)}$ inequalities (14) must hold. This observation leads to the following.

*Corollary 2.* If $X^1 \in \mathscr{A}^{\hat{x}(\cdot)}$ then there exist $X(t) = [\underline{x}(t), \overline{x}(t)]$ and $\xi(\cdot) = (\overline{\xi}(\cdot), \underline{\xi}(\cdot)) \in L^\infty([0,T], \mathbb{R}^{2n_x})$ with $\overline{\xi}(t) \succeq 0$, $\underline{\xi}(t) \preceq 0$ satisfying equations

$$\begin{aligned} \dot{\overline{x}}_i &= \begin{cases} f_i(t,\overline{x},\underline{u}_i,\overline{w}) + \overline{\xi}_i(t), & \hat{x}_i(t) < \overline{x}_i, \\ \max\{f_i(t,\overline{x},\underline{u}_i,\overline{w}) + \overline{\xi}_i(t), \dot{\hat{x}}_i(t)\}, & \hat{x}_i(t) \ge \overline{x}_i, \end{cases} \\ \dot{\underline{x}}_i &= \begin{cases} f_i(t,\underline{x},\overline{u}_i,\underline{w}) + \underline{\xi}_i(t), & \underline{x}_i < \hat{x}_i(t), \\ \min\{f_i(t,\underline{x},\overline{u}_i,\underline{w}) + \underline{\xi}_i(t), \dot{\hat{x}}_i(t)\}, & \underline{x}_i \ge \hat{x}_i(t) \end{cases} \end{aligned} \tag{15}$$

a.e. on $[0,T]$, initial conditions

$$\underline{x}(0) = \underline{x}^0, \quad \overline{x}(0) = \overline{x}^0 \tag{16}$$

and such that $X(T) = X^1$.

This result gives a useful description of the considered class $\mathscr{A}^{\hat{x}(\cdot)}$. Intuitively, the interval-valued map $X(t)$ which satisfies differential equations (15) with $\xi(t) \equiv 0$ should produce the minimal element of the respective class $\mathscr{A}^{\hat{x}(\cdot)}$. To formally establish it, we need to prove that (15) is monotone in state $(\overline{x}, \underline{x})$ and input $(\overline{\xi}, \underline{\xi})$ and has a solution for $\xi(t) \equiv 0$. First, we provide the following two lemmas.

*Lemma 3.* (1) System of equations (15) has a unique solution on $[0,T]$ in the sense of Filippov (see Filippov (1988), §4, definition a)). Moreover, the solution is Lipschitz continuous.

(2) For any solution of (15), (16), the following relation holds:

$$\underline{x}(t) \preceq \hat{x}(t) \preceq \overline{x}(t).$$

*Lemma 4.* The function $V$ defined by (13), (15), (16) is a viscosity supersolution of (4), (6) in backward time.

Thus, for every solution of (15), (16) the corresponding set $X(T) \in \mathscr{A}^{\hat{x}(\cdot)}$. Now we present the main result of this subsection.

*Theorem 1.* Consider the solution $(\underline{x}(\cdot), \overline{x}(\cdot))$ of (15), (16) with $\xi(t) \equiv 0$. The set $X^1 = [\underline{x}(T), \overline{x}(T)]$ is the unique minimal element of class $\mathscr{A}^{\hat{x}(\cdot)}$.

*Corollary 3.* For $X(t) = [\underline{x}(t), \overline{x}(t)]$ defined by (15), (16) with $\xi(t) \equiv 0$ and for any interval-valued map $X^+(t)$ such that $X^{\hat{u}(\cdot)}(t;0,X^0) \subseteq X^+(t)$, the inclusion holds

$$X(t) \subseteq X^+(t), \quad t \in [0,T].$$

### *4.2 Controller construction*

Let us now consider the interval-valued map $X(t) = [\underline{x}(t), \overline{x}(t)]$ defined by (15), (16) with $\xi(t) \equiv 0$. We define the following controller:

$$\begin{gathered} x_i^c(t) = (\underline{x}_i(t) + \overline{x}_i(t))/2, \quad x_i^r(t) = (\overline{x}_i(t) - \underline{x}_i(t))/2, \\ u_i^c = (\underline{u}_i + \overline{u}_i)/2, \quad u_i^r = (\overline{u}_i - \underline{u}_i)/2, \end{gathered}$$

$$u_i(t,x) = \begin{cases} \underline{u}_i, \ x_i > \overline{x}_i(t), \\ u_i^c + u_i^r \dfrac{x_i - x_i^c(t)}{x_i^r(t)}, \ \underline{x}_i(t) \le x \le \overline{x}_i(t), \\ \overline{u}_i, \ x_i < \underline{x}_i(t). \end{cases} \tag{17}$$

If $\underline{x}_i(t) = \overline{x}_i(t)$ we formally put $u_i(t,\hat{x}(t)) = \hat{u}_i(t)$.

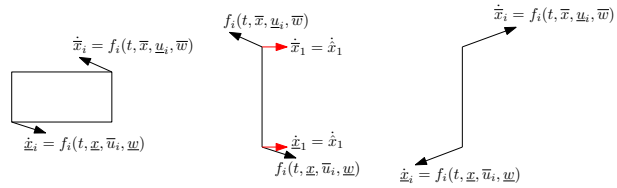


Fig. 1. Different scenarios of evolution of $X(t)$.

*Theorem 2.* The following propositions hold.

(1) Closed-loop system (2) has a unique solution on $[0,T]$ (in the sense of Filippov) for all admissible disturbances $w(\cdot)$. Every solution $x(\cdot)$ emanating from $X^0 = [\underline{x}^0, \overline{x}^0]$ satisfies the inclusions $x(t) \in [\underline{x}(t), \overline{x}(t)]$ for all $t \in [0,T]$;

(2) If the interior of $[\underline{x}(t), \overline{x}(t)]$ is not empty for all $t \in [0,T]$ then the closed-loop system (2) has a solution (in the sense of Carathéodory) for all admissible disturbances $w(\cdot)$. Every solution $x(\cdot)$ emanating from $X^0 = [\underline{x}^0, \overline{x}^0]$ satisfies the inclusions $x(t) \in [\underline{x}(t), \overline{x}(t)]$ for all $t \in [0,T]$.

## 5. ABSTRACTION ALGORITHM

In this section we consider the time-invariant version of system (1):

$$\dot{x}_i = f_i(x,u_i,w), \quad i = 1,\dots,n_x. \tag{18}$$

Here $u \in U = [\underline{u}, \overline{u}]$, $w \in W = [\underline{w}, \overline{w}]$ as before.

Given a controller $u$: $[0,T] \times \mathbb{R}^{n_x} \to U$, let $\mathbf{x}(t;x,u,w(\cdot))$ denote the set of all solution endpoints (in the sense of Filippov) of the closed-loop system satisfying the initial condition $x(0) = x$ and corresponding to the disturbance $w(\cdot) \in \mathcal{W}(0,T)$.

Let us denote $\mathscr{U}_T^0(x)$ the set of all controllers such that for $x(0) = x$ and for every $w(\cdot) \in \mathcal{W}(0,T)$ there is at least one Filippov solution of the closed-loop system and every such solution exists on $[0,T]$.

Let us consider a set $X \subseteq \mathbb{R}^{n_x}$, which we call the state space, and restrict the dynamics of system (18) to this set. Let the state space $X$ be covered by a finite set of intervals $(X_q)_{q\in Q}$: $X = \cup_{q\in Q} X_q$, $X_q = [\underline{x}^q, \overline{x}^q]$.

*Definition 1.* A *transition system* $S$ is a tuple $(X,U,Y,\Delta,H)$, where

- $X$ is a set of states;

- $U$ is a set of inputs;
- $Y$ is a set of outputs;
- $\Delta : X \times U \rightrightarrows X$ is a set-valued transition map;
- $H$: $X \longrightarrow Y$ is an output map.

An input $u \in U$ is called *enabled* at $x \in X$ if $\Delta(x,u) \neq \emptyset$. Let $\mathrm{enab}_\Delta(x) \subseteq U$ denote the set of all inputs enabled at $x$. If $\mathrm{enab}_\Delta(x) = \emptyset$ the state $x$ is called *blocking*.

Given the cover $(X_q)_{q\in Q}$, system (18) may be written as a transition system as follows:

$$S = (X, \mathscr{U}, Q, \delta, H)$$

where

$$\mathscr{U} = \{(T,u),\ T \in [0,+\infty),\ u:\ [0,T] \times \mathbb{R}^{n_x} \to U\},$$
$$q = H(x) \quad \Leftrightarrow \quad x \in X_q$$

and transition relation $\delta$ is defined as follows:

$$x' \in \delta(x,T,u), \quad (T,u) \in \mathrm{enab}_\delta(x)$$

if and only if there exists $w \in \mathcal{W}(0,T)$ such that $x' \in \mathbf{x}(T;0,x,u,w(\cdot))$. Here the set of enabled inputs is defined as follows

$$\mathrm{enab}_\delta(x) = \{(T,u) \in \mathscr{U} \mid u \in \mathscr{U}_T^0(x) \quad \text{and}$$
$$\forall w \in \mathcal{W}(0,T), \forall t \in [0,T],\ \mathbf{x}(t;0,x,u,w(\cdot)) \subseteq X\}.$$

We now define an abstract transition system $S_a$ using the cover $(X_q)_{q\in Q}$, a sampling parameter $\tau > 0$ and a finite set of control inputs $\mathscr{V}$:

$$S_a = (Q, \mathscr{V}, Q, \Delta, \mathrm{Id}).$$

Here Id is the identity map on $Q$. In a state $q \in Q$ a symbolic control $v \in \mathscr{V}$ corresponds to a pair $(\tau, u^{(q,v)}) \in \mathscr{U}$ such that $u^{(q,v)}$ is defined by (17) and the corresponding interval $X^{(q,v)}(t) = [\underline{x}^{(q,v)}(t), \overline{x}^{(q,v)}(t)]$ is defined by (15) with $\xi(t) \equiv 0$ and the initial conditions

$$\underline{x}^{(q,v)}(0) = \underline{x}^q \preceq \overline{x}^q = \overline{x}^{(q,v)}(0).$$

The corresponding reference trajectories $\hat{x}(\cdot)$ and reference controls $\hat{u}(\cdot)$ in (15) and (17) depend on the pair $(q,v)$. Below we provide a particular choice of those which guarantees the comparison result in Theorem 4.

Observe that $u^{(q,v)} \in \mathscr{U}_\tau^0(x)$. Transition relation $\Delta$ is defined as follows: $q' \in \Delta(q,v)$ for $v \in \mathscr{V}$ if and only if

$$X_{q'} \cap [\underline{x}^{(q,v)}(\tau), \overline{x}^{(q,v)}(\tau)] \neq \emptyset$$

and

$$[\underline{x}^{(q,v)}(t), \overline{x}^{(q,v)}(t)] \subseteq X$$

for all $t \in [0,\tau]$.

*Definition 2.* Let $S_a = (X_a, U_a, Y_a, \Delta_a, H_a)$ and $S_b = (X_b, U_b, Y_b, \Delta_b, H_b)$ be two transition systems with $Y_a = Y_b$. A relation $R \subseteq X_a \times X_b$ is an alternating simulation relation from $S_a$ to $S_b$ if the following conditions are satisfied:

(1) for every $(x_a, x_b) \in R$ we have $H_a(x_a) = H_b(x_b)$;
(2) for every $(x_a, x_b) \in R$ and for every $u_a \in \mathrm{enab}_{\Delta_a}(x_a)$ there exists $u_b \in \mathrm{enab}_{\Delta_b}(x_b)$ such that for every $x_b' \in \Delta_b(x_b, u_b)$ there exists $x_a' \in \Delta_a(x_a, u_a)$ satisfying $(x_a', x_b') \in R$.

It is said that $S_b$ alternatingly simulates $S_a$, denoted by $S_a \preceq_{AS} S_b$, if there exists an alternating simulation relation $R \neq \emptyset$ from $S_a$ to $S_b$.

*Theorem 3.* Transition system $S$ alternatingly simulates abstract system $S_a$: $S_a \preceq_{AS} S$.

Let us now introduce the standard abstract system $S_{std}$. Consider a finite approximation $\hat{U}$ of the control space: $\hat{U} \subset U$. We define the abstraction

$$S_{std} = (Q, \hat{U}, Q, \hat{\Delta}, \mathrm{Id})$$

where transition relation $\hat{\Delta}$ is defined as follows: $q' \in \hat{\Delta}(q,\hat{u})$ for $\hat{u} \in \hat{U}$ if and only if

$$X_{q'} \cap [\mathbf{x}(\tau; \underline{x}^q, \hat{u}, \underline{w}), \mathbf{x}(\tau; \overline{x}^q, \hat{u}, \overline{w})] \neq \emptyset$$

and

$$[\mathbf{x}(t; \underline{x}^q, \hat{u}, \underline{w}), \mathbf{x}(t; \overline{x}^q, \hat{u}, \overline{w})] \subseteq X$$

for all $t \in [0,\tau]$.

To provide a comparison result between $S_a$ and $S_{std}$, let us specify the set $\mathscr{V}$ and the corresponding controls $u^{(q,v)}$. Let $\mathscr{V} = \hat{U}$ and $u^{(q,v)}$ corresponds to the reference trajectory $\hat{x}(\cdot)$ defined by the following:

$$\dot{\hat{x}}_i = f(\hat{x}, \hat{u}_i, (\overline{w} - \underline{w})/2), \quad \hat{x}(0) = (\overline{x}^q - \underline{x}^q)/2.$$

*Theorem 4.* Transition system $S_a$ alternatingly simulates $S_{std}$: $S_{std} \preceq_{AS} S_a$.

Theorems 3 and 4 give us the relation

$$S_{std} \preceq_{AS} S_a \preceq_{AS} S.$$

Given an arbitrary control specification, every symbolic state $q \in Q$, which is controllable for $S_{std}$, is also controllable for $S_a$. We emphasize that by construction the number of transitions in the new abstraction does not exceed the number of transitions in the standard abstraction.

## 6. EXAMPLE

Let us consider a temperature regulation model of a circular $n_x$ room building which was adapted from Girard et al. (2015). The system is given by equations:

$$\dot{\mathbf{T}}_i(t) = \alpha(\mathbf{T}_{i+1}(t) + \mathbf{T}_{i-1}(t) - 2\mathbf{T}_i(t))$$
$$+\beta(\mathbf{T}_e(t) - \mathbf{T}_i(t)) + \gamma(\mathbf{T}_h - \mathbf{T}_i(t))u_i(t).$$

Here $\mathbf{T}_i$ is the temperature in room $i$, $\mathbf{T}_e(t) \in [\mathbf{T}_e^{\min}, \mathbf{T}_e^{\max}]$ is the outside temperature which is considered as disturbance, $\alpha$, $\beta$ and $\gamma$ are the corresponding conduction factors. The heater powers $u_i(t) \in [0,1]$ are the control parameters whereas the maximal heater temperature is $\mathbf{T}_h$. We utilize the following values for conduction factors: $\alpha = 0.05$, $\beta = 0.005$, $\gamma = 0.01$. The system is monotone in state and inputs.

We consider this system on the following state space:

$$X = [\mathbf{T}_1^{\min}, \mathbf{T}_1^{\max}] \times \cdots \times [\mathbf{T}_{n_x}^{\min}, \mathbf{T}_{n_x}^{\max}].$$

Let us introduce a partition for each coordinate ($i = 1, \ldots, n_x$):

$$[\mathbf{T}_i^{\min}, \mathbf{T}_i^{\min} + \frac{1}{N_i}(\mathbf{T}^{\max} - \mathbf{T}_i^{\min})], \ldots,$$
$$[\mathbf{T}_i^{\min} + \frac{N_i - 1}{N_i}(\mathbf{T}_i^{\max} - \mathbf{T}_i^{\min}), \mathbf{T}^{\max}].$$

Based on this partition we construct regions $X_q$ as Cartesian products of the elements from those partitions. The set of all such $X_q$ covers the whole state space $X$. For both abstraction algorithm we use sampling paramter $\tau$. We compare the two algorithms for $\tau = 1, 5, 40$.

We utilize $\hat{U} = \{0, \frac{1}{2}, 1\}^{n_x}$ as a finite approximation of $U$ in the standard abstraction algorithm. In the new algorithm

| $\tau$ | Algorithm | # of transitions | # of cont. states |
|---|---|---|---|
| 1 | Standard | 235221 | 1000 |
| 1 | New | (< 34%) 79245 | 1000 |
| 5 | Standard | 404167 | 1000 |
| 5 | New | (< 7.4%) 29592 | 1000 |
| 40 | Standard | 548940 | 0 |
| 40 | New | (< 5%) 27081 | 1000 |

Table 1. Comparison between the two abstraction algorithms for 3 different values of $\tau$.

we use $|\hat{U}| = 3^{n_x}$ reference trajectories each component of which is chosen according to one of the following three conditions ($i = 1, \ldots, n_x$):

$$\dot{\hat{\mathbf{T}}}_i = \alpha(\hat{\mathbf{T}}_{i+1}(t) + \hat{\mathbf{T}}_{i-1}(t) - 2\hat{\mathbf{T}}_i(t)) + \beta(\mathbf{T}_e^{\min} - \hat{\mathbf{T}}_i(t)) + \gamma(\mathbf{T}_h - \hat{\mathbf{T}}_i(t)),$$

or

$$\dot{\hat{\mathbf{T}}}_i = \alpha(\hat{\mathbf{T}}_{i+1}(t) + \hat{\mathbf{T}}_{i-1}(t) - 2\hat{\mathbf{T}}_i(t)) + \beta(\mathbf{T}_e^{\max} - \hat{\mathbf{T}}_i(t)),$$

or

$$\dot{\hat{\mathbf{T}}}_i = \alpha(\hat{\mathbf{T}}_{i+1}(t) + \hat{\mathbf{T}}_{i-1}(t) - 2\hat{\mathbf{T}}_i(t)).$$

For the simulations below we choose the following parameters: $n_x = 3$, $\mathbf{T}_i^{\min} = 19°\text{C}$, $\mathbf{T}_i^{\max} = 23°\text{C}$, $\mathbf{T}_e^{\min} = -1°\text{C}$, $\mathbf{T}_e^{\max} = 10°\text{C}$, $\mathbf{T}_h = 50°\text{C}$, $N_i = 10$. Here we consider a simple safety problem of keeping trajectories of the system in $X$ at all times.

Table 1 gives the total count of transitions and controllable states for the standard and the new abstraction algorithms. Both abstract systems utilize the same number of symbolic controls but the overall number of transitions is greatly reduced for the new abstraction. The higher reduction is achieved for bigger values of sampling parameter $\tau$. Coincidentally, for big enough values of $\tau$ the standard abstract system in this example becomes completely uncontrollable while the new abstract system is still controllable.

## 7. CONCLUSION

In this paper we introduced a new abstraction algorithm for a certain subclass of continuous-time monotone control systems. This algorithm produces more efficient symbolic systems with fewer number of transitions than a standard algorithm used in the literature for such systems. The improvement is achieved by considering interval-to-interval feedback controllers instead of open-loop (or constant) controls. The extension of the method to more general classes of systems, including mixed-monotone systems, is planned for future research.

## REFERENCES


Belta, C. and Habets, L.C. (2006). Controlling a class of nonlinear systems on rectangles. *IEEE Transactions on Automatic Control*, 51(11), 1749–1759.

Belta, C., Yordanov, B., and Gol, E.A. (2017). *Formal Methods for Discrete-Time Dynamical Systems.* Springer.

Ben Sassi, M.A. and Girard, A. (2013). Control of polynomial dynamical systems on rectangles. In *2013 European Control Conference (ECC)*, 658–663. IEEE.

Caines, P.E. and Wei, Y.J. (1998). Hierarchical hybrid control systems: A lattice theoretic formulation. *IEEE Transactions on Automatic Control*, 43(4), 501–508.

Crandall, M.G. and Lions, P.L. (1983). Viscosity solutions of hamilton-jacobi equations. *Transactions of the American Mathematical Society*, 277(1), 1–42.

Filippov, A.F. (1988). *Differential Equations with Discontinuous Righthand Sides*, volume 18. Springer Netherlands.

Fleming, W.H. and Soner, H.M. (1995). *Controlled Markov Processes and Viscosity Solutions.* Springer-Verlag, New York.

Girard, A. and Martin, S. (2012). Synthesis for constrained nonlinear systems using hybridization and robust controllers on simplices. *IEEE Transactions on Automatic Control*, 57(4), 1046–1051.

Girard, A., Gössler, G., and Mouelhi, S. (2015). Safety controller synthesis for incrementally stable switched systems using multiscale symbolic models. *IEEE Transactions on Automatic Control*, 61(6), 1537–1549.

Kostousova, E.K. (2014). On control synthesis for uncertain differential systems using a polyhedral technique. In I. Lirkov, S. Margenov, and J. Waśniewski (eds.), *Large-Scale Scientific Computing*, 98–106. Springer Berlin Heidelberg.

Kurzhanski, A.B. and Varaiya, P. (2001). Dynamic optimization for reachability problems. *Journal of Optimization Theory and Applications*, 108(2), 227–251.

Kurzhanski, A.B. and Varaiya, P. (2014). *Dynamics and Control of Trajectory Tubes.* Birkhäuser Basel.

Meyer, P.J., Girard, A., and Witrant, E. (2016). Robust controlled invariance for monotone systems: application to ventilation regulation in buildings. *Automatica*, 70, 14–20.

Meyer, P.J., Devonport, A., and Arcak, M. (2019). TIRA: Toolbox for interval reachability analysis. In $22^{nd}$ *ACM International Conference on Hybrid Systems: Computation and Control.*

Saoud, A., Girard, A., and Fribourg, L. (2018). Contract based design of symbolic controllers for interconnected multiperiodic sampled-data systems. In *57th IEEE Conference on Decision and Control.*

Scott, J.K. and Barton, P.I. (2013). Bounds on the reachable sets of nonlinear control systems. *Automatica*, 49(1), 93–100.

Sinyakov, V. and Girard, A. (2019). Controller synthesis for nonlinear systems with reachability specifications using monotonicity. In *58th Conference on Decision and Control (CDC'19).* IEEE.

Sinyakov, V.V. (2015). Method for computing exterior and interior approximations to the reachability sets of bilinear differential systems. *Differential Equations*, 51(8), 1097–1111.

Sloth, C. and Wisniewski, R. (2014). Control to facet for polynomial systems. In *Proceedings of the 17th international conference on Hybrid systems: computation and control*, 123–132. ACM.

Subbotin, A.I. (1995). *Generalized Solutions of First Order PDEs: The Dynamical Optimization Perspective.* Birkhäuser Basel.

Tabuada, P. (2008). *Verification and Control of Hybrid Systems: a symbolic approach.* Springer.

Wui Seah, S. (1982). Existence of solutions and asymptotic equilibrium of multivalued differential systems. *Journal of Mathematical Analysis and Applications*, 89, 648–663.

## 8. APPENDIX

**Proof of Lemma 1**. *Necessity.* In property (b) of the definition of class $\mathscr{A}^{\hat{x}(\cdot)}$ let us take $\tau = t+h$, $h>0$. We obtain

$$V(t+h, \mathbf{x}(t+h; t, x, u(\cdot), w(\cdot))) - V(t,x) \le 0$$

for all $x \in X(t)$. Dividing by $h$ and passing to the limit $h \to +0$ gives us inequality (11) for all $x \in X(t)$ (see, e.g., Fleming and Soner (1995), Lemma XI.6.2 for details). The Lipschitz condition on $f$ then ensures that (11) holds on $[0,T] \times \mathbb{R}^{n_x}$:

$$\begin{gathered} q + H(t,x,p) = \\ -LV(t,x) + \tilde{q} + \min_{u\in U}\max_{w\in W}\big\langle \tilde{p}, f(t,x,u,w)\big\rangle \le \\ \tilde{q} + \min_{u\in U}\max_{w\in W}\big\langle \tilde{p}, f(t,\tilde{x},u,w)\big\rangle + \\ L\big[e^{-Lt}\|x-\tilde{x}\|_\infty - V(t,x)\big] \le 0. \end{gathered}$$

Here $\tilde{x}$ is a projection of $x$ onto $X(t)$ and $(\tilde{q},\tilde{p}) \in D^- V(t,\tilde{x})$. We utilized that $q = \tilde{q} - LV(t,x)$, $p = \tilde{p}$ and the expression in brackets is equal to zero.

*Sufficiency.* If $V$ is a backward viscosity supersolution then it is known that its level set is weakly invariant (see Subbotin (1995)). Therefore, property (b) holds. □

**Proof of Corollary 1**. Since $X(\cdot)$ is convex-valued, $V$ is convex in $x$. Using the representation

$$D^+V(t,x) = \Bigg\{(q,p) \mid \limsup_{(s,y)\to(t,x)} \frac{V(s,y) - V(t,x) - q(s-t) - \big\langle p, x-y\big\rangle}{|s-t| + \|x-y\|} \le 0\Bigg\}$$

we may infer that for all $(q,p) \in D^+V(t,x)$, $V(t,y) \le V(t,x) + \big\langle p, x-y\big\rangle$ for all $y$ in some neighborhood of $x$. Therefore, if $(q,p)$, $(q',p') \in D^+V(t,x)$ then $p = p'$ due to convexity of $V$ in $x$. Thus, from the definition of $V$ it follows that either $D^+V(t,x) = \emptyset$ or $V(t,\cdot)$ is continuously differentiable at $x$ and $V_x$ is continuous in the neighborhood of $(t,x)$.

Let $D^+V(t,x) \ne \emptyset$. Then $D^+V(t,x) \subseteq \partial_C V(t,x)$ where $\partial_C V(t,x)$ denotes the Clarke generalized gradient. Since the left-hand side of (8) is linear in $q$, its maximum over $\partial_C V(t,x)$ is achieved at a corner point. Using Lipschitz continuity of $X(\cdot)$, we obtain that $V$ is Lipschitz in $(t,x)$. For such function $V$, for all $(t,x)$, for any corner point $(q_c,p_c) \in \partial_C V(t,x)$, there exists a sequence of points $\{(t^k,x^k)\}$ converging to $(t,x)$ such that $V$ is differentiable at $(t^k,x^k)$ and $(V_t(t^k,x^k), V_x(t^k,x^k))$ converges to $(q_c,p_c)$. Note that $p_c = V_x(t,x)$ since $V(t,\cdot)$ is continuously differentiable at $x$ and $V_x(s,y)$ is continuous in the neighborhood of $(t,x)$. Therefore, for condition (8) to be satisfied it is necessary and sufficient that

$$V_t + H(t,x,V_x) \le 0$$

a.e. on $[0,T] \times \mathbb{R}^{n_x}$. Here we also utilized continuity of $H$ in its variables. The inequality $V_t + H(t,x,V_x) \le 0$ is true since $V$ is a backward supersolution. □

**Proof of Lemma 2**. *Sufficiency* can be established by adapting the proof of Proposition 1 in Sinyakov and Girard (2019).

*Necessity.* Let $V$ be a viscosity subsolution of (4), (7) in forward time. For all $t \in [0,T]$ and all indices $i$ there exists a sequence of points $\{(t^k,x^k)\}_{k=0}^\infty$ converging to $(t,\overline{x}(t))$ such that $V$ is differentiable at $(t^k,x^k)$ and $V(t^k,x^k) = (x_i^k - \overline{x}_i(t^k))e^{-Lt^k}$. Then plugging $(t^k,x^k)$ into inequality (8) and passing to the limit we establish the first condition in (14). The second condition holds by the same argument. □

**Proof of Corollary 2**. The statement directly follows from Remark 2, Lemmas 1 and 2. □

**Proof of Lemma 3**. According to the definition from Filippov (1988), a pair of absolutely continuous functions $(\overline{x}(\cdot), \underline{x}(\cdot))$ is a solution of (15) if and only if it satisifies the initial condition $X(0) = X^0$ and is a solution of the corresponding differential inclusion which is defined by (15) when $\overline{x}_i \ne \hat{x}_i(t)$, $\underline{x}_i \ne \hat{x}_i(t)$ and

$$\dot{\overline{x}}_i \in \big[f_i(t,\overline{x},\underline{u}_i,\overline{w}) + \overline{\xi}_i(t), \max\{f_i(t,\overline{x},\underline{u}_i,\overline{w}) + \overline{\xi}_i(t), \dot{\hat{x}}_i(t)\}\big]$$

for $\overline{x}_i = \hat{x}_i(t)$,

$$\dot{\underline{x}}_i \in \big[\min\{f_i(t,\underline{x},\overline{u}_i,\underline{w}) + \underline{\xi}_i(t), \dot{\hat{x}}_i(t)\}, f_i(t,\underline{x},\overline{u}_i,\underline{w}) + \underline{\xi}_i(t)\big]$$

for $\underline{x}_i = \hat{x}_i(t)$. The right-hand side $F(t,\overline{x},\underline{x})$ of this differential inclusion is nonempty, compact, convex and for some $\alpha$, $\beta > 0$ satisfies the bound

$$\|F(t,\overline{x},\underline{x})\| \le \alpha(\|\overline{x}\| + \|\underline{x}\|) + \beta \tag{19}$$

for all $(t,\overline{x},\underline{x})$. The set-valued map $F$ is measurable in $t$ and upper semicontinuous in $(t,\overline{x},\underline{x})$. Thus, applying Theorem 3.3 of Wui Seah (1982) we obtain global existence of a solution of (15). From relation (19) it follows that all solutions of (15) are bounded: $\|\overline{x}(t)\| \le M$, $\|\underline{x}(t)\| \le M$, $t \in [0,T]$. Therefore, the solution is Lipschitz continuous (see Filippov (1988), §7, Lemma 2 and Theorem 2).

Next, one may verify that for all $(\overline{x},\underline{x}), (\overline{y},\underline{y}) \in \mathbb{R}^{2n_x}$ such that $\underline{y} \preceq \overline{y}$ and $\underline{x} \preceq \overline{x}$, the following estimate holds:

$$\frac{d}{dt}\big[\|\overline{x}-\overline{y}\|_2^2 + \|\underline{x}-\underline{y}\|_2^2\big] \le L'\big[\|\overline{x}-\overline{y}\|_\infty^2 + \|\underline{x}-\underline{y}\|_\infty^2\big] \tag{20}$$

a.e. on $[0,T]$ for $L' = 2Ln_x$. Therefore, the uniqueness follows from Filippov (1988), §10, Theorem 1.

To prove the second statement, let us assume that there exists a solution of (15), a number $i$ and a time instant $t_2 \in (0,T]$ such that

$$\hat{x}_i(t_2) > \overline{x}_i(t_2).$$

Then there exists $t_1 \in [0,T)$, $t_1 < t_2$ such that $\hat{x}_i(t_1) = \overline{x}_i(t_1)$ and

$$\hat{x}_i(t) > \overline{x}_i(t) \quad \forall t \in (t_1,t_2]$$

On the other hand, from (15) we have

$$\dot{\hat{x}}_i(t) \le \dot{\overline{x}}_i(t) \quad \text{a. e. on } [t_1,t_2].$$

Integrating this on $[t_1,t_2]$ we arrrive at

$$\hat{x}_i(t_2) - \hat{x}_i(t_1) \le \overline{x}_i(t_2) - \overline{x}_i(t_1)$$

which contradicts the assumption above. Similarly, one may obtain $\underline{x}_i(t) \le \hat{x}_i(t)$ on $[0,T]$. Thus, we obtained

$$\underline{x}_i(t) \le \hat{x}_i(t) \le \overline{x}_i(t)$$

for all $t \in [0,T]$. □

**Proof of Lemma 4**. Let us first assume that $\underline{x}^0 \prec \overline{x}^0$ and let $\varepsilon > 0$ be such that $\underline{x}^0 \preceq \overline{x}^0 - \varepsilon$. Consider now the following modification of system (15):

$$\dot{\overline{x}}_i = \begin{cases} f_i(t,\overline{x},\underline{u}_i,\overline{w}) + \overline{\xi}_i(t),\ \hat{x}_i(t) < \overline{x}_i - \varepsilon, \\ \max\{f_i(t,\overline{x},\underline{u}_i,\overline{w}) + \overline{\xi}_i(t),\ \dot{\hat{x}}_i(t)\},\ \hat{x}_i(t) \geq \overline{x}_i - \varepsilon, \end{cases}$$
$$\dot{\underline{x}}_i = \begin{cases} f_i(t,\underline{x},\overline{u}_i,\underline{w}) + \underline{\xi}_i(t),\ \underline{x}_i < \hat{x}_i(t) - \varepsilon, \\ \min\{f_i(t,\underline{x},\overline{u}_i,\underline{w}) + \underline{\xi}_i(t),\ \dot{\hat{x}}_i(t)\},\ \underline{x}_i \geq \hat{x}_i(t) - \varepsilon \end{cases}$$

Let us denote a solution of this system by $(\overline{x}^\varepsilon(\cdot), \underline{x}^\varepsilon(\cdot))$. Repeating the argument of Lemma 3, we obtain the global existence of the solution as well as the following inequalities:

$$\dot{\overline{x}}^\varepsilon(t) \geq f_i(t,\overline{x}^\varepsilon(t),\underline{u}_i,\overline{w}),$$
$$\dot{\underline{x}}^\varepsilon(t) \leq f_i(t,\underline{x}^\varepsilon(t),\overline{u}_i,\underline{w}),$$
$$\underline{x}^\varepsilon(t) \prec \overline{x}^\varepsilon(t).$$

Let us first prove the statement of this lemma for the approximation $V^\varepsilon$ of $V$ which is defined as follows:

$$V^\varepsilon(t,x) = e^{-Lt} \max_i \max\{x_i - \overline{x}_i^\varepsilon(t),\ \underline{x}_i^\varepsilon(t) - x_i, 0\}.$$

Let us consider an arbitrary point $(t,x)$. Without loss of generality let us assume that

$$x_j - \overline{x}_j^\varepsilon(t) \geq \underline{x}_j^\varepsilon(t) - x_j,\ 1 \leq j \leq j^*,$$
$$x_j - \overline{x}_j^\varepsilon(t) \leq \underline{x}_j^\varepsilon(t) - x_j,\ j^* < j \leq n_x$$

for some $j^*$. Let us then approximate the subdifferential:

$$D^- V^\varepsilon(t,x) \subseteq \{(q,p) \mid p_j = \lambda_j e^{-Lt}\mathrm{sgn}(j - j^* + \tfrac{1}{2}),$$
$$q = -\sum_{1\leq j\leq j^*} \lambda_j q_j^\varepsilon e^{-Lt} + \sum_{j^*<j\leq n_x} \lambda_j q_j^\varepsilon e^{-Lt} -$$
$$LV^\varepsilon(t,x),\ \sum_j \lambda_j \leq 1,\ \lambda_j \geq 0,$$
$$q_j^\varepsilon \in \partial_C \overline{x}_j^\varepsilon(t) \text{ for } j \leq j^* \text{ and } q_j^\varepsilon \in \partial_C \underline{x}_j^\varepsilon(t) \text{ for } j > j^*\}. \tag{21}$$

Here we utilized the strict relation $\underline{x}^\varepsilon(t) \prec \overline{x}^\varepsilon(t)$. By plugging this into (11), we obtain

$$-LV^\varepsilon(t,x)e^{Lt} - \sum_{1\leq j\leq j^*} \lambda_j q_j^\varepsilon + \sum_{j^*<j\leq n_x} \lambda_j q_j^\varepsilon +$$
$$\min_{u\in U}\max_{w\in W} \sum_{1\leq j\leq j^*} \lambda_j f_j(t,x,u_j,w) +$$
$$\min_{u\in U}\max_{w\in W} \sum_{j^*<j\leq n_x} (-\lambda_j) f_j(t,x,u_j,w) \leq 0.$$

Since the left-hand side is decreasing in $q_j^\varepsilon$ for $j \leq j^*$, increasing in $q_j^\varepsilon$ for $j > j^*$, function $f$ is continuous and function $V^\varepsilon$ is Lipschitz, it is sufficient to consider this inequality only a. e. on $[0,T]$. Therefore, after doing some rearrangements, we have

$$\max_{w\in W}\Big\{ \sum_{1\leq j\leq j^*} \lambda_j \big[ -\dot{\overline{x}}_j^\varepsilon(t) + f_j(t,x,\underline{u}_j,w)\big] +$$
$$\sum_{j^*<j\leq n_x} \lambda_j \big[\dot{\underline{x}}_j^\varepsilon(t) - f_j(t,x,\overline{u}_j,w)\big]\Big\} - LV^\varepsilon(t,x)e^{Lt} \leq 0.$$

For this relation to hold, it is sufficient that

$$\sum_{1\leq j\leq j^*} \lambda_j \big[ -\dot{\overline{x}}_j^\varepsilon(t) + f_j(t,x,\underline{u}_j,\overline{w})\big] +$$
$$\sum_{j^*<j\leq n_x} \lambda_j \big[\dot{\underline{x}}_j^\varepsilon(t) - f_j(t,x,\overline{u}_j,\underline{w})\big] - LV^\varepsilon(t,x)e^{Lt} \leq 0.$$

Now we take the maximum over all $(q,p)$ in the right-hand side of (21) which is the same as maximizing over $\lambda$ from

$$\Lambda = \{\lambda \in \mathbb{R}^{n_x} \mid \sum_j \lambda_j \leq 1,\ \lambda_j \geq 0\}.$$

Since the expression which is being maximized depends linearly on $\lambda$, the maximum is achieved at a corner point. For instance, let $i \leq j^*$ be such that

$$\lambda_i = 1,\ \lambda_j = 0 \quad \text{for} \quad i \neq j$$

is a maximizer. Then

$$-\dot{\overline{x}}_i^\varepsilon(t) + f_i(t,x,\underline{u}_i,\overline{w}) - LV^\varepsilon(t,x)e^{Lt} \leq 0.$$

By a similar reasoning as in Lemma 2, for this to hold it is sufficient that

$$\dot{\overline{x}}_i^\varepsilon(t) \geq f_i(t,\overline{x}^\varepsilon(t),\underline{u}_i,\overline{w})$$

a. e. on $[0,T]$. For the case $i > j^*$ we obtain the sufficient condition

$$\dot{\underline{x}}_i^\varepsilon(t) \leq f_i(t,\underline{x}^\varepsilon(t),\overline{u}_i,\underline{w})$$

a. e. on $[0,T]$. Thus, $V^\varepsilon$ is a viscosity supersolution of (4), (6) in backward time.

Let us now consider a sequence $(\varepsilon_k)_{k=0}^\infty$ such that $\varepsilon_k > 0$, $\varepsilon_k \longrightarrow 0$. Let $(\overline{x}^{\varepsilon_k}(\cdot), \underline{x}^{\varepsilon_k}(\cdot))$ be a sequence of the corresponding solutions. Note that for any $\delta > 0$ there exists $K \in \mathbb{N}$ such that for all $k \geq K$ the pair $(\overline{x}^{\varepsilon_k}(\cdot), \underline{x}^{\varepsilon_k}(\cdot))$ is also a *$\delta$-solution* of (15) (see Filippov (1988), §7). Just as in Lemma 3 it follows that every solution of $\varepsilon$-equation exists on $[0,T]$. Therefore, the set of $\delta$-solutions of (15) is compact in $C([0,T],\mathbb{R}^{2n})$. Hence, there is a converging subsequence of solutions whose limit is a solution of the limiting system (15). The corresponding subsequence of functions $V^{\varepsilon_{k_m}}$ then converges uniformly to function $V$. The statement of the lemma then follows from the stability property of the HJBI equation (see Fleming and Soner (1995), Section II.6, Lemma 6.2). □

**Proof of Theorem 1**. According to Lemma 3, for every $\xi(\cdot)$ there is a unique solution of (15) on $[0,T]$. Therefore, to prove the statement of the theorem it is then sufficient to establish monotonicity of system (15) with respect to state $(\overline{x},\underline{x})$ and input $(\overline{\xi},\underline{\xi})$.

For any $\varepsilon > 0$ one may construct a continuous monotone approximation of the right-hand side of (15) such that

$$\dot{\overline{x}}_i = \begin{cases} f_i(t,\overline{x},\underline{u}_i,\overline{w}) + \overline{\xi}_i(t),\ \hat{x}_i(t) + \varepsilon_k < \overline{x}_i, \\ \max\{f_i(t,\overline{x},\underline{u}_i,\overline{w}) + \overline{\xi}_i(t), \dot{\hat{x}}_i(t)\},\ \hat{x}_i(t) \geq \overline{x}_i, \end{cases}$$
$$\dot{\underline{x}}_i = \begin{cases} f_i(t,\underline{x},\overline{u}_i,\underline{w}) + \underline{\xi}_i(t),\ \underline{x}_i < \hat{x}_i(t) - \varepsilon_k, \\ \min\{f_i(t,\underline{x},\overline{u}_i,\underline{w}) + \underline{\xi}_i(t), \dot{\hat{x}}_i(t)\},\ \underline{x}_i \geq \hat{x}_i(t). \end{cases}$$

Consider a pair of solutions $(\overline{x}^{j,\varepsilon_k}(\cdot), \underline{x}^{j,\varepsilon_k}(\cdot))$, $j = 1,2$ of the approximation system corresponding to inputs $\overline{\xi}^1(t) \preceq \overline{\xi}^2(t)$, $\underline{\xi}^2(t) \preceq \underline{\xi}^1(t)$. Then $\underline{x}^{2,\varepsilon_k}(t) \preceq \underline{x}^{1,\varepsilon_k}(t) \preceq \overline{x}^{1,\varepsilon_k}(t) \preceq \overline{x}^{2,\varepsilon_k}(t)$ due to monotonicity. For a sequence $\varepsilon_k \to +0$ there is a subsequence of pairs of solutions which converges to some pair of solutions of (15). Since the solution of (15) is unique for every $\xi(\cdot)$, we obtain the monotonicity property: $\underline{x}^2(t) \preceq \underline{x}^1(t) \preceq \overline{x}^1(t) \preceq \overline{x}^2(t)$. □

**Proof of Corollary 3**. An arbitrary over-approximation $X^+(t)$ of the reachable set $X^{\hat{u}(\cdot)}(t;0,X^0)$ includes the minimal one due to monotonicity:

$$[\mathbf{x}(t;0,\underline{x}_0,\hat{u}(.),\underline{w}), \mathbf{x}(t;0,\overline{x}_0,\hat{u}(.),\overline{w})] \subseteq X^+(t).$$

Therefore, the statement of this corollary is a direct consequence of Theorem 1 and Remark 1. □

**Proof of Theorem 2**. Consider a partition of time interval $[0,T]$:

$$0 = t_0 < t_1 < \cdots < t_N = T.$$

We define piecewise-constant approximate controller as follows ($k = 1, \dots, N$)

$$u^N(t,x) = u_i(t_{k-1}, x(t_{k-1})), \quad t \in [t_{k-1}, t_k).$$

The number $\delta = \max_k |t_k - t_{k-1}|$ is called the diameter of the partition. The corresponding closed-loop system

$$\dot{x}_i = f_i(t, x, u_i^N(t,x), w(t)), \quad t \in [0,T] \tag{22}$$

has a solution (in the sense of Carathéodory) for all admissible disturbances $w(\cdot)$.

As in Lemma 3, one may observe that the right-hand side of the differential inclusion corresponding to (2) is nonempty, compact, convex and satisfy linear growth bound in $x$. The set-valued map is also measurable in $t$ and upper semicontinuous in $x$. Therefore, at least one solution of (2) exists and every solution can be extended on the whole interval $[0,T]$ and is bounded on it. Consider now a converging sequence of solutions of (22) with the diameter $\delta \to 0$. Then from Filippov (1988), §7, Lemma 3, it follows that the limiting function is a solution of (2). The uniqueness follows from Filippov (1988), §10, Theorem 1 since for some $L'' > 0$ we have

$$(x_i - y_i)(f_i(t,x,u(t,x),w(t)) - f_i(t,u,u(t,y),w(t))) \le L''\|x-y\|_2^2 \quad t \in [0,T],\ x,y \in \mathbb{R}^{n_x}.$$

One may check that $u(t,x)$ is an extremal aiming controller for the weakly invariant set-valued function $X^-(t) = [\underline{x}(t), \overline{x}(t)]$. Therefore, by Theorem 13.3 of Subbotin (1995), we obtain the first statement.

To prove the second statement we note that in this case $u(t,x)$ is Lipschitz in $(t,x)$. Thus, the Carathéodory solution of the closed-loop system exists. The result then follows from the previous statement. □

**Proof of Theorem 3**. Consider a relation $R \subset X \times Q$ defined by

$$(x,q) \in R \quad \Leftrightarrow \quad x \in X_q.$$

Let us prove that it is an alternating simulation relation. Condition 1) of the definition does obviously hold. Condition 2) reads: for every $q \in Q$, $x \in X_q$ and every $v \in \mathrm{enab}_\Delta(q)$ there exists $(T,u) \in \mathrm{enab}_\delta(x)$ such that for every $x' \in \delta(x,T,u)$ there exists $q' \in \Delta(q,v)$, $x' \in X_{q'}$. By Theorem 1, this condition holds for $T = \tau$ and $u = u^{(q,v)}$. □

**Proof of Theorem 4**. For the statement to hold it is sufficient that for every $q \in Q$ and every $\hat{u} \in \hat{U}$ there exists $v \in \mathscr{V}$ such that for every $q' \in \Delta(q,v)$ the inclusion $q' \in \hat{\Delta}(q,\hat{u})$ holds. Let us take $v = \hat{u}$. From Corollary 3 it follows that

$$[\underline{x}^{(q,\hat{u})}(t), \overline{x}^{(q,\hat{u})}(t)] \subseteq [\mathbf{x}(t;\underline{x}^q,\hat{u},\underline{w}), \mathbf{x}(t;\overline{x}^q,\hat{u},\overline{w})]$$

for all $t \in [0,\tau]$. Therefore, $\emptyset \ne \Delta(q,\hat{u}) \subseteq \hat{\Delta}(q,\hat{u})$ for all $q \in Q$, $\hat{u} \in \mathrm{enab}_{\hat{\Delta}}(q) \subseteq \hat{U}$. □