



**HAL**  
open science

**Traduction anglaise de trois textes de L. E. J. Brouwer  
avec les notes de A .Heyting: 1. "Remarks on the notion  
of order"; 2. "On the possibility of ordering the  
continuum"; 3. "Final discourse of Mr Brouwer**

Fabrice Pataut

► **To cite this version:**

Fabrice Pataut. Traduction anglaise de trois textes de L. E. J. Brouwer avec les notes de A .Heyting: 1. "Remarks on the notion of order"; 2. "On the possibility of ordering the continuum"; 3. "Final discourse of Mr Brouwer. 2020. hal-02898155

**HAL Id: hal-02898155**

**<https://hal.science/hal-02898155>**

Submitted on 13 Jul 2020

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

**Fabrice Pataut**  
(CNRS, UMR 8011 *Sciences, Normes, Démocratie*)

**TRADUCTION ANGLAISE DE TROIS TEXTES DE  
L. E. J. BROUWER AVEC LES NOTES DE A. HEYTING**

\*

**LUITZEN EGBERTUS JAN BROUWER**

**1. REMARKS ON THE NOTION OF ORDER [1950a]**

**2. ON THE POSSIBILITY OF ORDERING  
THE CONTINUUM [1950b]**

**3. FINAL DISCOURSE OF MR BROUWER [1950c]**

*English translation by Fabrice Pataut*

The translation should appear in a forthcoming volume, co-edited by Mark van Atten and Fabrice Pataut, to be published by Springer in its Logic, Epistemology and the Unity of Science series. (No publication date is yet available as of July 2020.) The volume will gather English translations of texts by Brouwer which have hitherto been available only in French or in Dutch, namely the Groningen lectures of 1933, translated from the Dutch by Mark van Atten, and the Geneva lectures of 1934, translated from the French by Fabrice Pataut.

The projected volume will also include the three short texts to be found below in their English translation. The three papers “Remarques sur la notion d’ordre”, “Sur la possibilité d’ordonner le continu” and “Discours final de M. BROUWER” have appeared in French in L. E. J. Brouwer: *Collected Works*, vol. I, *Philosophy and Foundations of Mathematics*, A.

Heyting, ed., North-Holland/American Elsevier, Amsterdam, New York, 1975, at pages 499-503.

Heyting's endnotes in English for *1950a* and *1950b* appear in *Brouwer 1975*: 604-605. The references to these endnotes are placed in double square brackets in the margins. In the translation below, the corresponding footnotes appear without these brackets and are referenced consecutively from *1950a* to *1950b*.

Footnote 2 to *1950a* sends the reader back to *Brouwer 1927c: endnote [[4]]*, at pages 596-597, which in turn sends the reader to *Brouwer 1926a* and to some unpublished notes in German also by Brouwer. I have translated this note. For this note, Heyting's editorial additions are placed in double square brackets just like in *Brouwer 1975* and the translator's remarks are placed in single square brackets. I must thank Véronique Cebal and Peter Lillie for their precious help in translating both Heyting's quotes from *Brouwer 1926a* and Brouwer's unpublished handwritten note, also quoted by Heyting, from the German.

## REMARKS ON THE NOTION OF ORDER

(Note by L. E. J. Brouwer, presented by Mr Émile Borel)

(*Brouwer 1950a*: 263-265 ; *Brouwer 1975*: 499-500. Notes 2 to 4 by Heyting in  
*Brouwer 1975*: 596-597, 604)

A species  $S$  of mathematical entities is said to be *partially ordered* if, according to certain criteria  $y$ , for certain pairs of elements of  $S$ , each being composed of a first element  $X$  and of a second element  $Y$ , there exist *order relations*  $X \equiv Y$ <sup>1</sup> ( $X$  and  $Y$  are of the same rank),  $X < Y$  ( $X$  is inferior to  $Y$ ), or  $X > Y$  ( $X$  is superior to  $Y$ ) satisfying the following conditions— $a, b, c, r, s$  designating arbitrary elements of  $S$ :

1.  $a \equiv a$
2.  $a \equiv b$  implies [*entraîne*]  $b \equiv a$
3. the union of  $a \equiv b$  and  $b \equiv c$  implies  $a \equiv c$
4.  $a < b$  and  $b > a$  imply each other
5. the union of  $a \equiv r, b \equiv s$  and  $a < b$  implies  $r < s$
6. the union of  $a < b$  and  $b < c$  implies  $a < c$
7.  $a < b$  precludes  $a > b$  (therefore, from what has been said so far,  $a \equiv b$  precludes  $a < b$ ).

Two elements  $a$  and  $b$  of a partially ordered species will be said to be *of different ranks* if the relation  $a \equiv b$  is found contradictory.

If, in a partially ordered species, two arbitrary elements  $a$  and  $b$  are either of the same rank or of different ranks, the species will be said to be *discretely ordered*.

---

1. The symbol used by Brouwer for sameness of rank is different: the dot appears between the two lines, not below as it does here. [Translator's note]

If, in a partially ordered species, one has for two arbitrary elements  $a$  and  $b$  of different ranks, either  $a < b$  or  $a > b$ , the species will be said to be *quasi-completely* ordered.

A partial order which is at once discrete and quasi-complete will be said to be *complete*.

A partial order will be said to be *natural* if its criteria  $y$  constitute an extension of those of some simple and intuitive complete order.

A relation  $u$ , which is a relation  $p \equiv q$ ,  $p < q$  or  $p > q$ , and which we imagine for two elements  $p$  and  $q$  of the partially ordered species  $S$ , will be said to be *compatible with the partial order of  $S$*  if, for the species of relations resulting from the union of the relation  $u$  and of the order relations following [*découlant*] from criteria  $y$ , the non-contradiction of the system of conditions 1, 2, ..., 7 still holds [*reste en vigueur*]. If it happens to be established that each order relation compatible with  $\pi$  follows itself from criteria  $y$ , and is therefore a part of  $\pi$ , the partial order  $\pi$  of  $S$  will be said to be *saturated*.

It has been proved [*démontré*] that the saturated order is equivalent to the *virtual* order<sup>2</sup>, i.e., to the partial order which, apart from conditions 1, 2, ..., 7, satisfies the following additional conditions :

---

2. See Brouwer 1927c, in particular note [[4]] to that paper. [Heyting's note]\*

\* Note [[4]] reads as follows:

Among Brouwer's papers there was the following handwritten note, dated March 25 1933, which was never published:

The *virtual order of the continuum  $C$*  introduced in *Die Math. Annalen*, vol. 95, p. 467[[1926a: 335]], has the property of *saturatedness* [*Unerweiterbarkeit*], which means that any relation  $a = b$  or  $c < d$  which may be consistently added to the relations of this kind, which already exist as a consequence of the order criterion, is already satisfied as a consequence of the order criterion. Namely, if the relation  $a = b$  may be consistently added to the existing relations, absurdity follows from the order criterion, from  $a \circ > b$  as well as from  $a < \circ b$ , therefore the absurdity of both  $a > b$  and  $a < b$ , hence the relation  $a = b$ . And if the relation  $c < d$  may be consistently added to the existing relations, then absurdity follows from the order criterion, from  $c = d$  as

8. The simultaneous absurdity of  $a \equiv b$  and  $a < b$  implies  $a > b$ .

9. The simultaneous absurdity of  $a < b$  and  $a > b$  implies  $a \equiv b$ .

One immediately notices that for each order relation that we imagine with respect to two elements of the virtually ordered species  $S$ , non-contradiction is equivalent to existence, and that for each pair of elements of this species  $S$ , the partaking in the virtual order is non-contradictory.

Virtual order finds its most important realization in the natural partial order of the intuitionistic continuum.

---

well as from  $c \circ > d$ , therefore the absurdity of  $c = d$  as well as of  $c > d$ , hence the relation  $c < d$ .

The fact that every saturated order is a virtual order has been proved in 1927c. The proof of the converse [*Umkherung*] of this proposition is however not well-founded [*stichhaltig*] since the incompatibility of the simultaneous existence of two arbitrary relations  $r = s$ ,  $r < s$  or  $r > s$ , does not imply the incompatibility of the simultaneously consistent adjunctability of two arbitrary relations to those aforementioned relations. In fact, the species of the four elements  $a, b, c, d$  with the order relation  $a < b$ ,  $a < c$ ,  $a < d$ ,  $b < d$ ,  $c < d$ , is virtually ordered although not unsaturatedly [*unerweiterbar*] ordered.

[[In the proof of Satz 3 the inference from  $\neg ((r < s) \in \alpha)$  to  $\alpha \rightarrow \neg (r < s)$  is not allowed.]]

In the margin of his copy Brouwer tried to avoid this difficulty by changing the definition of virtual order. He proposed to read axioms 6 and 7 as follows :

6'. From the inconsistency [*Ungereimtheit*], deriving [*herzuleiten*] one of the relations  $r < s$  and  $r > s$  from the definition of ordered projection, it follows [*folgt*] that  $r \equiv s$ .

7'. From the inconsistency, deriving one of the relations  $r > s$  and  $r \equiv s$  from the definition of ordered projection, it follows that  $r < s$ .

He adds the following remark :

In this case, for example, it is ruled out that for an element pair  $(a, b)$ , regarding all three relations  $a \equiv b$ ,  $a < b$ , and  $a > b$ , one may verify the impossibility of the derivation from the definition of the ordered projection.

With this definition of virtual order the theorems in the paper hold. The order relations in the continuum satisfy both definitions of virtual order. However, in the general case, the new definition seems less interesting because it involves the metamathematical notion of derivability. In (III), Chapter 3 [of *Brouwer 1975*, which gathers all papers related to the notion of order], Brouwer mentioned explicitly that conditions 6 and 7 ought to be interpreted according to 6' and 7' above. In 1950a, he asserted again that every saturated (*unerweiterbare*) order is a virtual order.

Obviously, complete order implies [*implique*] virtual order. By contrast, *virtual order does not imply complete order ; it does not even imply quasi-complete order*. With respect to natural partial orders, this failure of implication is brought to light by means of the example of an essentially negative property which I have presented at the Academy of Sciences of Amsterdam at its meeting of September 25 1948.<sup>3</sup>

*On the other hand, with respect to natural partial orders, quasi-complete order also fails to imply virtual order*. To see this, let us say two mathematical entities are *different* in case their equality is absurd, and let us say of two species of mathematical entities  $a$  and  $b$  that  $a$  is *contained* in  $b$  if each element of  $a$  is equal to some element of  $b$ , that  $a$  and  $b$  are *identical* if they contain each other, and that  $a$  *goes out of*  $b$  [*a sort de b*] if  $a$  contains an element which is different from each element of  $b$ . Let us then base the *natural partial order*  $\pi_i$  of some arbitrary species  $E$  of species of mathematical entities on the convention that, for two elements  $a$  and  $b$  of  $E$ , we shall write [*on mettra que*]  $a = b$  if  $a$  and  $b$  are identical, and  $a < b$  if  $a$  is contained in  $b$  while  $b$  goes out of  $a$ .

Let us consider in particular the partial order  $\pi_i$  of the species  $R$  containing as elements the species  $a_\omega$  and  $a_r$  defined as follows:  $a_\omega$  is composed of the infinite series of rational numbers  $0, 1, 2^{-1}, 2^{-2}, \dots$ , whereas  $a_r$  contains as elements those of  $a_\omega$  plus a real number  $r$  satisfying the condition that, although it is not contained in  $a_\omega$ , its belonging to  $a_\omega$  is non-contradictory. Intuitionism has several means at its disposal to construct [*construire*] such numbers.<sup>4</sup> So the difference in

---

3. *Brouwer 1948a*. [Heyting's note]

4. For example:  $r$  is the limit of the sequence  $\{a_n\}$ , where  $a_n = 2^{-n}$  if there are no 10 consecutive 7's among the first  $n$  digits in the decimal expansion of  $\pi$  and  $a_n = 2^{-k}$  if  $k$

rank of the two elements of  $R$  with respect to  $\pi_1$  being contradictory, the only pair of elements contained in  $R$  remains untouched by the condition of the quasi-complete order, which, therefore, is indeed realized. Furthermore, condition 9 of the virtual order isn't fulfilled, the relations  $a_r < a_\omega$  and  $a_r > a_\omega$  being both contradictory without the relation  $a_r \equiv a_\omega$  thereby obtaining. It follows that, as far as species  $R$  is concerned, the partial order  $\pi_1$  isn't virtual.

Leaving aside the domain of natural partial orders, the mutual independence of the quasi-complete order and of the virtual order is easier to make clear. Let us indeed consider a species  $S$  composed solely of two mathematical entities  $a$  and  $b$ . Let  $\alpha$  be a *non-testable* [*non-éprouvable*] mathematical assertion, which means that we do not know so far of any algorithm with which we may deduce [*déduire*] either the non-contradictory character or the absurdity of  $\alpha$ . Let us define a partial order of  $S$  by choosing as respective criteria for the relations  $a > b$ ,  $a < b$  and  $a \equiv b$ , the establishing of the non-contradiction, of the absurdity, and of the conjunction of [*et de l'ensemble de*] the non-contradiction and the absurdity of  $\alpha$ . This partial order is virtual without being quasi-complete. Furthermore, if these three criteria are successively replaced by the establishing of the conjunction of the truth and of the absurdity of  $\alpha$ , of the absurdity of  $\alpha$ , and of the truth of  $\alpha$ , we obtain a quasi-complete order which is not virtual.

---

is the smallest number such that the  $(k-9)$ th- $k$ th digits are 7 and  $k < n$ . [Heyting's note]



## ON THE POSSIBILITY OF ORDERING THE CONTINUUM

(Note by L. E. J. Brouwer, presented by Mr Emile Borel)

(*Brouwer 1950b*: 349-350 ; *Brouwer 1975*: 501-502. Notes 5 to 6 by Heyting in  
*Brouwer 1975*: 605)

In a previous communication <sup>5</sup>, I have brought to light the idea that the natural order of the intuitionistic continuum is not quasi-complete, i.e., that it does not include an order relation for each pair of distinct real numbers. The question arises as to whether there are means of quasi-completely ordering the intuitionistic continuum in some other way, given, of course, that equality remains a condition for equality of rank. The answer is negative, as the following argument establishes.

Let us suppose the existence of some quasi-complete order  $\pi$  for the intuitionistic continuum. *In what follows, the signs  $<$  and  $>$  will relate to the order  $\pi$ .* Let  $p_1$  and  $u_1$  be two real numbers whose *natural distance*  $a$  (in other words, the absolute value of their arithmetical difference) is greater than  $2^{-n}$  for some given natural number  $n$ . Let  $p_1 < u_1$  and let  $w_1$  be the real number which is the *natural mean* of  $p_1$  and  $u_1$ . We shall then have either  $w_1 < p_1 < u_1$ , or  $p_1 < u_1 < w_1$ , or  $p_1 < w_1 < u_1$ . In the first case, we shall set down that [*nous mettrons que*]  $p_2 = w_1$  and  $u_2 = u_1$ ; in the second and third cases that  $p_2 = p_1$  and  $u_2 = w_1$ . So, in every case, the natural distance from  $p_2$  to  $u_2$  will be  $2^{-1}a$ , whereas  $p_2 < u_2$ . Likewise, for any natural number  $\rho$ , let  $p_\rho$  and  $u_\rho$  be two real numbers with natural distance  $2^{-\rho+1}a$  such that  $p_\rho < u_\rho$ . Let then  $w_\rho$  be the natural mean of  $p_\rho$

---

5. Comptes rendus 230, 1950, p. 263. [This is most probably a note of the editor of the *Comptes rendus hebdomadaires des séances de l'Académie des Sciences*. Heyting's endnote to that footnote just gives the reference as *Brouwer 1950a*. [Translator's note]]

and  $u_\rho$ . We shall set down that  $p_{\rho+1} = w_\rho$  and  $u_{\rho+1} = u_\rho$  in case  $w_\rho < p_\rho < u_\rho$ , and that  $p_{\rho+1} = p_\rho$  and  $u_{\rho+1} = w_\rho$  in case either  $p_\rho < u_\rho < w_\rho$  or  $p_\rho < w_\rho < u_\rho$ . Therefore, in all cases, the natural distance from  $p_{\rho+1}$  to  $u_{\rho+1}$  will be  $2^{-\rho} a$ , whereas  $p_{\rho+1} < u_{\rho+1}$ . This procedure yields a fundamental series of pairs of real numbers  $(p_v, u_v)$  in which, for each  $v$ , we have  $p_v < u_v$ , whereas the natural distance from  $p_v$  to  $u_v$  is  $2^{-v+1} a$  and each *natural interval*  $(p_{v+1}, u_{v+1})$  is contained in the natural interval  $(p_v, u_v)$ . Consequently, the series converges towards a unique real number  $q$ .

Let  $\alpha$  be a *non-testable* mathematical assertion.<sup>6</sup> Let us consider an infinite series  $(h_1, k_1), (h_2, k_2), \dots$  of pairs of real numbers, unfolding [*avançant*] according to the following instructions :

As long as, during the successive creations of  $(h_n, k_n)$ , neither the absurdity, nor the non-contradiction of  $\alpha$  will have been established,  $h_n$  will be chosen equal to  $p_n$  and  $k_n$  equal to  $u_n$ . But as soon as the truth of  $\alpha$  will have been revealed by means of a choice between some given  $(h_{r-1}, k_{r-1})$  and  $(h_r, k_r)$ , each  $h_v$  ( $v \geq r$ ) will be chosen equal to  $p_r$  and each  $k_v$  ( $v \geq r$ ) equal to  $u_r$ . And as soon as the absurdity of  $\alpha$  will have been discovered by means of a choice between some given  $(h_{s-1}, k_{s-1})$  and  $(h_s, k_s)$ , each  $h_v$  ( $v \geq s$ ) will be chosen equal to  $u_s$  and each  $k_v$  ( $v \geq s$ ) equal to  $p_s$ . In any event, the infinite series  $(h_1, k_1), (h_2, k_2), \dots$  will converge towards a pair of real numbers  $(h, k)$ .

Let us suppose for a moment that  $h$  and  $k$  are equal. It would follow that the relations  $h < k$  and  $h > k$  are both impossible. So the equality of  $(h, k)$  to some  $(p_r, u_r)$ , and the equality of  $(h, k)$  to some  $(u_s, p_s)$  would be both impossible, and the assertion  $\alpha$  would at the same time be absurd and non-contradictory. Our supposition has therefore been refuted ; we

---

6. éprouvable = testable. See Brouwer 1948c. [Heyting's note]

have established that  $h$  and  $k$  are different, and that the quasi-complete order  $\pi$  will have to include an order relation for the pair  $(h, k)$ .

Now, this order relation will either be :

- $h < k$ , from which the impossibility of  $h > k$  follows, successively entailing [*entraînant*] the impossibility of the equality of  $(h, k)$  to some  $(u_s, p_s)$  and the non-contradictoriness of  $\alpha$ , and so the testability of  $\alpha$ ,

or

- $h > k$ , from which the impossibility of  $h < k$  follows, successively entailing the impossibility of the equality of  $(h, k)$  to some  $(p_r, u_r)$  and the absurdity of  $\alpha$ , and so, once again, the testability of  $\alpha$ .

We have come across a contradiction. This contradiction establishes that, as long as there will be untestable assertions, the intuitionistic continuum will not admit a quasi-complete order.

With respect to the quasi-complete order of the reduced continuum, in other words of the species of the predetermined real numbers, there exists an analogous impossibility, which is conditional from a logical point of view but absolute from a practical point of view, and which merely requires a longer formulation than in the case of the intuitionistic continuum.

(Originally published in *Comptes rendus des séances de l'Académie des Sciences*, vol. 230, pp. 349-350, meeting of January 23 1950.)

### FINAL DISCOURSE OF Mr BROUWER

(*Brouwer 1950c: 75 ; Brouwer1975: 503*)

The programme of the colloquium endowing me with the great honour of pronouncing a few words in way of a conclusion, I would like to ask you to consider in its essence the whole of mathematics and logic, the two sciences which are so intimately linked and with which we have been concerned. In order to do this, let us go to the deepest seat of our consciousness and let us notice that, originally, there is nothing there but a world-dream [*un monde-rêve*], and that in this world-dream, a pragmatic world comes to life only by means of the *phenomenon of discernment* [*discernement*], which creates *the thinking man*, and of the *phenomenon of shrewdness* [*astuce*], which creates *the acting man*, the two phenomena generating together the external world and the objects. Then, calling a « game » any activity carried out for its own sake and not triggered by either fear, constraint, desire or calling, let us pick out the existence of the *logical game* which, through discernment, replaces perceived objects by fictitious and purely indicative objects, and of the *mathematical game* which, through discernment, leaves objects aside entirely. Let us remark that these two games both originate in a *primary phenomenon of bi-unicity*, capable of a spontaneous and infinite pluralization which creates, in the realms [*champs*] of the mind, an unlimited and lush vegetation, considerably richer in the mathematical case than in the logical case, following the complete freeing from the ballast of objects which is indeed the privilege of mathematics [*dont jouissent les mathématiques*].

In virtue of their origin, the two games act upon each other. Because of their nature, they should not interfere in social life. Having nevertheless being thus called for, they endure the influence of the pragmatic sciences while contributing, against their nature, to the transformations of the social life we call progress. Fortunately, their most beautiful developments will probably never relate to technical, economical, or political matters.

All of us here who have attended the debates which have only now come to a close, have been able to hear, and to see that logic, today, when cultivated for its own sake, brings up fascinating problems and leads to discoveries which are as ingenious as they are surprising. A picture has unfolded, which will contribute to silence those who would like to deny the game of pure logic the right to existence.

Let us then hope that our colloquium has inaugurated a long series of resumptions, and let us be thankful to those who have initiated it and have so eminently organized it, in particular the Dean of the Faculty of Science and Professor Destouches, from the UNESCO, who has given us his precious support ; to our referees, who have shed so much light and opened such far-reaching vistas, and finally to all those who have taken part in such animated and instructive discussions.

## REFERENCES

Brouwer (L. E. J.), 1926a, “Zur Begründung der intuitionistischen Mathematik II”, *Mathematische Annalen*, vol. 95, pp. 453-472. Reprinted in *Brouwer 1975*: 321-340.

- 1927c, “Virtuelle Ordnung und unerweiterbare Ordnung”, *Journal für die reine und angewandte Mathematik*, vol. 151, pp. 255-257. Reprinted in *Brouwer 1975*: 406-408.

- 1948a, “Essentieel negatieve eigenschappen”, *Indagationes Mathematicae*, vol. 10, pp. 322-323. English translation by A. Heyting, with corrections by Dr. Van Stigt (as noted by Heyting in his introduction to *Brouwer 1975* at p. xv) as “Essentially negative properties” in *Brouwer 1975*: 478-479.

- 1948c, “Consciousness, Philosophy and Mathematics”, *Proceedings of the Tenth International Congress of Philosophy* (Amsterdam, August 11-18 1948), vol. 2, E. W. Beth, H. J. Pos and J. H. A. Hollak, eds., North-Holland Publishing Company, Amsterdam, pp. 1235-1249. [The reference indicates the year of the congress ; the volume was published in 1949.] Reprinted in *Brouwer 1975*: 480-494.

- 1950a, “Remarque sur la notion d’ordre”, *Comptes rendus hebdomadaires des séances de l’Académie des Sciences*, vol. 230, Paris, pp. 263-265. Reprinted in *Brouwer 1975*: 499-500.

- 1950b, “Sur la possibilité d’ordonner le continu”, *Comptes rendus hebdomadaires des séances de l’Académie des Sciences*, vol. 230, Paris, pp. 349-350. Reprinted in *Brouwer 1975*: 501-502.

- 1950c, “Discours final de M. BROUWER”, *Méthodes formelles en axiomatique : Colloques internationaux du Centre National de la Recherche Scientifique*, CNRS, Paris, p. 75. Reprinted in *Brouwer 1975*: 503.

- 1975, *Collected Works*, vol. I, *Philosophy and Foundations of Mathematics*, A. Heyting, ed., North-Holland/American Elsevier, Amsterdam, New York.