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A theoretical look at ELECTRE TRI-nB ^a

Denis Bouyssou ^b Thierry Marchant ^c Marc Pirlot ^d

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Abstract

ELECTRE TRI is a set of methods designed to sort alternatives evaluated on several attributes into ordered categories. The original ELECTRE TRI-B method uses one limiting profile per category. A more recent method, ELECTRE TRI-nB, allows one to use several limiting profiles for each category. We investigate the properties of ELECTRE TRI-nB. When the number of limiting profiles used to define each category is not restricted, ELECTRE TRI-nB is easy to characterize axiomatically and is found to be equivalent to several other methods proposed in the literature. We extend this result in various directions.

Keywords: Multiple criteria analysis, Sorting models, ELECTRE TRI.

1 Introduction

An additional ELECTRE method, called ¹ ELECTRE TRI-nB, for sorting alternatives evaluated on several attributes into ordered categories was proposed in Fernández, Figueira, Navarro, and Roy (2017). It is an extension of ETRI-B (Roy and Bouyssou, 1993, Yu, 1992). Whereas ETRI-B uses one limiting profile per category, ETRI-nB allows one to use several limiting profiles for each category. For an overview of ELECTRE methods, we refer to Roy and Bouyssou (1993, Ch. 5 & 6), Figueira, Greco, Roy, and Słowiński (2013), and Figueira, Mousseau, and Roy (2016). ETRI-nB deserves close attention for at least two reasons.

^aAuthors are listed alphabetically. They have contributed equally.

^bLAMSADE, UMR 7243, CNRS, Université Paris-Dauphine, PSL Research University, 75016 Paris, France, e-mail: bouyssou@lamsade.dauphine.fr. **Corresponding author.**

^cGhent University, Department of Data Analysis, H. Dunantlaan, 1, B-9000 Gent, Belgium, e-mail: thierry.marchant@UGent.be.

^dUniversité de Mons, rue de Houdain 9, 7000 Mons, Belgium, e-mail: marc.pirlot@umons.ac.be.

¹We often abbreviate ELECTRE TRI as ETRI in what follows.

First, as explained in Bouyssou and Marchant (2015), ETRI can be considered as a real success story within the ELECTRE family of methods. It has received a fairly complete axiomatic analysis in Bouyssou and Marchant (2007a,b). This method has been applied to a large variety of real world problems (see the references in Almeida-Dias, Figueira, and Roy, 2010, Sect. 6, as well as Bisdorff, Dias, Meyer, Mousseau, and Pirlot, 2015, Ch. 6, 10, 12, 13, 15, 16). Many techniques have been proposed for the elicitation of the parameters of this method (see the references in Bouyssou and Marchant, 2015, Sect. 1).

Second, the extension presented with ETRI-nB is most welcome. Since out-ranking relations are not necessarily complete, one may easily argue that it is natural to try to characterize a category using several limiting profiles, instead of just one. Moreover, compared to ETRI-B, ETRI-nB gives more flexibility to the decision-maker to define categories using limiting profiles².

In this paper, we analyze ETRI-nB from a theoretical point of view. Our aim is to characterize this method in the same way Bouyssou and Marchant (2007a,b) have characterized the original ETRI-B method. The usefulness of such axiomatic analyses has been discussed elsewhere and will not be repeated here (Bouyssou and Marchant, 2015, Bouyssou and Pirlot, 2015a, Gilboa, Postlewaite, Samuelson, and Schmeidler, 2019).

Our main finding is that, if the number of profiles used to characterize each category is not restricted, the axiomatic analysis of ETRI-nB is easy and rests on a condition, linearity, that is familiar in the analysis of sorting models (Bouyssou and Marchant, 2007a,b, 2010, Goldstein, 1991, Greco, Matarazzo, and Słowiński, 2001b, 2004, Słowiński, Greco, and Matarazzo, 2002). Our simple result shows the equivalence between ETRI-nB and many other sorting models proposed in the literature. It could also allow one to use elicitation techniques developed for these other models for the application of ETRI-nB. This is useful since Fernández et al. (2017) did not propose any elicitation technique (an elicitation technique was suggested afterwards in Fernández, Figueira, and Navarro, 2019).

The rest of this text is organized as follows. Section 2 introduces our notation and framework. Section 3 presents our main results about the pseudo-conjunctive version of ETRI-nB. Section 4 presents various extensions of these results. Section 5 deals with the case of the pseudo-disjunctive version of ETRI-nB. A final section discusses our findings. An appendix, containing supplementary material to this paper, will allow us to keep the text of manageable length. Its content will be detailed when needed.

²Let us also mention that Fernández et al. (2017) is the last paper on ELECTRE methods published by Bernard Roy, the founding father of ELECTRE methods, before he passed away at the end of 2017.

2 Notation and framework

Although the analyses presented in this paper can easily be extended to cover the case of several ordered categories, we will mostly limit ourselves to the study of the case of *two* ordered categories. This will allow us to keep things simple, while giving us a sufficiently rich framework to present our main points.

Similarly, we suppose throughout that the set of objects to be sorted is *finite*. This is hardly a limitation with applications of sorting methods in mind. The extension to the general case is not difficult but calls for developments that would obscure our main messages³.

2.1 The setting

Let $n \geq 2$ be an integer and $X = X_1 \times X_2 \times \dots \times X_n$ be a *finite* set of objects. Elements x, y, z, \dots of X will be interpreted as alternatives evaluated on a set $N = \{1, 2, \dots, n\}$ of attributes⁴. For any nonempty subset J of the set of attributes N , we denote by X_J (resp. X_{-J}) the set $\prod_{i \in J} X_i$ (resp. $\prod_{i \notin J} X_i$). With customary abuse of notation, (x_J, y_{-J}) will denote the element $w \in X$ such that $w_i = x_i$ if $i \in J$ and $w_i = y_i$ otherwise. When $J = \{i\}$ we shall simply write X_{-i} and (x_i, y_{-i}) .

Our primitives consist in a twofold *partition* $\langle \mathcal{A}, \mathcal{U} \rangle$ of the set X . This means that the sets \mathcal{A} and \mathcal{U} are nonempty and disjoint and that their union is the entire set X . Our central aim is to study various models allowing to represent the information contained in $\langle \mathcal{A}, \mathcal{U} \rangle$. We interpret the partition $\langle \mathcal{A}, \mathcal{U} \rangle$ as the result of a sorting method applied to the alternatives in X . Although the ordering of the categories is not part of our primitives, it is useful to interpret the set \mathcal{A} as containing *s*atisfactory objects, while \mathcal{U} contains *U*nsatisfactory ones.

We say that an attribute $i \in N$ is influential for $\langle \mathcal{A}, \mathcal{U} \rangle$ if there are $x_i, y_i \in X_i$ and $a_{-i} \in X_{-i}$ such that $(x_i, a_{-i}) \in \mathcal{A}$ and $(y_i, a_{-i}) \in \mathcal{U}$. We say that an attribute is degenerate if it is not influential. Note that the fact that $\langle \mathcal{A}, \mathcal{U} \rangle$ is a partition implies that there is at least one influential attribute in N . A degenerate attribute has no influence whatsoever on the sorting of the alternatives and may be suppressed from N . Hence, we suppose henceforth that all attributes are influential for $\langle \mathcal{A}, \mathcal{U} \rangle$.

³In fact our framework allow us to deal with some infinite sets of objects: all that is really required is that the set of equivalence classes of each set X_i under the equivalence \sim_i is finite, see below.

⁴We use a standard vocabulary for binary relations. For the convenience of the reader, all terms that are used in the main text are defined in Appendix A, given as supplementary material. See also, se.g., Aleskerov, Bouyssou, and Monjardet (2007), Doignon, Monjardet, Roubens, and Vincke (1988), Pirlot and Vincke (1992), Roubens and Vincke (1985).

A twofold partition $\langle \mathcal{A}, \mathcal{U} \rangle$ induces on each $i \in N$ a binary relation defined letting, for all $i \in N$ and all $x_i, y_i \in X_i$,

$$x_i \sim_i y_i \text{ if } [\forall a_{-i} \in X_{-i}, (y_i, a_{-i}) \in \mathcal{A} \Leftrightarrow (x_i, a_{-i}) \in \mathcal{A}].$$

This relation is always reflexive, symmetric and transitive, i.e., is an equivalence.

We omit the simple proof of the following (see Bouyssou and Marchant, 2007a, Lemma 1, p. 220).

Lemma 1

For all $x, y \in X$,

1. $[y \in \mathcal{A} \text{ and } x_i \sim_i y_i] \Rightarrow (x_i, y_{-i}) \in \mathcal{A}$,
2. $[x_i \sim_i y_i, \text{ for all } i \in N] \Rightarrow [x \in \mathcal{A} \Leftrightarrow y \in \mathcal{A}]$.

This lemma will be used to justify the convention made later in Section 3.1.

2.2 A general measurement framework

Goldstein (1991) suggested the use of conjoint measurement techniques for the analysis of twofold and threefold partitions of a set of multi-attributed alternatives. His analysis was rediscovered and developed in Greco et al. (2001b) and Słowiński et al. (2002). We briefly recall here the main points of the analysis in the above papers for the case of twofold partitions. We follow Bouyssou and Marchant (2007a).

Let $\langle \mathcal{A}, \mathcal{U} \rangle$ be a partition of X . Consider a measurement model in which, for all $x \in X$,

$$x \in \mathcal{A} \Leftrightarrow F(u_1(x_1), u_2(x_2), \dots, u_n(x_n)) > 0, \tag{D1}$$

where u_i is a real-valued function on X_i and F is a real-valued function on $\prod_{i=1}^n u_i(X_i)$ that is *nondecreasing* in each argument⁵. The special case of Model (D1) in which F is supposed to be *increasing* in each argument, is called Model (D2). Model (D2) contains as a particular case the additive model for sorting in which, for all $x \in X$,

$$x \in \mathcal{A} \Leftrightarrow \sum_{i=1}^n u_i(x_i) > 0, \tag{Add}$$

that is at the heart of the UTADIS technique (Jacquet-Lagrèze, 1995) and its variants (Greco, Mousseau, and Słowiński, 2010, Zopounidis and Doumpos, 2000a,b).

⁵In Model (D1), notice that we could have chosen to replace the strict inequality by a nonstrict one. The two versions of the model are equivalent, as shown in Bouyssou and Marchant (2007a, Rem. 8, p. 222). The same is true for Model (D2).

It is easy to check⁶ that there are twofold partitions that can be obtained in Model (D2) but that cannot be obtained in Model (Add).

In order to analyze Model (D1), we define on each X_i the binary relation \succsim_i letting, for all $x_i, y_i \in X_i$,

$$x_i \succsim_i y_i \text{ if [for all } a_{-i} \in X_{-i}, (y_i, a_{-i}) \in \mathcal{A} \Rightarrow (x_i, a_{-i}) \in \mathcal{A}].$$

It is not difficult to see that, by construction, \succsim_i is reflexive and transitive. We denote by \succ_i (resp. \sim_i) the asymmetric (resp. symmetric) part of \succsim_i (hence, the relation \sim_i coincides with the one used above).

We say that the partition $\langle \mathcal{A}, \mathcal{U} \rangle$ is linear on attribute $i \in N$ (condition i -linear) if, for all $x_i, y_i \in X_i$ and all $a_{-i}, b_{-i} \in X_{-i}$,

$$\left. \begin{array}{l} (x_i, a_{-i}) \in \mathcal{A} \\ \text{and} \\ (y_i, b_{-i}) \in \mathcal{A} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (y_i, a_{-i}) \in \mathcal{A}, \\ \text{or} \\ (x_i, b_{-i}) \in \mathcal{A}. \end{array} \right. \quad (i\text{-linear})$$

The partition is said to be linear if it is i -linear, for all $i \in N$. This condition was first proposed in Goldstein (1991) and generalized in Greco et al. (2001b) and Słowiński et al. (2002). The adaptation of this condition to the study of binary relations, adaptation first suggested by Goldstein (1991), is central in the analysis of the nontransitive decomposable models analyzed in Bouyssou and Pirlot (1999, 2002, 2004).

The following lemma takes note of the consequences of condition i -linear on the relation \succsim_i and shows that linearity is necessary for Model (D1). Its proof can be found in Bouyssou and Marchant (2007a, Lemma 5, p. 221).

Lemma 2

1. Condition i -linear holds iff \succsim_i is complete,
2. If a partition $\langle \mathcal{A}, \mathcal{U} \rangle$ has a representation in Model (D1) then it is linear.

The following proposition is due to Goldstein (1991, Theorem 2) and Greco et al. (2001b, Theorem 2.1, Part 2).

Proposition 3

Let $\langle \mathcal{A}, \mathcal{U} \rangle$ be a twofold partition of a set X . Then:

- (i) there is a representation of $\langle \mathcal{A}, \mathcal{U} \rangle$ in Model (D1) if and only if (iff) it is linear,

⁶When X is finite, it is clear that the variant of Model (Add) in which the strict inequality is replaced by a nonstrict one is equivalent to Model (Add).

(ii) if $\langle \mathcal{A}, \mathcal{U} \rangle$ has a representation in Model (D1), it has a representation in which, for all $i \in N$, u_i is a numerical representation of \succsim_i ,

(iii) moreover, Models (D1) and (D2) are equivalent.

PROOF

See, e.g., Bouyssou and Marchant (2007a, Proposition 6, p. 222) □

2.3 Interpretations of Model (D1)

The framework offered by Model (D1) is quite flexible. It contains many other sorting models as particular cases. We already observed that it contains Model (Add) as a particular case. Bouyssou and Marchant (2007a) have reviewed the various possible interpretations of Model (D1). In particular, they have shown that both the pseudo-conjunctive and the pseudo-disjunctive variants of ETRI-B enter into this framework⁷.

Greco et al. (2001b, Theorem 2.1, Parts 3 and 4) (see also Słowiński et al., 2002, Theorem 2.1) have proposed two equivalent reformulations of Model (D1). The first one uses “at least” decision rules. The second one uses a binary relation to compare alternatives to a profile. We refer to Bouyssou and Marchant (2007a) and to the original papers for details.

2.4 ETRI-B and ETRI-nB

2.4.1 Construction of the outranking relation

For the ease of future reference, we very briefly recall here the main points of ETRI-B and ETRI-nB with two categories. For a more detailed description, we refer the reader to Fernández et al. (2017), Mousseau, Słowiński, and Zielniewicz (2000), Roy and Bouyssou (1993), Yu (1992). We refer to Bouyssou and Marchant (2015) for an analysis of the importance of ETRI within the set of all ELECTRE methods.

A valued (in the $[0, 1]$ interval) outranking relation is built as in ELECTRE III (see Roy and Bouyssou, 1993, p. 284–289). This valued relation submitted to a *cutting level* (between 0.5 and 1) to obtain a crisp outranking relation S (see Roy and Bouyssou, 1993, p. 389). We use the symbol P (resp. I) to denote the asymmetric part (resp. symmetric part) of S .

An important property of this way of building the relation S is that it is compatible with the order induced on each dimension by the criteria (see, e.g., Bouyssou and Marchant, 2015, p. 202).

⁷Note that this does not mean that the pseudo-disjunctive variant enters into the framework of noncompensatory sorting models, as defined in Bouyssou and Marchant (2007a)

2.4.2 ETRI-B

The sorting of an alternative $x \in X$ is based upon the comparison of x with a limiting profile⁸ $p \in X$ using the relation S . In the pessimistic version of ETRI-B, now known, following Almeida-Dias et al. (2010), as the pseudo-conjunctive version (ETRI-B-pc), we have, for all $x \in X$,

$$x \in \mathcal{A} \Leftrightarrow x S p.$$

In the optimistic version of ELECTRE TRI, now known as the pseudo-disjunctive version (ETRI-B-pd), we have, for all $x \in X$,

$$x \in \mathcal{A} \Leftrightarrow \text{Not}[p P x],$$

where P is the asymmetric part of S .

It is shown in Bouyssou and Marchant (2007a) that ETRI-B-pc, is a particular case of Model ($D1$) in which each function u_i takes few values (two, when there is no discordance and three values when there is discordance, the smallest value forbidding to have a positive value for F). This is what Bouyssou and Marchant (2007a) have called the *noncompensatory sorting model* (with veto). They also showed that ETRI-B-pd also enters the framework of Model ($D1$) but in a more complex way that does not fit into the framework of the noncompensatory sorting model (with veto).

2.4.3 ETRI-nB

We now have a set of k limiting profiles $\mathcal{P} = \{p^1, p^2, \dots, p^k\} \subseteq X$. This set of limiting profiles must be such that, for all $p, q \in \mathcal{P}$, we have $\text{Not}[p P q]$.

In the pseudo-conjunctive version of ETRI-nB (ETRI-nB-pc, for short), we have, for all $x \in X$,

$$x \in \mathcal{A} \Leftrightarrow \begin{cases} x S p & \text{for some } p \in \mathcal{P}, \text{ and} \\ \text{Not}[q P x] & \text{for all } q \in \mathcal{P}, \end{cases}$$

and $x \in \mathcal{U}$, otherwise.

In the pseudo-disjunctive version of ETRI-nB (ETRI-nB-pd, for short), we have, for all $x \in X$,

$$x \in \mathcal{U} \Leftrightarrow \begin{cases} p P x & \text{for some } p \in \mathcal{P}, \text{ and} \\ \text{Not}[x P q] & \text{for all } q \in \mathcal{P}, \end{cases}$$

and $x \in \mathcal{A}$, otherwise.

⁸We suppose here that the profile is an element of X . Bouyssou and Marchant (2007a, Remarks 31 and 37) have shown that this is w.l.o.g..

Remark 4

It is easy to check that if we apply ETRI-nB-pc and ETRI-nB-pd using the same settings (relations S_i and V_i , weights w_i and threshold λ and profiles $p \in \mathcal{P}$), an alternative that belongs to \mathcal{A} in ETRI-nB-pc will also belong to \mathcal{A} in ETRI-nB-pd (see Fernández et al., 2017, Prop. 2, p. 217, for details). •

Remark 5

We do not deal here with ETRI-C and ETRI-nC (Almeida-Dias et al., 2010, Almeida-Dias, Figueira, and Roy, 2012), where one or several central profiles are used. This is because these two methods do not sort each alternative into a unique category but into an interval of categories, due to the conjoint use of the ascending and descending procedures. Moreover, Bouyssou and Marchant (2015) have shown that the logic behind ETRI-C is at some variance with the one underlying ETRI-nB. •

3 Main Results

3.1 Definitions

The following definition synthesizes the main features of ETRI-nB-pc, as described above. The main differences are that: (i) we do not suppose that the real-valued functions g_i are given beforehand and (ii) we do not use additive weights combined with a threshold to determine the winning coalitions. This is in line with the analysis in Bouyssou and Marchant (2007a).

Definition 6

We say that a partition $\langle \mathcal{A}, \mathcal{U} \rangle$ has a representation in Model (E) if:

- for all $i \in N$, there is a semiorder S_i on X_i (with asymmetric part P_i and symmetric part I_i),
- for all $i \in N$, there is a strict semiorder V_i on X_i that is included in P_i and is the asymmetric part of a semiorder U_i ,
- (S_i, U_i) is a *homogeneous* nested chain of semiorders and $W_i = S_i^{wo} \cap U_i^{wo}$ is a weak order that is compatible with both S_i and U_i ,
- there is a set of subsets of attributes $\mathcal{F} \subseteq 2^N$ such that, for all $I, J \in 2^N$, $[I \in \mathcal{F} \text{ and } I \subseteq J] \Rightarrow J \in \mathcal{F}$,
- there is a binary relation S on X (with symmetric part I and asymmetric part P) defined by

$$x S y \Leftrightarrow [S(x, y) \in \mathcal{F} \text{ and } V(y, x) = \emptyset],$$

- there is a set $\mathcal{P} = \{p^1, \dots, p^k\} \subseteq X$ of k limiting profiles, such that for all $p, q \in \mathcal{P}$, $\text{Not}[p P q]$,

such that

$$x \in \mathcal{A} \Leftrightarrow \begin{cases} x S p & \text{for some } p \in \mathcal{P} \text{ and} \\ \text{Not}[q P x] & \text{for all } q \in \mathcal{P}, \end{cases} \quad (E)$$

where

$$S(x, y) = \{i \in N : x_i S_i y_i\},$$

and

$$V(x, y) = \{i \in N : x_i V_i y_i\}.$$

We then say that $\langle (S_i, V_i)_{i \in N}, \mathcal{F}, \mathcal{P} \rangle$ is a representation of $\langle \mathcal{A}, \mathcal{U} \rangle$ in Model (E) . Model (E^c) is the particular case of Model (E) , in which there is a representation that shows no discordance effects, i.e., in which all relations V_i are empty. Model (E^u) is the particular case of Model (E) , in which there is a representation that requires unanimity, i.e., such that $\mathcal{F} = \{N\}$. \lrcorner

(E)	$\langle (S_i, V_i)_{i \in N}, \mathcal{F}, \mathcal{P} \rangle$	General model
(E^c)	$\langle (S_i, \emptyset)_{i \in N}, \mathcal{F}, \mathcal{P} \rangle$	Based on concordance
(E^u)	$\langle (S_i, \emptyset)_{i \in N}, \mathcal{F} = \{N\}, \mathcal{P} \rangle$	Based on unanimity

Table 1: Model (E) and its variants.

Table 1 summarizes the models defined above. It should be clear that (E) is closely related to ETRI-nB-pc. It does not use criteria but uses attributes, as is traditional in conjoint measurement. Moreover, it does not use an additive weighting scheme combined with a threshold to determine winning coalitions but uses instead a general family \mathcal{F} of subsets of attributes that is compatible with inclusion (see also Bouyssou and Marchant, 2007a).

Remark 7

It is clear that Model (E^u) is a particular case of Model (E^c) : if unanimity is required to have $x S y$, the veto relations V_i play no role and can always be taken to be empty. \bullet

Remark 8

Let us investigate in detail the obvious links existing between Model (E) and ETRI-nB-pc. Deeper links will be noticed after the characterization of Model (E) .

It is clear that Model (E^u) is a particular case of ETRI-nB-pc that is obtained taking the cutting level to be 1 and, on all criteria, the preference and indifference threshold to be equal. Since unanimity is required, veto thresholds play no role.

The relations between Model (E^c) and ETRI-nB-pc are more complex. On the one hand, Model (E) generalizes ETRI-nB-pc because winning coalitions are not necessarily obtained with an additive weighting scheme. On the other hand, Model (E^c) does not consider a “weak preference” zone as in ETRI-nB-pc. This is obtained in ETRI-nB-pc when preference and indifference thresholds are equal.

Finally, the relations between Model (E) and ETRI-nB-pc are even more complex. This is because, in model (E), discordance occurs in an “all or nothing” manner, following what was done in Bouyssou and Marchant (2007a,b). Even if this is clearly in the spirit of ETRI-nB-pc, the modelling of discordance is more complex with the original way of building the outranking relation. •

The following lemma takes note of elementary consequences of the fact that (S_i, U_i) is a *homogeneous* nested chain of semiorders.

Lemma 9

Let $\langle \mathcal{A}, \mathcal{U} \rangle$ be a twofold partition of X . If $\langle \mathcal{A}, \mathcal{U} \rangle$ is representable in (E) then, for all $a = (a_i, a_{-i}), b = (b_i, b_{-i}) \in X$, all $i \in N$ and all $c_i \in X_i$,

$$a S b \text{ and } b_i W_i c_i \Rightarrow a S (c_i, b_{-i}), \quad (1a)$$

$$a P b \text{ and } b_i W_i c_i \Rightarrow a P (c_i, b_{-i}), \quad (1b)$$

$$a S b \text{ and } c_i W_i a_i \Rightarrow (c_i, a_{-i}) S b, \quad (1c)$$

$$a P b \text{ and } c_i W_i a_i \Rightarrow (c_i, a_{-i}) P b, \quad (1d)$$

where W_i denotes a weak order that is compatible with the homogeneous nested chain of semiorders (S_i, U_i) .

PROOF

Let $a' = (c_i, a_{-i})$ and $b' = (c_i, b_{-i})$. Let us show that (1a) holds. Suppose that $a S b$, so that $S(a, b) \in \mathcal{F}$ and $V(b, a) = \emptyset$. Because $b_i W_i c_i$, we know that $S(a, b') \supseteq S(a, b)$. Hence, we have $S(a, b') \in \mathcal{F}$. Similarly, we know that $V(b, a) = \emptyset$, so that $\text{Not}[b_i V_i a_i]$. It is therefore impossible that $c_i V_i a_i$ since $b_i W_i c_i$ would imply $b_i V_i a_i$, a contradiction. Hence, $V(b', a) = \emptyset$ and we have $a S b'$.

Let us show that (1b) holds. Because $a P b$ implies $a S b$, we know from (1a) that $a S b'$. Suppose now that $b' S a$ so that $S(b', a) \in \mathcal{F}$ and $V(a, b') = \emptyset$. Because $b_i W_i c_i$, $c_i S_i a_i$ implies $b_i S_i a_i$, so that $S(b, a) \supseteq S(b', a)$, implying $S(b, a) \in \mathcal{F}$. Similarly, we know that $V(a, b') = \emptyset$, so that $\text{Not}[a_i V_i c_i]$. It is therefore impossible that $a_i V_i b_i$, since $b_i W_i c_i$ would imply $a_i V_i c_i$, a contradiction. Hence, we must have $V(a, b) = \emptyset$, so that we have $b S a$, a contradiction.

The proof of (1c) and (1d) is similar. □

The next lemma shows that Model (E) implies linearity.

Lemma 10

Let $\langle \mathcal{A}, \mathcal{U} \rangle$ be a twofold partition of $X = \prod_{i=1}^n X_i$. If $\langle \mathcal{A}, \mathcal{U} \rangle$ has a representation in Model (E) then it is linear.

PROOF

Suppose that we have $(x_i, a_{-i}) \in \mathcal{A}$, $(y_i, b_{-i}) \in \mathcal{A}$. We have either $x_i W_i y_i$ or $y_i W_i x_i$. Suppose that $x_i W_i y_i$. Because $(y_i, b_{-i}) \in \mathcal{A}$, we know that $(y_i, b_{-i}) S p$, for some $p \in \mathcal{P}$, and $\text{Not}[q P (y_i, b_{-i})]$ for all $q \in \mathcal{P}$, Lemma 9 implies that $(x_i, b_{-i}) S p$ and $\text{Not}[q P (x_i, b_{-i})]$ for all $q \in \mathcal{P}$. Hence, $(x_i, b_{-i}) \in \mathcal{A}$. The case $y_i W_i x_i$ is similar: we start with $(x_i, a_{-i}) \in \mathcal{A}$ to conclude that $(y_i, a_{-i}) \in \mathcal{A}$. Hence, linearity holds. \square

In view of Lemma 10, we therefore know from Lemma 2 that in Model (E) there is, on each attribute $i \in N$, a weak order \succsim_i on X_i that is compatible with the partition $\langle \mathcal{A}, \mathcal{U} \rangle$.

Convention

For the analysis of $\langle \mathcal{A}, \mathcal{U} \rangle$ on $X = \prod_{i=1}^n X_i$, it is not useful to keep in X_i elements that are equivalent w.r.t. the equivalence relation \sim_i . Indeed, if $x_i \sim_i y_i$ then $(x_i, a_{-i}) \in \mathcal{A}$ iff $(y_i, a_{-i}) \in \mathcal{A}$ (see Lemma 1).

In order to simplify the analysis, it is not restrictive to suppose that we work with X_i/\sim_i instead of X_i and, thus, on $\prod_{i=1}^n [X_i/\sim_i]$ instead of $\prod_{i=1}^n X_i$. This amounts to supposing that the equivalence \sim_i becomes the identity relation. *We systematically make this hypothesis below.* This is w.l.o.g. since the properties of a partition on $\prod_{i=1}^n [X_i/\sim_i]$ can immediately be extended to a partition on $\prod_{i=1}^n X_i$ (see Lemma 1) and is done for convenience only. In order to simplify notation, we suppose below that we are dealing with partitions on $\prod_{i=1}^n X_i$ for which all relations \sim_i are trivial. Our convention implies that each relation \succsim_i is antisymmetric, so that the sets X_i are linearly ordered by \succsim_i .

Let us define the relation \succsim on X letting, for all $x, y \in X$,

$$x \succsim y \Leftrightarrow x_i \succsim_i y_i, \text{ for all } i \in N.$$

It is clear that the relation \succsim plays the role of a dominance relation in our framework. It is a partial order on X , being reflexive, antisymmetric, and transitive. This partial order is obtained as a “direct product of chains” (the relations \succsim_i on each X_i) as defined in Caspard, Leclerc, and Monjardet (2012, p. 119).

Before we turn to our main results, it will be useful to take note of a few elementary observations about maximal and minimal elements in partially ordered sets (posets), referring to Davey and Priestley (2002), for more details.

3.2 Minimal and maximal elements in posets

Let T be a binary relation on a set Z . An element $x \in B \subseteq Z$ is maximal (resp. minimal) in B for T if there is no $y \in B$ such that $y T^\alpha x$ (resp. $x T^\alpha y$), where T^α denotes the asymmetric part of T . The set of all maximal (resp. minimal) elements in $B \subseteq Z$ for T is denoted by $\text{Max}(T, B)$ (resp. $\text{Min}(T, B)$).

For the record, the following proposition recalls some well-known facts about maximal and minimal elements of partial orders on finite sets (Davey and Priestley, 2002, p. 16). We sketch its proof in Appendix B for completeness.

Proposition 11

Let T be a partial order (i.e., a reflexive, antisymmetric and transitive relation) on a nonempty set Z . Let B be a finite nonempty subset of Z . Then the set of maximal elements, $\text{Max}(T, B)$, and the set of minimal elements, $\text{Min}(T, B)$, in B for T are both nonempty. For all $x, y \in \text{Max}(T, B)$ (resp. $\text{Min}(T, B)$) we have $\text{Not}[x T^\alpha y]$. Moreover, for all $x \in B$, there is $y \in \text{Max}(T, B)$ and $z \in \text{Min}(T, B)$ such that $y T x$ and $x T z$.

We will apply the above proposition to the finite subset \mathcal{A} of the set $X = \prod_{i=1}^n X_i$, partially ordered by \succsim .

3.3 A characterization of Model (E)

We know that \succsim is a partial order on $X = \prod_{i=1}^n X_i$. Because $\langle \mathcal{A}, \mathcal{U} \rangle$ is a twofold partition of X , we know that $\mathcal{A} \neq \emptyset$. Because we have supposed X to be finite, so is \mathcal{A} . Hence, we can apply Proposition 11 to conclude that the set $\mathcal{A}_* = \text{Min}(\succsim, \mathcal{A})$ is nonempty.

We are now fully equipped to present our main result.

Theorem 12

Let $X = \prod_{i=1}^n X_i$ be a finite set and $\langle \mathcal{A}, \mathcal{U} \rangle$ be a twofold partition of X . The partition $\langle \mathcal{A}, \mathcal{U} \rangle$ has a representation in Model (E) iff it is linear. This representation can always be taken to be $\langle (\succsim_i, V_i = \emptyset)_{i \in N}, \mathcal{F} = \{N\}, \mathcal{P} = \mathcal{A}_ \rangle$.*

PROOF

We know from Lemma 10 that Model (E) implies linearity. Let us prove the converse implication. Take, for each $i \in N$, $S_i = \succsim_i$ and $V_i = \emptyset$. Take $\mathcal{F} = \{N\}$. Hence, we have $S = \succsim$. Take $\mathcal{P} = \mathcal{A}_*$. Using Proposition 11, we know that \mathcal{A}_* is nonempty and that, for all $p, q \in \mathcal{A}_*$, we have $\text{Not}[p \succ q]$. Hence, taking $\mathcal{P} = \mathcal{A}_*$ leads to an admissible set of profiles in Model (E).

If $x \in \mathcal{A}$, we use Proposition 11 to conclude that there is $y \in \mathcal{A}_*$ such that $x \succsim y$, so that we have $x \succsim p$, for some $p \in \mathcal{P}$. Suppose now that, for some $q \in \mathcal{P}$, we have $q \succ x$. Using the fact that \succsim is a partial order, we obtain $q \succ p$,

contradicting the fact that $p, q \in \mathcal{A}_*$, in view of Proposition 11. Suppose now that $x \in \mathcal{U}$. Supposing that $x \succsim p$, for some $p \in \mathcal{P} = \mathcal{A}_*$, would lead to $x \in \mathcal{A}$, a contradiction. This completes the proof. \square

Remark 13

In the representation in Model (E) built in Theorem 12, the relation S is a partial order. When this is so, the condition stating that $\text{Not}[q P x]$ for all $q \in \mathcal{P}$ is automatically verified. Indeed, suppose that, for some $q \in \mathcal{P}$ and some $x \in \mathcal{A}$, we have $q P x$. Because $x \in \mathcal{A}$, there is $p \in \mathcal{P}$ such that $x S p$. Transitivity leads to $q P p$, violating the condition on the set of profiles. \bullet

Remark 14

Under our convention that \succsim_i is antisymmetric, for all $i \in N$, it is clear that, if we are only interested in representations with $\mathcal{F} = \{N\}$, the set \mathcal{P} must be taken equal to \mathcal{A}_* . Hence, the representation built above is unique, under our convention about antisymmetry and the constraint that $\mathcal{F} = \{N\}$. Without the constraint that $\mathcal{F} = \{N\}$, uniqueness does not obtain any more, as shown by simple examples. Since this is not important for our purposes, we do not investigate this point further in this text. \bullet

3.4 Example

We illustrate Theorem 12 with the example below.

Example 15

Let $X = \prod_{i=1}^3 X_i$ with $X_1 = X_2 = X_3 = \{39, 37, 34, 30, 25\}$. Hence, X contain $5^3 = 125$ objects.

Define the twofold partition $\langle \mathcal{A}, \mathcal{U} \rangle$ letting:

$$(x_1, x_2, x_3) \in \mathcal{A} \Leftrightarrow x_1 + x_2 + x_3 \geq 106.$$

In this twofold partition, the set \mathcal{A} contains 32 objects, while \mathcal{U} contains the remaining 93 objects.

It is easy to check that all attributes are influential for this partition and that On each attribute $i \in N$, it is easy to check that we have $39 \succ_i 37 \succ_i 34 \succ_i 30 \succ_i 25$. For instance, for attribute 1, we have:

$$\begin{aligned} (39, 37, 30) \in \mathcal{A} & \quad (37, 37, 30) \notin \mathcal{A}, \\ (37, 39, 30) \in \mathcal{A} & \quad (34, 39, 30) \notin \mathcal{A}, \\ (34, 39, 34) \in \mathcal{A} & \quad (30, 39, 34) \notin \mathcal{A}, \\ (30, 39, 37) \in \mathcal{A} & \quad (25, 39, 37) \notin \mathcal{A}. \end{aligned}$$

This twofold partition has an obvious representation in Model (*Add*). Hence it is linear and also has a representation in Model (*E*). Considering the representation built in Theorem 12 with $S_i = \succsim_i$, $V_i = \emptyset$, $\mathcal{P} = \mathcal{A}_*$ and $\mathcal{F} = \{N\}$, we obtain⁹ a representation that uses the following 12 profiles:

$$\begin{array}{cccc}
(37, 37, 34) & (39, 34, 34) & (39, 37, 30) & (37, 30, 39) \\
(37, 34, 37) & (34, 39, 34) & (39, 30, 37) & (30, 39, 37) \\
(34, 37, 37) & (34, 34, 39) & (37, 39, 30) & (30, 37, 39).
\end{array} \tag{2}$$

It is clear that these twelve profiles are pairwise incomparable w.r.t. $S = \succsim$.

For instance, the object (39, 30, 39) belongs to \mathcal{A} , because $39 + 30 + 39 = 108 \geq 106$. This object outranks (meaning here, dominates) the two profiles (39, 30, 37) and (37, 30, 39), but no other. \diamond

4 Remarks and extensions

4.1 Positioning Model (*E*) w.r.t. existing models

Theorem 12 gives a simple characterization of Model (*E*). It makes no restriction on the size of the set of profiles \mathcal{P} , except that it is finite.

The proof of Theorem 12 builds a representation of any partition $\langle \mathcal{A}, \mathcal{U} \rangle$ satisfying linearity in a special case of Model (*E*), Model (E^u). This shows that Models (E^u) and (*E*) are equivalent. Because, (E^u) is a particular case of Model (E^c), this also shows that Models (*E*), (E^u) and (E^c) are equivalent.

In view of Proposition 3, Model (*E*) is equivalent to Model (*D1*) and, hence, to Model (*D2*).

Because Model (*D2*) contains Model (*Add*) as a particular case, the same is true for Model (*E*).

We summarize our observations in the following.

Proposition 16

1. Models (*E*), (E^c), and (E^u) are equivalent.
2. Models (*E*), (*D1*), and (*D2*) are equivalent.
3. Model (*Add*) is a particular case of Model (*E*) but not vice versa.

The above proposition allows us to position rather precisely Model (*E*) within the family of all sorting models.

⁹We omit details and the reader is invited to check this example using, e.g., his/her favorite spreadsheet software.

Remark 17

We can now sharpen the observations made above in Remark 8. We know that Model (E) is equivalent to Model (E^u) . These models are characterized by Linearity. But all partitions obtained with the original method ETRI-nB-pc satisfy Linearity. This is because the way the outranking is built makes it compatible with the dominance relation generated by the family of criteria (see, e.g., Bouyssou and Marchant, 2015, p. 202).

Hence, the above result shows that a partition can be obtained with ETRI-nB-pc iff it can be obtained with Model (E^u) (and, hence, with Model (E)). •

4.2 The condition $\mathcal{F} = \{N\}$

The reader may be perplexed by the fact that the proof of Theorem 12 builds a representation in Model (E) in which $\mathcal{F} = \{N\}$. This is indeed a very particular form of representation. With the example below, we show that there are “reasonable” partitions for which the *only* representation in Model (E) has $\mathcal{F} = \{N\}$. Hence, the situation with $\mathcal{F} = \{N\}$ is perhaps not as bizarre as one may think.

Example 18

The example has $n = 4$ and $X_1 = X_2 = X_3 = X_4 = \{0, 5, 10\}$. We let $\mathcal{A} = \{x \in X : \sum_{i=1}^4 x_i \geq 30\}$. There are $3^4 = 81$ objects in X , 15 are in \mathcal{A} , while 66 are in \mathcal{U} .

Observe first that, on all attributes, we have $10 \succ_i 5 \succ_i 0$. Indeed, with $i = 1$, we have:

$$\begin{aligned} (10, 0, 10, 10) &\in \mathcal{A}, & (5, 0, 10, 10) &\in \mathcal{U}, \\ (5, 5, 10, 10) &\in \mathcal{A}, & (0, 5, 10, 10) &\in \mathcal{U}. \end{aligned}$$

The same relations clearly hold on all attributes since the problem is symmetric.

This partition clearly has a representation in Model (E) with $\mathcal{F} = \{N\}$ and a set of profiles consisting of all 10 objects in the class 30 (i.e., having a sum of components equal to 30). By construction, these 10 profiles are not linked by dominance (this is a representation in model (E^c)).

Our objective is to try obtaining a representation in Model (E) using a set \mathcal{F} that is not reduced to $\{N\}$. Notice first that bringing the veto relations into play will not help us do so. Indeed, it is easy to check that if a representation exists in model (E) , a representation exists in Model (E^c) (because whatever x_i , we can find a_{-i} such that $(x_i, a_{-i}) \in \mathcal{A}$). Hence, let us try to find a representation in Model (E^c) .

This clearly excludes to take any object in the class 30 as a profile. Indeed, a family \mathcal{F} that is not reduced to $\{N\}$ would then imply that some object in a class strictly lower than 30 belongs to \mathcal{A} , which is false. Hence, we must take as profiles objects belonging to the class 35 or 40.

Because profiles cannot dominate themselves, if we take the object $(10, 10, 10, 10)$ as a profile, it must be the only one. We know that $(10, 10, 5, 5) \in \mathcal{A}$. Hence, we must have $\{1, 2\} \in \mathcal{F}$. This is contradictory. Indeed, since $\{1, 2\} \in \mathcal{F}$, we should have $(10, 10, 0, 0) \in \mathcal{A}$, a contradiction.

Hence the set of profiles must consist exclusively of objects belonging to the class 35.

Suppose that there is a unique profile, e.g., $(10, 10, 10, 5)$. It is clear that the set $\{1, 2, 3\}$ must be included in all elements of \mathcal{F} (otherwise we would have an object in the class 25 belonging to \mathcal{A}). Because $(10, 10, 10, 0) \in \mathcal{A}$, it must be true that $\{1, 2, 3\}$ is an element of \mathcal{F} , which must therefore be equal to $\{\{1, 2, 3\}, \{1, 2, 3, 4\}\}$. This is contradictory since we know that $(0, 10, 10, 10) \in \mathcal{A}$. It is easy to see that, the problem being symmetric, it is therefore impossible to have a representation using a single profile from the class 35.

A similar reasoning can be made if we consider the cases of two or three profiles from the class 35 as profiles.

Suppose finally that we choose all four profiles from the class 35: $(5, 10, 10, 10)$, $(10, 5, 10, 10)$, $(10, 10, 5, 10)$, and $(10, 10, 10, 5)$. Using the same reasoning as above, the set \mathcal{F} must contain the sets $\{2, 3, 4\}$, $\{1, 3, 4\}$, $\{1, 2, 4\}$, and $\{1, 2, 3\}$, since $(0, 10, 10, 10)$, $(10, 0, 10, 10)$, $(10, 10, 0, 10)$ and $(10, 10, 10, 0)$ are all in \mathcal{A} . But this is contradictory since this would imply that $(0, 5, 10, 10) \in \mathcal{A}$ (since $(10, 5, 10, 10)$ is a profile and $\{2, 3, 4\} \in \mathcal{F}$).

Therefore, the only possible representation of this partition in Model (E) must use as profiles all 10 elements in the class 30 together with $\mathcal{F} = \{N\}$. \diamond

Hence, representations with $\mathcal{F} = \{N\}$ are sometimes quite useful and even, may be the only possible ones. As shown below, it is quite easy to transform a representation based on concordance into a unanimous representation.

4.3 Obtaining $\mathcal{F} = \{N\}$

Any representation in Model (E) can easily be transformed into a representation with $\mathcal{F} = \{N\}$. This is a direct consequence of Theorem 12. When the representation is without discordance, we show below that the process of building \mathcal{A} and then deriving \mathcal{A}_* can be avoided.

Suppose we know a representation $\langle (S_i, \emptyset)_{i \in N}, \mathcal{F}, \mathcal{P} \rangle$ of the partition $\langle A, U \rangle$ in (E) (in fact, in (E^c)) and that $\mathcal{F} \neq \{N\}$.

We want to find a representation such that $\mathcal{F} = \{N\}$. Theorem 12 ensures that such a representation exists. It can be built quite efficiently, independently of the construction used in Theorem 12.

Let $\mathcal{F}_* = \text{Min}(\mathcal{F}, \supseteq)$. For each $i \in N$, let x_i^0 be the unique element in X_i that

is minimal for the linear order \succsim_i . Define x^0 accordingly. Moreover, let:

$$\mathcal{P}' = \{(x_{-I}^0, p_I), \text{ for all } p \in \mathcal{P} \text{ and } I \in \mathcal{F}_*\}.$$

It is clear that $\langle (S_i, \emptyset)_{i \in N}, \{N\}, \mathcal{P}' \rangle$ is a representation of the partition $\langle A, U \rangle$ in (E) .

4.4 Variable set of winning coalitions \mathcal{F}

A rather natural generalization of Model (E) , called (\tilde{E}) , is as follows. Instead of considering a single family of winning coalitions \mathcal{F} that is used to build the relation S and compare each alternative in X to all profiles in \mathcal{P} , we could use a family \mathcal{F}^p that would be specific to each profile $p \in \mathcal{P}$, with a relation S^p that now depends on the profile.

The analysis of Model (\tilde{E}) is easy. It is simple to check that Model (\tilde{E}) implies linearity. Indeed, for each relation S^p , Lemma 9 holds and, hence, linearity cannot be violated. This shows that Model (\tilde{E}) is a particular case of Model (E) and, hence, is equivalent to it.

4.5 More than two categories

Our analysis of Model (E) can easily be extended to cover the case of an arbitrary number of categories. Because this would require the introduction of a rather cumbersome framework, without adding much to the analysis of two categories, we do not formalize this point. We briefly indicate how this can be done, leaving the details to the interested readers.

Linearity has been generalized to cope with more than two categories. This is done in Słowiński et al. (2002) and Bouyssou and Marchant (2007b). The intuition behind this generalization is simple. It guarantees that there is a weak order on each attribute that is compatible with the ordered partition.

When X is finite, this condition is necessary and sufficient to characterize the obvious generalization of Model $(D1)$ that uses more than one threshold, instead of the single threshold 0 (see, e.g., Bouyssou and Marchant, 2007b, Prop. 7, p. 250). Moreover, it is easy to check that this condition is satisfied by the natural generalization of Model (E) that uses more than two categories (this involves working with of a set of profiles \mathcal{P} for each of the induced twofold partitions).

Now, the technique used in the proof of Theorem 12 easily allows one to define a family of profiles for each of the induced twofold partitions. It just remains to check that these families of profiles satisfy the constraints put forward in Fernández et al. (2017, Condition 1, p. 216). This is immediate.

5 ETRI-nB, pseudo-disjunctive and Model (\overline{E})

Up to this point, we have investigated the properties of Model (E) which is closely linked to ETRI-nB-pc. We now turn to ETRI-nB-pd.

5.1 Definitions

We define Model (\overline{E}) that is to ETRI-nB-pd what Model (E) is to ETRI-nB-pc.

Definition 19

Model (\overline{E}) is defined exactly as Model (E) , except that now, we have:

$$x \in \mathcal{U} \Leftrightarrow \begin{cases} p P x & \text{for some } p \in \mathcal{P} \quad \text{and} \\ \text{Not}[x P q] & \text{for all } q \in \mathcal{P}. \end{cases} \quad (\overline{E})$$

Models (\overline{E}^c) and (\overline{E}^u) are defined as above. ┘

It is simple to check that in Model (\overline{E}) all profiles $p \in \mathcal{P}$ must belong to \mathcal{A} . It is also simple to check, that Lemma 9 holds for Model (\overline{E}) . Indeed, the outranking relation S is built in Model (\overline{E}) as it is built in Model (E) . This leads to the following observation¹⁰ that is proved exactly as Lemma 10.

Lemma 20

Let $\langle \mathcal{A}, \mathcal{U} \rangle$ be a twofold partition of $X = \prod_{i=1}^n X_i$. If $\langle \mathcal{A}, \mathcal{U} \rangle$ has a representation in Model (\overline{E}) then it is linear.

Hence, combining Lemma 20 with Theorem 12, we know that Model (\overline{E}) is a particular case of Model (E) . This is illustrated by the following example.

Example 21

As in Example 15 Let $X = \prod_{i=1}^3 X_i$ with $X_1 = X_2 = X_3 = \{39, 37, 34, 30, 25\}$.

We take the 12 profiles given in Eq. (2). We now apply model (\overline{E}) , still using a unanimous model $(\mathcal{F} = \{N\})$.

It is easy to check that in the resulting partition $\langle \mathcal{A}, \mathcal{U} \rangle$, the set \mathcal{A} contains 35 objects consisting of the 32 objects in \mathcal{A} in Example 15 plus the following three objects!

$$(39, 39, 25) \quad (39, 25, 39) \quad (25, 39, 30). \quad (3)$$

¹⁰It might also be interesting to observe that, when the relation S is complete, Models (E) and (\overline{E}) are equivalent. Also note that (the variant of) Model (Add) for which we have $x \in \mathcal{A} \Leftrightarrow \sum_{i=1}^n u_i(x_i) \geq 0$, can be obtained in Model (E) using the complete relation S such that $x S y \Leftrightarrow \sum_{i=1}^n u_i(x_i) \geq \sum_{i=1}^n u_i(y_i)$ and a unique profile p such that $\sum_{i=1}^n u_i(p_i) = 0$. This shows that Model (\overline{E}) contains (the variant of) Model (Add) as a particular case.

These three objects were in \mathcal{U} in Example (2) since

$$39 + 39 + 25 = 103 < 106.$$

These three objects now belong to \mathcal{A} . Indeed, it is clear that there is none of the 12 profiles given in Eq. (2) strictly dominates any of the above three alternatives.

Notice finally that the partition that we have obtained using the 12 profiles given in Eq. (2) together with $\mathcal{F} = \{N\}$ and Model (\overline{E}) also has a representation in Model (E) , since it is linear. It is easy to check that the representation in Model (E) as built in Theorem 12 consists in taking $\mathcal{F} = \{N\}$ together with the 12 profiles in Eq. (2) to which we add the three profiles in Eq. (3). \diamond

5.2 Remarks

We have observed above that Models (E) , (E^c) , and (E^u) are equivalent. The situation is far more complex with Model (\overline{E}) .

The following example shows that Model (\overline{E}^u) is not equivalent to Model (\overline{E}^c) .

Example 22

Let $N = \{1, 2, 3, 4\}$ and $X_i = \{0, 1\}$ for all $i \in N$, so that X has 16 elements. Consider the partition $\langle \mathcal{A}, \mathcal{U} \rangle$ such that $\mathcal{A} = \{1111, 1110, 1101, 0111, 1011, 1100, 0011\}$ and $\mathcal{U} = \{1010, 0110, 1001, 0101, 1000, 0100, 0010, 0001, 0000\}$, abusing notation in an obvious way. It is simple to check that all attributes are influential for $\langle \mathcal{A}, \mathcal{U} \rangle$ and that, for all $i \in N$, we have $1_i \succ_i 0_i$. Notice that we have $\mathcal{A}_* = \text{Min}(\mathcal{A}, \succ) = \{1100, 0011\}$ and $\mathcal{U}^* = \text{Max}(\mathcal{U}, \succ) = \{1010, 0110, 1001, 0101\}$.

Following Theorem 12, this partition can be represented in (E^u) , taking $\mathcal{P} = \mathcal{A}_* = \text{Min}(\mathcal{A}, \succ) = \{1100, 0011\}$ and $S = \succ$.

This partition can be represented in Model (\overline{E}^c) with a set of minimal winning coalitions $\mathcal{F}_* = \text{Min}(\mathcal{F}, \supseteq) = \{12, 34\}$, $S_i = \succ_i$, for all $i \in N$, and $\mathcal{P} = \{1111\}$.

Let us show that this partition *does not have* a representation in Model (\overline{E}^u) . First observe that because each X_i has only two elements, we must take $S_i = \succ_i$, for all $i \in N$. Because we are looking for a representation with $\mathcal{F} = \{N\}$, we have $S = \succ$, so that $P = \succ$.

For every element in \mathcal{U} , there must be a profile $p \in \mathcal{P} \subseteq \mathcal{A}$ that is strictly preferred to it (using the relation P). The candidates for being in \mathcal{P} are thus $\{1110, 1101, 1011, 0111, 1111\}$, i.e., all elements in \mathcal{A} , except 1100 and 0011. Indeed, e.g., it is clear that 1100 (resp. 0011) is not strictly preferred to 0110. It is therefore impossible to take \mathcal{P} equal to $\{1100\}$, $\{0011\}$ or to $\{1100, 0011\}$. But it is impossible to add to any of these sets another element from \mathcal{A} because this would violate the requirement that for all $p, q \in \mathcal{P}$, we have $\text{Not}[p P q]$.

If $1111 \in \mathcal{P}$, then it must be the unique profile. But this does not lead to the desired partition in Model (\overline{E}^u) because 1111 is strictly preferred to all other elements in X .

All other candidate profiles are strictly preferred to one of the elements in \mathcal{A}_* . Conversely, the elements in \mathcal{A}_* are not strictly preferred to any of the candidates. Therefore, putting some of these candidates in \mathcal{P} will not yield the right partition, using Model (\bar{E}^u) . \diamond

The situation is even more complex since the following example shows that there are linear partitions that cannot be represented in Model (\bar{E}^c) , together with the constraint that $S_i = \succsim_i$, for all $i \in N$.

Example 23

Let $n = 4$ and $X_1 = X_2 = X_3 = \{2, 1, 0\}$ and $X_4 = \{0, 1\}$, so that X has 54 elements. Consider the partition $\langle \mathcal{A}, \mathcal{U} \rangle$ such that $\mathcal{A} = \{2221, 2211, 2121, 1221, 2111, 1211, 1121, 1111, 2220\}$. It is easy to check that all attributes are influential for $\langle \mathcal{A}, \mathcal{U} \rangle$ and that, for all $i \in \{1, 2, 3\}$, we have $2_i \succ_i 1_i \succ_i 0_i$, while $1_4 \succ_4 0_4$. Hence, the partition is linear. It can be obtained using Model (E^u) with $S_i = \succsim_i$, for all $i \in N$, $\mathcal{P} = \mathcal{A}_* = \{1111, 2220\}$ and $\mathcal{F} = \{N\}$. Let us now show that this partition cannot be obtained using (\bar{E}^c) together with the constraint that $S_i = \succsim_i$, for all $i \in N$.

Note that attribute 4 is only included for the sake of having $2_i \succ_i 1_i$, for $i \in \{1, 2, 3\}$. All objects that we will compare below have 1 on attribute 4. We therefore drop this attribute in what follows. Once this is done, the problem becomes symmetric in the three remaining attributes¹¹.

If we take $\mathcal{F} = 2^N \setminus \{\emptyset\}$, it is impossible to recover the partition whatever the choice of the set of profiles \mathcal{P} , since, e.g., 220 bears S to all elements in \mathcal{A} .

If $\{1\} \in \mathcal{F}$, it is impossible to recover the partition, whatever the choice of the set of profiles \mathcal{P} , since 200 bears S to all elements in \mathcal{A} . A similar argument applies to $\{2\} \in \mathcal{F}$ (020 bears S to all elements in \mathcal{A}) and to $\{3\} \in \mathcal{F}$ (002 bears S to all elements in \mathcal{A}).

If $\{1, 2\} \in \mathcal{F}$, it is impossible to recover the partition, whatever the choice of the set of profiles \mathcal{P} , since 220 bears S to all elements in \mathcal{A} . A similar argument applies to $\{1, 3\} \in \mathcal{F}$ (202 bears S to all elements in \mathcal{A}) and to $\{2, 3\} \in \mathcal{F}$ (022 bears S to all elements in \mathcal{A}).

The only remaining choice consists in taking $\mathcal{F} = \{N\}$. If $222 \in \mathcal{P}$ then we must take $\mathcal{P} = \{222\}$ (because 222 bears P to all other elements in \mathcal{A}). But this

¹¹This is legitimate. Indeed, let $X' = X_1 \times X_2 \times X_3 \times \{1\}$, $\mathcal{A}' = \mathcal{A} \cap X'$, $\mathcal{U}' = \mathcal{U} \cap X'$. Note that $\mathcal{A} = \mathcal{A}' \cup \{2220\}$. Suppose that a representation of $\langle \mathcal{A}', \mathcal{U}' \rangle$ exists in model (\bar{E}^c) that uses a set \mathcal{P}' and a family \mathcal{F}' . It is not restrictive to suppose that, for all $I \in \mathcal{F}'$, we have $4 \in I$ since elements not containing attribute 4 would be useless. To obtain a representation of $\langle \mathcal{A}, \mathcal{U} \rangle$, it suffices to add the element 2220 to \mathcal{P}' and to add the set $\{1, 2, 3\}$ to \mathcal{F}' . Conversely, if a representation of $\langle \mathcal{A}, \mathcal{U} \rangle$ exists, it is easy to check there is a representation in which, $2220 \in \mathcal{P}$, $\{1, 2, 3\} \in \mathcal{F}$ and all other elements of \mathcal{F} include attribute 4. Hence, there is a representation of $\langle \mathcal{A}', \mathcal{U}' \rangle$ iff there is a representation of $\langle \mathcal{A}, \mathcal{U} \rangle$.

does not give the desired result. If $111 \in \mathcal{P}$ then we must take $\mathcal{P} = \{111\}$, since all other elements in \mathcal{A} bear P to 111. But, again, this does not give the desired partition (e.g., 220 is incomparable to 111). But all remaining candidate profiles (i.e., 221, 212, 122, 211, 121, and 112) bear P to 111. Hence, if we include any of them in the set of profiles \mathcal{P} , we would obtain that $111 \in \mathcal{U}$, a contradiction.

Hence, the partition cannot be obtained in Model (\overline{E}^c) together with the constraint that $S_i = \succsim_i$, for all $i \in N$. \diamond

Remark 24

In the above example, the constraint that $S_i = \succsim_i$, for all $i \in N$, is important. Without it, the example would not show that there are linear partitions that cannot be represented in Model (\overline{E}^c) . We have $2_i \succ_i 1_i \succ_i 0_i$ and $1_4 \succ_4 0_4$. Because X_4 has only two elements, we must take $S_4 = \succsim_4$. On the other attributes, we must take the semiorder S_i so that its induced weak order coincides with \succsim_i . Hence, on each of the attributes in $\{1, 2, 3\}$, we may take $S_i = \succsim_i$, but we may also take the semiorder S_i such that $2_i P_i 0_i$, $2_i I_i 1_i$ and $1_i I_i 0_i$. Let us suppose that we make the latter choice for $i = 1, 2, 3$. It is then simple to check that the partition in Example 23 can be obtained with Model (\overline{E}^c) , taking $\mathcal{F} = \{N\}$ and $\mathcal{P} = \{222\}$. According to Roy (1996) the indifference thresholds are not part of the parameters of an aggregation technique and are rather linked to the nature of the underlying information. Hence, the constraint used above seems rather innocuous.

Note finally that in the above Example, the constraint that any two profiles should not be linked by P is also clearly important. \bullet

We know that Model (\overline{E}) is a particular case of Model (E) . We also know that Model (\overline{E}^u) is a particular case of Model (\overline{E}^c) , but not vice versa. In turn, Model (\overline{E}^c) is a particular case of Model (E) , but not vice versa. It remains to investigate whether or not Model (\overline{E}) is equivalent to Model (E) . We will leave this question open. Indeed, the extra complexity that comes with Model (\overline{E}) when compared to Model (E) is a symptom of some difficulties already discussed in Bouyssou and Marchant (2015), to which we now turn.

5.3 Comments: transposition and duality

We will devote less effort to analyze Model (\overline{E}) than what we have done for Model (E) . The following arguments explain this choice.

First, we know that Model (\overline{E}) it is a particular case of Model (E) and we have at our disposal a simple characterization of the latter.

Second, Model (\overline{E}) makes central use of the relation P . But Bouyssou and Pirlot (2015a,b) have shown that the nature of the relation P is rather different from that of the relation S in the ELECTRE methods.

Third, the analysis in Bouyssou and Marchant (2015) concerning the comparison between ETRI-B-pc and ETRI-B-pd applies, *mutatis mutandis*, to the comparison of ETRI-nB-pc and ETRI-nB-pd. Bouyssou and Marchant (2015) show that ETRI-B-pd and ETRI-B-pc do not correspond via the *transposition operation* consisting in inverting the direction of preference on all criteria and permuting \mathcal{A} and \mathcal{U} (see Almeida-Dias et al., 2010, Bouyssou and Marchant, 2015, Roy, 2002). This is due to the fact that in both versions of ETRI-B, categories are closed below. This implies that in both versions, the unique profile belongs to \mathcal{A} . This is clearly incompatible with the transposition operation. The same analysis applies to ETRI-nB-pc and ETRI-nB-pd. It is easy to check that they do not correspond via the transposition operation.

Hence, following Bouyssou and Marchant (2015), we would rather like to put forward a different conception of profiles: they are at the *frontier* between categories and it makes sense to accept that they can belong to either of them. With this conception of profiles in mind, we can define the *dual* of ETRI-nB-pc, letting:

$$x \in \mathcal{U} \Leftrightarrow \begin{cases} p S x & \text{for some } p \in \mathcal{P} \quad \text{and} \\ \text{Not}[x P p] & \text{for all } q \in \mathcal{P}. \end{cases}$$

It is easy to see that ETRI-nB-pc and its dual now correspond via the transposition operation. Moreover, the dual model is also characterized by linearity¹².

6 Discussion

Using classical tools from conjoint measurement, we have proposed a complete axiomatic characterization of ETRI-nB-pc. This characterization is simple and shows the generality of the new model. However, this simplicity is obtained when there is no restriction on the set of profiles, except that it is finite. Our results nevertheless allow us to position rather precisely ETRI-nB-pc within the larger picture of models aiming at sorting multi-attributed objects between ordered categories.

A somewhat unexpected conclusion emerges. While Bernard Roy had always championed outranking approaches as an alternative to the classical additive value function model, it turns out that the last ELECTRE method that he published before he passed away, ELECTRE TRI-nB, contains the additive value function model for sorting as a particular case. We think that this unexpected conclusion is a plea for the development of axiomatic studies in the field of decision with multiple attributes, as already advocated in Bouyssou, Perny, Pirlot, Tsoukiàs, and Vincke (1993), more than 25 years ago.

¹²If we accept the arguments in Bouyssou and Marchant (2015), it should also be noted that ETRI-nB-pc and its dual should be used *conjointly*. This would call for a conjoint characterization. We do not develop this point here.

Theorem 12 is a simple result. It leaves open a number of interesting problems that we intend to deal with in later studies. Among them, let us mention the following sets of questions.

Algorithmic questions. Is it easy to test if a partition $\langle \mathcal{A}, \mathcal{U} \rangle$ satisfies linearity? Are there efficient algorithms to find a linear partition close to a partition that is not linear? Is it easy to test whether it is possible to build a linear partition on the basis of partial information about \mathcal{A} and \mathcal{U} ? Similar questions arise, when there is a supplementary constraint on the size of the set of profiles.

Elicitation questions. Does the structure of Model (E) lead to new elicitation algorithms besides the ones envisaged for Model (D1) and its variants (see Greco, Matarazzo, Słowiński, and Stefanowski, 2001c, Sobrie, Mousseau, and Pirlot, 2019)?

Combinatorial questions. Given a set $X = \prod_{i=1}^n X_i$, can we devise (easy to evaluate) formulae for the maximal number of objects in \mathcal{A}_* (this is related to the largest antichain¹³ in a direct product of chains, see Sander, 1993, for the case of the direct product of chains of the same length)? What is the number of twofold partitions of $X = \prod_{i=1}^n X_i$ that can be represented in Model (E) (this is related to the famous problem of Dedekind numbers, see Ersek Uyanık, Sobrie, Mousseau, and Pirlot, 2017, Kahn, 2002, Kisielewicz, 1988)?

Questions linked to the number of profiles. With practical applications in mind, in order to elicit Model (E), one can use the techniques that were designed to elicit the decision rule model (See Greco, Matarazzo, and Słowiński, 1999, 2001a, Greco et al., 2001c, Greco, Matarazzo, and Słowiński, 2016, Słowiński et al., 2002, and, for a recent application of these techniques Abastante, Bottero, Greco, and Lami, 2014). Because we are dealing here with quite flexible models, these techniques are not entirely straightforward and, e.g., may lead, in the decision rule model, to a large number of rules. Hence, it is important to investigate particular cases of Model (E), in which the cardinality of the set of profiles \mathcal{P} is restricted. Unfortunately, the problem seems to be difficult. This is the subject of a companion paper that deals with these more technical issues (Bouyssou, Marchant, and Pir-

¹³In Bouyssou, Marchant, and Pirlot (2019), we give the proof of a result that has already appeared in the grey literature but for which no proof was available. It states that if the chain on X_i has m_i elements, the maximal size of an antichain in $X = \prod_{i=1}^n X_i$ partially ordered by \succsim is

$$\sum_{I \subseteq N: m_I < h-n} \binom{h - m_I - 1}{n - 1} (-1)^{|I|},$$

where $m_I = \sum_{i \in I} m_i$ and

$$h = \left\lfloor \frac{n + \sum_{i \in N} m_i}{2} \right\rfloor.$$

The reader will check that this number grows fast with the vector $(m_i)_{i \in N}$.

lot, 2020). This companion paper only analyzes the particular case of two profiles coupled with unanimity. Even in this apparently simple case, the problem is not easy. Hence, our analysis also leaves open the study of the gain of expressiveness brought by increasing the size of the set of profiles. Going from a single profile, the case studied in Bouyssou and Marchant (2007a), to an arbitrarily large number of profiles, the case implicitly studied in Section 3, leads to a *huge* gain in expressiveness. Is this gain already present when going from a single profile to a small number of profiles? This question is clearly important as a guide to the elicitation of the parameters of ETRI-nB. Our analysis of the case of two profiles coupled with unanimity shows that it is unlikely that a purely axiomatic investigation will allow us to obtain clear answers to this question. Hence, this is also a plea to combine axiomatic work with other types of work, e.g., based on computer simulations.

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Appendix: To appear as supplementary material

A Binary relations

We use a standard vocabulary for binary relations. For the convenience of the reader and in order to avoid any misunderstanding, we detail our vocabulary here. A binary relation T on a set Z is a subset of $Z \times Z$. For $x, y \in Z$, as is usual, we will often write $x T y$ instead of $(x, y) \in T$.

Let T be a binary relation on Z . We define:

- the asymmetric part T^α of T as $x T^\alpha y \Leftrightarrow [x T y \text{ and } \text{Not}[y T x]]$,
- the symmetric part T^ι of T as $x T^\iota y \Leftrightarrow [x T y \text{ and } y T x]$,
- the symmetric complement T^σ of T as $x T^\sigma y \Leftrightarrow [\text{Not}[x T y] \text{ and } \text{Not}[y T x]]$,

for all $x, y \in Z$.

A binary relation T on Z is said to be:

- (i) *reflexive* if $x T x$,
- (ii) *irreflexive* if $\text{Not}[x T x]$,
- (iii) *complete* if $x T y$ or $y T x$,
- (iv) *symmetric* if $x T y$ implies $y T x$,
- (v) *asymmetric* if $x T y$ implies $\text{Not}[y T x]$,
- (vi) *antisymmetric* if $[x T y \text{ and } y T x] \Rightarrow x = y$,
- (vii) *transitive* if $[x T y \text{ and } y T z] \Rightarrow x T z$,
- (viii) *Ferrers* if $[x T y \text{ and } z T w] \Rightarrow [x T w \text{ or } z T y]$,
- (ix) *semitransitive* if $[x T y \text{ and } y T z] \Rightarrow [x T w \text{ or } w T z]$,

for all $x, y, z, w \in Z$.

We list below a number of remarkable structures. A binary relation T on Z is said to be:

- (i) a *weak order* (or *complete preorder*) if it is complete and transitive,

- (ii) a *linear order* if it is an antisymmetric weak order,
- (iii) a *semiorder* if it is reflexive, Ferrers and semitransitive,
- (iv) a *strict semiorder* if it is irreflexive, Ferrers and semitransitive,
- (v) an *equivalence* if it is reflexive, symmetric, and transitive,
- (vi) a *partial order* if it is reflexive, antisymmetric and transitive.

Notice that a reflexive and Ferrers relation must be complete. Similarly an irreflexive and Ferrers relation must be asymmetric.

When T is an equivalence relation on Z , Z/T will denote the set of equivalence classes of T on Z . A *partition* of Z is a collection of nonempty subsets of Z that are pairwise disjoint and such that the union of the elements in this collection is Z . It is clear that, when T is an equivalence relation on Z , Z/T is a partition of Z .

When T on Z is a semiorder, its asymmetric part T^α is irreflexive, Ferrers and semitransitive, i.e., a strict semiorder.

Any Ferrers and semitransitive T on Z (which includes semiorders and strict semiorders) induces a weak order T^{wo} on Z that is defined as follows:

$$a T^{wo} b \text{ if } \forall c \in Z, [b T c \Rightarrow a T c] \text{ and } [c T a \Rightarrow c T b]. \quad (4)$$

If T is a semiorder and V is its asymmetric part, it follows that $T^{wo} = V^{wo}$. The weak order induced by a semiorder is identical to the one induced by its asymmetric part.

Let T and V be two semiorders on Z such that $T \subseteq V$. We say that (T, V) is a *nested chain* of semiorders. Let T^{wo} (resp. V^{wo}) be the weak order on Z induced by T (resp. V). If a nested chain of semiorders $T \subseteq V$ is such that the relation $T^{wo} \cap V^{wo}$ is complete (and therefore is a weak order), we say that the nested chain of semiorders (T, V) is *homogeneous* (Doignon et al., 1988).

Finally, let us note that T is a semiorder on a finite set Z iff there are a real-valued function f on Z and a positive number $s > 0$ such that, for all $a, b \in Z$, $a T b \Leftrightarrow f(a) \geq f(b) - s$

B Sketch of the proof of Proposition 11

Suppose that $\text{Max}(T, B)$ is empty. Let $x \in B$. By hypothesis, x does not belong to $\text{Max}(T, B)$. This implies that there is $w_1 \in B$ such that $w_1 T^\alpha x$. Clearly, this implies that w_1 is distinct from x , because T^α is irreflexive. But w_1 does not belong to $\text{Max}(T, B)$. This implies that there is $w_2 \in B$ such that $w_2 T^\alpha w_1$. Clearly,

this implies that w_2 is distinct from both w_1 and x . Continuing the reasoning leads to postulating the existence of a chain of elements $w_i, i \in \mathbb{N}^+$, that are all distinct (otherwise, the transitivity of T^α will lead to violate irreflexivity). This violates the finiteness of B . Hence, $\text{Max}(T, B)$ must be nonempty. The proof that $\text{Min}(T, B)$ must be nonempty is similar. The fact that, for all $x, y \in \text{Max}(T, B)$, we have $\text{Not}[x T^\alpha y]$ is clear from the definition of $\text{Max}(T, B)$. The same is clearly true with $\text{Min}(T, B)$.

Suppose now that $x \in B$ and there is no $y \in \text{Max}(T, B)$ such that $y T x$. If $x \in \text{Max}(T, B)$, the contradiction is established, because T is reflexive. Suppose, hence, that $x \notin \text{Max}(T, B)$. There is $w_1 \in B$ such that $w_1 T^\alpha x$. But it is impossible that w_1 belongs to $\text{Max}(T, B)$. This implies that there is $w_2 \in B$ such that $w_2 T^\alpha w_1$. Because, T^α is transitive, it is impossible that $w_2 \in \text{Max}(T, B)$. Because T^α is asymmetric and transitive, it is impossible that w_2 is identical to w_1 or to x . Continuing the same reasoning, leads to postulating the existence of a chain of elements $w_i, i \in \mathbb{N}^+$, that are all distinct. This violates the finiteness of B . Hence, there exists $y \in \text{Max}(T, B)$ such that $y T x$. The proof that if $x \in B$, there is $z \in \text{Min}(T, B)$ such that $x T z$ is similar. \square