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# ON THE UNIVERSAL REGULAR HOMOMORPHISM IN CODIMENSION 2

BRUNO KAHN

ABSTRACT. We point out a gap in Murre's proof of the existence of a universal regular homomorphism for codimension 2 cycles on a smooth projective variety, and offer two arguments to fill this gap.

In [11], Jacob Murre shows the existence of a universal regular homomorphism for algebraically trivial cycles of codimension 2 on a smooth projective variety over an algebraically closed field. This theorem has been largely used in the literature, most lately in [1], [7] and [2]; for example, it is essential in [2] for descending the method of Clemens and Griffiths [6] to non-algebraically closed fields, thus allowing Benoist and Wittenberg to obtain new examples of geometrically rational non-rational 3-folds.

Unfortunately its proof contains a gap, but fortunately this gap can be filled, actually by two different methods. This is the purpose of this note, which is a slight modification of a letter to Murre on December 5, 2018.

Recall the set-up, with the notation of [11]:  $V$  is a smooth projective variety over an algebraically closed field  $k$  and  $A^n(V)$  denotes the group of codimension  $n$  cycles algebraically equivalent to 0 on  $V$ , modulo rational equivalence. Following Samuel, given an abelian  $k$ -variety  $A$ , a homomorphism

$$\phi : A^n(V) \rightarrow A(k)$$

is said to be *regular* if, for any pointed smooth projective  $k$ -variety  $(T, t_0)$  and any correspondence  $Z \in CH^n(T \times V)$ , the composition

$$(1) \quad T(k) \xrightarrow{w_Z} A^n(V) \xrightarrow{\phi} A(k)$$

is induced by a morphism  $f : T \rightarrow A$ ; here  $w_Z$  is the composition

$$(2) \quad T(k) \rightarrow A_0(T) \xrightarrow{Z_*} A^n(V)$$

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where the first map sends  $t$  to  $[t] - [t_0]$ . (Note that  $f$  is then unique, by Zariski density of the rational points in  $T$ .)

Using fancy language, regular homomorphisms from  $A^n(V)$  form a category and a *universal regular homomorphism* is an initial object of this category, if it exists. This initial object is well-known to exist when  $n = 0$ ,  $n = 1$  (the Picard variety) and  $n = \dim X$  (the Albanese variety). Murre's theorem is:

**Theorem 1** ([11, Th. 1.9]). *A universal regular homomorphism  $\phi_0$  exists when  $n = 2$  for any  $V$  (of dimension  $\geq 2$ ).*

Recall the main steps of his proof. First, given a regular homomorphism  $\phi$ , its image in  $A(k)$  is given by the points of some sub-abelian variety  $A' \subseteq A$  [11, Lemma 1.6.2 i)]. From this, one deduces [11, Prop. 2.1] that  $\phi_0$  exists if and only if  $\dim A$  is bounded when  $\phi$  runs through the *surjective* regular homomorphisms. Now, Murre's key idea is to bound  $\dim A$  by the torsion of  $A^2(V)$ , which is controlled by the Merkurjev-Suslin theorem (Bloch's observation).

Let us elaborate a little on this point, to avoid the  $l$ -adic argument of *loc. cit.*: it suffices to prove that  $\phi$  induces a surjection

$$(3) \quad A^2(V)\{l\} \twoheadrightarrow A(k)\{l\}$$

for some prime  $l \neq \text{char } k$ , where  $M\{l\}$  denotes the  $l$ -primary torsion of an abelian group  $M$ : indeed,  $\text{corank } A(k)\{l\} = 2 \dim A$ . Mainly by Merkurjev-Suslin (Diagram in [11, Prop. 6.1])<sup>1</sup>,

$$\text{corank } CH^2(V)\{l\} \leq \text{corank } H_{\text{ét}}^3(V, \mathbb{Q}_l/\mathbb{Z}_l(2)) (= b_3(V))$$

so the same holds *a fortiori* for  $\text{corank } A^2(V)\{l\}$ .

Now, in [11, Lemma 1.6.2 ii)], Murre constructs an abelian variety  $B$  (pointed at 0) and a correspondence  $Z \in CH^2(B \times V)$  such that (1) is surjective for  $T = B$ . Since this map is induced by a morphism of abelian varieties sending 0 to 0 (hence a homomorphism), it restricts to a surjection

$$(4) \quad B\{l\} \twoheadrightarrow A\{l\}.$$

This allows me to explain

the gap:

A priori (4) does not imply (3), because  $w_Z$  is in general only a set-theoretic map, not a group homomorphism (see e.g. [4, Th. (3.1 a)]).

We now fix a surjective regular homomorphism  $\phi$  as above. We shall give two ways to fill this gap:

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<sup>1</sup>One could replace this diagram by the injection  $CH^2(V) \hookrightarrow H_{\text{ét}}^4(V, \Gamma(2))$  of [9, Th. 2.13 (c)], together with the surjection  $H_{\text{ét}}^3(V, \mathbb{Q}_l/\mathbb{Z}_l(2)) \twoheadrightarrow H_{\text{ét}}^4(V, \Gamma(2))\{l\}$ , cf. *loc. cit.*, proof of Th. 2.15; here,  $\Gamma(2)$  is Lichtenbaum's complex.

- (A) construct  $(B, Z)$  such that  $w_Z$  is a homomorphism;
- (B) prove that  $w_Z$  always sends torsion to torsion.

(A) was my initial idea, and (B) was inspired by a discussion with Murre.

**Explanation of (A).** We have

**Lemma 1.** *Take  $(T, t_0, z)$  with  $T$  of dimension 1 and  $z \in CH^2(T \times V)$ . Let  $J = J(T)$  be the jacobian of  $T$ . Then the homomorphism  $z_* : A_0(T) = J(k) \rightarrow A^2(V)$  is of the form  $w_\alpha$  for some correspondence  $\alpha \in CH^2(J \times V)$  (using  $0 \in J(k)$  as base point).*

*Proof.* Let  $g$  be the genus of  $T$ . Recall from [10, Ex. 3.12] the universal relative Cartier divisor  $D_{\text{can}}$  on  $T \times T^{(g)}/T^{(g)}$ , parametrising the effective divisors of degree  $g$  on  $T$ . It defines a correspondence  $D_{\text{can}} : T^{(g)} \rightarrow T$ . Composing with the graph of the birational map  $J \dashrightarrow T^{(g)}$  inverse to  $(t_1, \dots, t_g) \mapsto \sum t_i - gt_0$ , we find a (Chow) correspondence  $D : J \rightarrow T$ . I claim that  $\alpha = z \circ D$  answers the question. Indeed, one checks immediately that the homomorphism

$$D_* : A_0(J) \rightarrow A_0(T)$$

is the Albanese morphism for  $J$ ; hence the composition

$$J(k) \rightarrow A_0(J) \xrightarrow{D_*} A_0(T)$$

is the identity. □

*Remark 1.* On the other hand, the morphism  $T \rightarrow A$  given by the regularity of  $\phi$  factors through a homomorphism

$$(5) \quad J(T) \rightarrow A.$$

This homomorphism coincides with the one underlying  $\phi \circ z_*$  in view of Lemma 1. Indeed, by uniqueness, it suffices to see that (5) induces  $\phi \circ z_*$  on  $k$ -points; this is clear since  $T(k)$  generates  $J(T)(k)$  as an abelian group.

Consider all triples  $(T, t_0, z)$  with  $\dim T = 1$ . The homomorphism  $\bigoplus A_0(T) \xrightarrow{(z_*)} A^2(V)$  is surjective, hence so is  $\bigoplus A_0(T) \xrightarrow{(z_*)} A^2(V) \twoheadrightarrow A(k)$ . As in Remark 1, each summand of this homomorphism is induced by a homomorphism  $\rho_{T, t_0, z} : J(T) \rightarrow A$ , so

$$B := \prod_{(T, t_0, z) \in S} J(T) \xrightarrow{(\rho_{T, t_0, z})} A$$

is surjective (faithfully flat) for a suitable finite set  $S$ . For each  $(T, t_0, z)$ , let  $\alpha = \alpha_z$  be a correspondence given by Lemma 1. Write  $\pi_{T, t_0, z} : B \rightarrow$

$J(T)$  for the canonical projection, viewed as an algebraic correspondence. The pair given by  $B$  and  $Z = \sum_{(T, t_0, z)} \alpha_z \circ \pi_{T, t_0, z}$  yields (A).

**Explanation of (B).** It suffices to show that the map

$$f : B(k) \rightarrow A_0(B)$$

sends  $l$ -primary torsion to  $l$ -primary torsion. Let  $d = \dim B$ . By Bloch's theorem [4, Th. (0.1)], we have  $A_0(B)^{*(d+1)} = 0$ , where  $*$  denotes Pontrjagin product. In other words,  $f$  has “degree  $\leq d$ ” in the sense that its  $(d+1)$ -st deviation [8, §8] is identically 0. It remains to show:

**Lemma 2.** *Let  $f : M \rightarrow N$  be a map of degree  $\leq d$  between two abelian groups, such that  $f(0) = 0$ . Let  $m_0 \in M$  be an element such that  $am_0 = 0$  for some integer  $a > 0$ . Then*

$$a^{\binom{d+1}{2}} f(m_0) = 0.$$

*Proof.* Induction on  $d$ . The case  $d = 1$  is trivial. Assume  $d > 1$ . By hypothesis, the  $d$ -th deviation of  $f$  is multilinear, which implies that the map

$$g_a(m) = f(am) - a^d f(m)$$

is of degree  $\leq d - 1$ . By induction,  $a^{\binom{d}{2}} g_a(m_0) = 0$ , hence the conclusion.  $\square$

*Remark 2.* Of course, either argument proves more generally the following: the map  $\phi : A^n(V)\{l\} \rightarrow A(k)\{l\}$  is surjective for any integer  $n$ , any surjective regular homomorphism  $\phi : A^n(V) \rightarrow A(k)$  and any prime  $l \neq \text{char } k$ .

*Remark 3.* In [3, §6, Lemma and Prop. 11], Beauville gives a different proof that  $f$  sends torsion to torsion. Moreover, he observes that Roïtman's theorem [13] then implies that the restriction of  $f$  to torsion is actually an isomorphism, hence a homomorphism.

If we apply Roïtman's theorem together with Lemma 2, we obtain the following stronger result: if  $m, m_0 \in B(k)$  and  $m_0$  is torsion, then  $f(m + m_0) = f(m) + f(m_0)$ . (Fixing  $m$ , the map  $f_m : m' \mapsto f(m + m') - f(m) - f(m')$  is of degree  $< d$ , hence  $a^{\binom{d}{2}} f_m(m_0) = 0$  if  $am_0 = 0$  by Lemma 2, and therefore  $f_m(m_0) = 0$  by Roïtman's theorem.)

**Some expectation.** The landmark work of Bloch and Esnault [5] yields the existence of 4-folds  $V$  over fields  $k$  of characteristic 0 such that the  $l$ -torsion of  $A^3(V)$  is infinite (hence its  $l$ -primary torsion has infinite corank). One example, used by Rosenschon-Srinivas [14] and Totaro [16] and relying on Nori's theorem [12] and Schoen's results [15],

is the following: start from the generic abelian 3-fold  $A$ , whose field of constants  $k_0$  is finitely generated over  $\mathbb{Q}$ ; choose an elliptic curve  $E/k_0(t)$ , not isotrivial with respect to  $k_0$ , and take  $V = A_{k_0(t)} \times E$ ,  $k =$  algebraic closure of  $k_0(t)$ .

**Conjecture 1.** *For this  $V$ , a universal regular homomorphism on  $A^3(V)$  does not exist.*

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