On the universal regular homomorphism in codimension 2
Bruno Kahn

To cite this version:

HAL Id: hal-02897438
https://hal.science/hal-02897438
Submitted on 12 Jul 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
ON THE UNIVERSAL REGULAR HOMOMORPHISM
IN CODIMENSION 2

BRUNO KAHN

ABSTRACT. We point out a gap in Murre’s proof of the existence of a universal regular homomorphism for codimension 2 cycles on a smooth projective variety, and offer two arguments to fill this gap.

In [11], Jacob Murre shows the existence of a universal regular homomorphism for algebraically trivial cycles of codimension 2 on a smooth projective variety over an algebraically closed field. This theorem has been largely used in the literature, most lately in [1], [7] and [2]; for example, it is essential in [2] for descending the method of Clemens and Griffiths [6] to non-algebraically closed fields, thus allowing Benoist and Wittenberg to obtain new examples of geometrically rational non-rational 3-folds.

Unfortunately its proof contains a gap, but fortunately this gap can be filled, actually by two different methods. This is the purpose of this note, which is a slight modification of a letter to Murre on December 5, 2018.

Recall the set-up, with the notation of [11]: $V$ is a smooth projective variety over an algebraically closed field $k$ and $A^n(V)$ denotes the group of codimension $n$ cycles algebraically equivalent to 0 on $V$, modulo rational equivalence. Following Samuel, given an abelian $k$-variety $A$, a homomorphism

$$\phi : A^n(V) \to A(k)$$

is said to be regular if, for any pointed smooth projective $k$-variety $(T, t_0)$ and any correspondence $Z \in CH^n(T \times V)$, the composition

$$T(k) \xrightarrow{w_Z} A^n(V) \xrightarrow{\phi} A(k)$$

is induced by a morphism $f : T \to A$; here $w_Z$ is the composition

$$T(k) \to A_0(T) \xrightarrow{Z} A^n(V)$$

\textit{Date:} May 18, 2020.

2010 \textit{Mathematics Subject Classification.} 14C25, 14K30.
where the first map sends \( t \) to \([t] - [t_0]\). (Note that \( f \) is then unique, by Zariski density of the rational points in \( T \).)

Using fancy language, regular homomorphisms from \( A^n(V) \) form a category and a *universal regular homomorphism* is an initial object of this category, if it exists. This initial object is well-known to exist when \( n = 0 \), \( n = 1 \) (the Picard variety) and \( n = \dim X \) (the Albanese variety). Murre’s theorem is:

**Theorem 1** ([11, Th. 1.9]). A universal regular homomorphism \( \phi_0 \) exists when \( n = 2 \) for any \( V \) (of dimension \( \geq 2 \)).

Recall the main steps of his proof. First, given a regular homomorphism \( \phi \), its image in \( A(k) \) is given by the points of some sub-abelian variety \( A' \subseteq A \) [11, Lemma 1.6.2 i)]. From this, one deduces [11, Prop. 2.1] that \( \phi_0 \) exists if and only if \( \dim A \) is bounded when \( \phi \) runs through the surjective regular homomorphisms. Now, Murre’s key idea is to bound \( \dim A \) by the torsion of \( A^2(V) \), which is controlled by the Merkurjev-Suslin theorem (Bloch’s observation).

Let us elaborate a little on this point, to avoid the \( l \)-adic argument of *loc. cit.*: it suffices to prove that \( \phi \) induces a surjection

\[
A^2(V)\{l\} \twoheadrightarrow A(k)\{l\}
\]

for some prime \( l \neq \text{char } k \), where \( M\{l\} \) denotes the \( l \)-primary torsion of an abelian group \( M \): indeed, \( \text{corank } A(k)\{l\} = 2 \dim A \). Mainly by Merkurjev-Suslin (Diagram in [11, Prop. 6.1])\(^1\),

\[
\text{corank } CH^2(V)\{l\} \leq \text{corank } H^3_{\acute{e}t}(V, \mathbb{Q}_l/\mathbb{Z}_l(2)) = b_3(V)
\]

so the same holds a fortiori for \( \text{corank } A^2(V)\{l\} \).

Now, in [11, Lemma 1.6.2 ii)], Murre constructs an abelian variety \( B \) (pointed at 0) and a correspondence \( Z \in CH^2(B \times V) \) such that (1) is surjective for \( T = B \). Since this map is induced by a morphism of abelian varieties sending 0 to 0 (hence a homomorphism), it restricts to a surjection

\[
B\{l\} \twoheadrightarrow A\{l\}.
\]

This allows me to explain

the gap:

A priori (4) does not imply (3), because \( w_Z \) is in general only a set-theoretic map, not a group homomorphism (see e.g. [4, Th. (3.1) a)]).

We now fix a surjective regular homomorphism \( \phi \) as above. We shall give two ways to fill this gap:

\(^1\)One could replace this diagram by the injection \( CH^2(V) \hookrightarrow H^4_{\acute{e}t}(V, \Gamma(2)) \) of [9, Th. 2.13 (c)], together with the surjection \( H^3_{\acute{e}t}(V, \mathbb{Q}_l/\mathbb{Z}_l(2)) \twoheadrightarrow H^3_{\acute{e}t}(V, \Gamma(2))\{l\} \), cf. *loc. cit.*, proof of Th. 2.15; here, \( \Gamma(2) \) is Lichtenbaum’s complex.
(A) construct \((B, Z)\) such that \(w_Z\) is a homomorphism;
(B) prove that \(w_Z\) always sends torsion to torsion.

(A) was my initial idea, and (B) was inspired by a discussion with Murre.

**Explanation of (A).** We have

**Lemma 1.** Take \((T, t_0, z)\) with \(T\) of dimension 1 and \(z \in CH^2(T \times V)\). Let \(J = J(T)\) be the jacobian of \(T\). Then the homomorphism \(z_* : A_0(T) = J(k) \to A^2(V)\) is of the form \(w_{\alpha}\) for some correspondence \(\alpha \in CH^2(J \times V)\) (using \(0 \in J(k)\) as base point).

**Proof.** Let \(g\) be the genus of \(T\). Recall from [10, Ex. 3.12] the universal relative Cartier divisor \(D_{\text{can}}\) on \(T \times T^{(g)}\), parametrising the effective divisors of degree \(g\) on \(T\). It defines a correspondence \(D_{\text{can}} : T^{(g)} \to T\). Composing with the graph of the birational map \(J \to T^{(g)}\) inverse to \((t_1, \ldots, t_g) \mapsto \sum t_i - gt_0\), we find a (Chow) correspondence \(D : J \to T\). I claim that \(\alpha = z \circ D\) answers the question. Indeed, one checks immediately that the homomorphism \(D_* : A_0(J) \to A_0(T)\) is the Albanese morphism for \(J\); hence the composition

\[ J(k) \to A_0(J) \xrightarrow{D_*} A_0(T) \]

is the identity. \(\square\)

**Remark 1.** On the other hand, the morphism \(T \to A\) given by the regularity of \(\phi\) factors through a homomorphism

\[ J(T) \to A. \tag{5} \]

This homomorphism coincides with the one underlying \(\phi \circ z_*\) in view of Lemma 1. Indeed, by uniqueness, it suffices to see that (5) induces \(\phi \circ z_*\) on \(k\)-points; this is clear since \(T(k)\) generates \(J(T)(k)\) as an abelian group.

Consider all triples \((T, t_0, z)\) with \(\dim T = 1\). The homomorphism \(\bigoplus A_0(T) \xrightarrow{(z_*)} A^2(V)\) is surjective, hence so is \(\bigoplus A_0(T) \xrightarrow{(z_*)} A^2(V) \to A(k)\). As in Remark 1, each summand of this homomorphism is induced by a homomorphism \(\rho_{T, t_0, z} : J(T) \to A\), so

\[ B := \prod_{(T, t_0, z) \in S} J(T) \xrightarrow{(\rho_{T, t_0, z})} A \]

is surjective (faithfully flat) for a suitable finite set \(S\). For each \((T, t_0, z)\), let \(\alpha = \alpha_z\) be a correspondence given by Lemma 1. Write \(\pi_{T, t_0, z} : B \to \)
J(T) for the canonical projection, viewed as an algebraic correspondence. The pair given by B and Z = \( \sum_{(T,t_0,z)} \alpha_z \circ \pi_{T,t_0,z} \) yields (A).

**Explanation of (B).** It suffices to show that the map 
\[ f : B(k) \to A_0(B) \]
sends \( l \)-primary torsion to \( l \)-primary torsion. Let \( d = \dim B \). By Bloch’s theorem [4, Th. (0.1)], we have \( A_0(B)^{(d+1)} = 0 \), where * denotes Pontrjagin product. In other words, \( f \) has “degree \( \leq d \)” in the sense that its \((d+1)\)st deviation [8, §8] is identically 0. It remains to show:

**Lemma 2.** Let \( f : M \to N \) be a map of degree \( \leq d \) between two abelian groups, such that \( f(0) = 0 \). Let \( m_0 \in M \) be an element such that \( am_0 = 0 \) for some integer \( a > 0 \). Then
\[ a^{(d+1)} f(m_0) = 0. \]

**Proof.** Induction on \( d \). The case \( d = 1 \) is trivial. Assume \( d > 1 \). By hypothesis, the \( d \)-th deviation of \( f \) is multilinear, which implies that the map 
\[ g_a(m) = f(am) - a^d f(m) \]
is of degree \( \leq d - 1 \). By induction, \( a^{(d+1)/2} g_a(m_0) = 0 \), hence the conclusion. \( \square \)

**Remark 2.** Of course, either argument proves more generally the following: the map \( \phi : A^n(V)\{l\} \to A(k)\{l\} \) is surjective for any integer \( n \), any surjective regular homomorphism \( \phi : A^n(V) \to A(k) \) and any prime \( l \neq \text{char } k \).

**Remark 3.** In [3, §6, Lemma and Prop. 11], Beauville gives a different proof that \( f \) sends torsion to torsion. Moreover, he observes that Roïtmann’s theorem [13] then implies that the restriction of \( f \) to torsion is actually an isomorphism, hence a homomorphism.

If we apply Roïtmann’s theorem together with Lemma 2, we obtain the following stronger result: if \( m, m_0 \in B(k) \) and \( m_0 \) is torsion, then \( f(m + m_0) = f(m) + f(m_0) \). (Fixing \( m \), the map \( f_m : m' \mapsto f(m + m') - f(m) - f(m') \) is of degree \( < d \), hence \( a^d f_m(m_0) = 0 \) if \( am_0 = 0 \) by Lemma 2, and therefore \( f_m(m_0) = 0 \) by Roïtmann’s theorem.)

**Some expectation.** The landmark work of Bloch and Esnault [5] yields the existence of 4-folds \( V \) over fields \( k \) of characteristic 0 such that the \( l \)-torsion of \( A^3(V) \) is infinite (hence its \( l \)-primary torsion has infinite corank). One example, used by Rosenschon-Srinivas [14] and Totaro [16] and relying on Nori’s theorem [12] and Schoen’s results [15],
is the following: start from the generic abelian 3-fold $\mathcal{A}$, whose field of constants $k_0$ is finitely generated over $\mathbb{Q}$; choose an elliptic curve $E/k_0(t)$, not isotrivial with respect to $k_0$, and take $V = \mathcal{A}_{k_0(t)} \times E$, $k =$ algebraic closure of $k_0(t)$.

Conjecture 1. For this $V$, a universal regular homomorphism on $A^3(V)$ does not exist.

Acknowledgements. I am indebted to Jacob Murre for discussions around this problem, and for his encouragement to publish this note. I am also indebted to the referee for a careful reading, pointing out an incorrect earlier formulation of Lemma 1, as well as the referee to [3]. (The referee credits in turn Charles Vial for this reference.)

References


IMJ-PRG, Case 247, 4 place Jussieu, 75252 Paris Cedex 05, France
E-mail address: bruno.kahn@imj-prg.fr