



**HAL**  
open science

# On the universal regular homomorphism in codimension 2

Bruno Kahn

► **To cite this version:**

Bruno Kahn. On the universal regular homomorphism in codimension 2. *Annales de l'Institut Fourier*, 2021, 71, pp.843-848. hal-02897438

**HAL Id: hal-02897438**

**<https://hal.science/hal-02897438>**

Submitted on 12 Jul 2020

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# ON THE UNIVERSAL REGULAR HOMOMORPHISM IN CODIMENSION 2

BRUNO KAHN

ABSTRACT. We point out a gap in Murre's proof of the existence of a universal regular homomorphism for codimension 2 cycles on a smooth projective variety, and offer two arguments to fill this gap.

In [11], Jacob Murre shows the existence of a universal regular homomorphism for algebraically trivial cycles of codimension 2 on a smooth projective variety over an algebraically closed field. This theorem has been largely used in the literature, most lately in [1], [7] and [2]; for example, it is essential in [2] for descending the method of Clemens and Griffiths [6] to non-algebraically closed fields, thus allowing Benoist and Wittenberg to obtain new examples of geometrically rational non-rational 3-folds.

Unfortunately its proof contains a gap, but fortunately this gap can be filled, actually by two different methods. This is the purpose of this note, which is a slight modification of a letter to Murre on December 5, 2018.

Recall the set-up, with the notation of [11]:  $V$  is a smooth projective variety over an algebraically closed field  $k$  and  $A^n(V)$  denotes the group of codimension  $n$  cycles algebraically equivalent to 0 on  $V$ , modulo rational equivalence. Following Samuel, given an abelian  $k$ -variety  $A$ , a homomorphism

$$\phi : A^n(V) \rightarrow A(k)$$

is said to be *regular* if, for any pointed smooth projective  $k$ -variety  $(T, t_0)$  and any correspondence  $Z \in CH^n(T \times V)$ , the composition

$$(1) \quad T(k) \xrightarrow{w_Z} A^n(V) \xrightarrow{\phi} A(k)$$

is induced by a morphism  $f : T \rightarrow A$ ; here  $w_Z$  is the composition

$$(2) \quad T(k) \rightarrow A_0(T) \xrightarrow{Z_*} A^n(V)$$

---

*Date:* May 18, 2020.

*2010 Mathematics Subject Classification.* 14C25, 14K30.

where the first map sends  $t$  to  $[t] - [t_0]$ . (Note that  $f$  is then unique, by Zariski density of the rational points in  $T$ .)

Using fancy language, regular homomorphisms from  $A^n(V)$  form a category and a *universal regular homomorphism* is an initial object of this category, if it exists. This initial object is well-known to exist when  $n = 0$ ,  $n = 1$  (the Picard variety) and  $n = \dim X$  (the Albanese variety). Murre's theorem is:

**Theorem 1** ([11, Th. 1.9]). *A universal regular homomorphism  $\phi_0$  exists when  $n = 2$  for any  $V$  (of dimension  $\geq 2$ ).*

Recall the main steps of his proof. First, given a regular homomorphism  $\phi$ , its image in  $A(k)$  is given by the points of some sub-abelian variety  $A' \subseteq A$  [11, Lemma 1.6.2 i)]. From this, one deduces [11, Prop. 2.1] that  $\phi_0$  exists if and only if  $\dim A$  is bounded when  $\phi$  runs through the *surjective* regular homomorphisms. Now, Murre's key idea is to bound  $\dim A$  by the torsion of  $A^2(V)$ , which is controlled by the Merkurjev-Suslin theorem (Bloch's observation).

Let us elaborate a little on this point, to avoid the  $l$ -adic argument of *loc. cit.*: it suffices to prove that  $\phi$  induces a surjection

$$(3) \quad A^2(V)\{l\} \twoheadrightarrow A(k)\{l\}$$

for some prime  $l \neq \text{char } k$ , where  $M\{l\}$  denotes the  $l$ -primary torsion of an abelian group  $M$ : indeed,  $\text{corank } A(k)\{l\} = 2 \dim A$ . Mainly by Merkurjev-Suslin (Diagram in [11, Prop. 6.1])<sup>1</sup>,

$$\text{corank } CH^2(V)\{l\} \leq \text{corank } H_{\text{ét}}^3(V, \mathbb{Q}_l/\mathbb{Z}_l(2)) (= b_3(V))$$

so the same holds *a fortiori* for  $\text{corank } A^2(V)\{l\}$ .

Now, in [11, Lemma 1.6.2 ii)], Murre constructs an abelian variety  $B$  (pointed at 0) and a correspondence  $Z \in CH^2(B \times V)$  such that (1) is surjective for  $T = B$ . Since this map is induced by a morphism of abelian varieties sending 0 to 0 (hence a homomorphism), it restricts to a surjection

$$(4) \quad B\{l\} \twoheadrightarrow A\{l\}.$$

This allows me to explain

the gap:

A priori (4) does not imply (3), because  $w_Z$  is in general only a set-theoretic map, not a group homomorphism (see e.g. [4, Th. (3.1 a)]).

We now fix a surjective regular homomorphism  $\phi$  as above. We shall give two ways to fill this gap:

<sup>1</sup>One could replace this diagram by the injection  $CH^2(V) \hookrightarrow H_{\text{ét}}^4(V, \Gamma(2))$  of [9, Th. 2.13 (c)], together with the surjection  $H_{\text{ét}}^3(V, \mathbb{Q}_l/\mathbb{Z}_l(2)) \twoheadrightarrow H_{\text{ét}}^4(V, \Gamma(2))\{l\}$ , cf. *loc. cit.*, proof of Th. 2.15; here,  $\Gamma(2)$  is Lichtenbaum's complex.

- (A) construct  $(B, Z)$  such that  $w_Z$  is a homomorphism;
- (B) prove that  $w_Z$  always sends torsion to torsion.

(A) was my initial idea, and (B) was inspired by a discussion with Murre.

**Explanation of (A).** We have

**Lemma 1.** *Take  $(T, t_0, z)$  with  $T$  of dimension 1 and  $z \in CH^2(T \times V)$ . Let  $J = J(T)$  be the jacobian of  $T$ . Then the homomorphism  $z_* : A_0(T) = J(k) \rightarrow A^2(V)$  is of the form  $w_\alpha$  for some correspondence  $\alpha \in CH^2(J \times V)$  (using  $0 \in J(k)$  as base point).*

*Proof.* Let  $g$  be the genus of  $T$ . Recall from [10, Ex. 3.12] the universal relative Cartier divisor  $D_{\text{can}}$  on  $T \times T^{(g)}/T^{(g)}$ , parametrising the effective divisors of degree  $g$  on  $T$ . It defines a correspondence  $D_{\text{can}} : T^{(g)} \rightarrow T$ . Composing with the graph of the birational map  $J \dashrightarrow T^{(g)}$  inverse to  $(t_1, \dots, t_g) \mapsto \sum t_i - gt_0$ , we find a (Chow) correspondence  $D : J \rightarrow T$ . I claim that  $\alpha = z \circ D$  answers the question. Indeed, one checks immediately that the homomorphism

$$D_* : A_0(J) \rightarrow A_0(T)$$

is the Albanese morphism for  $J$ ; hence the composition

$$J(k) \rightarrow A_0(J) \xrightarrow{D_*} A_0(T)$$

is the identity. □

*Remark 1.* On the other hand, the morphism  $T \rightarrow A$  given by the regularity of  $\phi$  factors through a homomorphism

$$(5) \quad J(T) \rightarrow A.$$

This homomorphism coincides with the one underlying  $\phi \circ z_*$  in view of Lemma 1. Indeed, by uniqueness, it suffices to see that (5) induces  $\phi \circ z_*$  on  $k$ -points; this is clear since  $T(k)$  generates  $J(T)(k)$  as an abelian group.

Consider all triples  $(T, t_0, z)$  with  $\dim T = 1$ . The homomorphism  $\bigoplus A_0(T) \xrightarrow{(z_*)} A^2(V)$  is surjective, hence so is  $\bigoplus A_0(T) \xrightarrow{(z_*)} A^2(V) \twoheadrightarrow A(k)$ . As in Remark 1, each summand of this homomorphism is induced by a homomorphism  $\rho_{T, t_0, z} : J(T) \rightarrow A$ , so

$$B := \prod_{(T, t_0, z) \in S} J(T) \xrightarrow{(\rho_{T, t_0, z})} A$$

is surjective (faithfully flat) for a suitable finite set  $S$ . For each  $(T, t_0, z)$ , let  $\alpha = \alpha_z$  be a correspondence given by Lemma 1. Write  $\pi_{T, t_0, z} : B \rightarrow$

$J(T)$  for the canonical projection, viewed as an algebraic correspondence. The pair given by  $B$  and  $Z = \sum_{(T,t_0,z)} \alpha_z \circ \pi_{T,t_0,z}$  yields (A).

**Explanation of (B).** It suffices to show that the map

$$f : B(k) \rightarrow A_0(B)$$

sends  $l$ -primary torsion to  $l$ -primary torsion. Let  $d = \dim B$ . By Bloch's theorem [4, Th. (0.1)], we have  $A_0(B)^{*(d+1)} = 0$ , where  $*$  denotes Pontrjagin product. In other words,  $f$  has "degree  $\leq d$ " in the sense that its  $(d+1)$ -st deviation [8, §8] is identically 0. It remains to show:

**Lemma 2.** *Let  $f : M \rightarrow N$  be a map of degree  $\leq d$  between two abelian groups, such that  $f(0) = 0$ . Let  $m_0 \in M$  be an element such that  $am_0 = 0$  for some integer  $a > 0$ . Then*

$$a^{\binom{d+1}{2}} f(m_0) = 0.$$

*Proof.* Induction on  $d$ . The case  $d = 1$  is trivial. Assume  $d > 1$ . By hypothesis, the  $d$ -th deviation of  $f$  is multilinear, which implies that the map

$$g_a(m) = f(am) - a^d f(m)$$

is of degree  $\leq d - 1$ . By induction,  $a^{\binom{d}{2}} g_a(m_0) = 0$ , hence the conclusion.  $\square$

*Remark 2.* Of course, either argument proves more generally the following: the map  $\phi : A^n(V)\{l\} \rightarrow A(k)\{l\}$  is surjective for any integer  $n$ , any surjective regular homomorphism  $\phi : A^n(V) \rightarrow A(k)$  and any prime  $l \neq \text{char } k$ .

*Remark 3.* In [3, §6, Lemma and Prop. 11], Beauville gives a different proof that  $f$  sends torsion to torsion. Moreover, he observes that Roïtman's theorem [13] then implies that the restriction of  $f$  to torsion is actually an isomorphism, hence a homomorphism.

If we apply Roïtman's theorem together with Lemma 2, we obtain the following stronger result: if  $m, m_0 \in B(k)$  and  $m_0$  is torsion, then  $f(m + m_0) = f(m) + f(m_0)$ . (Fixing  $m$ , the map  $f_m : m' \mapsto f(m + m') - f(m) - f(m')$  is of degree  $< d$ , hence  $a^{\binom{d}{2}} f_m(m_0) = 0$  if  $am_0 = 0$  by Lemma 2, and therefore  $f_m(m_0) = 0$  by Roïtman's theorem.)

**Some expectation.** The landmark work of Bloch and Esnault [5] yields the existence of 4-folds  $V$  over fields  $k$  of characteristic 0 such that the  $l$ -torsion of  $A^3(V)$  is infinite (hence its  $l$ -primary torsion has infinite corank). One example, used by Rosenschon-Srinivas [14] and Totaro [16] and relying on Nori's theorem [12] and Schoen's results [15],

is the following: start from the generic abelian 3-fold  $A$ , whose field of constants  $k_0$  is finitely generated over  $\mathbb{Q}$ ; choose an elliptic curve  $E/k_0(t)$ , not isotrivial with respect to  $k_0$ , and take  $V = A_{k_0(t)} \times E$ ,  $k =$  algebraic closure of  $k_0(t)$ .

**Conjecture 1.** *For this  $V$ , a universal regular homomorphism on  $A^3(V)$  does not exist.*

**Acknowledgements.** I am indebted to Jacob Murre for discussions around this problem, and for his encouragement to publish this note. I am also indebted to the referee for a careful reading, pointing out an incorrect earlier formulation of Lemma 1, as well as the referee to [3]. (The referee credits in turn Charles Vial for this reference.)

#### REFERENCES

- [1] J. D. Achter, S. Casalaina-Martin, C. Vial *On descending cohomology geometrically*, Compos. Math. **153** (2017), 1446–1478.
- [2] O. Benoist, O. Wittenberg *The Clemens-Griffiths method over non-closed fields*, preprint, 2019.
- [3] A. Beauville *Quelques remarques sur la transformation de Fourier dans l’anneau de Chow d’une variété abélienne*, Algebraic geometry (Tokyo/Kyoto, 1982), Lecture Notes in Math. **1016**, Springer, 1983, 238–260.
- [4] S. Bloch *Some elementary theorems about algebraic cycles on Abelian varieties*, Invent. Math. **37** (1976), no. 3, 215–228.
- [5] S. Bloch, H. Esnault *The coniveau filtration and non-divisibility for algebraic cycles*, Math. Ann. **304** (1996), no. 2, 303–314.
- [6] C. Clemens, P. Griffiths, *The intermediate Jacobian of the cubic threefold*, Ann. of Math. **95** (1972), 281–356.
- [7] J.-L. Colliot-Thélène, A. Pirutka *Troisième groupe de cohomologie non ramifiée d’un solide cubique sur un corps de fonctions d’une variable*, Épijournal de géométrie algébrique **2** (2018), article no. 13.
- [8] S. Eilenberg, S. Mac Lane *On the groups  $H(\Pi, n)$ , II; methods of computation*, Ann. of Math. (2) **60** (1954), 49–139.
- [9] S. Lichtenbaum *New results on weight-two motivic cohomology*, The Grothendieck Festschrift, Vol. III, 35–55, Progr. Math., **88**, Birkhäuser Boston, Boston, MA, 1990.
- [10] J. Milne *Jacobian varieties*, Ch. VII of Arithmetic geometry (G. Cornell, J. H. Silverman, eds), revised second printing, Springer, 1998,
- [11] J. P. Murre *Applications of algebraic K-theory to the theory of algebraic cycles*, Algebraic geometry, Sitges (Barcelona), 1983, 216–261, Lecture Notes in Math., **1124**, Springer, Berlin, 1985.
- [12] M. Nori *Cycles on the generic abelian threefold*, Proc. Indian Acad. Sci. Math. Sci. **99** (1989), no. 3, 191–196.
- [13] A. Roitman *The torsion of the group of 0-cycles modulo rational equivalence*, Annals of Math. **111** (1980), 553–569.

- [14] A. Rosenschon, V. Srinivas *The Griffiths group of the generic abelian 3-fold*, Cycles, motives and Shimura varieties, 449–467, Tata Inst. Fund. Res. Stud. Math. **21**, Tata Inst. Fund. Res., Mumbai, 2010.
- [15] C. Schoen *On certain exterior product maps of Chow groups*, Math. Res. Lett. **7** (2000), 177–194.
- [16] B. Totaro *Complex varieties with infinite Chow groups modulo 2*, Ann. of Math. (2) **183** (2016), no. 1, 363–375.

IMJ-PRG, CASE 247, 4 PLACE JUSSIEU, 75252 PARIS CEDEX 05, FRANCE  
*E-mail address:* `bruno.kahn@imj-prg.fr`