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GENERALIZED PELL-FERMAT EQUATIONS AND PASCAL TRIANGLE

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Abstract: Using Pascal triangle, we give a simple generalization to a well-known problem of S.Ramanujan. Thus we are interested in computing the median of some integer valued distributions, arising naturally when extending arithmetic progressions to progressions of polynomial growth. We show this reduces to equations of Pell-Fermat type of higher order, which admit very few integer solutions, but for which, following S.Ramanujan's original idea, we can always find integer sequences of best approximation, in the Diophantine sense. Our procedure relies much on formal Calculus with Mathematica.

1/ Introduction

a) Pascal triangle and progressions

Pascal triangle identifies with an infinite lower triangular matrix with elements $c_{nk} = \binom{n}{k}$, $0 \leq k \leq n$, the binomial coefficients. It contains many remarkable sequences : the sum of binomial coefficients in n:th row equals 2^n, the sum of diagonal elements are the terms of Fibonacci sequence, and the columns C_k generalize arithmetic sequences. Namely, the first column C_0 is the constant sequence equal to 1 ; the second one C_1, which gives the partial sums of C_0 consists in the arithmetic sequence with ratio 1 and first term 1 ; the third one C_2 gives the partial sums of C_1 by using the relation $\binom{n}{1} + \binom{n}{2} = \binom{n+1}{2}$; the fourth one C_3 gives the partial sums of C_2 by $\binom{n}{2} + \binom{n}{3} = \binom{n+1}{3}$, etc... All sequences C_k have a polynomial growth $\mathcal{O}(n^k)$. Because of their simple geometric interpretation, numbers in C_2 are known as the *triangular numbers*, these in C_3 the *tetraedric numbers*, and those occuring in higher order columns C_k the *k-simplicial numbers*.

Pascal triangle also admits asymmetric generalizations, which amount to replace 1 by arbitrary numbers on the second diagonal. Asymmetric Pascal triangle are built up the same way as standard Pascal triangle, and also extend to complex values of n (and complex values of k as well by using Euler Γ-function). The simplest one consists in replacing 1 by $a \in \mathbf{C}$, so that the binomial coefficient c_{nk} reads

$$c^a_{nk} = \binom{n}{k}_a = \binom{n}{k} + a\binom{n-1}{k-1}$$

In particular in the second column C^a_1 we recognize the arithmetic sequence of ratio 1 and first term $1 + a$. The term $a\binom{n-1}{k-1}$ accounts for a "lower order term" as n becomes large.

b) The "Houses of Ramanujan"

This problem, leading to a Pell-Fermat equation was reportledly discovered in quaint circumstances by S.Ramanujan [Ran] and solved (exactly) by the method of continued fractions. The

history of this Diophantine equation goes to Archimedes, then proceeds to Bhaskara, Brahmagupta, Fermat, Wallis, Euler and Lagrange who almost brought the theory to its definite form.

The problem of the "Houses of Ramanujan" deals with column C_2 of triangular numbers. We say that $m \in \mathbf{N}^* = \{1, 2, 3, \cdots\}$ is a *House of Ramanujan of order* 2 (or simply a House of Ramanujan) iff the sum $\binom{m}{2}$ of the $m-1$ first integers "to the left of the House of Ramanujan" is equal to the sum of the next $n-m$ integers "to the right of the House of Ramanujan", for some $n > m$, with the convention that $n = 1$ when $m = 1$, that is

$$(C2) \qquad \binom{n+1}{2} = 2\binom{m}{2} + m$$

In the language of Statistics, m is the median of the cumulants $\sum_{i \leq j} i,\, j \leq n$. Geometrically this means that for some values $m < n$, the number of integer points in the triangle $x_1 + x_2 \leq n$, $x_1, x_2 \geq 0$ is twice the number of integer points in the triangle $x_1 + x_2 \leq m$, $x_1, x_2 \geq 0$, plus m.

We can consider the first quartile as well, defined as the integer m such that

$$(Q1) \qquad \binom{n+1}{2} = 4\binom{m}{2} + m$$

or the third quartile, defined as the integer m such that

$$(Q3) \qquad \binom{n+1}{2} = \frac{4}{3}\binom{m}{2} + m$$

The point of course is to find integer solutions (n, m). The quadratic case C_2 plays a peculiar role, and all solutions of (C2) are given by a sequence in $\mathbf{Z}[\sqrt{2}]$ with constant coefficients.

c) Main results

The situation becomes drastically different when trying to find the House of Ramanujan of order $k \geq 3$, i.e. to decompose simplicial numbers of higher order $k \geq 3$. For $k = 3$ this amounts to solve

$$(C3) \qquad \binom{n+2}{3} = 2\binom{m+1}{3} + \binom{m+1}{2}$$

If this equation has any solution, then the number of integer points in the tetrahedron (with positive coordinates) $x_1 + x_2 + x_3 \leq n$, would be twice the number of integer points in the smaller tetrahedron $x_1 + x_2 + x_3 \leq m$, plus the triangular number $\binom{m+1}{2}$. Our guess is that there are no solution to (C3), (C4) and (C5). We found only one solution to (C6), corresponding to $(m, n) = (10, 11)$.

The corresponding equations, after some affine change of coordinates, take the form $P_k(x, y) = 0$, where P_k is polynomial with integer coefficients (see formulae (A.2) below), i.e. $P_k(x, y) = x^k - Dy^k + \cdots$ (or permuting x with y) where the dots mean a polynomial of degree $\leq k-2$ without mixed terms, and containing only monomials with the same parity as k.

But it turns out that we can always compute asymptotic solutions in the Diophantine sense, generalizing Ramanujan's approach.

Next we address the problem of finding (exact) integer solutions, or pairs (m, n) with half-integer m. We also look at the first and third quartile of such statistical distributions.

In case C_2 the method relies on known results for Pell-Fermat equation see [Wo] and references therein. While $u^2 - Dv^2 = 1$, D not a perfect square, has always integer solutions given by a (unique) sequence in $\mathbf{Z}[\sqrt{D}]$, this holds for $u^2 - Dv^2 = -1$ only for some values of D, using continued fractions. Once we have solved $u^2 - Dv^2 = \pm 1$, we can consider the more general Eq. $x^2 - Dy^2 = c$. Whenever we guess at a particular solution, we can build a sequence of solutions (called a fundamental sequence), using the sequences for $u^2 - Dv^2 = \pm 1$. So there exists so many fundamental sequences as "fundamental particular solutions" we can find for $x^2 - Dy^2 = c$. But such a family of solutions does not necessarily generate all solutions.

Applying this method for C_2 we find all solutions of (C2) (which was of course known before), and one fundamental sequence for (Q3). Other (related) solutions could be also found by considering the action of some discrete groups acting on the hyperbola of equation $x^2 - Dy^2 = c$, as is the case in the problem of quasi-isoceles triangles with a square angle. But we could not find new solutions this way.

Since P_4 is a polynomial in x^2 and y^2, we can reduce C_4 to a Pell-Fermat Eq., then we are left to find the solutions which are perfect squares (quadrature). Within the range of values we have considered, i.e. $n \leq 10^8$, we have shown this way there are no integer solutions (or solutions with half-integer m), when using the fundamental sequences given by Mathematica.

We call equation $P_k(x, y) = 0$ a *generalized Pell-Fermat equation*. Actually nothing seems to be known about these equations, and since elementary technics, as the reduction of $P_k(x, y) = 0$ modulo prime numbers p have not led us to any significant issue (see however Sect.3) we tried to resort to formal calculus on Mathematica.

So the case C_3 and C_6 rely instead on the resolution of a 3:rd degree polynomial, which makes use of real radicals only. For C_3 it follows by inspection that there are no integer solutions (or solutions with half-integer m) in the range considered. The same method (together with a quadrature) applies to C_6, which this time yields the solution $(n, m) = (11, 10)$ in that range.

In case C_5 we could not find any analytical method, and the only way is the systematic search in the range $n \leq 10^8$, which doesn't reveal any integer solution. A fortiori, there seems to be no analytical methods for $k = 7$ and beyond.

Of course it could be very tempting to invoke higher Number Theory to insert this problem into a general framework, such as Shimura-Taniyama-Weil (STW) conjecture, which was used in the context of Fermat theorem, see e.g. [Da], [He]. We leave this for future investigations.

The paper is organized as follows: In Sect.2 we review the quadratic case of triangular numbers and compute also exact and asymptotic solutions for the 1:st and 3:rd quartiles. In Sect.3 we investigate the case of tetrahedric numbers and beyond, up to 6-simplicial numbers, and compute

the median, with the help of Mathematica. In Sect.4 we try to generalize these technics to the determination of the 1:st and 3:rd quartiles for (C3)-(C6), allowing also for $m \in \mathbf{N} + 1/4$. In Sect.5 we reduce the equations $P_k(x, y) = 0$, modulo $p = 3, 5, 7$ when $k = 3, \cdots, 6$, so to save some trials when searching for integer solutions without resorting to the algorithms set up in Sect.3. In Sect.6 we consider the asymmetric Pascal triangle. In Appendix we give a table of all relevant $P_k(x, y)$ obtained so far, together with the corresponding number to be approximated in the Diophantine sense.

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2/ Triangular numbers

Eq. (Q1) actually solves simply as $m = (n+1)/2$, and Eq. (C2), (Q3) are Pell-Fermat equations. For (C2) (the equation considered by Ramanujan) we find $n(n+1) = 2m^2$ or

$$(2n+1)^2 = 8m^2 + 1 \tag{2.1}$$

It is well-known that all "physical" solutions, besides this for which $n(n+1) = 2m^2 = 0$, are given by the sequence indexed over $\alpha \in \mathbf{N}^*$

$$n_\alpha = \frac{1}{4}\big((3-2\sqrt{2})^\alpha + (3+2\sqrt{2})^\alpha - 2\big), \quad m_\alpha = -\frac{\sqrt{2}}{8}\big((3-2\sqrt{2})^\alpha - (3+2\sqrt{2})^\alpha\big) \tag{2.1}$$

Here we list the first pairs (m, n)

$$m_\alpha = 1; 6; 35; 204; 1189; 6930; 40\,391; 235\,416; 1372\,105; 7997\,214; \cdots$$
$$n_\alpha = 1; 8; 49; 288; 1681; 9800; 57121; 332\,928; 1940\,449; 11\,309\,768; \cdots$$

They were obtained by Ramanujan by expanding $\sqrt{8}$ as a continued fraction (see the first approximations in Appendix). The point is that the error term introduced in replacing $\sqrt{8}$ by its Diophantine approximation $(2n+1)/m$ is exactly balanced by the remainder $1/8m^2$. This situation is actually exceptional. Here we recall the sequence of integers p, q is a Diophantine approximation for $x \in \mathbf{R}^+$, iff

$$|x - \frac{p}{q}| < \frac{1}{q^2} \tag{2.2}$$

For (Q3) we find $3n(n+1) = 2m(2m+1)$, and excluding the trivial solutions for which $3n(n+1) = 2m(2m+1) = 0$,

$$(4m+1)^2 = 3(2n+1)^2 - 2 \tag{2.3}$$

This is again a Pell-Fermat Eq. Let $y = 4m+1$, $x = 2n+1$. Here we need to allow for half-integer values of m. Formal Calculus with Mathematica gives a fundamental sequence:

Proposition 2.1: *Diophantine equation $y^2 = 3x^2 - 2$ has a sequence of solutions, indexed by $\alpha \in \mathbf{N}^*$, of the form*

$$\begin{aligned} y_\alpha &= \frac{1}{2}\big((2+\sqrt{3})^\alpha(-1+\sqrt{3}) - (1+\sqrt{3})(2-\sqrt{3})^\alpha\big) \\ x_\alpha &= \frac{1}{6}\big((2+\sqrt{3})^\alpha(3-\sqrt{3}) + (3+\sqrt{3})(2-\sqrt{3})^\alpha\big) \end{aligned} \tag{2.4}$$

and they are the only ones when $x < 10^8$.

Here we list the 12 first couples (m_α, n_α). Actually the m_α's come in consecutive pairs of integers and half-integers.

$$m_\alpha = 4.5; 17.5; 66; 247; 922.5; 3443.5; 12\,852; 47\,965; 179\,008.5; 668\,069.5; 2\,493\,270;$$
$$n_\alpha = 5; 20; 76; 285; 1065; 3976; 14\,840; 55\,385; 206\,701; 771\,420; 2\,878\,980;$$

We check that all such $\frac{y_\alpha}{x_\alpha}$ belong to the sequence of Diophantine approximation of $\sqrt{3}$. But using (2.2) we see that all approximants (not only (2.4)) give a sequence of quasi-solutions, in the sense

$$\left|\frac{(4m_\alpha+1)^2}{(2n_\alpha+1)^2} - 3 - \frac{2}{(2n_\alpha+1)^2}\right| \leq \frac{C}{n_\alpha^2} \tag{2.5}$$

that is

$$\binom{n_\alpha+1}{2} - \frac{4}{3}\binom{m_\alpha}{2} - m_\alpha = \mathcal{O}(1),\ \alpha \to \infty \tag{2.6}$$

3/ Tetrahedric numbers and beyond.

In this Section we restrict to the median of the distribution of the c_{nk}, the other quartiles being investigated in Sect.4.

a) Tetrahedric numbers.

We need to solve

$$\binom{n+2}{3} = 2\binom{m+1}{3} + \binom{m+1}{2} \tag{$C3$}$$

or $n(n+1)(n+2) = m(m+1)(2m+1)$

The condition $n, m \geq 2$ implies to remove the trivial non negative solutions $(m,n) = (0,0), (1,1)$. We need also to remove negative values of n, m satisfying (C3), in particular those for which $n(n+1)(n+2) = m(m+1)(2m+1) = 0$. This situation occurs for all k, and (Ck) will always have at least a finite number of rational points.

Eq. (C3) can be rewritten as

$$4(n+1)^3 - 4(n+1) = (2m+1)^3 - (2m+1) \tag{3.2}$$

This time, Mathematica gives no sequence (m_α, n_α), but suggests instead to solve an equation of degree 3. If $P(x)$ is a polynomial of degree 3, with a positive discriminant $4p^3+27q^2$, it is known that the equation $P(x)=0$ has only one real root, which moreover can be expressed with real radicals of degree 2 and 3. This is indeed the case.

Proposition 3.1: *Consider equation* $4(n+1)^3 = (2m+1)^3 - (2m+1) + 4(n+1)$ *with unknown* $n \in \mathbf{R}$. *For* $m+1 \in \mathbf{R}^+$, *let* $A = 27(m+1) - 81(m+1)^2 + 54(m+1)^3$, *and* $B = \sqrt{-108 + 729(m+1-3(m+1)^2+2(m+1)^3)^2}$. *Then* n *is given by*

$$n+1 = \frac{2^{1/3}}{(A+B)^{1/3}} + \frac{(A+B)^{1/3}}{3\cdot 2^{1/3}} \tag{3.3}$$

Still within the range $n \le 10^8$, it follows by inspection that there are no integer solutions. But Diophantine approximation of $4^{1/3}$ still gives sequences of integers (m_α, n_α), verifying (3.2) mod $\mathcal{O}(n)$. The first terms are

$$m+1 = 10; 14; 114; 391; 1903; 2407; 74\,098;$$

$$n+1 = 12; 17; 143; 492; 2397; 3032; 93\,357;$$

Look now for $m+1 = p+\frac{1}{2}$ half-integer, $p \in \mathbf{N}^*$, we rewrite (3.2) in the form

$$4n(n+1)(n+2) = (2p-1)2p(2p+1) \tag{3.4}$$

Let $q = (2p-1)2p(2p+1)$. Inserting into (3.3) we find

$$n+1 = \frac{4+3\big(q+\sqrt{-\frac{64}{27}+q^2}\big)^{2/3}}{6\big(q+\sqrt{-\frac{64}{27}+q^2}\big)^{1/3}} \tag{3.5}$$

One checks that (3.5) holds for any $n \ge 1$, with $q = q(n) = 4n(n+1)(n+2)$. So we just recover (3.4), which is neither fulfilled for integer p, in the range $n \le 10^8$. So there are no solution to (C3) with m half-integer.

Still again, Diophantine approximations of $2^{1/3}$ give sequences of integers (m_α, n_α), verifying (3.4) mod $\mathcal{O}(n)$. The first terms are

$$m+1 = 3.5; 4.5; 23.5; 27.5; 50.5; 227.5;$$

$$n+1 = 4; 5; 29; 34; 63; 286;$$

b) 4-simplicial numbers.

Consider 4-simplicial numbers C_4, i.e.

$$\binom{n+3}{4} = 2\binom{m+2}{4} + \binom{m+2}{3} \tag{C4}$$

which we rewrite as $(n+3)(n+2)(n+1)n = 2(m+2)(m+1)^2 m$, The condition $n, m \ge 2$ implies again to remove the trivial non negative solutions $(m,n) = (0,0), (1,1)$.

This simplifies to

$$(2n+3)^4 - 32(m+1)^4 + 32(m+1)^2 - 10(2n+3)^2 + 9 = 0 \tag{3.6}$$

a quadratic equation in $i = (2n+3)^2 \geq 25, j = (m+1)^2 \geq 9$, so we start to solve Pell-Fermat equation $(i-5)^2 = 32j^2 - 32j + 16$, or with $u = i - 5$

$$u^2 = 8(2j-1)^2 + 8 \tag{3.7}$$

from which we remove the solutions $(u, j) = (4, 1),\ (20, 4)$. Mathematica gives us four fundamental sequences.

Proposition 3.2: Diophantine Eq. $u^2 = 8(2j-1)^2 + 8$ has at least 4 sequences of integer solutions indexed by $\alpha \in \mathbf{N}^*$, namely

$$\begin{aligned} u_\alpha &= (577 - 408\sqrt{2})^\alpha(2+\sqrt{2}) - (-2+\sqrt{2})(577+408\sqrt{2})^\alpha \\ j_\alpha &= \frac{1}{4}\big(2 + (-1+\sqrt{2})(577+408\sqrt{2})^\alpha - (577-408\sqrt{2})^\alpha(1+\sqrt{2})\big) \\ (u,j) &= (4,1),\ (20,4),\ (676,120),\ (780\,100, 137\,904),\ \&c \end{aligned} \tag{3.8}$$

$$\begin{aligned} u_\alpha &= (577 - 408\sqrt{2})^\alpha(10+7\sqrt{2}) + (10-7\sqrt{2})(577+408\sqrt{2})^\alpha \\ j_\alpha &= \frac{1}{4}\big(2 - (7+5\sqrt{2})(577-408\sqrt{2})^\alpha - (577+408\sqrt{2})^\alpha(7-5\sqrt{2})\big) \\ (u,j) &= (4,1),\ (20,4),\ (116,21),\ (133\,844, 23\,661),\ \&c \end{aligned} \tag{3.9}$$

$$\begin{aligned} u_\alpha &= (577 + 408\sqrt{2})^\alpha(2+\sqrt{2}) - (-2+\sqrt{2})(577-408\sqrt{2})^\alpha \\ j_\alpha &= \frac{1}{4}\big(2 + (1+\sqrt{2})(577+408\sqrt{2})^\alpha - (577-408\sqrt{2})^\alpha(-1+\sqrt{2})\big) \\ (u,j) &= (4,1),\ (20,4),\ (3\,940, 697),\ (4\,546\,756, 803\,761),\ \&c \end{aligned} \tag{3.10}$$

$$\begin{aligned} u_\alpha &= (577 + 408\sqrt{2})^\alpha(10+7\sqrt{2}) - (-10+7\sqrt{2})(577-408\sqrt{2})^\alpha \\ j_\alpha &= \frac{1}{4}\big(2 + (7+5\sqrt{2})(577+408\sqrt{2})^\alpha - (577-408\sqrt{2})^\alpha(-7+5\sqrt{2})\big) \\ (u,j) &= (4,1),\ (20,4),\ (22\,964, 460),\ (26\,500\,436, 4\,684\,660),\ \&c \end{aligned} \tag{3.11}$$

Except for $(u, j) = (4, 1),\ (20, 4)$ we have excluded, it seems by inspection that $(u+5, j)$ are never perfect squares. We can consider also consider the equation generalizing (3.7)

$$u^2 = 8v^2 + 8 \tag{3.12}$$

which admits also a fundamental sequence of integer solutions

$$\begin{aligned} u_\alpha &= (3 - 2\sqrt{2})^\alpha(2+\sqrt{2}) - (-2+\sqrt{2})(3+2\sqrt{2})^\alpha \\ v_\alpha &= \frac{1}{2}\big((3+2\sqrt{2})^\alpha(-1+\sqrt{2}) - (1+\sqrt{2})(3-2\sqrt{2})^\alpha\big) \end{aligned} \tag{3.13}$$

but neither leads to any integer solution to (3.7) (up to $n \leq 10^8$). So we try $v = w - 1/2$ half-integer, which gives $2j = w + 1/2$. Condition $j = (m-1)^2$ for m half-integer, $m = p + 1/2$, implies $w + 1/2 = 2(p - 1/2)^2 = 2p^2 - 2p + 1/2$, this gives the quadratic Eq. $2p^2 - 2p - w = 0$, which has an integer solution iff $2w + 1$ is a perfect square. But when $v = w - 1/2$, (3.12) can be written as $u^2 = 8w^2 - 8w + 10$, or else

$$u^2 = 2(2w-1)^2 + 8 \tag{3.14}$$

which is again a Pell-Fermat Eq. But contrary to (3.7) or (3.12) there are no integer solution to (3.14), cf. [Chr,p.483].

Still again, Diophantine approximation gives sequences of integers or half-integers (m_α, n_α), verifying (C4) mod $\mathcal{O}(n^2)$.

c) 5-simplicial numbers.

Consider 5-simplicial numbers C_5, i.e.

$$\binom{n+4}{5} = 2\binom{m+3}{5} + \binom{m+3}{4} \tag{$C5$}$$

with the condition $n, m \geq 2$, which we rewrite as $(n+4)(n+3)(n+2)(n+1)n = (m+3)(m+2)(m+1)m(2m+3)$, and we remove again the non negative solutions $(m,n) = (0,0), (1,1)$. This we rewrite as $P_5(n+2, 2m+3) = 0$ where P_5 as in (A.2). Mathematica gives no hint at solving this 5:th degree equation, and systematic search up to $n \leq 10^8$ either gives no integer solutions. But Diophantine approximation gives sequences of integers or half-integers (m_α, n_α), verifying (C5) mod $\mathcal{O}(n^3)$.

d) 6-simplicial numbers

Consider now 6-simplicial numbers, i.e.

$$\binom{n+5}{6} = 2\binom{m+4}{6} + \binom{m+4}{5} \tag{$C6$}$$

with the condition $n, m \geq 2$. This can be rewritten as $(n+5)(n+4)(n+3)(n+2)(n+1)n = 2(m+4)(m+3)(m+2)^2(m+1)m$, and we remove the solutions $(m,n) = (0,0), (1,1)$. The methods elaborated for (C3) will give at least 2 non-trivial solutions in the range $n \leq 10^8$. Let $x = 2n+5$, $y = 2m+4$, (C6) leads to

$$x^6 - 2y^6 - 35x^4 + 40y^4 + 259x^2 - 128y^2 - 225 = 0$$

or if we let $u = x^2$, $v = y^2$

$$u^3 - 2v^3 - 35u^2 + 40v^2 + 259u - 128v - 225 = 0 \tag{3.17}$$

As in Proposition 3.1 for (C3) Mathematica solves (3.17) by real radicals.

Proposition 3.3 *For any $u \in \mathbf{R}_+$, let*

$$A = 5770 + 13\,986u - 1890u^2 + 54u^3,\ \ B = 2\sqrt{-143\,982\,592 + \big(2885 + 27u(259 + (-35+u)u\big)^2}$$

Then $u^3 - 2v^3 - 35u^2 + 40v^2 + 259u - 128v - 225 = 0$ has the unique solution $v \in \mathbf{R}_+$ given by

$$v = \frac{20}{3} + \frac{416}{3(A+B)^{1/3}} + \frac{1}{6}(A+B)^{1/3} \tag{3.18}$$

For $v \leq 10^8$, it follows by inspection of (3.18), that the only integer solutions of (3.17), which are also perfect squares, are $(u, v) = (49, 36), (729, 576)$. They correspond to $(n, m) = (1, 1)$ (which is excluded) and $(n, m) = (11, 10)$. There are no solution of (3.17) in this interval with m half-integer. Again Diophantine approximation gives sequences of integers or half-integers (m_α, n_α), verifying (C6) mod $\mathcal{O}(n^4)$.

4/ First and third quartiles for distributions of the 3,4,5,6-simplicial numbers.

Equations (Q1) and (Q3) generalize to higher orders, leading to generalized Pell-Fermat equations with additional lower order terms provided we allow also for $m \in \mathbf{N} + 1/4$. Even if there are no exact integer solution, we can still find approximations in the Diophantine sense.

For C_3 the 1:st quartile is defined by

$$\binom{n+2}{3} = 4\binom{m+1}{3} + \binom{m+1}{2} \tag{Q13}$$

with the condition $n, m \geq 2$. This can be rewritten as $(n+1)(n+2)n = (4m-1)(m+1)m$, and we remove again the solutions $(m, n) = (0, 0), (1, 1)$. With $m + 1 = k + 1/4$, the latter equation takes the form

$$8(n+1)^3 = 4(2k-1)^3 + 8(n+1) - 7(2k-1) + 3 \tag{4.1}$$

which is a generalized Pell-Fermat equation with a constant term.

The 3:rd quartile is defined by

$$\binom{n+2}{3} = \frac{4}{3}\binom{m+1}{3} + \binom{m+1}{2} \tag{Q33}$$

with the condition $n, m \geq 2$. With $m + 1 = k - 1/4$, a similar computation leads to

$$24(n+1)^3 = 4(2k-1)^3 + 24(n+1) - 7(2k-1) - 3 \tag{4.2}$$

which is a generalized Pell-Fermat equation with a constant term.

Proposition 4.1: *Consider the quartiles (Q13) and (Q33) characterized by (4.1) and (4.2) resp. Then as in Proposition 3.1, the 3:rd degree equations can be solved in the following form: (...)*

For C_4 the 1:st quartile is defined by

$$\binom{n+3}{4} = 4\binom{m+2}{4} + \binom{m+2}{3} \tag{Q14}$$

with the condition $n, m \geq 2$. Reducing to a generalized Pell-Fermat equation with lower order terms leads to

$$(2n+3)^4 - 10(2n+3)^2 + 9 = 4(2m+\frac{3}{2})^4 + 106(2m+\frac{3}{2})^2 + 2772(2m+\frac{3}{2}) + 23409/4$$

Trying $m+2 = k - \frac{1}{4}$, $k \in \mathbf{N}$, we get

$$(2n+3)^4 - 10(2n+3)^2 = 4(2k-3)^4 + 106(2k-3)^2 + 2772(2k-3) + 23409/4 \tag{4.10}$$

which clearly has no integer solution, so is the case $m+2 = k + \frac{1}{4}$.

The 3:rd quartile is defined by

$$\binom{n+3}{4} = \frac{4}{3}\binom{m+2}{4} + \binom{m+2}{3} \tag{Q34}$$

with the condition $n, m \geq 2$. Reducing to a generalized Pell-Fermat equation with lower order terms leads to

$$3\big[(2n+3)^4 - 10(2n+3)^2 + 9\big] = 4\big[(2m+\frac{5}{2})^4 - \frac{11}{2}(2m+\frac{5}{2})^2 - 3(2m+\frac{5}{2}) - \frac{27}{8}\big]$$

As above the substitution $m+2 = k + \frac{1}{4}$, $k \in \mathbf{N}$ gives

$$6(2n+3)^4 - 60(2n+3)^2 = 8(2k-1)^4 - 44(2k-1)^2 - 24(2k-1) - 81 \tag{4.11}$$

which clearly has no integer solution, so is the case $m+2 = k - \frac{1}{4}$.

For C_6 the 3:rd quartile is defined by

$$\binom{n+5}{6} = \frac{4}{3}\binom{m+4}{6} + \binom{m+4}{5} \tag{Q36}$$

with the condition $n, m \geq 2$. We proceed as before but the computation is somewhat more tedious. We find

$$\begin{aligned} &3\big[(2n+5)^6 - 35(2n+5)^4 + 259(2n+5)^2 - 225\big] = \\ &4\big[(2m+8)^6 - 21(2m+8)^5 + 160(2m+8)^4 - 540(2m+8)^3 + 784(2m+8)^2 - 384(2m+8)\big] \end{aligned}$$

To eliminate the 2:nd term on the RHS we expand $(2m+8-\alpha)^6$ and are led to choose $\alpha = 7/2$. With $m+8 = k \pm \frac{1}{4}$ we find $2m+8-7/2 = 2k-3$, resp. $2m+8-7/2 = 2(k-2)$. When $m+8 = k+\frac{1}{4}$, we substitute $2k-3$ to $2m+8$ several times and find eventually

$$\begin{aligned} &3(2n+5)^6 - 105(2n+5)^4 + 777(2n+5)^2 = \\ &4(2k-3)^6 - 95(2k-3)^4 - 60(2k-3)^3 + \frac{1939}{4}(2k-3)^2 + \frac{1500}{4}(2k-3) + \frac{6075}{16} \end{aligned} \tag{4.12}$$

which is a generalized Pell-Fermat equation with lower order terms, but without a integer solution. The same conclusion holds for $m+8=k-\frac{1}{4}$. However there are always asymptotic solutions in the Diophantine sense, for instance for (4.12) it suffices to compute the approximants of $(4/3)^{1/6}$ as a continued fraction.

Note that it could turn out that there could still be integer solutions to (Q36) but not in the form $m+8=k\pm\frac{1}{4}$ (which provides asymptotic solutions).

Consider at last the 1:st quartile

$$\binom{n+5}{6}=4\binom{m+4}{6}+\binom{m+4}{5} \tag{$Q16$}$$

5/ Some arithmetics.

A straighforward way to find solutions to (Ck) is by trial. Using Mathematica we can easily check all values of n less than 10^8 say. None of them is a solution in that range, except the one for (C6). However to restrain somewhat the domain of numbers n,m we are looking for, we can first reduce Eq. (Ck) modulo prime numbers p, using the simple fact that if $a\in\mathbf{N}^*$, then $a^p=a[p]$, and if moreover p does not divide a, then $a^{p-1}=1[p]$. This reduces sometimes up to 50% the range of trials.

Apply this to (C3) (tetrahedral numbers) with $p=3$, we let $x=n+1, y=2m+1$ so that (3.2) reads $4(x^3-x)=y^3-y$, which always holds in $\mathbf{F}_3$, so this gives no criterion at all.

Consider now (C4) with $p=5$. With $x=2n+3, y=m+1$, we recall from (3.6) the identity $x^4-10x^2=32y^4-32y^2-9$. When 5 does not divide x and y, we find $32y^2-10x^2=2$ in $\mathbf{F}_5$. By inspection we find that this equation has no solution in $\mathbf{F}_3$ when $(x,y)\in\{1,2,3,4\}\times\{2,3\}$. When x or y vanish in $\mathbf{F}_3$, we find either that there are no solutions such that $(x,y)\in\{(0,1),(0,4),(0,0)\}$. So we can save about 50% of trials.

Consider (C5) with $p=5$. With $x=n+2, y=2m+3$, we recall from (3.15) the identity $16x^5-80x^3+64x=y^5-10y^3+9y$. Reducing mod 5, we find $80(x^3-x)=10(y^3-y)$, which always holds in $\mathbf{F}_5$, so again this gives no criterion.

Consider (C6) with $p=7$. With $x=2n+5, y=2m+4$, recall from (3.16) the identity $x^6-35x^4+259x^2=2y^6-40y^4+128y^2+225$. When 7 does not divide x and y, we find $40y^4-128y^2-226=0$ in $\mathbf{F}_7$. By inspection we find that this equation has no solution when $(x,y)\in\mathbf{F}_7^*\times\{1,6\}$. When 7 divides y but not x, the equation reduces to $x^6-1=0$, which always holds. When 7 divides x, the equation reduces to $2y^6-40y^4+128y^2+225=0$ in $\mathbf{F}_7$ we find that there are no solutions whenever $y\in\{2,3,4,5\}$. So again we can save about 50% of trials. Note that (C6) reduces to a polynomial equation of only one variable in $\mathbf{F}_7$, which may give a hint at the special case played by the 6-simplicial numbers.

6/ Asymmetric Pascal triangle

We consider floors C_2^a and C_3^a only, and content with Diophantine approximation, using also that, as n becomes large, the leading term of $\binom{n}{k}_a$ is $\binom{n}{k}$.

For C_2^a consider $\binom{n+1}{2}_a = \binom{n+1}{2} + an$, which we take to be equal to $2\binom{m}{2}_b + m + b = 2\binom{m}{2} + 2b(m-1) + m + b$ mod $\mathcal{O}(1)$ as $n \to \infty$. This gives

$$\binom{n+1}{2} - 2\binom{m}{2} - m = 2bm - an - b + \mathcal{O}(1)$$

We know that when $\frac{2m+1}{m}$ approximates $\sqrt{8}$ at this order, the LHS is 0, so we may replace m by $\frac{2n}{\sqrt{8}}$ in the RHS which gives $\left(\frac{4}{\sqrt{8}}b - a\right)n + \mathcal{O}(1)$, so we choose $b = a2^{-1/2}$.

For C_3^a consider $\binom{n+1}{3}_a = \binom{n+1}{3} + a\binom{n}{2}$, which we take to be equal mod $\mathcal{O}(n)$ to

$$2\binom{m}{3}_b + \binom{m}{2}_b = 2\binom{m}{3} + 2b\binom{m-1}{2} + \binom{m}{2} + b(m-1)$$

Using again Diophantine approximation of $4^{1/3}$ by $\frac{2m-1}{n}$ to this order, we need as before to cancel the term $bm^2 - an^2/2$, which gives $b = a2^{-1/3}$. So we proved

Proposition 5.1: At floor C_2^a, for $b = a2^{-1/2}$, we have

$$\binom{n+1}{2}_a = 2\binom{m}{2}_b + m + b + \mathcal{O}(1)$$

for the sequence (m_α, n_α) given in (2.2), while for $b = a2^{-1/3}$ at floor C_3^a, we have

$$\binom{n+1}{3}_a = 2\binom{m}{3}_b + \binom{m}{2}_b + \mathcal{O}(n)$$

for the sequence given by Diophantine approximation (this holds for integer and half-integer m).

Appendix

a) Table of Diophantine approximations.

$$\begin{aligned}
&2^{3/2} = 3 - \frac{1}{6-}\,\frac{1}{6-}\,\frac{1}{6-}\,\cdots = 3;\ \frac{17}{6};\ \frac{99}{35};\ \frac{577}{204};\\
&2^{2/3} = 1 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{2+}\,\cdots = 1;\ \frac{3}{2};\ \frac{8}{5};\ \frac{19}{22};\ \frac{27}{17};\ \frac{100}{63};\ \frac{227}{143};\ \frac{781}{492};\ \frac{1008}{635};\ \frac{3805}{2397};\ \frac{4813}{3032};\ \frac{14\,8195}{93\,357};\\
&2^{1/3} = 1;\ \frac{4}{3};\ \frac{5}{4};\ \frac{29}{23};\ \frac{34}{27};\ \frac{63}{50};\ \frac{286}{227};\\
&2^{1/2} = 1 + \frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\,\cdots;\ \frac{17}{12};\ \frac{99}{70};\ \frac{577}{408};\\
&(4/3)^{1/6} =
\end{aligned} \tag{A.1}$$

b) Some plane algebraic curves.

We have met the following polynomials, for which we can always find approximate integer roots in the Diophantine sense, when we compute approximants of the irrational number to the right:

$$\begin{array}{rlll}
P_2(x,y) = x^2 - 8y^2 - 1 & x = 2n+1 & y = m & 2^{3/2}\\
Q_2(x,y) = x^2 - 3y^2 + 2 & x = 2n+1 & y = 4m+1 & 3^{1/2}\\
P_3(x,y) = 4x^3 - y^3 - 4x + y & x = n & y = 2m-1 & 2^{2/3}\\
P_4(x,y) = x^4 - 32y^4 + 32y^2 - 10x^2 + 9 & x = 2n-1 & y = m-1 & 2^{5/4}\\
P_5(x,y) = 16x^5 - y^5 + 10y^3 - 80x^3 + 64x - 9y & x = n-1 & y = 2m-3 & 2^{4/5}\\
P_6(x,y) = x^6 - 2y^6 - 35x^4 + 40y^4 + 259x^2 - 128y^2 - 225 & x = 2n-3 & y = 2m-4 & 2^{1/6}
\end{array} \tag{A.2}$$

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