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GENERALIZED PELL-FERMAT EQUATIONS AND PASCAL TRIANGLE

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Abstract: Using Pascal triangle, we give a simple generalization to a well-known problem of S.Ramanujan. Thus we are interested in computing the median of some integer valued distributions, arising naturally when extending arithmetic progressions to progressions of polynomial growth. We show this reduces to equations of Pell-Fermat type of higher order, which admit very few integer solutions, but for which, following S.Ramanujan's original idea, we can always find integer sequences of best approximation, in the Diophantine sense. Our procedure relies much on formal Calculus with Mathematica.

1/ Introduction

a) Pascal triangle and progressions

Pascal triangle identifies with an infinite lower triangular matrix with elements $c_{nk} = \binom{n}{k}$, $0 \leq k \leq n$, the binomial coefficients. It contains many remarkable sequences : the sum of binomial coefficients in n :th row equals 2^n , the sum of diagonal elements are the terms of Fibonacci sequence, and the columns C_k generalize arithmetic sequences. Namely, the first column C_0 is the constant sequence equal to 1 ; the second one C_1 , which gives the partial sums of C_0 consists in the arithmetic sequence with ratio 1 and first term 1 ; the third one C_2 gives the partial sums of C_1 by using the relation $\binom{n}{1} + \binom{n}{2} = \binom{n+1}{2}$; the fourth one C_3 gives the partial sums of C_2 by $\binom{n}{2} + \binom{n}{3} = \binom{n+1}{3}$, etc... All sequences C_k have a polynomial growth $\mathcal{O}(n^k)$.

Pascal triangle also admits an asymmetric generalization. Namely, for $a \in \mathbf{C}$, replace the binomial coefficient c_{nk} by

$$c_{nk}^a = \binom{n}{k}_a = \binom{n}{k} + a \binom{n-1}{k-1}$$

Asymmetric Pascal triangle is built up the same way as standard Pascal triangle, and also extends to complex values of n (and complex values of k as well by using Euler Γ -function). In particular in the second column C_1^a we recognize the arithmetic sequence of ratio 1 and first term $1+a$. The term $a \binom{n-1}{k-1}$ accounts for a "lower order term" as n becomes large.

b) The "Houses of Ramanujan"

This problem, leading to a Pell-Fermat equation was reportedly discovered in quaint circumstances by S.Ramanujan [Ran] and solved (exactly) by the method of continued fractions. The history of this Diophantine equation goes to Archimedes, then proceeds to Bhaskara, Brahmagupta, Fermat, Wallis, Euler and Lagrange who almost brought the theory to its definite form.

The problem of the "Houses of Ramanujan" deals with column C_2 (we prefer to call them now the "floors") of Pascal triangle. We say that $m \in \mathbf{N}^* = \{1, 2, 3, \dots\}$ is a *House of Ramanujan at floor*

C_2 (or simply a House of Ramanujan) iff the sum $\binom{m}{2}$ of the $m - 1$ first integers “to the left of the House of Ramanujan” is equal to the sum of the next $n - m$ integers “to the right of the House of Ramanujan”, for some $n > m$, with the convention that $n = 1$ when $m = 1$, that is

$$(C2) \quad \binom{n+1}{2} = 2\binom{m}{2} + m$$

In the language of Statistics, m is the median of the cumulants $\sum_{i \leq j} i, j \leq n$. We can consider the first quartile as well, defined as the integer m such that

$$(Q1) \quad \binom{n+1}{2} = 4\binom{m}{2} + m$$

or the third quartile, defined as the integer m such that

$$(Q3) \quad \binom{n+1}{2} = \frac{4}{3}\binom{m}{2} + m$$

The point of course is to find integer solutions (n, m) . The quadratic case C_2 plays a peculiar role, and all solutions of (C2) are given by a sequence in $\mathbf{Z}[\sqrt{2}]$ with constant coefficients.

c) Main results

The situation becomes drastically different when investigating higher floors $C_k, k \geq 3$. As above, we say that $m \in \mathbf{N}^*$ is a House of Ramanujan at floor C_k iff the sum $\binom{m}{k}$ of the $m - 1$ first elements of C_k “to the left of the House of Ramanujan” is equal to the sum of the next elements of C_k “to the right of the House of Ramanujan”, for some $n > m$.

The corresponding equations, after some affine change of coordinates, take the form $P_k(x, y) = 0$, where P_k is polynomial with integer coefficients (see formulae (A.2) below), i.e. $P_k(x, y) = x^k - Dy^k + \dots$ (or permuting x with y) where the dots mean a polynomial of degree $\leq k - 2$ without mixed terms, and containing only monomials with the same parity as k . It turns out that we can always compute asymptotic solutions in the Diophantine sense, generalizing Ramanujan’s approach.

Next we address the problem of finding (exact) integer solutions, or pairs (m, n) with half-integer m . Our analysis uses Formal Calculus on Mathematica.

In case C_2 the method relies on known results for Pell-Fermat equation see [Wo] and references therein. While $u^2 - Dv^2 = 1$, D not a perfect square, has always integer solutions given by a (unique) sequence in $\mathbf{Z}[\sqrt{D}]$, this holds for $u^2 - Dv^2 = -1$ only for some values of D , using continued fractions. Once we have solved $u^2 - Dv^2 = \pm 1$, we can consider the more general Eq. $x^2 - Dy^2 = c$. Whenever we guess at a particular solution, we can build a sequence of solutions (called a fundamental sequence), using the sequences for $u^2 - Dv^2 = \pm 1$. So there exists so many fundamental sequences as “fundamental particular solutions” we can find for $x^2 - Dy^2 = c$. But such a family of solutions does not necessarily generate all solutions.

Applying this method for C_2 we find all solutions of (C2) (which was of course known before), and one fundamental sequence for (Q3). Other (related) solutions could be also found by considering the action of some discrete groups acting on the hyperbola of equation $x^2 - Dy^2 = c$, as is the case in the problem of quasi-isocetes triangles with a square angle. But we could not find new solutions this way.

Since P_4 is a polynomial in x^2 and y^2 , we can reduce C_4 to a Pell-Fermat Eq., then we are left to find the solutions which are perfect squares (quadrature). Within the range of values we have considered, i.e. $n \leq 10^8$, we have shown this way there are no integer solutions (or solutions with half-integer m), when using the fundamental sequences given by Mathematica.

The case C_3 and C_6 rely instead on the resolution of a 3:rd degree polynomial, which makes use of real radicals only. For C_3 it follows by inspection that there are no integer solutions (or solutions with half-integer m) in the range considered. The same method (together with a quadrature) applies to C_6 , which this time yields two solutions $(n, m) = (10, 5), (15, 14)$ (and only two) in that range.

In case C_5 we could not find any analytical method, and the only way is the systematic search in the range $n \leq 10^8$, which doesn't reveal any integer solution. A fortiori, there seems to be no analytical methods for $k = 7$ and beyond.

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2/ Floor C_2

Eq. (Q1) actually solves simply as $m = (n+1)/2$, and Eq. (C2), (Q3) are Pell-Fermat equations. For (C2) (the equation considered by Ramanujan) we find $n(n+1) = 2m^2$ or

$$(2.1) \quad (2n+1)^2 = 8m^2 + 1$$

It is well-known that all "physical" solutions, besides this for which $n(n+1) = 2m^2 = 0$, are given by the sequence indexed over $\alpha \in \mathbf{N}^*$

$$(2.1) \quad n_\alpha = \frac{1}{4}((3-2\sqrt{2})^\alpha + (3+2\sqrt{2})^\alpha - 2), \quad m_\alpha = -\frac{\sqrt{2}}{8}((3-2\sqrt{2})^\alpha - (3+2\sqrt{2})^\alpha)$$

Here we list the first pairs (m, n)

$$m_\alpha = 1; 6; 35; 204; 1189; 6930; 40\,391; 235\,416; 1372\,105; 7997\,214; \dots$$

$$n_\alpha = 1; 8; 49; 288; 1681; 9800; 57121; 332\,928; 1940\,449; 11\,309\,768; \dots$$

They were obtained by Ramanujan by expanding $\sqrt{8}$ as a continued fraction (see the first approximations in Appendix). The point is that the error term introduced in replacing $\sqrt{8}$ by its Diophantine approximation $(2n+1)/m$ is exactly balanced by the remainder $1/8m^2$. This situation is actually exceptional. Here we recall the sequence of integers p, q is a Diophantine approximation for $x \in \mathbf{R}^+$, iff

$$(2.2) \quad \left| x - \frac{p}{q} \right| < \frac{1}{q^2}$$

For (Q3) we find $3n(n+1) = 2m(2m+1)$, and excluding the trivial solutions for which $3n(n+1) = 2m(2m+1) = 0$,

$$(2.3) \quad (4m+1)^2 = 3(2n+1)^2 - 2$$

This is again a Pell-Fermat Eq. Let $y = 4m+1$, $x = 2n+1$. Here we need to allow for half-integer values of m . Formal Calculus with Mathematica gives a fundamental sequence:

Proposition 2.1: *Diophantine equation $y^2 = 3x^2 - 2$ has a sequence of solutions, indexed by $\alpha \in \mathbf{N}^*$, of the form*

$$(2.4) \quad \begin{aligned} y_\alpha &= \frac{1}{2}((2 + \sqrt{3})^\alpha(-1 + \sqrt{3}) - (1 + \sqrt{3})(2 - \sqrt{3})^\alpha) \\ x_\alpha &= \frac{1}{6}((2 + \sqrt{3})^\alpha(3 - \sqrt{3}) + (3 + \sqrt{3})(2 - \sqrt{3})^\alpha) \end{aligned}$$

and they are the only ones when $x < 10^8$.

Here we list the 12 first couples (m_α, n_α) . Actually the m_α 's come in consecutive pairs of integers and half-integers.

$$\begin{aligned} m_\alpha &= 4.5; 17.5; 66; 247; 922.5; 3443.5; 12\,852; 47\,965; 179\,008.5; 668\,069.5; 2\,493\,270; \\ n_\alpha &= 5; 20; 76; 285; 1065; 3976; 14\,840; 55\,385; 206\,701; 771\,420; 2\,878\,980; \end{aligned}$$

We check that all such $\frac{y_\alpha}{x_\alpha}$ belong to the sequence of Diophantine approximation of $\sqrt{3}$. But using (2.2) we see that all approximants (not only (2.4)) give a sequence of quasi-solutions, in the sense

$$(2.5) \quad \left| \frac{(4m_\alpha + 1)^2}{(2n_\alpha + 1)^2} - 3 - \frac{2}{(2n_\alpha + 1)^2} \right| \leq \frac{C}{n_\alpha^2}$$

that is

$$(2.6) \quad \binom{n_\alpha + 1}{2} - \frac{4}{3} \binom{m_\alpha}{2} - m_\alpha = \mathcal{O}(1), \quad \alpha \rightarrow \infty$$

3/ Upstairs

For simplicity we restrict to the median of the distribution of the c_{nk} , for the other quartiles give raise to more complicated polynomials.

a) *Floor C_3 .*

We need to solve

$$(C3) \quad \binom{n+1}{3} = 2 \binom{m}{3} + \binom{m}{2}$$

or $n(n-1)(n+1) = m(m-1)(2m-1)$, excluding the trivial solutions for which $n(n-1)(n+1) = m(m-1)(2m-1) = 0$. Eq. (C3) can be rewritten as

$$(3.2) \quad 4n^3 = (2m-1)^3 - (2m-1) + 4n$$

This time, Mathematica gives no sequence (m_α, n_α) , but suggests instead to solve an equation of degree 3. If $P(x)$ is a polynomial of degree 3, with a positive discriminant $4p^3 + 27q^2$, it is known that the equation $P(x) = 0$ has only one real root, which moreover can be expressed with real radicals of degree 2 and 3. This is indeed the case.

Proposition 3.1: *Consider equation $4n^3 = (2m - 1)^3 - (2m - 1) + 4n$ with unknown $n \in \mathbf{R}$. For $m \in \mathbf{R}^+$, let $A = 27m - 81m^2 + 54m^3$, and $B = \sqrt{-108 + 729(m - 3m^2 + 2m^3)^2}$. Then n is given by*

$$(3.3) \quad n = \frac{2^{1/3}}{(A+B)^{1/3}} + \frac{(A+B)^{1/3}}{3 \cdot 2^{1/3}}$$

Still within the range $n \leq 10^8$, it follows by inspection that there are no integer solutions. But Diophantine approximation of $4^{1/3}$ still gives sequences of integers (m_α, n_α) , verifying (3.2) mod $\mathcal{O}(n)$. The first terms are

$$\begin{aligned} m &= 10; 14; 114; 391; 1903; 2407; 74\,098; \\ n &= 12; 17; 143; 492; 2397; 3032; 93\,357; \end{aligned}$$

Look now for $m = p + \frac{1}{2}$ half-integer, $p \in \mathbf{N}^*$, we rewrite (3.2) in the form

$$(3.4) \quad 4(n-1)n(1+n) = (2p-1)2p(2p+1)$$

Let $q = (2p-1)2p(2p+1)$. Inserting into (3.3) we find

$$(3.5) \quad n = \frac{4 + 3(q + \sqrt{-\frac{64}{27} + q^2})^{2/3}}{6(q + \sqrt{-\frac{64}{27} + q^2})^{1/3}}$$

One checks that (3.5) holds for any $n \geq 2$, with $q = q(n) = 4(n-1)n(n+1)$. So we just recover (3.4), which is neither fulfilled for integer p , in the range $n \leq 10^8$. So there are no solution to (C3) with m half-integer.

Still again, Diophantine approximations of $2^{1/3}$ give sequences of integers (m_α, n_α) , verifying (3.4) mod $\mathcal{O}(n)$. The first terms are

$$\begin{aligned} m &= 3.5; 4.5; 23.5; 27.5; 50.5; 227.5; \\ n &= 4; 5; 29; 34; 63; 286; \end{aligned}$$

b) *Floor C_4 .*

Consider floor C_4 , i.e.

$$(C4) \quad \binom{n+1}{4} = 2 \binom{m}{4} + \binom{m}{3}$$

or $(n+1)n(n-1)(n-2) = 2m(m-1)^2(m-2)$, from which we remove as usual the trivial solutions of $(n+1)n(n-1)(n-2) = 2m(m-1)^2(m-2) = 0$. This simplifies to

$$(3.6) \quad (2n-1)^4 - 32(m-1)^4 + 32(m-1)^2 - 10(2n-1)^2 + 9 = 0$$

a quadratic equation in $i = (2n-1)^2, j = (m-1)^2$, so we start to solve Pell-Fermat equation $(i-5)^2 = 32j^2 - 32j + 16$, or

$$(3.7) \quad u^2 = 8(2j-1)^2 + 8$$

Mathematica gives us four fundamental sequences.

Proposition 3.2: Diophantine Eq. $u^2 = 8(2j-1)^2 + 8$ has at least 4 sequences of integer solutions indexed by $\alpha \in \mathbf{N}^*$, namely

$$(3.8) \quad \begin{aligned} u_\alpha &= (577 - 408\sqrt{2})^\alpha (2 + \sqrt{2}) - (-2 + \sqrt{2})(577 + 408\sqrt{2})^\alpha \\ j_\alpha &= \frac{1}{4} (2 + (-1 + \sqrt{2})(577 + 408\sqrt{2})^\alpha - (577 - 408\sqrt{2})^\alpha (1 + \sqrt{2})) \\ (u, j) &= (4, 1), (20, 4), (676, 120), (780\,100, 137\,904), \&c \end{aligned}$$

$$(3.9) \quad \begin{aligned} u_\alpha &= (577 - 408\sqrt{2})^\alpha (10 + 7\sqrt{2}) + (10 - 7\sqrt{2})(577 + 408\sqrt{2})^\alpha \\ j_\alpha &= \frac{1}{4} (2 - (7 + 5\sqrt{2})(577 - 408\sqrt{2})^\alpha - (577 + 408\sqrt{2})^\alpha (7 - 5\sqrt{2})) \\ (u, j) &= (4, 1), (20, 4), (116, 21), (133\,844, 23\,661), \&c \end{aligned}$$

$$(3.10) \quad \begin{aligned} u_\alpha &= (577 + 408\sqrt{2})^\alpha (2 + \sqrt{2}) - (-2 + \sqrt{2})(577 - 408\sqrt{2})^\alpha \\ j_\alpha &= \frac{1}{4} (2 + (1 + \sqrt{2})(577 + 408\sqrt{2})^\alpha - (577 - 408\sqrt{2})^\alpha (-1 + \sqrt{2})) \\ (u, j) &= (4, 1), (20, 4), (3\,940, 697), (4\,546\,756, 803\,761), \&c \end{aligned}$$

$$(3.11) \quad \begin{aligned} u_\alpha &= (577 + 408\sqrt{2})^\alpha (10 + 7\sqrt{2}) - (-10 + 7\sqrt{2})(577 - 408\sqrt{2})^\alpha \\ j_\alpha &= \frac{1}{4} (2 + (7 + 5\sqrt{2})(577 + 408\sqrt{2})^\alpha - (577 - 408\sqrt{2})^\alpha (-7 + 5\sqrt{2})) \\ (u, j) &= (4, 1), (20, 4), (22\,964, 460), (26\,500\,436, 4\,684\,660), \&c \end{aligned}$$

Except for trivial solutions $(u, j) = (4, 1), (20, 4)$ it seems by inspection that $(u+5, j)$ are never perfect squares.

We can consider also consider the equation generalizing (3.7)

$$(3.12) \quad u^2 = 8v^2 + 8$$

which admits also a fundamental sequence of integer solutions

$$(3.13) \quad \begin{aligned} u_\alpha &= (3 - 2\sqrt{2})^\alpha (2 + \sqrt{2}) - (-2 + \sqrt{2})(3 + 2\sqrt{2})^\alpha \\ v_\alpha &= \frac{1}{2} ((3 + 2\sqrt{2})^\alpha (-1 + \sqrt{2}) - (1 + \sqrt{2})(3 - 2\sqrt{2})^\alpha) \end{aligned}$$

but neither leads to any integer solution to (3.7) (up to $n \leq 10^8$). So we try $v = w - 1/2$ half-integer, which gives $2j = w + 1/2$. Condition $j = (m - 1)^2$ for m half-integer, $m = p + 1/2$, implies $w + 1/2 = 2(p - 1/2)^2 = 2p^2 - 2p + 1/2$, this gives the quadratic Eq. $2p^2 - 2p - w = 0$, which has an integer solution iff $2w + 1$ is a perfect square. But when $v = w - 1/2$, (3.12) can be written as $u^2 = 8w^2 - 8w + 10$, or else

$$(3.14) \quad u^2 = 2(2w - 1)^2 + 8$$

which is again a Pell-Fermat Eq. But contrary to (3.7) or (3.12) there are no integer solution to (3.14), cf. [Chr,p.483].

Still again, Diophantine approximation gives sequences of integers or half-integers (m_α, n_α) , verifying (C4) mod $\mathcal{O}(n^2)$.

c) Floor C_5

(C5) can be rewritten as $(n + 1)n(n - 1)(n - 2)(n - 3) = m(m - 1)(m - 2)(m - 3)(2m - 3)$, or $P_5(n - 1, 2m - 3) = 0$ where P_5 as in (A.2). Mathematica gives no hint at solving this 5:th degree equation, and systematic search up to $n \leq 10^8$ either gives no integer solutions. But Diophantine approximation gives sequences of integers or half-integers (m_α, n_α) , verifying (C5) mod $\mathcal{O}(n^3)$.

d) Floor C_6

Consider now floor C_6 , i.e.

$$(C6) \quad \binom{n+1}{6} = 2\binom{m}{6} + \binom{m}{5}$$

or $(n+1)n(n-1)(n-2)(n-3)(n-4) = 2m(m-1)(m-2)^2(m-3)(m-4)$. As usual we exclude the non-physical solutions such that $(n+1)n(n-1)(n-2)(n-3)(n-4) = 2m(m-1)(m-2)^2(m-3)(m-4) = 0$. The methods elaborated for (C3) will give at least 2 non-trivial solutions in the range $n \leq 10^8$. Let $x = 2n - 3$, $y = 2m - 4$, (C6) leads to

$$x^6 - 2y^6 - 35x^4 + 40y^4 + 259x^2 - 128y^2 - 225 = 0$$

or if we let $u = x^2$, $v = y^2$

$$(3.17) \quad u^3 - 2v^3 - 35u^2 + 40v^2 + 259u - 128v - 225 = 0$$

As in Proposition 3.1 for (C3) Mathematica solves (3.17) by real radicals.

Proposition 3.3 For any $u \in \mathbf{R}_+$, let

$$A = 5770 + 13986u - 1890u^2 + 54u^3, \quad B = 2\sqrt{-143982592 + (2885 + 27u(259 + (-35 + u)u))^2}$$

Then $u^3 - 2v^3 - 35u^2 + 40v^2 + 259u - 128v - 225 = 0$ has the unique solution $v \in \mathbf{R}_+$ given by

$$(3.18) \quad v = \frac{20}{3} + \frac{416}{3(A+B)^{1/3}} + \frac{1}{6}(A+B)^{1/3}$$

For $v \leq 10^8$, it follows by inspection of (3.18), that the only integer solutions of (3.17), which are also perfect squares, are $(u, v) = (49, 36), (729, 576)$. They correspond to $(n, m) = (10, 5), (15, 14)$. There are no solution of (3.17) in this interval with m half-integer. Again Diophantine approximation gives sequences of integers or half-integers (m_α, n_α) , verifying (C6) mod $\mathcal{O}(n^4)$.

4/ Asymmetric Pascal triangle

We consider floors C_2^a and C_3^a only, and content with Diophantine approximation, using also that, as n becomes large, the leading term of $\binom{n}{k}_a$ is $\binom{n}{k}$.

For C_2^a consider $\binom{n+1}{2}_a = \binom{n+1}{2} + an$, which we take to be equal to $2\binom{m}{2}_b + m + b = 2\binom{m}{2} + 2b(m-1) + m + b \bmod \mathcal{O}(1)$ as $n \rightarrow \infty$. This gives

$$\binom{n+1}{2} - 2\binom{m}{2} - m = 2bm - an - b + \mathcal{O}(1)$$

We know that when $\frac{2m+1}{m}$ approximates $\sqrt{8}$ at this order, the LHS is 0, so we may replace m by $\frac{2n}{\sqrt{8}}$ in the RHS which gives $(\frac{4}{\sqrt{8}}b - a)n + \mathcal{O}(1)$, so we choose $b = a2^{-1/2}$.

For C_3^a consider $\binom{n+1}{3}_a = \binom{n+1}{3} + a\binom{n}{2}$, which we take to be equal mod $\mathcal{O}(n)$ to

$$2\binom{m}{3}_b + \binom{m}{2}_b = 2\binom{m}{3} + 2b\binom{m-1}{2} + \binom{m}{2} + b(m-1)$$

Using again Diophantine approximation of $4^{1/3}$ by $\frac{2m-1}{n}$ to this order, we need as before to cancel the term $bm^2 - an^2/2$, which gives $b = a2^{-1/3}$. So we proved

Proposition 4.1: At floor C_2^a , for $b = a2^{-1/2}$, we have

$$\binom{n+1}{2}_a = 2\binom{m}{2}_b + m + b + \mathcal{O}(1)$$

for the sequence (m_α, n_α) given in (2.2), while for $b = a2^{-1/3}$ at floor C_3^a , we have

$$\binom{n+1}{3}_a = 2\binom{m}{3}_b + \binom{m}{2}_b + \mathcal{O}(n)$$

for the sequence given by Diophantine approximation (this holds for integer and half-integer m).

Appendix

a) Table of Diophantine approximations.

$$(A.1) \quad \begin{aligned} 2^{3/2} &= 3 - \frac{1}{6-} \frac{1}{6-} \frac{1}{6-} \cdots = 3; \frac{17}{6}; \frac{99}{35}; \frac{577}{204}; \\ 2^{2/3} &= 1 + \frac{1}{1+} \frac{1}{1+} \frac{1}{2+} \frac{1}{2+} \cdots = 1; \frac{3}{2}; \frac{8}{5}; \frac{19}{22}; \frac{27}{17}; \frac{100}{63}; \frac{227}{143}; \frac{781}{492}; \frac{1008}{635}; \frac{3805}{2397}; \frac{4813}{3032}; \frac{148195}{93357}; \\ 2^{1/3} &= 1; \frac{4}{3}; \frac{5}{4}; \frac{29}{23}; \frac{34}{27}; \frac{63}{50}; \frac{286}{227}; \\ 2^{1/2} &= 1 + \frac{1}{2+} \frac{1}{2+} \frac{1}{2+} \cdots; \frac{17}{12}; \frac{99}{70}; \frac{577}{408}; \end{aligned}$$

b) *Some plane algebraic curves.*

We have met the following polynomials, for which we can always find approximate integer roots in the Diophantine sense, when we compute approximants of the irrational number to the right:

$$\begin{array}{llll}
 P_2(x, y) = x^2 - 8y^2 - 1 & x = 2n + 1 & y = m & 2^{3/2} \\
 Q_2(x, y) = x^2 - 3y^2 + 2 & x = 2n + 1 & y = 4m + 1 & 3^{1/2} \\
 P_3(x, y) = 4x^3 - y^3 - 4x + y & x = n & y = 2m - 1 & 2^{2/3} \\
 P_4(x, y) = x^4 - 32y^4 + 32y^2 - 10x^2 + 9 & x = 2n - 1 & y = m - 1 & 2^{5/4} \\
 P_5(x, y) = 16x^5 - y^5 + 10y^3 - 80x^3 + 64x - 9y & x = n - 1 & y = 2m - 3 & 2^{4/5} \\
 P_6(x, y) = x^6 - 2y^6 - 35x^4 + 40y^4 + 259x^2 - 128y^2 - 225 & x = 2n - 3 & y = 2m - 4 & 2^{1/6}
 \end{array}
 \tag{A.2}$$

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