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## - To cite this version:

Fabien Brosset, Thierry Klein, Agnès Lagnoux, P Petit. Probabilistic proofs of large deviation results for sums of semiexponential random variables and explicit rate function at the transition. 2020. hal-02895684v1

HAL Id: hal-02895684
https://hal.science/hal-02895684v1
Preprint submitted on 15 Jul 2020 (v1), last revised 19 Jan 2021 (v2)

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# Probabilistic proofs of large deviation results for sums of semiexponential random variables and explicit rate function at the transition 

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July 15, 2020


#### Abstract

Asymptotics deviation probabilities of the sum $S_{n}=X_{1}+\cdots+X_{n}$ of independent and identically distributed real-valued random variables have been extensively investigated, in particular when $X_{1}$ is not exponentially integrable. For instance, A.V. Nagaev formulated exact asymptotics results for $\mathbb{P}\left(S_{n}>x_{n}\right)$ when $x_{n}>n^{1 / 2}$ (see, [13, 14). In this paper, we derive rough asymptotics results (at logarithmic scale) with shorter proofs relying on classical tools of large deviation theory and expliciting the rate function at the transition.


Key words: large deviations, sums of independent and identically distributed random variables, semiexponential random variables, Gärtner-Ellis theorem, contraction principle.
AMS subject classification: 60F10, 60G50.

## 1 Introduction

Moderate and large deviations of the sum of independent and identically distributed (i.i.d.) real-valued random variables have been investigated since the beginning of the 20th century. Kinchin [9] in 1929 was the first to give a result on large deviations of i.i.d. Bernoulli distributed random variables. In 1933, Smirnov [19] improved this result and in 1938 Cramér [4] gave a generalization to i.i.d. random variables satisfying the eponymous Cramér's condition which requires the Laplace transform of the common distribution of the random variables to be finite in a neighborhood of zero. Cramér's result was extended by Feller [7] to sequences of not necessarily identically distributed random variables under restrictive conditions (Feller considered only random variables taking values in bounded intervals), thus Cramér's result does not follow from Feller's result. A strengthening of Cramér's theorem was given by Petrov in [16] together with a generalization to the case of non-identically distributed random variables. Improvements of Petrov's result can be found in 17. When Cramér's condition does not hold, an early result is due to Linnik 11 in 1961 and concerns polynomial-tailed random variables. The case where the tail decreases faster than all power functions (but not enough for Cramér's condition to be satisfied) has been considered by Petrov [16] and by S.V. Nagaev [15]. In [13, 14, A.V. Nagaev studied the case where the commom distribution of the i.i.d. random variables is absolutely continuous with respect to the Lebesgue measure with density $p(t) \sim e^{-|t|^{1-\epsilon}}$ as $|t|$ tends to
infinity, with $\epsilon \in(0,1)$. He distinguished five exact-asymptotics results corresponding to five types of deviation speeds. In [2, 3], Borovkov investigated exact asymptotics of the deviations probability for semiexponentially-tailed random variables (i.e. the tail writes as $e^{-t^{1-\epsilon} L(t)}$, where $\epsilon \in(0,1)$ and $L$ is a suitably slowly varying function at infinity).
Now, let us present the setting of this article. Let $\epsilon \in(0,1)$ and let $X$ be a real-valued random variable with a density $p$ with respect to the Lebesgue measure verifying:

$$
\begin{equation*}
p(x) \sim e^{-x^{1-\epsilon}}, \quad \text { as } \quad x \rightarrow+\infty \tag{1}
\end{equation*}
$$

Unlike in [13, 14], this unilateral assumption is motivated by the fact that we focus on upper deviations of the sum. Observe that (11) implies that the Laplace transform of $X$ is not defined on the right side of zero. Nevertheless, all moments of $X_{+}:=\max (X, 0)$ are finite. A weaker assumption on the left tail is required:

$$
\begin{equation*}
\exists \gamma \in(0,1] \quad \rho:=\mathbb{E}\left[|X|^{2+\gamma}\right]<\infty . \tag{2}
\end{equation*}
$$

Finally, we will assume that $X$ is centered $(\mathbb{E}[X]=0)$ and will denote by $\sigma$ the standard deviation of $X\left(\operatorname{Var}(X)=\sigma^{2}\right)$.
For all $n \in \mathbb{N}^{*}$, let $X_{1}, X_{2}, \ldots, X_{n}$ be i.i.d. copies of $X$. We set $S_{n}=X_{1}+\cdots+X_{n}$ and $P_{n}(x)=\mathbb{P}\left(S_{n} \geqslant x\right)$. In this paper, we are interested in the asymptotic behavior of $\log P_{n}\left(x_{n}\right)$ for any positive sequence $x_{n} \gg n^{1 / 2}$. Not only does the logarithmic scale allow us to use the modern theory of large deviations and provide simpler proofs than in [2, 33, 13, 14], but we also obtain more explicit results. According to the asymptotics of $x_{n}$, only three logarithmic asymptotic ranges appear. First, the Gaussian range: when $x_{n} \ll n^{1 /(1+\epsilon)}, \log P_{n}\left(x_{n}\right) \sim \log \left(1-\phi\left(n^{-1 / 2} x_{n}\right)\right)$, $\phi$ being the distribution function of the standard Gaussian law. Next, the domain of validity of the maximal jump principle: when $x_{n} \gg n^{1 /(1+\epsilon)}, \log P_{n}\left(x_{n}\right) \sim \log \mathbb{P}\left(\max \left(X_{1}, \ldots, X_{n}\right) \geqslant x_{n}\right)$. Finally, the transition $\left(x_{n}=\Theta\left(n^{1 /(1+\epsilon)}\right)\right)$ appears to be an interpolation between the Gaussian range and the maximal jump one.
Logarithmic asymptotics were also considered in [10] for a wider class of distributions than in the present paper. Nevertheless, the setting was restricted to the particular sequence $x_{n}=n$ (that lies in the maximum jump range). In [6], the authors gave a necessary and sufficient condition on the logarithmic tails of the sum of i.i.d. real-valued random variables to satisfy a large deviation principle which covers the Gaussian range. In [1], Arcones proceeded analogously and covered the maximum jump range. Observe that, in the two previous papers, additional restrictions on $x_{n}$ were required. In [8], the author studied the more general case of semi-exponential upper tails, in a particular domain of the maximum jump range.
The transition at $x_{n}=\Theta\left(n^{1 /(1+\epsilon)}\right)$ is not considered in [2, 3]. It is treated in [13, 14, Theorems 2 and 4]. Nevertheless, the rate function is given through a non explicit inductive formula and hence is difficult to interpret. One of the main contributions of this work is to provide an explicit formula for the rate function thanks tools coming from large deviation techniques.
The paper is organized as follows. In Section 2, we state the main results. Section 3 is devoted to useful results, the proof of which are postponed to Section [5. In particular, we give unilateral versions of Gärtner-Ellis theorem inspired from [18] (see Theorem (10) and of the contraction principle (see Proposition [8). The proof of the main results can be found in Section (4).

## 2 Main results

In this section, Theorem $\mathbb{1}$ puts forward the domain of validity of the maximal jump principle at the logarithmic scale. In this range, the main contribution of the deviations of $S_{n}$ is due to one summand, meaning $\log P_{n}\left(x_{n}\right) \sim \log \mathbb{P}\left(X \geqslant x_{n}\right)$. A Gaussian range, determined in Theorem 2, occurs when all summands contribute to the deviations of $S_{n}$ in the sense that $\log P_{n}\left(x_{n}\right) \sim \log \mathbb{P}\left(S_{n} \geqslant x_{n}, \forall i \in \llbracket 1, n \rrbracket \quad X_{i}<x_{n}^{\epsilon}\right)$, unlike in the previous range. Finally, the transition range is studied in Theorem 3.

Theorem 1 (Maximal jump range). For $x_{n} \gg n^{1 /(1+\epsilon)}$, setting $M_{n}:=\max \left(X_{1}, \ldots, X_{n}\right)$,

$$
\lim _{n \rightarrow \infty} \frac{1}{x_{n}^{1-\epsilon}} \log P_{n}\left(x_{n}\right)=\lim _{n \rightarrow \infty} \frac{1}{x_{n}^{1-\epsilon}} \log \mathbb{P}\left(M_{n} \geqslant x_{n}\right)=\lim _{n \rightarrow \infty} \frac{1}{x_{n}^{1-\epsilon}} \log \mathbb{P}\left(X \geqslant x_{n}\right)=-1 .
$$

Theorem 2 (Gaussian range). For $n^{1 / 2} \ll x_{n} \ll n^{1 /(1+\epsilon)}$, we have:

$$
\lim _{n \rightarrow \infty} \frac{n}{x_{n}^{2}} \log P_{n}\left(x_{n}\right)=-\frac{1}{2 \sigma^{2}} .
$$

The sequence $n^{1 /(1+\epsilon)}$ is the solution (up to a scalar factor) of the equation in $x_{n}: x_{n}^{2} /\left(2 n \sigma^{2}\right)=$ $x_{n}^{1-\epsilon}$, equalizing the asymptotics of $\log P_{n}\left(x_{n}\right)$ obtained in the Gaussian range and in the maximal jump range. In Theorem 3, we give the asymptotic of $\log P_{n}\left(x_{n}\right)$ when $x_{n}=C n^{1 /(1+\epsilon)}, C>0$.
Theorem 3 (Transition). For $C>0$ and $x_{n}=C n^{1 /(1+\epsilon), ~}$

$$
\lim _{n \rightarrow \infty} \frac{1}{x_{n}^{1-\epsilon}} \log P_{n}\left(x_{n}\right)=-\inf _{0 \leqslant \leqslant 1}\left\{(1-t)^{1-\epsilon}+\frac{t^{2} C^{1+\epsilon}}{2 \sigma^{2}}\right\}=:-J(C) .
$$

Remark 4. Let $f(t)=(1-t)^{1-\epsilon}+t^{2} C^{1+\epsilon} /\left(2 \sigma^{2}\right)$. An easy computation shows that, if $C \leqslant$ $C_{\epsilon}^{\prime}:=(1+\epsilon)\left((1-\epsilon) \sigma^{2} \epsilon^{-\epsilon}\right)^{1 /(1+\epsilon)}, f$ is decreasing and its minimum $C^{1+\epsilon} /\left(2 \sigma^{2}\right)$ is attained at $t=1$. If $C>C_{\epsilon}^{\prime}, f$ has two local minima, at 1 and at $t(C)$ : the latter corresponds to the smallest of the two roots in $[0,1]$ of $f^{\prime}(t)=0$, equation equivalent to

$$
\begin{equation*}
t(1-t)^{\epsilon}=\frac{(1-\epsilon) \sigma^{2}}{C^{1+\epsilon}} . \tag{3}
\end{equation*}
$$

If $C_{\epsilon}^{\prime}<C \leqslant C_{\epsilon}:=(1+\epsilon)\left(\sigma^{2}(2 \epsilon)^{-\epsilon}\right)^{1 /(1+\epsilon)}$, then $f(t(C)) \geqslant f(1)$. And, if $C>C_{\epsilon}$, then $f(t(C))<f(1)$. As a consequence, for all $C>0$,

$$
J(C)= \begin{cases}\frac{C^{1+\epsilon}}{2 \sigma^{2}} & \text { if } C \leqslant C_{\epsilon}  \tag{4}\\ (1-t(C))^{1-\epsilon}+\frac{t(C)^{2} C^{1+\epsilon}}{2 \sigma^{2}} & \text { if } C>C_{\epsilon}\end{cases}
$$

Remark 5. The transition interpolates between the Gaussian range and the maximal jump one. First, when $x_{n}=C n^{1 /(1+\epsilon)}$, we get $x_{n}^{2} / n=C^{1+\epsilon} x_{n}^{1-\epsilon}$, thus the asymptotics of the Gaussian range coincide with the one of the transition for $C \leqslant C_{\epsilon}$. Moreover, $t\left(C_{\epsilon}\right)=(1-\epsilon) /(1+\epsilon)$ and one can check that $-C_{\epsilon}^{1+\epsilon} /\left(2 \sigma^{2}\right)=-\left(1-t\left(C_{\epsilon}\right)\right)^{1-\epsilon}-t\left(C_{\epsilon}\right)^{2} C_{\epsilon}^{1+\epsilon} /\left(2 \sigma^{2}\right)$. Finally, for $C>C_{\epsilon}$, by the definition of $t(C)$, we deduce that, as $C \rightarrow \infty, t(C) \rightarrow 0$ leading to $t(C) \sim(1-\epsilon) \sigma^{2} C^{-(1+\epsilon)}$. Consequently, $J(C) \rightarrow 1$ as $C \rightarrow \infty$, and we recover the asymptotic of the maximal jump range.

Before proving Theorems [12, and 3in Section [4, we establish several intermediate results useful in the sequel.

## 3 Preliminary results

First, we present a classical result, known as the principle of the largest term, that will allow us to consider the maximum of several quantities rather than their sum.

Lemma 6. Let $\left(v_{n}\right)_{n \geqslant 0}$ be a positive sequence converging to $0, N$ be a positive integer, and, for $i=1, \ldots, N,\left(a_{n, i}\right)_{n \geqslant 0}$ be a sequence of non-negative numbers. Then,

$$
\varlimsup_{n \rightarrow \infty} v_{n} \log \left(\sum_{i=1}^{N} a_{n, i}\right)=\max _{i=1, \ldots, N}\left(\overline{\lim }_{n \rightarrow \infty} v_{n} \log a_{n, i}\right) .
$$

Proof. See [5, Lemma 1.2.15].
We will often need the asymptotics of the tail of $X$, given in the following lemma.
Lemma 7. Under assumption (11), one has: $\log \mathbb{P}(X \geqslant x) \sim-x^{1-\epsilon}$ as $x$ goes to infinity.
Proof. An integration by parts leads to

$$
\begin{aligned}
\mathbb{P}(X \geqslant x) \underset{x \rightarrow \infty}{\sim} \int_{x}^{\infty} e^{-t^{1-\epsilon}} d t & =\frac{x^{\epsilon} e^{-x^{1-\epsilon}}}{1-\epsilon}+\frac{\epsilon}{1-\epsilon} \int_{x}^{\infty} t^{-1+\epsilon} e^{-t^{1-\epsilon}} d t \\
& =\frac{x^{\epsilon} e^{-x^{1-\epsilon}}}{1-\epsilon}+o\left(\int_{x}^{\infty} e^{-t^{1-\epsilon}} d t\right)=\frac{x^{\epsilon} e^{-x^{1-\epsilon}}}{1-\epsilon}(1+o(1))
\end{aligned}
$$

Now, we present a unilateral version of the contraction principle for a sequence of random variables in $\mathbb{R}^{2}$ with independent coordinates where the function considered is the sum of the coordinates. Observe that only unilateral assumptions are required.

Proposition 8. Let $\left(\left(Y_{n, 1}, Y_{n, 2}\right)\right)_{n \geqslant 0}$ be a sequence of $\mathbb{R}^{2}$-valued random variables such that, for each $n, Y_{n, 1}$ and $Y_{n, 2}$ are independent. Let $\left(\epsilon_{n}\right)_{n \geqslant 0}$ be a positive sequence converging to 0 . For all $a \in \mathbb{R}$ and $i \in\{1,2\}$, let us define

$$
\underline{I}_{i}(a)=-\inf _{u<a} \underset{n \rightarrow \infty}{\lim } \epsilon_{n} \log \mathbb{P}\left(Y_{n, i}>u\right) \quad \text { and } \quad \bar{I}_{i}(a)=-\inf _{u<a} \varlimsup_{n \rightarrow \infty} \epsilon_{n} \log \mathbb{P}\left(Y_{n, i}>u\right)
$$

## Assume that:

(H) for all $M>0$, there exists $d>0$ such that

$$
\varlimsup_{n \rightarrow \infty} \epsilon_{n} \log \mathbb{P}\left(Y_{n, 1}>d, \quad Y_{n, 2}<-d\right)<-M \quad \text { and } \quad \varlimsup_{n \rightarrow \infty} \epsilon_{n} \log \mathbb{P}\left(Y_{n, 1}<-d, Y_{n, 2}>d\right)<-M
$$

Then, for all $c \in \mathbb{R}$, one has

$$
-\inf _{t>c} \underline{I}(t) \leqslant \underline{\lim }_{n \rightarrow \infty} \epsilon_{n} \log \mathbb{P}\left(Y_{n, 1}+Y_{n, 2}>c\right) \leqslant \varlimsup_{n \rightarrow \infty} \epsilon_{n} \log \mathbb{P}\left(Y_{n, 1}+Y_{n, 2} \geqslant c\right) \leqslant-\inf _{t \geqslant c} \bar{I}(t)
$$

where, for all $t \in \mathbb{R}$,

$$
\underline{I}(t):=\inf _{\substack{a, b \in \mathbb{R} \\ a+b=t}} \underline{I}_{1}(a)+\underline{I}_{2}(b) \quad \text { and } \quad \bar{I}(t):=\inf _{\substack{a, b \in \mathbb{R} \\ a+b=t}} \bar{I}_{1}(a)+\bar{I}_{2}(b)
$$

Moreover $\underline{I}$ and $\bar{I}$ are nondecreasing functions.
Proof. See Section 5,
Remark 9. A sufficient condition for assumption (H) is: for $i \in\{1,2\}$,

$$
\lim _{a \rightarrow \infty} \bar{I}_{i}(a)=\infty
$$

The next theorem is inspired from [18]. Its proof is omitted to lighten the present paper.
Theorem 10. Let $\left(Y_{n}\right)_{n \geqslant 0}$ be a sequence of real random variables and a positive sequence $\left(\epsilon_{n}\right)_{n \geqslant 0}$ converging to 0 . Suppose that there exists a differentiable function $\Lambda$ defined on $\mathbb{R}_{+}$such that $\Lambda^{\prime}$ is a (increasing) bijective function from $\mathbb{R}_{+}$to $\mathbb{R}_{+}$and, for all $\lambda \geqslant 0$ :

$$
\epsilon_{n} \log \mathbb{E}\left[e^{\lambda Y_{n} / \epsilon_{n}}\right] \underset{n \rightarrow \infty}{ } \Lambda(\lambda)
$$

Then, for all $c \geqslant 0$,

$$
-\inf _{t>c} \Lambda^{*}(t) \leqslant \underline{\lim }_{n \rightarrow \infty} \epsilon_{n} \log \mathbb{P}\left(Y_{n}>c\right) \leqslant \varlimsup_{n \rightarrow \infty} \epsilon_{n} \log \mathbb{P}\left(Y_{n} \geqslant c\right) \leqslant-\inf _{t \geqslant c} \Lambda^{*}(t)
$$

where, for all $t \geqslant 0, \Lambda^{*}(t):=\sup \{\lambda t-\Lambda(\lambda) ; \lambda \geqslant 0\}$.

## 4 Proofs of the main results

From now on, all non explicitly mentioned asymptotics are taken as $n \rightarrow \infty$.

### 4.1 Proof of Theorem 1

Applying Lemma 7 , we first obtain

$$
\mathbb{P}\left(M_{n} \geqslant x_{n}\right)=1-\left[1-\mathbb{P}\left(X \geqslant x_{n}\right)\right]^{n} \sim n \mathbb{P}\left(X \geqslant x_{n}\right),
$$

and then

$$
\lim \frac{1}{x_{n}^{1-\epsilon}} \log \mathbb{P}\left(M_{n} \geqslant x_{n}\right)=\lim \frac{1}{x_{n}^{1-\epsilon}} \log \mathbb{P}\left(X \geqslant x_{n}\right)=-1 .
$$

Now, let us turn to $P_{n}\left(x_{n}\right)$. We observe that:

$$
\frac{1}{x_{n}^{1-\epsilon}} \log \mathbb{P}\left(S_{n} \geqslant x_{n}\right) \geqslant \frac{1}{x_{n}^{1-\epsilon}} \log \mathbb{P}\left(\frac{S_{n-1}}{\sqrt{n-1}} \geqslant 0\right)+\frac{1}{x_{n}^{1-\epsilon}} \log \mathbb{P}\left(X_{n} \geqslant x_{n}\right) .
$$

By the central limit theorem, the first term in the right hand side converges to 0 , while the second one converges to -1 . As a result,

$$
\underline{\lim } \frac{1}{x_{n}^{1-\epsilon}} \log P_{n}\left(x_{n}\right) \geqslant-1 .
$$

To obtain the upper bound, let us introduce the following decomposition:

$$
\mathbb{P}\left(S_{n} \geqslant x_{n}\right)=\mathbb{P}\left(S_{n} \geqslant x_{n}, \forall i \quad X_{i}<x_{n}\right)+\mathbb{P}\left(S_{n} \geqslant x_{n}, \exists i \quad X_{i} \geqslant x_{n}\right)=: P_{n, 0}\left(x_{n}\right)+R_{n, 0}\left(x_{n}\right) .
$$

Lemma 11. If $x_{n} \gg n^{1 / 2}$, then

$$
\begin{equation*}
\overline{\lim } \frac{1}{x_{n}^{1-\epsilon}} \log R_{n, 0}\left(x_{n}\right) \leqslant-1 . \tag{5}
\end{equation*}
$$

Proof. Just apply the union bound and Lemma 7 :

$$
\frac{1}{x_{n}^{1-\epsilon}} \log R_{n, 0}\left(x_{n}\right) \leqslant \frac{1}{x_{n}^{1-\epsilon}} \log \left(n \mathbb{P}\left(X \geqslant x_{n}\right)\right) \rightarrow-1
$$

To conclude the proof of Theorem [1, it remains to prove that, if $x_{n} \gg n^{1 /(1+\epsilon)}$,

$$
\begin{equation*}
\varlimsup \overline{\lim } \frac{1}{x_{n}^{1-\epsilon}} \log P_{n, 0}\left(x_{n}\right) \leqslant-1, \tag{6}
\end{equation*}
$$

and to apply Lemma 6. To prove inequality (6), we apply the standard Chebyshev's exponential inequality, which amounts to $\mathbb{1}_{a \geqslant 0} \leqslant e^{a}$ : for all $u \in(0,1)$,

$$
\begin{align*}
\frac{1}{x_{n}^{1-\epsilon}} \log P_{n, 0}\left(x_{n}\right) & =\frac{1}{x_{n}^{1-\epsilon}} \log \mathbb{E}\left[\mathbb{1}_{S_{n}-x_{n} \geqslant 0} \mathbb{1}_{\forall i} X_{i}<x_{n}\right] \\
& \leqslant \frac{1}{x_{n}^{1-\epsilon}} \log \mathbb{E}\left[e^{u x_{n}^{-\epsilon}\left(S_{n}-x_{n}\right)} \mathbb{1}_{\forall i X_{i}<x_{n}}\right] \\
& =-u+\frac{n}{x_{n}^{1-\epsilon}} \log \mathbb{E}\left[e^{u x_{n}^{-\epsilon} X_{1}} \mathbb{1}_{X<x_{n}}\right] . \tag{7}
\end{align*}
$$

Now, there exists $c>0$ such that, for all $t \leqslant u, e^{t} \leqslant 1+t+c t^{2}$. Therefore,

$$
\mathbb{E}\left[e^{u x_{n}^{-\epsilon} X_{1}} \mathbb{1}_{X x_{n}^{\epsilon}}\right] \leqslant \mathbb{E}\left[1+u x_{n}^{-\epsilon} X+c\left(u x_{n}^{-\epsilon} X\right)^{2}\right]=1+O\left(x_{n}^{-2 \epsilon}\right)
$$

And, using (1), for $n$ large enough,

$$
\mathbb{E}\left[e^{u x_{n}^{-\epsilon} X_{1}} \mathbb{1}_{x_{n}^{\epsilon} \leqslant X<x_{n}}\right]=\int_{x_{n}^{\epsilon}}^{x_{n}} e^{u x_{n}^{-\epsilon} y} p(y) d y \leqslant \int_{x_{n}^{\epsilon}}^{x_{n}} 2 e^{u x_{n}^{-\epsilon} y-y^{1-\epsilon}} d y
$$

The convex function $f_{n}(y)=u x_{n}^{-\epsilon} y-y^{1-\epsilon}$ attains its maximum on $\left[x_{n}^{\epsilon}, x_{n}\right.$ ] on the boundary. Since $f_{n}\left(x_{n}^{\epsilon}\right) \sim-x_{n}^{\epsilon(1-\epsilon)}, f_{n}\left(x_{n}\right)=(u-1) x_{n}^{1-\epsilon}$, and $u \in(0,1), f_{n}\left(x_{n}\right) \leqslant f_{n}\left(x_{n}^{\epsilon}\right)$ for $n$ large enough, whence

$$
\mathbb{E}\left[e^{u x_{n}^{-\epsilon} X_{1}} \mathbb{1}_{x_{n}^{\epsilon} \leqslant X<x_{n}}\right] \leqslant 2 x_{n} e^{-\frac{1}{2} x_{n}^{\epsilon(1-\epsilon)}}=O\left(x_{n}^{-2 \epsilon}\right)
$$

Consequently, for $x_{n} \gg n^{1 /(1+\epsilon)}$,

$$
\begin{equation*}
\frac{n}{x_{n}^{1-\epsilon}} \log \mathbb{E}\left[e^{u x_{n}^{-\epsilon} X^{1}} \mathbb{1}_{X<x_{n}}\right]=\frac{n}{x_{n}^{1-\epsilon}} \log \left(1+O\left(x_{n}^{-2 \epsilon}\right)\right)=O\left(\frac{n}{x_{n}^{1+\epsilon}}\right) \rightarrow 0 \tag{8}
\end{equation*}
$$

Combining (7) and (8), and letting $u \rightarrow 1$ completes the proof of inequality (6).

### 4.2 Proof of Theorem 2

By assumption (1) the Laplace transform $\Lambda_{X}$ of $X$ is not defined at the right of zero. That is why, we consider a new partition of $\mathbb{R}^{n}$, and define truncated random variables in order to get a nondegenerated limit of their log-Laplace transforms (see Theorem 10). For this, we introduce, for all $a \in \mathbb{R}$ and $m \in \llbracket 0, n \rrbracket$,

$$
\Pi_{n, m}(a):=\mathbb{P}\left(S_{n} \geqslant a, \forall i \in \llbracket 1, m \rrbracket \quad x_{n}^{\epsilon} \leqslant X_{i}<x_{n}, \forall i \in \llbracket m+1, n \rrbracket \quad X_{i}<x_{n}^{\epsilon}\right)
$$

so that $P_{n}\left(x_{n}\right)$ writes as

$$
P_{n}\left(x_{n}\right)=P_{n, 0}\left(x_{n}\right)+R_{n, 0}\left(x_{n}\right)=\sum_{m=0}^{n}\binom{n}{m} \Pi_{n, m}\left(x_{n}\right)+R_{n, 0}\left(x_{n}\right)
$$

By Lemma 11 and the fact that, for $x_{n} \ll n^{1 /(1+\epsilon)}, 1 / x_{n}^{1-\epsilon} \ll n / x_{n}^{2}$, we get

$$
\begin{equation*}
\lim \frac{n}{x_{n}^{2}} \log R_{n, 0}\left(x_{n}\right)=-\infty \tag{9}
\end{equation*}
$$

Lemma 12. Let $C>0$. If $n^{1 / 2} \ll x_{n} \leqslant C n^{1 /(1+\epsilon)}$ and $t>0$, then

$$
\begin{equation*}
\lim \frac{n}{x_{n}^{2}} \log \Pi_{n, 0}\left(t x_{n}\right)=-\frac{t}{2 \sigma^{2}} \tag{10}
\end{equation*}
$$

Proof. Let us introduce $\bar{X}$ with distribution $\mathcal{L}\left(X \mid X<x_{n}^{\epsilon}\right)$. For all $n \in \mathbb{N}^{*}$, let $\bar{X}_{1}, \bar{X}_{2}, \ldots, \bar{X}_{n}$ be i.i.d. copies of $\bar{X}$ and let $\bar{S}_{n}=\bar{X}_{1}+\cdots+\bar{X}_{n}$, so that

$$
\Pi_{n, 0}\left(t x_{n}\right)=\mathbb{P}\left(S_{n} \geqslant t x_{n}, X_{1}, \ldots, X_{n}<x_{n}^{\epsilon}\right)=\mathbb{P}\left(\bar{S}_{n} \geqslant t x_{n}\right) \mathbb{P}\left(X<x_{n}^{\epsilon}\right)^{n} \sim \mathbb{P}\left(\bar{S}_{n} \geqslant t x_{n}\right)
$$

by Lemma [7. We want to apply Theorem 10 to the random variables $\bar{S}_{n} / x_{n}$ with $\epsilon_{n}=n / x_{n}^{2}$. For $u>0$,

$$
\begin{equation*}
\frac{n}{x_{n}^{2}} \log \mathbb{E}\left[e^{u \frac{x_{n}^{2}}{n} \frac{\bar{S}_{n}}{x_{n}}}\right]=\frac{n^{2}}{x_{n}^{2}} \log \mathbb{E}\left[e^{\frac{u x_{n} X}{n}} \mathbb{1}_{X<x_{n}^{\epsilon}}\right]-\frac{n^{2}}{x_{n}^{2}} \log \mathbb{P}\left(X<x_{n}^{\epsilon}\right) \tag{11}
\end{equation*}
$$

The second term in the right side of the above equation goes to 0 as $n \rightarrow \infty$ since $\log \mathbb{P}(X<$ $\left.x_{n}^{\epsilon}\right) \sim-\mathbb{P}\left(X>x_{n}^{\epsilon}\right)=O\left(e^{-x_{n}^{\epsilon(1-\epsilon)} / 2}\right)$, by Lemma[7, As for the first term, if $y<x_{n}^{\epsilon}$, then $x_{n} y / n \leqslant$
$x_{n}^{1+\epsilon} / n \leqslant C^{1+\epsilon}$. Now, there exists $c>0$ such that, for all $s \leqslant C^{1+\epsilon},\left|e^{s}-\left(1+s+s^{2} / 2\right)\right| \leqslant c|s|^{2+\gamma}$. Hence,

$$
\begin{align*}
& \left|\mathbb{E}\left[e^{\frac{u x_{n} X}{n}} \mathbb{1}_{X<x_{n}^{\epsilon}}\right]-e^{\frac{u^{2} x_{n}^{2} \sigma^{2}}{2 n^{2}}}\right| \\
& \leqslant\left|\mathbb{E}\left[e^{\frac{u x_{n} X}{n}} \mathbb{1}_{X<x_{n}^{\epsilon}}\right]-\mathbb{E}\left[\left(1+\frac{u x_{n} X}{n}+\frac{u^{2} x_{n}^{2} X^{2}}{2 n^{2}}\right) \mathbb{1}_{X<x_{n}^{\epsilon}}\right]\right| \\
& +\left|\mathbb{E}\left[\left(1+\frac{u x_{n} X}{n}+\frac{u^{2} x_{n}^{2} X^{2}}{2 n^{2}}\right) \mathbb{1}_{X<x_{n}^{\epsilon}}\right]-\left(1+\frac{u^{2} x_{n}^{2} \sigma^{2}}{2 n^{2}}\right)\right| \\
& +\left|\left(1+\frac{u^{2} x_{n}^{2} \sigma^{2}}{2 n^{2}}\right)-e^{\frac{u^{2} x_{n}^{2} \sigma^{2}}{2 n^{2}}}\right| \\
& \leqslant c \rho\left(\frac{u x_{n}}{n}\right)^{2+\gamma}+\left|\mathbb{E}\left[\left(1+\frac{u x_{n} X}{n}+\frac{u^{2} x_{n}^{2} X^{2}}{2 n^{2}}\right) \mathbb{1}_{X \geqslant x_{n}^{\epsilon}}\right]\right|+o\left(\frac{x_{n}^{2}}{n^{2}}\right) . \tag{12}
\end{align*}
$$

For $n$ large enough, applying Hölder's inequality,

$$
\begin{align*}
\left|\mathbb{E}\left[\left(1+\frac{u x_{n} X}{n}+\frac{u^{2} x_{n}^{2} X^{2}}{2 n^{2}}\right) \mathbb{1}_{X \geqslant x_{n}^{\epsilon}}\right]\right| & \leqslant \mathbb{E}\left[X^{2} \mathbb{1}_{X \geqslant x_{n}^{\epsilon}}\right] \\
& \leqslant \mathbb{E}\left[X^{2+\gamma}\right]^{2 /(2+\gamma)} \mathbb{P}\left(X \geqslant x_{n}^{\epsilon}\right)^{\gamma /(2+\gamma)} \\
& =o\left(\frac{x_{n}^{2}}{n^{2}}\right) \tag{13}
\end{align*}
$$

by Lemma 7. Combining (11), (12), and (13), we get

$$
\frac{n}{x_{n}^{2}} \log \mathbb{E}\left[e^{u \frac{x_{n}^{2}}{n} \frac{\overline{S_{n}}}{x_{n}}}\right] \rightarrow \frac{u^{2} \sigma^{2}}{2}:=\Lambda(u)
$$

Since $\Lambda^{*}(t)=t / 2 \sigma^{2}$, (10) stems from Theorem 10 ,
Theorem 2 stems from Lemma 6, (19), (10) (with $t=1$ ), and the fact that, for $n^{1 / 2} \ll x_{n} \ll$ $n^{1 /(1+\epsilon)}$,

$$
\begin{equation*}
\varlimsup \frac{n}{x_{n}^{2}} \log \sum_{m=1}^{n}\binom{n}{m} \Pi_{n, m}\left(x_{n}\right) \leqslant-\frac{1}{2 \sigma^{2}} \tag{14}
\end{equation*}
$$

the proof of which is given now. By (1), we have, for all $n$ large enough and for all $m \in \llbracket 1, n \rrbracket$,

$$
\begin{align*}
\Pi_{n, m}\left(x_{n}\right) & =\mathbb{P}\left(S_{n} \geqslant x_{n}, x_{n}^{\epsilon} \leqslant X_{1}, \ldots, X_{m}<x_{n}, X_{m+1}, \ldots, X_{n}<x_{n}^{\epsilon}\right) \\
& =\int_{\left[x_{n}^{\epsilon}, x_{n}\right]^{m}} \Pi_{n-m, 0}\left(x_{n}-u_{1}-\ldots-u_{m}\right) \mathrm{d}_{X_{1}}\left(u_{1}\right) \ldots \mathrm{d} \mathbb{P}_{X_{m}}\left(u_{m}\right) \\
& \leqslant 2^{m} \int_{\left[x_{n}^{\epsilon}, x_{n}\right]^{m}} \Pi_{n-m, 0}\left(x_{n}-u_{1}-\ldots-u_{m}\right) e^{-\sum_{i=1}^{m} u_{i}^{1-\epsilon}} d u_{1} \ldots d u_{m} \tag{15}
\end{align*}
$$

Set $m_{n}=\left\lfloor x_{n}^{(1-\epsilon)^{2}}\right\rfloor$ so that $m_{n} x_{n}^{\epsilon(1-\epsilon)} \sim x_{n}^{1-\epsilon}$ and $m_{n}=o(n)$.
Lemma 13. If $x_{n} \gg n^{1 / 2}$, then

$$
\sum_{m=2 m_{n}+1}^{n}\binom{n}{m} \Pi_{n, m}\left(x_{n}\right)=O\left(e^{-m_{n} x_{n}^{\epsilon(1-\epsilon)}}\right)
$$

Proof. Using $\Pi_{n, m}\left(x_{n}-u\right) \leqslant 1$ in (15) yields

$$
\begin{aligned}
\sum_{m=2 m_{n}+1}^{n}\binom{n}{m} \Pi_{n, m}\left(x_{n}\right) & \leqslant \sum_{m=2 m_{n}+1}^{n}\binom{n}{m} 2^{m}\left(\int_{x_{n}^{\epsilon}}^{x_{n}} e^{-u^{1-\epsilon}} d u\right)^{m} \\
& \leqslant \sum_{m=2 m_{n}+1}^{n}\binom{n}{m}\left(2 x_{n}\right)^{m} e^{-m x_{n}^{\epsilon(1-\epsilon)}} \\
& \leqslant e^{-m_{n} x_{n}^{\epsilon(1-\epsilon)}} \sum_{m=2 m_{n}+1}^{n}\binom{n}{m}\left(2 x_{n} e^{-\frac{1}{2} x_{n}^{\epsilon(1-\epsilon)}}\right)^{m},
\end{aligned}
$$

and the conclusion holds, since the latter sum is bounded.
As $n^{1 / 2} \ll x_{n} \ll n^{1 /(1+\epsilon)}$, we conclude that

$$
\begin{equation*}
\overline{\lim } \frac{n}{x_{n}^{2}} \log \sum_{m=2 m_{n}+1}^{n}\binom{n}{m} \Pi_{n, m}\left(x_{n}\right) \leqslant \overline{\lim }-\frac{n}{x_{n}^{1+\epsilon}}=-\infty . \tag{16}
\end{equation*}
$$

Now we consider the values of $m \in \llbracket 1,2 m_{n} \rrbracket$. To get a lower bound on $\left[x_{n}^{\epsilon}, x_{n}\right]^{m}$ of the function

$$
s_{m}:\left(u_{1}, \ldots, u_{m}\right) \mapsto \sum_{i=1}^{m} u_{i}^{1-\epsilon}
$$

we define

$$
A_{1, m}:=\left\{\left(u_{1}, \ldots, u_{m}\right) \in\left[x_{n}^{\epsilon}, x_{n}\right]^{m}, \sum_{i=1}^{m} u_{i} \geqslant x_{n}\right\}
$$

and

$$
A_{2, m}:=\left\{\left(u_{1}, \ldots, u_{m}\right) \in\left[x_{n}^{\epsilon}, x_{n}\right]^{m}, \sum_{i=1}^{m} u_{i}<x_{n}\right\}
$$

in such a way that $\left[x_{n}^{\epsilon}, x_{n}\right]^{m}=A_{1, m} \cup A_{2, m}$ and we introduce, for $j \in\{1,2\}$,

$$
\begin{equation*}
I_{j, m}:=\int_{A_{j, m}} \Pi_{n-m, 0}\left(x_{n}-u_{1}-\ldots-u_{m}\right) e^{-s_{m}\left(u_{1}, \ldots, u_{m}\right)} d u_{1} \ldots d u_{m} \tag{17}
\end{equation*}
$$

so that (15) leads to

$$
\begin{equation*}
\Pi_{n, m}\left(x_{n}\right) \leqslant 2^{m}\left(I_{1, m}+I_{2, m}\right) . \tag{18}
\end{equation*}
$$

Lemma 14. If $x_{n} \gg n^{1 / 2}$, then

$$
\sum_{m=1}^{2 m_{n}}\binom{n}{m} 2^{m} I_{1, m}=O\left(e^{-x_{n}^{1-\epsilon}}\right)
$$

Proof. Notice that $I_{1,1}=0$ since $A_{1,1}=\left\{x_{n}\right\}$. Now, consider $m \in\left\{2, \ldots, 2 m_{n}\right\}$. By Lemma 16 (see Section (5), the function $s_{m}$ reaches its minimum on $A_{1, m}$ at the points with all coordinates equal to $x_{n}^{\epsilon}$ except one equal to $x_{n}-(m-1) x_{n}^{\epsilon}$. By a Taylor expansion of $u \mapsto(1-u)^{1-\epsilon}$ near the origin $\left((m-1) x_{n}^{-(1-\epsilon)} \leqslant 2 x_{n}^{-\epsilon(1-\epsilon)} \rightarrow 0\right)$, there exists $\eta_{n} \rightarrow 0$ such that

$$
\left(x_{n}-(m-1) x_{n}^{\epsilon}\right)^{1-\epsilon}=x_{n}^{1-\epsilon}\left(1-(m-1)\left(1-\epsilon+\eta_{n}\right) x_{n}^{-(1-\epsilon)}\right) \geqslant x_{n}^{1-\epsilon}-2(m-1)
$$

for $n$ large enough. Using $\Pi_{n-m, 0}\left(x_{n}-u_{1}-\cdots-u_{m}\right) \leqslant 1$ in (17), one gets

$$
I_{1, m} \leqslant x_{n}^{m} e^{-(m-1) x_{n}^{\epsilon(1-\epsilon)}-\left(x_{n}-(m-1) x_{n}^{\epsilon}\right)^{1-\epsilon}} \leqslant x_{n}^{m} e^{-x_{n}^{1-\epsilon}} e^{-(m-1)\left(x_{n}^{\epsilon(1-\epsilon)}-2\right)},
$$

for $n$ large enough (and uniformly in $m$ ). Thus,

$$
\sum_{m=1}^{2 m_{n}}\binom{n}{m} 2^{m} I_{1, m} \leqslant e^{-x_{n}^{1-\epsilon}} \sum_{m=2}^{2 m_{n}}\binom{n}{m}\left(2 x_{n} e^{2}\right)^{m} e^{-(m-1) x_{n}^{\epsilon(1-\epsilon)}} \leqslant e^{-x_{n}^{1-\epsilon}} \sum_{m=2}^{2 m_{n}}\left(2 x_{n} n e^{-\frac{1}{4} x_{n}^{\epsilon(1-\epsilon)}}\right)^{m}
$$

and the conclusion holds, since the latter sum is bounded.

If $n^{1 / 2} \ll x_{n} \ll n^{1 /(1+\epsilon)}$, we get

$$
\begin{equation*}
\varlimsup \frac{n}{x_{n}^{2}} \log \sum_{m=1}^{2 m_{n}}\binom{n}{m} 2^{m} I_{1, m}=-\infty \tag{19}
\end{equation*}
$$

Lemma 15. Let $C>0$. If $n^{1 / 2} \ll x_{n} \leqslant C n^{1 /(1+\epsilon)}$, then, for $n$ large enough, for all $m \in$ $\left\{1, \ldots, 2 m_{n}\right\}$,

$$
\begin{equation*}
I_{2, m} \leqslant x_{n}^{m} e^{-(m-1) x_{n}^{\epsilon(1-\epsilon)}} \exp \left\{\sup _{m x_{n}^{\epsilon} \leqslant u<x_{n}} \phi_{m}(u)\right\} \tag{20}
\end{equation*}
$$

where

$$
\phi_{m}(u):=-\frac{\left(x_{n}-u\right)^{2}}{2 \sigma^{2}(n-m)\left(1+c_{n}\right)}-\left(u-(m-1) x_{n}^{\epsilon}\right)^{1-\epsilon}
$$

Proof. Here, we use Chebyshev's exponential inequality to control $\Pi_{n-m, 0}\left(x_{n}-u\right)$ in (17), introducing $u=u_{1}+\cdots+u_{m}$. For all $l \in \mathbb{N}^{*}$ and for all $\lambda>0$,

$$
\Pi_{l, 0}\left(x_{n}-u\right) \leqslant e^{-\lambda\left(x_{n}-u\right)} \mathbb{E}\left[e^{\lambda X_{1}} \mathbb{1}_{X<x_{n}^{\epsilon}}\right]^{l}=\exp \left\{-\lambda\left(x_{n}-u\right)+l \log \mathbb{E}\left[e^{\lambda X_{X}} \mathbb{1}_{X<x_{n}^{\epsilon}}\right]\right\}
$$

Let $M>C^{1+\epsilon} / \sigma^{2}$. There exists $c>0$ such that, for all $s \leqslant M$, we have $e^{s} \leqslant 1+s+s^{2} / 2+c|s|^{2+\gamma}$. Hence, as soon as $\lambda \leqslant M x_{n}^{-\epsilon}$,

$$
\mathbb{E}\left[e^{\lambda X} \mathbb{1}_{X<x_{n}^{\epsilon}}\right] \leqslant 1+\frac{\lambda^{2} \sigma^{2}}{2}+c \rho \lambda^{2+\gamma} \leqslant 1+\frac{\lambda^{2} \sigma^{2}}{2}\left(1+c_{n}\right)
$$

where

$$
c_{n}:=2 c \rho \sigma^{-2} M^{\gamma} x_{n}^{-\gamma \epsilon}=o(1)
$$

Thus, for $l=n-m$ and $\lambda \leqslant M x_{n}^{-\epsilon}, \Pi_{n-m, 0}\left(x_{n}-u\right) \leqslant \exp \left\{-\lambda\left(x_{n}-u\right)+(n-m) \lambda^{2} \sigma^{2}\left(1+c_{n}\right) / 2\right\}$. For $n$ large enough, for all $m \in\left\{1, \ldots, 2 m_{n}\right\}$, the minimum of $\lambda \mapsto-\lambda\left(x_{n}-u\right)+(n-m) \lambda^{2} \sigma^{2}(1+$ $\left.c_{n}\right) / 2$ is reached at

$$
\lambda^{*}=\frac{x_{n}-u}{(n-m) \sigma^{2}\left(1+c_{n}\right)} \leqslant \frac{x_{n}}{\left(n-m_{n}\right) \sigma^{2}\left(1+c_{n}\right)} \leqslant M x_{n}^{-\epsilon}
$$

and is equal to $-\left(x_{n}-u\right)^{2} /\left(2 \sigma^{2}(n-m)\left(1+c_{n}\right)\right)$. Then we obtain, for $n$ large enough and for all $m \in\left\{1, \ldots, 2 m_{n}\right\}$ :

$$
\begin{equation*}
\Pi_{n-m, 0}\left(x_{n}-u\right) \leqslant \exp \left\{-\frac{\left(x_{n}-u\right)^{2}}{2 \sigma^{2}(n-m)\left(1+c_{n}\right)}\right\} \tag{21}
\end{equation*}
$$

By Lemma 16, on $A_{2, m, u}:=\left\{\left(u_{1}, \ldots, u_{m}\right) \in\left[x_{n}^{\epsilon}, x_{n}\right]^{m}, u_{1}+\cdots+u_{m}=u\right\}$, the function $s_{m}$ reaches its minimum at the points with all coordinates equal to $x_{n}^{\epsilon}$ except one equal to $u-(m-1) x_{n}^{\epsilon}$. This, together with (21) and (17), yields

$$
\begin{aligned}
I_{2, m} & \leqslant x_{n}^{m} \sup _{m x_{n}^{\epsilon} \leqslant u<x_{n}} \exp \left\{-\frac{\left(x_{n}-u\right)^{2}}{2 \sigma^{2}(n-m)\left(1+c_{n}\right)}-\left(u-(m-1) x_{n}^{\epsilon}\right)^{1-\epsilon}-(m-1) x_{n}^{\epsilon(1-\epsilon)}\right\} \\
& \leqslant x_{n}^{m} e^{-(m-1) x_{n}^{\epsilon(1-\epsilon)}} \exp \left\{\sup _{m x_{n}^{\epsilon} \leqslant u<x_{n}} \phi_{m}(u)\right\}
\end{aligned}
$$

The function $\phi_{m}$ is differentiable on $\left[m x_{n}^{\epsilon}, x_{n}\right)$ and

$$
\phi_{m}^{\prime}(u) \leqslant 0 \Longleftrightarrow\left(x_{n}-u\right)\left(u-(m-1) x_{n}^{\epsilon}\right)^{\epsilon}-(1-\epsilon) \sigma^{2}(n-m)\left(1+c_{n}\right) \leqslant 0
$$

For $n$ large enough, for all $m \in\left\{1, \ldots, 2 m_{n}\right\}$, for all $u \in\left[m x_{n}^{\epsilon}, x_{n}\right)$, one has

$$
\left(x_{n}-u\right)\left(u-(m-1) x_{n}^{\epsilon}\right)^{\epsilon} \leqslant x_{n}^{1+\epsilon} \leqslant(1-\epsilon) \sigma^{2}\left(n-m_{n}\right)\left(1+c_{n}\right) \leqslant(1-\epsilon) \sigma^{2}(n-m)\left(1+c_{n}\right)
$$

since $x_{n}^{1+\varepsilon} \ll n \sim n-m_{n}$. We deduce that, for $n$ large enough, $\phi_{m}$ is nonincreasing on $\left[m x_{n}^{\epsilon}, x_{n}\right)$ and

$$
\sup _{m x_{n}^{\epsilon} \leqslant u<x_{n}} \phi_{m}(u)=\phi_{m}\left(m x_{n}^{\epsilon}\right)=-\frac{\left(x_{n}-m x_{n}^{\epsilon}\right)^{2}}{2 \sigma^{2}(n-m)\left(1+c_{n}\right)}-x_{n}^{\epsilon(1-\epsilon)} \leqslant-\frac{\left(x_{n}-m_{n} x_{n}^{\epsilon}\right)^{2}}{2 n \sigma^{2}\left(1+c_{n}\right)}-x_{n}^{\epsilon(1-\epsilon)} .
$$

It follows that

$$
\sum_{m=1}^{2 m_{n}}\binom{n}{m} 2^{m} I_{2, m} \leqslant e^{-\frac{\left(x_{n}-m_{n} x_{n}^{\epsilon}\right)^{2}}{2 n \sigma^{2}\left(1+c_{n}\right)}} \sum_{m=1}^{2 m_{n}}\binom{n}{m}\left(2 x_{n} e^{-x_{n}^{\epsilon(1-\epsilon)}}\right)^{m}=O\left(e^{-\frac{\left(x_{n}-m_{n} x_{k}\right)^{2}}{2 n \sigma^{2}\left(1+c_{n}\right)}}\right)
$$

since the latter sum is bounded and, finally, one gets

$$
\begin{equation*}
\overline{\lim } \frac{n}{x_{n}^{2}} \log \sum_{m=1}^{2 m_{n}}\binom{n}{m} 2^{m} I_{2, m} \leqslant-\frac{1}{2 \sigma^{2}}, \tag{22}
\end{equation*}
$$

as $m_{n} x_{n}^{\epsilon}=o\left(x_{n}\right)$ and $c_{n}=o(1)$. By (16), (18), (19), (22), and Lemma 6] we get (14) as required.

### 4.3 Proof of Theorem 3

We consider $x_{n}=C n^{1 /(1+\epsilon)}$ with $C>0$. As in the proof of Theorem [2, we consider the decomposition

$$
P_{n}\left(x_{n}\right)=P_{n, 0}\left(x_{n}\right)+R_{n, 0}\left(x_{n}\right)=\sum_{m=0}^{n}\binom{n}{m} \Pi_{n, m}\left(x_{n}\right)+R_{n, 0}\left(x_{n}\right) .
$$

By Lemmas 11 and (12, and the very definition of $J$ in (4), we have

$$
\begin{equation*}
\overline{\lim } \frac{1}{x_{n}^{1-\epsilon}} \log R_{n, 0}\left(x_{n}\right) \leqslant-1 \leqslant-J(C) \tag{23}
\end{equation*}
$$

and

$$
\begin{equation*}
\overline{\lim } \frac{1}{x_{n}^{1-\epsilon}} \log \Pi_{n, 0}\left(x_{n}\right) \leqslant-\frac{C^{1+\epsilon}}{2 \sigma^{2}} \leqslant-J(C) . \tag{24}
\end{equation*}
$$

To complete the proof of Theorem 3, it remains to prove that

$$
\begin{align*}
& \lim \frac{1}{x_{n}^{1-\epsilon}} \log \left(n \Pi_{n, 1}\left(x_{n}\right)\right)=-J(C),  \tag{25}\\
& \overline{\lim } \frac{1}{x_{n}^{1-\epsilon}} \log \sum_{m=2}^{n}\binom{n}{m} \Pi_{n, m}\left(x_{n}\right) \leqslant-J(C), \tag{26}
\end{align*}
$$

and to apply Lemma 6 .
Proof of (25). Recall that, for $n \in \mathbb{N}^{*}$,

$$
\Pi_{n, 1}\left(x_{n}\right)=\mathbb{P}\left(S_{n} \geqslant x_{n} \mid x_{n}^{\epsilon} \leqslant X_{1} \leqslant x_{n}, X_{2}, \ldots, X_{n}<x_{n}^{\epsilon}\right) \mathbb{P}\left(x_{n}^{\epsilon} \leqslant X_{1} \leqslant x_{n}\right) \mathbb{P}\left(X<x_{n}^{\epsilon}\right)^{n-1}
$$

Using Lemma 7, it suffices to prove that

$$
\frac{1}{x_{n}^{1-\epsilon}} \log \mathbb{P}\left(S_{n} \geqslant x_{n} \mid x_{n}^{\epsilon} \leqslant X_{1}<x_{n}, X_{2}, \ldots, X_{n}<x_{n}^{\epsilon}\right) \rightarrow-J(C)
$$

To do so, we apply the contraction principle of Proposition 8 to $\left(Y_{n, 1}, Y_{n, 2}\right)$ with
$\mathcal{L}\left(Y_{n, 1}\right)=\mathcal{L}\left(x_{n}^{-1} X_{1} \mid x_{n}^{\epsilon} \leqslant X_{1}<x_{n}\right) \quad$ and $\quad \mathcal{L}\left(Y_{n, 2}\right)=\mathcal{L}\left(x_{n}^{-1}\left(X_{2}+\ldots+X_{n}\right) \mid X_{2}, \ldots, X_{n}<x_{n}^{\epsilon}\right)$, and $\epsilon_{n}=x_{n}^{-(1-\epsilon)}$. First, one has obviously, $\mathbb{P}\left(X \geqslant u x_{n} \mid x_{n}^{\epsilon} \leqslant X<x_{n}\right)=1$ for $u \leqslant 0$ and $\mathbb{P}\left(X \geqslant u x_{n} \mid x_{n}^{\epsilon} \leqslant X<x_{n}\right)=0$ for $u \geqslant 1$. In addition, for $u \in(0,1)$, applying Lemma 7 ,

$$
\log \mathbb{P}\left(X \geqslant u x_{n} \mid x_{n}^{\epsilon} \leqslant X<x_{n}\right) \sim-\left(u x_{n}\right)^{1-\epsilon} .
$$

Using the notation of Proposition 团 it follows that $\underline{I}_{1}(a)=\bar{I}_{1}(a)=I_{1}(a)$, where

$$
I_{1}(a)=\sup _{u<a}\left\{-\lim \frac{1}{x_{n}^{1-\epsilon}} \log \mathbb{P}\left(X_{1} \geqslant u x_{n} \mid x_{n}^{\epsilon} \leqslant X_{1} \leqslant x_{n}\right)\right\}= \begin{cases}0 & \text { if } a<0,  \tag{27}\\ a^{1-\epsilon} & \text { if } a \in[0,1], \\ \infty & \text { if } a>1\end{cases}
$$

Moreover, for all $u>0$,

$$
\frac{1}{x_{n}^{1-\epsilon}} \log \mathbb{P}\left(X_{2}+\ldots+X_{n} \geqslant u x_{n} \mid X_{2}, \ldots, X_{n}<x_{n}^{\epsilon}\right)=\frac{1}{x_{n}^{1-\epsilon}} \log \Pi_{n-1,0}\left(u x_{n}\right) \rightarrow-\frac{u^{2} C^{1+\epsilon}}{2 \sigma^{2}}
$$

by Lemma 12. Thus, we have $I_{2}(b)=b^{2} C^{1+\epsilon} /\left(2 \sigma^{2}\right)$ for all $b>0$ and, since $I_{2}$ is a nondecreasing and nonnegative function, we get $I_{2}(b)=0$ for all $b \leqslant 0$. This, together with (27), leads to: for all $t \in \mathbb{R}$,

$$
I(t)=\inf _{\substack{a+b=t \\(a, b) \in \mathbb{R}^{2}}}\left\{I_{1}(a)+I_{2}(b)\right\}=\inf _{t-1 \leqslant b \leqslant t}\left\{I_{1}(t-b)+I_{2}(b)\right\}=\inf _{t-1 \leqslant b \leqslant t}\left\{(t-b)^{1-\epsilon}+\frac{b^{2} C^{1+\epsilon}}{2 \sigma^{2}}\right\},
$$

since $b<t-1$ entails $I_{1}(t-b)=\infty$ and $b>t$ entails $I_{1}(t-b)+I_{2}(b)>I_{1}(0)+I_{2}(t)$.
It is a standard result (see, e.g., [12, 4.c.]) that $I$ is upper semicontinuous. Since $I$ is also nondecreasing, $I$ is right continuous and we get

$$
\inf _{t \geqslant 1} I(t)=\inf _{t>1} I(t)=I(1) .
$$

Applying Proposition [8, this completes the proof of (25).
Proof of (26). We follow the same lines as in the proof of (14) in the Gaussian range. With the same definition of $m_{n}=\left\lfloor x_{n}^{(1-\epsilon)^{2}}\right\rfloor$, Lemmas 13 and 14 entail

$$
\left.\varlimsup \frac{1}{x_{n}^{1-\epsilon}} \sum_{m=2 m_{n}+1}^{n}\binom{n}{m} \Pi_{n, m} \leqslant-1 \quad \text { and } \quad \varlimsup \frac{1}{\lim } \frac{2 m_{n}}{x_{n}^{1-\epsilon}} \log \sum_{m=2}^{n} \begin{array}{c}
n  \tag{28}\\
m
\end{array}\right) 2^{m} I_{1, m} \leqslant-1 .
$$

To complete the proof, it suffices to show that

$$
\begin{equation*}
\varlimsup \frac{1}{x_{n}^{1-\epsilon}} \log \sum_{m=2}^{2 m_{n}} 2^{m}\binom{n}{m} I_{2, m} \leqslant-J(C) . \tag{29}
\end{equation*}
$$

Recall that Lemma 15 provides $I_{2, m} \leqslant x_{n}^{m} e^{-(m-1) x_{n}^{\epsilon(1-\epsilon)}} e^{M_{n}}$ ，where

$$
\begin{aligned}
M_{n} & =\sup _{\left.m x_{n}^{\epsilon} \leqslant u<x_{n}\right)}\left\{-\frac{\left(x_{n}-u\right)^{2}}{2 \sigma^{2}(n-m)\left(1+c_{n}\right)}-\left(u-(m-1) x_{n}^{\epsilon}\right)^{1-\epsilon}\right\} \\
& \leqslant \sup _{m x_{n}^{-1+\epsilon} \leqslant t<1}\left\{-\frac{\left(x_{n}-t x_{n}\right)^{2}}{2 \sigma^{2} n\left(1+c_{n}\right)}-\left(t x_{n}-(m-1) x_{n}^{\epsilon}\right)^{1-\epsilon}\right\} \\
& =-x_{n}^{1-\epsilon} \inf _{m x_{n}^{-1+\epsilon} \leqslant t<1}\left\{\frac{(1-t)^{2} C^{1+\epsilon}}{2 \sigma^{2}\left(1+c_{n}\right)}+t^{1-\epsilon}\left(1-\frac{(m-1) x_{n}^{-1+\epsilon}}{t}\right)^{1-\epsilon}\right\}
\end{aligned}
$$

Let us fix $\eta \in(0,1)$ ．For $n$ large enough，for all $m \in\left\{2, \ldots, 2 m_{n}\right\}$ ，

$$
\begin{aligned}
\inf _{m x_{n}^{-1+\epsilon} \leqslant t<\eta}\left\{\frac{(1-t)^{2} C^{1+\epsilon}}{2 \sigma^{2}\left(1+c_{n}\right)}+t^{1-\epsilon}\left(1-\frac{(m-1) x_{n}^{-1+\epsilon}}{t}\right)^{1-\epsilon}\right\} & \geqslant \frac{(1-\eta)^{2} C^{1+\epsilon}}{2 \sigma^{2}}(1-\eta) \\
& \geqslant(1-\eta)^{3} J(C)
\end{aligned}
$$

and

$$
\begin{aligned}
\inf _{\eta \leqslant t<1}\left\{\frac{(1-t)^{2} C^{1+\epsilon}}{2 \sigma^{2}\left(1+c_{n}\right)}+t^{1-\epsilon}\left(1-\frac{(m-1) x_{n}^{-1+\epsilon}}{t}\right)^{1-\epsilon}\right\} & \geqslant \inf _{\eta \leqslant t<1}\left\{\frac{(1-t)^{2} C^{1+\epsilon}}{2 \sigma^{2}}+t^{1-\epsilon}\right\}(1-\eta) \\
& \geqslant(1-\eta) J(C)
\end{aligned}
$$

So $M_{n} \leqslant-x_{n}^{1-\epsilon}(1-\eta)^{3} J(C)$ and

$$
\begin{aligned}
\varlimsup & \frac{1}{x_{n}^{1-\epsilon}} \log \sum_{m=2}^{2 m_{n}}\binom{n}{m} 2^{m} I_{2, m}
\end{aligned} \leqslant-(1-\eta)^{3} J(C)+\overline{\lim } \frac{1}{x_{n}^{1-\epsilon}} \log \sum_{m=2}^{2 m_{n}}\binom{n}{m}\left(2 x_{n} e^{-x_{n}^{\epsilon(1-\epsilon)}}\right)^{m}{ }^{m},(1-\eta)^{3} J(C), ~ \$
$$

and（29）follows，letting $\eta \rightarrow 0$ ．

## 5 Proofs of the intermediate results

Proof of Proposition［8．Obviously，the functions $\underline{I}_{1}, \bar{I}_{1}, \underline{I}_{2}$ ，and $\bar{I}_{2}$ are nondecreasing．Let us prove that $\underline{I}$ is nondecreasing，the proof for $\bar{I}$ being similar．Let $t_{1}<t_{2}$ ，let $\eta>0$ ，and let $a \in \mathbb{R}$ be such that $\underline{I}\left(t_{2}\right) \geqslant \underline{I}_{1}(a)+\underline{I}_{2}\left(t_{2}-a\right)-\eta$ ．Since $\underline{I}_{2}$ is nondecreasing，we have

$$
\underline{I}\left(t_{1}\right) \leqslant \underline{I}_{1}(a)+\underline{I}_{2}\left(t_{1}-a\right) \leqslant \underline{I}_{1}(a)+\underline{I}_{2}\left(t_{2}-a\right) \leqslant \underline{I}\left(t_{2}\right)+\eta,
$$

which completes the proof of the monotony of $\underline{I}$ ，letting $\eta \rightarrow 0$ ．
Lower bound．Let $c \in \mathbb{R}$ ，let $t>c$ ，and let $\delta>0$ be such that $0<2 \delta<t-c$ ．For all $(a, b) \in \mathbb{R}^{2}$ such that $a+b=t$ ，we have

$$
\begin{aligned}
\varliminf & \epsilon_{n} \log \mathbb{P}\left(Y_{n, 1}+Y_{n, 2}>c\right)
\end{aligned} \geqslant \underline{\lim \epsilon_{n} \log \mathbb{P}\left(Y_{n, 1}>a-\delta\right)+\underline{\lim } \epsilon_{n} \log \mathbb{P}\left(Y_{n, 2}>b-\delta\right)} ⿻ 土 一 \underline{I}_{1}(a)-\underline{I}_{2}(b) .
$$

Therefore，

$$
\underline{\lim } \epsilon_{n} \log \mathbb{P}\left(Y_{n, 1}+Y_{n, 2}>c\right) \geqslant \sup _{t>c} \sup _{\substack{(a, b) \in \mathbb{R}^{2} \\ a+b=t}}\left(-\underline{I}_{1}(a)-\underline{I}_{2}(b)\right)=-\inf _{t>c} \underline{I}(t) .
$$

Upper bound. Let $c \in \mathbb{R}$ and let $M>0$. Let $d>0$ be given by assumption (H). Define

$$
Z=\left\{(a, b) \in \mathbb{R}^{2} ; a+b \geqslant c, a \geqslant-d, b \geqslant-d\right\}
$$

and

$$
K=\left\{(a, b) \in \mathbb{R}^{2} ; a+b=c, a \geqslant-d, b \geqslant-d\right\} .
$$

Write

$$
\begin{aligned}
\mathbb{P}\left(Y_{n, 1}+Y_{n, 2} \geqslant c\right) & \leqslant \mathbb{P}\left(Y_{n, 1}>d, Y_{n, 2}<-d\right)+\mathbb{P}\left(Y_{n, 1}<-d, Y_{n, 2}>d\right)+\mathbb{P}\left(\left(Y_{n, 1}, Y_{n, 2}\right) \in Z\right) \\
& =: Q_{n, 1}+Q_{n, 2}+Q_{n, 3} .
\end{aligned}
$$

By assumption,

$$
\overline{\lim } \epsilon_{n} \log \left(Q_{n, 1}\right)<-M \quad \text { and } \quad \overline{\lim } \epsilon_{n} \log \left(Q_{n, 2}\right)<-M .
$$

Let us estimate $\overline{\lim } \epsilon_{n} \log Q_{n, 3}$. For all $(a, b) \in K$,

$$
\begin{aligned}
& -\inf _{u<a} \overline{\lim } \epsilon_{n} \log \mathbb{P}\left(Y_{n, 1}>u, Y_{n, 2}>v\right) \\
& \quad \geqslant-\inf _{u<a} \overline{\lim } \epsilon_{n} \log \mathbb{P}\left(Y_{n, 1}>u\right)-\inf _{v<b} \overline{\lim } \epsilon_{n} \log \mathbb{P}\left(Y_{n, 2}>v\right) \\
& \quad=\bar{I}_{1}(a)+\bar{I}_{2}(b) .
\end{aligned}
$$

Defining $\theta^{[\delta]}:=\min \left(\theta-\delta, \delta^{-1}\right)$ for all $\delta>0$ and for all $\theta \in(-\infty, \infty]$, there exists $u_{a}<a$ and $v_{b}<b$ such that

$$
\begin{equation*}
-\overline{\lim } \epsilon_{n} \log \mathbb{P}\left(Y_{n, 1}>u_{a}, Y_{n, 2}>v_{b}\right) \geqslant\left(\bar{I}_{1}(a)+\bar{I}_{2}(b)\right)^{[\delta]} \tag{30}
\end{equation*}
$$

From the cover $\left(\left(u_{a}, \infty\right) \times\left(v_{b}, \infty\right)\right)_{(a, b) \in K}$ of the compact subset $K$, we can extract a finite subcover $\left(\left(u_{a_{i}}, \infty\right) \times\left(v_{b_{i}}, \infty\right)\right)_{1 \leqslant i \leqslant p}$. Since

$$
Z \subset \bigcup_{i=1}^{p}\left(u_{a_{i}}, \infty\right) \times\left(v_{b_{i}}, \infty\right),
$$

we obtain, thanks to Lemma 6 and (30),

$$
\begin{aligned}
\overline{\lim } \epsilon_{n} \log Q_{3} & \leqslant \varlimsup_{\lim } \epsilon_{n} \log \sum_{i=1}^{p} \mathbb{P}\left(Y_{n, 1}>u_{a_{i}}, Y_{n, 2}>v_{b_{i}}\right) \\
& =\max _{1 \leqslant i \leqslant p}\left\{{\overline{\lim } \epsilon_{n}} \log \mathbb{P}\left(Y_{n, 1}>u_{a_{i}}, Y_{n, 2}>v_{b_{i}}\right)\right\} \\
& \leqslant \max _{1 \leqslant i \leqslant p}\left\{-\left(\bar{I}_{1}\left(a_{i}\right)+\bar{I}_{2}\left(b_{i}\right)\right)^{[\delta]}\right\} \\
& \leqslant-\inf _{\substack{(a, b) \in \mathbb{R}^{2} \\
a+b=c}}\left\{\left(\bar{I}_{1}(a)+\bar{I}_{2}(b)\right)^{[\delta]}\right\} .
\end{aligned}
$$

Letting $\delta \rightarrow 0$ and using the definition of $\bar{I}$, we deduce that

$$
\begin{equation*}
\overline{\lim } \epsilon_{n} \log Q_{3} \leqslant-\inf _{\substack{(a, b) \in \mathbb{R}^{2} \\ a+b=c}}\left(\bar{I}_{1}(a)+\bar{I}_{2}(b)\right)=-\bar{I}(c)=-\inf _{t \geqslant c} \bar{I}(t) . \tag{31}
\end{equation*}
$$

Letting $M \rightarrow \infty$, we get the desired upper bound.
Lemma 16. Let the real-valued function $s_{m}$ be defined on $(0, \infty)^{m}$ by

$$
s_{m}\left(u_{1}, \ldots, u_{m}\right)=\sum_{i=1}^{m} u_{i}^{1-\epsilon}
$$

and the sets $A_{1, m}$ and $A_{2, m, u}$, as defined in the proof of (14). Assume that $m x^{\epsilon}<x$.

1. On $A_{1, m}$, the function $s_{m}$ reaches its minimum at points with one coordinate equal to $x-(m-1) x^{\epsilon}$ and the other coordinates equal to $x^{\epsilon}$.
2. On $A_{2, m, u}$, the function $s_{m}$ reaches its minimum at points with one coordinate equal to $u-(m-1) x^{\epsilon}$ and the other coordinates equal to $x^{\epsilon}$.

Proof of Lemma 16. 1. $A_{1, m}=L_{x}^{\geqslant} \cap\left(\bigcap_{i=1}^{m} H_{i}^{\geqslant} \cap G_{i}^{\leqslant}\right)$is a compact polyhedron defined through the following half-spaces:

$$
H_{i}^{\geqslant}=\left\{v \in \mathbb{R}^{m}, v_{i} \geqslant x^{\epsilon}\right\}, \quad G_{i}^{\leqslant}=\left\{v \in \mathbb{R}^{m}, v_{i} \leqslant x\right\}, \quad L_{x}^{\geqslant}=\left\{v \in \mathbb{R}^{m}, \sum_{i=1}^{m} v_{i} \geqslant x\right\}
$$

the boundaries of which are

$$
H_{i}=\left\{v \in \mathbb{R}^{m}, v_{i}=x^{\epsilon}\right\}, \quad G_{i}=\left\{v \in \mathbb{R}^{m}, v_{i}=x\right\}, \quad L_{x}=\left\{v \in \mathbb{R}^{m}, \sum_{i=1}^{m} v_{i}=x\right\}
$$

Now, $s_{m}$ being a continuous and concave function, its minimum is reached at an extremal point of $A_{1, m}$, that is, an intersection of $m+1$ supporting hyperplanes, the equations of which are linearly independent and compatible. Nevertheless, for all $i \in\{1, \ldots, m\}, G_{i} \cap L_{x} \cap A_{1, m}=\emptyset$, and the assumption $m x^{\epsilon}<x$ excludes that all coordinates of an extremal point are equal to $x^{\epsilon}$. Consequently, an extremal point $t=\left(t_{1}, \ldots, t_{m}\right)$ is such that $m-1$ coordinates are equal to $x^{\epsilon}$ and $t_{1}+\cdots+t_{m}=x$ or $k$ coordinates are equal to $x^{\epsilon}$ and $(m-k)$ equal to $x$, with $k \in\{1, \ldots, m-1\}$. Setting, for $v \geqslant 0, p(v)=v^{1-\epsilon}$, the first case gives

$$
s_{m}(t)=(m-1) p\left(x^{\epsilon}\right)+p\left(x-(m-1) x^{\epsilon}\right)
$$

while the second gives

$$
s_{m}(t)=k p\left(x^{\epsilon}\right)+(m-k) p(x)
$$

As $p$ is an increasing function, one has, for all $k \in\{1, \ldots, m-1\}$,

$$
(m-k) p(x)=(m-k-1) p(x)+p(x) \geqslant(m-k-1) p\left(x^{\epsilon}\right)+p\left(x-(m-1) x^{\epsilon}\right)
$$

then

$$
k p\left(x^{\epsilon}\right)+(m-k) p(x) \geqslant(m-1) p\left(x^{\epsilon}\right)+p\left(x-(m-1) x^{\epsilon}\right)
$$

which proves the claim.
2. We prove the result in a similar way than in the previous point. With the same notation, we get

$$
A_{2, m, u}=L_{u}^{\geqslant} \cap L_{u}^{\leqslant} \cap\left(\bigcap_{i=1}^{m} H_{i}^{\geqslant} \cap G_{i}^{\leqslant}\right)
$$

As $u \in\left[m x^{\epsilon}, x\right)$, an extremal point of $A_{2, m, u}$ has no coordinate equal to $x$. Then such a point has $(m-1)$ coordinates equal to $x^{\epsilon}$ and one equal to $u-(m-1) x^{\epsilon}$ which completes the proof.

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