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Probabilistic proofs of large deviation results for sums of semiexponential random variables and explicit rate function at the transition

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Abstract

Asymptotics deviation probabilities of the sum $S_n = X_1 + \dots + X_n$ of independent and identically distributed real-valued random variables have been extensively investigated, in particular when X_1 is not exponentially integrable. For instance, A.V. Nagaev formulated exact asymptotics results for $\mathbb{P}(S_n > x_n)$ when $x_n > n^{1/2}$ (see, [13, 14]). In this paper, we derive rough asymptotics results (at logarithmic scale) with shorter proofs relying on classical tools of large deviation theory and expliciting the rate function at the transition.

Key words: large deviations, sums of independent and identically distributed random variables, semiexponential random variables, Gärtner-Ellis theorem, contraction principle.

AMS subject classification: 60F10, 60G50.

1 Introduction

Moderate and large deviations of the sum of independent and identically distributed (i.i.d.) real-valued random variables have been investigated since the beginning of the 20th century. Kinchin [9] in 1929 was the first to give a result on large deviations of i.i.d. Bernoulli distributed random variables. In 1933, Smirnov [19] improved this result and in 1938 Cramér [4] gave a generalization to i.i.d. random variables satisfying the eponymous Cramér's condition which requires the Laplace transform of the common distribution of the random variables to be finite in a neighborhood of zero. Cramér's result was extended by Feller [7] to sequences of not necessarily identically distributed random variables under restrictive conditions (Feller considered only random variables taking values in bounded intervals), thus Cramér's result does not follow from Feller's result. A strengthening of Cramér's theorem was given by Petrov in [16] together with a generalization to the case of non-identically distributed random variables. Improvements of Petrov's result can be found in [17]. When Cramér's condition does not hold, an early result is due to Linnik [11] in 1961 and concerns polynomial-tailed random variables. The case where the tail decreases faster than all power functions (but not enough for Cramér's condition to be satisfied) has been considered by Petrov [16] and by S.V. Nagaev [15]. In [13, 14], A.V. Nagaev studied the case where the common distribution of the i.i.d. random variables is absolutely continuous with respect to the Lebesgue measure with density $p(t) \sim e^{-|t|^{1-\epsilon}}$ as $|t|$ tends to

infinity, with $\epsilon \in (0, 1)$. He distinguished five exact-asymptotics results corresponding to five types of deviation speeds. In [2, 3], Borovkov investigated exact asymptotics of the deviations probability for semiexponentially-tailed random variables (i.e. the tail writes as $e^{-t^{1-\epsilon}L(t)}$, where $\epsilon \in (0, 1)$ and L is a suitably slowly varying function at infinity).

Now, let us present the setting of this article. Let $\epsilon \in (0, 1)$ and let X be a real-valued random variable with a density p with respect to the Lebesgue measure verifying:

$$p(x) \sim e^{-x^{1-\epsilon}}, \quad \text{as } x \rightarrow +\infty. \quad (1)$$

Unlike in [13, 14], this unilateral assumption is motivated by the fact that we focus on upper deviations of the sum. Observe that (1) implies that the Laplace transform of X is not defined on the right side of zero. Nevertheless, all moments of $X_+ := \max(X, 0)$ are finite. A weaker assumption on the left tail is required:

$$\exists \gamma \in (0, 1] \quad \rho := \mathbb{E}[|X|^{2+\gamma}] < \infty. \quad (2)$$

Finally, we will assume that X is centered ($\mathbb{E}[X] = 0$) and will denote by σ the standard deviation of X ($\text{Var}(X) = \sigma^2$).

For all $n \in \mathbb{N}^*$, let X_1, X_2, \dots, X_n be i.i.d. copies of X . We set $S_n = X_1 + \dots + X_n$ and $P_n(x) = \mathbb{P}(S_n \geq x)$. In this paper, we are interested in the asymptotic behavior of $\log P_n(x_n)$ for any positive sequence $x_n \gg n^{1/2}$. Not only does the logarithmic scale allow us to use the modern theory of large deviations and provide simpler proofs than in [2, 3, 13, 14], but we also obtain more explicit results. According to the asymptotics of x_n , only three logarithmic asymptotic ranges appear. First, the Gaussian range: when $x_n \ll n^{1/(1+\epsilon)}$, $\log P_n(x_n) \sim \log(1 - \phi(n^{-1/2}x_n))$, ϕ being the distribution function of the standard Gaussian law. Next, the domain of validity of the maximal jump principle: when $x_n \gg n^{1/(1+\epsilon)}$, $\log P_n(x_n) \sim \log \mathbb{P}(\max(X_1, \dots, X_n) \geq x_n)$. Finally, the transition ($x_n = \Theta(n^{1/(1+\epsilon)})$) appears to be an interpolation between the Gaussian range and the maximal jump one.

Logarithmic asymptotics were also considered in [10] for a wider class of distributions than in the present paper. Nevertheless, the setting was restricted to the particular sequence $x_n = n$ (that lies in the maximum jump range). In [6], the authors gave a necessary and sufficient condition on the logarithmic tails of the sum of i.i.d. real-valued random variables to satisfy a large deviation principle which covers the Gaussian range. In [1], Arcones proceeded analogously and covered the maximum jump range. Observe that, in the two previous papers, additional restrictions on x_n were required. In [8], the author studied the more general case of semi-exponential upper tails, in a particular domain of the maximum jump range.

The transition at $x_n = \Theta(n^{1/(1+\epsilon)})$ is not considered in [2, 3]. It is treated in [13, 14, Theorems 2 and 4]. Nevertheless, the rate function is given through a non explicit inductive formula and hence is difficult to interpret. One of the main contributions of this work is to provide an explicit formula for the rate function thanks tools coming from large deviation techniques.

The paper is organized as follows. In Section 2, we state the main results. Section 3 is devoted to useful results, the proof of which are postponed to Section 5. In particular, we give unilateral versions of Gärtner-Ellis theorem inspired from [18] (see Theorem 10) and of the contraction principle (see Proposition 8). The proof of the main results can be found in Section 4.

2 Main results

In this section, Theorem 1 puts forward the domain of validity of the maximal jump principle at the logarithmic scale. In this range, the main contribution of the deviations of S_n is due to one summand, meaning $\log P_n(x_n) \sim \log \mathbb{P}(X \geq x_n)$. A Gaussian range, determined in Theorem 2, occurs when all summands contribute to the deviations of S_n in the sense that $\log P_n(x_n) \sim \log \mathbb{P}(S_n \geq x_n, \forall i \in \llbracket 1, n \rrbracket \quad X_i < x_n^\epsilon)$, unlike in the previous range. Finally, the transition range is studied in Theorem 3.

Theorem 1 (Maximal jump range). For $x_n \gg n^{1/(1+\epsilon)}$, setting $M_n := \max(X_1, \dots, X_n)$,

$$\lim_{n \rightarrow \infty} \frac{1}{x_n^{1-\epsilon}} \log P_n(x_n) = \lim_{n \rightarrow \infty} \frac{1}{x_n^{1-\epsilon}} \log \mathbb{P}(M_n \geq x_n) = \lim_{n \rightarrow \infty} \frac{1}{x_n^{1-\epsilon}} \log \mathbb{P}(X \geq x_n) = -1.$$

Theorem 2 (Gaussian range). For $n^{1/2} \ll x_n \ll n^{1/(1+\epsilon)}$, we have:

$$\lim_{n \rightarrow \infty} \frac{n}{x_n^2} \log P_n(x_n) = -\frac{1}{2\sigma^2}.$$

The sequence $n^{1/(1+\epsilon)}$ is the solution (up to a scalar factor) of the equation in x_n : $x_n^2/(2n\sigma^2) = x_n^{1-\epsilon}$, equalizing the asymptotics of $\log P_n(x_n)$ obtained in the Gaussian range and in the maximal jump range. In Theorem 3, we give the asymptotic of $\log P_n(x_n)$ when $x_n = Cn^{1/(1+\epsilon)}$, $C > 0$.

Theorem 3 (Transition). For $C > 0$ and $x_n = Cn^{1/(1+\epsilon)}$,

$$\lim_{n \rightarrow \infty} \frac{1}{x_n^{1-\epsilon}} \log P_n(x_n) = -\inf_{0 \leq t \leq 1} \left\{ (1-t)^{1-\epsilon} + \frac{t^2 C^{1+\epsilon}}{2\sigma^2} \right\} =: -J(C).$$

Remark 4. Let $f(t) = (1-t)^{1-\epsilon} + t^2 C^{1+\epsilon}/(2\sigma^2)$. An easy computation shows that, if $C \leq C'_\epsilon := (1+\epsilon)((1-\epsilon)\sigma^2\epsilon^{-\epsilon})^{1/(1+\epsilon)}$, f is decreasing and its minimum $C^{1+\epsilon}/(2\sigma^2)$ is attained at $t = 1$. If $C > C'_\epsilon$, f has two local minima, at 1 and at $t(C)$: the latter corresponds to the smallest of the two roots in $[0, 1]$ of $f'(t) = 0$, equation equivalent to

$$t(1-t)^\epsilon = \frac{(1-\epsilon)\sigma^2}{C^{1+\epsilon}}. \quad (3)$$

If $C'_\epsilon < C \leq C_\epsilon := (1+\epsilon)(\sigma^2(2\epsilon)^{-\epsilon})^{1/(1+\epsilon)}$, then $f(t(C)) \geq f(1)$. And, if $C > C_\epsilon$, then $f(t(C)) < f(1)$. As a consequence, for all $C > 0$,

$$J(C) = \begin{cases} \frac{C^{1+\epsilon}}{2\sigma^2} & \text{if } C \leq C_\epsilon, \\ (1-t(C))^{1-\epsilon} + \frac{t(C)^2 C^{1+\epsilon}}{2\sigma^2} & \text{if } C > C_\epsilon. \end{cases} \quad (4)$$

Remark 5. The transition interpolates between the Gaussian range and the maximal jump one. First, when $x_n = Cn^{1/(1+\epsilon)}$, we get $x_n^2/n = C^{1+\epsilon}x_n^{1-\epsilon}$, thus the asymptotics of the Gaussian range coincide with the one of the transition for $C \leq C_\epsilon$. Moreover, $t(C_\epsilon) = (1-\epsilon)/(1+\epsilon)$ and one can check that $-C_\epsilon^{1+\epsilon}/(2\sigma^2) = -(1-t(C_\epsilon))^{1-\epsilon} - t(C_\epsilon)^2 C_\epsilon^{1+\epsilon}/(2\sigma^2)$. Finally, for $C > C_\epsilon$, by the definition of $t(C)$, we deduce that, as $C \rightarrow \infty$, $t(C) \rightarrow 0$ leading to $t(C) \sim (1-\epsilon)\sigma^2 C^{-(1+\epsilon)}$. Consequently, $J(C) \rightarrow 1$ as $C \rightarrow \infty$, and we recover the asymptotic of the maximal jump range.

Before proving Theorems 1, 2, and 3 in Section 4, we establish several intermediate results useful in the sequel.

3 Preliminary results

First, we present a classical result, known as the principle of the largest term, that will allow us to consider the maximum of several quantities rather than their sum.

Lemma 6. Let $(v_n)_{n \geq 0}$ be a positive sequence converging to 0, N be a positive integer, and, for $i = 1, \dots, N$, $(a_{n,i})_{n \geq 0}$ be a sequence of non-negative numbers. Then,

$$\overline{\lim}_{n \rightarrow \infty} v_n \log \left(\sum_{i=1}^N a_{n,i} \right) = \max_{i=1, \dots, N} \left(\overline{\lim}_{n \rightarrow \infty} v_n \log a_{n,i} \right).$$

Proof. See [5, Lemma 1.2.15]. □

We will often need the asymptotics of the tail of X , given in the following lemma.

Lemma 7. *Under assumption (1), one has: $\log \mathbb{P}(X \geq x) \sim -x^{1-\epsilon}$ as x goes to infinity.*

Proof. An integration by parts leads to

$$\begin{aligned} \mathbb{P}(X \geq x) &\underset{x \rightarrow \infty}{\sim} \int_x^\infty e^{-t^{1-\epsilon}} dt = \frac{x^\epsilon e^{-x^{1-\epsilon}}}{1-\epsilon} + \frac{\epsilon}{1-\epsilon} \int_x^\infty t^{-1+\epsilon} e^{-t^{1-\epsilon}} dt \\ &= \frac{x^\epsilon e^{-x^{1-\epsilon}}}{1-\epsilon} + o\left(\int_x^\infty e^{-t^{1-\epsilon}} dt\right) = \frac{x^\epsilon e^{-x^{1-\epsilon}}}{1-\epsilon} (1 + o(1)). \end{aligned}$$

□

Now, we present a unilateral version of the contraction principle for a sequence of random variables in \mathbb{R}^2 with independent coordinates where the function considered is the sum of the coordinates. Observe that only unilateral assumptions are required.

Proposition 8. *Let $((Y_{n,1}, Y_{n,2}))_{n \geq 0}$ be a sequence of \mathbb{R}^2 -valued random variables such that, for each n , $Y_{n,1}$ and $Y_{n,2}$ are independent. Let $(\epsilon_n)_{n \geq 0}$ be a positive sequence converging to 0. For all $a \in \mathbb{R}$ and $i \in \{1, 2\}$, let us define*

$$\underline{I}_i(a) = -\inf_{u < a} \liminf_{n \rightarrow \infty} \epsilon_n \log \mathbb{P}(Y_{n,i} > u) \quad \text{and} \quad \overline{I}_i(a) = -\inf_{u < a} \limsup_{n \rightarrow \infty} \epsilon_n \log \mathbb{P}(Y_{n,i} > u).$$

Assume that:

(H) *for all $M > 0$, there exists $d > 0$ such that*

$$\limsup_{n \rightarrow \infty} \epsilon_n \log \mathbb{P}(Y_{n,1} > d, Y_{n,2} < -d) < -M \quad \text{and} \quad \limsup_{n \rightarrow \infty} \epsilon_n \log \mathbb{P}(Y_{n,1} < -d, Y_{n,2} > d) < -M.$$

Then, for all $c \in \mathbb{R}$, one has

$$-\inf_{t > c} \underline{I}(t) \leq \liminf_{n \rightarrow \infty} \epsilon_n \log \mathbb{P}(Y_{n,1} + Y_{n,2} > c) \leq \limsup_{n \rightarrow \infty} \epsilon_n \log \mathbb{P}(Y_{n,1} + Y_{n,2} \geq c) \leq -\inf_{t \geq c} \overline{I}(t)$$

where, for all $t \in \mathbb{R}$,

$$\underline{I}(t) := \inf_{\substack{a, b \in \mathbb{R} \\ a+b=t}} \underline{I}_1(a) + \underline{I}_2(b) \quad \text{and} \quad \overline{I}(t) := \inf_{\substack{a, b \in \mathbb{R} \\ a+b=t}} \overline{I}_1(a) + \overline{I}_2(b).$$

Moreover \underline{I} and \overline{I} are nondecreasing functions.

Proof. See Section 5. □

Remark 9. A sufficient condition for assumption **(H)** is: for $i \in \{1, 2\}$,

$$\lim_{a \rightarrow \infty} \overline{I}_i(a) = \infty.$$

The next theorem is inspired from [18]. Its proof is omitted to lighten the present paper.

Theorem 10. *Let $(Y_n)_{n \geq 0}$ be a sequence of real random variables and a positive sequence $(\epsilon_n)_{n \geq 0}$ converging to 0. Suppose that there exists a differentiable function Λ defined on \mathbb{R}_+ such that Λ' is a (increasing) bijective function from \mathbb{R}_+ to \mathbb{R}_+ and, for all $\lambda \geq 0$:*

$$\epsilon_n \log \mathbb{E}[e^{\lambda Y_n / \epsilon_n}] \xrightarrow{n \rightarrow \infty} \Lambda(\lambda).$$

Then, for all $c \geq 0$,

$$-\inf_{t > c} \Lambda^*(t) \leq \liminf_{n \rightarrow \infty} \epsilon_n \log \mathbb{P}(Y_n > c) \leq \limsup_{n \rightarrow \infty} \epsilon_n \log \mathbb{P}(Y_n \geq c) \leq -\inf_{t \geq c} \Lambda^*(t),$$

where, for all $t \geq 0$, $\Lambda^*(t) := \sup\{\lambda t - \Lambda(\lambda) ; \lambda \geq 0\}$.

4 Proofs of the main results

From now on, all non explicitly mentioned asymptotics are taken as $n \rightarrow \infty$.

4.1 Proof of Theorem 1

Applying Lemma 7, we first obtain

$$\mathbb{P}(M_n \geq x_n) = 1 - [1 - \mathbb{P}(X \geq x_n)]^n \sim n\mathbb{P}(X \geq x_n),$$

and then

$$\lim \frac{1}{x_n^{1-\epsilon}} \log \mathbb{P}(M_n \geq x_n) = \lim \frac{1}{x_n^{1-\epsilon}} \log \mathbb{P}(X \geq x_n) = -1.$$

Now, let us turn to $P_n(x_n)$. We observe that:

$$\frac{1}{x_n^{1-\epsilon}} \log \mathbb{P}(S_n \geq x_n) \geq \frac{1}{x_n^{1-\epsilon}} \log \mathbb{P}\left(\frac{S_{n-1}}{\sqrt{n-1}} \geq 0\right) + \frac{1}{x_n^{1-\epsilon}} \log \mathbb{P}(X_n \geq x_n).$$

By the central limit theorem, the first term in the right hand side converges to 0, while the second one converges to -1 . As a result,

$$\underline{\lim} \frac{1}{x_n^{1-\epsilon}} \log P_n(x_n) \geq -1.$$

To obtain the upper bound, let us introduce the following decomposition:

$$\mathbb{P}(S_n \geq x_n) = \mathbb{P}(S_n \geq x_n, \forall i \ X_i < x_n) + \mathbb{P}(S_n \geq x_n, \exists i \ X_i \geq x_n) =: P_{n,0}(x_n) + R_{n,0}(x_n).$$

Lemma 11. *If $x_n \gg n^{1/2}$, then*

$$\overline{\lim} \frac{1}{x_n^{1-\epsilon}} \log R_{n,0}(x_n) \leq -1. \quad (5)$$

Proof. Just apply the union bound and Lemma 7:

$$\frac{1}{x_n^{1-\epsilon}} \log R_{n,0}(x_n) \leq \frac{1}{x_n^{1-\epsilon}} \log(n\mathbb{P}(X \geq x_n)) \rightarrow -1.$$

□

To conclude the proof of Theorem 1, it remains to prove that, if $x_n \gg n^{1/(1+\epsilon)}$,

$$\overline{\lim} \frac{1}{x_n^{1-\epsilon}} \log P_{n,0}(x_n) \leq -1, \quad (6)$$

and to apply Lemma 6. To prove inequality (6), we apply the standard Chebyshev's exponential inequality, which amounts to $\mathbb{1}_{a \geq 0} \leq e^a$: for all $u \in (0, 1)$,

$$\begin{aligned} \frac{1}{x_n^{1-\epsilon}} \log P_{n,0}(x_n) &= \frac{1}{x_n^{1-\epsilon}} \log \mathbb{E}[\mathbb{1}_{S_n - x_n > 0} \mathbb{1}_{\forall i \ X_i < x_n}] \\ &\leq \frac{1}{x_n^{1-\epsilon}} \log \mathbb{E}\left[e^{ux_n^{-\epsilon}(S_n - x_n)} \mathbb{1}_{\forall i \ X_i < x_n}\right] \\ &= -u + \frac{n}{x_n^{1-\epsilon}} \log \mathbb{E}\left[e^{ux_n^{-\epsilon}X} \mathbb{1}_{X < x_n}\right]. \end{aligned} \quad (7)$$

Now, there exists $c > 0$ such that, for all $t \leq u$, $e^t \leq 1 + t + ct^2$. Therefore,

$$\mathbb{E}\left[e^{ux_n^{-\epsilon}X} \mathbb{1}_{X < x_n^\epsilon}\right] \leq \mathbb{E}\left[1 + ux_n^{-\epsilon}X + c(ux_n^{-\epsilon}X)^2\right] = 1 + O(x_n^{-2\epsilon}).$$

And, using (1), for n large enough,

$$\mathbb{E} \left[e^{ux_n^{-\epsilon} X} \mathbb{1}_{x_n^\epsilon \leq X < x_n} \right] = \int_{x_n^\epsilon}^{x_n} e^{ux_n^{-\epsilon} y} p(y) dy \leq \int_{x_n^\epsilon}^{x_n} 2e^{ux_n^{-\epsilon} y - y^{1-\epsilon}} dy.$$

The convex function $f_n(y) = ux_n^{-\epsilon} y - y^{1-\epsilon}$ attains its maximum on $[x_n^\epsilon, x_n]$ on the boundary. Since $f_n(x_n^\epsilon) \sim -x_n^{\epsilon(1-\epsilon)}$, $f_n(x_n) = (u-1)x_n^{1-\epsilon}$, and $u \in (0, 1)$, $f_n(x_n) \leq f_n(x_n^\epsilon)$ for n large enough, whence

$$\mathbb{E} \left[e^{ux_n^{-\epsilon} X} \mathbb{1}_{x_n^\epsilon \leq X < x_n} \right] \leq 2x_n e^{-\frac{1}{2}x_n^{\epsilon(1-\epsilon)}} = O(x_n^{-2\epsilon}).$$

Consequently, for $x_n \gg n^{1/(1+\epsilon)}$,

$$\frac{n}{x_n^{1-\epsilon}} \log \mathbb{E} \left[e^{ux_n^{-\epsilon} X} \mathbb{1}_{X < x_n} \right] = \frac{n}{x_n^{1-\epsilon}} \log(1 + O(x_n^{-2\epsilon})) = O\left(\frac{n}{x_n^{1+\epsilon}}\right) \rightarrow 0. \quad (8)$$

Combining (7) and (8), and letting $u \rightarrow 1$ completes the proof of inequality (6).

4.2 Proof of Theorem 2

By assumption (1) the Laplace transform Λ_X of X is not defined at the right of zero. That is why, we consider a new partition of \mathbb{R}^n , and define truncated random variables in order to get a nondegenerated limit of their log-Laplace transforms (see Theorem 10). For this, we introduce, for all $a \in \mathbb{R}$ and $m \in \llbracket 0, n \rrbracket$,

$$\Pi_{n,m}(a) := \mathbb{P}(S_n \geq a, \forall i \in \llbracket 1, m \rrbracket \quad x_n^\epsilon \leq X_i < x_n, \forall i \in \llbracket m+1, n \rrbracket \quad X_i < x_n^\epsilon),$$

so that $P_n(x_n)$ writes as

$$P_n(x_n) = P_{n,0}(x_n) + R_{n,0}(x_n) = \sum_{m=0}^n \binom{n}{m} \Pi_{n,m}(x_n) + R_{n,0}(x_n).$$

By Lemma 11 and the fact that, for $x_n \ll n^{1/(1+\epsilon)}$, $1/x_n^{1-\epsilon} \ll n/x_n^2$, we get

$$\lim \frac{n}{x_n^2} \log R_{n,0}(x_n) = -\infty. \quad (9)$$

Lemma 12. *Let $C > 0$. If $n^{1/2} \ll x_n \leq Cn^{1/(1+\epsilon)}$ and $t > 0$, then*

$$\lim \frac{n}{x_n^2} \log \Pi_{n,0}(tx_n) = -\frac{t}{2\sigma^2}. \quad (10)$$

Proof. Let us introduce \bar{X} with distribution $\mathcal{L}(X | X < x_n^\epsilon)$. For all $n \in \mathbb{N}^*$, let $\bar{X}_1, \bar{X}_2, \dots, \bar{X}_n$ be i.i.d. copies of \bar{X} and let $\bar{S}_n = \bar{X}_1 + \dots + \bar{X}_n$, so that

$$\Pi_{n,0}(tx_n) = \mathbb{P}(S_n \geq tx_n, X_1, \dots, X_n < x_n^\epsilon) = \mathbb{P}(\bar{S}_n \geq tx_n) \mathbb{P}(X < x_n^\epsilon)^n \sim \mathbb{P}(\bar{S}_n \geq tx_n),$$

by Lemma 7. We want to apply Theorem 10 to the random variables \bar{S}_n/x_n with $\epsilon_n = n/x_n^2$. For $u > 0$,

$$\frac{n}{x_n^2} \log \mathbb{E} \left[e^{u \frac{x_n^2}{n} \frac{\bar{S}_n}{x_n}} \right] = \frac{n^2}{x_n^2} \log \mathbb{E} \left[e^{\frac{ux_n X}{n}} \mathbb{1}_{X < x_n^\epsilon} \right] - \frac{n^2}{x_n^2} \log \mathbb{P}(X < x_n^\epsilon). \quad (11)$$

The second term in the right side of the above equation goes to 0 as $n \rightarrow \infty$ since $\log \mathbb{P}(X < x_n^\epsilon) \sim -\mathbb{P}(X > x_n^\epsilon) = O(e^{-x_n^{\epsilon(1-\epsilon)}/2})$, by Lemma 7. As for the first term, if $y < x_n^\epsilon$, then $x_n y/n \leq$

$x_n^{1+\epsilon}/n \leq C^{1+\epsilon}$. Now, there exists $c > 0$ such that, for all $s \leq C^{1+\epsilon}$, $|e^s - (1+s+s^2/2)| \leq c|s|^{2+\gamma}$. Hence,

$$\begin{aligned}
& \left| \mathbb{E} \left[e^{\frac{ux_n X}{n}} \mathbf{1}_{X < x_n^\epsilon} \right] - e^{\frac{u^2 x_n^2 \sigma^2}{2n^2}} \right| \\
& \leq \left| \mathbb{E} \left[e^{\frac{ux_n X}{n}} \mathbf{1}_{X < x_n^\epsilon} \right] - \mathbb{E} \left[\left(1 + \frac{ux_n X}{n} + \frac{u^2 x_n^2 X^2}{2n^2} \right) \mathbf{1}_{X < x_n^\epsilon} \right] \right| \\
& \quad + \left| \mathbb{E} \left[\left(1 + \frac{ux_n X}{n} + \frac{u^2 x_n^2 X^2}{2n^2} \right) \mathbf{1}_{X < x_n^\epsilon} \right] - \left(1 + \frac{u^2 x_n^2 \sigma^2}{2n^2} \right) \right| \\
& \quad + \left| \left(1 + \frac{u^2 x_n^2 \sigma^2}{2n^2} \right) - e^{\frac{u^2 x_n^2 \sigma^2}{2n^2}} \right| \\
& \leq c\rho \left(\frac{ux_n}{n} \right)^{2+\gamma} + \left| \mathbb{E} \left[\left(1 + \frac{ux_n X}{n} + \frac{u^2 x_n^2 X^2}{2n^2} \right) \mathbf{1}_{X \geq x_n^\epsilon} \right] \right| + o\left(\frac{x_n^2}{n^2} \right). \tag{12}
\end{aligned}$$

For n large enough, applying Hölder's inequality,

$$\begin{aligned}
\left| \mathbb{E} \left[\left(1 + \frac{ux_n X}{n} + \frac{u^2 x_n^2 X^2}{2n^2} \right) \mathbf{1}_{X \geq x_n^\epsilon} \right] \right| & \leq \mathbb{E}[X^2 \mathbf{1}_{X \geq x_n^\epsilon}] \\
& \leq \mathbb{E}[X^{2+\gamma}]^{2/(2+\gamma)} \mathbb{P}(X \geq x_n^\epsilon)^{\gamma/(2+\gamma)} \\
& = o\left(\frac{x_n^2}{n^2} \right), \tag{13}
\end{aligned}$$

by Lemma 7. Combining (11), (12), and (13), we get

$$\frac{n}{x_n^2} \log \mathbb{E} \left[e^{u \frac{x_n^2 \bar{S}_n}{x_n}} \right] \rightarrow \frac{u^2 \sigma^2}{2} := \Lambda(u).$$

Since $\Lambda^*(t) = t/2\sigma^2$, (10) stems from Theorem 10. \square

Theorem 2 stems from Lemma 6, (9), (10) (with $t = 1$), and the fact that, for $n^{1/2} \ll x_n \ll n^{1/(1+\epsilon)}$,

$$\overline{\lim} \frac{n}{x_n^2} \log \sum_{m=1}^n \binom{n}{m} \Pi_{n,m}(x_n) \leq -\frac{1}{2\sigma^2}, \tag{14}$$

the proof of which is given now. By (1), we have, for all n large enough and for all $m \in \llbracket 1, n \rrbracket$,

$$\begin{aligned}
\Pi_{n,m}(x_n) & = \mathbb{P}(S_n \geq x_n, x_n^\epsilon \leq X_1, \dots, X_m < x_n, X_{m+1}, \dots, X_n < x_n^\epsilon) \\
& = \int_{[x_n^\epsilon, x_n]^m} \Pi_{n-m,0}(x_n - u_1 - \dots - u_m) d\mathbb{P}_{X_1}(u_1) \dots d\mathbb{P}_{X_m}(u_m) \\
& \leq 2^m \int_{[x_n^\epsilon, x_n]^m} \Pi_{n-m,0}(x_n - u_1 - \dots - u_m) e^{-\sum_{i=1}^m u_i^{1-\epsilon}} du_1 \dots du_m. \tag{15}
\end{aligned}$$

Set $m_n = \lfloor x_n^{(1-\epsilon)^2} \rfloor$ so that $m_n x_n^{\epsilon(1-\epsilon)} \sim x_n^{1-\epsilon}$ and $m_n = o(n)$.

Lemma 13. *If $x_n \gg n^{1/2}$, then*

$$\sum_{m=2m_n+1}^n \binom{n}{m} \Pi_{n,m}(x_n) = O\left(e^{-m_n x_n^{\epsilon(1-\epsilon)}} \right).$$

Proof. Using $\Pi_{n,m}(x_n - u) \leq 1$ in (15) yields

$$\begin{aligned} \sum_{m=2m_n+1}^n \binom{n}{m} \Pi_{n,m}(x_n) &\leq \sum_{m=2m_n+1}^n \binom{n}{m} 2^m \left(\int_{x_n^\epsilon}^{x_n} e^{-u^{1-\epsilon}} du \right)^m \\ &\leq \sum_{m=2m_n+1}^n \binom{n}{m} (2x_n)^m e^{-mx_n^{\epsilon(1-\epsilon)}} \\ &\leq e^{-m_n x_n^{\epsilon(1-\epsilon)}} \sum_{m=2m_n+1}^n \binom{n}{m} \left(2x_n e^{-\frac{1}{2}x_n^{\epsilon(1-\epsilon)}} \right)^m, \end{aligned}$$

and the conclusion holds, since the latter sum is bounded. \square

As $n^{1/2} \ll x_n \ll n^{1/(1+\epsilon)}$, we conclude that

$$\overline{\lim} \frac{n}{x_n^2} \log \sum_{m=2m_n+1}^n \binom{n}{m} \Pi_{n,m}(x_n) \leq \overline{\lim} -\frac{n}{x_n^{1+\epsilon}} = -\infty. \quad (16)$$

Now we consider the values of $m \in \llbracket 1, 2m_n \rrbracket$. To get a lower bound on $[x_n^\epsilon, x_n]^m$ of the function

$$s_m : (u_1, \dots, u_m) \mapsto \sum_{i=1}^m u_i^{1-\epsilon},$$

we define

$$A_{1,m} := \left\{ (u_1, \dots, u_m) \in [x_n^\epsilon, x_n]^m, \sum_{i=1}^m u_i \geq x_n \right\}$$

and

$$A_{2,m} := \left\{ (u_1, \dots, u_m) \in [x_n^\epsilon, x_n]^m, \sum_{i=1}^m u_i < x_n \right\},$$

in such a way that $[x_n^\epsilon, x_n]^m = A_{1,m} \cup A_{2,m}$ and we introduce, for $j \in \{1, 2\}$,

$$I_{j,m} := \int_{A_{j,m}} \Pi_{n-m,0}(x_n - u_1 - \dots - u_m) e^{-s_m(u_1, \dots, u_m)} du_1 \dots du_m, \quad (17)$$

so that (15) leads to

$$\Pi_{n,m}(x_n) \leq 2^m (I_{1,m} + I_{2,m}). \quad (18)$$

Lemma 14. *If $x_n \gg n^{1/2}$, then*

$$\sum_{m=1}^{2m_n} \binom{n}{m} 2^m I_{1,m} = O(e^{-x_n^{1-\epsilon}}).$$

Proof. Notice that $I_{1,1} = 0$ since $A_{1,1} = \{x_n\}$. Now, consider $m \in \{2, \dots, 2m_n\}$. By Lemma 16 (see Section 5), the function s_m reaches its minimum on $A_{1,m}$ at the points with all coordinates equal to x_n^ϵ except one equal to $x_n - (m-1)x_n^\epsilon$. By a Taylor expansion of $u \mapsto (1-u)^{1-\epsilon}$ near the origin ($(m-1)x_n^{-(1-\epsilon)} \leq 2x_n^{-\epsilon(1-\epsilon)} \rightarrow 0$), there exists $\eta_n \rightarrow 0$ such that

$$(x_n - (m-1)x_n^\epsilon)^{1-\epsilon} = x_n^{1-\epsilon} \left(1 - (m-1)(1-\epsilon + \eta_n)x_n^{-\epsilon(1-\epsilon)} \right) \geq x_n^{1-\epsilon} - 2(m-1),$$

for n large enough. Using $\Pi_{n-m,0}(x_n - u_1 - \dots - u_m) \leq 1$ in (17), one gets

$$I_{1,m} \leq x_n^m e^{-(m-1)x_n^{\epsilon(1-\epsilon)} - (x_n - (m-1)x_n^\epsilon)^{1-\epsilon}} \leq x_n^m e^{-x_n^{1-\epsilon}} e^{-(m-1)(x_n^{\epsilon(1-\epsilon)} - 2)},$$

for n large enough (and uniformly in m). Thus,

$$\sum_{m=1}^{2m_n} \binom{n}{m} 2^m I_{1,m} \leq e^{-x_n^{1-\epsilon}} \sum_{m=2}^{2m_n} \binom{n}{m} (2x_n e^2)^m e^{-(m-1)x_n^{\epsilon(1-\epsilon)}} \leq e^{-x_n^{1-\epsilon}} \sum_{m=2}^{2m_n} \left(2x_n n e^{-\frac{1}{4}x_n^{\epsilon(1-\epsilon)}} \right)^m,$$

and the conclusion holds, since the latter sum is bounded. \square

If $n^{1/2} \ll x_n \ll n^{1/(1+\epsilon)}$, we get

$$\overline{\lim} \frac{n}{x_n^2} \log \sum_{m=1}^{2m_n} \binom{n}{m} 2^m I_{1,m} = -\infty. \quad (19)$$

Lemma 15. *Let $C > 0$. If $n^{1/2} \ll x_n \leq Cn^{1/(1+\epsilon)}$, then, for n large enough, for all $m \in \{1, \dots, 2m_n\}$,*

$$I_{2,m} \leq x_n^m e^{-(m-1)x_n^{\epsilon(1-\epsilon)}} \exp \left\{ \sup_{mx_n^\epsilon \leq u < x_n} \phi_m(u) \right\}, \quad (20)$$

where

$$\phi_m(u) := -\frac{(x_n - u)^2}{2\sigma^2(n-m)(1+c_n)} - (u - (m-1)x_n^\epsilon)^{1-\epsilon}.$$

Proof. Here, we use Chebyshev's exponential inequality to control $\Pi_{n-m,0}(x_n - u)$ in (17), introducing $u = u_1 + \dots + u_m$. For all $l \in \mathbb{N}^*$ and for all $\lambda > 0$,

$$\Pi_{l,0}(x_n - u) \leq e^{-\lambda(x_n - u)} \mathbb{E} \left[e^{\lambda X} \mathbf{1}_{X < x_n^\epsilon} \right]^l = \exp \left\{ -\lambda(x_n - u) + l \log \mathbb{E} \left[e^{\lambda X} \mathbf{1}_{X < x_n^\epsilon} \right] \right\}.$$

Let $M > C^{1+\epsilon}/\sigma^2$. There exists $c > 0$ such that, for all $s \leq M$, we have $e^s \leq 1 + s + s^2/2 + c|s|^{2+\gamma}$. Hence, as soon as $\lambda \leq Mx_n^{-\epsilon}$,

$$\mathbb{E} \left[e^{\lambda X} \mathbf{1}_{X < x_n^\epsilon} \right] \leq 1 + \frac{\lambda^2 \sigma^2}{2} + c\rho \lambda^{2+\gamma} \leq 1 + \frac{\lambda^2 \sigma^2}{2} (1 + c_n),$$

where

$$c_n := 2c\rho\sigma^{-2}M^\gamma x_n^{-\gamma\epsilon} = o(1).$$

Thus, for $l = n - m$ and $\lambda \leq Mx_n^{-\epsilon}$, $\Pi_{n-m,0}(x_n - u) \leq \exp\{-\lambda(x_n - u) + (n - m)\lambda^2\sigma^2(1 + c_n)/2\}$. For n large enough, for all $m \in \{1, \dots, 2m_n\}$, the minimum of $\lambda \mapsto -\lambda(x_n - u) + (n - m)\lambda^2\sigma^2(1 + c_n)/2$ is reached at

$$\lambda^* = \frac{x_n - u}{(n - m)\sigma^2(1 + c_n)} \leq \frac{x_n}{(n - m_n)\sigma^2(1 + c_n)} \leq Mx_n^{-\epsilon},$$

and is equal to $-(x_n - u)^2/(2\sigma^2(n - m)(1 + c_n))$. Then we obtain, for n large enough and for all $m \in \{1, \dots, 2m_n\}$:

$$\Pi_{n-m,0}(x_n - u) \leq \exp \left\{ -\frac{(x_n - u)^2}{2\sigma^2(n - m)(1 + c_n)} \right\}. \quad (21)$$

By Lemma 16, on $A_{2,m,u} := \{(u_1, \dots, u_m) \in [x_n^\epsilon, x_n]^m, u_1 + \dots + u_m = u\}$, the function s_m reaches its minimum at the points with all coordinates equal to x_n^ϵ except one equal to $u - (m - 1)x_n^\epsilon$. This, together with (21) and (17), yields

$$\begin{aligned} I_{2,m} &\leq x_n^m \sup_{mx_n^\epsilon \leq u < x_n} \exp \left\{ -\frac{(x_n - u)^2}{2\sigma^2(n - m)(1 + c_n)} - (u - (m - 1)x_n^\epsilon)^{1-\epsilon} - (m - 1)x_n^{\epsilon(1-\epsilon)} \right\} \\ &\leq x_n^m e^{-(m-1)x_n^{\epsilon(1-\epsilon)}} \exp \left\{ \sup_{mx_n^\epsilon \leq u < x_n} \phi_m(u) \right\}. \end{aligned}$$

□

The function ϕ_m is differentiable on $[mx_n^\epsilon, x_n)$ and

$$\phi'_m(u) \leq 0 \iff (x_n - u)(u - (m-1)x_n^\epsilon)^\epsilon - (1-\epsilon)\sigma^2(n-m)(1+c_n) \leq 0.$$

For n large enough, for all $m \in \{1, \dots, 2m_n\}$, for all $u \in [mx_n^\epsilon, x_n)$, one has

$$(x_n - u)(u - (m-1)x_n^\epsilon)^\epsilon \leq x_n^{1+\epsilon} \leq (1-\epsilon)\sigma^2(n-m_n)(1+c_n) \leq (1-\epsilon)\sigma^2(n-m)(1+c_n)$$

since $x_n^{1+\epsilon} \ll n \sim n - m_n$. We deduce that, for n large enough, ϕ_m is nonincreasing on $[mx_n^\epsilon, x_n)$ and

$$\sup_{mx_n^\epsilon \leq u < x_n} \phi_m(u) = \phi_m(mx_n^\epsilon) = -\frac{(x_n - mx_n^\epsilon)^2}{2\sigma^2(n-m)(1+c_n)} - x_n^{\epsilon(1-\epsilon)} \leq -\frac{(x_n - m_n x_n^\epsilon)^2}{2n\sigma^2(1+c_n)} - x_n^{\epsilon(1-\epsilon)}.$$

It follows that

$$\sum_{m=1}^{2m_n} \binom{n}{m} 2^m I_{2,m} \leq e^{-\frac{(x_n - m_n x_n^\epsilon)^2}{2n\sigma^2(1+c_n)}} \sum_{m=1}^{2m_n} \binom{n}{m} (2x_n e^{-x_n^{\epsilon(1-\epsilon)}})^m = O\left(e^{-\frac{(x_n - m_n x_n^\epsilon)^2}{2n\sigma^2(1+c_n)}}\right)$$

since the latter sum is bounded and, finally, one gets

$$\overline{\lim} \frac{n}{x_n^2} \log \sum_{m=1}^{2m_n} \binom{n}{m} 2^m I_{2,m} \leq -\frac{1}{2\sigma^2}, \quad (22)$$

as $m_n x_n^\epsilon = o(x_n)$ and $c_n = o(1)$. By (16), (18), (19), (22), and Lemma 6, we get (14) as required.

4.3 Proof of Theorem 3

We consider $x_n = Cn^{1/(1+\epsilon)}$ with $C > 0$. As in the proof of Theorem 2, we consider the decomposition

$$P_n(x_n) = P_{n,0}(x_n) + R_{n,0}(x_n) = \sum_{m=0}^n \binom{n}{m} \Pi_{n,m}(x_n) + R_{n,0}(x_n).$$

By Lemmas 11 and 12, and the very definition of J in (4), we have

$$\overline{\lim} \frac{1}{x_n^{1-\epsilon}} \log R_{n,0}(x_n) \leq -1 \leq -J(C) \quad (23)$$

and

$$\overline{\lim} \frac{1}{x_n^{1-\epsilon}} \log \Pi_{n,0}(x_n) \leq -\frac{C^{1+\epsilon}}{2\sigma^2} \leq -J(C). \quad (24)$$

To complete the proof of Theorem 3, it remains to prove that

$$\lim \frac{1}{x_n^{1-\epsilon}} \log(n\Pi_{n,1}(x_n)) = -J(C), \quad (25)$$

$$\overline{\lim} \frac{1}{x_n^{1-\epsilon}} \log \sum_{m=2}^n \binom{n}{m} \Pi_{n,m}(x_n) \leq -J(C), \quad (26)$$

and to apply Lemma 6.

Proof of (25). Recall that, for $n \in \mathbb{N}^*$,

$$\Pi_{n,1}(x_n) = \mathbb{P}(S_n \geq x_n \mid x_n^\epsilon \leq X_1 \leq x_n, X_2, \dots, X_n < x_n^\epsilon) \mathbb{P}(x_n^\epsilon \leq X_1 \leq x_n) \mathbb{P}(X < x_n^\epsilon)^{n-1},$$

Using Lemma 7, it suffices to prove that

$$\frac{1}{x_n^{1-\epsilon}} \log \mathbb{P}(S_n \geq x_n \mid x_n^\epsilon \leq X_1 < x_n, X_2, \dots, X_n < x_n^\epsilon) \rightarrow -J(C).$$

To do so, we apply the contraction principle of Proposition 8 to $(Y_{n,1}, Y_{n,2})$ with

$$\mathcal{L}(Y_{n,1}) = \mathcal{L}(x_n^{-1}X_1 \mid x_n^\epsilon \leq X_1 < x_n) \quad \text{and} \quad \mathcal{L}(Y_{n,2}) = \mathcal{L}(x_n^{-1}(X_2 + \dots + X_n) \mid X_2, \dots, X_n < x_n^\epsilon),$$

and $\epsilon_n = x_n^{-(1-\epsilon)}$. First, one has obviously, $\mathbb{P}(X \geq ux_n \mid x_n^\epsilon \leq X < x_n) = 1$ for $u \leq 0$ and $\mathbb{P}(X \geq ux_n \mid x_n^\epsilon \leq X < x_n) = 0$ for $u \geq 1$. In addition, for $u \in (0, 1)$, applying Lemma 7,

$$\log \mathbb{P}(X \geq ux_n \mid x_n^\epsilon \leq X < x_n) \sim -(ux_n)^{1-\epsilon}.$$

Using the notation of Proposition 8, it follows that $\underline{I}_1(a) = \overline{I}_1(a) = I_1(a)$, where

$$I_1(a) = \sup_{u < a} \left\{ -\lim_{x_n^{1-\epsilon}} \frac{1}{x_n^{1-\epsilon}} \log \mathbb{P}(X_1 \geq ux_n \mid x_n^\epsilon \leq X_1 \leq x_n) \right\} = \begin{cases} 0 & \text{if } a < 0, \\ a^{1-\epsilon} & \text{if } a \in [0, 1], \\ \infty & \text{if } a > 1. \end{cases} \quad (27)$$

Moreover, for all $u > 0$,

$$\frac{1}{x_n^{1-\epsilon}} \log \mathbb{P}(X_2 + \dots + X_n \geq ux_n \mid X_2, \dots, X_n < x_n^\epsilon) = \frac{1}{x_n^{1-\epsilon}} \log \Pi_{n-1,0}(ux_n) \rightarrow -\frac{u^2 C^{1+\epsilon}}{2\sigma^2},$$

by Lemma 12. Thus, we have $I_2(b) = b^2 C^{1+\epsilon} / (2\sigma^2)$ for all $b > 0$ and, since I_2 is a nondecreasing and nonnegative function, we get $I_2(b) = 0$ for all $b \leq 0$. This, together with (27), leads to: for all $t \in \mathbb{R}$,

$$I(t) = \inf_{\substack{a+b=t \\ (a,b) \in \mathbb{R}^2}} \{I_1(a) + I_2(b)\} = \inf_{t-1 \leq b \leq t} \{I_1(t-b) + I_2(b)\} = \inf_{t-1 \leq b \leq t} \left\{ (t-b)^{1-\epsilon} + \frac{b^2 C^{1+\epsilon}}{2\sigma^2} \right\},$$

since $b < t-1$ entails $I_1(t-b) = \infty$ and $b > t$ entails $I_1(t-b) + I_2(b) > I_1(0) + I_2(t)$.

It is a standard result (see, e.g., [12, 4.c.]) that I is upper semicontinuous. Since I is also nondecreasing, I is right continuous and we get

$$\inf_{t \geq 1} I(t) = \inf_{t > 1} I(t) = I(1).$$

Applying Proposition 8, this completes the proof of (25). \square

Proof of (26). We follow the same lines as in the proof of (14) in the Gaussian range. With the same definition of $m_n = \lfloor x_n^{(1-\epsilon)^2} \rfloor$, Lemmas 13 and 14 entail

$$\overline{\lim} \frac{1}{x_n^{1-\epsilon}} \sum_{m=2m_n+1}^n \binom{n}{m} \Pi_{n,m} \leq -1 \quad \text{and} \quad \overline{\lim} \frac{1}{x_n^{1-\epsilon}} \log \sum_{m=2}^{2m_n} \binom{n}{m} 2^m I_{1,m} \leq -1. \quad (28)$$

To complete the proof, it suffices to show that

$$\overline{\lim} \frac{1}{x_n^{1-\epsilon}} \log \sum_{m=2}^{2m_n} 2^m \binom{n}{m} I_{2,m} \leq -J(C). \quad (29)$$

Recall that Lemma 15 provides $I_{2,m} \leq x_n^m e^{-(m-1)x_n^\epsilon} e^{M_n}$, where

$$\begin{aligned} M_n &= \sup_{mx_n^\epsilon \leq u < x_n} \left\{ -\frac{(x_n - u)^2}{2\sigma^2(n-m)(1+c_n)} - (u - (m-1)x_n^\epsilon)^{1-\epsilon} \right\} \\ &\leq \sup_{mx_n^{-1+\epsilon} \leq t < 1} \left\{ -\frac{(x_n - tx_n)^2}{2\sigma^2 n(1+c_n)} - (tx_n - (m-1)x_n^\epsilon)^{1-\epsilon} \right\} \\ &= -x_n^{1-\epsilon} \inf_{mx_n^{-1+\epsilon} \leq t < 1} \left\{ \frac{(1-t)^2 C^{1+\epsilon}}{2\sigma^2(1+c_n)} + t^{1-\epsilon} \left(1 - \frac{(m-1)x_n^{-1+\epsilon}}{t} \right)^{1-\epsilon} \right\}. \end{aligned}$$

Let us fix $\eta \in (0, 1)$. For n large enough, for all $m \in \{2, \dots, 2m_n\}$,

$$\begin{aligned} \inf_{mx_n^{-1+\epsilon} \leq t < \eta} \left\{ \frac{(1-t)^2 C^{1+\epsilon}}{2\sigma^2(1+c_n)} + t^{1-\epsilon} \left(1 - \frac{(m-1)x_n^{-1+\epsilon}}{t} \right)^{1-\epsilon} \right\} &\geq \frac{(1-\eta)^2 C^{1+\epsilon}}{2\sigma^2} (1-\eta) \\ &\geq (1-\eta)^3 J(C) \end{aligned}$$

and

$$\begin{aligned} \inf_{\eta \leq t < 1} \left\{ \frac{(1-t)^2 C^{1+\epsilon}}{2\sigma^2(1+c_n)} + t^{1-\epsilon} \left(1 - \frac{(m-1)x_n^{-1+\epsilon}}{t} \right)^{1-\epsilon} \right\} &\geq \inf_{\eta \leq t < 1} \left\{ \frac{(1-t)^2 C^{1+\epsilon}}{2\sigma^2} + t^{1-\epsilon} \right\} (1-\eta) \\ &\geq (1-\eta) J(C). \end{aligned}$$

So $M_n \leq -x_n^{1-\epsilon} (1-\eta)^3 J(C)$ and

$$\begin{aligned} \overline{\lim}_{x_n^{1-\epsilon}} \frac{1}{x_n^{1-\epsilon}} \log \sum_{m=2}^{2m_n} \binom{n}{m} 2^m I_{2,m} &\leq -(1-\eta)^3 J(C) + \overline{\lim}_{x_n^{1-\epsilon}} \frac{1}{x_n^{1-\epsilon}} \log \sum_{m=2}^{2m_n} \binom{n}{m} (2x_n e^{-x_n^\epsilon})^m \\ &= -(1-\eta)^3 J(C), \end{aligned}$$

and (29) follows, letting $\eta \rightarrow 0$. □

5 Proofs of the intermediate results

Proof of Proposition 8. Obviously, the functions \underline{I}_1 , \bar{I}_1 , \underline{I}_2 , and \bar{I}_2 are nondecreasing. Let us prove that \underline{I} is nondecreasing, the proof for \bar{I} being similar. Let $t_1 < t_2$, let $\eta > 0$, and let $a \in \mathbb{R}$ be such that $\underline{I}(t_2) \geq \underline{I}_1(a) + \underline{I}_2(t_2 - a) - \eta$. Since \underline{I}_2 is nondecreasing, we have

$$\underline{I}(t_1) \leq \underline{I}_1(a) + \underline{I}_2(t_1 - a) \leq \underline{I}_1(a) + \underline{I}_2(t_2 - a) \leq \underline{I}(t_2) + \eta,$$

which completes the proof of the monotony of \underline{I} , letting $\eta \rightarrow 0$.

Lower bound. Let $c \in \mathbb{R}$, let $t > c$, and let $\delta > 0$ be such that $0 < 2\delta < t - c$. For all $(a, b) \in \mathbb{R}^2$ such that $a + b = t$, we have

$$\begin{aligned} \underline{\lim} \epsilon_n \log \mathbb{P}(Y_{n,1} + Y_{n,2} > c) &\geq \underline{\lim} \epsilon_n \log \mathbb{P}(Y_{n,1} > a - \delta) + \underline{\lim} \epsilon_n \log \mathbb{P}(Y_{n,2} > b - \delta) \\ &\geq -\underline{I}_1(a) - \underline{I}_2(b). \end{aligned}$$

Therefore,

$$\underline{\lim} \epsilon_n \log \mathbb{P}(Y_{n,1} + Y_{n,2} > c) \geq \sup_{t > c} \sup_{\substack{(a,b) \in \mathbb{R}^2 \\ a+b=t}} (-\underline{I}_1(a) - \underline{I}_2(b)) = -\inf_{t > c} \underline{I}(t).$$

Upper bound. Let $c \in \mathbb{R}$ and let $M > 0$. Let $d > 0$ be given by assumption **(H)**. Define

$$Z = \{(a, b) \in \mathbb{R}^2 ; a + b \geq c, a \geq -d, b \geq -d\}$$

and

$$K = \{(a, b) \in \mathbb{R}^2 ; a + b = c, a \geq -d, b \geq -d\}.$$

Write

$$\begin{aligned} \mathbb{P}(Y_{n,1} + Y_{n,2} \geq c) &\leq \mathbb{P}(Y_{n,1} > d, Y_{n,2} < -d) + \mathbb{P}(Y_{n,1} < -d, Y_{n,2} > d) + \mathbb{P}((Y_{n,1}, Y_{n,2}) \in Z) \\ &=: Q_{n,1} + Q_{n,2} + Q_{n,3}. \end{aligned}$$

By assumption,

$$\overline{\lim} \epsilon_n \log(Q_{n,1}) < -M \quad \text{and} \quad \overline{\lim} \epsilon_n \log(Q_{n,2}) < -M.$$

Let us estimate $\overline{\lim} \epsilon_n \log Q_{n,3}$. For all $(a, b) \in K$,

$$\begin{aligned} & - \inf_{\substack{u < a \\ v < b}} \overline{\lim} \epsilon_n \log \mathbb{P}(Y_{n,1} > u, Y_{n,2} > v) \\ & \geq - \inf_{u < a} \overline{\lim} \epsilon_n \log \mathbb{P}(Y_{n,1} > u) - \inf_{v < b} \overline{\lim} \epsilon_n \log \mathbb{P}(Y_{n,2} > v) \\ & = \overline{I}_1(a) + \overline{I}_2(b). \end{aligned}$$

Defining $\theta^{[\delta]} := \min(\theta - \delta, \delta^{-1})$ for all $\delta > 0$ and for all $\theta \in (-\infty, \infty]$, there exists $u_a < a$ and $v_b < b$ such that

$$- \overline{\lim} \epsilon_n \log \mathbb{P}(Y_{n,1} > u_a, Y_{n,2} > v_b) \geq (\overline{I}_1(a) + \overline{I}_2(b))^{[\delta]}. \quad (30)$$

From the cover $((u_a, \infty) \times (v_b, \infty))_{(a,b) \in K}$ of the compact subset K , we can extract a finite subcover $((u_{a_i}, \infty) \times (v_{b_i}, \infty))_{1 \leq i \leq p}$. Since

$$Z \subset \bigcup_{i=1}^p (u_{a_i}, \infty) \times (v_{b_i}, \infty),$$

we obtain, thanks to Lemma 6 and (30),

$$\begin{aligned} \overline{\lim} \epsilon_n \log Q_3 &\leq \overline{\lim} \epsilon_n \log \sum_{i=1}^p \mathbb{P}(Y_{n,1} > u_{a_i}, Y_{n,2} > v_{b_i}) \\ &= \max_{1 \leq i \leq p} \{ \overline{\lim} \epsilon_n \log \mathbb{P}(Y_{n,1} > u_{a_i}, Y_{n,2} > v_{b_i}) \} \\ &\leq \max_{1 \leq i \leq p} \{ -(\overline{I}_1(a_i) + \overline{I}_2(b_i))^{[\delta]} \} \\ &\leq - \inf_{\substack{(a,b) \in \mathbb{R}^2 \\ a+b=c}} \{ (\overline{I}_1(a) + \overline{I}_2(b))^{[\delta]} \}. \end{aligned}$$

Letting $\delta \rightarrow 0$ and using the definition of \overline{I} , we deduce that

$$\overline{\lim} \epsilon_n \log Q_3 \leq - \inf_{\substack{(a,b) \in \mathbb{R}^2 \\ a+b=c}} (\overline{I}_1(a) + \overline{I}_2(b)) = -\overline{I}(c) = - \inf_{t \geq c} \overline{I}(t). \quad (31)$$

Letting $M \rightarrow \infty$, we get the desired upper bound. \square

Lemma 16. Let the real-valued function s_m be defined on $(0, \infty)^m$ by

$$s_m(u_1, \dots, u_m) = \sum_{i=1}^m u_i^{1-\epsilon}$$

and the sets $A_{1,m}$ and $A_{2,m,u}$, as defined in the proof of (14). Assume that $mx^\epsilon < x$.

1. On $A_{1,m}$, the function s_m reaches its minimum at points with one coordinate equal to $x - (m - 1)x^\epsilon$ and the other coordinates equal to x^ϵ .
2. On $A_{2,m,u}$, the function s_m reaches its minimum at points with one coordinate equal to $u - (m - 1)x^\epsilon$ and the other coordinates equal to x^ϵ .

Proof of Lemma 16. 1. $A_{1,m} = L_x^\geq \cap \left(\bigcap_{i=1}^m H_i^\geq \cap G_i^\leq \right)$ is a compact polyhedron defined through the following half-spaces:

$$H_i^\geq = \{v \in \mathbb{R}^m, v_i \geq x^\epsilon\}, \quad G_i^\leq = \{v \in \mathbb{R}^m, v_i \leq x\}, \quad L_x^\geq = \left\{ v \in \mathbb{R}^m, \sum_{i=1}^m v_i \geq x \right\},$$

the boundaries of which are

$$H_i = \{v \in \mathbb{R}^m, v_i = x^\epsilon\}, \quad G_i = \{v \in \mathbb{R}^m, v_i = x\}, \quad L_x = \left\{ v \in \mathbb{R}^m, \sum_{i=1}^m v_i = x \right\}.$$

Now, s_m being a continuous and concave function, its minimum is reached at an extremal point of $A_{1,m}$, that is, an intersection of $m + 1$ supporting hyperplanes, the equations of which are linearly independent and compatible. Nevertheless, for all $i \in \{1, \dots, m\}$, $G_i \cap L_x \cap A_{1,m} = \emptyset$, and the assumption $mx^\epsilon < x$ excludes that all coordinates of an extremal point are equal to x^ϵ . Consequently, an extremal point $t = (t_1, \dots, t_m)$ is such that $m - 1$ coordinates are equal to x^ϵ and $t_1 + \dots + t_m = x$ or k coordinates are equal to x^ϵ and $(m - k)$ equal to x , with $k \in \{1, \dots, m - 1\}$. Setting, for $v \geq 0$, $p(v) = v^{1-\epsilon}$, the first case gives

$$s_m(t) = (m - 1)p(x^\epsilon) + p(x - (m - 1)x^\epsilon),$$

while the second gives

$$s_m(t) = kp(x^\epsilon) + (m - k)p(x).$$

As p is an increasing function, one has, for all $k \in \{1, \dots, m - 1\}$,

$$(m - k)p(x) = (m - k - 1)p(x) + p(x) \geq (m - k - 1)p(x^\epsilon) + p(x - (m - 1)x^\epsilon)$$

then

$$kp(x^\epsilon) + (m - k)p(x) \geq (m - 1)p(x^\epsilon) + p(x - (m - 1)x^\epsilon)$$

which proves the claim.

2. We prove the result in a similar way than in the previous point. With the same notation, we get

$$A_{2,m,u} = L_u^\geq \cap L_u^\leq \cap \left(\bigcap_{i=1}^m H_i^\geq \cap G_i^\leq \right).$$

As $u \in [mx^\epsilon, x)$, an extremal point of $A_{2,m,u}$ has no coordinate equal to x . Then such a point has $(m - 1)$ coordinates equal to x^ϵ and one equal to $u - (m - 1)x^\epsilon$ which completes the proof. \square

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