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# Entropy solutions in $BV^s$ for a class of triangular systems involving a transport equation

Christian Bourdarias <sup>\*</sup>, Anupam Pal Choudhury <sup>†</sup>,  
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ABSTRACT. – In this article, we consider a class of strictly hyperbolic triangular systems involving a transport equation. Such systems are known to create measure solutions for the initial value problem. Adding a stronger transversality assumption on the fields, we are able to obtain solutions in  $L^\infty$  under optimal fractional  $BV$  regularity of the initial data. Our results show that the critical fractional regularity is  $s = 1/3$ . We also construct an initial data that is not in  $BV^{1/3}$  but for which a blow-up in  $L^\infty$  occurs, proving the optimality of our results.

**AMS Classification:** 35L65, 35L67, 35Q35, 26A45, 76N10.

**Key words:** conservation laws; triangular systems; entropy solution; fractional  $BV$  spaces; linear transport; chromatography; Cauchy problem; non convex flux; regularity.

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# 1 Introduction

We consider triangular systems of the form,  $t > 0$ ,  $x \in \mathbb{R}$ ,

$$\partial_t u + \partial_x f(u) = 0, \tag{1.1}$$

$$\partial_t v + \partial_x (a(u)v) = 0. \tag{1.2}$$

Here  $f$  is the scalar flux function (which we shall henceforth refer to as the flux) for the equation (1.1) and the function  $a$  denotes the velocity of the linear equation with respect to  $v$  (1.2). The system has a decoupled nonlinear conservation law and a coupled “linear” transport equation with a discontinuous velocity. The above system is complemented by a set of initial data,  $x \in \mathbb{R}$ ,

$$u(x, 0) = u_0(x), \tag{1.3}$$

$$v(x, 0) = v_0(x). \tag{1.4}$$

The Pressure Swing Adsorption process (PSA) in chemistry [9] has such a triangular structure after a change of variables from Euler to Lagrange [12, 42]. Such systems are already of mathematical interest due to the coupling of the theory of scalar conservation laws with the theory of transport equations. This system was studied in [34] in a non hyperbolic setting,  $f' = a$ , with measure solutions for  $v$ . That even a strictly hyperbolic setting is not enough to avoid measure solutions was shown in [24]. Here, strengthening the hyperbolicity of the system by a stronger transversality condition, global weak bounded entropy solutions are provided with an optimal fractional  $BV$  regularity for the initial data  $u_0$ .

At the first sight, such a system seems easy to solve in the “triangular” manner, that is solving the first equation (1.1) to get  $u$  and then solving the linear equation (1.2) keeping  $u$  fixed. This method works well for smooth solutions [22], but, when a shock wave appears in  $u$  the velocity  $a(u)$  becomes discontinuous. The theory of linear transport equations with discontinuous velocity is a delicate topic yielding measure solutions and a loss of uniqueness [5, 6, 43].

In this paper, we propose a different approach to obtain global weak solutions in  $L^\infty$ . A main idea is to consider the system (1.1), (1.2) not as a triangular system, but, as a  $2 \times 2$  hyperbolic system as in [3]. If  $f$  is nonlinear, one field is nonlinear and the other one is linearly degenerate. When  $f$  is uniformly convex it is a particular case of  $2 \times 2$  systems with one genuinely nonlinear field and a linearly degenerate one [25]. An example of such a system when the flux  $f$  is piecewise convex or concave arises from gas-liquid chromatography [12], The presence of linearly degenerate fields can simplify the study of solutions [41, 42]. but can also produce blow-up behaviours [12, 39].

We show that the behaviour of the proposed entropy solutions is linked to the fractional  $BV$  regularity of  $u$ , indeed,  $u_0 \in BV^s$ ,  $0 < s \leq 1$ . The  $BV^s$  framework is optimal to study the regularity of entropy solutions of scalar conservation laws [16, 36]. This framework is recalled in Section 1.3. For the scalar case, the theory works well for all  $s > 0$ . For the triangular system (1.1), (1.2), we prove that the regularity  $s = 1/3$  of  $u_0$  is critical for the existence of  $L^\infty$  entropy solutions. The exponent  $s = 1/3$  is directly linked to the cubic estimate on the Lax curves [32]. For nonconvex fields, it is known that the Lax curves are less regular [4, 2, 35]. However, for the triangular system (1.1), (1.2) with the flux  $f \in C^4$ , the velocity  $a \in C^3$ , and satisfying a uniformly strict hyperbolicity assumption, we prove that the exponent  $s = 1/3$  is optimal for the existence theory of bounded entropy solutions. Our proof is based on a new cubic estimate that holds even when the flux  $f$  is non-convex. Our estimate generalizes the well known cubic estimates for genuinely nonlinear (convex) systems by Lax [32].

The paper is organized as follows. The hyperbolicity of the triangular system and the key assumptions are given in Section 1.1. The definitions of weak and entropy solutions are stated in Section 1.2. The  $BV^s$  framework is recalled in Section 1.3. In Section 2, the two main results are stated: existence for  $s > 1/3$  and blow-up for  $s < 1/3$ . In Section 3, the Riemann invariants and the Riemann problem are studied. The Lax curves and the key cubic estimates are studied in Section 4. The proof of existence is discussed in Section 5 and the blow-up is studied in Section 6. In the appendices A and B, we discuss on the uniqueness of the Riemann problem in the class of bounded weak entropy solutions.

## 1.1 The hyperbolic triangular system

The system (1.1), (1.2) of conservation laws is hyperbolic when  $f' \neq a$  and can be rewritten, using the vectorial flux  $\mathbf{F}$ , as  $\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = 0$ ,  $\mathbf{U} = (u, v)^\top$ . The matrix of the linearized system has a triangular structure,

$$D\mathbf{F}(\mathbf{U}) = \begin{pmatrix} f'(u) & 0 \\ a'(u)v & a(u) \end{pmatrix}.$$

This system has the unbounded invariant region  $[-M, M]_u \times \mathbb{R}_v$ , where  $M = \|u_0\|_\infty$ . In this paper, the system is assumed to be strictly hyperbolic through the condition

$$\forall u \in [-M, M], \quad f'(u) > a(u). \quad (1.5)$$

Of course, the symmetric assumption:  $\forall u \in [-M, M], f'(u) < a(u)$ , yields a similar study. For large data, the strict hyperbolicity condition (1.5) has to be strengthened on the set

$[-M, M]$ , by the following uniformly strict hyperbolicity (USH) condition:

$$\inf_{|u| \leq M} f'(u) > \sup_{|u| \leq M} a(u). \quad (1.6)$$

An interesting case is already when  $f$  is convex,  $f'' > 0$  everywhere, or concave  $f'' < 0$ . This case occurs for a chromatography system with a convex isotherm written in appropriate Lagrangian coordinates [12].

In this paper, the flux  $f(\cdot)$  belongs to  $C^4$  and the velocity  $a(\cdot)$  belongs to  $C^3$ . Moreover, the flux is locally piecewise convex or concave as a consequence of the following assumption,

$$\mathcal{Z} = \{u, f''(u) = 0, |u| \leq M\} \text{ is finite.} \quad (1.7)$$

## 1.2 Weak and entropy solutions

A weak solution of the system (1.1)-(1.2) satisfying the initial conditions (1.3)-(1.4) is defined as follows.

**Definition 1.1** (weak solution). *The pair  $(u, v)$  is a weak solution of the system (1.1)-(1.2) with initial data (1.3)-(1.4) if for all compactly supported test functions  $\varphi, \psi \in C_c^1(\mathbb{R} \times [0, +\infty[, \mathbb{R})$ , the following integral identities hold:*

$$\int_0^{+\infty} \int_{\mathbb{R}} (u \partial_t \varphi + f(u) \partial_x \varphi) dx dt + \int_{\mathbb{R}} u_0(x) \varphi(x, 0) dx = 0, \quad (1.8)$$

$$\int_0^{+\infty} \int_{\mathbb{R}} (v \partial_t \psi + a(u) v \partial_x \psi) dx dt + \int_{\mathbb{R}} v_0(x) \psi(x, 0) dx = 0. \quad (1.9)$$

The following regularity is usually required for  $(u, v)$ :  $u \in L_{loc}^\infty$ ,  $v \in L_{loc}^1$  or  $v$  is a measure. In the case when  $v$  is a measure, there are some issues in defining the product  $a(u) \times v$  [5]. But our main focus in this paper is on bounded weak solutions  $u, v \in L^\infty$ .

We propose below a notion of entropy solutions for the system (1.1)-(1.2). As in [8, 29] the entropy condition is only tested on the nonlinear component  $u$ . Contact discontinuity waves linked to a linearly degenerate field are well known not to affect the entropy inequality which, therefore, remains an equality [20].

**Definition 1.2** (entropy solution). *The pair  $(u, v)$  is an entropy solution of the system (1.1)-(1.2) with initial data (1.3)-(1.4) if it is a weak solution and for all convex function  $\eta$  and all non-negative test functions  $\varphi \in C_c^1(\mathbb{R} \times ]0, +\infty[, \mathbb{R})$ , with  $q' = \eta' f'$ ,  $u$  satisfies the following inequality:*

$$\int_0^{+\infty} \int_{\mathbb{R}} (\eta(u) \partial_t \varphi + q(u) \partial_x \varphi) dx dt + \int_{\mathbb{R}} \eta(u_0(x)) \varphi(x, 0) dx \geq 0. \quad (1.10)$$

Thus, it suffices only to have a weak solution of the system and the entropy solution of (1.1).

Uniqueness of entropy solutions with a fixed initial data  $(u_0, v_0)$  is a delicate matter [14]. It is due to the lack of uniqueness of weak solutions for the linear transport equation

with a discontinuous velocity ([5, 6]).

For triangular systems as discussed in this paper, one can have measure solutions. A prototypical example of such triangular systems [24] is given by

$$\begin{aligned} \partial_t u + \partial_x \left[ \frac{u^2}{2} \right] &= 0, \\ \partial_t v + \partial_x [(u-1)v] &= 0, \end{aligned} \tag{1.11}$$

with  $f(u) = \frac{u^2}{2}$  and  $a(u) = u - 1$ . Clearly this satisfies the strict hyperbolicity condition (1.5) but not the more restrictive uniform strict hyperbolicity condition (1.6) for too large data. Similar phenomenon is well known for transport equations [5] and is also widely observed in the case of non-strictly hyperbolic systems [6].

The main approach in solving the system (1.11) in [24] was to observe that the first equation in  $u$  can be solved independently. The second equation can then be considered as a transport equation in  $v$  with a discontinuous coefficient  $u$  (see also [5]), but this idea of looking at the two equations separately might not be a good one. Our aim in this article is to prove the existence of solutions to systems of type (1.11) in the class of fractional BV functions and hence one need not appeal to  $\delta$ -shock wave type solutions. Moreover, to obtain  $L^\infty$  solution with a wave front algorithm, we suppose that the system (1.1), (1.2) is uniformly strictly hyperbolic (1.6). Note that we already need the uniform strict hyperbolicity condition (1.6) to solve the Riemann problem appropriately.

### 1.3 $BV^s$ functions

For one dimensional scalar conservation laws, the spaces  $BV^s$  are known to give optimal results for weak entropy solutions [11, 16, 36, 37], first on the fractional Sobolev regularity, and second on the structure of such solutions. Such results on the maximal regularity have been extended for some systems and the multidimensional case [17, 22].

In this section, basic facts on  $BV^s$  functions are recalled.

**Definition 1.3.** [38] *A function  $u$  is said to be in  $BV^s(\mathbb{R})$  with  $0 < s \leq 1$  if  $TV^s u < +\infty$ , where*

$$TV^s u := \sup_{n \in \mathbb{N}, x_1 < \dots < x_n} \sum_{i=1}^n |u(x_{i+1}) - u(x_i)|^{1/s}.$$

*The  $BV^s$  semi-norm is defined by*

$$|u|_{BV^s} := (TV^s u)^s,$$

*and a norm on this space is defined by*

$$\|u\|_{BV^s} := \|u\|_{L^\infty} + |u|_{BV^s}.$$

Fractional  $BV$  functions have traces like  $BV$  functions. This is a fundamental property to define the Rankine-Hugoniot condition for shock waves. Moreover, this property is not true for the Sobolev functions in  $W^{s,1/s}$ , the Sobolev space nearest to  $BV^s$  [11].

**Theorem 1.1.** [38] For all  $s \in ]0, 1[$ ,  $BV^s$  functions are regulated functions.

The fractional total variation only depends on the local extrema of the function and the order of this extrema.

**Lemma 1.1.** [11, 25] If  $u$  is a piecewise monotonous function and if its local extrema are located in the increasing sequence  $(x_i)_{i \in I}$ , then  $TV^s u$  only depends on the sequence  $(u(x_i))_{i \in I}$ . Moreover, there exists an ordered subset  $J$  of  $I$  such that

$$TV^s u = \sum_{j \in J, j \neq \max J} |u(x_{\text{suc}(j)}) - u(x_j)|^{1/s},$$

where  $\text{suc}(j)$  denotes the successor of  $j$  in  $J$ ,  $\text{suc}(j) = \min\{k \in J, k > j\}$ .

Moreover, it can be dangerous to refine the mesh to compute the fractional total variation [11, example 2.1], [15]. Consider  $u(x) \equiv x$  on  $[0, R]$ ,  $p = 1/s > 1$ . Then  $TV^s u[0, R] = R^p$  but, when  $n \rightarrow +\infty$ ,

$$\sum_{i=1}^n |u(iR/n) - u((i-1)R/n)|^p = n(R/n)^p = R^p/n^{p-1} \rightarrow 0.$$

This property, which is not true in  $BV$ , is used later to prove the existence of weak solutions in  $BV^{1/3}$  for the triangular system (1.1), (1.2) when the flux  $f$  is convex.

## 2 Main results

The main results are stated in the  $BV^s$  framework. The basic facts on this setting were recalled in Section 1.3. The critical space for the existence theory is  $BV^{1/3}$  when the flux  $f$  is convex. For the nonconvex case, the existence is proved in the slightly smaller space

$$BV^{1/3+0} := \bigcup_{s > 1/3} BV^s \subsetneq BV^{1/3}.$$

**Theorem 2.1** (Existence in  $L^\infty$  with  $(u_0, v_0) \in BV^{1/3+0} \times L^\infty$ ). Suppose that the flux  $f(\cdot)$  and the transport velocity  $a(\cdot)$  satisfy the following assumptions on  $[-M, M]$  where  $M = \|u_0\|_\infty$ :

1.  $f'(\cdot)$  and  $a(\cdot)$  belong to  $C^3([-M, M], \mathbb{R})$ ,
2.  $f'(u) > a(u)$  and satisfy the uniform strict hyperbolicity assumption (USH) (1.6),
3.  $f(\cdot)$  has at most a finite number of inflection points (1.7).

Then, if  $(u_0, v_0)$  belongs to  $BV^s \times L^\infty$  and  $s > 1/3$ , there exists an entropy solution  $(u, v)$  of the system (1.1), (1.2),  $u \in L^\infty([0, +\infty)_t, BV^s(\mathbb{R}_x, \mathbb{R}))$ ,  $v \in L^\infty((0, +\infty)_t \times \mathbb{R}_x, \mathbb{R})$ . Moreover, if the flux  $f$  is convex then the existence result remains true for  $s = 1/3$ . In addition, the positivity of the initial data is preserved, that is, if  $\inf v_0 > 0$  then  $\inf v > 0$ .

Two mandatory conditions are required to avoid singular phenomena at  $t = 0+$ . The strong transversality condition (USH) (1.6) is the key assumption to avoid instantaneous  $\delta$ -shock. The  $BV^s$  regularity of  $u_0$  is the other key condition to avoid instantaneous blow-up in  $L^\infty$ .

The positivity of  $v$  is the key tool to get the uniform strict hyperbolicity of the (PSA) system [8]. The complete proof of the positivity of  $v$  for the (PSA) system is obtained in [13] without using the triangular structure of the (PSA) system [12].

Notice that the sign of  $v$  is preserved a.e. by Theorem 2.1. Of course, by linearity of the equation (1.2) with respect to  $v$ , the negativity of the initial data is also preserved, that is, if  $\sup v_0 < 0$  then  $\sup v < 0$ .

The triangular approach, that means solving first (1.1) with the unique Krushkov entropy solutions [31] and then (1.2), usually yields non-unique measure solutions  $v$  [5, 43]. It is the reason why the uniqueness of entropy solutions for the whole triangular system is an open problem. Of course, we have uniqueness for  $u$  but the problem of uniqueness remains for  $v$ . For a weakly coupled system with some linearly degenerate fields, the entropy condition only on the nonlinear fields is enough to ensure uniqueness in [29]. For the triangular system, the coupling by the transport velocity is too nonlinear to achieve uniqueness in the same way.

For the existence result, a Wave Front Tracking algorithm (WFT) is proposed as in [3] for the whole system. As a matter of fact the approximate solution  $u$  of the scalar equation satisfies the  $BV^s$  uniform bounds as in [11, 28]. The difficult point is to bound  $v$  in  $L^\infty$ . Since the system is linear with respect to  $v$ , an  $L^\infty$  bound for  $v$  is enough to get an existence theorem as in [12].

Theorem (2.1) is optimal and, in general, we cannot reduce the  $BV^{1/3}$  regularity of  $u_0$ , else a blow-up can occur. For this purpose, we build a sharp example in a space very near to  $BV^{1/3}$ , namely,  $BV^{1/3-0}$

$$BV^{1/3} \subsetneq BV^{1/3-0} := \bigcap_{s < 1/3} BV^s.$$

The simple example (1.11) from [24] with a Burgers' flux  $f$  and a linear velocity  $a$  is enough to provide a blow-up.

**Theorem 2.2** (Blow-up in  $L^\infty$  at  $t = 0+$  for  $u_0 \in BV^{1/3-0}$ ). *For the system (1.11) with the Burgers' flux  $f(u) = u^2/2$  and the velocity  $a(u) = u - 1$ , there exists  $u_0 \in BV^{1/3-0}$ ,  $v_0 \in L^\infty$  such that*

- *the system (1.1), (1.2) is uniformly strictly hyperbolic (USH) (1.6), and*
- *there doesn't exist any bounded entropy solution of the system (1.1), (1.2) with the initial data  $u_0, v_0$ .*

Of course  $u_0 \notin BV^{1/3}$ , otherwise the existence theorem 2.1 gives a bounded entropy solution. There is no blow-up for  $u$  since the entropy solution  $u$  of the scalar conservation



law satisfies the maximum principle. Indeed, only the function  $v$  has a blow-up at time  $t = 0^+$ . Notice also that the blow-up depends only on the regularity of  $u_0$  and not of  $v_0$ .

The example presented below follows the construction of particular explicit solutions [1] which provide optimal example in  $BV^s$  for convex scalar conservation laws [22]. On any compact set avoiding the blow-up point, the entropy solution  $u$  is locally Lipschitz except on a finite number of lines. For initial data  $u_0$  leading to such an entropy solution  $u$ , we show that a uniqueness result for the entropy solution  $(u, v)$  holds at least upto the time of first interaction of waves in  $u$ . Thus there is no way to avoid the blow-up. In general, a possible dense jump set in plane  $(t, x)$  is possible for the entropy solution  $u$  of a scalar conservation law [23].

Another consequence of this blow-up example is that for general nonlinear  $2 \times 2$  strictly hyperbolic systems with a genuinely nonlinear eigenvalue and a linearly degenerate one, the existence result proven in [25] is optimal.

### 3 The uniformly strictly hyperbolic system

In this section, we study the Riemann invariants and the solution of the Riemann problem for the triangular system (1.1)-(1.2).

#### 3.1 Riemann invariants

A  $2 \times 2$  strictly hyperbolic system admits, at least locally, a set of coordinates which diagonalizes the hyperbolic system for smooth solutions [45]. The knowledge of this coordinate system, given by the Riemann invariants, is often useful in understanding the structure of the system. Next we study the Riemann invariants for the system (1.1)-(1.2).

The eigenvalues of the system (1.1)-(1.2) are

$$\lambda_1 = f'(u) > a(u) = \lambda_2. \quad (3.1)$$

Notice that the eigenvalues are functions of  $u$  only. Let  $r_1$  and  $r_2$  denote the corresponding right eigenvectors.

Clearly,  $u$  is a 2-Riemann invariant associated to the right eigenvector  $r_2 = (0, 1)^\top$  and satisfies

$$\partial_t u + f'(u) \partial_x u = 0.$$

A 1-Riemann invariant, which we denote as  $z(u, v)$ , corresponding to the right eigenvector  $r_1$ , can be computed in the following manner. We note that a right eigenvector of the matrix  $D\mathbf{F}(u)$  corresponding to the eigenvalue  $f'(u)$  is given by

$$r_1 = \begin{pmatrix} 1 \\ \frac{a'(u)v}{f'(u) - a(u)} \end{pmatrix}.$$

Then  $z$  satisfies

$$\frac{1}{v} \partial_u z = \frac{a'(u)}{a(u) - f'(u)} \partial_v z.$$

This can be solved using a separation of variables.

$$\partial_u z = \frac{a'(u)}{a(u) - f'(u)}, \quad \partial_v z = \frac{1}{v}.$$

For instance, a Riemann invariant is given by

$$z(u, v) = A(u) + \ln(|v|), \quad A'(u) = \frac{a'(u)}{a(u) - f'(u)}.$$

To avoid the singularity at  $v = 0$ , it suffices to take the exponential

$$Z = \exp(z) = v \exp(A(u)), \quad (3.2)$$

which satisfies the equation

$$\partial_t Z + a(u) \partial_x Z = 0.$$

### 3.2 The Riemann problem

We study the Riemann problem for the system (1.1)-(1.2) with initial data:

$$u_0(x) = u_{\pm}, \pm x > 0, \quad (3.3)$$

$$v_0(x) = v_{\pm}, \pm x > 0. \quad (3.4)$$

A direct and somewhat naive approach is to solve the conservation law (1.1) first and then the second equation (1.2) using the solution  $u$ . In such an approach, one faces the difficulty of solving the linear transport equation with a discontinuous coefficient.

Instead, we consider the two equations together as a system. This is a key point as in [3]. The solution of the Riemann problem consists of two waves separated by an intermediary state  $(u_m, v_m)$  where  $u_m = u_-$  and  $v_m$  is unknown. In accordance with the labeling of the eigenvalues (3.1), a wave associated to  $u$  is called a 1-wave and a wave associated to  $Z$  is called a 2-wave.

1. A wave associated to the eigenvalue  $\lambda_1 = f'$  is a shock wave or a rarefaction wave if  $f$  is convex or concave. For a non-convex flux  $f$ , this wave is a composite wave.
2. A linearly degenerate wave associated to the eigenvalue  $\lambda_2 = a$  is called a contact discontinuity. The speed of this 2-wave is  $a(u_-)$ .

The intermediate value  $v_m$  has to be computed through the 1-wave. Now, the various 1-waves that can occur are detailed. For this purpose, we consider Riemann problems yielding only a 1-wave.

#### Shock waves

Let us denote  $U = (u, v)$  and  $\tilde{U} = (u, Z)$ .

The Rankine-Hugoniot condition gives

$$s[u] = [f(u)], \quad s[v] = [a(u)v], \quad (3.5)$$

where  $s$  denotes the speed of the discontinuity (or the slope of the jump).

Thus, the slope of the jump is determined by  $u_{\pm}$  and the flux  $f$ ,

$$s = \frac{[f(u)]}{[u]}. \quad (3.6)$$

Since entropy solutions are considered, the Oleinik-entropy condition [14] for (1.1) enforces the Lax-entropy conditions,

$$f'(u_-) \geq s \geq f'(u_+), \quad (3.7)$$

whence

$$s \geq f'(u_+) > a(u_+). \quad (3.8)$$

Using the second equation of (3.5) yields

$$(s - a(u_+))v_+ = (s - a(u_-))v_-, \quad (3.9)$$

$$v_+ = v_- \frac{s - a(u_-)}{s - a(u_+)} := \mathcal{S}_-(u_+; U_-). \quad (3.10)$$

The equation (3.10) can be interpreted in terms of the Lax shock curve. For a fixed  $U_- = (u_-, v_-)$ , the right hand side of (3.10) is only a function of  $u_+$  as  $s$  is given by (3.6) as a function of  $u_+$ . In the plane  $U = (u, v)$ , (3.10) describes the set of  $U_+$  such that the Riemann problem with initial data  $U_{\pm}$  is solved by a shock. On this curve, only  $U_+$  satisfying the Oleinik condition (3.7) are considered to allow an entropic shock.

Conversely, if  $U_+$  is fixed, the Lax shock curve is parametrized by  $u_-$  and reads

$$v_- = v_+ \frac{s - a(u_+)}{s - a(u_-)} := \mathcal{S}_+(u_-; U_+). \quad (3.11)$$

Proceeding similarly with  $Z = v \exp(A(u))$ ,  $\tilde{U}_- = (u_-, Z_-)$  and keeping the notation  $\mathcal{S}$  in coordinates  $(u, Z)$  yields,

$$Z_+ = Z_- \frac{s - a(u_-)}{s - a(u_+)} \exp[A(u_+) - A(u_-)] := \mathcal{S}_-(u_+; \tilde{U}_-),$$

and

$$Z_- = Z_+ \frac{s - a(u_+)}{s - a(u_-)} \exp[A(u_-) - A(u_+)] := \mathcal{S}_+(u_-; \tilde{U}_+).$$

### 1-Contact discontinuity

This is a limiting case of the preceding one, when  $f'$  is constant on the interval  $[u_-, u_+]$ . The same formula for the shock wave follows.

$$s = f'(u_{\pm}) = f'(u) > a(u_{\pm}), \quad u \in [u_-, u_+], \quad (3.12)$$

$$v_+ = v_- \frac{f'(u_-) - a(u_-)}{f'(u_+) - a(u_+)} = \mathcal{S}_-(u_+; U_-), \quad (3.13)$$

$$Z_+ = Z_- \frac{f'(u_-) - a(u_-)}{f'(u_+) - a(u_+)} \exp[A(u_+) - A(u_-)] := \mathcal{S}_-(u_+; \tilde{U}_-). \quad (3.14)$$

Such waves arise in the wave-front tracking (WFT) method when the flux is approximated by piecewise linear functions.

### Rarefaction waves

Since  $Z$  is a 1-Riemann invariant, the Lax rarefaction curve  $\mathcal{R}_-(\tilde{U}_-)$  is simply,

$$Z_+ = Z_-. \quad (3.15)$$

Using the definition of  $Z$ , this implies  $v_+ \exp(A(u_+)) = v_- \exp(A(u_-))$  and thus the Lax rarefaction curve can be written explicitly,

$$v_+ = \mathcal{R}_-(u_+; U_-) = v_- \exp(A(u_-) - A(u_+)), \quad (3.16)$$

$$v_- = \mathcal{R}_+(u_-; U_+) = v_+ \exp(A(u_+) - A(u_-)). \quad (3.17)$$

Notice that  $v_+$  and  $v_-$  have the same signs. In particular, if  $v_- = 0$ ,  $v = 0$  is constant through the rarefaction wave.

### 1-Composite waves

The composite waves only occur for the waves associated to the first eigenvalue  $f'(u)$ . In general, the entropy solution  $u$  is a composite wave [26]. If  $f$  has a finite number  $N$  of inflection points, then there are at most  $N$  contact-shock waves [20, 35]. The Lax curves associated to such waves are studied below in Section 4.

### 2-Contact discontinuity

$u$  is a 2-Riemann invariant and hence is constant along the 2-contact discontinuity. Thus the Lax curve is simply a vertical line in the plane  $U = (u, v)$  or the plane  $\tilde{U} = (u, Z)$ ,

$$u_- = u_+. \quad (3.18)$$

## 4 The Lax curves

A fundamental theorem due to Lax [32] states that the shock curve and the rarefaction curve emanating from a constant state  $U_-$  in the plane  $(u, Z)$  or  $(u, v)$  have a contact of the second order for a genuinely nonlinear eigenvalue. This means that the shock curve can be replaced by the rarefaction curve up to an error of order  $[u]^3$  where  $[u] = u_+ - u_-$  [14, 20, 44]. For the triangular system, a genuinely nonlinear eigenvalue means  $f'' > 0$  (or  $f'' < 0$ ) everywhere. For nonconvex cases, typically  $f''$  locally has a finite number of roots where  $f''$  changes its sign and the Lax curves are less regular due to the occurrence of contact-shocks. Under a concave-convex assumption, which means here that  $f'''$  does not vanish, the regularity of the Lax curves is only piecewise  $C^2$  [35], see also [2]. As a consequence, the error becomes of order  $[u]^2$  for the variation of  $Z$  through a contact-shock

[35]. The situation is worse in general, the Lax curves are only Lipschitz [4]. However, we prove that cubic estimates are still valid for the triangular system (1.1), (1.2). It is mainly due to the existence of Riemann invariant coordinates.

In this section, cubic estimates on the Lax curves for the triangular system in the plane  $(u, Z)$  are proved.

Let  $(u_-, v_-)$ ,  $(u_+, v_+)$  be two constant states connected by a rarefaction or a shock wave. It is more convenient to use the Riemann invariant coordinates  $(u, Z)$ .

The rarefaction curves in the plane  $(u, Z)$  are simply,

$$Z_- = Z_+, \quad (4.1)$$

which means that in the  $(u, v)$  plane

$$v_- \exp(A(u_-)) = v_+ \exp(A(u_+)). \quad (4.2)$$

For the shock curve, the Rankine-Hugoniot condition is written in the conservative variables  $(u, v)$ ,

$$s = \frac{[f(u)]}{[u]} = \frac{f(u_+) - f(u_-)}{u_+ - u_-}, \quad (4.3)$$

$$v_-(s - a(u_-)) = v_+(s - a(u_+)). \quad (4.4)$$

Since  $f \in C^4$ ,  $s$  is a  $C^3$  function of its arguments. Moreover, fixing  $(u_+, v_+)$  and considering  $u_-$  as a variable, the Lax shock curve is  $C^3$  with respect to  $u_-$ ,

$$v_- = v_+ \frac{s - a(u_+)}{s - a(u_-)}. \quad (4.5)$$

Indeed, the denominator never vanishes due to the uniformly strict hyperbolicity assumption (USH) (1.6). The same regularity of the shock curve holds in the variables  $(u, Z)$

$$Z_- = Z_+ \frac{s - a(u_+)}{s - a(u_-)} \exp(A(u_-) - A(u_+)). \quad (4.6)$$

Of course, the Lax rarefaction curve and the Lax shock curve has to be restricted on the subset satisfying entropy conditions. Nevertheless, we use these curves for all range of  $u_-$  in  $\mathbb{R}$  (at least for  $-M \leq u_- \leq M$ ) to obtain a generalized Lax cubic estimate for the nonconvex case.

## 4.1 The Lax cubic estimate on the Rankine-Hugoniot curve

The Lax cubic estimate [32] can be written as follows for a shock wave connecting  $(u_-, Z_-)$  to  $(u_+, Z_+)$  for the triangular system (1.1),(1.2), as soon as  $Z$  is bounded,

$$[Z] = \mathcal{O}([u])^3, \quad [Z] = Z_+ - Z_-, \quad [u] = u_+ - u_-. \quad (4.7)$$

The Riemann invariant  $Z$  is constant along the rarefaction curves. The Lax cubic estimate means that the shock curve and the rarefaction curve have a contact of order 2. The Lax

cubic estimate was written in a genuinely nonlinear framework [32]. This means that  $f''$  does not vanish. Indeed, the Lax computations are still valid without this convex assumption and without only considering the entropic part of the Rankine-Hugoniot curve. Here, we also use the non-entropic part of the Rankine-Hugoniot curve. It is a useful tool used many times in this paper, first in the next section 4.2 to get a cubic estimate for the entropy solution of the Riemann problem for the triangular system with a non-convex flux  $f$ .

For the triangular system, the Rankine-Hugoniot curve is global, explicit and well defined thanks to the uniformly strict hyperbolicity assumption (1.6). For  $2 \times 2$  systems, such a global curve **does** not always exist [30].

Now, to prove cubic estimates, we have to write the Rankine-Hugoniot curve. Here, we choose to write  $Z_-$  as a function of  $u_-$  when  $(u_+, Z_+)$  are fixed for the following reasons.

1. To solve the Riemann problem and compute the intermediary state  $v_m$  which corresponds to  $Z_m$  and here  $Z_-$ .
2. To obtain the cubic estimate on the global Rankine-Hugoniot curve below.
3. To obtain the existence result, bounding  $Z$  along the 2-characteristics.
4. To build a blow-up, again computing  $Z$  from the right to the left on the 2-characteristics.

The Rankine-Hugoniot curve  $\mathcal{RH}_+$  when  $\tilde{U}_+ = (u_+, Z_+)$  is fixed and  $Z_-$  is a function of  $u_-$ , is given by

$$Z_- = Z_+ \frac{s - a(u_+)}{s - a(u_-)} \exp [A(u_-) - A(u_+)] = Z_+ r(u_-, u_+) := \mathcal{RH}_+(u_-; u_+, Z_+). \quad (4.8)$$

The classic Lax cubic estimate on the shock curve is generalized on the global Rankine-Hugoniot curve.

**Lemma 4.1** (Cubic flatness of the global Rankine-Hugoniot curve). *If  $f'$  and  $a$  belong to  $C^3(\mathbb{R}, \mathbb{R})$  and satisfy the uniform strict hyperbolicity condition (1.6), then*

$$\begin{aligned} s &= s(u_-, u_+) = \frac{[f]}{[u]} = \frac{f(u_+) - f(u_-)}{u_+ - u_-} \in C^3([-M, M]^2, \mathbb{R}), \\ r &= r(u_-, u_+) = \frac{s - a(u_+)}{s - a(u_-)} \exp [A(u_-) - A(u_+)] \in C^3([-M, M]^2, \mathbb{R}), \\ r &= 1 + \mathcal{O}(1) [u]^3 > 0, \quad \forall (u_-, u_+) \in [-M, M]^2, \\ Z_- &= Z_+ (1 + \mathcal{O}(1) [u]^3), \quad \forall (u_-, u_+, Z_+) \in [-M, M]^2 \times \mathbb{R}, \end{aligned} \quad (4.9)$$

$$(4.10)$$

where the constant  $\mathcal{O}(1)$  depends only on the derivatives of  $f'$  and  $a$  on  $[-M, M]$ . Moreover,  $Z_-$  has the same sign as  $Z_+$ , more precisely,

$$\begin{aligned} Z_+ = 0 &\Rightarrow Z_- = 0, \\ Z_+ \neq 0 &\Rightarrow Z_- Z_+ > 0. \end{aligned}$$

The cubic flatness of the Rankine-Hugoniot curve  $\mathcal{RH}_+$  is expressed in (4.10). It does not depend on the convexity of the fields, although in the classical textbooks [14, 20, 26, 35, 44, 45] some nonlinearity assumptions on the fields are given. The reason of these nonlinear assumptions is usually to introduce the rarefaction curve and the shock curve. Here, the global Rankine-Hugoniot curve defined for all  $u_- \in [-M, M]$  is the main subject without looking at the entropic parts of this curve. A careful reading of the classical proof of the cubic estimate in textbooks shows that it is a geometric property of the Rankine-Hugoniot curve itself, without referring at the entropic or nonlinearity assumptions. This geometric property is a consequence of the symmetry of the Rankine-Hugoniot condition with respect to  $U_-$  and  $U_+$  as explained in [44]. It is very important in this paper to prove the cubic estimate for the non-entropic part of the Rankine-Hugoniot curve for two reasons.

1. The cubic estimates on the Lax curve (which is not piecewise  $C^3$  for nonconvex  $f$  [35]) uses the global Rankine-Hugoniot curve, as discussed in the next section 4.2.
2. Rarefaction wave fans are replaced by weak non-entropic jumps in the wave front tracking algorithm [14]. The error in the weak formulation is controlled by the cubic estimate to pass to the limit and get a weak solution of (1.1), (1.2).

The proof appears as a direct consequence of (4.9). An elementary and self-contained proof using only Taylor's expansions is proposed. A more tedious computation can give the more precise result

$$r(u_-, u_+) = 1 + E[u]^3 + \mathcal{O}([u]^3), \quad (4.11)$$

where  $E$  depends in a quite complicated way on the derivatives of  $f'$  and  $a$  at  $u = u_+$ . The computation of  $E$  is quite intricate and not useful here, except in the last section 6 on the blow-up where a direct computation of  $E$  at  $u_+ = 0$  is given when  $f$  is quadratic and  $a$  is linear.

Now, Lemma 4.1 is proven.

*Proof.* The positivity on  $r$  is a consequence of the assumption (1.6). This positivity implies that  $Z_-$  has the same sign as  $Z_+$ .

To obtain (4.9), we use Taylor expansion. Here  $u_+$  is fixed and  $u_-$  is the variable near  $u_+$ . The notations  $a_- = a(u_-)$ ,  $a_+ = a(u_+)$  and so on are used to shorten the expressions.

$$\begin{aligned} a_- &= a_+ - a'_+[u] + (a''_+/2)[u]^2 + \mathcal{O}([u]^3), \\ s &= \frac{[f]}{[u]} = \frac{f_- - f_+}{-[u]} = f'_+ - (f''_+/2)[u] + (f'''_+/6)[u]^2 + \mathcal{O}([u]^3). \end{aligned}$$

The hyperbolic quantity  $h$  is used,

$$h := f' - a > 0. \quad (4.12)$$

The sign of  $h$  is the consequence of the strict hyperbolicity assumption (1.6). We start with the fraction part of  $r$  and  $h_+ = f'_+ - a_+ \neq 0$ .

$$\frac{s - a_+}{s - a_-} = \frac{(f'_+ - a_+) - (f''_+/2)[u] + (f'''_+/6)[u]^2}{(f'_+ - a_+) - (f''_+/2 - a'_+)[u] + (f'''_+/6 - a''_+/2)[u]^2} + \mathcal{O}([u]^3) \quad (4.13)$$

$$= 1 - \frac{a'_+}{h_+}[u] + \frac{h_+ a''_+ - a'_+ f''_+ + 2a_+^2}{2h_+^2}[u]^2 + \mathcal{O}([u]^3). \quad (4.14)$$

For the term  $\exp(-[A])$  in  $r$ , the Taylor expansion of  $A'_-$  is used at the first order.

$$\begin{aligned} -A'_- &= \frac{a'_-}{h_-} = \frac{a'_+ - a''_+[u]}{h_+ - h'_+[u]} + \mathcal{O}([u]^2) \\ &= \frac{a'_+}{h_+} + \frac{a'_+ h'_+ - a''_+ h_+}{h_+^2}[u] + \mathcal{O}([u]^2). \end{aligned}$$

Integrating with respect to  $u_-$  yields,

$$A_- - A_+ = \frac{a'_+}{h_+}[u] + \frac{a'_+ h'_+ - a''_+ h_+}{2h_+^2}[u]^2 + \mathcal{O}([u]^3).$$

Since  $\exp(x) = 1 + x + x^2/2 + \mathcal{O}(x)^3$ , it yields

$$\exp[A_- - A_+] = 1 + \frac{a'_+}{h_+}[u] + \frac{a'_+ h'_+ - a''_+ h_+ + a_+^2}{2h_+^2}[u]^2 + \mathcal{O}([u]^3). \quad (4.15)$$

Now, multiplying (4.14) and (4.15) yields  $r = 1 + \mathcal{O}([u]^3)$ .  $\square$

## 4.2 Cubic estimates for the Riemann problem

Now, the intermediary state  $Z_m$  of a Riemann problem has to be estimated. For this purpose, the variation of  $Z$  along a composite wave is studied. When the flux is convex, Lax proved that the variation of  $Z$  is a cubic order of the variation of  $u$  [32]. For a non-convex flux, it is well known that the Lax curve is less regular, piecewise  $C^2$  [35] or only Lipschitz [4]. However, we prove that for our triangular system we are able to keep a cubic order. This is mainly due to the existence of Riemann coordinates for  $2 \times 2$  systems and the cubic estimate for the global Rankine-Hugoniot locus, Lemma 4.1. As a consequence, we can prove a similar estimate for the variation of  $Z$  over a composite 1-wave. This improves the well known square root estimate for concave-convex eigenvalues [35], which correspond to cubic degeneracies for  $f$ . That means that for the triangular system (1.1), (1.2), the estimate is as precise as for the convex case [32].

**Theorem 4.1** (Variation of  $Z$  through a composite wave).

Let the states  $\tilde{U}_i = (u_i, Z_i)$ ,  $i = 1, \dots, m$ , where  $u_0 < u_1 < \dots < u_m$  ( $m \leq N_{infl} + 1$ ) comprise a composite 1-wave, and  $Z_- = Z_0$  and  $Z_+ = Z_m$ . Then the total variation of  $Z$  through a 1-wave is,

$$\|Z\|_\infty \leq |Z_+| \exp(\mathcal{O}(|u_+ - u_-|^3)), \quad (4.16)$$

$$TVZ \leq \mathcal{O}(1)|Z_+||u_+ - u_-|^3. \quad (4.17)$$



*Proof.* We note that  $Z$  is a 1-Riemann invariant and therefore it remains constant over a rarefaction wave. Moreover, if the states  $\tilde{U}_i$  and  $\tilde{U}_{i+1}$  are joined by a jump, by Lemma 4.1 and the classical inequality  $1 + x \leq \exp(x)$ , we have the estimate

$$|Z_i| = |Z_{i+1}r(u_i, u_{i+1})| = |Z_{i+1}| (1 + \mathcal{O}(|u_i - u_{i+1}|^3)) \leq |Z_{i+1}| \exp(\mathcal{O}(|u_i - u_{i+1}|^3)).$$

Summing up and noting that  $u_0, u_1, \dots, u_m$  are ordered, the estimate (4.16) follows,

$$\max_i |Z_i| \leq |Z_+| \exp(\mathcal{O}(|u_+ - u_-|^3)).$$

Now, the  $BV$  bound for  $Z$  through the composite wave is computed.

$$\begin{aligned} Z_i - Z_{i+1} &= Z_{i+1}(r(u_i, u_{i+1}) - 1) = Z_{i+1}\mathcal{O}(|u_i - u_{i+1}|^3), \\ TVZ &\leq \max_i |Z_i| \sum_i \mathcal{O}(|u_i - u_{i+1}|^3) \leq |Z_+| \mathcal{O}(|u_+ - u_-|^3) \exp(\mathcal{O}(|u_+ - u_-|^3)). \end{aligned}$$

Since  $u$  satisfies the maximum principle, we further have  $\exp(\mathcal{O}(|u_+ - u_-|^3)) = \mathcal{O}(1)$  which only depends on the  $L^\infty$  bound of the initial data  $u_0$ . Hence the inequality (4.17) follows.  $\square$

## 5 Existence in $BV^s$

In this section, Theorem 2.1 is proved using a simplified Wave Front Tracking (WFT) [14, 26] algorithm for such a triangular system (1.1), (1.2). The  $BV^s$  estimates for  $u$  are a consequence of such estimates for scalar conservation laws [11, 27, 28]. The  $L^\infty$  bound for  $v$  and the proof of existence of a weak solution for the triangular system are based on an approach using the  $BV^{1/3}$  regularity for  $u$ .

### 5.1 The Wave Front Tracking algorithm

The WFT depends on an integer parameter  $\nu > 0$ . The approximate solutions will be denoted by  $u_\nu, v_\nu, Z_\nu$ . We shall mostly use the Riemann invariant coordinates  $(u_\nu, Z_\nu)$  except when passing to the limit in the weak formulation.

This algorithm is explained in many books [14, 20, 26] on hyperbolic systems. Taking advantage of the structure of the triangular system (1.1), (1.2), we will mix the WFT for the scalar case [19] and for systems [14, 26]. The main principle is to work with piecewise constant approximations.

As in the scalar case [14, 26, 27, 28], the values of  $u_\nu$  are taken on a uniform grid parametrized by the integer  $\nu$ , that is,

$$u_\nu \in \nu^{-1}\mathbb{Z}. \tag{5.1}$$

On the other hand,  $v_\nu$ , or equivalently  $Z_\nu$ , is not required to stay on the uniform grid, that is,

$$v_\nu, Z_\nu \in \mathbb{R}, \tag{5.2}$$

as we solve the Riemann problem with the exact flux  $f$  to compute  $v_\nu$ , or equivalently  $Z_\nu$ . In this way, we can use the cubic estimate on the global Rankine-Hugoniot curves of Lemma 4.1. A similar approach for a convex case was used in [25].

The initial data are approximated as follows. Let  $N_0 + 1$  be the number of constant states for the approximate initial data  $u_{0,\nu}, v_{0,\nu}$  and the corresponding  $Z_{0,\nu}$ . Here  $u_{0,\nu}$  takes values on the grid and  $N_0$  is a function of  $\nu$ .

To ensure that  $(u_{0,\nu}, v_{0,\nu})$  converges towards the initial data  $(u_0, v_0)$ , we need the condition that  $N_0$  goes to infinity as  $\nu$  tends to infinity. We shall later see that for suitably chosen  $N_0$  satisfying the above condition, we can prove that the approximate solutions  $(u_\nu, v_\nu)$  converge to a weak solution of the triangular system.

The approximate initial data can be chosen to satisfy the following uniform estimates with respect to  $\nu$  [27, 28]:

$$\|u_{0,\nu}\|_\infty \leq \|u_0\|_\infty, \quad (5.3)$$

$$TV^s u_{0,\nu} \leq TV^s u_0, \quad (5.4)$$

$$\|v_{0,\nu}\|_\infty \leq \|v_0\|_\infty. \quad (5.5)$$

Moreover,  $(u_{0,\nu}, v_{0,\nu})$  converges pointwise almost everywhere to  $(u_0, v_0)$  when  $\nu \rightarrow +\infty$  and therefore, the previous inequalities become equalities at the limit  $\nu \rightarrow +\infty$ .

The flux  $f$  is replaced by a piecewise linear continuous flux  $f_\nu$  [19] that coincides with  $f$  on the uniform grid [14, Ch. 6], [26, p. 70],

$$f_\nu(k/\nu) = f(k/\nu), \quad \forall k \in \mathbb{Z}. \quad (5.6)$$

At  $t = 0+$ ,  $N_0$  Riemann problem are solved. The approximate solution  $u_\nu$  is the weak entropy solution of

$$\partial_t u_\nu + \partial_x f_\nu(u_\nu) = 0, \quad u_\nu(x, 0) = u_{0,\nu}(x). \quad (5.7)$$

With strong compactness on  $u_\nu$ , the Kruzkov entropy solution is recovered [14, 26].

For  $v_\nu$ , the situation is less simple than for  $u_\nu$  as the exact vectorial flux  $F$  is used. As discussed in Section 5.3 below, there is a consistency error in the weak formulation. The Wave Front tracking used here mixes the initial idea of Dafermos [19] using an approximate flux, and the idea of Bressan using the exact flux but with approximated rarefaction waves by many contact discontinuities. Here the exact vectorial flux is  $F(u) = (f(u), a(u))$  and the approximate flux is  $F_\nu(u) = (f_\nu(u), a(u))$ . The approximate flux  $F_\nu$  is used to get  $u_\nu$  and the exact Flux  $F$  is used to compute  $Z_\nu$ . The reason for this choice is twofold. First, to have a simpler wave front tracking for the scalar equation for  $u$  with a non-convex flux. Secondly, to use the new generalized cubic estimates for  $Z_\nu$  which holds for the exact solution  $(u, v)$  of the system with the exact flux  $F$ .

After some time, some of the Riemann problems interact and hence, new Riemann problems have to be solved. The non-linear interactions for the triangular system are described in detail below. The process continues until the second time of interactions and so on.

This process can be continued for all time because there is only a finite number of interactions for all time. For  $u_\nu$  this follows from the existing results for the scalar case. An explicit bound on the number of interactions is given in [26, p. 71-72]. The 2-waves, associated to the linearly degenerate eigenvalue  $a(u_\nu)$ , never interact together since they are contact discontinuities. Due to the transversality assumption (USH), a 2-wave can interact only once with a 1-wave and creates a new 2-wave (and does not modify the 1-wave). Thus, the number of such interactions and of 2-waves is finite. This proves that the WFT is well defined for all time.

Now, the approximate Riemann solver and the nonlinear interactions of the waves are detailed.

### Approximate Riemann solver

In this short section, the approximate Riemann solver is detailed. Let  $(u_-, Z_-)$ ,  $(u_+, Z_+)$  denote the left and right states in the initial data. For  $u_\nu$  the solution is a series of entropic jumps  $u_- = u_0 < u_1 < \dots < u_m = u_+$  for the piecewise linear flux  $f_\nu$ . For  $Z_\nu$ , there are many possibilities. We want to approximate the exact solution of the Riemann problem and keep the cubic estimates (Proposition 4.1) which generalize the Lax cubic estimates for genuinely nonlinear waves. So, we use the exact solution of the Riemann problem and the exact Lax curve to determine  $Z_{m-1}, \dots, Z_1$ . The states  $Z_i$ ,  $i = m, m-1, \dots, 2$  are built as follows:

- If the jump between  $u_{i-1}$  and  $u_i$  corresponds to a shock for  $f_\nu$  (and hence for the exact flux  $f$ ), then  $Z_{i-1}$  is given by the Rankine-Hugoniot curve (4.8),

$$Z_{i-1} = \mathcal{RH}_+(u_{i-1}; u_i, Z_i). \quad (5.8)$$

Thus, the jump between  $(u_{i-1}, Z_{i-1})$  and  $(u_i, Z_i)$  satisfies the Rankine-Hugoniot condition and there is no error in the weak formulation of the exact triangular system (1.1), (1.2).

- If the jump between  $u_{i-1}$  and  $u_i$  corresponds to a contact discontinuity for  $f_\nu$  then, necessarily,  $|u_i - u_{i-1}| = \nu^{-1}$  and we keep  $Z$  constant as for the exact solution,

$$Z_i = Z_{i-1}. \quad (5.9)$$

The chosen approximate solution of the Riemann problem is a jump which does not satisfy the Rankine-Hugoniot condition and leads to a consistency error in the weak formulation of the exact system. This is controlled in a cubic way thanks to Lemma 4.1. This case corresponds to a rarefaction wave for the exact flux  $F$ .

### Nonlinear wave interactions

We briefly describe the different possible nonlinear wave interactions and the details of the interactions are given in the  $(u, Z)$  plane.

In all the cases, we consider three states  $\tilde{U}_-, \tilde{U}_0, \tilde{U}_+$  before the interaction. Here  $\tilde{U}_0$  denotes the intermediary state which disappears after the interaction. The states  $\tilde{U}_-, \tilde{U}_0$

are connected by an elementary wave and similarly the states  $\tilde{U}_0, \tilde{U}_+$  are connected by an elementary wave. An elementary wave is either a 1-wave: shock or a small jump, or a 2-wave: a contact discontinuity. At a time of interaction  $t_{interact}$ , the Riemann problem is solved with a new intermediary state  $\tilde{U}_m$  (see Figure 1).

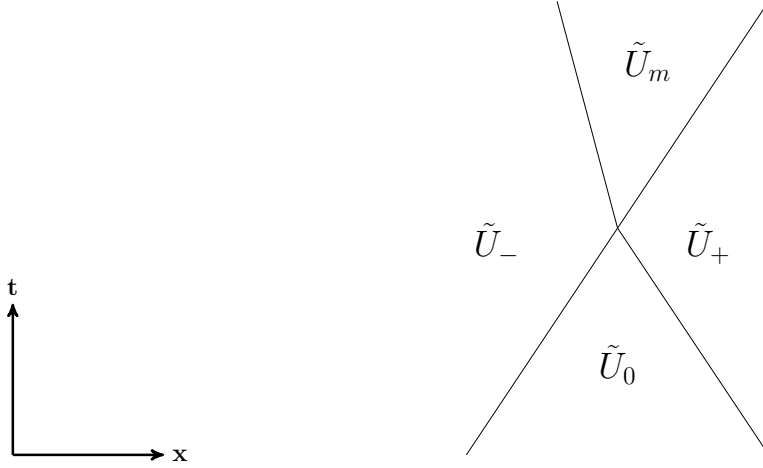


Figure 1: Nonlinear interaction of two waves. The 1-wave on the left crosses the interaction point with the same speed and the same value  $u_-$  on the left of the 1-wave ( $u_m = u_-$ ) and  $u_+$  on the right. On the other hand, the second wave is affected by the interaction: the speed of the 2-wave and a new value  $Z_m$  appears.

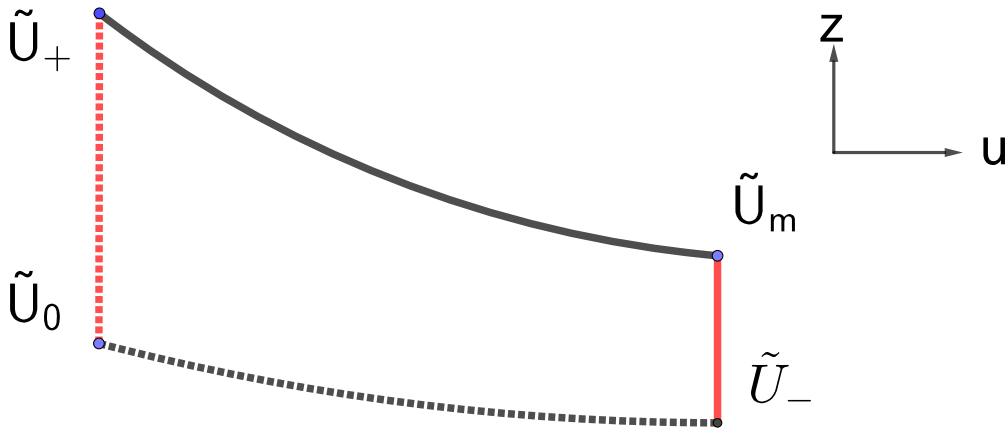
We shall use the following notations.

- $S$  or  $S_1$  stands for a shock wave which is always a 1-wave.
- $C_1$  or  $C_2$  stands for a contact discontinuity associated to  $\lambda_1$  or  $\lambda_2$ , a 1-wave or a 2-wave.  $C_1$  is considered as a degenerate shock  $S_1$ .

The key point here is to understand the effect of the  $L^\infty$  norm of  $Z_\nu$  after the interaction. First of all, we note that there is no self interaction for the second family since there are only  $C_2$  waves. Also the case of interactions between the 1- waves of the first family have already been well-studied [14], [20],[26] for the component  $u_\nu$  and are not presented here. The new feature is the effect of this interaction on the second component  $Z_\nu$ . We have seen in Lemma 4.1 and Proposition 4.1 that the change in  $L^\infty$  norm of  $Z_\nu$  is of the order of the cube of the change in  $u$  (or  $u_\nu$ ).

Finally, we consider the interaction of a 1-wave with a 2-wave.

1. In the case of an interaction of the form  $S_1 - C_2$  (which means that a 1-shock interacts with a 2-contact discontinuity), the outgoing wave is of the form  $C_2 - S_1$ . The shock continues with the same slope and the same value  $u_-, u_+$  and still satisfies the entropy condition (3.7). Thus, for  $u$  there is no change before and after



## Interaction Shock & Contact Discontinuity

Figure 2: Interaction of a shock with a 2-contact discontinuity. The interacting waves are represented by dotted lines, a 1-wave in black followed by a 2-wave in red. The full lines represent the resulting waves.

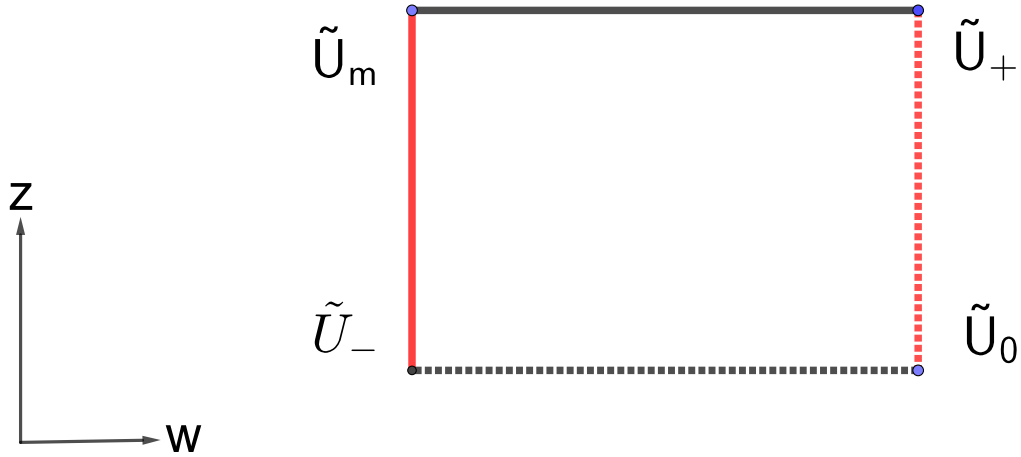
the interaction. Roughly speaking, the interaction of a 1-shock with a 2-contact discontinuity is transparent for  $u$ . On the contrary, there is a change for  $Z_\nu$  and following [25], it can be shown that the change in  $L^\infty$  norm of  $Z_\nu$  is of the order of cube of the change in  $u_\nu$ .

2. The interaction  $C_1 - C_2$  (that is, when a 1-contact discontinuity interacts with a 2-contact discontinuity) generates outgoing waves of the form  $C_2 - C_1$ . This case can be dealt in a similar manner as in the case of  $S_1 - C_2$  and it follows that the change in  $L^\infty$  norm of  $Z_\nu$  is of the order of cube of the change in  $u_\nu$ .

The reader interested by all the cases  $S_1 - C_1$ ,  $C_1 - S_1$ ,  $S_1 - C_2$ ,  $C_1 - C_2$  can consult [25] (and different notations where the 1 and 2 waves have to be exchanged) in the plane of Riemann invariant, a technique used in [45] for the isentropic Euler system. Here we present the last two cases for the triangular system, the case  $S_1 - C_2$  in figure 2 and the case  $C_1 - C_2$  in figure 3.

### 5.2 Uniform estimates

The  $BV^s$  estimates for  $u_\nu$  are already known since  $u$  is the entropy solution of the scalar conservation law (1.1). These estimates are recalled briefly in the first paragraph. The only difficulty in this section is to obtain the  $L^\infty$  estimates for  $v$ . For this purpose, we generalize the approach first used in [12] and recently in [25]. The approach consists of bounding  $v_\nu$ , indeed  $Z_\nu$ , along the 2-characteristics. For the chromatography system [12], the 2-characteristics are simply straight lines. In general, here, the 2-characteristics are



## Interaction 1 and 2 Contact Discontinuities

Figure 3: Interaction between a 1-contact discontinuity and a 2-contact discontinuity. The interacting waves are represented by dotted lines, a 1-wave in black (horizontal line) followed by a 2-wave in red (vertical line). The full lines represent the resulting waves.

piecewise linear. They are uniquely defined since the second eigenvalue  $a(u)$  is linearly degenerate and its integral curves are transverse to the discontinuity lines of the first field (assumption (USH)). The precise definition of such characteristics is given in the second paragraph. Then the estimate on  $Z_\nu$  along 2-characteristics, as in [12, 25], is given in the last paragraph using our generalized estimates on the Lax curves.

### *BV<sup>s</sup> estimates for $u$*

The  $TV^s$  decay is known for the Glimm scheme, the Godunov scheme and the Wave Front Tracking algorithm [11, 12]. Another argument is that  $u_\nu$  is also the exact entropy solution of the scalar conservation law (5.7) with the piecewise-constant initial data  $u_{0,\nu}$ , and the decay of  $TV^s u$  gives the uniform estimates with respect to  $\nu$ ,

$$TV^s u_\nu(\cdot, t) \leq TV^s u_{0,\nu} \leq TV^s u_0. \quad (5.10)$$

### The approximate 2-characteristics

Essentially, for the WFT, an approximate  $i$ -characteristic is a continuous curve which is piecewise linear following the velocity  $\lambda_i$ ,  $i = 1, 2$ . Since the eigenvalues depend only on  $u$ , there is a problem to define an  $i$ -characteristic where  $u$  is not defined. For  $i = 2$ , there is no problem of uniqueness, since a 2-characteristic is always transverse to the discontinuity lines of  $u$ . Thus, the 2-characteristic crosses the  $u$  discontinuity with a kink.

Let  $\gamma_\nu(x_0, t)$  be the forward generalized 2-characteristic starting at the point  $x_0$ , that is

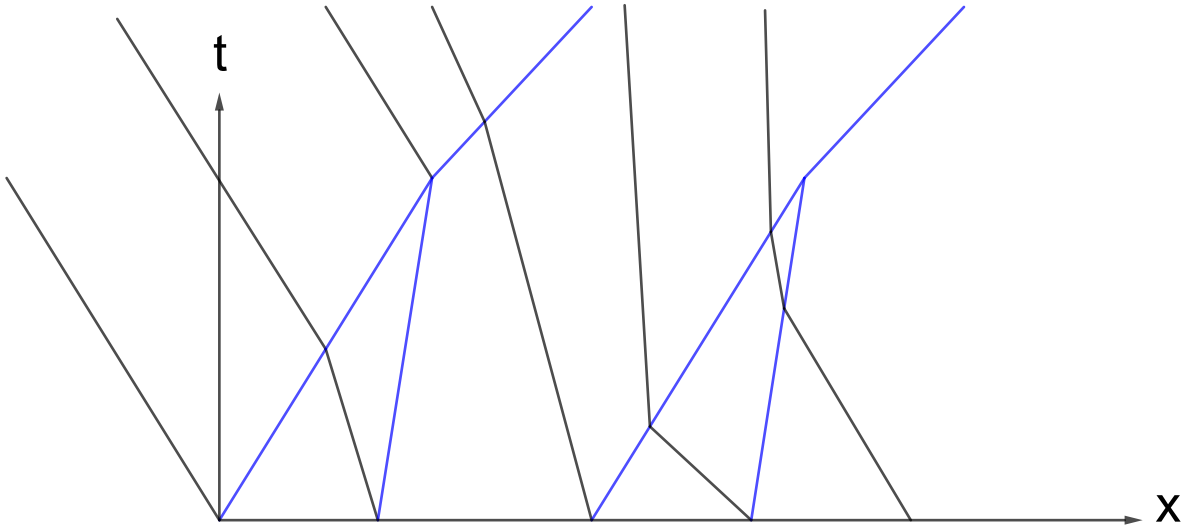


Figure 4: Wave front tracking algorithm. For the picture,  $f'(u) > 0 > a(u)$ . Thus the 1-waves in blue go to the right and the 2-waves in black to the left. Notice that these 1-waves are not affected by the interaction with the 2-waves, but the 2-waves are affected by the interaction.

$\gamma_\nu(x_0, t)$  is a solution of the differential inclusion

$$\frac{d}{dt}\gamma_\nu(x_0, t) \in [a(u_\nu(\gamma_\nu(x_0, t) - 0, t)), a(u_\nu(\gamma_\nu(x_0, t) + 0, t))], \quad \gamma_\nu(x_0, 0) = x_0.$$

For the wave front tracking, these 2-characteristics are uniquely determined and are piecewise linear continuous curves (thanks to the transversality assumption (USH)) and satisfy the differential equation

$$\frac{d}{dt}\gamma_\nu(x_0, t) = a(u_\nu(\gamma_\nu(x_0, t), t)), \quad \gamma_\nu(x_0, 0) = x_0, \quad (5.11)$$

except for a finite number of times which correspond to a jump of the piecewise constant function  $u_\nu$ .

### $L^\infty$ estimates for $v$

The  $L^\infty$  estimate on  $v_\nu$  is first obtained on the approximate Riemann invariant  $Z_\nu$ .  $Z_\nu$  is easy to bound through a rarefaction wave as it is constant through it. However,  $Z_\nu$  is not constant through a shock wave. But, we know that the variation of  $Z_\nu$  is of order of the cube of the variation of  $u_\nu$ , by Proposition 4.1. When there is no shock, the simple curve to bound  $Z_\nu$  is the 2-characteristic. In this “smooth” case,  $Z_\nu(\gamma_\nu(x_0), t) = Z_\nu(x_0, 0)$ .

Note that the 2-characteristics starting at time  $t = 0$  span the whole  $(t, x)$  half-plane but  $Z_\nu$  is only defined outside a finite number of 2-characteristics.

For example,  $Z_\nu$  is not well-defined on a 2-characteristic that contains a 2-wave (contact discontinuity for the second family). But these are only finite in number as the number of 2-wave fronts is finite.

Also  $Z_\nu$  is not well-defined on 2-characteristics passing through the points of intersections of waves corresponding to the first family. Again there are only finite number of such curves.

$Z_\nu$  is also not well-defined on a finite number of 2-characteristics originating from the points on  $t = 0$  where the Riemann problems are solved initially.

But since there are only finite number of such 2-characteristics, this is enough to estimate  $\|Z_\nu\|_\infty$ .

Now, due to the transversality conditions, a 2-characteristic meets many 1-waves. A 1-wave is usually a composite wave and thanks to Theorem 4.1, we have a cubic estimate for the  $L^\infty$  norm and the total variation of  $Z$  through a 1-wave. More precisely, if  $t_1$  is a time just before the 2-characteristic meets a 1-wave and  $t_2$  is the time just after the 2-characteristic crosses the 1-wave, we have the following estimate,

$$\begin{aligned} \|Z_\nu\|_{\infty,(\gamma_\nu(x_0,\cdot),\cdot)[t_1,t_2]} &\leq |Z_\nu(t_1)| \exp\left(\mathcal{O}\left(TV^{1/3}u_\nu(\gamma_\nu(x_0,t_2),t_2)\right)\right), \\ TVZ_\nu(\gamma_\nu(x_0,\cdot),\cdot)[t_1,t_2] &\leq \|Z_\nu\|_{\infty,(\gamma_\nu(x_0,\cdot),\cdot)[t_1,t_2]} \left(\mathcal{O}\left(TV^{1/3}u_\nu(\gamma_\nu(x_0,\cdot),\cdot)[t_1,t_2]\right)\right). \end{aligned}$$

That means that the  $L^\infty$  norm and the total variation of  $Z_\nu$  along the piece of curve  $\{(\gamma_\nu(x_0,t),t), t \in [t_1,t_2]\}$  is controlled by  $TV^{1/3}u_\nu$  along the same piece of curve. Notice also that the sign of  $Z_\nu$  is constant along a 2-characteristic by Lemma 4.1.

First, the  $L^\infty$  norm of  $Z$  is bounded on the 2-characteristic and, second, the estimate on  $TV Z_\nu$  follows. The total variation is additive and the fractional total variation is sub-additive [15]. So adding all these estimates, on the whole 2-characteristic  $\Gamma(x_0) = \{(\gamma_\nu(x_0,t),t), t > 0\}$  starting at  $x = x_0$ , we have the estimates

$$\begin{aligned} \|Z\|_{\infty,\Gamma(x_0)} &\leq |Z_{0,\nu}(x_0)| \exp\left(\mathcal{O}\left(TV^{1/3}u_\nu[\Gamma(x_0)]\right)\right), \\ &\leq \|Z_0\|_\infty \exp\left(\mathcal{O}\left(TV^{1/3}u_\nu[\Gamma(x_0)]\right)\right), \\ TVZ_\nu[\Gamma(x_0)] &\leq \|Z_0\|_\infty \Psi\left(\mathcal{O}\left(TV^{1/3}u_\nu[\Gamma(x_0)]\right)\right), \\ \Psi(x) &= x \exp(x). \end{aligned}$$

Now note that  $\Gamma(x_0)$  is a space like curve for the 1-characteristics of (5.7) ([20]). Hence

$$TV^{1/3}u_\nu[\Gamma(x_0)] \leq TV^{1/3}u_{0,\nu} \leq TV^{1/3}u_0.$$

This follows from the fact the wave-fans corresponding to  $u_\nu$  are monotone with respect to the left and right end states. Therefore no extremal values are added when we measure the fractional variation along  $\Gamma(x_0)$ .

Together these yield  $L^\infty$  and  $BV$  bounds for  $Z_\nu$  (uniform with respect to  $\nu$ ) along the 2-characteristics,

$$\|Z_\nu\|_\infty \leq \|Z_0\|_\infty \exp\left(\mathcal{O}(1)TV^{1/3}u_0\right), \quad (5.12)$$

$$TVZ_\nu[\Gamma(x_0)] \leq \|Z_0\|_\infty \Psi\left(\mathcal{O}\left(TV^{1/3}u_0\right)\right). \quad (5.13)$$



Notice also that the positivity is preserved. If  $\inf v_0 > 0$ , then  $\inf Z_0 > 0$  and by similar arguments presented above, we have  $\inf Z_\nu > \inf Z_0 \exp(\mathcal{O}(1)TV^{1/3}u_0) > 0$ , where the constant  $\mathcal{O}(1)$  is negative. Finally, a  $L^\infty$  bound (respectively a positivity) for  $Z_\nu$  yields a  $L^\infty$  bound (respectively a positivity) for  $v_\nu$ .

For the triangular system, the  $L^\infty$  bound of  $Z_\nu$  and hence of  $v_\nu$  is enough to pass to the weak limit in (5.15) since the left hand side is linear with respect to  $v_\nu$ . The uniform BV bound on  $Z_\nu$  along the 2-characteristics can be used to recover a strong trace at  $t = 0$ , like in [8] (at  $x = 0$ ).

### 5.3 Passage to the limit in the weak formulation

Passing to the strong-weak limit in the equation (1.2) which is linear with respect to  $v$  allows us to get a weak solution. But to do so, we need to understand the error of consistency of the scheme. There is no error of consistency in the independent scalar equation for  $u_\nu$  (5.14) [19]. As already noted earlier, the error of consistency arises only for the transport equation. Note that the error of consistency is uniquely determined by the WFT.

**Lemma 5.1.** *The error of consistency  $E_\nu$  of the scheme satisfies*

$$\partial_t u_\nu + \partial_x f_\nu(u_\nu) = 0, \quad (5.14)$$

$$\partial_t v_\nu + \partial_x (a(u_\nu)v_\nu) = \text{div}_{x,t} E_\nu. \quad (5.15)$$

Moreover, the estimate  $E_\nu = \mathcal{O}(TV^{1/3}u_0)$  holds.

This error  $E_\nu$  converges towards 0 if  $u_0$  belongs to  $BV^s$  with  $s > 1/3$  or, if the flux  $f$  is strictly convex and  $s \geq 1/3$ .

The convergence towards 0 of the error of consistency for systems is usually done in BV, [14, p. 126], [26, p. 305]. With less regularity in  $BV^s$ ,  $s < 1$ , new features in the estimate of the error of consistency appear.

*Proof.* That  $E_\nu$  satisfies the equations (5.14)-(5.15) immediately follows from the WFT. Now the approximate solution is piecewise constant, presenting only contact discontinuities or shock waves for  $u_\nu$ . Since  $f_\nu = f$  on the grid  $\nu^{-1}\mathbb{Z}$ , a weak jump solution of (5.14) is also a weak solution of the scalar conservation law with the exact flux  $f$ . Moreover, the approximate speed  $s_\nu$  equals the exact speed. If the jump is entropic for the exact flux  $f$ , as in the case of a shock wave, then  $E_\nu = 0$  because  $v_\nu$  is chosen on the exact Lax curve. The problem of consistency occurs only in the case of a contact discontinuity corresponding to  $u_\nu$ .

In this case, let  $u_0$  be the left state,  $u_m$  the right one and  $u_i$  the intermediary states,  $i = 1, \dots, m-1$ . Localising the error at a contact discontinuity front  $i$  between  $u_i$  and  $u_{i+1}$ , we see that  $E_\nu = \mathcal{O}(|u_{i+1} - u_i|^3) = \mathcal{O}(\nu^{-3})$ , due to Lemma 4.1. Adding these local errors near the 1-contact discontinuities yield the estimate  $E_\nu = \mathcal{O}(TV^{1/3}u_0)$ .

Now if  $u_0 \in BV^s$  with  $s > 1/3$ , then we can split the local error term as

$$|u_{i+1} - u_i|^3 = |u_{i+1} - u_i|^p |u_{i+1} - u_i|^\eta = |u_{i+1} - u_i|^p \nu^{-\eta}, \quad (5.16)$$

where  $p = 1/s = 3 - \eta$ ,  $\eta > 0$ . Therefore

$$\int_{\mathbb{R}} |E_\nu| dx = \mathcal{O}(1) \cdot \sum_{\text{1-contact discontinuities}} |u_{i+1} - u_i|^p \nu^{-\eta} \leq \mathcal{O}(1) \nu^{-\eta} TV^s u_0. \quad (5.17)$$

The sum is taken over all 1-contact discontinuities. Thus the error of consistency converges towards zero in  $L^1$  when  $\nu \rightarrow +\infty$ .

Now, consider the case  $s = 1/3$  when the exact flux  $f$  is strictly convex. In this case, we have blocks of consecutive contact discontinuities (with monotonic values) for the approximate flux corresponding to a rarefaction for the exact flux.

Let  $\bar{m}$  denote the number of consecutive contact discontinuities in a maximal such block. Then  $\bar{m} \leq 2(\nu M + 1)$ , where  $M = \|u_0\|_\infty$  and for any such block, we have

$$\sum |u_{i+1} - u_i|^3 \leq \bar{m} \nu^{-3} \leq 2(\nu M + 1) \nu^{-3} = \mathcal{O}(\nu^{-2}). \quad (5.18)$$

The number of such blocks is not increasing with time and is bounded by the number  $N_0$  of initial Riemann problems at  $t = 0$  in the WFT. Therefore

$$\int_{\mathbb{R}} |E_\nu| dx \leq \mathcal{O}(1) \sum_{\text{blocks}} 2(\nu M + 1) \nu^{-3} \leq 2N_0 (\nu M + 1) \nu^{-3} \cdot \mathcal{O}(1). \quad (5.19)$$

Therefore if we choose  $N_0 = N_0' = \nu$  to begin with, we see that the error of consistency vanishes as  $\nu$  tends to  $\infty$ .  $\square$

## 6 Blow-up of an entropy solution with $u_0 \notin BV^{1/3}$

In this section, we provide a proof of Theorem 2.2. For this purpose, we construct initial data  $u_0, Z_0$  that satisfy

$$\begin{cases} u_0 \in BV^{1/3-0}(\mathbb{R}), \\ Z_0 \in L^\infty(\mathbb{R}), \end{cases}$$

and exhibit blow-up in  $L^\infty$  norm of  $Z$  at  $t = 0+$  for the system (1.1)-(1.2).

Recall that the notation  $u_0 \in BV^{1/3-0}(\mathbb{R})$  means that  $u_0$  is in  $BV^s$  for all  $s \in [0, 1/3)$ . Our idea of construction is motivated by similar examples studied in [1, 17, 21, 22].

Notice that when  $v_0 \equiv 0$ , i.e.  $Z_0 \equiv 0$ , no blow-up occurs since  $(u, v) \equiv (u, 0)$  gives a global entropy solution where  $u$  the entropy solution associated to the  $L^\infty$  initial data  $u_0$ . Thus, in the following construction, we have to avoid the value 0 for  $Z$ .

Let us consider the system (1.1)-(1.2) with flux  $f(u) = \frac{u^2}{2}$ ,  $a(u) = u - 1$  and initial data  $u_0$  as described below and  $Z_0 \equiv 1$ .

Let  $x_0 = 0$  and  $x_n = 1 - \frac{1}{2^n}$ ,  $n \geq 1$ . Let  $B_n$  be chosen such that

$$x_n = x_{n-1} + 2B_n,$$

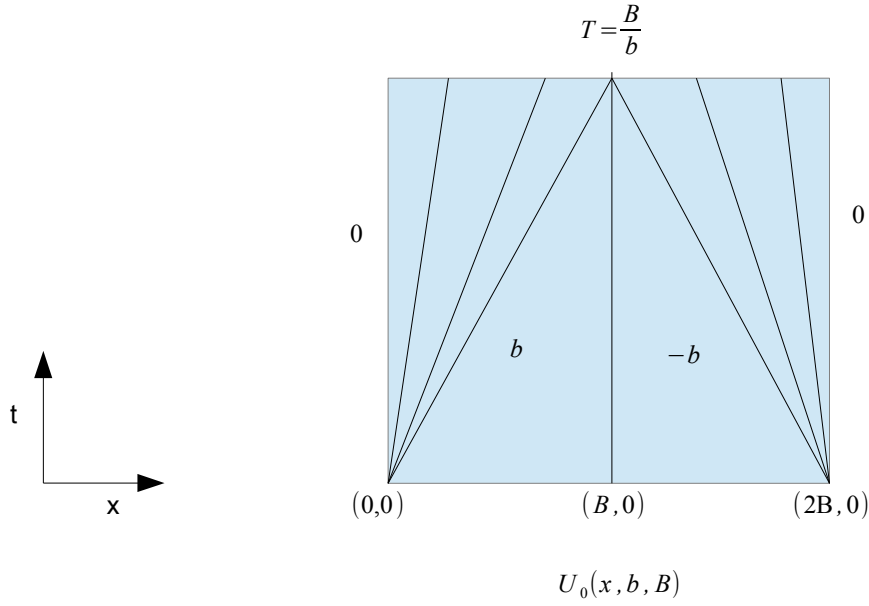


Figure 5: A typical building-block

that is,  $B_n = \frac{1}{2^{n+1}}$ . Let  $b_n = \frac{1}{(n+26)^{\frac{1}{3}}}$  and we define

$$U_0(x, b, B) = b\mathbb{1}_{[0,B)}(x) - b\mathbb{1}_{[B,2B)}(x).$$

Using this we define the initial data  $u_0$  as

$$u_0(x) = \sum_{n \geq 1} U_0(x - x_{n-1}, b_n, B_n).$$

Note that  $\|u_0\|_\infty \leq \frac{1}{3}$  and therefore, the condition of uniform strict hyperbolicity is satisfied.

The first interaction times  $T_n$  for the Riemann problems for the equation

$$\partial_t u + \partial_x f(u) = 0$$

with initial data  $u_0$  is given by

$$T_n = \frac{B_n}{b_n} = \frac{(n+26)^{\frac{1}{3}}}{2^{n+1}}.$$

Note that the first interaction time  $T_n$  satisfies the relation

$$T_n > 1 - x_n. \tag{6.1}$$

The initial data  $u_0$ , described above, clearly does not belong to  $BV^{\frac{1}{3}}(\mathbb{R})$ , but, as in [12, 17] we can conclude that  $u_0 \in BV^{1/3-0}(\mathbb{R})$ .

We use the forward generalized characteristic for  $Z$ . Since  $a(u) = u - 1$  and the first

interaction times of the Riemann problems for  $u$  satisfy (6.1), it follows that the forward generalized characteristic for  $Z$  starting at the point  $x_\infty := 1$  crosses infinitely many shocks before the first interaction of the waves in  $u$ .

As we have already seen from the nonlinear wave interactions, the  $L^\infty$  norm of  $Z$  does not change when it interacts with a 1-rarefaction wave.

Now, let us consider a left state  $u_-$  and a right state  $u_+$  connected by a 1-shock wave. In our example, we have  $u_- = b > 0$  and  $u_+ = -b < 0$ . Hence, the speed of the shock  $s = 0$ . Also by construction  $0 < b_n < \frac{1}{2} < 1$  and hence we assume that the prototype  $b$  satisfies the same.

Now

$$\begin{aligned} \frac{Z_-}{Z_+} &= \frac{s - a(u_+)}{s - a(u_-)} \exp [A(u_-) - A(u_+)] \\ &= \frac{a(u_+)}{a(u_-)} \exp [A(u_-) - A(u_+)] \\ &= \frac{u_+ - 1}{u_- - 1} \exp [A(u_-) - A(u_+)] \\ &= \frac{1 + b}{1 - b} \exp [A(u_-) - A(u_+)]. \end{aligned} \tag{6.2}$$

Also

$$A'(u) = \frac{a'(u)}{a(u) - f'(u)} = \frac{1}{(u - 1) - u} = -1 < 0,$$

and hence

$$A(u_-) - A(u_+) = \int_{u_+}^{u_-} A'(u) du = -(u_- - u_+) = -2b.$$

Therefore from (6.2), we have

$$\frac{Z_-}{Z_+} = \frac{1 + b}{1 - b} e^{-2b}. \tag{6.3}$$

We show that for  $b$  positive and small enough,

$$\frac{1 + b}{1 - b} e^{-2b} > 1, \tag{6.4}$$

and therefore  $Z_- > Z_+$ . Thus,  $Z$  increases in strength as the forward 2-generalized characteristic crosses a shock (from right to left).

Notice that we simply need that  $Z$  increases when  $b \sim 0$ . It is for very small oscillations that  $Z$  blows-up. Inequality (6.4) follows from a Taylor expansion up to the third order:

$$\begin{aligned} (1 + b) \exp(-2b) &= (1 + b)(1 - 2b + 1/2 \cdot (2b)^2 - 1/6 \cdot (2b)^3 + \mathcal{O}(b)^4) \\ &= 1 + (1 - 2)b + (-2 + 2)b^2 + (2 - 4/3)b^3 + \mathcal{O}(b)^4 \\ &= 1 - b + 2/3b^3 + \mathcal{O}(b)^4 \\ &> 1 - b, \end{aligned}$$

for  $b$  sufficiently small and hence  $Z_- > Z_+ > 0$ .

Thus, due to an interaction with a 1–shock wave, there is a change of order  $[u]^3$  in the  $L^\infty$  norm of  $Z$ . Since

$$\sum_{n \geq 1} |[u]^3| = \sum_{n \geq 1} \left( \frac{2}{(n+26)^{\frac{1}{3}}} \right)^3 = 8 \sum_{n \geq 1} \frac{1}{(n+26)} = +\infty,$$

we find that the  $L^\infty$  norm for  $Z$  blows up.

**Remark 6.1.** *Let  $\Gamma(x_0) = \{(\gamma(x_0, t), t), t \geq 0\}$  be the 2-characteristic issued from  $x = x_0$  at  $t = 0$ . The solution is well defined under  $\Gamma(1)$  that is on the set  $\{(x, t), x < \gamma(1, t), t \geq 0\}$ . But, over  $\Gamma(1)$ ,  $\{(x, t), 1 > x > \gamma(1, t), t > 0\}$ ,  $Z$  and  $v$  blow up,  $v = +\infty$ .*

**Remark 6.2.** *This example does not contradict the Lax-Oleinik smoothing effect ([32]) as the blow-up for  $Z$  occurs only at  $t = 0$ , that is, there is an immediate blow-up. Such a blow-up is not possible for a time  $t_0 > 0$  due to the BV smoothing of  $u$ .*

Now suppose that there exists a bounded weak solution for the system (1.1)-(1.2) with the above initial data. Then since the solution of the Riemann problem in the class of bounded weak solutions is unique upto the time of first interaction of the waves (see appendix B), any bounded weak solution will satisfy the construction discussed above. This, in turn, implies that the bounded weak solution exhibits a blow-up in  $L^\infty$  norm, which contradicts its definition.

Thus, there cannot exist a bounded weak solution for the system (1.1)-(1.2) with the above initial data, which proves Theorem 2.2. This also shows that the existence Theorem 2.1 is optimal in the class of bounded weak solutions.

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## A On Lipschitz test functions

Here we briefly discuss the use of Lipschitz test functions instead of  $C^1$  test functions for bounded weak entropy solutions for triangular systems (1.1)-(1.2). We show that if  $u, v \in L^\infty$  is a weak bounded solution (and therefore the solution constructed in Section 5) for the triangular system, then the  $C^1$  test functions can be enlarged to Lipschitz functions. This can be proved using the following observations. Note that  $(u, v)$  is a weak solution and hence satisfies the integral identities (1.8)-(1.9) for compactly supported smooth test functions. Now if we consider a compactly supported Lipschitz continuous test function  $\psi$ , we can construct (using a standard mollifier) a sequence of compactly supported smooth test functions  $\{\phi_n\}$  that converges pointwise almost everywhere to  $\psi$ . An application of dominated convergence theorem shows that  $\partial_t \phi_n$  and  $\partial_x \phi_n$  converge to

$\partial_t \psi$  and  $\partial_x \psi$  respectively at points of continuity (and therefore almost everywhere) of  $\psi$ . Since the integral identities (1.8)-(1.9) are satisfied for each  $\phi_n$ , we can use the above facts to pass to the limit in the identities to prove the integral identities for the Lipschitz continuous test function  $\psi$ .

## B Uniqueness of Riemann problem in $L^\infty$

We next show that in the case of strictly convex fluxes  $f$ , the solution of the Riemann problem is unique in the class of bounded entropy solutions at least upto the time of first interaction of waves. In this direction, we first note the following result.

**Lemma B.1.** *If  $(u, v)$  is a bounded entropy solution to the system (1.1)-(1.2) with initial data  $(u_0, v_0) \in L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R})$ , then  $Z := v \exp(A(u))$  satisfies*

$$\partial_t Z + a(u) \partial_x Z \stackrel{E}{=} 0, \quad (\text{B.1})$$

in open sets where  $u$  is Lipschitz continuous.

Here  $E := Lip'_c(\mathbb{R} \times [0, +\infty[, \mathbb{R})$  denotes the dual of the space of compactly supported Lipschitz continuous functions.

*Proof.* Since  $u$  is Lipschitz continuous and  $A \in C^1(\mathbb{R})$ , we can write any  $\psi \in Lip_c$  in the form  $\psi = \phi \exp(-A(u))$  for a suitable  $\phi \in Lip_c$ . Therefore, for  $\psi \in Lip_c(\mathbb{R} \times [0, +\infty[, \mathbb{R})$ , we can write,

$$\begin{aligned} & - \int_0^\infty \int_{\mathbb{R}} [Z \partial_t \psi + Z \partial_x (a(u) \psi)] \, dx dt + \int_{\mathbb{R}} Z_0(x) \psi(x, 0) \, dx \\ &= - \int_0^\infty \int_{\mathbb{R}} [Z \partial_t (\phi \exp(-A(u))) + Z \partial_x (a(u) \phi \exp(-A(u)))] \, dx dt \\ & \quad + \int_{\mathbb{R}} Z_0(x) \phi(x, 0) \exp(-A(u_0)) \, dx \\ &= - \int_0^\infty \int_{\mathbb{R}} [Z \exp(-A(u)) \partial_t \phi + Z a(u) \exp(-A(u)) \partial_x \phi] \, dx dt \\ & \quad + \int_{\mathbb{R}} Z_0(x) \phi(x, 0) \exp(-A(u_0)) \, dx \\ & \quad + \int_0^\infty \int_{\mathbb{R}} [Z A'(u) \partial_t u \exp(-A(u)) \phi + Z A'(u) \partial_x u \phi a(u) \exp(-A(u))] \, dx dt \\ & \quad - \int_0^\infty \int_{\mathbb{R}} Z \exp(-A(u)) \phi a'(u) \partial_x u \, dx dt \\ &= \int_0^\infty \int_{\mathbb{R}} Z \phi \exp(-A(u)) (A'(u) \partial_t u + A'(u) a(u) \partial_x u - a'(u) \partial_x u) \, dx dt, \end{aligned} \quad (\text{B.2})$$

where we use the fact that  $v$  satisfies the integral identity (1.9) for any  $\phi \in Lip_c(\mathbb{R} \times [0, +\infty[, \mathbb{R})$ .

Now  $\partial_t u$  and  $\partial_x f(u)$  belong to  $L^\infty$  and hence in  $L^1_{loc}$ , and also satisfy the integral identity (1.8) for all compactly supported smooth functions. Therefore  $\partial_t u = -\partial_x f(u)$  almost

everywhere. Also using the fact that  $f \in C^1$  and  $u$  is continuous, using Volpert's formula [46], it follows that  $-\partial_x f(u) = f'(u)\partial_x u$  almost everywhere.

Using these facts and the definition of  $A$  in (B.2), it follows that

$$-\int_0^\infty \int_{\mathbb{R}} [Z\partial_t \psi + Z\partial_x(a(u)\psi)] dxdt + \int_{\mathbb{R}} Z_0(x)\psi(x, 0) dx = 0,$$

for  $\psi \in Lip_c(\mathbb{R} \times [0, +\infty[, \mathbb{R})$ . □

Now suppose that  $f$  is strictly convex and consider the Riemann problem with left and right states given by  $(u_-, v_-)$  and  $(u_+, v_+)$  respectively. Let the corresponding states in  $(u, Z)$  coordinates be given by  $(u_-, Z_-)$  and  $(u_+, Z_+)$  respectively. Without loss of generality, we assume that the Riemann problem is based at the point  $(0, 0)$ .

First we consider the case when there is a rarefaction wave in  $u$ . Note that in this case  $u$  is a Lipschitz continuous function away from a ball  $B(0, r)$  of radius  $r$  around  $(0, 0)$ , for any  $r > 0$ .

From the previous lemma, we conclude that away from the ball  $B(0, r)$ ,  $Z$  (corresponding to any bounded weak solution  $(u, v)$ ) satisfies the identity (B.1) and by unicity of  $u$  and the unicity of weak solutions of a transport equation with Lipschitz continuous velocity field,  $Z$  is uniquely determined.

Since this is true for any  $r > 0$ , it follows that in the class of bounded weak solutions,  $(u, Z)$  (and therefore  $(u, v)$ ) is unique away from the set  $\{x = a(u_-)t\}$  of measure zero.

Next let us consider the case when there is a shock wave in  $u$ . In this case, using the previous lemma, we see that  $Z$  (corresponding to any bounded weak solution  $(u, v)$ ) satisfies the transport equation (B.1) with velocity  $a(u_-)$  in the region

$$\left\{ (x, t) : -\infty < \frac{x}{t} < s := \frac{[f(u)]}{[u]} \right\},$$

and therefore is uniquely determined in this region and has a trace on the line  $x = st$ .

Similarly,  $Z$  satisfies the transport equation (B.1) with velocity  $a(u_+)$  in the region

$$\left\{ (x, t) : s < \frac{x}{t} < +\infty \right\},$$

and therefore is uniquely determined in this region as well.

But any bounded weak solution  $(u, v)$  satisfies the same Rankine-Hugoniot condition across the shock curve, and therefore, the left trace of  $Z$  on the line  $x = st$  is uniquely determined by the states  $u_-, u_+, v_+$ , which implies the unicity of  $Z$  as well.

Therefore, for a strictly convex flux  $f$ , **at least** upto the time of first interaction of waves, the solution of the Riemann problem is unique in the class of bounded entropy solutions.

Note that the above proof holds for initial data  $u_0$  which gives rise to entropy solutions  $u$  that are locally Lipschitz except for a finite number of lines, and hence applies to the initial data that we construct in Section 6. Uniqueness for the slightly smaller class of piecewise  $C^1$  entropy solutions are well known [33] and has been already obtained for the (PSA) system [7].

Notice also that for the Kruzkov solution  $u$  a continuity in time with respect to  $L^1_{loc}$  is space is required [31], but for a nonlinear flux this property is automatically fulfilled [18].

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