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Entropy solutions in BV^s for a class of triangular systems involving a transport equation

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ABSTRACT. –Strictly hyperbolic triangular systems with a decoupled nonlinear conservation law and a coupled “linear” transport equation with a discontinuous velocity are known to create measure solutions for the initial value problem [40]. Adding a uniform strictly hyperbolic assumption on such systems we are able to obtain bounded solutions in L^∞ under optimal fractional BV regularity of the initial data. The Pressure Swing Adsorption process (PSA) [16] is an example coming from chemistry which, after a change of variables from Euler to Lagrange, has such a triangular structure [20]. Here, we provide global weak L^∞ entropy solutions in the framework of fractional BV spaces: BV^s , $1/3 < s < 1$, when the zero set of the second derivative of the decoupled flux is locally finite. In addition, for some initial data not in $BV^{1/3}$, a blow-up in L^∞ may occur.

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Key words: conservation laws; triangular systems; entropy solution; fractional BV spaces; linear transport; chromatography; Cauchy problem; non convex flux; regularity.

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1 Introduction

We consider triangular systems of the form

$$\partial_t u + \partial_x f(u) = 0, \tag{1.1}$$

$$\partial_t v + \partial_x (a(u)v) = 0. \tag{1.2}$$

Here f is the scalar flux function (which we shall henceforth refer to as the flux) for the equation (1.1) and the function $a(\cdot)$ denotes the velocity of the linear (with respect to v) equation (1.2). The above system is complemented by a set of initial data

$$u(x, 0) = u_0(x), \tag{1.3}$$

$$v(x, 0) = v_0(x). \tag{1.4}$$

Such systems are already of mathematical interest due to the coupling of the theory of scalar conservation laws with the theory of transport equations. This system was studied in [51] in a non hyperbolic setting, $f' = a$, with measure solutions for v . It is classically used before the apparition of shock waves [5, 37]. Here, we study this in a strictly hyperbolic setting obtaining global weak entropic bounded solutions in L^∞ with a minimal fractional BV regularity for u_0 . Surprisingly, if the fractional regularity for u_0 is not satisfied, a blow-up in L^∞ can occur. For 2×2 strictly hyperbolic and genuinely nonlinear systems, the existence of solutions in L^∞ with only small L^∞ initial data [8] or large L^∞ initial data for Euler equations [35] is well known. Here, with the linear degeneracy of the continuity equation (1.2), an existence result in L^∞ with only L^∞ initial data is not expected.

Another class of triangular system is possible [42], for instance the 2×2 system studied in [6]. The system (1.1), (1.2) appears in chemical engineering [36, 62, 63] for two chemical species [16, 20] after rewriting the system in Lagrangian coordinates [60, 68].

At the first sight, such a system seems easy to solve in the “triangular” manner, that is solve the first equation (1.1) to get u and then solve the linear equation (1.2) keeping u fixed. This method works well for smooth solutions [5, 37], but, when a shock wave appears in u the velocity $a(u)$ becomes discontinuous. The theory of linear transport equations with discontinuous velocity is a delicate topic yielding measure solutions and a loss of uniqueness [11, 12, 13, 61].

In this paper, we propose a different approach to obtain global weak solutions bounded in L^∞ . A main idea is to consider the system (1.1), (1.2) not as a triangular system, but, really, as a 2×2 hyperbolic system. If f is nonlinear, one field is nonlinear and the other one is linearly degenerate. When f is uniformly convex it is a particular case of 2×2 systems with one genuinely nonlinear field and a linearly degenerate one [41]. When the flux f is piecewise convex or concave, the system corresponds to the most important case in gas-liquid chromatography [16, 17, 36].

A linearly degenerate field is known to simplify the study of a 2×2 system [59, 60]. Such systems are known to propagate one component with less regularity, see [28, 31, 39] where one component has a large total variation. In [19, 20], one component is only L^∞ . On the contrary such linearly degenerate field can also produce blow-up behaviors [18, 20, 57].

We show that the behavior of the proposed entropy solutions is linked to the fractional BV regularity of u , indeed, $u_0 \in BV^s$, $0 < s \leq 1$. The BV^s framework seems optimal to study the optimal regularity of entropy solutions of scalar conservation laws [19, 54]. This framework and more references are recalled in Section 1.4. For the scalar case, the theory works well for all $s > 0$. For the triangular system (1.1), (1.2), we prove that the regularity $s = 1/3$ is critical for the existence of L^∞ entropy solutions. As a consequence, it proves that for general nonlinear 2×2 systems with a genuinely nonlinear eigenvalue and a linearly degenerate one, the existence result obtained in [41] is optimal. The exponent $s = 1/3$ is directly linked to the cubic estimate on the Lax curves [20, 50]. For nonconvex fields, it is known that the Lax curves are less regular [7, 3, 4, 52]. However, for the triangular system (1.1), (1.2), with a uniform hyperbolicity assumption, $f \in C^4$, $a \in C^3$, we prove that the exponent $s = 1/3$ is still critical for the existence theory.

The paper is organized as follows. The hyperbolicity of the triangular system and the key assumptions are given in Section 1.1. The definitions of weak and entropy solutions are stated in Section 1.2. Explicit occurrences of measure solutions are discussed in Section 1.3. The BV^s framework is recalled in Section 1.4. In Section 2 the two main results are stated: existence for $s > 1/3$, blow-up for $s < 1/3$. In Section 3 the Riemann invariants and the Riemann problem are studied. The Lax curves and the key cubic estimates are studied in Section 4. The existence proof is done in Section 5 and the blow-up in Section 6.

1.1 The hyperbolic triangular system

The system (1.1), (1.2) of conservation laws is clearly hyperbolic and can be rewritten, using the vectorial flux \mathbf{F} , as $\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = 0$, $\mathbf{U} = (u, v)^\top$. The matrix of the linearized system has a triangular structure,

$$D\mathbf{F}(\mathbf{U}) = \begin{pmatrix} f'(u) & 0 \\ a'(u)v & a(u) \end{pmatrix}.$$

This system has the unbounded invariant region $[-M, M]_u \times \mathbb{R}_v$, where $M = \|u_0\|_\infty$. In this paper, the system is assumed to be strictly hyperbolic through the condition

$$\forall u \in [-M, M], \quad f'(u) > a(u). \quad (1.5)$$

Of course, the symmetric assumption: $\forall u \in [-M, M], f'(u) < a(u)$, yields a similar study. For large data, the strict hyperbolicity condition (1.5) has to be strengthened on the set $[-M, M]$, by the following uniformly strict hyperbolicity (USH) condition:

$$\inf_{|u| \leq M} f'(u) > \sup_{|u| \leq M} a(u). \quad (1.6)$$

An interesting case is already when f is convex, $f'' > 0$ everywhere, or concave $f'' < 0$. This case occurs for a chromatography system with a convex isotherm written in appropriate Lagrangian coordinates [20].

In this paper, the flux $f(\cdot)$ belongs to C^4 and the velocity $a(\cdot)$ belongs to C^3 . Moreover, the flux is locally piecewise convex or concave as a consequence of the following assumption,

$$\mathcal{Z} = \{u, f''(u) = 0, |u| \leq M\} \text{ is finite.} \quad (1.7)$$

1.2 Weak and entropy solutions

A weak solution of the system (1.1)-(1.2) satisfying the initial conditions (1.3)-(1.4) is defined as follows.

Definition 1.1 (Weak solutions). *The pair (u, v) is a weak solution of the system (1.1)-(1.2) with initial data (1.3)-(1.4) if for all compactly supported test functions $\varphi, \psi \in C_c^1(\mathbb{R} \times [0, +\infty[, \mathbb{R})$, the following integral identities hold:*

$$\int_0^{+\infty} \int_{\mathbb{R}} (u \partial_t \varphi + f(u) \partial_x \varphi) dx dt + \int_{\mathbb{R}} u_0(x) \varphi(x, 0) dx = 0, \quad (1.8)$$

$$\int_0^{+\infty} \int_{\mathbb{R}} (v \partial_t \psi + a(u) v \partial_x \psi) dx dt + \int_{\mathbb{R}} v_0(x) \psi(x, 0) dx = 0. \quad (1.9)$$

In a quite general framework, the following regularity is required for (u, v) : $u \in L_{loc}^\infty$, $v \in L_{loc}^1$ or more generally v is a measure. In the case when v is a measure, there are some issues in defining the product $a(u) \times v$ [12]. We shall briefly touch upon the issue of measure solutions. But our main focus in this paper is on entropy solutions $u, v \in L_{loc}^\infty$.

We propose below a notion of entropy solutions for the system (1.1)-(1.2). As in [15, 20, 47], the entropy condition is only tested on the nonlinear component u . Contact discontinuity waves linked to a linearly degenerate field are well known not to affect the entropy inequality which remains an equality [30].

Definition 1.2 (entropy solutions). *The pair (u, v) is an entropy solution of the system (1.1)-(1.2) with initial data (1.3)-(1.4) if it is a weak solution and for all convex function η and all non-negative test functions $\varphi \in C_c^1(\mathbb{R} \times]0, +\infty[, \mathbb{R})$, with $q' = \eta' f'$, u satisfies the following inequality:*

$$\int_0^{+\infty} \int_{\mathbb{R}} (\eta(u) \partial_t \varphi + q(u) \partial_x \varphi) dx dt + \int_{\mathbb{R}} \eta(u_0(x)) \varphi(x, 0) dx \geq 0. \quad (1.10)$$

Thus, it suffices only to have a weak solution of the system and the entropy solution of (1.1). We do not need to use another family of entropies associated to v [15, 20]. This

is due to the linear degeneracy of the second equation. For some diagonal systems with linearly degenerate eigenvalues [47], such definition is enough to get uniqueness of entropy solutions.

Uniqueness of entropy solutions with a fixed initial data (u_0, v_0) is wrong without additive assumptions. It is due to the lack of uniqueness of weak solutions for the linear transport equation with a discontinuous velocity.

1.3 Measure solutions

For triangular systems as discussed in this paper, one can have measure solutions. A prototypical example of such triangular systems is given by

$$\begin{aligned} \partial_t u + \partial_x \left[\frac{u^2}{2} \right] &= 0, \\ \partial_t v + \partial_x [(u-1)v] &= 0, \end{aligned} \tag{1.11}$$

with $f(u) = \frac{u^2}{2}$ and $a(u) = u - 1$. Clearly this satisfies the strict hyperbolicity condition (1.5) but not the more restrictive uniform strict hyperbolicity condition (1.6). In [40], it was shown that for strong shocks the solutions must be searched in the class of Radon measures. In particular, let us consider the Riemann problem with initial data

$$u(x, 0) = \begin{cases} u_l, & x < 0, \\ u_r, & x > 0, \end{cases} \quad v(x, 0) = \begin{cases} v_l, & x < 0, \\ v_r, & x > 0, \end{cases}$$

where u_l, u_r, v_l and v_r are constants. Then for $u_l \geq u_r + 2$ (see Section 3, [40]), the system (1.11) has a solution given by

$$u(x, t) = \begin{cases} u_l, & x < \bar{u}t, \\ u_r, & x > \bar{u}t, \end{cases} \quad \bar{u} = \frac{u_l + u_r}{2},$$

which is the entropy solution of the Burgers equation satisfied by u , and

$$v(x, t) = \begin{cases} v_l, & x < \bar{u}t, \\ v_r, & x > \bar{u}t, \end{cases}$$

with a Dirac mass propagating with the δ -shock in v on the line $x = \bar{u}t$, whose strength is given by

$$\int_{\bar{u}t^-}^{\bar{u}t^+} v(x, t) dx = \frac{t}{2} [v_r(u_l - u_r + 2) - v_l(u_r - u_l + 2)].$$

In other words, v is of the form

$$v(x, t) = v_l \chi_{x < \bar{u}t}(x, t) + v_r \chi_{x > \bar{u}t}(x, t) + \frac{t}{2} [v_r(u_l - u_r + 2) - v_l(u_r - u_l + 2)] \delta(x - \bar{u}t),$$

where χ_A is the characteristic function of the set A .

Thus, corresponding to a strong shock in u , v lies in the class of bounded Radon measures. Similar phenomenon is well known for transport equations [12] and is also widely observed in the case of non-strictly hyperbolic systems [21, 13].

The main approach in solving the system (1.11) in [40] was to observe that the first equation in u can be solved independently. The second equation can then be considered as a transport equation in v with a discontinuous coefficient u (see also [12]).

But this idea of looking at the two equations separately might not be a good one. Our aim in this article is to prove the existence of solutions to systems of type (1.11) in the class of fractional BV functions and hence one need not appeal to δ -shock wave type solutions. Moreover, to obtain L^∞ solution with a wave front algorithm, we suppose that the system (1.1), (1.2) is a strictly hyperbolic system (1.5). Notice that we also need the uniform strict hyperbolicity condition (1.5), else we are not able to solve the Riemann problem appropriately.

For instance, consider the example with $f'(u) = u$ and $a(u) = u - d$, $d > 0$ and $M = 1$. This system is strictly hyperbolic for $(u, v) \in [-1, 1] \times \mathbb{R}$. For $d \leq 2$, the uniform strict hyperbolicity condition (1.6) is not fulfilled. Consider for instance the Riemann problem with initial data $u_0(x) = \pm 1$, $\pm x > 0$ and $v_0(x) \equiv 1$. For $d > 1$, the Riemann problem can be solved by a contact discontinuity with speed $a(1) = 1 - d < 0$ and a stationary shock wave with 0 speed. The intermediary constant state between the two waves is computed thanks to the Rankine-Hugoniot condition along the shock at $x = 0$ which says that $a(u)v$ is constant through the shock (since the shock speed is 0). The intermediary constant state is $(u, v) = (-1, v^*)$ with

$$v^* = 1 \times \frac{a(1)}{a(-1)} = \frac{1 - d}{-1 - d} = \frac{d - 1}{d + 1} < 0.$$

For $d = 1$ there is no room to put this intermediary state. It is even worse for $0 < d < 1$. The velocity $a(u)$ becomes positive, so the contact discontinuity wave has to be on the right of the shock. But its speed $a(u)$ corresponds to the left state. Thus, it is impossible to solve it by this way.

Transport equations and δ -shock waves

The phenomenon of Dirac mass already appears for the linear transport equation with a discontinuous coefficient [12, 61].

$$\partial_t v + \partial_x(\alpha(x)v) = 0, \quad v(x, 0) = v_0(x). \quad (1.12)$$

For instance, if $\alpha(x) = -\text{sign}(x)$ and $v_0(x) \equiv 1$ a Dirac mass appears immediately at $x = 0$: $v(0, t) = 2t\delta(x - 0)$.

The point is that the linear transport equation (1.2) is the more difficult part of the triangular system (1.1)-(1.2). In this paper, a construction of L_{loc}^∞ weak solution is proposed.

It is tempting to solve the singular transport equation (1.12) with the triangular system (1.1)-(1.2), choosing a flux f and α such that $\alpha(x, t) = a(u(x, t))$ but it is impossible. $u(x, t)$ has a stationary shock so $0 \in f'([-M, M])$ and the uniform strict hyperbolicity is lost.

1.4 BV^s functions

For one dimensional scalar conservation laws, the spaces BV^s are known to give optimal results for weak entropy solutions [19, 25, 54, 55], first on the fractional Sobolev regularity [53, 45] and second on the structure of such solutions [1, 2, 9, 10, 32, 33]. Such results on the maximal regularity has been extended for some systems and the multidimensional case [26, 37, 46]. For less regular fluxes more generalized BV spaces are also used in [27, 54, 55, 38]. In this section basic facts on BV^s functions are recalled.

Definition 1.3. [56] *A function u is said to be in $BV^s(\mathbb{R})$ with $0 < s \leq 1$ if $TV^s u < +\infty$ where*

$$TV^s u := \sup_{n \in \mathbb{N}, x_1 < \dots < x_n} \sum_{i=1}^n |u(x_{i+1}) - u(x_i)|^{1/s}.$$

The BV^s semi-norm is defined by

$$|u|_{BV^s} := (TV^s u)^s$$

and a norm on this space is defined by

$$\|u\|_{BV^s} := \|u\|_{L^\infty} + |u|_{BV^s}.$$

Fractional BV functions have traces like BV functions. This is a fundamental property to define the Rankine-Hugoniot condition for shock waves. Moreover, this property is not true for the Sobolev functions in $W^{s,1/s}$, the Sobolev space nearest to BV^s [19].

Theorem 1.1. [56] *For all $s \in]0, 1[$, BV^s functions are regulated functions.*

The fractional total variation only depends on the local extrema of the function and the order of this extrema.

Lemma 1.1. [19, 41] *If u is a piecewise monotonous function and if its local extrema are located in the increasing sequence $(x_i)_{i \in I}$, then $TV^s u$ only depends on the sequence $(u(x_i))_{i \in I}$. Moreover, there exists an ordered subset J of I such that*

$$TV^s u = \sum_{j \in J, j \neq \max J} |u(x_{suc(j)}) - u(x_j)|^{1/s},$$

where $suc(j)$ denotes the successor of j in J , $suc(j) = \min\{k \in J, k > j\}$.

Moreover, it is dangerous to refine the mesh to compute the fractional total variation [19, example 2.1], [23, 24]. Consider $u(x) \equiv x$ on $[0, R]$, $p = 1/s > 1$. Then $TV^s u[0, R] = R^p$ but, when $n \rightarrow +\infty$,

$$\sum_{i=1}^n |u(iR/n) - u((i-1)R/n)|^p = n(R/n)^p = R^p/n^{p-1} \rightarrow 0.$$

This property, which is not true in BV , is used later to prove the existence of weak solutions in $BV^{1/3}$ for the triangular system (1.1), (1.2) when the flux f is convex.

2 Main results

The main results are stated in the BV^s framework. The basic facts on this setting were recalled in Section 1.4. The critical space for the existence theory is $BV^{1/3}$ when the flux f is convex. For the nonconvex case, the existence is proved in the little smaller space than $BV^{1/3}$ [19],

$$BV^{1/3+0} := \bigcup_{s>1/3} BV^s \subsetneq BV^{1/3}.$$

Theorem 2.1 (Existence in $BV^{1/3+0} \times L^\infty$). *The following assumptions are required on the flux $f(\cdot)$ and the transport velocity $a(\cdot)$ on $[-M, M]$ where $M = \|u_0\|_\infty$:*

1. $f'(\cdot)$ and $a(\cdot)$ belong to $C^3([-M, M], \mathbb{R})$,
2. $f'(u) > a(u)$ and satisfy the uniform strict hyperbolicity assumption (USH) (1.6),
3. $f(\cdot)$ has at most a finite number of inflections points (1.7).

If (u_0, v_0) belongs to $BV^s \times L^\infty$ and $s > 1/3$, then there exists an entropy solution (u, v) of system (1.1), (1.2), which belongs to $L^\infty([0, +\infty)_t, BV^s(\mathbb{R}_x, \mathbb{R})) \times L^\infty((0, +\infty)_t \times \mathbb{R}_x, \mathbb{R})$.

In addition, the positivity of the initial data is preserved, if $\inf v_0 > 0$ then $\inf v > 0$.

Moreover, if the flux f is convex then the existence result remains true for $s = 1/3$.

Such a result requires many comments since such systems are known to produce δ -shocks instantly [66]. To have a bounded solution in L^∞ , we forget the triangular structure and do not solve the first equation (1.1) independently and then the second one (1.2). We consider the triangular system (1.1), (1.2) as a whole system. For a Riemann problem involving a shock wave we write the Rankine-Hugoniot condition for the whole system. The first equation (1.1) fixes the speed of the shock wave $s = [f(u)]/[u]$ independently of v where $[u]$ is the jump of u . Then we solve the second equation $s[v] = |a(u)v|$ with s already fixed. This process is possible under the uniformly strictly hyperbolicity hypothesis (USH) (1.6), as explained later in Section 3.2. Thus, (USH) is a key assumption to use such a process and to avoid a blow-up as in [18, 57]. The philosophy of using a whole system to solve a singular scalar equation is already in the smart paper [6]. For the gas-liquid chromatography system, (USH) is automatically fulfilled if the incoming gas-velocity is positive [15]. Notice that the positivity of v is preserved a.e. by Theorem 2.1. Of course, by linearity of the equation (1.2) with respect to v , the negativity of the initial data is also preserved, if $\sup v_0 < 0$ then $\sup v < 0$.

The triangular approach, that means solving first (1.1) with the unique Krushkov entropy solutions [49] and then (1.2), usually yields a measure solution v . It is the reason why we do not expect an uniqueness result. Of course, we have uniqueness for u but the problem of uniqueness remains for v . For a weakly coupled system with some linearly degenerate fields, the entropy condition only on the nonlinear field is enough to ensure the uniqueness in [47]. For the triangular system the coupling by the transport velocity is too nonlinear and certainly needs another condition to enforce the uniqueness.

For the existence result, a Wave Front Tracking algorithm (WFT) is proposed following the approach of solving a Riemann problem for the whole system. As a matter of fact the approximate solution u satisfies the BV^s uniform bounds [19]. The difficult point is to bound v in L^∞ . Since the system is linear with respect to v , an L^∞ bound for v is enough to get an existence theorem as in [15].

Theorem (2.1) is optimal and, in general, we cannot reduce the $BV^{1/3}$ regularity of u_0 , else a blow-up can occur. For this purpose, we build a sharp example in a space very near to $BV^{1/3}$, namely, $BV^{1/3-0}$

$$BV^{1/3} \subsetneq BV^{1/3-0} := \bigcap_{s < 1/3} BV^s.$$

Theorem 2.2 (Blow-up in $BV^{1/3-0}$). *There exist $u_0 \notin BV^{1/3}$, but $u_0 \in BV^{1/3-0}$, $v_0 \in L^\infty$, a and f such that the solution of the initial value problem provided by our construction has a blow-up immediately at time $t = 0^+$.*

There is no blow-up for u since the entropy solution u of the scalar conservation law satisfies the maximum principle. Indeed, only the function v has a blow-up at time $t = 0^+$. For the flux f , only a nonlinear flux is needed. We provide a simple example with a Burgers' flux f and a linear velocity a . Notice also that the blow-up depends only on the regularity of u_0 and not of v_0 .

Another way to present this blow-up is the following.

Remark 2.1. *For the Burgers' flux $f(u) = u^2/2$ and the velocity $a(u) = u - 3$, there exists a sequence of initial data $(u_{0,n}, v_{0,n})$ in BV such that the system (1.1), (1.2) is uniformly strictly hyperbolic (USH) and, the associated solutions (u_n, v_n) provided by our wave front tracking algorithm satisfy*

$$\lim_{n \rightarrow +\infty} TV^{1/3} u_{0,n} = +\infty \tag{2.1}$$

$$\lim_{n \rightarrow +\infty} t_n = 0 \tag{2.2}$$

$$\lim_{n \rightarrow +\infty} \|v_n(\cdot, t_n)\|_\infty = +\infty. \tag{2.3}$$

As a consequence, it proves that for general nonlinear 2×2 systems with a genuinely nonlinear eigenvalue and a linearly degenerate one the existence result proven in [41] is optimal.

3 The uniformly strictly hyperbolic system

In this section, we get the Riemann invariant and solve the Riemann problem study for the triangular system (1.1)-(1.2).

3.1 Riemann invariants

A 2×2 strictly hyperbolic system admits, at least locally, a set of coordinates which diagonalizes the hyperbolic system for smooth solutions [65]. The knowledge of this

coordinate system, given by the Riemann invariants is often useful in understanding the structure of the system. Next we study the Riemann invariants for the system (1.1)-(1.2).

The eigenvalues of the system (1.1)-(1.2) are:

$$\lambda_1 = f'(u) > a(u) = \lambda_2. \quad (3.1)$$

Notice that the eigenvalues are functions of u only. Let r_1 and r_2 denote the corresponding right eigenvectors.

Clearly, u is a 2-Riemann invariant associated to the right eigenvector $r_2 = (0, 1)^\top$ and satisfies

$$\partial_t u + f'(u)\partial_x u = 0.$$

A 1-Riemann invariant, which we denote as $z(u, v)$, corresponding to the right eigenvector r_1 , can be computed in the following manner. We note that a right eigenvector of the matrix $D\mathbf{F}(u)$ corresponding to the eigenvalue $f'(u)$ is given by

$$r_1 = \begin{pmatrix} 1 \\ \frac{a'(u)v}{f'(u) - a(u)} \end{pmatrix}.$$

Then z satisfies

$$\frac{1}{v}\partial_u z = \frac{a'(u)}{a(u) - f'(u)}\partial_v z.$$

This can be solved using a separation of variables.

$$\partial_u z = \frac{a'(u)}{a(u) - f'(u)}, \quad \partial_v z = \frac{1}{v}.$$

For instance, a Riemann invariant is

$$z(u, v) = A(u) + \ln(|v|), \quad A'(u) = \frac{a'(u)}{a(u) - f'(u)}.$$

To avoid the singularity at $v = 0$, it suffices to take the exponential,

$$Z = \exp(z) = v \exp(A(u)), \quad (3.2)$$

which satisfies the equation

$$\partial_t Z + a(u)\partial_x Z = 0.$$

Z is exactly the Riemann invariant discovered by the first author in [14] for a chromatography system studied also in [15, 17, 18, 20].

3.2 The Riemann problem

We study the Riemann problem for the system (1.1)-(1.2) with initial data:

$$u_0(x) = u_\pm, \pm x > 0, \quad (3.3)$$

$$v_0(x) = v_\pm, \pm x > 0. \quad (3.4)$$

A direct and somewhat naive approach is to solve the conservation law (1.1) first and then the second equation (1.2) using the solution u . In such an approach, one faces the difficulty of solving the linear transport equation with a discontinuous coefficient.

Instead, we consider the two equations together as a system. This is a key point as in [6]. The solution of the Riemann problem consists of two waves separated by an intermediary state (u_m, v_m) where $u_m = u_-$ and v_m is unknown. In accordance with the labeling of the eigenvalues (3.1), a wave associated to u is called a 1-wave and a wave associated to Z is called a 2-wave.

1. A wave associated to the eigenvalue $\lambda_1 = f'$: this wave is a shock wave or a rarefaction wave if f is convex or concave. For a non convex flux f , this wave is a composite wave.
2. A linearly degenerate wave associated to the eigenvalue $\lambda_2 = a$: the speed of this 2-wave is $a(u_-)$.

The intermediate value v_m has to be computed through the 1-wave. Now, the various 1-waves that can occur are detailed. For this purpose, we consider Riemann problems yielding only a 1-wave.

Shock waves

Let us denote $U = (u, v)$ and $\tilde{U} = (u, Z)$.

The Rankine-Hugoniot condition gives

$$s[u] = [f(u)], \quad s[v] = [a(u)v], \quad (3.5)$$

where s denotes the speed of the discontinuity (or the slope of the jump).

Thus, the slope of the jump is determined by u_{\pm} and the flux f ,

$$s = \frac{[f(u)]}{[u]}. \quad (3.6)$$

Since entropy solutions are considered, the Oleinik-entropy condition [22] for (1.1) enforces the Lax-entropy conditions,

$$f'(u_-) \geq s \geq f'(u_+), \quad (3.7)$$

whence

$$s \geq f'(u_+) > a(u_+). \quad (3.8)$$

Using the second equation of (3.5), yields

$$(s - a(u_+))v_+ = (s - a(u_-))v_-, \quad (3.9)$$

$$v_+ = v_- \frac{s - a(u_-)}{s - a(u_+)} := \mathcal{S}_-(u_+; U_-). \quad (3.10)$$

The equation (3.10) can be interpreted in terms of the Lax shock curve. For a fixed $U_- = (u_-, v_-)$, the right hand side of (3.10) is only a function of u_+ as s is given by (3.6) as a function of u_+ . In the plane $U = (u, v)$, (3.10) describes the set of U_+ such that the Riemann problem with initial data U_\pm is solved by a shock. On this curve, only U_+ satisfying the Oleinik condition (3.7) are considered to allow an entropic shock.

Conversely, if U_+ is fixed, the Lax shock curve is parametrized by u_- and reads

$$v_- = v_+ \frac{s - a(u_+)}{s - a(u_-)} := \mathcal{S}_+(u_-; U_+). \quad (3.11)$$

Proceeding similarly with $Z = v \exp(A(u))$, $\tilde{U}_- = (u_-, Z_-)$ and keeping the notation \mathcal{S} in coordinates (u, Z) yields,

$$Z_+ = Z_- \frac{s - a(u_-)}{s - a(u_+)} \exp[A(u_+) - A(u_-)] := \mathcal{S}_-(u_+; \tilde{U}_-),$$

and

$$Z_- = Z_+ \frac{s - a(u_+)}{s - a(u_-)} \exp[A(u_-) - A(u_+)] := \mathcal{S}_+(u_-; \tilde{U}_+).$$

1-Contact discontinuity

This is a limiting case of the preceding one, when f' is constant on the interval $[u_-, u_+]$. The same formula for the shock wave follows.

$$s = f'(u_\pm) = f'(u) > a(u_\pm), \quad u \in [u_-, u_+], \quad (3.12)$$

$$v_+ = v_- \frac{f'(u_-) - a(u_-)}{f'(u_+) - a(u_+)} = \mathcal{S}_-(u_+; U_-), \quad (3.13)$$

$$Z_+ = Z_- \frac{f'(u_-) - a(u_-)}{f'(u_+) - a(u_+)} \exp[A(u_+) - A(u_-)] := \mathcal{S}_-(u_+; \tilde{U}_-). \quad (3.14)$$

Such waves arise in the wave-front tracking (WFT) method when the flux is approximated by piecewise linear functions.

Let us suppose that $f''(u) \neq 0$ in the interval $(u_-, u_+) = \{(1 - \theta)u_- + \theta u_+, \theta \in [0, 1]\}$. Indeed, [22, 30]:

- if $f'' > 0$ then $u_- < u_+$,
- if $f'' < 0$ then $u_- > u_+$.

Since Z is a 1-Riemann invariant, the Lax rarefaction curve $\mathcal{R}_-(\tilde{U}_-)$ is simply,

$$Z_+ = Z_-. \quad (3.15)$$

Using the definition of Z , this implies $v_+ \exp(A(u_+)) = v_- \exp(A(u_-))$ and thus the Lax rarefaction curve can be written explicitly,

$$v_+ = \mathcal{R}_-(u_+; U_-) = v_- \exp(A(u_-) - A(u_+)), \quad (3.16)$$

$$v_- = \mathcal{R}_+(u_-; U_+) = v_+ \exp(A(u_+) - A(u_-)). \quad (3.17)$$

Notice that v_+ and v_- have the same signs. In particular, if $v_- = 0$, $v = 0$ is constant through the rarefaction wave.

Composite waves

In general, the entropy solution u is a composite wave [15]. If f has a finite number N of inflection points, then there are at most N contact-shock waves [30, 52, 58]. The Lax curves associated to such waves are studied below in Section 4.

2-Contact discontinuity

u is a 2-Riemann invariant and hence is constant along the 2-contact discontinuity. Thus the Lax curve is simply a vertical line in the plane $U = (u, v)$ or the plane $\tilde{U} = (u, Z)$,

$$u_- = u_+. \quad (3.18)$$

4 The Lax curves

A fundamental theorem due to Lax [50] states that the shock curve and the rarefaction curve emanating from a constant state U_- in the plane (u, Z) or (u, v) have a contact of the second order for a genuinely nonlinear eigenvalue. This means that the shock curve can be replaced by the rarefaction curve up to an error of order $[u]^3$ where $[u] = u_+ - u_-$ [22, 30, 64]. For the triangular system, a genuinely nonlinear eigenvalue means $f'' > 0$ (or $f'' < 0$) everywhere. For nonconvex cases, typically f'' locally has a finite number of roots where f'' changes its sign and the Lax curves are less regular due to the occurrence of contact-shocks. Under a concave-convex assumption, which means here that f''' does not vanish, the regularity of the Lax curves is only piecewise C^2 [52], see also [3, 4]. As a consequence, the error becomes of order $[u]^2$ for the variation of Z through a contact-shock [52]. The situation is worse in general, the Lax curves are only Lipschitz [7, 44]. However, we prove that cubic estimates are still valid for the triangular system (1.1), (1.2). It is mainly due of the existence of Riemann invariant coordinates.

In this section, cubic estimates on the Lax curves for the triangular system in the plane (u, Z) are done.

Let (u_-, v_-) , (u_+, v_+) be two constant states connected by a rarefaction or a shock wave. It is more convenient to use the Riemann invariant coordinates (u, Z) . The rarefaction curves in the plane (u, Z) are simply,

$$Z_- = Z_+, \quad (4.1)$$

which means that in the (u, v) plane

$$v_- \exp(A(u_-)) = v_+ \exp(A(u_+)). \quad (4.2)$$

For the shock curve, the Rankine-Hugoniot condition is written in the conservative variables (u, v) ,

$$s = \frac{[f(u)]}{[u]} = \frac{f(u_+) - f(u_-)}{u_+ - u_-}, \quad (4.3)$$

$$v_-(s - a(u_-)) = v_+(s - a(u_+)). \quad (4.4)$$

Since $f \in C^4$, s is a C^3 function of its arguments. Moreover, fixing (u_+, v_+) and considering u_- as a variable, the Lax shock curve is C^3 with respect to u_- ,

$$v_- = v_+ \frac{s - a(u_+)}{s - a(u_-)}. \quad (4.5)$$

Indeed, the denominator never vanishes due to the uniform strictly hyperbolic assumption (USH) (1.6). The same regularity of the shock curve holds in the variables (u, Z)

$$Z_- = Z_+ \frac{s - a(u_+)}{s - a(u_-)} \exp(A(u_-) - A(u_+)). \quad (4.6)$$

Of course, the Lax rarefaction curve and the Lax shock curve has to be restricted on the subset satisfying entropy conditions. Nevertheless, we use these curves for all range of u_- in \mathbb{R} (at least for $-M \leq u_- \leq M$) to obtain a generalized Lax cubic estimate for the nonconvex case.

4.1 The Lax cubic estimate on the Rankine-Hugoniot curve

The Lax cubic estimate [50] can be written as follows for a shock wave connecting (u_-, Z_-) to (u_+, Z_+) for the triangular system (1.1),(1.2), as soon as Z is bounded,

$$[Z] = \mathcal{O}([u])^3, \quad [Z] = Z_+ - Z_-, \quad [u] = u_+ - u_-. \quad (4.7)$$

The Riemann invariant Z is constant along the rarefaction curves. The Lax' cubic estimate means that the shock curve and the rarefaction curve have a contact of order 2. The Lax' cubic estimate was written in a genuinely nonlinear framework [50]. This means that f'' does not vanish. Indeed, the Lax' computations are still valid without this convex assumption and without only considering the entropic part of the Rankine-Hugoniot curve. We are also used the non entropic part of the Rankine curve. It is a key point in this paper, used many times in this paper, first in the next section 4.2 to get a cubic estimate for the Riemann problem for a non convex flux.

For the triangular system, the Rankine-Hugoniot curve is global and well defined thanks to the uniformly strictly hyperbolic assumption (1.6). For 2×2 systems, there is not always such a global curve [48].

Now, to prove cubic estimates, we have to write the Rankine-Hugoniot curve. Here, we choose to write Z_- as a function of u_- when (u_+, Z_+) are fixed for the following reasons.

1. To solve the Riemann problem and compute the intermediary state v_m which corresponds to Z_m and here Z_- .
2. To check that the triangular system (1.1),(1.2) is not a Temple system [67].
3. To obtain the cubic estimate on the global Rankine-Hugoniot curve below.
4. To obtain the existence result, bounding Z along the 2-characteristics.
5. To build a blow-up, again computing Z from the right to the left on the 2-characteristics.

The Rankine-Hugoniot curve \mathcal{RH}_+ when $(\tilde{U}_+) = (u_+, Z_+)$ is fixed and Z_- is a function of u_- is then,

$$Z_- = Z_+ \frac{s - a(u_+)}{s - a(u_-)} \exp [A(u_-) - A(u_+)] = Z_+ r(u_-, u_+) := \mathcal{RH}_+(u_-; u_+, Z_+). \quad (4.8)$$

The classic Lax' cubic estimate on the shock curve is generalized on the global Rankine-Hugoniot curve.

Proposition 4.1 (Cubic flatness of the global Rankine-Hugoniot curve). *If f' and a belong to $C^3(\mathbb{R}, \mathbb{R})$ and satisfy the uniform strict hyperbolicity condition (1.6) then*

$$s = s(u_-, u_+) = \frac{[f]}{[u]} = \frac{f(u_+) - f(u_-)}{u_+ - u_-} \in C^3([-M, M]^2, \mathbb{R}),$$

$$r = r(u_-, u_+) = \frac{s - a(u_+)}{s - a(u_-)} \exp [A(u_-) - A(u_+)] \in C^3([-M, M]^2, \mathbb{R}),$$

$$r = 1 + \mathcal{O}(1) [u]^3 > 0, \quad \forall (u_-, u_+) \in [-M, M]^2, \quad (4.9)$$

$$Z_- = \mathcal{O}(1) Z_+ [u]^3, \quad \forall (u_-, u_+, Z_+) \in [-M, M]^2 \times \mathbb{R}, \quad (4.10)$$

where the constant $\mathcal{O}(1)$ depends only on the derivatives of f' and a on $[-M, M]$. Moreover, Z_- has the same sign as Z_+ , more precisely,

$$\begin{aligned} Z_+ = 0 &\Rightarrow Z_- = 0, \\ Z_+ \neq 0 &\Rightarrow Z_- Z_+ > 0. \end{aligned}$$

The cubic flatness of the Rankine-Hugoniot curve \mathcal{RH}_+ is expressed in (4.10). It does not depend on the convexity of the fields, although in the classical textbooks [22, 31, 43, 52, 64, 65] some nonlinearity assumptions on the fields are given. The reason of these nonlinear assumptions on the textbooks is to introduce the rarefaction curve and the shock curve. Here, the global Rankine-Hugoniot defined for all $u_- \in [-M, M]$ is the main subject without looking at the entropic parts of this curve. A careful reading of the classical proof of the cubic estimate in textbooks shows that it is a geometric property of the Rankine-Hugoniot curve itself, without referring at the entropic or nonlinearity assumptions. This geometric property is a consequence of the symmetry of the Rankine-Hugoniot condition with respect to U_- and U_+ [64]. It is very important in this paper to prove the cubic estimate for the non entropic part of the Rankine-Hugoniot curve for two reasons.

1. The cubic estimates on the Lax' curve which is not piecewise C^3 for nonconvex f [52] uses the global Rankine-Hugoniot curve in the next section 4.2.
2. Rarefaction wave fans are replaced by weak non entropic jumps in the wave front tracking algorithm [22]. The error in the weak formulation has to be controlled by the cubic estimate to pass to the limit and get a weak solution of (1.1), (1.2).

The proof appears as a direct consequence of (4.9). An elementary and self-contained proof using only Taylor's expansions are proposed. A more tedious computation can give the more precise result.

$$r(u_-, u_+) = 1 + E[u]^3 + \mathcal{O}([u])^3, \quad (4.11)$$

where E depends in a quite complicate way on the derivatives of f' and a at $u = u_+$. The computations of E is quite intricate and not useful here, except in the last section 6 on the blow-up where a direct computation of E at $u_+ = 0$ is given when f is quadratic and a is linear.

Now, Proposition 4.1 is proven.

Proof. The positivity on r is a consequence of the assumption (1.6). This positivity implies that Z_- has the same sign as Z_+ .

Many Taylor expansions are used to obtain (4.9). u_+ is fixed and u_- is the variable near u_+ . The notations $a_- = a(u_-)$, $a_+ = a(u_+)$ and so on are used to shorten the expressions.

$$\begin{aligned} a_- &= a_+ - a'_+[u] + (a''_+/2)[u]^2 + \mathcal{O}([u])^3, \\ s &= \frac{[f]}{[u]} = \frac{f_- - f_+}{-[u]} = f'_+ - (f''_+/2)[u] + (f'''_+/6)[u]^2 + \mathcal{O}([u])^3. \end{aligned}$$

The hyperbolic quantity h is used,

$$h := f' - a > 0. \quad (4.12)$$

The sign of h is the consequence of the strict hyperbolicity assumption (1.6). We start with the fraction part of r and $h_+ = f'_+ - a_+ \neq 0$.

$$\frac{s - a_+}{s - a_-} = \frac{(f'_+ - a_+) - (f''_+/2)[u] + (f'''_+/6)[u]^2}{(f'_+ - a_+) - (f''_+/2 - a'_+)[u] + (f'''_+/6 - a''_+/2)[u]^2} + \mathcal{O}([u])^3 \quad (4.13)$$

$$= 1 - \frac{a'_+}{h_+}[u] + \frac{h_+a''_+ - a'_+f''_+ + 2a_+^2}{2h_+^2}[u]^2 + \mathcal{O}([u])^3 \quad (4.14)$$

For the term $\exp(-[A])$ in r , the Taylor expansion of A'_- is used at the first order.

$$\begin{aligned} -A'_- &= \frac{a'_-}{h_-} = \frac{a'_+ - a''_+[u]}{h_+ - h'_+[u]} + \mathcal{O}([u])^2 \\ &= \frac{a'_+}{h_+} + \frac{a'_+h'_+ - a''_+h_+}{h_+^2}[u] + \mathcal{O}([u])^2 \end{aligned}$$

Integrating with respect to u_- yields,

$$A_- - A_+ = \frac{a'_+}{h_+}[u] + \frac{a'_+h'_+ - a''_+h_+}{2h_+^2}[u]^2 + \mathcal{O}([u])^3$$

Since $\exp(x) = 1 + x + x^2/2 + \mathcal{O}(x)^3$, it yields

$$\exp[A_- - A_+] = 1 + \frac{a'_+}{h_+}[u] + \frac{a'_+h'_+ - a''_+h_+ + a_+^2}{2h_+^2}[u]^2 + \mathcal{O}([u])^3 \quad (4.15)$$

Now, multiplying (4.14) and (4.15) yields $r = 1 + \mathcal{O}([u])^3$. \square

4.2 Cubic estimates for the Riemann problem

Now, the intermediary state Z_m of a Riemann problem has to be estimated. For this purpose, the variation of Z along a composite wave is studied. When the flux is convex, Lax proved that the variation of Z is a cubic order of the variation of u [50]. For a non convex flux, it is well known that the Lax curve is less regular, piecewise C^2 [52] or only Lipschitz [7]. However, we prove that for our triangular system we are able to keep a cubic order. This is mainly due to the existence of Riemann coordinates for 2×2 systems and the cubic estimate for the global Rankine-Hugoniot locus, Proposition 4.1. As a consequence, we can prove a similar estimate for the variation of Z over a composite 1-wave. This improves the well known square root estimate for concave-convex eigenvalues [52], which correspond to cubic degeneracies for f . That means that for the triangular system (1.1), (1.2) the estimate is as precise as for the convex case [50].

Proposition 4.2 (Variation of Z through a composite wave). *Let the states $\tilde{U}_i = (u_i, Z_i)$, $i = 1, \dots, m$, where $u_0 < u_1 < \dots < u_m$ ($m \leq N_{infl} + 1$), comprise a composite 1-wave, $Z_- = Z_0$ and $Z_+ = Z_m$, then, the total variation of Z through a 1-wave is,*

$$\|Z\|_\infty \leq |Z_+| \exp(\mathcal{O}(u_+ - u_-)^3), \quad (4.16)$$

$$TVZ \leq \mathcal{O}(1)|Z_+||u_+ - u_-|^3. \quad (4.17)$$

Proof. We note that Z is a 1-Riemann invariant and therefore it remains constant over a rarefaction wave. Moreover, if the states \tilde{U}_i and \tilde{U}_{i+1} are joined by a jump, by Proposition 4.1 and the classic inequality $1 + x \leq \exp(x)$ we have the estimate

$$|Z_i| = |Z_{i+1}r(u_i, u_{i+1})| = |Z_{i+1}|(1 + \mathcal{O}(u_i - u_{i+1})^3) \leq |Z_{i+1}| \exp(\mathcal{O}(u_i - u_{i+1})^3).$$

Summing up and noting that u_0, u_1, \dots, u_m are ordered, the estimate (4.16) follows,

$$\max_i |Z_i| \leq |Z_+| \exp(\mathcal{O}(u_+ - u_-)^3).$$

Now, the BV bound for Z through the composite wave is computed.

$$\begin{aligned} Z_i - Z_{i+1} &= Z_{i+1}(r(u_i, u_{i+1}) - 1) = Z_{i+1}\mathcal{O}(u_i - u_{i+1})^3 \\ TVZ &\leq \max_i |Z_i| \sum_i \mathcal{O}(u_i - u_{i+1})^3 \leq |Z_+| \mathcal{O}(u_+ - u_-)^3 \exp(\mathcal{O}(u_+ - u_-)^3). \end{aligned}$$

u satisfies the maximum principle, so $\exp(\mathcal{O}(u_+ - u_-)^3) = \mathcal{O}(1)$ which only depends on the L^∞ bound of the initial data, $\|u_0\|_\infty$, so the inequality (4.17) is proved. \square

5 Existence in BV^s

In this section, Theorem 2.1 is proved using a simplified Wave Front Tracking (WFT) [22, 30, 43] algorithm for such a triangular system (1.1), (1.2). The BV^s estimates for u are a consequence of such estimates for scalar conservation laws [19]. The L^∞ bound for v and the proof that a weak solution is obtained for the triangular system are based on an approach using a $BV^{1/3}$ regularity for u .

5.1 The Wave Front Tracking algorithm

The (WFT) depends on an integer parameter $\nu > 0$. The approximate solutions will be denoted by u_ν, v_ν, Z_ν . We use mostly the Riemann invariant coordinates (u_ν, Z_ν) except when passing to the limit in the weak formulation.

This algorithm is explained in many books [22, 30, 43] on hyperbolic systems. Taking advantage of the structure of the triangular system (1.1), (1.2), we will mix the (WFT) for the scalar case [29] and for systems [22, 43]. The principle is to work with piecewise constant approximations.

As in the scalar case [22, 43], the values of u_ν is taken on a uniform grid parametrized by the integer ν ,

$$u_\nu \in \nu^{-1}\mathbb{Z}. \quad (5.1)$$

v_ν , or equivalently Z_ν , is not required to stay on the uniform grid,

$$v_\nu, Z_\nu \in \mathbb{R}, \quad (5.2)$$

because, we solve the exact Riemann problem to compute v_ν , or equivalently Z_ν . In this way, we can use the cubic estimate on the global Rankine-Hugoniot curves of Proposition 4.1. A similar approach for a convex case was used in [41].

The initial data are approximated as follows. Let $N_0 + 1$ be the number of constant states for the approximate initial data $u_{0,\nu}, v_{0,\nu}$ and the corresponding $Z_{0,\nu}$. N_0 is related to the uniform grid $\nu^{-1}\mathbb{Z}$. Later, to prove that (u_ν, v_ν) converges towards a weak solution, the following requirement is imposed on N_0 ,

$$N_0 = N_{0,\nu} = \mathcal{O}(\nu). \quad (5.3)$$

The approximate initial data can be chosen to satisfy the following uniform estimates with respect to ν [19]:

$$\|u_{0,\nu}\|_\infty \leq \|u_0\|_\infty, \quad (5.4)$$

$$TV^s u_{0,\nu} \leq TV^s u_0, \quad (5.5)$$

$$\|v_{0,\nu}\|_\infty \leq \|v_0\|_\infty. \quad (5.6)$$

Moreover, $(u_{0,\nu}, v_{0,\nu}) \rightarrow (u_0, v_0)$ a.e. when $\nu \rightarrow +\infty$ so, the previous inequalities become equalities at the limit $\nu \rightarrow +\infty$.

Now, at $t = 0+$, N_0 Riemann problem are solved. In general, the flux is non convex, so there are composite waves built with a succession of shock waves (or jump waves) and rarefaction waves. The shock waves are solved with the exact shock speed. The rarefaction waves are not piecewise constant, thus, they must be approximated by a series of small jumps. These jumps are non entropic shocks but are still weak solutions. The built u_ν is a weak solution, but not an entropic one, to the scalar conservation law (1.1). Moreover, the entropy condition is recovered if the flux f is replaced by a piecewise-linear continuous

flux [29]. The piecewise-linear continuous flux f_ν coincides with f on the uniform grid [22, Ch. 6], [43, p. 70],

$$f_\nu(k/\nu) = f(k/\nu), \quad \forall k \in \mathbb{Z}. \quad (5.7)$$

That means that u_ν is the weak entropy solution of

$$\partial_t u_\nu + \partial_x f_\nu(u_\nu) = 0, \quad u_\nu(x, 0) = u_{0,\nu}(x). \quad (5.8)$$

With strong compactness on u_ν , the Kruzkov entropy solution is recovered [22, 43].

For v_ν the situation is less simple than for u_ν . There is an error in the weak formulation discussed in Section 5.3. This is due to the approximation of rarefaction waves.

$$\partial_t v_\nu + \partial_x (a(u_\nu)v_\nu) = \text{Error}_\nu, \quad v_\nu(x, 0) = v_{0,\nu}(x). \quad (5.9)$$

As in the case of systems of conservation laws, a consistency error remains in the weak formulation [22, 43].

At the same time, some Riemann problems interact and new Riemann problems have to be solved. We follow the approach of the scalar case, many interactions can appear at the same time. The detailed nonlinear interactions are given for the triangular system below. The process continues until the second time of interactions and so on.

Now, the well posedness of the WFT is proved. That means that the process can be continued for all time because there is only a finite number of interactions for all time. For u_ν this is clear since this result is known for the scalar case. An explicit bound of the number of interactions is given in [43, p. 71-72]. The 2-waves, associated to the linearly degenerate eigenvalue $a(u_\nu)$, never interact together since they are contact discontinuities. Due to the transversality assumption (USH), a 2-wave can interact only once with a 1-wave and creates a new 2-wave (and not modifies the 1-wave). Thus, the number of such interactions and of 2-waves is finite. This proves that the WFT is well defined for all time.

Now, the approximate Riemann solver and the nonlinear interaction of waves are detailed.

Approximate Riemann solver

In this short section, the approximate Riemann solver is detailed. The initial data are (u_-, Z_-) , (u_+, Z_+) . For u_ν the solution is a series of entropic jumps $u_0 < u_1 < \dots < u_m$ for the piecewise linear flux f_ν . For Z_ν , there are many possibilities. We want to approximate the exact solution of the Riemann problem and keep the cubic estimates (Proposition 4.2) which generalize the Lax cubic estimates for genuinely nonlinear waves. So, we use the exact solution of the Riemann problem and the exact Lax curve to determine Z_{m-1}, \dots, Z_1 . Let $u_0 = u_-$ and $u_m = u_+$. Then Z_i , $i = m, m-1, \dots, 2$ are built as follows:

- If the jump between u_{i-1} and u_i corresponds to an entropic jump for the exact flux f then Z_{i-1} is given by the Rankine-Hugoniot curve (4.8),

$$Z_{i-1} = \mathcal{RH}_+(u_{i-1}; u_i, Z_i). \quad (5.10)$$

Thus, the jump between (u_{i-1}, Z_{i-1}) and (u_i, Z_i) satisfies the Rankine-Hugoniot condition and there is no error in the weak formulation of the exact triangular system (1.1), (1.2).

- If the jump between u_{i-1} and u_i corresponds to a rarefaction for f then, necessarily, $|u_i - u_{i-1}| = \nu^{-1}$ and we keep Z constant as for the exact solution,

$$Z_i = Z_{i-1}. \quad (5.11)$$

The chosen approximate solution of the Riemann problem is a jump which does not satisfy the Rankine-Hugoniot condition. An error in the weak formulation of the exact system appears and it is controlled in a cubic way thanks to Proposition 4.1.

Nonlinear wave interactions

We briefly describe the different possible nonlinear wave interactions and the details of the interactions are given in the (u, Z) plane.

In all the cases, we consider three states $\tilde{U}_-, \tilde{U}_0, \tilde{U}_+$ before the interaction. Here \tilde{U}_0 denotes the intermediary state which disappears after the interaction. The states \tilde{U}_-, \tilde{U}_0 are connected by an elementary wave and similarly the states \tilde{U}_0, \tilde{U}_+ are connected by an elementary wave. An elementary wave is either a 1-wave: shock or a small jump, or a 2-wave: a contact discontinuity. At a time of interaction $t_{interact}$, the Riemann problem is solved with a new intermediary state \tilde{U}_0^* (see Figure 1).

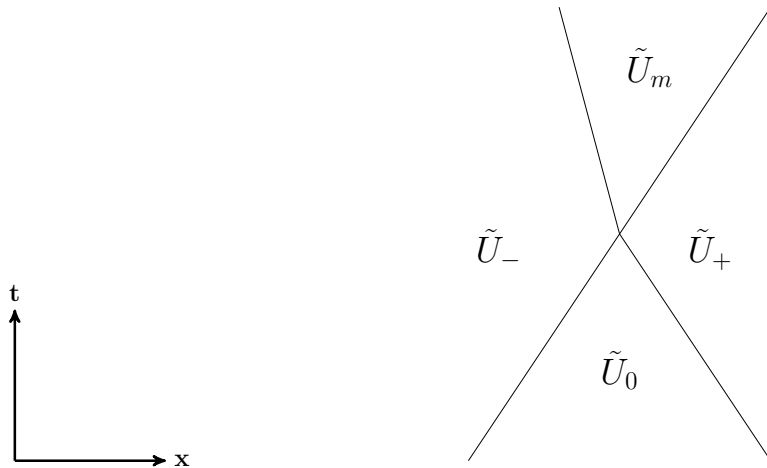


Figure 1: Nonlinear interaction of two waves. The 1-wave on the right crosses the interaction point with the same speed and the same value u_- on the left of the 1-wave ($u_m = u_-$) and u_+ on the right. On the other hand, the second wave is affected by the interaction: the speed of the 2-wave and a new value Z_m appears.

We shall use the following notations.

- S or S_1 stands for a shock wave which is always a 1-wave.
- R or R_1 stands for a rarefaction wave which is always a 1-wave.
- C_1 or C_2 stands for a contact discontinuity associated to λ_1 or λ_2 , a 1-wave or a 2-wave. C_1 is considered as a degenerate shock S_1 or as a non-entropic jump when it is used to approximate a rarefaction R_1 .

The key point here is to understand the effect of the L^∞ norm of Z after the interaction. First of all, we note that there is no self interaction for the second family since there are only C_2 waves. Also the case of interactions between the waves of the first family have already been well-studied (see [30],[22],[43]). We have seen in Proposition 4.1 and Proposition 4.2 that the change in L^∞ norm of Z is of the order of the cube of the change in u (or u_ν).

Finally, we consider the interaction of a 1-wave with a 2-wave.

1. In the case of an interaction of the form $S_1 - C_2$ (which means that a 1-shock interacts with a 2-contact discontinuity), the outgoing wave is of the form $C_2 - S_1$. The shock continues with the same slope and the same value u_-, u_+ and still satisfies the entropy condition (3.7). Thus, for u there is no change before and after the interaction. Roughly speaking, the interaction of a 1-shock with a 2-contact discontinuity is transparent for u . On the contrary, there is a change for Z and following [41], it can be shown that the change in L^∞ norm of Z is of the order of cube of the change in u .
2. The interaction $C_1 - C_2$ (that is, when a 1-contact discontinuity interacts with a 2-contact discontinuity) generates outgoing waves of the form $C_2 - C_1$. This case can be dealt in a similar manner as in the case of $S_1 - C_2$ and it follows that the change in L^∞ norm of Z is of the order of cube of the change in u .

5.2 Uniform estimates

The BV^s estimates for u_ν are already known since u is the entropy solution of the scalar conservation law (1.1). These estimates are recalled briefly in the first paragraph. The only difficulty in this section is to obtain the L^∞ estimates for v . For this purpose, we generalize the approach first used in [14, 15, 20] and recently in [41]. The approach consists in bounding v_ν , indeed Z_ν , along the 2-characteristics. For the chromatography system [15], the 2-characteristics are simply lines. In general, here, the 2-characteristics are piecewise lines. They are uniquely defined since the second eigenvalue $a(u)$ is linearly degenerate and its integral curves are transverse to the discontinuity lines of the first field (assumption (USH)). The precise definition of such characteristics is given in the second paragraph. Then the estimate on Z_ν along 2-characteristics, as in [14, 15, 20, 41], is given in the last paragraph using our generalized estimates on the Lax curves.

BV^s estimates for u

The TV^s decay is known for the Glimm scheme, the Godunov scheme and the Wave Front Tracking algorithm [19, 20]. Another argument is that u_ν is also the exact entropy

solution of the scalar conservation law (5.8) with the piecewise-constant initial data $u_{0,\nu}$, so the decay of $TV^s u$ gives the uniform estimates with respect to ν ,

$$TV^s u_\nu(\cdot, t) \leq TV^s u_{0,\nu} \leq TV^s u_0. \quad (5.12)$$

The 2-approximate characteristics

Essentially, for the WFT, an approximate i -characteristic is a continuous curve which is piecewise linear following the velocity λ_i , $i = 1, 2$. Since the eigenvalues depend only on u , there is a problem to define an i -characteristic where u is not defined. For $i = 2$, there is no problem of uniqueness, since a 2-characteristic is always transverse to the discontinuity lines of u . Thus, the 2-characteristic crosses the u discontinuity with a kink.

Let $\gamma_\nu(x_0, t)$ be the forward generalized 2-characteristic starting at the point x_0 , that is $\gamma_\nu(x_0, t)$ is a solution of the differential inclusion

$$\frac{d}{dt} \gamma_\nu(x_0, t) \in [a(u_\nu(\gamma_\nu(x_0, t) - 0, t)), a(u_\nu(\gamma_\nu(x_0, t) + 0, t))], \quad \gamma_\nu(x_0, 0) = x_0.$$

For the wave front tracking, these 2-characteristics are uniquely determined and piecewise linear continuous curves thanks to the transversality assumption (USH) and satisfy the differential equation, except for a finite number of times which correspond to a jump of the piecewise constant function u_ν ,

$$\frac{d}{dt} \gamma_\nu(x_0, t) = a(u_\nu(\gamma_\nu(x_0, t), t)), \quad \gamma_\nu(x_0, 0) = x_0. \quad (5.13)$$

L^∞ estimates for v

The L^∞ estimate on v_ν is first obtained on the approximate Riemann invariant Z_ν . Z_ν is easy to bound through a rarefaction wave as it is constant through it. However, Z_ν is not constant through a shock wave. But, we know that the variation of Z_ν is of order of the cube of the variation of u_ν , Proposition 4.2. When there is no shock, the simple curve to bound Z_ν is the 2-characteristic. In this “smooth” case $Z_\nu(\gamma_\nu(x_0), t) = Z_\nu(x_0, 0)$.

Notice that Z_ν is well defined on a 2-characteristic if the 2-characteristic does not contain a 2-wave where Z has a jump. Since the number of 2-wave fronts is finite, the number of such 2-characteristics is also finite. Hence, only 2-characteristics not touching a 2-wave is considered. This choice of 2-characteristics is enough to estimate $\|Z_\nu\|_\infty$.

Now, due to the transversality conditions, a 2-characteristic meets many 1-waves. A 1-wave is usually a composite wave and thanks to Proposition 4.2 we have a cubic estimate for the L^∞ norm and the total variation of Z through a 1-wave. More precisely, if t_1 is a time just before the 2-characteristic meets a 1-wave and t_2 is the time just after the 2-characteristic crosses all the 1-wave, we have the following estimate,

$$\begin{aligned} \|Z_\nu\|_{\infty, (\gamma_n u(x_0, \cdot), \cdot)[t_1, t_2]} &\leq |Z_\nu(t_1)| \exp(\mathcal{O}(TV^{1/3} u_\nu(\gamma_n u(x_0, t_2), t_2))), \\ TV Z_\nu(\gamma_n u(x_0, \cdot), \cdot)[t_1, t_2] &\leq \|Z_\nu\|_{\infty, (\gamma_n u(x_0, \cdot), \cdot)[t_1, t_2]} \mathcal{O}(TV^{1/3} u_\nu(\gamma_n u(x_0, \cdot), \cdot)[t_1, t_2]). \end{aligned}$$

That means that the L^∞ norm and the total variation of Z_ν along the piece of curve $\{(\gamma_n u(x_0, t), t), t \in [t_1, t_2]\}$ is controlled by $TV^{1/3} u_\nu$ along the same piece of curve. Notice also that the sign of Z_ν is constant along a 2-characteristic by Proposition 4.10.

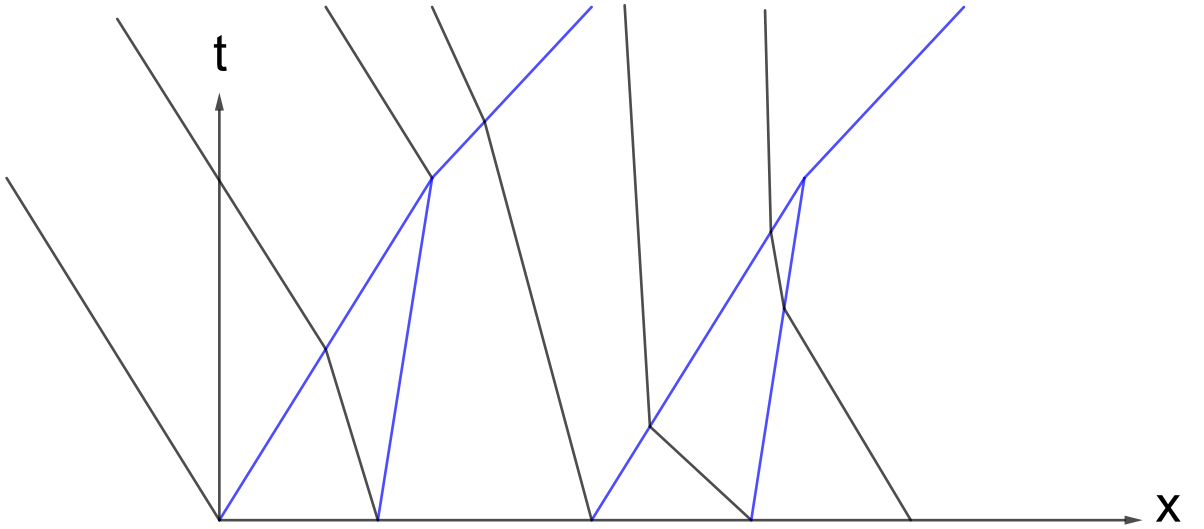


Figure 2: Wave front tracking algorithm. For the picture, $f'(u) > 0 > a(u)$, thus the 1-waves go to the right and the 2-waves to the left. The 1-wave are blue. Notice that these are not affected by the interaction with the 2-waves. The 2-waves are black. The 2-waves are affected by the interaction.

First, the L^∞ norm of Z is bounded on the 2-characteristic and, second, the estimation on $TV Z_\nu$ follows. The total variation is additive and the fractional total variation is sub-additive [24], so adding all these estimates, on the whole 2-characteristic starting at $x = x_0$ $\Gamma(x_0) = \{(\gamma_n u(x_0, t), t), t > 0\}$ we have the estimate,

$$\begin{aligned} \|Z\|_{\infty, \Gamma(x_0)} &\leq |Z_{0,\nu}(x_0)| \exp(\mathcal{O}(TV^{1/3} u_\nu[\Gamma(x_0)])), \\ &\leq \|Z_0\|_\infty \exp(\mathcal{O}(TV^{1/3} u_\nu[\Gamma(x_0)])), \\ TV Z_\nu[\Gamma(x_0)] &\leq \|Z_0\|_\infty \Psi(\mathcal{O}(TV^{1/3} u_\nu[\Gamma(x_0)])), \\ \Psi(x) &= x \exp(x). \end{aligned}$$

$\Gamma(x_0)$ is the space like curve for the 1-characteristics of (5.8) so

$$TV^{1/3} u_\nu[\Gamma(x_0)] \leq TV^{1/3} u_{0,\nu} \leq TV^{1/3} u_0.$$

This yields an L^∞ and BV bound for Z_ν along 2-characteristics uniform with respect to ν ,

$$\|Z_\nu\|_\infty \leq \|Z_0\|_\infty \exp(\mathcal{O}(1)TV^{1/3} u_0), \quad (5.14)$$

$$TV Z_\nu[\Gamma(x_0)] \leq \|Z_0\|_\infty \Psi(\mathcal{O}(TV^{1/3} u_0)). \quad (5.15)$$

Notice also that the positivity is preserved. If $\inf v_0 > 0$ then $\inf Z_0 > 0$ and by similar arguments presented above $\inf Z > \inf Z_0 \exp(\mathcal{O}(1)TV^{1/3} u_0) > 0$ where the constant

$\mathcal{O}(1)$ is negative. Finally, a L^∞ bound (respectively a positivity) for Z_ν , yields a L^∞ bound (respectively a positivity) for v_ν .

We can say more about the stratified structure of Z along the 2-characteristics [17, 41]. However, for the triangular system, the L^∞ bound of Z_ν and hence of v_ν is enough to pass to the weak limit in (5.17) since the left hand side is linear with respect to v_ν . The uniform BV bound on Z along the 2-characteristics can be used to recover a strong trace at $t = 0$, like in [15] (at $x = 0$).

5.3 Passage to the limit in the weak formulation

Passing to the strong-weak limit in the equation (1.2) which is linear with respect to v allows to get a weak solution. The error of consistency of the scheme has to be studied.

Lemma 5.1. *The error of consistency E_ν of the scheme is only on the transport equation (5.17) and satisfies,*

$$\partial_t u_\nu + \partial_x f_\nu(u_\nu) = 0, \quad (5.16)$$

$$\partial_t v_\nu + \partial_x (a(u_\nu)v_\nu) = \text{div}_{x,t} E_\nu. \quad (5.17)$$

Moreover, the following estimate holds,

$$E_\nu = \mathcal{O}(TV^{1/3}u_0) \in L^\infty((0, +\infty)_t, L^1(\mathbb{R}_x, \mathbb{R}^2)) \quad (5.18)$$

This error E_ν converges towards 0 if u_0 belongs to BV^s with $s > 1/3$ or, if the flux f is strictly convex and $s \geq 1/3$.

There is no error of consistency in the independant scalar equation for u_ν (5.16) [29]. The consistency error is hidden in the flux f_ν . If we replace the piecewise-linear flux f_ν by the exact flux f , then the consistency error $\partial_x(f(u_\nu) - f_\nu(u_\nu))$ appears at the right hand side of (5.16).

The convergence towards 0 of the error of consistency for systems is usually done in BV , [22, p. 126], [43, p. 305]. With less regularity in BV^s , $s < 1$, new features in the estimate of the error of consistency occur.

Proof. The approximate solution is piecewise constant, presenting only contact discontinuities or shock waves for u_ν . Since $f_\nu = f$ on the grid $\nu^{-1}\mathbb{Z}$ a weak jump solution of (5.16) is also a weak solution of the scalar conservation law with the exact flux f . Moreover, the approximate speed s_ν equals the exact speed. If the jump is entropic for the exact flux f , then $E_\nu = 0$ because v is chosen on the exact Lax curve. The problem of consistency occurs only when the jump is not entropic for the exact flux. That is the jump corresponds to a piecewise constant approximation of a rarefaction. In this case, let u_0 be the left state, u_m the right one and u_i the intermediary states, $i = 1, \dots, m-1$. Then, due to the approximation of a rarefaction wave fan by many non entropic small jumps with the size ν^{-1} we have $|u_{i+1} - u_i| = \nu^{-1}$. Localising the error at the front i between u_i and u_{i+1} , $E_\nu = \mathcal{O}(u_{i+1} - u_i)^3 = \mathcal{O}(\nu^{-3})$, due to the error between the rarefaction curve and the Rankine-Hugoniot curve, Proposition 4.1. Thus, adding this local error yields the estimate (5.18).

Now, if $u_0 \in BV^s$ with $s > 1/3$ then we can split the local error term with the notation $p = 1/s = 3 - \eta$, $\eta > 0$.

$$|u_{i+1} - u_i|^3 = |u_{i+1} - u_i|^p |u_{i+1} - u_i|^\eta = |u_{i+1} - u_i|^p \nu^{-\eta}, \quad (5.19)$$

$$\int_{\mathbb{R}} |E_\nu| dx = \mathcal{O}(1) \sum_{\text{rarefaction fronts}} |u_{i+1} - u_i|^p \nu^{-\eta} \leq \mathcal{O}(1) \nu^{-\eta} TV^s u_0. \quad (5.20)$$

The sum is taken over all rarefaction wave fronts. Thus the error of consistency converges towards zero in L^1 when $\nu \rightarrow +\infty$.

Now, consider the case $s = 1/3$ when the exact flux is strictly convex. The number of rarefaction wave fans is not increasing and bounded by the number N_0 of initial Riemann problems at $t = 0$ in the WFT. A rarefaction wave fan with amplitude R is splitted in many fronts $m/\nu = R$, i.e.

$$m = \nu R.$$

The local estimate in L^1 of the error over all the wave fan is, up to a constant,

$$\sum |u_{i+1} - u_i|^3 = m \nu^{-3} = R^3 m^{-2}. \quad (5.21)$$

Enumerating the rarefaction wave fan with the amplitude R_i and its number of fronts m_i yields,

$$\begin{aligned} \int_{\mathbb{R}} |E_\nu| dx &\leq \mathcal{O}(1) \sum_{i \leq N_0} m_i \nu^{-3} \leq \nu^{-2} \mathcal{O}(1) \sum_{i \leq N_0} R_i \leq \mathcal{O}(1) \nu^{-2} N_0 \max_i R_i. \\ &\leq \mathcal{O}(1) \nu^{-2} N_0. \end{aligned} \quad (5.22)$$

The amplitude of the rarefaction are bounded uniformly due to the previous L^∞ estimates. Thus, it suffices to take $N_0 = N'_0 = \nu$ to have a vanishing error of consistency. \square

6 L^∞ blow-up for v when $u_0 \notin BV^{1/3}$

In this section, we provide a proof of Theorem 2.2 by constructing an example of blow-up at $t = 0+$ for the system (1.1)-(1.2). For this purpose, initial data u_0, Z_0 satisfy

$$\begin{cases} u_0 \in BV^{1/3-0}(\mathbb{R}), \\ Z_0 \in L^\infty(\mathbb{R}). \end{cases}$$

That means that the u_0 provided is in all BV^s for $s < 1/3$. Thus, the existence Theorem 2.1 is optimal. Our idea of construction is motivated by similar examples studied in [1, 5, 27, 34, 37].

Notice that when $v_0 \equiv 0$, i.e. $Z_0 \equiv 0$, no blow-up occurs since $(u, v) \equiv (u, 0)$ gives a global entropy solution where u the entropy solution associated to the L^∞ initial data u_0 . Thus, in the following construction, we have to avoid the value 0 for Z .

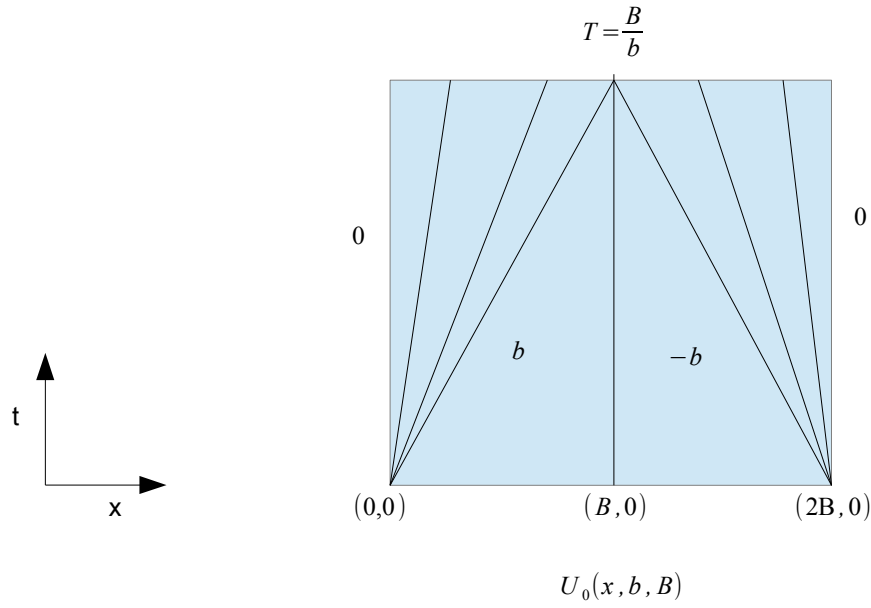


Figure 3: A typical building-block

Let us consider the system (1.1)-(1.2) with flux $f(u) = \frac{u^2}{2}$, $a(u) = u - 3$ and initial data u_0 as described below and $Z_0 \equiv 1$.

Let $x_0 = 0$ and $x_n = 1 - \frac{1}{2^n}$, $n \geq 1$. Let B_n be chosen such that

$$x_n = x_{n-1} + 2B_n,$$

that is, $B_n = \frac{1}{2^{n+1}}$. Let $b_n = \frac{1}{(n+8)^{\frac{1}{3}}}$ and we define

$$U_0(x, b, B) = b\mathbb{1}_{[0,B)}(x) - b\mathbb{1}_{[B,2B)}(x).$$

Using this we define the initial data u_0 as

$$u_0(x) = \sum_{n \geq 1} U_0(x - x_{n-1}, b_n, B_n).$$

The first interaction times T_n for the Riemann problems for the equation

$$\partial_t u + \partial_x f(u) = 0$$

with initial data u_0 is given by

$$T_n = \frac{B_n}{b_n} = \frac{(n+8)^{\frac{1}{3}}}{2^{n+1}}.$$

Note that the first interaction time T_n satisfies the relation

$$T_n > 1 - x_n. \tag{6.1}$$

The initial data u_0 , described above, clearly does not belong to $BV^{\frac{1}{3}}(\mathbb{R})$, but using Proposition 10 in [18], we can conclude that $u_0 \in BV^{1/3-0}(\mathbb{R})$.

Now as in the case of WFT described in the last section, we use the forward generalized characteristic for Z . Since $a(u) = u - 3$ and the first interaction times of the Riemann problems for u satisfy (6.1), it follows that the forward generalized characteristic for Z starting at the point $x_\infty := 1$ crosses infinitely many shocks before the first interaction of the waves in u .

As we have already seen from the nonlinear wave interactions, the L^∞ norm of Z does not change when it interacts with a 1-rarefaction wave.

Now, let us consider a left state u_- and a right state u_+ connected by a 1-shock wave. In our example, we have $u_- = b > 0$ and $u_+ = -b < 0$. Hence, the speed of the shock $s = 0$. Also by construction $0 < b_n < \frac{1}{2} < 1$ and hence we assume that the prototype b satisfies the same.

Now

$$\begin{aligned} \frac{Z_-}{Z_+} &= \frac{s - a(u_+)}{s - a(u_-)} \exp [A(u_-) - A(u_+)] \\ &= \frac{a(u_+)}{a(u_-)} \exp [A(u_-) - A(u_+)] \\ &= \frac{u_+ - 3}{u_- - 3} \exp [A(u_-) - A(u_+)] \\ &= \frac{3 + b}{3 - b} \exp [A(u_-) - A(u_+)]. \end{aligned} \tag{6.2}$$

Also

$$A'(u) = \frac{a'(u)}{a(u) - f'(u)} = \frac{1}{(u - 3) - u} = -\frac{1}{3} < 0,$$

and hence

$$A(u_-) - A(u_+) = \int_{u_+}^{u_-} A'(u) du = -\frac{1}{3}(u_- - u_+) = -\frac{2b}{3}.$$

Therefore from (6.2), we have

$$\frac{Z_-}{Z_+} = \frac{3 + b}{3 - b} e^{-\frac{2b}{3}}. \tag{6.3}$$

We show that for b positive small enough,

$$\frac{3 + b}{3 - b} e^{-\frac{2b}{3}} > 1. \tag{6.4}$$

and therefore $Z_- > Z_+$. Thus, Z increases in strength as the forward 2-generalized characteristic crosses a shock (from right to left).

Notice that we simply need that Z increases when $b \sim 0$. It is for very small oscillations

that Z blows-up. Inequality (6.4) follows from a Taylor expansion up to the third order:

$$\begin{aligned}
(3+b)\exp(-2b/3) &= (3+b)(1 - 2b/3 + 1/2(2b/3)^2 - 1/6(2b/3)^3 + \mathcal{O}(b^4)) \\
&= 3 + (1-2)b + (-2/3 + 2/3)b^2 + (2/9 - 4/3^4)b^3 + \mathcal{O}(b^4) \\
&= 3 - b + 14/81b^3 + \mathcal{O}(b^4) \\
&> 3 - b
\end{aligned}$$

for b sufficiently small and hence $Z_- > Z_+ > 0$.

Moreover, Inequality (6.4) is valid for all $b \in (0, 1)$. To see this we consider

$$\frac{3+b}{3-b}e^{-\frac{2b}{3}} - 1 = \frac{(3+b)e^{-\frac{2b}{3}} - (3-b)}{3-b}.$$

Define the function $p(b) := (3+b)e^{-\frac{2b}{3}} - (3-b)$. Then $p(0) = 0$ and

$$\begin{aligned}
p'(b) &= e^{-\frac{2b}{3}} + (3+b)\left(-\frac{2b}{3}\right)e^{-\frac{2b}{3}} + 1 \\
&= e^{-\frac{2b}{3}} + 1 - 2be^{-\frac{2b}{3}} - \frac{2b^2}{3}e^{-\frac{2b}{3}} \\
&= \underbrace{(1-2b)e^{-\frac{2b}{3}}}_{>0} + \underbrace{\left[1 - \frac{2b^2}{3}e^{-\frac{2b}{3}}\right]}_{:=g(b)}
\end{aligned}$$

Now $g(b) = 1 - \frac{2b^2}{3}e^{-\frac{2b}{3}}$ and so $g(0) = 1$.

And $g'(b) = -\frac{4b}{3}e^{-\frac{2b}{3}} - \frac{2b^2}{3}\left(-\frac{2b}{3}\right)e^{-\frac{2b}{3}} = e^{-\frac{2b}{3}}\left[-\frac{4b}{3} + \frac{4b^3}{9}\right] < 0$, as $0 < b < 1$. Also

$$g(1) = 1 - \frac{2}{3}e^{-\frac{2}{3}} = \frac{3 - 2e^{-\frac{2}{3}}}{3} > \frac{2 - 2e^{-\frac{2}{3}}}{3} = \frac{2}{3}(1 - e^{-\frac{2}{3}}) > 0.$$

Thus, $g(b) > 0$ for $b \in (0, 1)$. Using this, we see that

$$p'(b) > 0, \quad b \in (0, 1)$$

and hence $p(b) > 0$, $b \in (0, 1)$.

Therefore Inequality (6.4) is satisfied for all $b \in (0, 1)$ and so

$$Z_- > Z_+.$$

Thus, due to an interaction with a 1-shock wave, there is a change of order $[u]^3$ in the L^∞ norm of Z . Since

$$|[u]^3| = \sum_{n \geq 1} \left(\frac{2}{(n+8)^{\frac{1}{3}}} \right)^3 = 8 \sum_{n \geq 1} \frac{1}{(n+8)} = +\infty,$$

we find that the L^∞ norm for Z blows up.

Remark 6.1. We can also replace $n + 8$ by n in the construction above. This would mean that the interaction time for the first few waves in u would be before the forward generalized characteristic for Z reaches them, but the generalized characteristic would still have to cross infinitely many shock waves and hence our conclusion remains valid.

Remark 6.2. Let $\Gamma(x_0) = \{(\gamma(x_0, t), t), t \geq 0\}$ be the 2-characteristic issued from $x = x_0$ at $t = 0$. The solution is well defined under $\Gamma(1)$ that is on the set $\{(x, t), x < \gamma(1, t), t \geq 0\}$. But, over $\Gamma(1)$, $\{(x, t), 1 > x > \gamma(1, t), t > 0\}$, Z and v blow up, $v = +\infty$.

Remark 6.3. This example does not contradict the Lax-Oleinik smoothing effect ([50]) as the blow-up for Z occurs only at $t = 0$, that is, there is an immediate blow-up. Such a blow-up is not possible for a time $t_0 > 0$ due to the BV smoothing of u .

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