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DESCRIPTION OF THE SMOOTHING EFFECTS OF SEMIGROUPS GENERATED BY FRACTIONAL ORNSTEIN-UHLENBECK OPERATORS AND SUBELLIPTIC ESTIMATES

PAUL ALPHONSE

ABSTRACT. We study semigroups generated by general fractional Ornstein-Uhlenbeck operators acting on $L^2(\mathbb{R}^n)$. We characterize geometrically the partial Gevrey-type smoothing properties of these semigroups and we sharply describe the blow-up of the associated seminorms for short times, generalizing the hypoelliptic and the quadratic cases. As a byproduct of this study, we establish partial subelliptic estimates enjoyed by fractional Ornstein-Uhlenbeck operators on the whole space by using interpolation theory.

1. INTRODUCTION

We study semigroups generated by fractional Ornstein-Uhlenbeck operators. These are non-local and non-selfadjoint operators in general, sum of a fractional diffusion and a linear transport operator. More precisely, the fractional Ornstein-Uhlenbeck operator associated with the positive real number $s > 0$ and the real $n \times n$ matrices B and Q , with Q symmetric positive semidefinite, is given by

$$(1.1) \quad \mathcal{P} = \frac{1}{2} \operatorname{Tr}^s(-Q\nabla_x^2) + \langle Bx, \nabla_x \rangle, \quad x \in \mathbb{R}^n.$$

In this definition, the operator $\operatorname{Tr}^s(-Q\nabla_x^2)$ stands for the Fourier multiplier associated with the symbol $\langle Q\xi, \xi \rangle^s$, and $\langle Bx, \nabla_x \rangle$ denotes the following differential operator

$$\langle Bx, \nabla_x \rangle = \sum_{i=1}^n \sum_{j=1}^n B_{i,j} x_j \partial_{x_i}, \quad B = (B_{i,j})_{1 \leq i,j \leq n}.$$

The operator \mathcal{P} equipped with the domain

$$(1.2) \quad D(\mathcal{P}) = \{u \in L^2(\mathbb{R}^n) : \mathcal{P}u \in L^2(\mathbb{R}^n)\},$$

is known from the work [2] (Theorem 1.1) to generate a strongly continuous semigroup $(e^{-t\mathcal{P}})_{t \geq 0}$ on $L^2(\mathbb{R}^n)$. The purpose of this work is to understand how the possible non-commutation phenomena between the diffusion and the transport parts of the operator \mathcal{P} allow the associated evolution operators $e^{-t\mathcal{P}}$ to enjoy smoothing properties in specific directions of the space we aim at sharply and completely describing. More precisely, our objective is to exhibit the vector spaces $\Sigma \subset \mathbb{R}^n$ containing the smoothing directions of the operators $e^{-t\mathcal{P}}$, i.e. such that for all $t > 0$, $m \geq 1$ and $\xi_1, \dots, \xi_m \in \Sigma$, there exists a positive constant $c_{t,m,\xi_1,\dots,\xi_m} > 0$ such that for all $u \in L^2(\mathbb{R}^n)$,

$$(1.3) \quad \|\langle \xi_1, \nabla_x \rangle \dots \langle \xi_m, \nabla_x \rangle e^{-t\mathcal{P}} u\|_{L^2(\mathbb{R}^n)} \leq c_{t,m,\xi_1,\dots,\xi_m} \|u\|_{L^2(\mathbb{R}^n)},$$

and to sharply describe the dependence of the constant $c_{t,m,\xi_1,\dots,\xi_m}$ with respect to $m \geq 1$ and $\xi_1, \dots, \xi_m \in \Sigma$, at least for short times $0 < t \ll 1$. This kind of estimates have already been obtained with a precise control of the short times asymptotics in the case $s = 1$ in the work [3] (Example 2.7). On the other hand, the study of the smoothing properties of semigroups generated by fractional Ornstein-Uhlenbeck operators has already been performed in the work [2] (Theorem 1.2) in a hypoelliptic setting, i.e. when the

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matrices B and Q satisfy an algebraic condition known as the Kalman rank condition and defined just after (1.4). The present work aims at unifying these two results in a general setting for the matrices B and Q , and for general positive real numbers $s > 0$. An application to the study of the partial subelliptic estimates enjoyed by the operator P on the whole space will also be given, generalizing results from the two same works [3] (Example 2.11) and [2] (Corollary 1.15).

The Ornstein-Uhlenbeck operators (case $s = 1$) and their associated semigroups have been very much studied in the last two decades. The structure of these operators was analyzed in [11], while their spectral properties were investigated in [16, 17]. The smoothing properties of the associated semigroups were studied in [6, 7, 8, 13, 16, 17] and some global hypoelliptic estimates were derived in [4, 7, 8, 17]. We also refer the reader to [5, 14] where the Ornstein-Uhlenbeck operators are studied while acting on spaces of continuous functions. We recall from these works that among the different characterizations of the hypoellipticity of these operators is the Kalman rank condition

$$(1.4) \quad \text{Rank} [B \mid \sqrt{Q}] = n,$$

where the $n \times n^2$ matrix

$$[B \mid \sqrt{Q}] = [\sqrt{Q}, B\sqrt{Q}, \dots, B^{n-1}\sqrt{Q}],$$

is obtained by writing consecutively the columns of the matrices $\sqrt{Q}, B\sqrt{Q}, \dots, B^{n-1}\sqrt{Q}$, in addition to different assertions derived from the classical Hörmander's work on the hypoellipticity of differential operators, see e.g. the introduction of [2]. The Kalman rank condition will naturally appear in the following.

Outline of the work. In Section 2, we present the main results contained in this work. Section 3 is devoted to the study of the smoothing properties of semigroups generated by fractional Ornstein-Uhlenbeck operators. Section 4 is an Appendix containing the proofs of some technical results.

Notations. The following notations and conventions will be used all over the work:

1. The canonical Euclidean scalar product of \mathbb{R}^n is denoted by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ stands for the associated canonical Euclidean norm. The orthogonality with respect to this scalar product, as for it, is denoted F^\perp for all $F \subset \mathbb{R}^n$.
2. The inner product of $L^2(\mathbb{R}^n)$ is defined for all $u, v \in L^2(\mathbb{R}^n)$ by

$$\langle u, v \rangle_{L^2} = \int_{\mathbb{R}^n} u(x) \overline{v(x)} \, dx,$$

while $\|\cdot\|_{L^2}$ stands for the associated norm.

3. For all function $u \in \mathcal{S}(\mathbb{R}^n)$, the Fourier transform of u is denoted by \widehat{u} and defined by

$$\widehat{u}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} u(x) \, dx.$$

With this convention, Plancherel's theorem states that

$$\forall u \in \mathcal{S}(\mathbb{R}^n), \quad \|\widehat{u}\|_{L^2}^2 = (2\pi)^n \|u\|_{L^2}^2.$$

4. The Japanese bracket $\langle \cdot \rangle$ is defined for all $\xi \in \mathbb{R}^n$ by $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$.
5. For all continuous function $F : \mathbb{R}^n \rightarrow \mathbb{C}$, the notation $F(D_x)$ is used to denote the Fourier multiplier associated with the symbol $F(\xi)$.

2. MAIN RESULTS

This section is devoted to present the main results contained in this work.

2.1. Regularizing effects. We begin by sharply describing the smoothing properties of the evolution operators $e^{-t\mathcal{P}}$ generated by the fractional Ornstein-Uhlenbeck operator \mathcal{P} . To that end, we consider the finite-dimensional vector space $S \subset \mathbb{R}^n$ defined by the following intersection of kernels

$$(2.1) \quad S = \bigcap_{j=0}^{n-1} \text{Ker} (\sqrt{Q}(B^T)^j).$$

The structure of this vector space S , intrinsically linked to the matrices B and Q , will play a key role in the study of fractional Ornstein-Uhlenbeck operators and their associated semigroups. Among others, we will prove that its canonical Euclidean orthogonal contains the smoothing directions of the operators $e^{-t\mathcal{P}}$, i.e. $\Sigma = S^\perp$ is the unique maximal vector space answering the problem presented in the introduction. By definition, we may consider the smallest integer $0 \leq r \leq n-1$ satisfying

$$(2.2) \quad S = \bigcap_{j=0}^r \text{Ker} (\sqrt{Q}(B^T)^j).$$

This integer $0 \leq r \leq n-1$ will also play a key role in the following.

First, we begin by checking that any smoothing direction of the operators $e^{-t\mathcal{P}}$ is contained in S^\perp the canonical Euclidean orthogonal of the vector space S .

Theorem 2.1. *Let \mathcal{P} be the fractional Ornstein-Uhlenbeck operator defined in (1.1) and equipped with the domain (1.2). We consider S the vector space associated with the matrices B and Q defined in (2.1). If there exist $t > 0$ and $\xi_0 \in \mathbb{R}^n$ such that the operator $\langle \xi_0, \nabla_x \rangle e^{-t\mathcal{P}}$ is bounded on $L^2(\mathbb{R}^n)$, then $\xi_0 \in S^\perp$.*

Notice that the directions of \mathbb{R}^n in which the evolution operators $e^{-t\mathcal{P}}$ can be differentiated depend on the matrices B and Q but not on the positive real number $s > 0$.

In order to sharply describe the constant $c_{t,m,\xi_1,\dots,\xi_m} > 0$ appearing in (1.3) with respect to the vectors ξ_1, \dots, ξ_m in short times $0 < t \ll 1$, we need to introduce the notion of index of any vector of S^\perp . To that end, we consider the finite-dimensional vector spaces $V_0, \dots, V_r \subset \mathbb{R}^n$ defined for all $k \in \{0, \dots, r\}$ by

$$(2.3) \quad V_k = \bigcap_{j=0}^k \text{Ker} (\sqrt{Q}(B^T)^j).$$

Notice that the orthogonal complement of these vector spaces $V_0^\perp, \dots, V_r^\perp$ form an increasing family which satisfies, according to (2.2),

$$(2.4) \quad V_0^\perp \subsetneq \dots \subsetneq V_r^\perp = S^\perp.$$

This stratification of S^\perp allows to define the *index* of any vector $\xi_0 \in S^\perp$ by

$$(2.5) \quad k_{\xi_0} = \min \{0 \leq k \leq r : \xi_0 \in V_k^\perp\}.$$

This notion of index plays a key role in the understanding of the blow-up for short times of the seminorms associated with the smoothing effects of the evolution operators $e^{-t\mathcal{P}}$, as illustrated in the following theorem which is the main result contained in this work.

Theorem 2.2. *Let \mathcal{P} be the fractional Ornstein-Uhlenbeck operator defined in (1.1) and equipped with the domain (1.2). We consider S the vector space associated with the matrices B and Q , and $0 \leq r \leq n-1$ the smallest integer such that (2.2) holds. There exist some positive constants $c > 1$ and $T > 0$ such that for all $m \geq 1$, $\xi_1, \dots, \xi_m \in S^\perp$, $0 < t < T$ and $u \in L^2(\mathbb{R}^n)$,*

$$\| \langle \xi_1, \nabla_x \rangle \dots \langle \xi_m, \nabla_x \rangle e^{-t\mathcal{P}} u \|_{L^2} \leq \frac{c^m e^{\frac{1}{2} \text{Tr}(B)t}}{t^{k_{\xi_1} + \dots + k_{\xi_m} + \frac{m}{2s}}} \left(\prod_{j=1}^m |\xi_j| \right) (m!)^{\frac{1}{2s}} \|u\|_{L^2},$$

where $0 \leq k_{\xi_j} \leq r$ denotes the index of the vector $\xi_j \in S^\perp$.

This result shows that the semigroup $(e^{-t\mathcal{P}})_{t \geq 0}$ enjoys partial Gevrey-type smoothing properties, the short-time asymptotics of m differentiations of this semigroup in the directions generated by the vectors $\xi_1, \dots, \xi_m \in S^\perp$ being given by $\mathcal{O}(t^{-k_{\xi_1} - \dots - k_{\xi_m} - \frac{m}{2s}})$, and therefore depend on the indexes $k_{\xi_1}, \dots, k_{\xi_m}$ of those vectors. The justification of the terminology ‘‘partial Gevrey-type smoothing properties’’ is the following. In the particular case where the vector space S is reduced to $\{0\}$, i.e. when the Kalman rank condition (1.4) holds according to Lemma 4.2, Theorem 2.2 implies that the evolution operators $e^{-t\mathcal{P}}$ can be differentiated in any direction of \mathbb{R}^n . The behavior of the estimates given by Theorem 2.2 with respect to $m \geq 1$ shows that the semigroup $(e^{-t\mathcal{P}})_{t \geq 0}$ enjoys Gevrey-type regularity for all $t > 0$, i.e. Gevrey regularity when $0 < s < 1/2$, analytic regularity when $s = 1/2$ and ultra-analytic regularity when $s > 1/2$. In the general case, the evolution operators $e^{-t\mathcal{P}}$ can be differentiated only in specific directions of \mathbb{R}^n and one can say that the semigroup $(e^{-t\mathcal{P}})_{t \geq 0}$ only enjoys ‘‘partial Gevrey-type regularity’’ for all $t > 0$.

Theorem 2.2 is the exact extension of [3] (Example 2.11) in the particular case of semigroups generated by Ornstein-Uhlenbeck operators (case $s = 1$). More generally, one of the objectives of the work [3] is to describe the smoothing properties of the evolution operators e^{-tq^w} generated by accretive quadratic operators $q^w(x, D_x)$. Theorem 2.6 in [3], which is quite a lot similar to Theorem 2.2, states that there exist some positive constants $c > 1$ and $T > 0$ such that for all $0 < t \leq T$, $m \geq 1$, $X_1, \dots, X_m \in S^\perp$ and $u \in L^2(\mathbb{R}^n)$,

$$\|\langle X_1, X \rangle^w \dots \langle X_m, X \rangle^w e^{-tq^w} u\|_{L^2} \leq \frac{c^m}{t^{k_{\xi_1} + \dots + k_{\xi_m} + \frac{m}{2}}} \left(\prod_{j=1}^m |\xi_j| \right) \sqrt{m!} \|u\|_{L^2},$$

with

$$\langle X_j, X \rangle^w = \langle x_j, x \rangle + \langle \xi_j, D_x \rangle, \quad X_j = (x_j, \xi_j) \in S^\perp, \quad D_x = -i\nabla_x,$$

where $S \subset \mathbb{R}^{2n}$ is a vector subspace of the phase space intrinsically linked the accretive quadratic operator $q^w(x, D_x)$, and k_{X_1}, \dots, k_{X_m} are the indexes of the vectors $X_1, \dots, X_m \in S^\perp$, similarly defined than (2.5). These estimates for accretive quadratic semigroups are obtained by exploiting the polar decomposition of the evolution operators e^{-tq^w} , i.e. a splitting formula for these operators as the product of a selfadjoint operator and a unitary operator in $L^2(\mathbb{R}^n)$, which the main result of the paper [3] (Theorem 2.1). The strength of Theorem 2.2 is to obtain the same kind of estimates for all $s > 0$ and to track the various constants with respect to this parameter. This theorem will be proven by using the following explicit formula for the operators $e^{-t\mathcal{P}}$, coming from the work [2] (Theorem 1.1), which can also be seen as a polar decomposition:

$$e^{-t\mathcal{P}} = \exp\left(-\frac{t}{2} \int_0^1 |\sqrt{Q} e^{\alpha t B^T} D_x|^{2s} d\alpha\right) e^{-t\langle Bx, \nabla_x \rangle}.$$

The strategy of using representations of the above type has already been used in the literature. For example, in the work [9], it allowed to derive some L^p estimates for the maximal fractional regularity for degenerate non-local Kolmogorov operators.

Assuming that the vector space S is reduced to zero anew, we notice that Theorem 2.2 and a straightforward induction imply in particular that for all $\alpha \in \mathbb{N}^n$, $0 < t < T$ and $u \in L^2(\mathbb{R}^n)$,

$$(2.6) \quad \|\partial_x^\alpha (e^{-t\mathcal{P}} u)\|_{L^2(\mathbb{R}^n)} \leq \frac{c^{1+|\alpha|}}{t^{|\alpha|(\frac{1}{2s} + r)}} (\alpha!)^{\frac{1}{2s}} \|u\|_{L^2(\mathbb{R}^n)},$$

since the index of any vector $\xi_0 \in S^\perp$ satisfies $0 \leq k_{\xi_0} \leq r$. These Gevrey-type regularizing effect estimates have already been proven in [2] (Theorem 1.1), with the same blow-up of the associated seminorms for short times $t \rightarrow 0^+$.

In a general setting on the matrices B and Q , Theorem 2.2 does not provide any long time behavior of the norm of the linear operators $\langle \xi_1, \nabla_x \rangle \dots \langle \xi_m, \nabla_x \rangle e^{-t\mathcal{P}}$. However, this long-time study can be performed in particular cases. For example, notice from Plancherel’s

theorem and a straightforward study of function that for all $m \geq 1$, $\xi_1, \dots, \xi_m \in \mathbb{R}^n$, $t > 0$ and $u \in L^2(\mathbb{R}^n)$,

$$\|\langle \xi_1, \nabla_x \rangle \dots \langle \xi_m, \nabla_x \rangle e^{-t(-\Delta_x)^s} u\|_{L^2} \leq \frac{2^{-\frac{m}{2s}}}{t^{\frac{m}{2s}}} \left(\prod_{j=1}^m |\xi_j| \right) (m!)^{\frac{1}{2s}} \|u\|_{L^2}.$$

We can therefore completely describe the long-time asymptotics of the smoothing properties of semigroups generated by fractional Laplacians. By adapting the proof of Theorem 2.2, one can obtain estimates in the more general case where the matrix B is nilpotent, which apply in particular to semigroups generated by fractional Kolmogorov operators, some operators naturally appearing in the theory of kinetic equations defined as follows. Given $s > 0$, the associated fractional Kolmogorov operator \mathcal{K} is the operator defined by

$$(2.7) \quad \mathcal{K} = (-\Delta_v)^s + v \cdot \nabla_x, \quad (x, v) \in \mathbb{R}^{2n}.$$

This is the fractional Ornstein-Uhlenbeck operator associated with the matrices B and Q respectively given by

$$B = \begin{pmatrix} 0_n & I_n \\ 0_n & 0_n \end{pmatrix} \quad \text{and} \quad Q = 2^{\frac{1}{s}} \begin{pmatrix} 0_n & 0_n \\ 0_n & I_n \end{pmatrix}.$$

We study the operator \mathcal{K} equipped with the domain

$$(2.8) \quad D(\mathcal{K}) = \{u \in L^2(\mathbb{R}^{2n}) : \mathcal{K}u \in L^2(\mathbb{R}^{2n})\}.$$

A straightforward calculus shows that the vector space S associated with the matrices B and Q is given by

$$S = \text{Ker}(\sqrt{Q}) \cap \text{Ker}(\sqrt{Q}B^T) = \{0\}.$$

The associated integer $0 \leq r \leq 2n - 1$ defined in (2.2) is therefore equal to 1. The smoothing properties of the semigroup generated by the operator \mathcal{K} are presented in the following result.

Proposition 2.3. *Let \mathcal{K} be the fractional Kolmogorov operator defined in (2.7) and equipped with the domain (2.8). There exists a positive constant $c > 1$ such that for all $(\alpha, \beta) \in \mathbb{N}^{2n}$, $t > 0$ and $u \in L^2(\mathbb{R}^{2n})$,*

$$\|\partial_x^\alpha \partial_v^\beta (e^{-t\mathcal{K}}u)\|_{L^2} \leq \frac{c^{|\alpha|+|\beta|}}{t^{(1+\frac{1}{2s})|\alpha|+\frac{|\beta|}{2s}}} (\alpha!)^{\frac{1}{2s}} (\beta!)^{\frac{1}{2s}} \|u\|_{L^2}.$$

The long-time study of the smoothing properties of semigroups generated by fractional Ornstein-Uhlenbeck operators is an interesting question that remains open.

2.2. Subelliptic estimates. The study of the smoothing properties of semigroups generated by fractional Ornstein-Uhlenbeck operators allows to obtain partial subelliptic estimates enjoyed by these operators on the whole space.

Theorem 2.4. *Let \mathcal{P} be the fractional Ornstein-Uhlenbeck operator defined in (1.1) and equipped with the domain (1.2). We consider $0 \leq r \leq n - 1$ the smallest integer such that (2.2) holds. There exists a positive constant $c > 0$ such that for all $u \in D(\mathcal{P})$,*

$$\sum_{k=0}^r \|\langle \sqrt{Q}(B^T)^k D_x \rangle^{\frac{2s}{1+2ks}} u\|_{L^2} \leq c(\|\mathcal{P}u\|_{L^2} + \|u\|_{L^2}).$$

The entire proof of Theorem 2.4 will not be presented in the present paper since it is completely detailed in the author's thesis work [1] (Chapter 4, Subsection 4.3). Nevertheless, let us present its main steps. Theorem 2.4 can be directly obtained by exploiting the following estimates, established in (3.21) in Subsection 3.2 while proving Theorem 2.2, holding for all $k \in \{0, \dots, r\}$, $q > 0$, $0 < t \ll 1$ and $u \in L^2(\mathbb{R}^n)$,

$$(2.9) \quad \|\langle \sqrt{Q}(B^T)^k D_x \rangle^q e^{-t\mathcal{P}}u\|_{L^2} \leq \left(\frac{c}{t^{k+\frac{1}{2s}}} \right)^q e^{\frac{1}{2} \text{Tr}(B)t} q^{\frac{q}{2s}} \|u\|_{L^2},$$

and elements of interpolation theory. Precisely, the link between the above smoothing estimates and the subelliptic estimates stated in Theorem 2.4 is made by the following result

Proposition 2.5 (Proposition 2.7 in [8]). *Let X be a Hilbert space and $A : D(A) \subset X \rightarrow X$ be a maximal accretive operator such that $(-A, D(A))$ is the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$. Assume that there exists a Banach space $E \subset X$, $\rho > 1$ and $C > 0$ such that*

$$\forall t > 0, \quad \|T(t)\|_{\mathcal{L}(X,E)} \leq \frac{C}{t^\rho},$$

and that $t \mapsto T(t)u$ is measurable with values in E for each $u \in X$. Then, the following continuous inclusion holds

$$D(A) \subset (X, E)_{\frac{1}{\rho}, 2},$$

where $(X, E)_{\frac{1}{\rho}, 2}$ denotes the space obtained by real interpolation.

The estimate (2.9) used with $q = 1 + \lfloor 2s \rfloor$ and Proposition 2.5 applied with the Hilbert spaces $X = L^2(\mathbb{R}^n)$ and

$$E = \mathcal{H}_k = \{u \in L^2(\mathbb{R}^n) : \Lambda_k^q u \in L^2(\mathbb{R}^n)\}, \quad \text{with } \Lambda_k = \langle \sqrt{Q}(B^T)^k D_x \rangle,$$

allow to obtain that for all $k \in \{0, \dots, r\}$, the following continuous inclusion holds

$$(2.10) \quad D(\mathcal{P}) \subset (L^2(\mathbb{R}^n), \mathcal{H}_k)_{\theta, 2} \quad \text{with } \theta = \frac{2s}{q(1+2ks)} \in (0, 1).$$

By setting \mathcal{H}_k as the domain of the Fourier multiplier Λ_k^q , another interpolation result, namely [15] (Theorem 4.36), justifies that

$$(2.11) \quad (L^2(\mathbb{R}^n), \mathcal{H}_k)_{\theta, 2} = (D((\Lambda_k^q)^0), D((\Lambda_k^q)^1))_{\theta, 2} = D(\Lambda_k^{q\theta}).$$

The combinaison of the continuous inclusion (2.10) and (2.11) ends the proof of Theorem 2.4. Notice that this proof follows line to line the one presented in Subsections 5.1 and 5.2 in [2], where subelliptic estimates similar to the ones states in Theorem 2.4 are established when the Kalman rank condition (1.4) holds. This strategy of proof, inspired by the work [8], is also used in the paper [3] to establish subelliptic estimates enjoyed by accretive quadratic operators on the whole space, of which Ornstein-Uhlenbeck operators (case $s = 1$) are a particular case, see [3] (Example 2.11).

Let us assume that the Kalman rank condition (1.4) holds. In this case, the vector space S is reduced to zero as we check in Lemma 4.1 in Appendix, and it follows that

$$\forall \xi \in \mathbb{R}^n, \quad |\xi|^2 \lesssim \sum_{k=0}^r |\sqrt{Q}(B^T)^k \xi|^2.$$

We therefore deduce from this estimate, Theorem 2.4 and Plancherel's theorem that for all $u \in D(\mathcal{P})$,

$$\begin{aligned} \|\langle D_x \rangle^{\frac{2s}{1+2rs}} u\|_{L^2} &\lesssim \sum_{k=0}^r \|\langle \sqrt{Q}(B^T)^k D_x \rangle^{\frac{2s}{1+2rs}} u\|_{L^2} \\ &\lesssim \sum_{k=0}^r \|\langle \sqrt{Q}(B^T)^k D_x \rangle^{\frac{2s}{1+2ks}} u\|_{L^2} \lesssim \|\mathcal{P}u\|_{L^2} + \|u\|_{L^2}. \end{aligned}$$

When $s \geq 1$ is a positive integer, the operator \mathcal{P} therefore enjoys a global subelliptic estimate on the whole space

$$\|\langle D_x \rangle^{2s(1-\delta)} u\|_{L^2} \lesssim \|\mathcal{P}u\|_{L^2} + \|u\|_{L^2},$$

with a loss of

$$\delta = \frac{2rs}{1+2rs} > 0,$$

derivatives with respect to the elliptic case. The above estimate has already been established in [2] (Corollary 1.15).

3. REGULARIZING EFFECTS OF SEMIGROUPS GENERATED BY FRACTIONAL ORNSTEIN-UHLENBECK OPERATORS

Let \mathcal{P} be the fractional Ornstein-Uhlenbeck operator defined in (1.1) and equipped with the domain (1.2). We consider the vector space $S \subset \mathbb{R}^n$ defined in (2.1) and $0 \leq r \leq n-1$ the smallest integer such that (2.2) holds. This section is devoted to the study of the smoothing properties of the semigroup $(e^{-t\mathcal{P}})_{t \geq 0}$ generated by the operator \mathcal{P} . To that end, we will use the fact that the evolution operators $e^{-t\mathcal{P}}$ are explicitly given by the following explicit formula

$$(3.1) \quad e^{-t\mathcal{P}} = \exp \left(-\frac{t}{2} \int_0^1 |\sqrt{Q} e^{\alpha t B^T} D_x|^{2s} d\alpha \right) e^{-t\langle Bx, \nabla_x \rangle}.$$

This formula is a straightforward reformulation of [2] (Theorem 1.1). Indeed, this result states that the evolution operators $e^{-t\mathcal{P}}$ are given through their Fourier transform for all $t \geq 0$ by

$$\forall u \in L^2(\mathbb{R}^n), \quad \widehat{e^{-t\mathcal{P}}u} = \exp \left(-\frac{1}{2} \int_0^t |\sqrt{Q} e^{\tau B^T} \cdot|^{2s} d\tau \right) e^{\text{Tr}(B)t} \widehat{u}(e^{tB^T} \cdot).$$

Moreover, $e^{-t\langle Bx, \nabla_x \rangle} u = u(e^{-tB} \cdot)$ for all $u \in L^2(\mathbb{R}^n)$ and a straightforward calculus of inverse Fourier transform shows that for all $\xi \in \mathbb{R}^n$,

$$\widehat{u(e^{-tB} \cdot)}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} u(e^{-tB} x) dx = e^{\text{Tr}(B)t} \widehat{u}(e^{tB^T} \xi).$$

This proves that formula (3.1) actually holds. It shows that the operators $e^{-t\mathcal{P}}$ can be split as the product of a selfadjoint Fourier multiplier and an operator which is a similitude on $L^2(\mathbb{R}^n)$, since

$$(3.2) \quad \forall u \in L^2(\mathbb{R}^n), \quad \|e^{-t\langle Bx, \nabla_x \rangle} u\|_{L^2} = \|u(e^{-tB} \cdot)\|_{L^2(\mathbb{R}^n)} = e^{\frac{1}{2} \text{Tr}(B)t} \|u\|_{L^2}.$$

The splitting formula (3.1) can somehow be seen as the polar decomposition of the evolution operators $e^{-t\mathcal{P}}$.

3.1. Smoothing directions. The first part of this section is devoted to the proof of Theorem 2.1. Let $t > 0$ and $\xi_0 \in \mathbb{R}^n$. We assume that the linear operator $\langle \xi_0, \nabla_x \rangle e^{-t\mathcal{P}}$ is bounded on $L^2(\mathbb{R}^n)$. We aim at proving that $\xi_0 \in S^\perp$. Let us introduce the time-dependent real-valued symbol a_t defined for all $\xi \in \mathbb{R}^n$ by

$$(3.3) \quad a_t(\xi) = \frac{1}{2} \int_0^1 |\sqrt{Q} e^{\alpha t B^T} \xi|^{2s} d\alpha.$$

With this notation, the polar decomposition (3.1) can be written in the following way

$$(3.4) \quad e^{-t\mathcal{P}} = e^{-ta_t^w} e^{-t\langle Bx, \nabla_x \rangle},$$

where $e^{-ta_t^w}$ denotes the time-dependent Fourier multiplier associated with the time-dependent real-valued symbol e^{-ta_t} . We first notice from the splitting formula (3.4) that the boundedness on $L^2(\mathbb{R}^n)$ of the operator $\langle \xi_0, \nabla_x \rangle e^{-t\mathcal{P}}$ is equivalent to the boundedness of the Fourier multiplier $\langle \xi_0, \nabla_x \rangle e^{-ta_t^w}$, since the evolution operator $e^{-t\langle Bx, \nabla_x \rangle}$ is invertible on $L^2(\mathbb{R}^n)$. As a consequence, there exists a positive constant $c_{t, \xi_0} > 0$ depending on $t > 0$ and $\xi_0 \in \mathbb{R}^n$ such that

$$(3.5) \quad \forall u \in L^2(\mathbb{R}^n), \quad \|\langle \xi_0, \nabla_x \rangle e^{-ta_t^w} u\|_{L^2} \leq c_{t, \xi_0} \|u\|_{L^2}.$$

According to the orthogonal decomposition $\mathbb{R}^n = S \oplus S^\perp$, we write $\xi_0 = \xi_{0,S} + \xi_{0,S^\perp}$, with $\xi_{0,S} \in S$ and $\xi_{0,S^\perp} \in S^\perp$. For all $\lambda \geq 0$, we consider the Gaussian function $u_\lambda \in \mathcal{S}(\mathbb{R}^n)$ defined for all $x \in \mathbb{R}^n$ by

$$(3.6) \quad u_\lambda(x) = e^{i\lambda \langle \xi_{0,S}, x \rangle} e^{-|x|^2}.$$

The strategy will be to obtain upper and lower bounds for the norm

$$\|\langle \xi_0, \nabla_x \rangle e^{-ta_t^w} u_\lambda\|_{L^2},$$

and to consider the asymptotics when λ goes to $+\infty$ in order to conclude that $\xi_{0,S} \in S$ has to be equal to zero. An upper bound can be directly obtained since it follows from (3.5), (3.6) and the Cauchy-Schwarz inequality that for all $\lambda \geq 0$,

$$(3.7) \quad \|\langle \xi_0, \nabla_x \rangle e^{-ta_i^w} u_\lambda\|_{L^2} \leq c_{t,\xi_0} \|u_\lambda\|_{L^2} = c_{t,\xi_0} \|u_0\|_{L^2}.$$

Notice that the right-hand side of the above estimate does not depend on the parameter $\lambda \geq 0$. On the other hand, by using that for all $\lambda \geq 0$, the Fourier transform of the function u_λ is given for all $\xi \in \mathbb{R}^n$ by

$$(3.8) \quad \widehat{u}_\lambda(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} e^{i\lambda \langle \xi_{0,S}, x \rangle} e^{-|x|^2} dx = \int_{\mathbb{R}^n} e^{-i\langle x, \xi - \lambda \xi_{0,S} \rangle} e^{-|x|^2} dx = \widehat{u}_0(\xi - \lambda \xi_{0,S}),$$

we deduce from Plancherel's theorem that for all $\lambda \geq 0$,

$$(3.9) \quad \|\langle \xi_0, \nabla_x \rangle e^{-ta_i^w} u_\lambda\|_{L^2} = \frac{1}{(2\pi)^{\frac{n}{2}}} \|\langle \xi_0, \xi \rangle e^{-ta_i(\xi)} \widehat{u}_0(\xi - \lambda \xi_{0,S})\|_{L^2}.$$

Moreover, notice that for all $\xi \in \mathbb{R}^n$ and $\eta \in S$,

$$(3.10) \quad a_t(\xi + \eta) = \frac{1}{2} \int_0^1 |\sqrt{Q} e^{\alpha t B^T} (\xi + \eta)|^{2s} d\alpha = \frac{1}{2} \int_0^1 |\sqrt{Q} e^{\alpha t B^T} \xi|^{2s} d\alpha = a_t(\xi).$$

Indeed, the Cayley-Hamilton theorem applied to the matrix B^T shows that

$$\forall j \in \mathbb{N}, \forall \xi \in \mathbb{R}^n, \quad (B^T)^j \xi \in \text{Span}(\xi, B^T \xi, \dots, (B^T)^{n-1} \xi).$$

This proves that the space S is also given by the following infinite intersection of kernels

$$(3.11) \quad S = \bigcap_{j=0}^{+\infty} \text{Ker}(\sqrt{Q}(B^T)^j).$$

As a consequence of this fact, we get that

$$(3.12) \quad \forall \alpha \in [0, 1], \forall \eta \in S, \quad \sqrt{Q} e^{\alpha t B^T} \eta = \sum_{j=0}^{+\infty} \frac{(\alpha t)^j}{j!} \sqrt{Q} (B^T)^j \eta = 0.$$

This implies that (3.10) actually holds. We therefore deduce from (3.8), (3.9), (3.10), a change of variable in the norm and the triangle inequality that for all $\lambda \geq 0$,

$$(3.13) \quad \begin{aligned} \|\langle \xi_0, \nabla_x \rangle e^{-ta_i^w} u_\lambda\|_{L^2} &= \frac{1}{(2\pi)^{\frac{n}{2}}} \|\langle \xi_0, \xi + \lambda \xi_{0,S} \rangle e^{-ta_i(\xi + \lambda \xi_{0,S})} \widehat{u}_0(\xi)\|_{L^2}, \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \|\langle \xi_0, \xi + \lambda \xi_{0,S} \rangle e^{-ta_i(\xi)} \widehat{u}_0\|_{L^2} \\ &\geq \|\langle \xi_0, \lambda \xi_{0,S} \rangle e^{-ta_i^w} u_0\|_{L^2} - \|\langle \xi_0, \nabla_x \rangle e^{-ta_i^w} u_0\|_{L^2} \\ &\geq \lambda |\xi_{0,S}|^2 \|e^{-ta_i^w} u_0\|_{L^2} - c_{t,\xi_0} \|u_0\|_{L^2}, \end{aligned}$$

since $\lambda \xi_{0,S} \in S$. It follows from (3.7) and (3.13) that for all $\lambda \geq 0$,

$$\lambda |\xi_{0,S}|^2 \|e^{-ta_i^w} u_0\|_{L^2} \leq 2c_{t,\xi_0} \|u_0\|_{L^2}.$$

Since the function $e^{-ta_i^w} u_0$ is not equal to zero (u_0 is a Gaussian function and the symbol of the bounded Fourier multiplier $e^{-ta_i^w}$ is not equal to zero) and that the right-hand side of the above estimate does not depend on the parameter $\lambda \geq 0$, we conclude that $\xi_{0,S} = 0$, that is, $\xi_0 \in S^\perp$. This ends the proof of Theorem 2.1.

3.2. Regularizing effects. In the second part of this section, we prove Theorem 2.2. We keep the notations introduced in the previous subsection, in particular the time-dependent real-valued symbol a_t defined in (3.3).

Since the operator $e^{-t\langle Bx, \nabla_x \rangle}$ is a similitude on $L^2(\mathbb{R}^n)$ for all $t \geq 0$, we notice from (3.4) that it is sufficient to obtain the regularizing effects of the operators $e^{-ta_t^w}$ to derive the ones of the operators e^{-tP} . Let $m \geq 1$ be a positive integer and $\xi_1, \dots, \xi_m \in S^\perp$. We are interested in studying the operators

$$\langle \xi_1, \nabla_x \rangle \dots \langle \xi_m, \nabla_x \rangle e^{-ta_t^w}, \quad t > 0.$$

Since the operators $\langle \xi_j, \nabla_x \rangle$ and $e^{-ta_t^w}$ are Fourier multipliers, they commute, and the following factorization holds for all $t > 0$,

$$(3.14) \quad \begin{aligned} \langle \xi_1, \nabla_x \rangle \dots \langle \xi_m, \nabla_x \rangle e^{-ta_t^w} &= \langle \xi_1, \nabla_x \rangle \dots \langle \xi_m, \nabla_x \rangle \underbrace{e^{-\frac{t}{m}a_t^w} \dots e^{-\frac{t}{m}a_t^w}}_{m \text{ factors}} \\ &= \langle \xi_1, \nabla_x \rangle e^{-\frac{t}{m}a_t^w} \dots \langle \xi_m, \nabla_x \rangle e^{-\frac{t}{m}a_t^w}, \end{aligned}$$

where we used the semigroup property of the family of contraction operators $(e^{-sa_t^w})_{s \geq 0}$. The initial problem is therefore reduced to the analysis of the operators

$$\langle \xi_j, \nabla_x \rangle e^{-\frac{t}{m}a_t^w}, \quad j \in \{1, \dots, m\}, \quad t > 0.$$

In the following proposition, we prove a first regularizing effect for the semigroups $(e^{-\tau a_t^w})_{\tau \geq 0}$ for all $t > 0$.

Proposition 3.1. *There exist some positive constants $c > 1$ and $T > 0$ such that for all $k \in \{0, \dots, r\}$, $q > 0$, $0 < t < T$, $\tau > 0$ and $u \in L^2(\mathbb{R}^n)$,*

$$\| |\sqrt{Q}(B^T)^k D_x|^q e^{-\tau a_t^w} u \|_{L^2} \leq \left(\frac{c}{t^k \tau^{\frac{1}{2s}}} \right)^q q^{\frac{q}{2s}} \|u\|_{L^2}.$$

Proof. Let us begin by noticing that to prove Proposition 3.1, it is sufficient to obtain the existence of some positive constants $c_1 > 0$ and $t_1 > 0$ such that for all $q > 0$, $0 < t < t_1$, $\tau > 0$ and $\xi \in \mathbb{R}^n$,

$$(3.15) \quad \left(\int_0^1 |\sqrt{Q} e^{\alpha t B^T} \xi|^2 d\alpha \right)^{\frac{q}{2}} \exp \left(-\frac{\tau}{2} \int_0^1 |\sqrt{Q} e^{\alpha t B^T} \xi|^{2s} d\alpha \right) \leq \left(\frac{c_1}{\tau^{\frac{1}{2s}}} \right)^q \left(\frac{q}{es} \right)^{\frac{q}{2s}}.$$

Indeed, we check in Proposition 4.2 in Appendix, by using finite-dimensional and compactness arguments, that there exist some positive constants $c_0 > 0$ and $t_0 > 0$ such that for all $0 < t < t_0$ and $\xi \in \mathbb{R}^n$,

$$(3.16) \quad \int_0^1 |\sqrt{Q} e^{\alpha t B^T} \xi|^2 d\alpha \geq c_0 \sum_{k=0}^r t^{2k} |\sqrt{Q}(B^T)^k \xi|^2.$$

Once the estimates (3.15) and (3.16) are established, we deduce from the definition (3.3) of the symbols a_t and Plancherel's theorem that for all $k \in \{0, \dots, r\}$, $q > 0$, $0 < t < \min(t_0, t_1)$, $\tau > 0$ and $u \in L^2(\mathbb{R}^n)$,

$$\begin{aligned} \| |\sqrt{Q}(B^T)^k D_x|^q e^{-\tau a_t^w} u \|_{L^2(\mathbb{R}^n)} &\leq \left(\frac{1}{\sqrt{c_0} t^k} \right)^q \left\| \left(\int_0^1 |\sqrt{Q} e^{\alpha t B^T} D_x|^2 d\alpha \right)^{\frac{q}{2}} e^{-\tau a_t^w} u \right\|_{L^2(\mathbb{R}^n)} \\ &\leq \left(\frac{1}{\sqrt{c_0} t^k} \right)^q \left(\frac{c_1}{\tau^{\frac{1}{2s}}} \right)^q \left(\frac{q}{es} \right)^{\frac{q}{2s}} \|u\|_{L^2(\mathbb{R}^n)}, \end{aligned}$$

which is the desired estimate. We therefore focus on establishing the estimate (3.15). In order to simplify the notations, let us consider the real-valued symbol $\Gamma_{q,t,\tau}$ defined for all $q > 0$, $t > 0$, $\tau > 0$ and $\xi \in \mathbb{R}^n$ by

$$(3.17) \quad \Gamma_{q,t,\tau}(\xi) = \left(\int_0^1 |\sqrt{Q} e^{\alpha t B^T} \xi|^2 d\alpha \right)^{\frac{q}{2}} \exp \left(-\frac{\tau}{2} \int_0^1 |\sqrt{Q} e^{\alpha t B^T} \xi|^{2s} d\alpha \right).$$

First of all, observe from (3.12) and (3.17) that for all $q > 0$, $t > 0$ and $\tau > 0$, the symbol $\Gamma_{q,t,\tau}$ satisfies

$$\forall \xi \in \mathbb{R}^n, \quad \Gamma_{q,t,\tau}(\xi) = \Gamma_{q,t,\tau}(\xi_{S^\perp}),$$

where $\xi_{S^\perp} \in \mathbb{R}^n$ denotes the coordinate of the vector $\xi \in \mathbb{R}^n$ with respect to the decomposition $\mathbb{R}^n = S \oplus S^\perp$, the orthogonality being taken with respect to the canonical Euclidean structure of \mathbb{R}^n . Thus, we only need to establish the estimate (3.15) when $\xi \in S^\perp$. To that end, we will take advantage of the homogeneity property with respect to the ξ -variable of the two integrals

$$\int_0^1 |\sqrt{Q}e^{\alpha t B^T} \xi|^2 d\alpha \quad \text{and} \quad \int_0^1 |\sqrt{Q}e^{\alpha t B^T} \xi|^{2s} d\alpha.$$

Let $\xi \in S^\perp \setminus \{0\}$ and (ρ, σ) be the polar coordinates of ξ , i.e. $\xi = \rho\sigma$ with $\rho > 0$ and $\sigma \in \mathbb{S}^{n-1}$. It follows from (3.17) and the estimate

$$\forall q > 0, \forall x \geq 0, \quad x^q e^{-x^{2s}} \leq \left(\frac{q}{2es} \right)^{\frac{q}{2s}},$$

that for all $q > 0$, $t > 0$ and $\tau > 0$,

$$\begin{aligned} (3.18) \quad \Gamma_{q,t,\tau}(\xi) &= \left(\int_0^1 |\sqrt{Q}e^{\alpha t B^T} \sigma|^2 d\alpha \right)^{\frac{q}{2}} \rho^q \exp \left(- \left(\frac{\tau}{2} \int_0^1 |\sqrt{Q}e^{\alpha t B^T} \sigma|^{2s} d\alpha \right) \rho^{2s} \right) \\ &\leq \left(\int_0^1 |\sqrt{Q}e^{\alpha t B^T} \sigma|^2 d\alpha \right)^{\frac{q}{2}} \left(\frac{\tau}{2} \int_0^1 |\sqrt{Q}e^{\alpha t B^T} \sigma|^{2s} d\alpha \right)^{-\frac{q}{2s}} \left(\frac{q}{2es} \right)^{\frac{q}{2s}} \\ &\leq \left(\frac{M_t}{\tau^{\frac{1}{2s}}} \right)^q \left(\frac{q}{es} \right)^{\frac{q}{2s}}, \end{aligned}$$

where we set

$$(3.19) \quad M_t = \sup_{\eta \in \mathbb{S}^{n-1} \cap S^\perp} \left(\int_0^1 |\sqrt{Q}e^{\alpha t B^T} \eta|^2 d\alpha \right)^{\frac{1}{2}} \left(\int_0^1 |\sqrt{Q}e^{\alpha t B^T} \eta|^{2s} d\alpha \right)^{-\frac{1}{2s}}.$$

The study of the term M_t is made by the author and J. Bernier in [2] (Proposition 3.6) when the vector space S is reduced to zero, that is, when the Kalman rank condition (1.4) holds according to Lemma 4.1. The same strategy of proof as the one used in this paper (finite-dimensional and compactness arguments) turns out to work in the general case. By using these arguments, it is proved in Proposition 4.4 in Appendix that there exist some positive constants $c_1 > 0$ and $t_1 > 0$ such that for all $0 < t < t_1$,

$$(3.20) \quad M_t \leq c_1.$$

We therefore deduce from (3.17), (3.18) and (3.20) that for all $q > 0$, $0 < t < t_1$, $\tau > 0$ and $\xi \in S^\perp$,

$$\left(\int_0^1 |\sqrt{Q}e^{\alpha t B^T} \xi|^2 d\alpha \right)^{\frac{q}{2}} \exp \left(- \frac{\tau}{2} \int_0^1 |\sqrt{Q}e^{\alpha t B^T} \xi|^{2s} d\alpha \right) \leq \left(\frac{c_1}{\tau^{\frac{1}{2s}}} \right)^q \left(\frac{q}{es} \right)^{\frac{q}{2s}}.$$

This ends the proof of the estimate (3.15) and hence, the one of Proposition 3.1. \square

The above result also provides estimates for the operators $e^{-t\mathcal{P}}$. Indeed, we deduce from (3.1), (3.2), Proposition 3.1 and Plancherel's theorem that for all $k \in \{0, \dots, r\}$, $q > 0$, $0 < t < T$ and $u \in L^2(\mathbb{R}^n)$,

$$\begin{aligned} (3.21) \quad \left\| \left| \sqrt{Q}(B^T)^k D_x \right|^q e^{-t\mathcal{P}} u \right\|_{L^2} &\leq \left(\frac{c}{t^{k+\frac{1}{2s}}} \right)^q q^{\frac{q}{2s}} \left\| e^{-t\langle Bx, \nabla_x \rangle} u \right\|_{L^2} \\ &= \left(\frac{c}{t^{k+\frac{1}{2s}}} \right)^q e^{\frac{1}{2} \text{Tr}(B)t} q^{\frac{q}{2s}} \|u\|_{L^2}. \end{aligned}$$

As explained in Subsection 2.2, these estimates are key to establish the partial subelliptic estimates enjoyed by fractional Ornstein-Uhlenbeck operators on the whole space.

Corollary 3.2. *There exist some positive constants $c > 0$ and $0 < T < 1$ such that for all $\xi_0 \in S^\perp$, $0 < t < T$, $\tau > 0$ and $u \in L^2(\mathbb{R}^n)$,*

$$\|\langle \xi_0, \nabla_x \rangle e^{-\tau a_t^w} u\|_{L^2} \leq c |\xi_0| t^{-k_{\xi_0}} \tau^{-\frac{1}{2s}} \|u\|_{L^2},$$

where $0 \leq k_{\xi_0} \leq r$ denotes the index of the vector $\xi_0 \in S^\perp$ defined in (2.5).

Proof. First, let us check that there exists a positive constant $M_0 > 0$ such that for all $\xi_0 \in S^\perp$ and $\xi \in \mathbb{R}^n$,

$$(3.22) \quad |\langle \xi_0, \xi \rangle|^2 \leq M_0^2 |\xi_0|^2 \sum_{j=0}^{k_{\xi_0}} |\sqrt{Q}(B^T)^j \xi|^2.$$

For all non-negative integer $k \in \{0, \dots, r\}$, we consider \mathbb{P}_k the orthogonal projection onto the vector subspace $V_k^\perp \subset \mathbb{R}^n$, the orthogonality being taken with respect to the canonical Euclidean structure of \mathbb{R}^n , where the vector subspace V_k is defined in (2.3) by

$$V_k = \bigcap_{j=0}^k \text{Ker}(\sqrt{Q}(B^T)^j).$$

Notice from the definition of the vector spaces V_k that for all $k \in \{0, \dots, r\}$, there exists a positive constant $c_k > 0$ such that for all $\xi \in \mathbb{R}^n$,

$$\sum_{j=0}^k |\sqrt{Q}(B^T)^j \xi|^2 = \sum_{j=0}^k |\sqrt{Q}(B^T)^j \mathbb{P}_k \xi|^2 \geq c_k |\mathbb{P}_k \xi|^2.$$

Let $\xi_0 \in S^\perp$. The definition (2.5) of index implies that $\xi_0 \in V_{k_{\xi_0}}^\perp$. It follows from the above estimate and the Cauchy-Schwarz inequality that for all $\xi \in \mathbb{R}^n$,

$$|\langle \xi_0, \xi \rangle|^2 = |\langle \xi_0, \mathbb{P}_{k_{\xi_0}} \xi \rangle|^2 \leq |\xi_0|^2 |\mathbb{P}_{k_{\xi_0}} \xi|^2 \leq \frac{|\xi_0|^2}{c_{k_{\xi_0}}} \sum_{k=0}^{k_{\xi_0}} |\sqrt{Q}(B^T)^k \xi|^2.$$

This proves that the estimate (3.22) holds, with $M_0^2 = 1/\min_{k \in \{0, \dots, r\}} c_k > 0$. We therefore deduce from (3.22) and Plancherel's theorem that for all $\xi_0 \in S^\perp$, $t \geq 0$, $\tau \geq 0$ and $u \in L^2(\mathbb{R}^n)$,

$$(3.23) \quad \|\langle \xi_0, \nabla_x \rangle e^{-\tau a_t^w} u\|_{L^2} \leq M_0 |\xi_0| \sum_{k=0}^{k_{\xi_0}} \|\sqrt{Q}(B^T)^k D_x |e^{-\tau a_t^w} u\|_{L^2}.$$

On the other hand, Proposition 3.1 provides the existence of some positive constants $M_1 > 1$ and $t_1 > 0$ such that for all $k \in \{0, \dots, r\}$, $0 < t < t_1$, $\tau > 0$ and $u \in L^2(\mathbb{R}^n)$,

$$(3.24) \quad \|\sqrt{Q}(B^T)^k D_x |e^{-\tau a_t^w} u\|_{L^2} \leq \frac{c'}{t^k \tau^{\frac{1}{2s}}} \|u\|_{L^2}.$$

It follows from (3.23) and (3.24) that for all $\xi_0 \in S^\perp$, $0 < t < \min(t_1, 1)$, $\tau > 0$ and $u \in L^2(\mathbb{R}^n)$,

$$\begin{aligned} \|\langle \xi_0, \nabla_x \rangle e^{-\tau a_t^w} u\|_{L^2} &\leq M_0 |\xi_0| \left(\sum_{k=0}^{k_{\xi_0}} \frac{M_1}{t^k \tau^{\frac{1}{2s}}} \right) \|u\|_{L^2} \\ &\leq M_0 M_1 (r+1) |\xi_0| t^{-k_{\xi_0}} \tau^{-\frac{1}{2s}} \|u\|_{L^2}, \end{aligned}$$

since $0 \leq k_{\xi_0} \leq r$. This ends the proof of Corollary 3.2. \square

We can now tackle the proof of Theorem 2.2. To that end, we implement the strategy presented in the beginning of this subsection. Let $m \geq 1$ and $\xi_1, \dots, \xi_m \in S^\perp$. We denote by $0 \leq k_{\xi_j} \leq r$ the index of the vector $\xi_j \in S^\perp$ defined in (2.5). It follows from (3.14) that for all $t \geq 0$,

$$(3.25) \quad \langle \xi_1, \nabla_x \rangle \dots \langle \xi_m, \nabla_x \rangle e^{-t a_t^w} = \langle \xi_1, \nabla_x \rangle e^{-\frac{t}{m} a_t^w} \dots \langle \xi_m, \nabla_x \rangle e^{-\frac{t}{m} a_t^w}.$$

According to Corollary 3.2, there exist some positive constants $c > 0$ and $0 < T < 1$ such that for all $\xi_0 \in S^\perp$, $m \geq 1$ and $0 < t < T$,

$$(3.26) \quad \|\langle \xi_0, \nabla_x \rangle e^{-\frac{t}{m} a_t^w} u\|_{L^2} \leq \frac{c}{t^{k_{\xi_0} + \frac{1}{2s}}} |\xi_0| m^{\frac{1}{2s}} \|u\|_{L^2},$$

where $0 \leq k_{\xi_0} \leq r$ denotes the index of the vector $\xi_0 \in S^\perp$ defined in (2.5). We deduce from (3.25) and (3.26) that for all $0 < t < T$ and $u \in L^2(\mathbb{R}^n)$,

$$\begin{aligned} \|\langle \xi_1, \nabla_x \rangle \dots \langle \xi_m, \nabla_x \rangle e^{-t a_t^w} u\|_{L^2} &\leq \frac{c^m}{t^{k_{\xi_1} + \dots + k_{\xi_m} + \frac{m}{2s}}} \left(\prod_{j=1}^m |\xi_j| \right) m^{\frac{m}{2s}} \|u\|_{L^2} \\ &\leq \frac{e^{\frac{m}{2s}} c^m}{t^{k_{\xi_1} + \dots + k_{\xi_m} + \frac{m}{2s}}} \left(\prod_{j=1}^m |\xi_j| \right) (m!)^{\frac{1}{2s}} \|u\|_{L^2}, \end{aligned}$$

where we used the factorial estimate $m^m \leq e^m m!$. It follows from (3.1), (3.2) and Plancherel's theorem that for all $0 < t < T$ and $u \in L^2(\mathbb{R}^n)$,

$$\begin{aligned} \|\langle \xi_1, \nabla_x \rangle \dots \langle \xi_m, \nabla_x \rangle e^{-t \mathcal{P}} u\|_{L^2} &\leq \frac{e^{\frac{m}{2s}} c^m}{t^{k_{\xi_1} + \dots + k_{\xi_m} + \frac{m}{2s}}} \left(\prod_{j=1}^m |\xi_j| \right) (m!)^{\frac{1}{2s}} \|e^{-\langle Bx, \nabla_x \rangle} u\|_{L^2} \\ &= \frac{e^{\frac{m}{2s}} c^m e^{\frac{1}{2} \text{Tr}(B)t}}{t^{k_{\xi_1} + \dots + k_{\xi_m} + \frac{m}{2s}}} \left(\prod_{j=1}^m |\xi_j| \right) (m!)^{\frac{1}{2s}} \|u\|_{L^2}. \end{aligned}$$

This ends the proof of Theorem 2.2.

3.3. The nilpotent case. To end this section, we explain how the proof of Theorem 2.2 can be adapted to obtain long-time asymptotics when the matrix B is nilpotent (of order $K \geq 1$ say), and to prove in particular the estimates stated in Proposition 2.3. In this case, we deduce from Propositions 4.2 and 4.4 in Appendix that there exists a positive constant $c_0 > 0$ such that

$$\forall t > 0, \forall \xi \in \mathbb{R}^n, \quad \int_0^1 |\sqrt{Q} e^{\alpha t B^T} \xi|^2 d\alpha \geq c_0 \sum_{k=0}^{K-1} t^{2k} |\sqrt{Q} (B^T)^k \xi|^2,$$

and

$$\forall t > 0, \quad \sup_{\xi \in \mathbb{S}^{n-1} \cap S^\perp} \left(\int_0^1 |\sqrt{Q} e^{\alpha t B^T} \xi|^2 d\alpha \right)^{\frac{1}{2}} \left(\int_0^1 |\sqrt{Q} e^{\alpha t B^T} \xi|^{2s} d\alpha \right)^{-\frac{1}{2s}} \leq c_0.$$

From the same proof than the one of Proposition 3.1, we deduce that there exists another positive constant $c_1 > 1$ such that for all $k \in \{0, \dots, K-1\}$, $q > 0$, $t > 0$, $\tau > 0$ and $u \in L^2(\mathbb{R}^n)$,

$$(3.27) \quad \|\sqrt{Q} (B^T)^k D_x |^q e^{-\tau a_t^w} u\|_{L^2} \leq \left(\frac{c_1}{t^k \tau^{\frac{1}{2s}}} \right)^q q^{\frac{q}{2s}} \|u\|_{L^2}.$$

We now assume that the fractional Ornstein-Uhlenbeck operator \mathcal{P} is the fractional Kolmogorov operator \mathcal{K} given by

$$\mathcal{K} = (-\Delta_v)^s + v \cdot \nabla_x, \quad (x, v) \in \mathbb{R}^{2n},$$

that is, that the matrices B and Q are respectively given by

$$B = \begin{pmatrix} 0_n & I_n \\ 0_n & 0_n \end{pmatrix} \quad \text{and} \quad Q = 2^{\frac{1}{s}} \begin{pmatrix} 0_n & 0_n \\ 0_n & I_n \end{pmatrix}.$$

In this particular case, the estimates (3.27) write in the following way for all $q > 0$, $t > 0$, $\tau > 0$ and $u \in L^2(\mathbb{R}^n)$,

$$(3.28) \quad \|\sqrt{D_x} |^q e^{-\tau a_t^w} u\|_{L^2} \leq \left(\frac{c_1}{t \tau^{\frac{1}{2s}}} \right)^q q^{\frac{q}{2s}} \|u\|_{L^2}, \quad \|\sqrt{D_v} |^q e^{-\tau a_t^w} u\|_{L^2} \leq \left(\frac{c_1}{\tau^{\frac{1}{2s}}} \right)^q q^{\frac{q}{2s}} \|u\|_{L^2}.$$

Since the operators $|D_x|$, $|D_v|$ and $e^{-\frac{t}{2}a_t^w}$ are Fourier multipliers, they commute, and by writing

$$|D_x|^{q_1}|D_v|^{q_2}e^{-ta_t^w} = |D_x|^{q_1}e^{-\frac{t}{2}a_t^w}|D_v|^{q_2}e^{-\frac{t}{2}a_t^w},$$

we therefore deduce from (3.1), (3.2) and (3.28) that for all $q_1, q_2 > 0$, $t > 0$ and $u \in L^2(\mathbb{R}^n)$,

$$\| |D_x|^{q_1}|D_v|^{q_2}e^{-t\mathcal{K}}u \|_{L^2} \leq \left(\frac{c_1}{t^{1+\frac{1}{2s}}} \right)^{q_1} (2q_1)^{\frac{q_1}{2s}} \left(\frac{c_1}{t^{\frac{1}{2s}}} \right)^{q_2} (2q_2)^{\frac{q_2}{2s}} \|u\|_{L^2}.$$

These estimates imply in particular that there exists a positive constant $c_2 > 1$ such that $(\alpha, \beta) \in \mathbb{N}^{2n}$, $t > 0$ and $u \in L^2(\mathbb{R}^{2n})$,

$$\| \partial_x^\alpha \partial_v^\beta (e^{-t\mathcal{K}}u) \|_{L^2} \leq \frac{c_2^{|\alpha|+|\beta|}}{t^{(1+\frac{1}{2s})|\alpha|+\frac{|\beta|}{2s}}} (\alpha!)^{\frac{1}{2s}} (\beta!)^{\frac{1}{2s}} \|u\|_{L^2},$$

where we used the factorial estimate $m^m \leq e^m m!$ holding for all $m \geq 1$. This ends the proof of Proposition 2.3.

4. APPENDIX

Let B and Q be $n \times n$ real matrices, with Q symmetric positive semidefinite. We consider the associated vector space S defined in (2.1) and $0 \leq r \leq n-1$ the smallest integer such that (2.2) holds. In this appendix, we give the proofs of some results involving the matrices B and Q .

4.1. About the Kalman rank condition. First, we prove the characterization of the Kalman rank condition in term of the space S .

Lemma 4.1. *The Kalman rank condition (1.4) holds if and only if the vector space S is reduced to $\{0\}$.*

Proof. Using the notation of (1.4), we have the following equivalences:

$$\begin{aligned} \text{Rank } [B \mid \sqrt{Q}] = n &\Leftrightarrow \text{Ran } [B \mid \sqrt{Q}] = \mathbb{R}^n \\ &\Leftrightarrow \text{Ker } \left([B \mid \sqrt{Q}]^T \right) = \left(\text{Ran } [B \mid \sqrt{Q}] \right)^\perp = \{0\} \\ &\Leftrightarrow S = \bigcap_{k=0}^{n-1} \text{Ker } (\sqrt{Q}(B^T)^k) = \text{Ker } \left([B \mid \sqrt{Q}]^T \right) = \{0\}, \end{aligned}$$

where \perp denotes the orthogonality with respect to the canonical Euclidean structure. This ends the proof of Lemma 4.1. \square

4.2. A technical estimate. In this subsection, we give the proof of the estimate (3.16) which has a key role in the proof of Proposition 3.1 in Section 3. To that end, we will exploit the equivalence of norms in finite-dimensional normed vector spaces applied with Hardy's norms, as well as compactness arguments, inspired by techniques used in [3] (Section 4).

Proposition 4.2. *In the general case, there exist some positive constants $c > 0$ and $T > 0$ such that for all $0 \leq t < T$ and $\xi \in \mathbb{R}^n$,*

$$(4.1) \quad \int_0^1 |\sqrt{Q}e^{\alpha t B^T} \xi|^2 \, d\alpha \geq c \sum_{k=0}^r t^{2k} |\sqrt{Q}(B^T)^k \xi|^2.$$

In the particular case where the matrix B is nilpotent of order $K \geq 1$, there exists another positive constant $c_0 > 0$ such that for all $t > 0$ and $\xi \in \mathbb{R}^n$,

$$(4.2) \quad \int_0^1 |\sqrt{Q}e^{\alpha t B^T} \xi|^2 \, d\alpha \geq c_0 \sum_{k=0}^{K-1} t^{2k} |\sqrt{Q}(B^T)^k \xi|^2.$$

Proof. In order to simplify the notations, we set for all $t \geq 0$ and $\xi \in \mathbb{R}^n$,

$$(4.3) \quad q_t(\xi) = \int_0^1 |\sqrt{Q}e^{\alpha t B^T} \xi|^2 d\alpha = \|\sqrt{Q}e^{\alpha t B^T} \xi\|_{L^2(0,1)}^2.$$

Notice that in the above $L^2(0,1)$ norm, the parameter $\alpha \in (0,1)$ stands for the variable of integration. This notation will be used throughout the proof each time a $L^2(0,1)$ norm appears. Moreover, the notation α will be used once as a polynomial variable for a Hardy's norm \mathcal{H}^1 on $(\mathbb{R}_r[\alpha])^n$, see (4.5) just after.

Minkowski's inequality implies that for all $t \geq 0$ and $\xi \in \mathbb{R}^n$,

$$(4.4) \quad \sqrt{q_t(\xi)} \geq \left\| \sum_{k=0}^r \frac{(\alpha t)^k}{k!} \sqrt{Q}(B^T)^k \xi \right\|_{L^2(0,1)} - \left\| \sum_{k \geq r+1} \frac{(\alpha t)^k}{k!} \sqrt{Q}(B^T)^k \xi \right\|_{L^2(0,1)}.$$

We focus on controlling the two terms appearing in the right-hand side of the above estimate. First, notice that on the finite-dimensional vector space $(\mathbb{R}_r[\alpha])^n$, the Hardy's norm $\|\cdot\|_{\mathcal{H}^1}$ defined by

$$(4.5) \quad \left\| \sum_{k=0}^r y_k \alpha^k \right\|_{\mathcal{H}^1} = \sum_{k=0}^r k! |y_k|, \quad y_1, \dots, y_r \in \mathbb{R}^n,$$

is equivalent to the standard Lebesgue's norm $\|\cdot\|_{L^2(0,1)}$ given by

$$\left\| \sum_{k=0}^r y_k \alpha^k \right\|_{L^2(0,1)}^2 = \int_0^1 \left| \sum_{k=0}^r y_k \alpha^k \right|^2 d\alpha, \quad y_1, \dots, y_r \in \mathbb{R}^n.$$

This implies that there exists a positive constant $c_1 > 0$ such that for all $t \geq 0$ and $\xi \in \mathbb{R}^n$,

$$(4.6) \quad \left\| \sum_{k=0}^r \frac{(\alpha t)^k}{k!} \sqrt{Q}(B^T)^k \xi \right\|_{L^2(0,1)} \geq c_1 \sum_{k=0}^r t^k |\sqrt{Q}(B^T)^k \xi|,$$

since

$$\forall t \geq 0, \forall \xi \in \mathbb{R}^n, \quad \sum_{k=0}^r \frac{(\alpha t)^k}{k!} \sqrt{Q}(B^T)^k \xi \in (\mathbb{R}_r[\alpha])^n.$$

In view of (4.1) and (4.4), it remains to check that the remainder term

$$\left\| \sum_{k \geq r+1} \frac{(\alpha t)^k}{k!} \sqrt{Q}(B^T)^k \xi \right\|_{L^2(0,1)},$$

can be controlled by $\sum_{k=0}^r t^k |\sqrt{Q}(B^T)^k \xi|$. Precisely, we will prove that there exist some positive constants $c_2 > 0$ and $t_1 > 0$ such that for all $0 \leq t < t_1$ and $\xi \in S^\perp$,

$$(4.7) \quad \left\| \sum_{k \geq r+1} \frac{(\alpha t)^k}{k!} \sqrt{Q}(B^T)^k \xi \right\|_{L^2(0,1)} \leq c_2 t \sum_{k=0}^r t^k |\sqrt{Q}(B^T)^k \xi|.$$

Once the estimate (4.7) is established, we deduce from (4.4) and (4.6) that for all $0 \leq t < t_1$ and $\xi \in S^\perp$,

$$\sqrt{q_t(\xi)} \geq (c_1 - c_2 t) \sum_{k=0}^r t^k |\sqrt{Q}(B^T)^k \xi|.$$

This estimate combined with the triangular inequality implies that there exist other positive constants $c > 0$ and $T > 0$ such that for all $0 \leq t < T$ and $\xi \in S^\perp$,

$$(4.8) \quad q_t(\xi) \geq c \sum_{k=0}^r t^{2k} |\sqrt{Q}(B^T)^k \xi|^2.$$

Moreover, for all $0 \leq t < T$, both quadratic forms q_t and $\sum_{k=0}^r t^{2k} |\sqrt{Q}(B^T)^k \cdot|^2$ vanish on the vector space S from (3.11), which proves that the estimate (4.8) can be extended to all $0 \leq t < T$ and $\xi \in \mathbb{R}^n$ since $\mathbb{R}^n = S \oplus S^\perp$, according to the following elementary lemma whose proof is straightforward and omitted here

Lemma 4.3. *Let E be a real finite-dimensional vector space and q_1, q_2 be two non-negative quadratic forms on E . If $E = F \oplus G$ is a direct sum of two vector subspaces such that $q_1 \leq q_2$ on F and q_1, q_2 both vanish on G , then $q_1 \leq q_2$ on E .*

We therefore need to check that the estimate (4.7) actually holds to end the proof of (4.1). First, notice that there exists a positive constant $c_3 > 0$ such that for all $0 \leq t \leq 1$ and $\xi \in \mathbb{R}^n$,

$$(4.9) \quad \left\| \sum_{k \geq r+1} \frac{(\alpha t)^k}{k!} \sqrt{Q}(B^T)^k \xi \right\|_{L^2(0,1)} = t^{r+1} \left\| \sum_{k \geq r+1} t^{k-r-1} \frac{\alpha^k}{k!} \sqrt{Q}(B^T)^k \xi \right\|_{L^2(0,1)} \leq c_3 t^{r+1} |\xi|.$$

On the other hand, it follows from the definition (2.2) of the integer $0 \leq r \leq n-1$ that there exists a positive constant $c_4 > 0$ such that for all $0 \leq t \leq 1$ and $\xi \in S^\perp$,

$$(4.10) \quad \sum_{k=0}^r t^k |\sqrt{Q}(B^T)^k \xi| \geq t^r \sum_{k=0}^r |\sqrt{Q}(B^T)^k \xi| \geq c_4 t^r |\xi|,$$

the orthogonality being taken with respect to the canonical Euclidean structure of \mathbb{R}^n . Combining (4.9) and (4.10), we obtain that for all $0 \leq t \leq 1$ and $\xi \in S^\perp$,

$$\left\| \sum_{k \geq r+1} \frac{(\alpha t)^k}{k!} \sqrt{Q}(B^T)^k \xi \right\|_{L^2(0,1)} \leq \frac{c_3 t}{c_4} \sum_{k=0}^r t^k |\sqrt{Q}(B^T)^k \xi|.$$

This ends the proof of the estimate (4.7).

When the matrix B is nilpotent of order $K \geq 1$, the matrix B^T is also nilpotent with the same order and we deduce that

$$\forall t \geq 0, \forall \xi \in \mathbb{R}^n, \quad \sqrt{Q} e^{\alpha t B^T} \xi \in (\mathbb{R}_{K-1}[\alpha])^n.$$

The finite-dimensional argument of the beginning of this proof and the triangle inequality then allow to obtain the existence of a positive constant $c_0 > 0$ such that for all $t \geq 0$ and $\xi \in \mathbb{R}^n$,

$$q_t(\xi) \geq c_0 \sum_{k=0}^{K-1} t^{2k} |\sqrt{Q}(B^T)^k \xi|^2.$$

This ends the proof of the estimate (4.2). \square

4.3. Study of the term M_t . To end this appendix, we establish the estimate (3.20). The proof of the following proposition follows quite line to line the ones of [2] (Subsection 3.2) and is presented for the sake of completeness of the present work.

Proposition 4.4. *There exist some positive constants $c > 0$ and $T > 0$ such that for all $0 < t < T$,*

$$(4.11) \quad \sup_{\xi \in \mathbb{S}^{n-1} \cap S^\perp} \left(\int_0^1 |\sqrt{Q} e^{\alpha t B^T} \xi|^2 d\alpha \right)^{\frac{1}{2}} \left(\int_0^1 |\sqrt{Q} e^{\alpha t B^T} \xi|^{2s} d\alpha \right)^{-\frac{1}{2s}} \leq c.$$

When the matrix B is nilpotent, the above estimate holds for all $t > 0$.

Proof. Keeping the notation (3.19) from the proof of Proposition 3.1, we consider the term M_t defined for all $t > 0$ by

$$(4.12) \quad M_t = \sup_{\xi \in \mathbb{S}^{n-1} \cap S^\perp} \left(\int_0^1 |\sqrt{Q} e^{\alpha t B^T} \xi|^2 d\alpha \right)^{\frac{1}{2}} \left(\int_0^1 |\sqrt{Q} e^{\alpha t B^T} \xi|^{2s} d\alpha \right)^{-\frac{1}{2s}}.$$

Given $t > 0$, let us first check that the term M_t is well-defined. If the vector $\xi \in \mathbb{S}^{n-1} \cap S^\perp$ satisfies

$$\int_0^1 |\sqrt{Q} e^{\alpha t B^T} \xi|^p d\alpha = 0,$$

for some $p \in \{2, 2s\}$, we deduce that

$$\forall \alpha \in [0, 1], \quad \sqrt{Q}e^{\alpha B^T} \xi = 0.$$

By differentiating this identity with respect to α and evaluating in $\alpha = 0$, we deduce that

$$\forall k \in \{0, \dots, r\}, \quad \sqrt{Q}(B^T)^k \xi = 0.$$

We obtain from (2.2) that $\xi \in S$, which is not possible since $\xi \in \mathbb{S}^{n-1} \cap S^\perp$. We therefore proved that

$$\forall p \in \{2, 2s\}, \forall \xi \in \mathbb{S}^{n-1} \cap S^\perp, \quad \int_0^1 |\sqrt{Q}e^{\alpha B^T} \xi|^p \, d\alpha > 0.$$

Moreover, for all $p \in \{2, 2s\}$, the functions

$$\xi \in \mathbb{S}^{n-1} \cap S^\perp \mapsto \int_0^1 |\sqrt{Q}e^{\alpha B^T} \xi|^p \, d\alpha,$$

are continuous on the compact set $\mathbb{S}^{n-1} \cap S^\perp$. This implies that the term M_t is well defined and satisfies $0 < M_t < +\infty$.

In the remaining of this proof, we will widely use the following lemma whose proof is straightforward and can be found e.g. in [2] (Lemma 3.4).

Lemma 4.5. *Let E be a real finite-dimensional vector space and $L_1, L_2 : E \rightarrow \mathbb{R}_+$ be two continuous functions satisfying for all $j \in \{1, 2\}$,*

$$\forall \lambda \geq 0, \forall P \in E, \quad L_j(\lambda P) = \lambda L_j(P),$$

and

$$\forall P \in E \setminus \{0\}, \quad L_j(P) > 0.$$

Then, there exists a positive constant $c > 0$ such that

$$\forall P \in E, \quad L_1(P) \leq cL_2(P).$$

We now tackle the proof of the estimate (4.11). For all $t > 0$ and $\xi \in \mathbb{S}^{n-1} \cap S^\perp$, we consider

$$M_t(\xi) = \left(\int_0^1 |\sqrt{Q}e^{\alpha B^T} \xi|^2 \, d\alpha \right)^{\frac{1}{2}} \left(\int_0^1 |\sqrt{Q}e^{\alpha B^T} \xi|^{2s} \, d\alpha \right)^{-\frac{1}{2s}}.$$

Let us first assume that the matrix B is nilpotent of order $K \geq 1$. Since the matrix B^T is also nilpotent with the same index K , we have that

$$\forall t > 0, \forall \xi \in \mathbb{S}^{n-1} \cap S^\perp, \quad \sqrt{Q}e^{\alpha B^T} \xi \in (\mathbb{R}_{K-1}[\alpha])^n.$$

It follows from Lemma 4.5 applied with the real finite-dimensional vector space $E = (\mathbb{R}_{K-1}[\alpha])^n$ and the homogeneous continuous functions

$$(4.13) \quad L_1(P) = \left(\int_0^1 |P(\alpha)|^2 \, d\alpha \right)^{\frac{1}{2}} \quad \text{and} \quad L_2(P) = \left(\int_0^1 |P(\alpha)|^{2s} \, d\alpha \right)^{\frac{1}{2s}},$$

that there exists a positive constant $c > 0$ such that for all $t > 0$ and $\xi \in \mathbb{S}^{n-1} \cap S^\perp$,

$$M_t(\xi) \leq c.$$

This ends the proof of Proposition 4.4 in this particular case.

Back to the case where the matrix B is general, we consider $P_{t,\xi}$ and $R_{t,\xi}$ the functions defined for all $t > 0$, $\xi \in \mathbb{S}^{n-1} \cap S^\perp$ and $\alpha \in [0, 1]$ by

$$(4.14) \quad P_{t,\xi}(\alpha) = \sum_{k=0}^r \alpha^k \frac{t^k}{k!} \sqrt{Q}(B^T)^k \xi \quad \text{and} \quad R_{t,\xi}(\alpha) = \sqrt{Q}e^{t\alpha B^T} \xi - P_{t,\xi}(\alpha).$$

It is fundamental in the following to notice that for all $t > 0$ and $\xi \in \mathbb{S}^{n-1} \cap S^\perp$, the coordinates of the function $P_{t,\xi}$ are polynomials of degree less than or equal to r , that is

$$(4.15) \quad \forall t > 0, \forall \xi \in \mathbb{S}^{n-1} \cap S^\perp, \quad P_{t,\xi} \in (\mathbb{R}_r[\alpha])^n.$$

By using (4.15) and Lemma 4.5 with the real finite-dimensional vector space $E = (\mathbb{R}_r[\alpha])^n$ and the homogeneous continuous functions defined in (4.13) anew, we obtain the existence of a constant $c > 0$ such that for all $t > 0$ and $\xi \in \mathbb{S}^{n-1} \cap S^\perp$,

$$\left(\int_0^1 |P_{t,\xi}(\alpha)|^2 d\alpha \right)^{\frac{1}{2}} \leq c \left(\int_0^1 |P_{t,\xi}(\alpha)|^{2s} d\alpha \right)^{\frac{1}{2s}}.$$

This estimate implies that for all $t > 0$ and $\xi \in \mathbb{S}^{n-1} \cap S^\perp$,

$$(4.16) \quad M_t(\xi) \leq c \left(\frac{\int_0^1 |\sqrt{Q}e^{t\alpha B^T} \xi|^2 d\alpha}{\int_0^1 |P_{t,\xi}(\alpha)|^2 d\alpha} \right)^{\frac{1}{2}} \left(\frac{\int_0^1 |P_{t,\xi}(\alpha)|^{2s} d\alpha}{\int_0^1 |\sqrt{Q}e^{\alpha t B^T} \xi|^{2s} d\alpha} \right)^{\frac{1}{2s}}.$$

We aim at establishing uniform upper bounds with respect to $t > 0$ and $\xi \in \mathbb{S}^{n-1} \cap S^\perp$ for these two factors. To that end, we equip the vector space $(\mathbb{R}_r[\alpha])^n$ of the Hardy's norm $\|\cdot\|_{\mathcal{H}^\infty}$ defined by

$$\forall P \in (\mathbb{R}_r[\alpha])^n, \quad \|P\|_{\mathcal{H}^\infty} = \max_{k \in \{0, \dots, r\}} \frac{|P^{(k)}(0)|}{k!}.$$

We deduce anew from (4.15) and Lemma 4.5 applied with the real finite-dimensional vector space $E = (\mathbb{R}_r[\alpha])^n$ and the homogeneous continuous functions $\|\cdot\|_{\mathcal{H}^\infty}$ and (4.13) that

$$(4.17) \quad \forall p \in \{2, 2s\}, \exists c_p > 0, \forall P \in (\mathbb{R}_r[\alpha])^n, \quad \|P\|_{\mathcal{H}^\infty} \leq c_p \left(\int_0^1 |P(\alpha)|^p d\alpha \right)^{\frac{1}{p}}.$$

According to the definition (2.2) of the vector space S , we notice that

$$\forall \xi \in \mathbb{S}^{n-1} \cap S^\perp, \quad \max_{k \in \{0, \dots, r\}} \frac{|\sqrt{Q}(B^T)^k \xi|}{k!} > 0.$$

Since the function

$$\xi \in \mathbb{S}^{n-1} \cap S^\perp \mapsto \max_{k \in \{0, \dots, r\}} \frac{|\sqrt{Q}(B^T)^k \xi|}{k!},$$

is continuous on the compact set $\mathbb{S}^{n-1} \cap S^\perp$, we deduce that there exists a positive constant $\varepsilon > 0$ such that

$$\forall \xi \in \mathbb{S}^{n-1} \cap S^\perp, \quad \|P_{1,\xi}\|_{\mathcal{H}^\infty} = \max_{k \in \{0, \dots, r\}} \frac{|\sqrt{Q}(B^T)^k \xi|}{k!} \geq \varepsilon.$$

It follows that

$$\forall t \in (0, 1], \forall \xi \in \mathbb{S}^{n-1} \cap S^\perp, \quad \|P_{t,\xi}\|_{\mathcal{H}^\infty} = \max_{k \in \{0, \dots, r\}} \frac{t^k |\sqrt{Q}(B^T)^k \xi|}{k!} \geq \varepsilon t^r,$$

and we deduce from (4.17) that

$$(4.18) \quad \forall p \in \{2, 2s\}, \forall t \in (0, 1], \forall \xi \in \mathbb{S}^{n-1} \cap S^\perp, \quad \varepsilon t^r \leq c_p \left(\int_0^1 |P_{t,\xi}(\alpha)|^p d\alpha \right)^{\frac{1}{p}}.$$

On the other hand, it follows from Taylor's formula with remainder term that

$$\forall t > 0, \forall \xi \in \mathbb{S}^{n-1} \cap S^\perp, \forall \alpha \in [0, 1], \quad R_{t,\xi}(\alpha) = \frac{(t\alpha)^{r+1}}{r!} \int_0^1 (1-\theta)^r \sqrt{Q}(B^T)^{r+1} e^{t\alpha\theta B^T} \xi d\theta.$$

Therefore, there exists a positive constant $M > 0$ such that

$$(4.19) \quad \forall t \in (0, 1], \forall \xi \in \mathbb{S}^{n-1} \cap S^\perp, \quad \|R_{t,\xi}\|_{L^\infty[0,1]} \leq Mt^{r+1}.$$

With these estimates, we can obtain upper bounds on the two factors of the right-hand side of the estimate (4.16).

1. By using (4.14) and applying the triangle inequality for the L^2 norm, we first obtain

$$\forall t > 0, \forall \xi \in \mathbb{S}^{n-1} \cap S^\perp, \quad \left(\frac{\int_0^1 |\sqrt{Q}e^{t\alpha B^T} \xi|^2 d\alpha}{\int_0^1 |P_{t,\xi}(\alpha)|^2 d\alpha} \right)^{\frac{1}{2}} \leq 1 + \left(\frac{\int_0^1 |R_{t,\xi}(\alpha)|^2 d\alpha}{\int_0^1 |P_{t,\xi}(\alpha)|^2 d\alpha} \right)^{\frac{1}{2}}.$$

According to (4.18) and (4.19), we therefore get that for all $0 < t \leq 1$ and $\xi \in \mathbb{S}^{n-1} \cap S^\perp$,

$$(4.20) \quad \left(\frac{\int_0^1 |\sqrt{Q}e^{t\alpha B^T} \xi|^2 d\alpha}{\int_0^1 |P_{t,\xi}(\alpha)|^2 d\alpha} \right)^{\frac{1}{2}} \leq 1 + \frac{c_2 M t^{r+1}}{\varepsilon t^r} = 1 + \frac{c_2 M}{\varepsilon} t \leq 1 + \frac{c_2 M}{\varepsilon}.$$

2. Notice that the classical estimate

$$\forall \xi, \eta \in \mathbb{R}^n, \quad |\xi + \eta|^{2s} \leq 2^{(2s-1)_+} (|\xi|^{2s} + |\eta|^{2s}),$$

with $(2s-1)_+ = \max(2s-1, 0)$, implies the following one

$$\forall \xi, \eta \in \mathbb{R}^n, \quad 2^{-(2s-1)_+} |\xi|^{2s} - |\eta|^{2s} \leq |\xi - \eta|^{2s}.$$

We therefore deduce from (4.14) that for all $0 < t \leq 1$ and $\xi \in \mathbb{S}^{n-1} \cap S^\perp$,

$$\frac{\int_0^1 |\sqrt{Q}e^{t\alpha B^T} \xi|^{2s} d\alpha}{\int_0^1 |P_{t,\xi}(\alpha)|^{2s} d\alpha} \geq 2^{-(2s-1)_+} - \frac{\int_0^1 |R_{t,\xi}(\alpha)|^{2s} d\alpha}{\int_0^1 |P_{t,\xi}(\alpha)|^{2s} d\alpha}.$$

Moreover, it follows from (4.18) and (4.19) that for all $0 < t \leq 1$ and $\xi \in \mathbb{S}^{n-1} \cap S^\perp$,

$$\frac{\int_0^1 |R_{t,\xi}(\alpha)|^{2s} d\alpha}{\int_0^1 |P_{t,\xi}(\alpha)|^{2s} d\alpha} \leq \left(\frac{c_{2s} M t^{r+1}}{\varepsilon t^r} \right)^{2s} = \left(\frac{c_{2s} M}{\varepsilon} t \right)^{2s},$$

from which we deduce that

$$\frac{\int_0^1 |\sqrt{Q}e^{t\alpha B^T} \xi|^{2s} d\alpha}{\int_0^1 |P_{t,\xi}(\alpha)|^{2s} d\alpha} \geq 2^{-(2s-1)_+} - \left(\frac{c_{2s} M}{\varepsilon} t \right)^{2s}.$$

It follows that there exist some positive constants $c_0 > 0$ and $0 < t_0 < 1$ such that for all $0 < t < t_0$ and $\xi \in \mathbb{S}^{n-1} \cap S^\perp$,

$$(4.21) \quad \left(\frac{\int_0^1 |\sqrt{Q}e^{t\alpha B^T} \xi|^{2s} d\alpha}{\int_0^1 |P_{t,\xi}(\alpha)|^{2s} d\alpha} \right)^{\frac{1}{2s}} \geq c_0.$$

As a consequence of (4.16), (4.20) and (4.21), there exists a positive constant $c_1 > 0$ such that

$$\forall t \in (0, t_0), \forall \xi \in \mathbb{S}^{n-1} \cap S^\perp, \quad M_t(\xi) \leq c_1.$$

This ends the proof of the estimate (4.11) in the general case. \square

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