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# Prudence and preference for flexibility gain \*

Daniel Danau<sup>†</sup>

## Abstract

We investigate the properties of the premium that a risk-averse individual is willing to pay to benefit from flexibility before an irreversible decision is made. Her decision concerns the quantity of some specific good to be acquired, knowing that the unit cost of the acquisition is either the mean value of a uniform distribution, or it is drawn from that distribution. The benefit of the latter technology, which we call a “flexibility gain”, is that after choosing the technology and before choosing the quantity, the individual learns the true unit cost. A richer individual values the flexibility more than a poorer individual if and only if her coefficient of absolute prudence is larger than a coefficient that is directly proportional to the rate at which the monetary benefit decreases when the state of nature drawn from the uniform distribution is above its mean value. We apply this result to an investment timing problem and show that the optimal waiting period increases with the ratio between the two coefficients. We next show that when the decision consists of exerting a preventive effort, the individual is more likely to prefer flexibility when she is more risk averse, but less likely to prefer flexibility when she is more prudent. Finally, we show how the identified link between the flexibility gain and degree of prudence determines the preference over specific lotteries. Moreover, when the individual is risk-neutral, the flexibility gain increases with the mean value of the distribution if and only if the marginal benefit function with respect to the unit cost is convex. Consequently, the individual has a stronger preference for an interval with lower mean, or lower mean and higher spread, when the marginal benefit function is concave rather than convex.

*Keywords:* Prudence; Flexibility gain; Utility premium; Choice between lotteries; Choice between distributions

*J.E.L. Classification Numbers:* D81

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<sup>†</sup>Université de Caen Normandie - Centre de Recherche en Economie et Management, Esplanade de la Paix, 14000 Caen (France). E-mail: daniel.danau@unicaen.fr

# 1 Introduction

In decision theory, much attention has been devoted to situations in which an individual prefers a certain utility to some expected utility with an uncertain outcome, in an expected utility framework. A core notion to measure this preference is the utility premium, introduced by Friedman and Savage [12]. The utility premium expresses the reduction induced in the expected utility of the individual, if a zero-mean lottery is added to her initial wealth. The higher the individual's wealth, the lower the utility premium if and only if the third derivative of the utility function is positive (Hanson and Menezes [16]), which means prudence, in the definition of Kimball [20]. In terms of behaviour, the implication is that when the utility premium decreases with the wealth (and so the individual is prudent), she saves more if she knows that a zero-mean lottery will be added to her initial wealth in the future (Kimball [20]). This kind of link between prudence and preference for a certain outcome over an uncertain outcome, as expressed by the utility premium, is systematically used in the literature when prudence is expressed within the expected utility framework (see Eeckhoudt and Schlesinger [10], for instance).

However, the utility premium alone cannot explain which individual characteristics determine investment choices under uncertainty. On one hand, the consumption-savings model captures the investor's problem, to the extent that investment opportunities can be associated with savings decisions (see, for instance, Eeckhoudt et alii [8], Chapter 4). On the other hand, in addition to deciding how much to invest, the investor has one more decision to make: namely, whether to take current or future investment opportunities, given that the states of nature (such as cost and demand), which the future investment opportunities depend upon, evolve stochastically over time (Dixit and Pindyck [5]). The investor may prefer the expected utility of future investment opportunities, given, say, the expected evolution of the technology, to the utility derived from an immediate investment, with the technology currently in use. To account for such situations, we investigate the link between the characteristics of the individual and the benefit she perceives as information is acquired. We apply the analysis to investment timing problems. Our ultimate goal is to understand how the notion of prudence extends to situations in which the individual benefits from information acquisition before making an irreversible decision.

To pursue this goal, we consider an individual who can choose to base her quantity decision on a unit cost of acquisition that is uniformly distributed over a range  $[\theta - \eta, \theta + \eta]$  rather than being equal to  $\theta$ . Because the technology is chosen before learning the true cost and the quantity is chosen after learning the true cost, the individual obviously prefers to draw the state from the interval, and we label this benefit as a flexibility gain. We first show that the initial wealth influences the value that the individual attaches to the enjoyed flexibility, in the same way that the utility premium depends on the initial wealth. Whether a richer individual values this flexibility more or less depends on both the characteristics of her preferences and the properties of the benefit function deriving from the decision. On one hand, the more prudent the risk-averse individual is, the more likely she is to prefer the set  $[\theta - \eta, \theta + \eta]$  to the singleton  $\{\theta\}$ . On the other hand, regardless of the characteristics of the individual preferences, when the set  $[\theta - \eta, \theta + \eta]$  is chosen, there is always a positive probability that the individual will be faced with a state worse than  $\theta$ , once the state becomes known. Clearly, this is a cost. For this

reason, a richer individual attaches a higher value to flexibility than a poorer individual, if and only if her coefficient of absolute prudence  $P(\cdot)$  is large enough relative to a coefficient  $R(x)$  that is directly proportional to the rate at which her benefit function decreases once a state of nature  $x > \theta$  is realized. We next apply our result to the classical investment timing problem of McDonald and Siegel [22]. The individual decides when to make a one-shot investment, knowing that the unit cost of operation evolves stochastically over time. Implicitly, the size of the operation is adjusted dynamically according to each realization of the state of nature once the investment is made. Referring to utility functions and surplus functions with constant  $P(\cdot)$  and  $R(\cdot)$ , we show that the optimal investment delay (if any) increases with the ratio between these two parameters. In line with our first result, the value of waiting (hence, the value of flexibility) is larger if the individual is more prudent, and her benefit decreases less if a bad state of nature is realized in the future. Our next finding is that when the individual has to exert a preventive effort, the above results no longer hold. Regardless of the initial wealth, a more prudent individual is less likely to prefer flexibility, because preventive effort is a necessity rather than an opportunity. Lastly, we apply the notion of flexibility gain to preferences over specific lotteries. Under risk aversion, we identify specific lotteries such that the preference of the individual depends on whether the flexibility gain increases or decreases with wealth, in the same way that Bigelow and Menezes [2] and Eeckhoudt and Schlesinger [10] relate choice over specific lotteries to the marginal utility premium. Finally, we show that under risk neutrality, the rate of decrease of the benefit function  $R(\cdot)$  determines the benefit/cost to the individual as the mean value  $\theta$  of the interval  $[\theta - \eta, \theta + \eta]$ , its spread  $\eta$ , or both, increase.

First, this paper is related to the studies of Hanson and Menezes [16] and Kimball [20]. Our contribution is to show that prudence is linked not only to the size of the utility premium but also to the size of the flexibility gain, if any.

Second, the paper is related to studies on decision theory, which focus on the link between prudence and irreversibility in decision making. This literature restricts attention to problems of preventive investment, which is aimed either at avoiding a catastrophe (Gollier et al. [15] and Eeckhoudt and Gollier [7], Eeckhoudt and Gollier [7], Eeckhoudt et al [6], Menegatti [23] and Peter [27]) or at decreasing the probability of an event against which insurance has been subscribed (Bourles [3]). However, these studies focus on situations in which no information can be acquired before deciding to invest. Our contribution is to show that the individual degree of prudence is also relevant when new information can be acquired over time, prior to investing.

Lastly, the paper is related to the studies on investment timing problems under risk aversion of Henderson and Hobson [17], Hugonnier, and Morellec [18] and Chronopoulos et al. [4]. The link is especially tight with the latter two studies, in which an optimal stopping time approach is followed. We contribute to this literature by showing that the optimal stopping time depends not only on risk aversion but also on prudence.

The outline of the paper is as follows. In Section 2, we provide the benchmark of the analysis, recalling the link between utility premium and prudence identified in previous studies. In Section 3, we develop a model of flexibility gain that is explored thereafter in specific contexts. In the three subsequent sections, the concept of flexibility gain is applied to an investment timing problem (Section 4); a problem of optimal prevention of a catastrophe (Section 5); the optimal

choice between specific lotteries and the preference over sets with different means and/or spreads (Section 6). Section 7 concludes the paper.

## 2 Benchmark: utility premium, prudence and intensity of prudence

Consider an individual who has an initial amount of money  $m$  and an indirect increasing and concave utility function  $u(m)$ . The utility premium measures the decrease that is induced in the individual utility as a zero-mean risk  $\tilde{\varepsilon}$  is added to the initial amount of money  $m$ . Therefore, the utility premium is given by  $-\hat{w}(m) > 0$ , where

$$\hat{w}(m) = \mathbb{E}[u(m + \tilde{\varepsilon})] - u(m). \quad (1)$$

Hanson and Menezes [16] show that  $\hat{w}'(m) > 0$  if and only if  $u'''(\cdot) > 0$ . This equivalence is obtained by applying Jensen's inequality to the marginal utility function. Indeed, by Jensen inequality,  $\mathbb{E}'[u(m + \tilde{\varepsilon})] > u'(m)$  if and only if  $u'''(\cdot) > 0$ . In other words, the reduction in the expected utility, which results from the introduction of the lottery, is higher for a lower initial amount of money  $m$ . If one thinks of this problem as a choice between consuming  $m$  in period zero and consuming  $m + \tilde{\varepsilon}$  in period one, then it can be said that the individual is better off with the initial consumption  $m$ , and the marginal benefit with respect to  $m$  depends on the sign of  $u'''(\cdot)$ .

The sign of the marginal utility premium has been shown to be very useful in identifying behavioral characteristics of the individual in a consumption-savings model. Kimball [20] considers a setting where the individual lives two periods, her utility is additively separable between periods, and she must decide how much to save during the first period. When a zero-mean lottery is added to the initial revenue during the second period, the individual saves more if and only if  $u'''(\cdot) > 0$  (Leland [21], Sandmo [28], Kimball [20]). Kimball [20] refers to this behavioral characteristic as prudence. Formally, if  $m$  is the initial revenue and a lottery  $\tilde{\varepsilon}$  (as previously defined) is added during the second period, then an individual who consumes  $C$  during the first period will be left with  $m - C + \tilde{\varepsilon}$  in the second period. If there were no risk, then the optimal consumption would be such that  $u'(C) = u'(m - C)$ . Taking the risk into account, the individual decides how much to consume according to rule  $u'(C) = \mathbb{E}[u'(m - C + \tilde{\varepsilon})]$ , which mirrors a trade-off between consumption in the first period and expected utility in the second period. Recall that  $\mathbb{E}[u'(m - C + \tilde{\varepsilon})] > u'(m - C)$  if and only if the individual is prudent. We see that the marginal utility is higher than in a risk-free situation. That is, the individual consumes less and saves more in the presence of risk. In substance, the reason why  $C$  is decreased is that, by doing so, the utility premium  $\hat{w}(m - C)$  is reduced in the second period. Hence, the trade-off experienced by the individual is a trade-off between the first-period consumption and utility premium, which arises because of uncertainty about the second period's revenues and decreases with  $C$  if and only if the individual is prudent.

In applications, scholars often refer to the coefficient of absolute prudence, namely,  $P(m) \equiv -u'''(m)/u''(m)$ , which was also introduced by Kimball, as a measure of the degree of prudence.

According to Eeckhoudt [6],  $P(m)$  measures the intensity of the prudence attitude, rather than the direction of prudence, which is expressed by the sign of  $u'''(m)$ . One benefit of the measure  $P(m)$  is that it permits one to identify the marginal value of the coefficient of absolute risk aversion  $A(m) \equiv -u''(m)/u'(m)$ . Indeed,  $A'(m) > 0$  if and only if  $A(m) > P(m)$ , hence, if and only if the degree of absolute risk aversion of the individual exceeds her degree of prudence. In the analysis developed hereafter, the results will depend mainly on the intensity of prudence.

### 3 Flexibility gain

Consider an individual who has an increasing and concave utility function  $u(m + \hat{\pi}(y, x))$ , where  $m$  is the initial wealth of the individual and  $\hat{\pi}(y, x)$  is a benefit function, such that  $\hat{\pi}(y, x) = S(y) - xy$ , with  $S(y)$  being the monetary value of the gross surplus derived from  $y$  units of some good and  $x$  being the unit cost of obtaining  $y$ . The surplus  $S(y)$  is such that  $S'(\cdot) > 0$ ,  $S''(\cdot) < 0$ , and  $S'''(\cdot)$  has a constant sign. If  $S(y)$  is a revenue function and  $x$  is the unit cost of production, then  $\hat{\pi}(y, x)$  is the profit obtained by the individual for producing  $y$  units of the good. If  $S(y)$  is the gross consumer surplus and  $x$  is the unit cost of acquisition, then  $\hat{\pi}(y, x)$  is the net consumer surplus. Denote  $y(x)$  as the unique level of  $y$  that maximizes  $\hat{\pi}(y, x)$ , and implicitly,  $u(\hat{\pi}(y(x), x))$ . Then,  $y'(x) < 0$ . The optimized benefit is  $\pi(x) \equiv \hat{\pi}(y(x), x)$ , so that the utility obtained by the individual, after deciding how many units  $y$  to obtain, is given by  $u(m + \pi(x))$ . For the sake of interpretation, we take both  $\pi'''(\cdot)$  and  $u'''(\cdot)$  to exist and have a constant sign over the relevant range of values.

The individual must decide which of the amounts between  $\pi(\theta)$  and  $\pi(\tilde{x})$  will be added to her initial wealth  $m$ , where  $\theta$  is known,  $\theta > 0$ , and  $\tilde{x}$  is unknown when this decision is made and uniformly distributed over  $[\theta - \eta, \theta + \eta]$ , for some  $\eta \in (0, \theta)$ .<sup>1</sup> The unknown benefit is preferable because

$$\mathbb{E}[u(m + \pi(\tilde{x}))] \geq u(m + \pi(\theta)).$$

Because  $\pi(\cdot)$  is determined endogenously the individual is better off with the expected utility of the unknown outcome than with the utility of the certain outcome. Henceforth, we will use the notation

$$w(m) = \mathbb{E}[u(m + \pi(\tilde{x}))] - u(m + \pi(\theta)) \quad (2)$$

to indicate the *flexibility gain* that the individual enjoys by conditioning her benefit on the realization of  $\tilde{x}$  rather than on  $\theta$ . Unlike the notion of the utility premium, flexibility gain refers to situations in which the risk-averse individual is better off with the expected utility than with the certain utility of money, when the amount of money  $\hat{\pi}(y, x)$  depends, in turn, on another decision variable  $y$ . In other words, the expected utility of the individual is not a downward shift of the certain utility, because, unlike in the benchmark, future brings about

<sup>1</sup>The reason why we take  $\tilde{x}$  to be uniformly distributed is that this allows us to focus exclusively on the effect that the changing support of the unit cost has on decision making. Otherwise, decision making would depend on a combination of the support of the unit cost and of its distribution function. The analysis and the results could easily extend to account for situations where the distribution function is not uniform.

information rather than risk. Defining

$$R(x) \equiv -\frac{y'(x)}{y(x)} \frac{1}{y(x)} \quad (3)$$

as the rate of decrease with respect to the unit cost  $x$  of the optimal production per production unit, the following result is obtained.

**Proposition 1**  $w'(m) \geq 0 (> 0)$  if

$$P(m + \pi(x)) \geq (>)R(x), \forall x \in [\theta - \eta, \theta + \eta], \quad (4)$$

and  $w'(m) \leq 0 (< 0)$  if the converse is true.

This result shows that a richer individual, who is also prudent, may value the flexibility more than a poorer individual, but only if she is prudent enough. The reason is that while opting for a flexible decision grants an obvious benefit to the individual, it also exposes her to the risk of facing a state of nature  $x$  that is above  $\theta$ , which is a cost. The higher the unit cost is, the lower the optimal production  $y(x)$ , so this cost is identified by  $R(x)$ , the rate of decrease of the optimal production with respect to  $x$  per production unit. When  $R(x)$  is large relative to the coefficient of absolute prudence, the individual values the flexibility less the richer she is. We also remark that, qualitatively, this result does not depend on the specific functional form of the profit function  $\hat{\pi}(y, x)$  that is adopted. More generally, we can write  $R(x) = \frac{\pi''(x)}{\pi'(x)} \frac{1}{\pi'(x)}$ , which here boils down to (3), because  $\pi'(x) = -y(x)$  and  $\pi''(x) = -y'(x)$ . Written in this way,  $R(x)$  represents the degree of convexity of the benefit function with respect to  $x$  per unit of marginal benefit. To interpret it, note that the reason why the benefit decreases with  $x$  at a decreasing rate, rather than at a linear rate, is that the quantity  $y(x)$  is optimally adjusted to  $x$ . The higher the degree of convexity of the benefit function, *i.e.*, the higher the  $-\frac{\pi''(x)}{\pi'(x)}$ , the costlier the quantity reduction is for the optimal benefit.

**Example 1** Consider the surplus function  $S(y) = a + \frac{1}{b} \ln(y)$ , for some  $a \geq 0$  and  $b > 1$ . This function is such that  $y(x) = \frac{1}{bx}$ ,

$$\pi(x_t) = a - \frac{1}{b} (\ln(b) + 1) - \ln\left(x_t^{\frac{1}{b}}\right) \quad (5)$$

and  $R(x) = b$ ; hence, the rate of decrease of the optimal production with respect to the unit cost  $x$  per production unit is constant over the interval  $[\theta - \eta, \theta + \eta]$ . Further, take the utility function  $u(n) = c_1 e^{-dn} + c_2$ , where  $c_1 < 0$ ,  $c_2 > 0$ , and  $n$  is an amount of money such that  $P(n) = d$ , namely, the coefficient of absolute prudence is constant. It is easy to observe that  $u(n)$  is a constant absolute risk aversion (CARA) utility function. Then, the preference of the individual for the set  $[\theta - \eta, \theta + \eta]$  over the singleton  $\theta$  is higher when the individual is richer if and only if  $d > b$ .

**Example 2** Now, take the surplus function considered in Example 1 and the CRRA utility function, namely,  $u(n) = \frac{n^{1-\gamma}-1}{1-\gamma}$ , if  $\gamma \neq 1$  and  $u(n) = \ln(\gamma)$  if  $\gamma = 1$ . With this function,

$A(n) = \gamma n^{-1}$  and  $P(n) = (\gamma + 1) n^{-1}$ . Therefore,  $P(n)$  increases with  $\gamma$ , and one can use the parameter  $\gamma$  to perform comparative statics not only on the coefficient of absolute risk aversion  $A(n)$  but also on the coefficient of absolute prudence  $P(n)$ . In this case, condition (4) is rewritten as

$$\frac{\gamma + 1}{m + \pi(x)} \geq b. \quad (6)$$

Recalling that the optimized benefit is given by (5), we can say that the individual values the set  $[\theta - \eta, \theta + \eta]$  more relative to  $\theta$  when she is richer, if and only if

$$\gamma \geq [b(m + a) - \ln(b) - 1 - \ln(x)] - 1.$$

When the individual is sufficiently risk-averse and, implicitly, with a CRRA utility function, sufficiently prudent, she is more likely to attach a higher value to the set  $[\theta - \eta, \theta + \eta]$  when she is richer. However, unlike in Example 1, it also depends on the size of her initial wealth. If the individual is rich enough, and  $m$  is large enough relative to  $\gamma$ , then (4) is violated.

### 3.1 Related concepts

Our definition of flexibility gain is related to the “value of information” that is usually considered in decision theory. In that definition, the utility obtained by the individual when the decision is to take the set of  $\tilde{x}$  rather than  $\theta$  is  $\max_y \mathbb{E}u(\hat{\pi}(y, \tilde{x}))$  rather than  $u(\pi(\theta))$  (see Gollier [13], chapter 24, for instance). However, even in that case, one may identify some degenerated variable  $\hat{\theta}$ , such that  $u(\pi(\hat{\theta})) = \max_y \mathbb{E}u(\hat{\pi}(y, \tilde{x}))$ .<sup>2</sup> In that case, one ends up again with the kind of analysis that we pursue in this study. Moreover, a shift from  $\theta$  to the interval  $[\theta - \eta, \theta + \eta]$  may be seen as a shift to a distribution with a higher spread, as considered by Meyer and Ormiston [26]. Unlike in their setting but similarly to that of Jones and Ostroy [19], this shift does not represent an increase in risk but a flexibility gain. Once again, this is due to the possibility for the decision maker to choose  $y$  only after the state of nature is realized. We obtain a result that is similar to that in Proposition 0 of Jones and Ostroy [19] if we assume that the individual has no wealth and calculates the marginal flexibility gain with respect to  $\theta$  rather than to her wealth. In the terminology of Jones and Ostroy [19],  $\theta$  represents the initial position. Without initial wealth,  $v(x) = u(\pi(x))$ , and the flexibility gain is given by  $f(\theta) = \mathbb{E}[v(\tilde{x})] - v(\theta)$ .

**Proposition 2** *The following equivalence holds*

$$f'(\theta) \geq (>)0 \Leftrightarrow v'''(\cdot) \geq (>)0. \quad (7)$$

One can read this result as follows. As the support  $[\theta - \eta, \theta + \eta]$  shifts rightwards, the individual has a higher value of flexibility if and only if the marginal value function  $v'(\cdot)$  is convex. Under risk neutrality, Proposition 2 would reduce to the following equivalence:

$$f'(\theta) \geq (>)0 \Leftrightarrow \pi'''(\cdot) \geq (>)0, \quad (8)$$

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<sup>2</sup>This approach is found, for instance, in Gollier [14].



which is similar to Proposition 0 of Jones and Ostroy [19], except that the shift from  $\theta$  to  $[\theta - \eta, \theta + \eta]$  does not represent a shift from an initial belief about the state of nature to a set of future beliefs. Let us first interpret (8). For this interpretation, we recall that  $\pi(x)$  is convex: namely, it decreases with the unit cost  $x$  at a decreasing rate. The reason why  $\pi(x)$  does not decrease linearly with  $x$  is that the optimal quantity  $y(x)$  is optimally adjusted to any realization  $x$  of the unit cost. In addition, when  $\pi'''(x) \geq 0$ , the decreasing rate of the marginal profit is more pronounced for higher values of  $x$  and thus for a rightward shift of the support  $[\theta - \eta, \theta + \eta]$ . This involves the individual gaining more in flexibility, because for any given  $\delta \in (0, \eta]$ , the profit increase of  $\pi(\theta - \delta) - \pi(\theta)$ , which is obtained when the state is  $\theta - \delta$  rather than  $\theta$ , is larger than the profit decrease of  $\pi(\theta) - \pi(\theta + \delta)$ , which is incurred when the state is  $\theta + \delta$  instead of  $\theta$ . Turning back to risk aversion the interpretation of Proposition 2 is similar because the important factor for the result is the shape of  $v'(x)$ , not the sign of  $v''(x)$ , which can be either positive or negative unlike  $\pi''(x)$ . When the individual is not very risk-averse, and the profit does not decrease sharply with the unit cost  $x$ , the marginal value function decreases smoothly with  $x$ , so  $v''(x) > 0$ . The sign of  $v''(x)$  matters if one needs to identify the properties of the utility function, which determine whether the flexibility gain increases or decreases, as in Proposition 1.

## 4 Investment timing

We now turn to apply the concept of flexibility gain, as previously defined, to the investment timing problem. The properties of the utility function play a role, because financial markets are incomplete. We take  $x$  to follow a geometric Brownian motion, such that

$$dx_t = \alpha x_t dt + \sigma x_t dz_t, \quad (9)$$

where  $z_t$  is a simple Brownian motion, with  $\alpha$  and  $\sigma$  being the drift and volatility parameters, respectively. We also specify the surplus function as in the two examples above and the CARA utility function as in Example 1 so that  $S(y) = a + \frac{1}{b} \ln(y)$  and  $u(n) = c_1 e^{-dn} + c_2$ .<sup>3</sup> Starting from an initial situation in which the individual's wealth is equal to  $m + K$ , where  $K$  is the fixed cost of some investment opportunity, the individual must decide at which instant  $T$  to spend  $K$  and obtain the profit  $\pi(x_t)$  for any  $t \geq T$ . Taking the initial value of the unit cost to be  $x_0$ , the individual invests immediately if  $x_0 < x^*$ ; otherwise, the individual delays the investment until the stochastic date  $T$ , defined as follows:

$$T = \inf \{t \geq 0, \text{ s.t. } x_t = x^*\}.$$

For simplicity, we do not allow for disinvestment.

We now proceed to solve the investment timing problem analytically. With the previously

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<sup>3</sup>We do not refer to the CRRA utility function, which is usually considered in investment timing problems under risk aversion, for technical reasons. A closed-form solution with a CRRA function could be obtained under the hypothesis that the individual has no initial wealth.

defined surplus function, we again obtain the optimized profit

$$\pi(x_t) = a - \frac{1}{b} (\ln(b) + 1) - \ln\left(x_t^{\frac{1}{b}}\right) \quad (10)$$

for any given  $t \geq T$ . We take  $\delta(b) > 0, \forall b$ , where

$$\delta(b) = a - \frac{1}{b} (\ln(b) + 1) - rK \quad (11)$$

and  $r$  is the risk-free interest rate. With this assumption, we can say that in the absence of any cost  $x_t$ , the investor obtains more in period  $t$  from the investment of cost  $K$  than she would obtain by investing the same amount in a riskless asset. We note also that  $\delta'(b) > 0$ . The higher  $b$  the higher such benefit because, as from (10), the non-stochastic part of the profit increases. The reason why is that the marginal surplus decreases with  $b$  at a lower rate. Indeed,  $S''(y) = -\frac{1}{by^2}$ . As in Chronopoulos et al. [4], the individual obtains a certain return of  $r(m + K)$  on her initial wealth  $m + K$  before making the investment, where  $r$  is the risk-free interest rate. With the previously specified utility function, her instantaneous utility is given by

$$u(r(m + K)) = c_2 + c_1 e^{-dr(m+K)}. \quad (12)$$

After making the investment, at each instant  $t \geq T$ , the individual obtains

$$\begin{aligned} u(rm + \pi(x_t)) &= c_2 + c_1 e^{-d(rm + \pi(x_t))} \\ &= c_2 + c_1 e^{-d(rm + a - \frac{1}{b}[\ln(b) + 1])} x_t^{\frac{d}{b}}. \end{aligned} \quad (13)$$

Further taking  $\rho$  as the subjective rate of time preference, such that  $\rho > \alpha$ , the investor solves the following maximization problem:

$$\underset{T}{Max} V_0 + \mathbb{E}_{x_0} e^{-\rho(t-T)} V_1, \quad (14)$$

where

$$V_0 = \int_0^\infty e^{-\rho t} \{u(r(m + K))\} dt$$

regroups the discounted utilities obtained by the individual with an initial wealth of  $m + K$  at each instant  $t \in (0, \infty)$ , as if the individual never invests  $K$  in the project, and

$$V_1 = \int_T^\infty e^{-\rho(t-T)} \{u(rm + \pi(x_t)) - u(r(m + K))\} dt - \int_T^\infty e^{-\rho(t-T)} \{u(r(m + K))\} dt$$

regroups the discounted utilities obtained by the individual after investing, net of the utility derived from the initial wealth at each instant  $t \geq T$ . Replacing (12) and (13) in  $V_1$  and rearranging, we obtain

$$V_1 = c_1 e^{-d(rm + a - \frac{1}{b}[\ln(b) + 1])} \int_T^\infty e^{-\rho(t-T)} x_t^{\frac{d}{b}} - e^{-dr(m+K)} \int_T^\infty e^{-\rho(t-T)} dt.$$

With standard calculations, we can write<sup>4</sup>

$$\int_T^\infty e^{-\rho(t-T)} x_t^{\frac{d}{b}} dt = \frac{(x_T)^{\frac{d}{b}}}{\rho - \frac{d}{b}\alpha - \frac{1}{2}\frac{d(d-b)}{b^2}\sigma^2}, \quad (15)$$

provided  $\rho > \frac{d}{b}\alpha + \frac{1}{2}\frac{d(d-b)}{b^2}\sigma^2$ . Moreover,  $\int_T^\infty e^{-\rho(t-T)} dt = \frac{1}{\rho}$ . Hence,

$$V_1 = \frac{c_1 e^{-d(rm+a-\frac{1}{b}[\ln(b)+1])}}{\rho - \frac{d}{b}\alpha - \frac{1}{2}\frac{d(d-b)}{b^2}\sigma^2} (x_T)^{\frac{d}{b}}.$$

Also,

$$V_0 = \frac{c_2 + c_1 e^{-dr(m+K)}}{\rho}.$$

Moreover,

$$\mathbb{E}_{x_0} (e^{-rT}) = \left( \frac{x_0}{x_T} \right)^{\beta_2}, \quad (16)$$

where  $\beta_2$  is the negative root of the quadratic equation<sup>5</sup>

$$\beta(\beta - 1) \frac{\sigma^2}{2} + \alpha\beta = \rho. \quad (17)$$

We can thus reformulate the maximization problem (14) as follows:

$$\text{Max}_{x_T} \frac{c_2 + c_1 e^{-dr(m+K)}}{\rho} + \left( \frac{x_0}{x_T} \right)^{\beta_2} c_1 \left[ \frac{e^{-d(rm+a-\frac{1}{b}[\ln(b)+1])}}{\rho - \frac{d}{b}\alpha - \frac{1}{2}\frac{d(d-b)}{b^2}\sigma^2} x_T^{\frac{d}{b}} - \frac{e^{-dr(m+K)}}{\rho} \right].$$

The optimal threshold  $x^*$  that triggers an immediate investment is found to be

$$x^* = \left( e^{d\delta(b)} \frac{\rho - \frac{d}{b} [\alpha + \frac{1}{2} (\frac{d}{b} - 1) \sigma^2]}{\rho} \right)^{\frac{b}{d}} \left( \frac{\beta_2}{\beta_2 - \frac{d}{b}} \right)^{\frac{b}{d}}, \quad (18)$$

whenever  $x^* > x_0$ . Otherwise, the investment should take place with no delay. Before presenting the next result, we note that the following condition

$$\delta(b) < \frac{\alpha + (\frac{d}{b} - \frac{1}{2}) \sigma^2}{b \left\{ \rho - \frac{d}{b} [\alpha + \frac{1}{2} (\frac{d}{b} - 1) \sigma^2] \right\}} \quad (19)$$

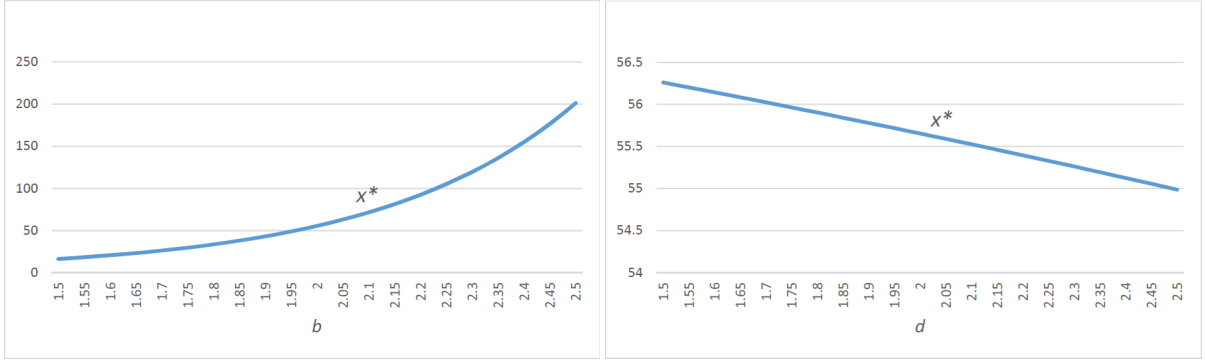
tightens as  $b$  increases.

**Proposition 3**  $x^*$  increases with  $b$ . If  $b$  is not too large, so that (19) is satisfied, then  $x^*$  decreases with  $d$ .

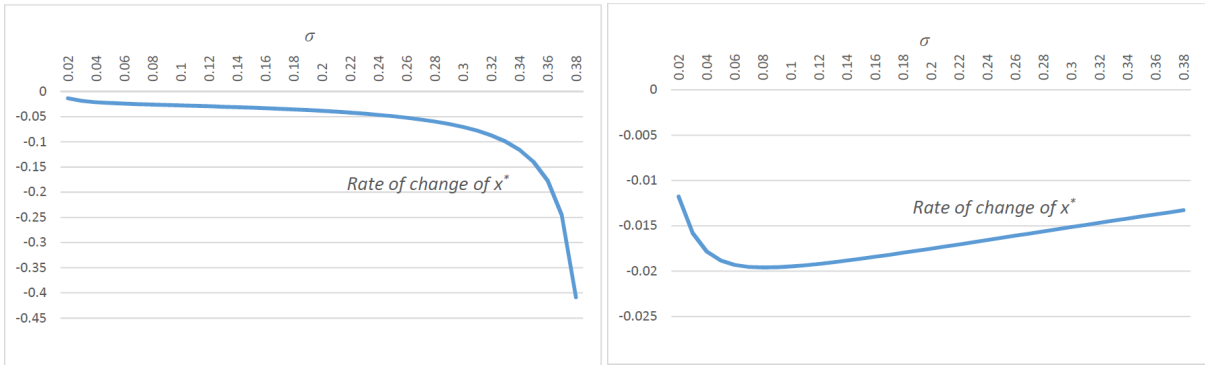
The reason why  $x^*$  increases with  $b$  is that, as already explained, the non-stochastic part of the instantaneous profit  $\pi(x_t)$  increases. Thereby there is less value to delaying the investment. When  $b$  is not too large, so that (19) is satisfied, the economic value of waiting is high, regardless

<sup>4</sup>See, for instance, Dixit and Pindyck [5], page 82.

<sup>5</sup>See, for instance, Dixit and Pindyck [5] pages 142-144.



**Figure 1:**  $x^*$  as a function of  $b$  for  $d = 2$  and  $\sigma = 0.1$  (left) and as a function of  $d$  for  $b = 2$  and  $\sigma = 0.1$  (right)



**Figure 2:** Rate of change of  $x^*$  as a function of  $\sigma$  for  $d = 1.5$  and  $b = 2.5$  (left); as a function of  $\sigma$  for  $d = 2.5$  and  $b = 1.5$  (right)

of the characteristics of the utility function of the individual. Important for our analysis is that, in that case,  $x^*$  decreases with  $d$ . The more prudent the individual is the more she prefers to delay the investment. This shows that there is a link between prudence and the optimal stopping time. Referring to the definition of flexibility gain and resting on Proposition 1, we can say that the individual is likely to delay the investment more when she is richer if and only if she is sufficiently prudent. In that case, the individual is prone to forego the benefit of an immediate investment and preserve flexibility in her investment decision. We further remark that  $x^*$  decreases with the cost of investment  $K$ , which is obvious, and that (18) is independent of  $m$ . Intuitively, this is due to the fact that the absolute degree of prudence is constant. The size of  $d$  is a measure of the degree of prudence of the individual, which in turn expresses how much the individual values flexibility when she is richer, as Proposition 1 shows. Indirectly, the threshold  $x^*$  at which the investment is triggered depends on the wealth of the individual.

## Numerical example

Take  $\rho = 0.1$ ,  $\alpha = 0.01$ ,  $r = 0.05$ ,  $K = 1000$  and  $a = 53$ . The graphs in Figure 1 and 2 are drawn for  $\sigma = 0.1$ . Taking  $d = 2$ , Figure 1 left highlights that  $x^*$  increases with  $b$ ; further taking  $b = 2$ , Figure 1 right highlights that  $x^*$  decreases with  $d$ . One can easily deduce that the investment is more delayed the higher the ratio  $d/b$  is. Figure 2 highlights how much the trigger value  $x^*$  decreases with the uncertainty parameter  $\sigma$ , for a case where  $d < b$  (graph

to the left) and for a case where  $d > b$  (graph to the right). Not surprisingly,  $x^*$  decreases with  $\sigma$  in either case, showing that the more uncertain the evolution of the stochastic variable is the higher the value of waiting. Hence, the rate of change of  $x^*$  is negative in either case. Interestingly, by comparing the two cases, we see that the decrease in  $x^*$  induced by an increase in the uncertainty parameter is more pronounced when  $d < b$  than otherwise, especially when the uncertainty parameter is high. This can also be deduced from (18). In particular, if  $\sigma$  increases from 0.09 to 0.1, then  $x^*$  decreases by 2.8% when  $d = 2.5$  and  $b = 1.5$ ; it decreases by 1.9% when  $d = 1.5$  and  $b = 2.5$ . If  $\sigma$  increases from 0.37 to 0.38, then  $x^*$  decreases by 40% when  $d = 2.5$  and  $b = 1.5$ ; it decreases by only 1.3% when  $d = 1.5$  and  $b = 2.5$ . In line with the explanation previously provided, the value of waiting is higher the more prudent the individual is, and/or the lower the rate of change of the benefit function is as  $x_t$  varies across periods. As the value of waiting increases with the uncertainty parameter, it is natural that the impact of uncertainty on the investment delay is more important the higher the ratio  $d/b$  is.

## 5 Optimal prevention

A number of studies in decision theory have investigate the optimal preventive effort to be made to reduce the probability of occurrence of an adverse effect (Gollier et al. [15] and Eeckhoudt and Gollier [7], Eeckhoudt and Gollier [7], Eeckhoudt et al [6], Menegatti [23] and Peter [27]). A link between the prudence attitude and the optimal preventive effort is usually identified.<sup>6</sup> A natural question is whether the flexibility gain depends on the prudence attitude when the individual makes an investment that consists of preventing a risk, as in the aforementioned literature.

An underlying assumption of the literature is that there is no benefit to information acquisition before the effort is chosen. To allow for such a possibility, we build on the model of Eeckhoudt and Gollier [7] and allow for the cost of the effort to depend on the technology that is chosen. One technology is such that the individual bears a cost of  $\psi(e, \theta) = e + \theta$ , where  $e$  is the amount of effort and  $\theta$  is part of the cost borne by the individual, which is independent of the size of the effort. A higher effort corresponds to a lower probability  $p(e)$  of incurring a loss, which amounts to  $L > 0$ . Hence,  $p'(e) < 0$ . The expected utility of the individual, if the preventive effort is made in period zero, is written as

$$p(e) u(m - L - e - \theta) + (1 - p(e)) u(m - e - \theta),$$

which we assume to be concave in effort.<sup>7</sup> Another technology is such that the cost is  $\psi(e, x)$ , where  $x$  is unknown when the technology is chosen and uniformly distributed over  $[\theta - \eta, \theta + \eta]$ .

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<sup>6</sup>In particular, Eeckhoudt and Gollier [7] find the counterintuitive result that a prudent individual makes less preventive effort than an imprudent individual. Menegatti [23] uses a different assumption that the cost of the effort is not incurred at the same time that the utility is obtained. With this assumption, Menegatti obtains an opposite result. Peter [27] further shows that the result obtained by Menegatti [23] holds if and only if the individual makes no savings between periods.

<sup>7</sup>In line with Eeckhoudt and Gollier [7] and to facilitate a comparison with their analysis, we assume that there is no inter-temporal separation in the utility function between the individual's initial wealth and the cost of the effort.

Hence, the cost of the effort with the second technology may be either above or below the cost of the effort corresponding to the first technology. If the second technology is adopted, then the expected utility is given by

$$p(e) u(m - L - e - x) + (1 - p(e)) u(m - e - x), \quad (20)$$

for some realized  $x$ . We let  $e(x, L)$  be the optimal effort, where  $x = \theta$  if the first technology is chosen. For convenience, we use the notation  $k(e, x) = m - e - x$ , together with

$$r(k(e, x), L) = p(e) [u(k(e, x)) - u(k(e, x) - L)].$$

Accordingly, the expected utility expressed in (20) is rewritten as

$$u(k(e, x)) - r(k(e, x), L),$$

where  $u(k(e, x))$  is the utility obtained in the good state and  $r(k(e, x), L)$  is the reduction in the expected utility, which is due to a disaster occurring with a positive probability. The optimal effort  $e(x, L)$  solves

$$u'(k(e, x)) = r_k(k(e, x), L), \quad (21)$$

where  $u'(k(e, x))$  is the marginal utility and  $r_k(k(e, x), L) \equiv \frac{dr(k(e, x), L)}{dk}$  is the marginal expected reduction in the utility due to the unfavorable event. We see that (21) expresses a trade-off between the utility in the good state and the expected reduction. Because  $r_k(k(e, x), L)$  shifts outwards when  $L$  is increased, one has  $e_L(x, L) > 0$ , where  $e_L(x, L) \equiv \frac{\partial e(x, L)}{\partial L}$ . Intuitively, as the expected cost of the disaster increases, the individual is ready to exert more effort. Moreover, as the cost of the effort  $x$  increases, for any given level of effort, both the marginal utility and the marginal expected reduction in the utility increase. As a result, the sign of  $e_x(x, L) \equiv \frac{\partial e(x, L)}{\partial x}$  is ambiguous. For simplicity, we hereafter focus on situations where  $e_x(x, L) > 0$ .

We denote

$$V(x) \equiv u(k^*(x, L)) - r^*(x, L), \quad (22)$$

the optimized utility of the individual, where  $k^*(x, L) \equiv k(e(x, L), x)$  and

$$r^*(x, L) \equiv r(k(e(x, L), x), L).$$

We will show that, unlike in the situations that we previously considered, it is not given that the individual benefits from a flexibility gain by choosing the set  $[\theta - \eta, \theta + \eta]$  instead of  $\theta$ .

As a first step, we present the following preliminary result.

**Lemma 1**  $V'(x) = p'(e(x)) [u(k^*(x, L)) - u(k^*(x, L) - L)] = \frac{p'(e(x, L))}{p(e(x, L))} r^*(x, L)$ .

The marginal expected utility with respect to the cost of the effort  $x$  is negative. To interpret this result, first consider the effect of a higher cost of the effort on the optimal effort  $e(x, L)$ . The value of  $V'(x)$  is given by the product between the rate of decrease of the probability of the disaster with respect to a marginal increase in effort  $e(x, L)$  (namely  $\frac{p'(e(x, L))}{p(e(x, L))} < 0$ ), and the

reduction in the expected utility. In this way, the expression of  $V'(x)$  represents the extent to which the total expected utility is decreased as the cost of the effort increases, and the optimal effort  $e(x, L)$  is adjusted accordingly.

As a second step, we express the flexibility gain of the delay as follows:

$$f_p(\theta) = \mathbb{E}[V(x)] - V(\theta) = \frac{1}{2\eta} \int_{-\eta}^0 \int_{\theta}^{\theta+\eta} V''(z) dz dx.$$

We see that whether or not the individual experiences a flexibility gain when choosing  $[\theta - \eta, \theta + \eta]$  over  $\theta$  depends on the sign of  $V''(z)$ . If the expected utility  $V(x)$  decreases with the cost of the effort at a decreasing rate, then  $f_p(\theta) > 0$ . In the opposite case, the individual should exert effort  $e(\theta, L)$ . Its cost is certain and lower than the cost she would face with the second technology if  $x > \theta$ . Given that the utility  $u(k(e, x))$  decreases with  $x$ , the set  $[\theta - \eta, \theta + \eta]$  is preferable to  $\theta$  if and only if  $u(k(e, x))$  decreases at a lower rate than  $r(k(e, x), L)$  increases with  $x$ . In this way, the individual gains more from experiencing a lower cost of effort ( $x < \theta$ ) than loses by facing a higher cost of effort ( $x > \theta$ ). Moreover, the sign of the flexibility gain is related to the risk aversion level of the individual, as the next result shows.

**Proposition 4** *The following conditions are equivalent:*

1.  $f_p(\theta) \geq 0$ ;
2.  $V''(\cdot) \geq 0$ ;
3.  $A(k^*(x, l) - l) \geq -\frac{p''(e(x))}{p'(e(x))} \frac{e_x(x, L)}{e_x(x, L)+1}, \forall l \in [0, L]$ .

Knowing that the expected cost of the effort drawn from  $[\theta - \eta, \theta + \eta]$  is equal to  $\theta$ , a risk-neutral individual is indifferent between exerting and delaying the preventive effort. Instead, if  $p''(e(x)) > 0$ , then a risk-averse individual decides to take the lottery if and only if her risk aversion is sufficiently pronounced. In this latter case, effort decreases the likelihood of a disaster at a decreasing rate. On one hand, flexibility is beneficial because it may lead to a lower cost of effort. The more risk-averse the individual, the more the individual dislikes taking the risk of sustaining the cost of the effort  $\theta$ , instead of obtaining more information and adjusting her effort accordingly. On the other hand, flexibility has a cost, given the effort that must be adjusted based on changes in  $x$ . The higher the rate at which the marginal likelihood of the disaster increases with the effort (*i.e.*, the higher  $-\frac{p''(e(x))}{p'(e(x))}$  is), and/or the more the effort must be increased with  $x$  (*i.e.*, the higher  $\frac{e_x(x, L)}{e_x(x, L)+1}$  is), the higher the cost of flexibility is, given the possibility that  $x$  will increase switching from one technology to the other technology.

In the same vein as in the previous analysis, the sign of the marginal flexibility gain depends on the degree of prudence of the individual. However, unlike in the situation where the investment is not a preventive effort, the more prudent the individual is, the less likely it is that the marginal flexibility gain is positive.

**Corollary 1** *The following conditions are equivalent:*

1.  $f'_p(\theta) \geq 0$ ;

2.  $V'''(\cdot) \geq 0$ ;

$$3. P(k^*(x, l) - l) \leq \frac{[p''(e(x, L))e_x(x, L)]'}{p'(e(x, L))(e_x(x, L)+1)^2} \frac{1}{A(k^*(x, l)-l)} + \frac{p''(e(x, L))}{p'(e(x, L))} + \frac{e_{xx}(x, L)}{(e_x(x, L)+1)^2}, \forall l \in [0, L].$$

Recall from Proposition 4 that the more risk-averse the individual, the more likely is that the individual will prefer flexibility. Taking, for instance,  $p''(e(x)) > 0$ , it is not surprising that condition 3. in the corollary is more relaxed the more risk averse the individual is. Indeed, in that case, the individual benefits more from flexibility, given that the support of costs  $[\theta - \eta, \theta + \eta]$  shifts rightwards and, hence, all states become less efficient. On the other hand, condition 3 in the corollary is tighter for a more prudent individual. Unlike in our previous analysis, this is exactly due to the fact that the effort is made to prevent a disaster rather than to seize an investment opportunity. Here, the investor cannot simply reduce her effort when it is more costly, because this behavior would make the disaster more likely.

## 6 Other related concepts and applications

### 6.1 Preference over specific lotteries

Bigelow and Menezes [2] and Eeckhoudt and Schlesinger [10] express prudence in terms of preference over lotteries  $\mathbf{A} = (0; \tilde{\varepsilon} - k)$  and  $\mathbf{B} = (-k; \tilde{\varepsilon})$ . In each lottery, the two outcomes are equally likely,  $k$  is known, and  $\tilde{\varepsilon}$  is unknown such that  $\mathbf{E}(\tilde{\varepsilon}) = 0$ . The individual is prudent if and only if lottery  $\mathbf{B}$  is preferred to lottery  $\mathbf{A}$ . In the words of Eeckhoudt and Schlesinger [10], lottery  $\mathbf{B}$  allows for a disaggregation of the two harms  $\tilde{\varepsilon}$  and  $-k$ , which explains why an individual with  $u'''(\cdot) > 0$  prefers  $\mathbf{B}$  to  $\mathbf{A}$ . The preference for  $\mathbf{B}$  over  $\mathbf{A}$  is proven to be equivalent to  $u'''(\cdot) \geq 0$  by assuming that the individual has some initial wealth  $m$ , which adds up to each of the potential outcomes of the selected lottery. Then,  $\mathbf{B}$  is preferred to  $\mathbf{A}$  if and only if

$$\mathbb{E}(u(m + \tilde{\varepsilon})) - \mathbb{E}(u(m + \tilde{\varepsilon} - k)) \geq u(m + 0) - u(m - k).$$

By taking  $k$  to be arbitrarily small, this reduces to  $\mathbb{E}(u'(m)) \geq u'(m)$ , which is equivalent to  $u'''(\cdot) \geq 0$  by Jensen's inequality.

Remarkably, the preference for  $\mathbf{B}$  over  $\mathbf{A}$ , as previously defined, can be reinterpreted by making specific considerations about the range of values from which  $\tilde{\varepsilon}$  is drawn. Suppose that  $\tilde{\varepsilon} \in [-\hat{\varepsilon}, \hat{\varepsilon}]$ , for some  $\hat{\varepsilon} > 0$ . If  $k < \hat{\varepsilon}$  and lottery  $\mathbf{A}$  is used, then the outcome is included in the interval  $[-\hat{\varepsilon} - k, \hat{\varepsilon} - k]$ . If lottery  $\mathbf{B}$  is used, then the outcome lies in  $[-k, \hat{\varepsilon}]$ . If  $k > \hat{\varepsilon}$ , then the outcomes of lottery  $\mathbf{A}$  are drawn from the interval  $[-\hat{\varepsilon} - k, 0]$ ; those of lottery  $\mathbf{B}$  are drawn again from the interval  $[-k, \hat{\varepsilon}]$ . In either case, we observe an *upward shift in the support* of the final outcomes when switching from lottery  $\mathbf{A}$  to lottery  $\mathbf{B}$ , despite earning less with lottery  $\mathbf{B}$  when  $\tilde{\varepsilon}$  turns out to be high. Hence, we can say that a prudent individual has a specific preference for a shift in that support.

In the literature, the preference for  $\mathbf{B}$  over  $\mathbf{A}$  is considered to be a transposition in terms of the behavior of the preference for a certain outcome over an uncertain outcome, as described in the benchmark (see Eeckhoudt and Schlesinger [10]). We hereafter show that the preference for



an unknown outcome over a certain outcome can be represented in terms of individual behavior, particularly, as a choice between two specific lotteries.

To this end, we proceed in two steps. First, instead of considering an individual who has an initial wealth of  $m$  and plays a lottery, as in the benchmark, we suppose that the individual has to choose between the two lotteries  $\mathbf{A}' = (g(\theta); g(\theta) + \tilde{\varepsilon} - k)$  and  $\mathbf{B}' = (g(\theta) - k; g(\theta) + \tilde{\varepsilon})$ , where  $f(\theta)$  is again an exogenously given profit. This profit includes the initial wealth and is net of any cost that must be incurred to obtain an outcome. As is the case of  $\mathbf{A}$  and  $\mathbf{B}$ , the outcomes of  $\mathbf{A}'$  and those of  $\mathbf{B}'$  are both equally likely. Now, observe that each of the lotteries  $\mathbf{A}'$  and  $\mathbf{B}'$  includes some given amount (say, a “known” outcome) and a lottery (say, an “unknown” outcome).  $\mathbf{B}'$  is preferable to  $\mathbf{A}'$  when  $u'''(\cdot) > 0$ , because the net gain for the unknown outcome in the former lottery exceeds the net gain for the known outcome in the latter lottery:

$$\mathbb{E}(u(m + g(\theta) + \tilde{\varepsilon})) - \mathbb{E}(u(m + g(\theta) + \tilde{\varepsilon} - k)) \geq u(m + g(\theta)) - u(m + g(\theta) - k).$$

As a second step, we return to our setting and let the profit be endogenous, namely  $\pi(\cdot) = \hat{\pi}(y(\cdot), \cdot)$ , as previously defined in the investment problem. The individual must choose between the lotteries  $\mathbf{A}''$  and  $\mathbf{B}''$ , instead of  $\mathbf{A}'$  and  $\mathbf{B}'$ , where  $\mathbf{A}'' = (\pi(\theta); \pi(x - \hat{k}))$ ,  $\mathbf{B}'' = (\pi(\theta - \hat{k}); \pi(x))$ ,  $\theta$  and  $\hat{k}$  are known,  $x$  is unknown,  $\mathbb{E}(x) = \theta$  and  $x \in [\theta - \eta, \theta + \eta]$ . Essentially, as in the previous cases, the individual gains more in terms of a certain outcome with lottery  $\mathbf{A}''$  than with lottery  $\mathbf{B}''$ , whereas the individual gains less in terms of an unknown outcome. Again, the outcomes of  $\mathbf{A}''$  and those of  $\mathbf{B}''$  are equally likely.

**Proposition 5** *Lottery  $\mathbf{B}''$  is preferred to lottery  $\mathbf{A}''$  if (4) is satisfied, and lottery  $\mathbf{A}''$  is preferred to lottery  $\mathbf{B}''$  if the converse of (4) is satisfied.*

## 6.2 Preference over distributions with different means and/or spreads

Let us define the utility function of the risk-neutral individual as  $S(y) - x_{ij}y$  and assume that  $x_{ij} = \theta_i + \tilde{\sigma}_j$ , where  $\theta_i$  is given,  $\tilde{\sigma}_j \in [-\eta_j, \eta_j]$ ,  $i \in \{1, 2\}$ ,  $j \in \{1, 2\}$ ,  $\theta_2 > \theta_1$ ,  $\eta_2 > \eta_1$  and  $\mathbb{E}(\tilde{\sigma}_j) = 0$ ,  $\forall j$ . The individual must select one of the four cost distributions indexed by  $ij$ . As the distributions are defined, two of the distributions have the same mean, and two distributions have the same spread. Hence, one can associate the former pair with first-order stochastic dominance and the latter pair with mean-preserving spread, which is a particular case of second-order stochastic dominance. In this sense, it is easy to check that the individual will prefer  $1j$  to  $2j$  and  $i2$  to  $i1$ . However, neither first- nor second-order stochastic dominance applies if the four distributions are considered altogether. As a result, one cannot apply the usual reasoning to identify the preference order of the four distributions. We will prove that the third derivative of the function  $S(\cdot)$  measures the extent to which the individual prefers to be faced with  $1j$  instead of  $2j$ , and with  $i2$  instead of  $i1$ . To this end, we define

$$D_{ij/i'j'} = \mathbb{E}[\pi_{ij} - \pi_{i'j'}],$$

where

$$\pi_{ij} = S(y(x_{ij})) - x_{ij}y(x_{ij}), \forall ij.$$

For instance,  $D_{1j/2j}$  measures the additional gain granted by a technology associated with an expected unit cost of  $\theta_1$  relative to one associated with an expected unit cost of  $\theta_2$ , for any given value  $\eta_j$  of the spread.

**Proposition 6**  $\forall j$ ,  $D_{1j/2j}$  and  $D_{11/22}$  both decrease with  $\pi'''(\cdot)$ .

Note also that the preference for a set  $[\theta_1 - \eta_j, \theta_1 + \eta_j]$  over  $[\theta_2 - \eta_j, \theta_2 + \eta_j]$  is further explained by the properties of the surplus function  $S(\cdot)$  of the individual. Denote

$$\mathcal{S}(x) = -\frac{S'''(y(x))}{S''(y(x))}$$

as the degree of convexity/concavity of the marginal surplus  $S'(y(x))$  in point  $x$ , depending on whether  $S'(y(x))$  is convex or concave. Rewriting

$$\begin{aligned} D_{1j/2j} &= -\mathbb{E} \left[ \int_{x_{1j}}^{x_{2j}} \pi'(z) dz \right] = \mathbb{E} \left[ \int_{x_{1j}}^{x_{2j}} y(z) dz \right] \\ &= \frac{1}{2\eta} \int_{\theta_1 - \eta_j}^{\theta_1} \int_0^{\Delta\theta} \int_0^{\eta_j} y''(x + z + t) dt dz dx + \frac{1}{2\eta} \int_{\theta_1}^{\theta_2} y(x) dx \end{aligned}$$

and

$$D_{11/22} = D_{11/21} + \frac{1}{2\eta} \left[ \int_{\theta_2 - \eta_1}^{\theta_2 - \eta_2} y(x) dx + \int_{\theta_2 + \eta_1}^{\theta_2 + \eta_2} y(x) dx \right],$$

and knowing that  $y'(x) = \frac{1}{S''(y(x))}$  and  $y''(x) = \frac{\mathcal{S}(x)}{[S''(y(x))]^2}$ ,  $\forall x$ , it is easy to observe that both  $D_{1j/2j}$  and  $D_{11/22}$  increase (decrease) with the degree of convexity (concavity) of the surplus function. If two of the four distributions were to share the same support and could be ordered in the sense of either first-order stochastic dominance or mean-preserving spread, then the preference for one distribution over the other would be unrelated to the properties of the surplus function.

## 7 Conclusion

The general lesson of our analysis is that in situations where the individual has to make an irreversible decision and can acquire useful information prior to the decision, it is necessary to know both her prudence attitude and the properties of her benefit function, in order to assess the economic value of the additional information. A more specific outcome of the analysis concerns the notion of prudence, which is related not only to the notion of the utility premium, as usually considered by the literature, but also to the notion of flexibility gain. A richer individual, who is also prudent, attaches a higher subjective value to flexibility, only if the economic value of that flexibility is *per se* sufficiently high. This finding is especially relevant in investment timing decisions, in which, as is well known, there is often an economic value to flexibility ensuing

from an investment delay. The link we identified between prudence and flexibility gain points to the conclusion that not only risk aversion but also the prudence attitude is relevant in optimal investment timing decisions. Moreover, one cannot look at decisions about the investments that represent financing opportunities in the same manner as decisions about investments aiming to prevent the occurrence of a negative event.

We highlighted the equivalence between a preference for flexibility and a preference over distributions that differ in terms of expected value or expected value and spread. This equivalence is very useful in applications. To provide one example, in principal-agent relationships in which the principal is risk-neutral and the agent holds private knowledge about the mean and spread of a variable that is relevant in the trade, the rent that the principal is prone to concede to the agent to elicit information depends on the properties of her benefit function, just as in the case of the flexibility gain. This link is essential for determining the characteristics of the optimal contract to be offered by the principal to the agent.

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## A Proof of Proposition 1

Using the notation  $\gamma(m, x) = m + \pi(x)$ , we calculate:

$$\begin{aligned}
w'(m) &= \mathbb{E} [u'(m + \pi(x))] - u'(m + \pi(\theta)) \\
&= \mathbb{E} [u'(\gamma(m, x))] - u'(\gamma(m, \theta)) \\
&= \frac{1}{2\eta} \left\{ \int_0^{+\eta} \{ [u'(\gamma(m, x))] - u'(\gamma(m, \theta)) \} dx - \int_{-\eta}^0 \{ u'(\gamma(m, \theta)) - u'(\gamma(m, x)) \} dx \right\} \\
&= \frac{1}{2\eta} \int_0^{+\eta} \left[ \int_{\theta}^{\theta+x} u''(\gamma(m, z)) \pi'(z) dz - \int_{\theta-x}^{\theta} u''(\gamma(m, z)) \pi'(z) dz \right] dx \\
&= \frac{1}{2\eta} \int_0^{+\eta} \int_{\theta-x}^{\theta} \int_z^{z+x} \left\{ u'''(\gamma(m, t)) [\pi'(t)]^2 + u''(\gamma(m, t)) \pi''(t) \right\} dt dz dx
\end{aligned}$$

Using also  $u''(\cdot) < 0$ , we deduce that  $w'(m) \geq (> 0)$  if

$$-\frac{u'''(\gamma(m, x))}{u''(\gamma(m, x))} \geq (>) \frac{\pi''(x)}{[\pi'(x)]^2}, \forall x, \quad (23)$$

and  $w'(m) < 0$  when the converse is true. Finally, recalling that  $y(x) = \operatorname{argmax} \hat{\pi}(y, x)$  and that  $\pi(x) = \hat{\pi}(y(x), x)$ , the envelope theorem implies that  $\pi'(x) = -y(x)$  and  $\pi''(x) = -y'(x)$ . Using them in (23) and then replacing  $\gamma(m, x) = m + \pi(x)$ , we obtain (4).

## B Proof of Proposition 2

Because  $v(x) = u[\pi(x)]$  we can rewrite (7) as

$$g'(\theta) = \frac{1}{2\eta} \int_{-\eta}^{+\eta} [v(\theta + x)]' dx - v'(\theta).$$

Proceeding as in the proof of Proposition 1, the above equality is rewritten as

$$g'(\theta) = \frac{1}{2\eta} \int_0^{+\eta} \int_{\theta-x}^{\theta} \int_z^{z+x} v'''(t) dt dz dx.$$

Because both  $u'''(\cdot)$  and  $\pi'''(\cdot)$  have a constant sign, so does  $v'''(\cdot)$ . It follows that  $g'(\theta) \geq (>) 0$  if and only if  $v'''(\cdot) \geq (>) 0$ .

## C Proof of Lemma 1

Using the first-order condition of the maximization of (20), we can write

$$\begin{aligned} V'(x) &= \frac{\partial V(x)}{\partial x} \\ &= p(e(x)) [u'(k(x, L)) - u'(k(x, L) - L)] - u'(k(x, L)). \end{aligned}$$

Moreover, writing the first-order condition as

$$\begin{aligned} p(e(x)) [u'(k(x, L)) - u'(k(x, L) - L)] - u'(k(x, L)) \\ = p'(e(x)) [u(k(x, L)) - u(k(x, L) - L)] \end{aligned}$$

we can reformulate  $V'(x)$  as in the lemma.

## D Proof of Proposition 3

Using  $\delta'(b) > 0$  and  $\beta_2 < 0$ , it is easy to deduce from (18) that  $x^*$  increases with  $b$ . Moreover,  $x^*$  decreases with  $d$  if the term

$$e^{d\delta(b)} \frac{\rho - \frac{d}{b} [\alpha + \frac{1}{2} (\frac{d}{b} - 1) \sigma^2]}{\rho}$$

decreases with  $d$ , which is true if and only if (19) is satisfied.

## E Proof of Proposition 4 and Corollary 1

Recalling

$$V'(x) = p'(e(x, L)) [u(k^*(x, L)) - u(k^*(x, L) - L)],$$

we calculate

$$\begin{aligned} V''(x) &= p''(e(x, L)) e_x(x, L) [u(k^*(x, L)) - u(k^*(x, L) - L)] \\ &\quad - p'(e(x, L)) [e_x(x, L) + 1] [u'(k^*(x, L)) - u'(k^*(x, L) - L)]. \end{aligned} \quad (24)$$

Taking integrals yields

$$\begin{aligned} V''(x) &= -p''(e(x, L)) e_x(x, L) \int_0^L u'(k^*(x, l) - l) [e_l(x, l) + 1] dl \\ &\quad + p'(e(x, L)) [e_x(x, L) + 1] \int_0^L u''(k^*(x, l) - l) [e_l(x, l) + 1] dl \end{aligned}$$

Hence,  $V''(x) \geq 0$  if condition 3. in the proposition is satisfied,  $\forall l$ , and the converse is true if condition 3. is violated,  $\forall l$ .

Using (24) we further calculate

$$\begin{aligned} V'''(x) &= -[p''(e(x, L)) e_x(x, L)]' \int_0^L u'(k^*(x, l) - l) [e_l(x, l) + 1] dl \\ &\quad + \Gamma(x, L) \int_0^L u''(k^*(x, l) - l) [e_l(x, l) + 1] dl \\ &\quad + p'(e(x, L)) [e_x(x, L) + 1]^2 \int_0^L u'''(k^*(x, l) - l) [e_l(x, l) + 1] dl, \end{aligned}$$

where  $\Gamma(x, L) \equiv p''(e(x, L)) [e_x(x, L) + 1]^2 + p'(e(x, L)) e_{xx}(x, L)$ .

Then,  $V'''(x) \geq 0$  if condition 3. in the corollary is satisfied,  $\forall l$ , and the converse is true if condition 3. is violated,  $\forall l$ .

## F Proof of Proposition 5

Same as in the proof of Proposition 1, use the notation  $\gamma(m, x) = m + \pi(x)$ . The difference between the expected value of lottery  $\mathbf{B}''$  and that of lottery  $\mathbf{A}''$  is written as

$$\begin{aligned} &\frac{1}{2} \{ \mathbb{E} [u(m + \pi(x))] - u(m + \pi(\theta)) \} - \frac{1}{2} \{ \mathbb{E} [u(m + \pi(x - k))] - u(m + \pi(\theta - k)) \} \\ &= \frac{1}{2} \{ \{ \mathbb{E} [u(\gamma(m, x))] - u(\gamma(m, \theta)) \} - \{ \mathbb{E} [u(\gamma(m, x - k))] - u(\gamma(m, \theta - k)) \} \} \\ &= \frac{1}{2} \frac{1}{2\eta} \int_0^\eta \int_{\theta-x}^\theta \int_{z-k}^{z+x-k} \int_t^{t+k} \left\{ u'''(\gamma(m, r)) [\pi'(r)]^2 + u''(\gamma(m, r)) \pi''(r) \right\} dr dt dz dx. \end{aligned}$$

The rest of the proof is identical to that of Proposition 1.

## G Proof of Proposition 6

Being based on the definition of  $U_{ij}$ , we compute

$$\begin{aligned} D_{1j/2j} &= -\frac{1}{2\eta_j} \left[ \int_{\theta_2-\eta_j}^{\theta_2+\eta_j} \pi(x) dx - \int_{\theta_1-\eta_j}^{\theta_1+\eta_j} \pi(x) dx \right] \\ &= -\frac{1}{2\eta_j} \left[ \int_{\theta_1+\Delta\theta-\eta_j}^{\theta_1+\Delta\theta+\eta_j} [\pi(x) - \pi(\theta_1 + \Delta\theta)] dx - \int_{\theta_1-\eta_j}^{\theta_1+\eta_j} [\pi(x) - \pi(\theta_1)] dx \right] \\ &\quad + \pi(\theta_1) - \pi(\theta_1 + \Delta\theta), \end{aligned}$$

where  $\Delta\theta = \theta_2 - \theta_1$ . We develop further the braked expression in the first term of  $D_{1j/2j}$  as follows:

$$\begin{aligned} &\frac{1}{2\eta_j} \left[ \int_{\theta_1+\Delta\theta-\eta_j}^{\theta_1+\Delta\theta+\eta_j} [\pi(x) - \pi(\theta_1 + \Delta\theta)] dx - \int_{\theta_1-\eta_j}^{\theta_1+\eta_j} [\pi(x) - \pi(\theta_1)] dx \right] \\ &= \frac{1}{2\eta_j} \left[ \int_{\theta_1-\eta_j}^{\theta_1+\eta_j} \left( \int_0^{\Delta\theta} [\pi'(x+z) - \pi'(\theta_1+z)] dz dx \right) \right] \\ &= \frac{1}{2\eta_j} \int_{\theta_1-\eta_j}^{\theta_1+\eta_j} \int_0^{\Delta\theta} \int_{\theta_1}^x [\pi''(t+z)] dt dz dx \\ &= \frac{1}{2\eta_j} \int_0^{\eta_j} \int_0^{\Delta\theta} \int_{\theta_1-x}^{\theta_1} \int_0^x [\pi'''(t+z+l)] dl dt dz dx. \end{aligned}$$

Putting together the two terms of  $D_{1j/2j}$ , we find:

$$D_{1j/2j} = -\frac{1}{2\eta_j} \int_0^{\eta_j} \int_0^{\Delta\theta} \int_{\theta_1-x}^{\theta_1} \int_0^x [\pi'''(t+z+l)] dl dt dz dx - \int_0^{\Delta\theta} \pi'(\theta_1+x) dx,$$

which decreases with  $\pi'''(\cdot)$ .

We further develop

$$\begin{aligned} D_{11/22} &= \mathbb{E} \left[ \int_{\theta_1+\tilde{\sigma}_1}^{\theta_2+\tilde{\sigma}_1} \pi'(x) dx \right] + \mathbb{E} \left[ \int_{\theta_2+\tilde{\sigma}_1}^{\theta_2+\tilde{\sigma}_2} \pi'(x) dx \right] \\ &= D_{11/21} - \frac{1}{2\eta} \left[ \int_{\theta_2-\eta_1}^{\theta_2-\eta_2} \pi'(x) dx - \int_{\theta_2+\eta_1}^{\theta_2+\eta_2} \pi'(x) dx \right], \end{aligned}$$

where  $D_{11/21}$  decreases with  $\pi'''(\cdot)$  as found above and the second term is unrelated to  $\pi'''(\cdot)$ .