Finite Element Approximation of the Neumann Eigenvalue Problem in Domains with Multiple Cracks

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Abstract

We study the Neumann-Laplacian eigenvalue problem in domains with multiple cracks. We derive a mixed variational formulation which holds on the whole geometric domain (including the cracks) and implement efficient finite element discretizations for the computation of eigenvalues. Optimal error estimates are given and several numerical examples are presented, confirming the efficiency of the method. As applications, we numerically investigate the behavior of the low eigenvalues in domains with a high number of cracks.

1 Introduction

The study of eigenvalues and eigenfunctions of partial differential operators both in theoretical and approximation grounds is very important in many branches of sciences: quantum mechanics, structural mechanics, acoustic, economy, biology, etc. In most applications, the knowledge of the eigenvalues allows, for example, to deduce stability of the physical system in neighborhoods of the equilibrium states. For mathematicians, the spectral theory is also a source of fascinating and "challenging" problems. Number of them is still unsolved, in particular several questions concerning the behavior of the spectrum for the geometric domain variations (see [8]).

The dependence of the eigenvalues on the geometric domain is usually a complex question in shape optimization. This dependence, which consists in properties such as stability, monotonicity or sensitivity, is linked both to the operator itself and to the geometry. In the case of the Dirichlet-Laplacian for example, several of the above properties are well understood, while for the Neumann-Laplacian this is not the case. We refer to [8, 9, 17] for a detailed description of this topic.

Specific difficulties arise in the study and approximation of eigenvalues in nonsmooth domains such as those with cracks. Typical examples are the comb domains or domains with rooms and passages like in [16]. In this article, we present an efficient method for approximating the eigenvalues of the Neumann-Laplacian in a plane domain with many cracks. This approximation method is based on a mixed formulation obtained by extending admissible solutions to the entire domain, while the conditions prescribed on the cracks are considered as functional constraints and are included in the functional spaces. Since this approach reduces greatly the dependence of the computations with respect to the geometric constraints (the cracks), it allows us to build efficient discretizations by finite element method preserving good approximation properties: high accuracy, free from spurious modes and low cost. As application of the method, we look for numerical evidences on the behavior of the spectrum in a cracked domain when the number of the cracks increases. This behavior is, in general, not well understood and plays an important role in shape optimization [4].

The outline of the paper is as follows: In Section 2, we describe the variational formulation of the eigenvalue problem and the principle of our method. Section 3 is devoted to the finite element discretizations and the discrete eigenvalue problem. In Section 4, we perform the convergence analysis and we give precise approximation results. In Section 5, we give the details of the implementation and we present several numerical results to confirm the efficiency of our approach. Further numerical experiments and applications are given in the last section.

2 Variational formulation and regularity

Let Ω be a bounded domain of $\mathbb{R}^2$ with smooth boundary Γ, and $(\gamma_i)_i$, $1 \leq i \leq I$ a given number of disjoint Lipschitz continuous curves in Ω without selfintersections. We assume that each γ_i can be extended up to a closed smooth curve $\Sigma_i \subset \overline{\Omega}$ and that the subdomains Ω_i with boundaries Σ_i, $1 \leq i \leq I$ form a partition of the entire domain Ω, i.e.

$$\overline{\Omega} = \bigcup_{i=1}^{I} \overline{\Omega}_i, \quad \Omega_i \cap \Omega_j = \emptyset,\ 1 \leq i < j \leq I. \tag{1}$$

When some γ_i touches the boundary, we assume that the angle between γ_i and Γ is not obtuse (to avoid cusps in subdomains).

Remark 2.1 Assumption (1) covers exactly the case of interest in this paper and it allows us to avoid additional technicalities. Let Ω_γ be the domain $\Omega \setminus (\bigcup_i^I \gamma_i)$, then under these assumptions the embedding of $H^1(\Omega_\gamma)$ into $L^2(\Omega_\gamma)$ is compact.

The Neumann eigenvalue problem for Laplace operator in Ω_γ reads: Find $\lambda \in \mathbb{R}$, and $u \neq 0$, such that

$$-\Delta u = \lambda u \qquad \text{in } \Omega_\gamma, \tag{2}$$

$$\frac{\partial u}{\partial n} = 0 \qquad \text{on } \Gamma, \tag{3}$$

$$(\frac{\partial u}{\partial n_i})^{\pm} = 0 \qquad \text{on } \gamma_i^{\pm}, \quad 1 \leq i \leq I, \tag{4}$$

where $(\frac{\partial u}{\partial n_i})^{\pm}$ denote the normal derivatives of the function u on the crack faces $\gamma_i^{\pm}$, $1 \leq i \leq I$.

In the case of Lipschitz domains, it is well known from the spectral theory of compact operators [11] that solutions (λ, u) of (2)-(3) are eigenpairs of the Neumann-Laplacian and the nonzero eigenvalues λ are isolated and of finite multiplicity (recall that the multiplicity of an eigenvalue λ for a compact operator T is the dimension of $\bigcup_{n=1}^{\infty} \ker(\lambda I - T)^n$ where $\ker A$ denotes the kernel of A). Therefore, the spectrum consists only of eigenvalues which can be ordered into an increasing sequence

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \ldots,$$

each λ_i, except zero, being isolated and of finite multiplicity. Moreover, there exists an orthonormal basis of $L^2(\Omega)$ made of the associated eigenfunctions.

In fact, the compactness of the injection $H^1(\Omega_\gamma) \subset L^2(\Omega_\gamma)$ is the key property for having a spectrum of eigenvalues. It is readily checked that problem (2) to (4) admits the following variational formulation: Find $\lambda \in \mathbb{R}$, such that there exists a solution $u \neq 0$, $u \in H^1(\Omega_\gamma)$ of

$$a(u, v) = \lambda(u, v),\ v \in H^1(\Omega_\gamma), \tag{5}$$

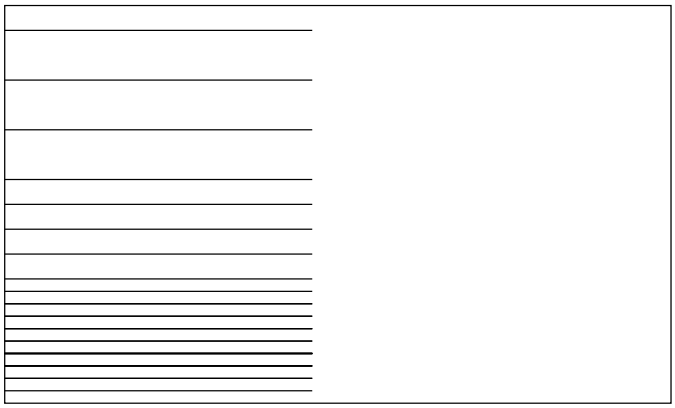

Figure 1: Example of multi-cracked domain

where $a(u,v) = \int_{\Omega_\gamma} \mathbf{grad}\, u \cdot \mathbf{grad}\, v\, dx$ and $(u,v) = \int_{\Omega_\gamma} u\, v\, dx$. The equivalence of problem (2)-(4) and problem(5) is a consequence of the density of $\mathcal{C}^\infty(\Omega_\gamma) \cap H^1(\Omega_\gamma)$ in $H^1(\Omega_\gamma)$. Moreover, denoting by $T : L^2(\Omega_\gamma) \longrightarrow H^1(\Omega_\gamma)$, the linear operator such that

$$a(Tf, v) = (f, v),\ v \in H^1(\Omega_\gamma), \tag{6}$$

it follows from the compactness of the embedding $H^1(\Omega_\gamma) \subset L^2(\Omega_\gamma)$ that T is compact. Therefore, classical existence results and properties of eigenpairs of compact operators in smooth domains extend to problem (2)-(4).

The method for approximating solutions of problem (2)-(4) developed in this article is based on a mixed variational formulation that we introduce here. We set in $\mathcal{D}'(\Omega_\gamma)$

$$\mathbf{p} = \mathbf{grad}\, u, \quad \text{in } \Omega_\gamma. \tag{7}$$

Next, we consider the space

$$\mathbf{X}(\Omega_\gamma) = \left\{ \mathbf{q} \in L^2(\Omega_\gamma)^2,\ \operatorname{div} \mathbf{q} \in L^2(\Omega_\gamma),\ \mathbf{q} \cdot \nu = 0, \text{ on } \Gamma \right\},$$

equipped with the norm

$$\|\mathbf{q}\|_{\mathbf{X}(\Omega_\gamma)} = \left(\|\mathbf{q}\|^2_{(L^2(\Omega_\gamma))^2} + \|\operatorname{div} \mathbf{q}\|^2_{L^2(\Omega_\gamma)} \right)^{\frac{1}{2}},$$

and the subspace

$$\mathbf{X}_\diamond(\Omega_\gamma) = \left\{ \mathbf{q} \in \mathbf{X}(\Omega_\gamma),\ \mathbf{q} \cdot \nu_i^{\pm} = 0, \text{ on } \gamma_i,\ 1 \le i \le I \right\}.$$

For each Σ_i, $1 \le i \le I$, we also introduce the space $H^{\frac{1}{2}}(\Sigma_i)$ equipped with the norm

$$\|\varphi\|^2_{H^{\frac{1}{2}}(\Sigma_i)} = \|\varphi\|^2_{L^2(\Sigma_i)} + \int_{\Sigma_i} \int_{\Sigma_i} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^2}\, dx\, dy,$$

and we denote by $H^{-\frac{1}{2}}(\Sigma_i)$ its dual. For $\mathbf{q} \in \mathbf{X}(\Omega_\gamma)$ the traces $(\mathbf{q}.\nu_i)^{\pm}$ can be defined as elements of $H^{-\frac{1}{2}}(\Sigma_i)$ and the trace operator is continuous from $\mathbf{X}(\Omega_\gamma)$ to $H^{-\frac{1}{2}}(\Sigma_i)$, [14]. Moreover, denoting by $H^{\frac{1}{2}}_{00}(\gamma_i)$ the subspace of $H^{\frac{1}{2}}(\gamma_i)$ made of functions (formally) vanishing at the endpoints of γ_i

(see [20]), we can also define $(\mathbf{q}.\nu_i)^{\pm} \in (H_{00}^{\frac{1}{2}}(\gamma_i))'$ and the constraints in the definition of $\mathbf{X}_\diamond(\Omega_\gamma)$ are to be understood in the following weak sense

$$< (\mathbf{q}.\nu_i)^{\pm}, \varphi >_{\frac{1}{2},\gamma_i} = 0 \quad \forall \varphi \in H_{00}^{\frac{1}{2}}(\gamma_i),\ 1 \le i \le I,$$

$$< (\mathbf{q}.\nu), \varphi >_{-\frac{1}{2},\frac{1}{2},\Gamma} = 0 \quad \forall \varphi \in H^{\frac{1}{2}}(\Gamma),$$

where $< .,. >_{\frac{1}{2},\gamma_i}$ stands for the duality pairing between $H_{00}^{\frac{1}{2}}(\gamma_i)$ and its dual $(H_{00}^{\frac{1}{2}}(\gamma_i))'$ and $< .,. >_{-\frac{1}{2},\frac{1}{2},\Gamma}$ is the usual duality product of $H^{\frac{1}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$.

The mixed variational formulation of problem (2)-(4) reads: Find $(\mathbf{p}, u) \in \mathbf{X}_\diamond(\Omega_\gamma) \times L^2(\Omega_\gamma)$ such that

$$\begin{cases} \int_{\Omega_\gamma} \mathbf{p}\,\mathbf{q}\,dx + \int_{\Omega_\gamma} u \,\mathrm{div}\,\mathbf{q}\,dx = 0, \quad \forall \mathbf{q} \in \mathbf{X}_\diamond(\Omega_\gamma), \\ \\ \int_{\Omega_\gamma} \mathrm{div}\,\mathbf{p}\, v\,dx + \lambda \int_{\Omega_\gamma} u\, v\,dx = 0, \quad \forall v \in L^2(\Omega_\gamma). \end{cases} \tag{8}$$

The study of such problems fits under the general theory of variationally posed spectral problems (see [1, 2, 12, 19, 21, 22]). Note that if λ and u is an eigenpair of (2)-(4) and $\mathbf{p} = \mathbf{grad}\, u$, then λ and $(u, \mathbf{p})$ is an eigenpair of problem (8) and conversely.

In order to perform the computation for problem (2)-(4), and since we intend to work with many cracks, we will derive a new variational formulation which will allow us to work in the entire domain $\Omega = \Omega_\gamma \cup \gamma$. This formulation, introduced in [18] for some elasticity problems (see also [3]), consists in extending the admissible solutions (the admissible displacement and its gradient in the case of the elastic membrane) to the crack faces. Therefore the new admissible solutions are defined in the entire domain Ω and the restrictions imposed on the cracks are expressed as internal constraints prescribed on the given subset $\bigcup_i \gamma_i$ of Ω while the cracks γ_i, $1 \le i \le I$ are removed from the formulation (8). Therefore, for numerical computations, only one global mesh is necessary in Ω which is a crucial for problem (2)-(4) from both practical and approximation points of view.

We still denote

$$\mathbf{X}_\diamond = \{\mathbf{q} \in \mathbf{X}(\Omega),\ \mathbf{q} \cdot \nu_i = 0, \text{ on } \gamma_i,\ 1 \le i \le I,\ \mathbf{q} \cdot \nu = 0, \text{ on } \Gamma\},$$

then, the new formulation consists in rewriting problem (8) by replacing Ω_γ with Ω (with obvious modifications of the spaces and integrals). For brevity, we will label the new problem also (8). Note that, if λ, $(u, \mathbf{p})$ is an eigenpair for the latest problem then λ, and the restrictions of $(u, \mathbf{p})$ to the domain Ω_γ is an eigenpair of the initial problem. The converse statement is obviously true under additional regularity assumptions on γ and solutions $(u, \mathbf{p})$.

In that follows, we denote

$$s(u, v) = \int_\Omega u\, v\,dx,\ a(\mathbf{p}, \mathbf{q}) = \int_\Omega \mathbf{p}\,\mathbf{q}\,dx,\ b(v, \mathbf{q}) = \int_\Omega v \,\mathrm{div}\,\mathbf{q}\,dx.$$

Let us introduce the bilinear forms A and B defined on the product space $\mathbf{X}_\diamond \times L^2(\Omega)$ by

$$\begin{aligned} A((u, \mathbf{p}), (v, \mathbf{q})) &= a(u, v) + b(v, \mathbf{p}) + b(u, \mathbf{q}), \\ B((u, \mathbf{p}), (v, \mathbf{q})) &= -s(u, v). \end{aligned}$$

We define the space

$$\mathbf{Y} = \left\{\mathbf{q} \in \mathbf{X}_\diamond,\ b(v, \mathbf{q}) = 0,\ \forall v \in L^2(\Omega)\right\}.$$

Problem (8) takes the form: Find $\lambda \in \mathbb{R}, 0 \neq U \in \mathbf{X}_\diamond \times L^2(\Omega)$ such that

$$A(U,V) = \lambda B(U,V), \quad \forall V \in \mathbf{X}_\diamond \times L^2(\Omega). \tag{9}$$

It is readily checked that the two following conditions hold

$$a(\mathbf{p},\mathbf{p}) \geq \alpha \|\mathbf{p}\|^2_{\mathbf{X}}, \ \forall \mathbf{p} \in \mathbf{Y}, \alpha > 0, \tag{10}$$

$$\sup_{\mathbf{q} \in \mathbf{X}_\diamond} \frac{|b(v,\mathbf{q})|}{\|\mathbf{q}\|_{\mathbf{X}}} \geq \beta \|u\|_{L^2(\Omega)}, \ \beta > 0. \tag{11}$$

Thus, it is well known ([7], [24]) that under conditions (10) and (11), the following positivity properties are satisfied

$$\inf_{\substack{U \in \mathbf{X}_\diamond \times L^2(\Omega) \\ \|U\|_{\mathbf{X}_\diamond \times L^2(\Omega)} = 1}} \sup_{\substack{V \in \mathbf{X}_\diamond \times L^2(\Omega) \\ \|V\|_{\mathbf{X}_\diamond \times L^2(\Omega)} = 1}} \frac{|A(U,V)|}{\|U\|_{\mathbf{X} \times L^2(\Omega)} \|V\|_{\mathbf{X} \times L^2(\Omega)}} \geq \beta > 0, \tag{12}$$

and

$$\sup_{U \in \mathbf{X}_\diamond \times L^2(\Omega)} |A(U,V)| > 0, \quad \forall 0 \neq V \in \mathbf{X}_\diamond \times L^2(\Omega). \tag{13}$$

We also have that the operator $T : L^2(\Omega) \longrightarrow L^2(\Omega)$, defined by

$$A((\mathbf{p},Tf),V) = B((\mathbf{p},-f),V), \quad V = (\mathbf{q},v) \in \mathbf{X}_\diamond \times L^2(\Omega),$$

is self-adjoint and compact. Therefore, we have (see for instance [1])

Theorem 2.2 *The triplet* $(\mathbf{p},u,\lambda)$ *is an eigensolution of problem* (8) *if and only if* $\lambda Tu = u$, $\mathbf{p} = \boldsymbol{grad} u$.

To achieve our aim of constructing a general framework for an efficient approximation method to problem (8), we now derive an unconstrained formulation where the zero Neumann conditions on γ_i, $1 \leq i \leq I$ are expressed via Lagrange multipliers. This yields the following hybrid formulation.

Notation Let γ be one of the cracks γ_i, and c_1, c_2 its endpoints. Let us denote by $H^1_0(\gamma,\{c_\ell\})$ the subspace of functions of $H^1(\gamma)$ vanishing at $\{c_\ell\})$, $\ell = 1,2$.

We define the space $H^{\frac{1}{2}}_*(\gamma)$ as

$$\begin{aligned} H^{\frac{1}{2}}_{00}(\gamma) &\quad \text{if } c_1 \text{ and } c_2 \in \partial\Omega, \\ H^{\frac{1}{2}}_{00}(\gamma,\{c_i\}) &\quad \text{if } c_i \in \partial\Omega, \ i = 1,2, \\ H^{\frac{1}{2}}(\gamma) &\quad \text{if } \overline{\gamma} \cap \partial\Omega = \emptyset, \end{aligned}$$

where $H^{\frac{1}{2}}_{00}(\gamma,\{c_i\})$ stands for the space obtained by Hilbertian interpolation of index $\frac{1}{2}$: $\left[H^1_0(\gamma,\{c_i\}), L^2(\gamma)\right]_{\frac{1}{2}}$ (see [20]).

We introduce the Lagrange multiplier space

$$\mathbf{M} = \left\{ \mu = (\mu_i)_{1 \leq i \leq I} \in \prod_{i=1}^{I} H^{\frac{1}{2}}_*(\gamma_i) \right\},$$

and we define the bilinear form $d(.,.)$ on the space $\mathbf{X}_\diamond \times (L^2(\Omega) \times \mathbf{M})$ such that

$$c(\mathbf{q},(v,\mu)) = b(v,\mathbf{q}) + \sum_{i=1}^{I} < \mu_i, \mathbf{q}\cdot\nu_i >_{\frac{1}{2},\gamma_i} . \tag{14}$$

We are now in position to define the (hybrid) variational formulation which reads:

$$\begin{cases} a(\mathbf{p},\mathbf{q}) + c(\mathbf{q},(u,\Lambda)) = 0, \quad \forall \mathbf{q} \in \mathbf{X}_\diamond, \\ \\ c(\mathbf{p},(v,\mu)) + \lambda s(u,v) = 0, \quad \forall (v,\mu) \in L^2(\Omega)\times \mathbf{M}. \end{cases} \tag{15}$$

This problem fits under the general theory of mixed variational formulation of spectral problems. Moreover, replacing the previous bilinear forms A and B by the similar ones defined on $\mathbf{X}_\diamond \times (L^2(\Omega) \times \mathbf{M})$ as

$$\begin{aligned} A_1((\mathbf{p},(u,\Lambda)),(\mathbf{q},(v,\mu))) &= a(\mathbf{p},\mathbf{q}) + c(\mathbf{p},(v,\mu)) + c(\mathbf{q},(u,\Lambda)), \\ B((\mathbf{p},(u,\Lambda)),(\mathbf{q},(v,\mu))) &= -s(u,v), \end{aligned}$$

properties (10), (11), and (12) hold for the new bilinear forms. Note that the inf-sup condition for $c(.,(.,.))$ between the space $\mathbf{X}_\diamond$ and the space $L^2(\Omega)\times\mathbf{M}$ equipped with the norm $\|.\|_{L^2(\Omega)} + \|.\|_{\mathbf{M}}$ follows easily from the separate inf-sup conditions on $b(.,.)$ and the form defined by

$$d(\mathbf{q},\mu) = < \mu, \mathbf{q}\cdot\nu >_{\frac{1}{2},\gamma} = \sum_{i=1}^{I} < \mu_i, \mathbf{q}\cdot\nu_i >_{\frac{1}{2},\gamma_i} . \tag{16}$$

Therefore the operator T defined by

$$\forall (\mathbf{q},(v,\mu)) \in \mathbf{X}_\diamond \times L^2(\Omega)\times\mathbf{M}, \qquad A_1(\mathbf{p},(Tf,\Lambda),(\mathbf{q},(v,\mu))) = B_1((\mathbf{p},(f,\Lambda)),(\mathbf{q},(v,\mu))), \tag{17}$$

is also self-adjoint and compact, consequently the result of Theorem 2.2 holds.

3 Discrete variational formulation and approximation results

We will assume that Ω is a polygonal domain and the cracks γ_i, $1 \le i \le I$ are polygonal lines with vertices which are also nodes of the triangulations. We denote by $(\mathcal{T}_h)_h$ a family of triangulations of Ω made of elements which are triangles (the extension to quadrilaterals is standard). The maximal size of elements is the parameter of discretization denoted by $h > 0$. In addition, we assume that each triangulation satisfies the usual admissibility assumptions, i.e., the intersection of two different elements is either empty, a vertex, or a whole edge, and $\mathcal{T}_h$ is assumed to be "regular", i.e., the ratio of the diameter of any element $K \in \mathcal{T}_h$ to the diameter of its largest inscribed ball is bounded by a constant σ independent of K and h. Note that the trace of the triangulation $\mathcal{T}_h$ on each γ_i define a 1D mesh $\mathcal{T}_h^{\gamma_i}$, and we assume that the endpoints c_1^i, c_2^i of γ_i are vertices of $\mathcal{T}_h^{\gamma_i}$, $1 \le i \le I$. Thus, the mesh on each γ_i is defined by the lattice $c_1^i = x_0^i, x_1^i, \ldots, x_{L_i-1}^i, x_{L_i}^i = c_2^i$, and we set $t_\ell^i = \left]x_{\ell-1}^i, x_\ell^i\right[$, $1 \le \ell \le L_i$. We will assume for simplicity that the triangulation $\mathcal{T}_h$ is quasi-uniform, i.e., there is a constant $\tau > 0$, such that

$$\frac{\max_K \ h_K}{\min_K \ h_K} \le \tau.$$

We also assume for technical reasons that no triangle has all its vertices on the boundary.

Remark 3.1 Since the spectral problem of the Neumann-Laplacian is highly depending on the geometry, one should be careful to the way of how to approach an arbitrary Lipschitz domain by a polygonal one. Indeed, in order to prevent pollution by spurious eigenmodes the approximation by polygonal domains should preserve a uniform cone condition. However, for arbitrary Lipschitz cracks the use of isoparametric finite elements seems more appropriate.

To approximate solutions of (15) we will consider a discretization based on the piecewise constant element for the approximation of the displacement u while for approximating the pressure $\mathbf{p} = \mathbf{grad}\, u$, we will consider two discretizations based on the so called Raviart-Thomas element. The two discretizations are close and they correspond to the two known ways for the implementation of the Raviart-Thomas element. The discrete Lagrange multiplier spaces M_h are built with affine, respectively piecewise constants, functions on the triangulations $\mathcal{T}_h^{\gamma_i}$ defined on the cracks γ_i, $1 \leq i \leq I$.

For $K \in \mathcal{T}_h$, let $RT_0(K)$ be the space

$$P_0^2 + P_0\mathbf{x}, \quad \text{for } \mathbf{x} \in K,$$

where P_0 stands for the space of constant functions. Let $\mathcal{E}_K$ denote the edges of $K \in T_h$, $\mathcal{E} = \bigcup_K \mathcal{E}_K$ and $\mathcal{E}_\Omega = \mathcal{E} \setminus \Gamma$. We define the following finite dimensional space

$$V_h = \left\{ v_h \in L^2(\Omega),\ v_{h|K} \in P_0(K), K \in \mathcal{T}_h \right\}.$$

Next we consider the following spaces

1. First discretization

$$\mathbf{X}_h^0 = \left\{ \mathbf{q}_h \in L^2(\Omega)^2,\ \mathbf{q}_h \in RT_0(K),\ K \in \mathcal{T}_h,\ \mathbf{q}_h.\nu = 0 \text{ on } \Gamma \right\},$$

$$M_h^{\text{int}} = \left\{ \eta_h,\ \mu_{h|e} \in P_0(e),\ \forall e \in \mathcal{E}_\Omega \right\},$$

and

$$M_h^0(\gamma_i) = \left\{ \mu_h,\ \mu_{h|t_\ell^i} \in P_0(t_\ell),\ t_\ell^i \in \mathcal{T}_h^{\gamma_i},\ 0 \leq \ell \leq L_i - 1 \right\}.$$

We set

$$\mathbf{M}_h^0 = \prod_{i=1}^{I} M_h^0(\gamma_i).$$

The discrete variational problem reads: Find $\lambda_h \in \mathbb{R}$ and $(u_h, \mathbf{p}_h, \varphi_h, \Lambda_h) \in \mathbf{X}_h^0 \times V_h \times M_h^{\text{int}} \times \mathbf{M}_h^0$, such that

$$\begin{cases} a(\mathbf{p}_h, \mathbf{q}_h) + c_0(\mathbf{q}_h, (u_h, \varphi_h, \Lambda_h)) = 0, \quad \forall \mathbf{q}_h \in \mathbf{X}_h^0, \\ \\ c_0(\mathbf{p}_h, (v_h, \eta_h, \mu_h)) + \lambda_h s(u_h, v_h) = 0, \quad \forall (v_h, \eta_h, \mu_h) \in V_h \times M_h^{\text{int}} \times \mathbf{M}_h^0. \end{cases} \tag{18}$$

The bilinear form $c_0(.,.)$ is defined as

$$c_0(\mathbf{q}_h, (v_h, \eta_h, \mu_h)) = c(\mathbf{q}_h, (v_h, \mu_h)) + \sum_{e \in \mathcal{E}_\Omega} \int_e (\mathbf{q}_h \cdot \nu_e) \eta_h \, de.$$

2. Second discretization

$$\mathbf{X}_h^1 = \{\mathbf{q}_h \in \mathbf{X}_\diamond,\ \mathbf{q}_h \in RT_0(K),\ K \in \mathcal{T}_h\},$$

for each γ_i, $1 \le i \le I$, we associate the space

$$M_h^1(\gamma_i) = \big\{\mu_h,\ \mu_{h|t_\ell} \in P_1(t_\ell),\ 0 \le \ell \le L_i - 1, \\ \mu_{h|t_{i^*}} \in P_0(t_{i^*}),\ \text{if}\, c_{i^*} \in \Gamma,\ i^* = 0, L_{i-1}\big\},$$

where P_1 stands for the space of affine functions, and we set

$$\mathbf{M}_h^1 = \prod_{i=1}^{I} M_h^1(\gamma_i).$$

The discrete variational problem reads: Find $\lambda_h \in \mathbb{R}$ and $(u_h, \mathbf{p}_h, \Lambda_h) \in \mathbf{X}_h^1 \times V_h \times \mathbf{M}_h^1$, such that

$$\begin{cases} a(\mathbf{p}_h, \mathbf{q}_h) + c_1(\mathbf{q}_h, (u_h, \Lambda_h)) = 0, \quad \forall \mathbf{q}_h \in \mathbf{X}_h^1, \\ \\ c_1(\mathbf{p}_h, (v_h, \mu_h)) + \lambda_h s(u_h, v_h) = 0, \quad \forall (v_h, \mu_h) \in V_h \times \mathbf{M}_h^1. \end{cases} \tag{19}$$

The bilinear form $c_1(.,.)$ is defined as

$$c_1(\mathbf{q}_h, (v_h, \mu_h)) = c(\mathbf{q}_h, (v_h, \mu_h)).$$

Let $T_h : L^2(\Omega) \longrightarrow L^2(\Omega)$ denote the discrete counterpart of T, defined for $\ell = 0, 1$ by:

$$\begin{cases} a(\mathbf{p}_h, \mathbf{q}_h) + c_\ell(\mathbf{q}_h, (T_h f, \Lambda_h)) = 0, \quad \forall \mathbf{q}_h \in \mathbf{X}_h^\ell, \\ \\ c_\ell(\mathbf{p}_h, (v_h, \Psi_h)) + \lambda_h s(f, v_h) = 0, \quad \forall (v_h, \Psi_h) \in V_h \times \mathcal{M}_h^\ell, \end{cases} \tag{20}$$

where $\mathcal{M}_h^1 = \mathbf{M}_h^1$ and $\mathcal{M}_h^0 = M_h^{\text{int}} \times \mathbf{M}_h^0$.

Then (T_h) is a family of self-adjoint compact operators in $L^2(\Omega)$. It is standard that $(\mathbf{p}_h, u_h, \lambda_h)$ is an eigensolution of problems (18) or (19) if and only if

$$\lambda_h T_h u_h = u_h, \quad \mathbf{p}_h = \mathbf{grad}_h\, u_h,$$

where $\mathbf{grad}_h$ is a discrete counterpart of grad.

Relying to the spectral approximation theory of variationally posed eigenvalue problems we can give abstract error estimates. Indeed, assume that properties (10), (11), and (12) still hold for the two discretization. Assume also that

$$\lim_{h \to 0} \|T - T_h\|_{\mathcal{L}(L^2(\Omega))} = 0. \tag{21}$$

Then, let λ be an eigenvalue of problem (15), with algebraic multiplicity m, there exists exactly m eigenvalues λ_{h1}, λ_{h2},..., λ_{hm} of problem (19) (counted according to the multiplicity m) which converge to λ when h goes to zero. Let $\hat{\lambda}_h = \frac{1}{m}\sum_{i=1}^{m} \lambda_{ih}$, we also denote by E the eigenspace corresponding to λ and by E_h the direct sum of the eigenspace corresponding to λ_{1h},..., λ_{mh}. Then, the following estimate holds for $\ell = 0, 1$

$$|\lambda - \lambda_{jh}| \le C\varepsilon_h^2, \quad j = 1, \ldots, m, \tag{22}$$

$$|\lambda - \hat{\lambda}_h| \leq C\varepsilon_h^2, \tag{23}$$

and for eigenfunctions

$$|u_j - u_{jh}| \leq C\varepsilon_h, \quad j = 1, \ldots, m, \tag{24}$$

where

$$\varepsilon_h = \sup_{\substack{(\mathbf{p},\, u,\, \Lambda) \in E \\ \|\mathbf{p}\|_{\mathbf{X}} + \|u\|_{L^2(\Omega)} + \|\Lambda\|_{\mathbf{M}} = 1}} \inf_{\substack{\mathbf{q}_h \in \mathbf{X}_h^\ell, \\ v_h \in V_h, \\ \Psi_h \in \mathcal{M}_h^\ell}} \big(\|\mathbf{p} - \mathbf{q}_h\|_{L^2(\Omega)^2} + \|u - v_h\|_{L^2(\Omega)} + \|\Lambda - \Psi_h\|_{\mathbf{M}}\big).$$

4 Error estimates

Both discretizations are close and their analysis is essentially similar and fits under the general theory of approximation of eigenvalue problems by mixed finite element method (see [1, 2, 6, 10, 21]). We only give a bridged analysis.

The main differences between the two discretization comes from the construction of the Lagrange multiplier spaces. In fact, problem (19) is a hybrid formulation in the usual sense ([24]), i.e. $\mathbf{M}_h^1$ is a subspace of $\mathbf{M}$, while $\mathbf{M}_h^0$ is not. Therefore, the discrete inf-sup condition, with respect to the natural norms, is uniform in h, in the first case, and not uniform in the second. The proof of these inf-sup conditions is standard (see [14], [23]). We have for a constant β, independent of h,

$$\forall \mu_h \in \mathbf{M}_h^1, \quad \sup_{\mathbf{q}_h \in \mathbf{X}_h^1} \frac{d(\mathbf{q}_h, \mu_h)}{\|\mathbf{q}_h\|_{\mathbf{X}}} \geq \beta \|\mu_h\|_{\mathbf{M}}, \tag{25}$$

and

$$\forall \mu_h \in \mathbf{M}_h^0, \quad \sup_{\mathbf{q}_h \in \mathbf{X}_h^0} \frac{d(\mathbf{q}_h, \mu_h)}{\|\mathbf{q}_h\|_{L^2(\Omega)^2}} \geq \beta h^{-\frac{1}{2}} \|\mu_h\|_{L^2(\gamma)}. \tag{26}$$

Recall also from [24] the usual discrete inf-sup conditions on the bilinear form $b(.,.)$

$$\forall v_h \in V_h, \quad \sup_{\mathbf{q}_h \in \mathbf{X}_h^\ell} \frac{b(v_h, \mathbf{q}_h)}{\|\mathbf{q}_h\|_{L^2(\Omega)^2}} \geq \beta \|v_h\|_{L^2(\Omega)}. \tag{27}$$

It is natural to ask whether the constants in the inf-sup conditions (26) and (25) depend on the number of cracks which could deteriorate the conditioning of the matrices of the discrete problems. It is readily checked by resorting to the proof of both inf-sup conditions in the framework of domain decomposition, that the constants are independent of the number of the subdomains Ω_i, thus of the number of the cracks γ_i. This is also confirmed by the numerical experiments.

In what follows, we need some approximation tools. For $K \in \mathcal{T}_h$, e will denote an edge of the triangle K. For $\mathbf{q} \in \mathbf{X}$ sufficiently regular, the interpolation operator for the Raviart-Thomas elements $\mathcal{J}_h$ is defined by

$$\begin{cases} \mathcal{J}_h \mathbf{q} \in \mathbf{X}_h, \\ \int_e \mu \mathcal{J}_h \mathbf{q}_h \cdot \nu \, d\, e = \int_e \mu \mathbf{q} \cdot \nu \, d\, e, \quad \forall e \in \mathcal{E}_\Omega, \ \forall \mu \in P_0. \end{cases} \tag{28}$$

It is well known that $\mathcal{J}_h$ satisfies the approximation properties (see. [23], [24])

$$\begin{aligned} \|\mathbf{q} - \mathcal{J}_h \mathbf{q}\|_{(L^2(\Omega)^2} &\leq Ch|\mathbf{q}|_{H^1(\Omega)^2}, \\ \|\mathrm{div}\,(\mathbf{q} - \mathcal{J}_h \mathbf{q})\|_{(L^2(\Omega)} &\leq Ch|\mathrm{div}\,\mathbf{q}|_{H^1(\Omega)}. \end{aligned} \tag{29}$$

Similarly, the L^2-orthogonal projection P_h from $L^2(\Omega)$ onto V_h, satisfies (see [13])

$$\|u - P_h u\|_{L^2(\Omega)} \leq Ch|u|_{H^1(\Omega)}. \tag{30}$$

We also denote by π_h^0 the L^2-projection operator $L^2(\gamma) \longrightarrow \mathbf{M}_h^0$, defined as follows: For each γ_i, $1 \leq i \leq I$,

$$\int_{\gamma_i} v\psi_h \, d\sigma = \int_{\gamma_i} \pi_h^0(v)\psi_h \, d\sigma, \quad \psi_h \in M_h^0(\gamma_i). \tag{31}$$

π_h^0 satisfies the following estimates (see [24]). Namely, for the functions $\varphi \in H^\nu(\gamma_i)$, $1 \leq i \leq I$, with $\nu = \frac{1}{2}$, or with $\nu = 1$, there exists a constant $c > 0$ independent of h such that

$$\|\varphi - \pi_h^0\varphi\|_{L^2(\gamma_i)} \leq ch^\nu \|\varphi\|_{H^\nu(\gamma_i)}. \tag{32}$$

Moreover, if $\varphi \in L^2(\gamma_i)$, then

$$\|\varphi - \pi_h^0\varphi\|_{H^{-\frac{1}{2}}(\gamma_i)} \leq ch^{\frac{1}{2}} \|\varphi - \pi_h^0\varphi\|_{L^2(\gamma_i)}. \tag{33}$$

Finally, we define the projection operator $\pi_h^1 \ : \ L^2(\gamma_i) \mapsto \ M_h^1(\gamma_i)$, $1 \leq i \leq I$, with respect to the scalar product in $L^2(\gamma_i)$, which satisfies the following properties (see [5]). Given $\mu \in [0, 1]$ and $\nu \in \left]\frac{1}{2}, 2\right]$, there exists a constant $c > 0$ which is independent of h, such that for all functions $\varphi \in H^\nu(\gamma_i)$,

$$\|\varphi - \pi_h^1\varphi\|_{H^{-\mu}(\gamma_i)} + h^{\mu+\frac{1}{2}} \|\varphi - \pi_h^1\varphi\|_{H^{\frac{1}{2}}(\gamma_i)} \leq ch^{\mu+\nu} \|\varphi\|_{H^\nu(\gamma_i)}. \tag{34}$$

4.1 The case of the second discretization

Since the space $\mathbf{X}_h^1 \times V_h \times \mathbf{M}_h^1$ is a subspace of $\mathbf{X}_\diamond \times L^2(\Omega) \times \mathbf{M}$, it can be checked that properties (10), (11), and (12) still hold in the discrete case.

We consider the source problem (20) and derive from the saddle-point approximation theory ([14, 7]) the following error estimates.

Proposition 4.1 *For every $f \in L^2(\Omega)$ the following estimates hold*

$$\|\mathbf{p} - \mathbf{p}_h\|_{L^2(\Omega)^2} \leq C\big(\inf_{\mathbf{q}_h \in \mathbf{X}_h} \|\mathbf{p} - \mathbf{q}_h\|_{L^2(\Omega)^2} + \inf_{(v_h, \Psi_h) \in V_h \times \mathbf{M}_h} (\|u - v_h\|_{L^2(\Omega)} + \|\Lambda - \Psi_h\|_{\mathbf{M}})\big), \tag{35}$$

and

$$\|Tf - T_h f\|_{L^2(\Omega)} + \|\Lambda - \Lambda_h\|_{\mathbf{M}} \leq C\big(\|\mathbf{p} - \mathbf{p}_h\|_{L^2(\Omega)^2} + \inf_{(v_h, \Psi_h) \in V_h \times \mathbf{M}_h} (\|u - v_h\|_{L^2(\Omega)} + \|\Lambda - \Psi_h\|_{\mathbf{M}})\big). \tag{36}$$

Remark 4.2 We also have in the case of the second discretization the estimate

$$\|\mathbf{p} - \mathbf{p}_h\|_{\mathbf{X}} \leq C\big(\inf_{\mathbf{q}_h \in \mathbf{X}_h} \|\mathbf{p} - \mathbf{q}_h\|_{\mathbf{X}} + \inf_{(v_h, \Psi_h) \in V_h \times \mathbf{M}_h} (\|u - v_h\|_{L^2(\Omega)} + \|\Lambda - \Psi_h\|_{\mathbf{M}})\big). \tag{37}$$

Proof The way to obtain these error estimates is standard. For the convenience of the reader we just point out the streamline of the proof. Using the inf-sup conditions (25) - (27) and the error equations for the source problems, we derive

$$\begin{aligned}\|T_hf - P_h(Tf)\|_{L^2(\Omega)} + \|\Lambda_h - \pi_h^1(\Lambda)\|_{\mathbf{M}} &\leq C\frac{1}{\|\mathbf{q}_h\|_{L^2(\Omega)^2}}\big(b(T_hf - Tf, \mathbf{q}_h)\\ &+ b(Tf - P_h(Tf), \mathbf{q}_h) + d(\Lambda_h - \Lambda, \mathbf{q}_h) + d(\Lambda - \pi_h^1(\Lambda), \mathbf{q}_h)\big)\\ &\leq C\frac{1}{\|\mathbf{q}_h\|_{L^2(\Omega)^2}}\big(a(\mathbf{p}_h - \mathbf{p}, \mathbf{q}_h) + b(Tf - P_h(Tf), \mathbf{q}_h) + d(\Lambda - \pi_h^1(\Lambda), \mathbf{q}_h)\big)\end{aligned} \tag{38}$$

The triangle inequality yields (36).

Using again the error equations for the source problems and the definitions of the projection operators $\mathcal{J}_h$ and π_h^1, we obtain

$$\begin{aligned}\|\mathcal{J}_h\mathbf{p} - \mathbf{p}_h\|^2_{L^2(\Omega)^2} &= a(\mathcal{J}_h\mathbf{p} - \mathbf{p}, \mathbf{p}_h - \mathcal{J}_h\mathbf{p}) + a(\mathbf{p} - \mathbf{p}_h, \mathbf{p}_h - \mathcal{J}_h\mathbf{p})\\ &= a(\mathcal{J}_h\mathbf{p} - \mathbf{p}, \mathbf{p}_h - \mathcal{J}_h\mathbf{p}) - c_1(\mathcal{J}_h\mathbf{p} - \mathbf{p}_h, (Tf - T_hf, \Lambda - \Lambda_h))\\ &= a(\mathcal{J}_h\mathbf{p} - \mathbf{p}, \mathbf{p}_h - \mathcal{J}_h\mathbf{p}) + b(\mathbf{p}_h - \mathcal{J}_h\mathbf{p}, Tf - P_h(Tf))\\ &\quad + d(\mathbf{p}_h - \mathcal{J}_h\mathbf{p}, \Lambda - \pi_h^1(\Lambda))\\ &\leq C\|\mathbf{p}_h - \mathcal{J}_h\mathbf{p}\|_{L^2(\Omega)^2}\big(\|\mathbf{p} - \mathbf{p}_h\|_{L^2(\Omega)^2} + \|Tf - \mathcal{P}_h(Tf)\|_{L^2(\Omega)}\\ &+ \|\Lambda - \pi_h^1(\Lambda)\|_{\mathbf{M}}\big).\end{aligned} \tag{39}$$

Thus, the triangle inequality leads to (35) □

It follows immediately from (36) and (29) that

$$\lim_{h\to 0}\|T - T_h\|_{\mathcal{L}(L^2(\Omega))} = 0. \tag{40}$$

Since, when $\Lambda \in H^{\frac{1}{2}+\mu}(\gamma) = \prod_{i=1}^I H^{\frac{1}{2}+\mu}(\gamma_i)$, $0 < \mu \leq 1$, we have

$$\|\Lambda - \pi_h^1(\Lambda)\|_{\mathbf{M}} \leq Ch^\mu\|\Lambda\|_{H^{\frac{1}{2}+\mu}(\gamma)}. \tag{41}$$

Assembling estimates (29), (30), (43) together with (35) and (36) and inserting in (22), we get the following error estimates

Theorem 4.3 *Assume that λ, $(u, \mathbf{p}, \Lambda)$ is an eigensolution of problem (15), with the algebraic multiplicity m for λ and assume that $u \in \prod_{i=1}^I H^1(\Omega_i)$, $\mathbf{p} \in (H^1(\Omega))^2$ and $\Lambda \in H^{\frac{1}{2}+\mu}(\gamma)$, $0 < \mu < 1$. Let λ_{ih}, $i = 1, \ldots, m$ be the eigenvalues associated to λ and obtained from problem (19). Then, the following error estimate holds*

$$|\lambda - \lambda_{ih}| \leq Ch^{2\mu}, \quad i = 1, \ldots, m, \tag{42}$$

where the constant C depends linearly on $(\|u\|_{H^1(\Omega_i)}, 1 \leq i \leq I, \|\mathbf{p}\|_{H^1(\Omega)^2})$.

Remark 4.4 Recall that $\Lambda_{|\gamma_i} = [u]_{|\gamma_i}$; thus in the case of high regularity, that is $\Lambda \in H^{\frac{3}{2}}$, (which is the case for smooth problems) we retrieve the well known $O(h^2)$ order of convergence for the eigenvalues.

4.2 The case of the first discretization

The same analysis as in the second discretization is valid. The only change is the use of a mesh-dependent norm for the Lagrange multiplier. More precisely, we will consider that $M_h^{\text{int}} \times \mathbf{M}_h^0$ is equipped with the norm: For $\Psi_h \in M_h^{\text{int}} \times \mathbf{M}_h^0$

$$h^{-\frac{1}{2}} \|\Psi_h\|_{L^2(\mathcal{E}_0)}.$$

Observing that when $\Lambda \in H^{\frac{1}{2}+\mu}(\gamma) = \prod_{i=1}^{I} H^{\frac{1}{2}+\mu}(\gamma_i)$, $0 < \mu \leq \frac{1}{2}$, we have for each i

$$h^{-\frac{1}{2}} \|\Lambda - \pi_h^0(\Lambda)\|_{L^2(\gamma_i)} \leq C h^{\mu} \|\Lambda\|_{H^{\mu}(\gamma_i)}. \tag{43}$$

As in the previous case, we get the following.

Theorem 4.5 *Assume that λ, $(u, \mathbf{p}, \Lambda)$ is an eigensolution of problem (15), with the algebraic multiplicity m for λ and assume that $u \in \prod_{i=1}^{I} H^1(\Omega_i)$, $\mathbf{p} \in (H^1(\Omega))^2$ and $\Lambda \in H^{\frac{1}{2}+\mu}(\gamma)$, $0 < \mu < \frac{1}{2}$. Let λ_{ih}, $i = 1, \ldots, m$ be the eigenvalues associated to λ and obtained from problem (19). Then, the following error estimate holds*

$$|\lambda - \lambda_{ih}| \leq C h^{2\mu}, \quad i = 1, \ldots, m, \tag{44}$$

where the constant C depends linearly on $(\|u\|_{H^1(\Omega_i)}, 1 \leq i \leq I, \|\mathbf{p}\|_{H^1(\Omega)^2})$.

Note that in this last case we obtain the convergence rate $O(h)$ when $\Lambda \in H^1(\gamma)$ (which is a reasonable regularity assumption). So the result is identical to the one of the previous discretization, however, here we cannot improve this convergence rate, even if Λ is more regular. In some sense, we are limited by the approximation of $\mathbf{M}$ by $L^2(\gamma)$ functions.

To finish this section, we point out the fact that the use of continuous finite elements for u or $\mathbf{p}$ do not give satisfactory results. In fact, such elements give rise to spurious modes because they do not satisfy the assumption (40) as the Raviart-Thomas type elements (see also [2], [6]).

5 Implementation details

In order to perform the computations, we derive the matrix formulation of discrete problems (18) and (19). Let us denote by $\mathcal{E}_C$, $\mathcal{E}_I$ and $\mathcal{E}_N$ the set of edges on the cracks, in the interior and on the boundary, respectively. We denote by x_p the components of $\mathbf{p}_h$ and x_u is related to the elementwise constant function u_h. The unknowns x_{λ_C} are the Lagrange multipliers corresponding to the Neumann condition on $\partial\Omega \cup \gamma$ and the unknowns x_{λ_I} are the Lagrange multipliers used to ensure the continuity of the normal components of $\mathbf{p}$ across interior edges in the first discretization.

5.1 The first discretization

The matrix formulation of discrete problem (18) reads:

$$\begin{pmatrix} \text{B} & \text{C} & \text{D} & \text{F} \\ C^t & 0 & 0 & 0 \\ D^t & 0 & 0 & 0 \\ F^t & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_p \\ x_u \\ x_{\lambda_I} \\ x_{\lambda_C} \end{pmatrix} = \lambda \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \text{-S} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_p \\ x_u \\ x_{\lambda_I} \\ x_{\lambda_C} \end{pmatrix} \tag{45}$$

The matrices are built locally and then assembled in a usual way. For all $K \in \mathcal{T}_h$, we define the three functions

$$\omega_1(x,y) = (1,0), \quad \omega_2(x,y) = (0,1), \quad \omega_3(x,y) = (x-\overline{x}, y-\overline{y}) \ \ \forall (x,y) \in \ K, \tag{46}$$

where $(\overline{x}, \overline{y})$ is the center of gravity of the triangle K. With these notations, the local matrices B_K and C_K are defined by

$$(B_K)_{ij} = \int_K \omega_i \omega_j dx \ \ \forall i,j = 1,2,3, \quad (C_K)_i = \int_K div(\omega_i) dx \ \ \forall i = 1,2,3.$$

Denoting $z_j = (x_j, y_j)$ the vertices of K, it is readily checked that

$$B_K = |K| \, \mathrm{diag}\,(1,1,\frac{s}{36}) \quad \text{and } C_K = \mathrm{diag}\,(0,0,2|K|),$$

where $s = |z_2 - z_1|^2 + |z_3 - z_2|^2 + |z_3 - z_1|^2$ and $|K|$ denotes the area of K.

Each interior edge e_i is shared by two triangles K_+ and K_-. The corresponding local matrix E_i (of $\mathbb{R}^{6,1}$) is

$$E_{ij} = -\int_{e_i} \omega_j . \nu_i ds \quad \forall j = 1, \ldots, 6,$$

where the ω_i $(i = 1, \ldots, 3)$ are the three functions of (4) defined for K_+ while ω_i $(i = 4, \ldots, 6)$ are related to K_-.

For each edge $e_k \in \mathcal{E}_N$, define

$$F_j = \int_{e_k} \omega_j \cdot \nu_k \, ds \quad \text{for } j = 1,2,3.$$

Finally, the matrix S_K is given by

$$S_K = |K| \mathrm{diag}\,(1,1,1).$$

Next, assembling all these local matrices yield the global system (47).

5.2 The second discretization

The matrix formulation of the discrete problem (18) reads with similar notations:

$$\begin{pmatrix} \mathrm{B} & \mathrm{C} & \mathrm{F} \\ C^t & 0 & 0 \\ F^t & 0 & 0 \end{pmatrix} \begin{pmatrix} x_p \\ x_u \\ x_{\lambda_C} \end{pmatrix} = \lambda \begin{pmatrix} 0 & 0 & 0 \\ 0 & \text{-S} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_p \\ x_u \\ x_{\lambda_C} \end{pmatrix} \tag{47}$$

Since functions of $\mathbf{X}_h$ have their divergence in $L^2(\Omega)$, we use now the edge-basis functions to derive the matrix form of (19). We recall the definition of these functions which form a basis of $\mathbf{X}_h$. For $e \in \mathcal{E}$, let $K_\pm = \mathrm{conv}(K_\pm, P_\pm)$ for the vertex $P_\pm$ opposite to e of $K_\pm$. Let ν_e denote the unit normal vector which points outward from K_+ to K_-. If e is an exterior edge then $\nu = \nu_e$. We set

$$\psi_e(x) = \begin{cases} \pm \dfrac{|e|}{2|K_\pm|}(x - P_\pm) & \text{for } x \in K_\pm, \\ 0 & \text{elsewhere.} \end{cases}$$

Then the local stiffness matrices are given by:

$$(B_K)_{ij} = \int_K \psi_j \cdot \psi_i \, dx \quad \text{for } j, i = 1, 2, 3,$$

and

$$(C_K) = \text{diag}\,(\int_K \text{div}\,\psi_1 \, dx, \int_K \text{div}\,\psi_2 \, dx, \int_K \text{div}\,\psi_3 \, dx).$$

For each γ_i, $1 \leq i \leq I$, we denote by w_ℓ, $0 \leq \ell \leq L_i - 1$ the basis functions of $M_h^1(\gamma_i)$, then we have

$$(F^i)_{\ell j} = \int_{\gamma_i} w_\ell \, (\psi_j . \nu) \, ds, \quad 0 \leq \ell \leq L_i - 1, \ 1 \leq j \leq N_T,$$

and $F = (F^i)$, $1 \leq i \leq I$. Assembling matrices B and C and computing S as before yield the global system for this discretization.

6 Numerical examples

We present some numerical results, first to underline the efficiency of the proposed discretizations, next to show the asymptotic behavior of the spectrum when the number of cracks increases (for a given geometry).

To verify the efficiency of the approach, we consider three examples: We set $\Omega = [-1, 1]^2$ and we solve the eigenvalue problems corresponding to the domain without cracks, the domain with 7 cracks, the domain with 15 cracks and finally with 31 cracks. In all these examples, the cracks are horizontal segments $[-1, 0] \times \{y_i\}$ for some values y_i on the y-axis. Table 1, Table 2, Table 3 and Table 4 summarize the results we obtain. For each example, we have reported the values of the first and the second (non-zero) eigenvalues and the error $|\lambda_{\text{ref}} - \lambda_h|$ in the log scale. λ_{ref} stands for the reference value computed on the finest mesh. These results confirm the rate of convergence $O(h^\alpha)$, $\alpha = 1.96$ for Table 1, and α equal 0.93, 0.96, 0.95 respectively, for Table 2, Table 3 and Table 4.

	λ_1	e_1	λ_2	e_2
$h_1 = 0.372678$ (160 elts)	2.46762745	-8.3934293	4.93919511	-5.4277635
$h_2 = 0.3292694$ (246 elts)	2.46537865	-6.2034455	4.93241818	-6.0389669
$h_3 = 0.1303899$ (1350 elts)	2.46732800	-9.5236785	4.93456371	-8.341180
$h_4 = 0.0937725$ (3116 elts)	2.46743449	-10.307262	4.93454894	-8.281091
	$\pi^2/4$		$\pi^2/2$	

Table 1:

	λ_1	e_1	λ_2	e_2
$h_1 = 0.3609942$ (336 elts)	0.80319711	-3.0013691	1.39193385	-2.1181228
$h_2 = 0.1786276$ (1368 elts)	0.82571782	-3.604603	1.44907129	-2.762722
$h_3 = 0.1219679$ (3248 elts)	0.83430518	-3.9840089	1.46969180	-3.158269
$h_4 = 0.0939467$ (6008 elts)	0.83880861	-4.261052	1.48020938	-3.4425936
$h_{ref} = 0.0130860$ (137960 elts)	0.85291606		1.51219101	

Table 2:

	λ_1	e_1	λ_2	e_2
$h_1 = 0.259579$ (1054 elts)	0.79196817	-3.4479164	1.37203684	-2.524477
$h_2 = 0.1786276$ (1292 elts)	0.79990106	-3.734757	1.39594167	-2.878921
$h_3 = 0.125$ (1968 elts)	0.80707215	-4.0918754	1.41584428	-3.3161376
$h_4 = 0.088849$ (5388 elts)	0.81244789	-4.480113	1.42640026	-3.659835
$h_{ref} = 0.0131982$ (136446 elts)	0.82378002		1.45213702	

Table 3:

For the second discretization, we give only the convergence results in the case of the domain with 7 cracks. All other results agree with the theoretical ones, in particular we have the rate of convergence $O(h^\alpha)$, $\alpha = 1.91$ as shown in Table 5. For the second non zero eigenvalue we have $\alpha = 1.49$ but it increases for more finer meshes. Note that the number of cracks do not affect the expected precision of the method, since the inf-sup conditions (26) and (25) do not depend on that number.

Another example, Table 6, shows that when the crack breaks the simple connectivity of the initial domain (topological change), the resulting spectrum is the union of the spectra of each subdomain.

We investigate now numerically the asymptotic behavior of the spectrum if the number of cracks increases. The first example that we consider consists in a square domain $\Omega = [-1, 1]^2$ with the following number of cracks $L = 0, 7, 15, 31, 63, 127$. The cracks are still horizontal equispaced straight segments $[-1, 0] \times \{y_i\}$. Let us recall that no a priori knowledge on the asymptotic behavior is available for the Neumann eigenvalue problem. In Table 7, we have reported the 20 first eigenvalues for each case. We observe that the first values of the spectrum decrease but remain bounded by a given value 0.8096075353. This asymptotic value is stable when L increases. The value $\frac{\pi^2}{4}$ which is the first non zero eigenvalue with no crack becomes a larger eigenvalue with L cracks, and this is explained by the choice of parallel cracks. In the example of the cracks of length equal to 1, $\frac{\pi^2}{4}$ becomes the $L + 1$ non zero eigenvalue. This fact seems to be a coincidence and it does not hold

	λ_1	e_1	λ_2	e_2
$h_1 = 0.1456065$ (1552 elts)	0.79553444	-4.263488	1.38726351	-3.342662
$h_2 = 0.1017412$ (3932 elts)	0.79707064	-4.3790766	1.39391769	-3.5512572
$h_3 = 0.0888499$ (4794 elts)	0.79962331	-4.6067454	1.39908678	-3.7499271
$h_4 = 0.0577491$ (6976 elts)	0.80289115	-4.9420593	1.40606327	-4.1017940
$h_{ref} = 0.0129046$ (147496 elts)	0.80960757		1.42260624	

Table 4:

	λ_1	e_1	λ_2	e_2
$h_1 = 0.1814387$ (1450 elts)	0.84091988	-4.4231653	1.47029416	-3.1725446
$h_2 = 0.0900933$ (3656 elts)	0.84766675	-5.249654	1.49261209	-3.933301
$h_3 = 0.0642886$ (7008 elts)	0.85049332	-6.0228438	1.50123763	-4.514107
$h_4 = 0.0501623$ (12244 elts)	0.85202546	-7.0235927	1.50648883	-5.1669067
$h_{ref} = 0.0130860$ (137960 elts)	0.85291606		1.51219101	

Table 5:

when the length of the cracks changes and we think also that it does not hold for very large number of cracks.

In Table 8, we have reported the smallest non zero eigenvalue as a function of the number of the cracks (rows) and their length L (columns). This value changes with this two parameters as it could be expected from the monotonicity property.

The example of the disk below Table 9, without cracks, with 15 and 24 cracks shows a similar asymptotic behavior. This can be explained by the fact that the cracks are chosen on the rays and are equispaced.

The last example is a rooms and passages domain (see Figure 2). In Table 10 we have written the low eigenvalues in the case without cracks and in the case with 17 cracks.

References

[1] I. Babuska, J. Osborn—Eigenvalue problems, in the Handbook of Numerical Analysis, Vol **II**, P.G. Ciarlet & J.-L. Lions eds, North-Holland, (1991), 641-788.

λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}
0.000	0.000	2.467	2.467	9.867	9.867	9.867	9.867	12.337	12.337
19.740	19.741	22.192	22.193	32.074	32.074	39.438	39.444	39.444	39.447

Table 6:

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}
0	0.0	**2.467**	**2.467**	4.935	9.869	9.869	12.337	12.337	19.740	22.204
7	0.0	0.850	1.506	1.806	1.977	2.082	2.146	2.181	**2.467**	4.101
15	0.0	0.824	1.446	1.732	1.902	2.013	2.091	2.149	2.192	2.225
31	0.0	0.810	1.421	1.700	1.865	1.974	2.052	2.110	2.155	2.191
63	0.0	0.802	1.407	1.681	1.844	1.951	2.028	2.085	2.130	2.165
127	0.0	0.799	1.401	1.675	1.836	1.943	2.019	2.076	2.121	2.156

	λ_{11}	λ_{12}	λ_{13}	λ_{14}	λ_{15}	λ_{16}	λ_{17}	λ_{18}	λ_{19}	λ_{20}
7	9.869	10.165	11.185	14.892	15.317	17.403	18.545	19.209	19.562	22.203
15	2.251	2.271	2.286	2.297	2.305	2.309	**2.467**	4.096	9.869	10.010
31	2.220	2.244	2.264	2.281	2.296	2.309	2.319	2.329	2.337	2.345
63	2.194	2.218	2.239	2.256	2.271	2.284	2.296	2.306	2.315	2.323
127	2.185	2.209	2.229	2.247	2.262	2.275	2.286	2.297	2.306	2.314

Table 7:

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	7	15	31	63	127
$L = \frac{1}{2}$	1.621	1.567	1.543	1.527	1.523
$L = 1$	0.850	0.824	0.809	0.802	0.799
$L = \frac{3}{2}$	0.434	0.416	0.408	0.404	0.402

Table 8:

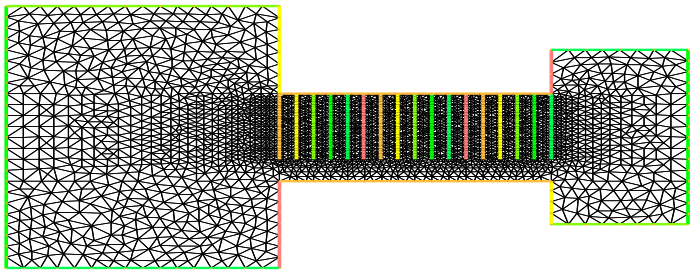

Figure 2: Rooms and passage

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	no cracks	15 cracks	24 cracks
λ_1	0.000	0.000	0.000
λ_2	3.390	1.967	1.916
λ_3	3.390	2.012	1.916
λ_4	9.329	3.356	3.256
λ_5	9.329	3.651	3.351
λ_6	**14.682**	4.409	4.237
λ_7	17.651	4.614	4.237
λ_8	17.651	5.207	4.817
λ_9	28.278	5.212	4.945
λ_{10}	28.278	5.528	5.328
λ_{11}	28.425	5.626	5.328
λ_{12}	28.425	6.111	5.562
λ_{13}	41.163	6.147	5.763
λ_{14}	41.163	6.182	5.885
λ_{15}		6.199	5.885
λ_{16}		6.152	5.954
λ_{17}		**14.682**	6.387
λ_{18}		27.805	6.408
λ_{19}		27.805	6.409
λ_{20}		40.763	6.434
λ_{21}		41.316	6.489
λ_{22}		49.217	6.503
λ_{23}		51.930	6.504
λ_{24}		51.930	6.512
λ_{25}		56.092	**14.682**

Table 9:

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}
No crack	0.0	0.052	0.287	0.384	0.698	0.721	0.915	1.098	1.425	1.966
17	0.0	0.016	0.129	0.274	0.341	0.516	0.627	0.681	0.736	0.788

Table 10: