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Abstract:	<p>We consider a telecommunications company expanding its network capacity to face an increasing demand. The company can also invest in marketing to incentivize clients to shift to more recent technologies, hopefully leading to cheaper overall costs. To model the effect of the marketing campaigns, previous works have relied on the Bass model. Since that model only provides a rough approximation of the actual shifting mechanism, the purpose of this work is to consider uncertainty in the shifting mechanism through the lens of robust optimization. We thus assume that the (discrete) shifting function can take any value in a given polytope and wish to optimize against the worst-case realization. The resulting robust optimization problem possesses integer recourse variables and non-linear dependencies on the uncertain parameters. We address these difficulties as follows. First, the integer recourse is tackled heuristically through a piece-wise constant policy dictated by a prior partition of the uncertainty polytope. Second, the non-linearities are handled by a careful analysis of the dominating scenarios. The scalability and economical relevance of our models are assessed through numerical experiments performed on real instances, underlining the benefit of using robust optimization.</p>
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Optimizing subscriber migrations for a telecommunication operator in uncertain context

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Abstract

We consider a telecommunications company expanding its network capacity to face an increasing demand. The company can also invest in marketing to incentivize clients to shift to more recent technologies, hopefully leading to cheaper overall costs. To model the effect of the marketing campaigns, previous works have relied on the Bass model. Since that model only provides a rough approximation of the actual shifting mechanism, the purpose of this work is to consider uncertainty in the shifting mechanism through the lens of robust optimization. We thus assume that the (discrete) shifting function can take any value in a given polytope and wish to optimize against the worst-case realization. The resulting robust optimization problem possesses integer recourse variables and non-linear dependencies on the uncertain parameters. We address these difficulties as follows. First, the integer recourse is tackled heuristically through a piece-wise constant policy dictated by a prior partition of the uncertainty polytope. Second, the non-linearities are handled by a careful analysis of the dominating scenarios. The scalability and economical relevance of our models are assessed through numerical experiments performed on real instances, underlining the benefit of using robust optimization.

Keywords: OR in telecommunications, Capacity Expansion, Bass model, Mixed-Integer Linear Programming, Robust Optimization

1. Introduction and problem description

New bandwidth-consuming usages and the increase in the number of users induce an exponential growth of mobile traffic CISCO (2017). Facing this traffic growth, telecommunications companies

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4 are hence pushed to expand their network through important investments (several billion € to
5 improve the mobile network in the last six years, see Orange (2018)). This network design problem
6 has two specific features. First, a fast roll-out of mobile networks generations and a highly compet-
7 itive environment that encourages operators to invest in the newest technology available. Second,
8 telecommunications companies are often both infrastructure operators, planning their network ex-
9 pansion, and service providers, designing the offers proposed to the subscribers. These two features
10 impact the investment decisions of the operator.
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17 Through marketing investments, the operator can control the demand on its different technolo-
18 gies and avoid over-dimensioning, hence reducing its investments in network design. Reversely, the
19 efficiency of such investments over a given year is also dependent on the network deployment per-
20 formed in the previous years: subscribers will easily accept to shift towards the newest technology
21 if it benefits from a high level of coverage, thanks to previous investments in the deployment of that
22 technology. Exploring the trade-off between network and subscribers dynamic can be financially
23 more interesting than a separate optimization of the two problems. Moreover, an operator fixes
24 strategical guidelines on its network to remain competitive. Some of these guidelines, for example
25 throughput, depend on both network and subscriber performances. Investments in subscribers and
26 networks should hence be jointly optimized over the whole time-horizon of strategical planning,
27 which is typically 5 years for a telecommunication operator.
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37 This enlightens that subscriber and network dynamics are intertwined. To the best of our
38 knowledge, the authors in Cambier et al. (2019) are the first to consider jointly the two dynamics. To
39 represent the subscriber dynamic, they consider discrete subsidies that represent different possible
40 marketing savings on a new phone required to access the new service. To model how the subscribers
41 react to such subsidies, they rely on the well-known Bass model: see Bass (1969) for the original
42 paper, while Norton and Bass (1987) provide a model with the notion of generations, and Bass
43 (2004) considers applications to telecommunications context. The authors of Cambier et al. (2019)
44 tackle only the deterministic framework where the number of subscribers deciding to shift to the
45 newest generation is a fully determined function of the subsidy offered and of an indicator of the
46 network deployment. However, that function is only a naive approximation, since many internal
47 and external factors intervene in operator marketing. We hence consider in this article uncertainty
48 on the values of the shifting function, focusing on the two-periods case.
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4 We model the shifting function uncertainty via a known polyhedral set, leading to a robust
5 optimization problem. Robust optimization is an efficient framework to handle uncertainty in
6 (mixed-integer) linear optimization problems, see Ben-Tal et al. (2009) and Gabrel et al. (2014),
7 among others. The framework can, in theory, address a wide variety of optimization problems,
8 including discrete variables (Buchheim and Kurtz, 2018), non-linear constraints (Ben-Tal et al.,
9 2015), and the multi-stage setting where some of the decision variables can adjust their value to
10 the realization of the uncertain parameters (see Delage and Iancu (2015) and İhsan Yanıkoğlu et al.
11 (2019)). In practice, however, the dimensions of the problems one can expect to solve, and the
12 optimality guarantees of the resulting algorithms, strongly depend on each problem characteristics
13 and on the difficulty of the underlying nominal optimization problem. Here one must bear in mind
14 that the underlying model considered, which was studied in Cambier et al. (2019), is already a
15 difficult MILP that can be solved exactly only for moderate-size instances. This means that the
16 robust counterpart we seek should not be much harder than its nominal variant.

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18 The specific application considered in this paper features two specific difficulties: some of the
19 integer variables are adjustable, and the constraints of the problem have non-linear dependencies
20 on the shifting function. While both specificities have been addressed in the recent robust opti-
21 mization literature, they are often addressed through heuristic solution procedures. On the one
22 hand, integer adjustable variables are typically tackled (approximately) by partitioning the uncer-
23 tainty polytope (Bertsimas and Caramanis, 2010; Bertsimas and Dunning, 2016; Postek and den
24 Hertog, 2016) or by introducing complex decision rules (Bertsimas and Georghiou, 2018). On the
25 other hand, robust optimization with non-linear dependency on the uncertainty parameters can,
26 in some cases, be reformulated efficiently, for instance, when these dependencies are concave func-
27 tions (Ben-Tal et al., 2015). The non-linear dependencies considered herein will be modeled by
28 (non-concave) bilinear functions for which reformulations exist only in very particular cases, e.g.,
29 when the uncertainty polytope is an ellipsoid (Ben-Tal et al., 2002) or if the function is the product
30 of two affine functions (Pessoa and Poss, 2015).

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32 In this paper, we address the adjustable variables through uncertainty partitioning, while the
33 non-linearities are handled by underlining dominances of the uncertainty polytope. Specifically,
34 our contribution is three-fold. First, we provide two formulations for the two-period deterministic
35 problem: one comes directly from Cambier et al. (2019) and the other one is a reformulation for the
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4 two period-case which will be more amenable to the derivation of the robust counterpart. These
5 formulations are strengthened with RLT inequalities (see Sherali and Adams (1998)). Second,
6 we provide two robust counterparts (static and adjustable) of the previous model, as well as a
7 solving method. This method analyzes the structure of the problem to reformulate it as a linear
8 mixed-integer program with a finite number of constraints and variables. Third, through numerical
9 experiments on realistic instances, we provide insights for the operator on the effects of uncertainty,
10 and the importance to adapt her decisions to the shifting function outcome in the first-stage.

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17 The remainder of this article is organized as follows. Section 2 presents the two mixed-integer
18 formulations for the deterministic framework. Section 3 introduces the static and adjustable robust
19 counterparts. Numerical experiments assess both models in Section 4. Concluding remarks are
20 given in Section 5.

25 26 **2. Deterministic model and formulations**

27
28 In this section, we present two formulations for the deterministic framework. Section 2.1 de-
29 scribes the problem and presents the formulation from Cambier et al. (2019). A reformulation
30 specific to the two period-case is provided in Section 2.2. This reformulation is linearized and
31 strengthened in Section 2.3.

32 33 *2.1. Current formulation*

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36 In this work, we focus on a framework with two network technologies (denoted by $g \in \mathcal{G} =$
37 $\{CG, NG\}$), the current one CG and the newest one NG that the telecommunication operator aims
38 to deploy. The time horizon is discretized in two equally-sized time periods, denoted by $t \in \mathcal{T} =$
39 $\{1, 2\}$, adding “0” to denote the beginning of the time-horizon. We consider a set $\mathcal{S} = \{1, \dots, N_S\}$
40 of N_S existing telecommunication sites. At the beginning of the time-horizon, all sites are equipped
41 with CG technology, while NG is deployed on some sites and can be deployed on the other ones.
42 These sites have an initial capacity (discretized in numbers of modules) which can be increased
43 by adding new modules. We also consider that we know an equivalent number of subscribers
44 associated with each site $s \in \mathcal{S}$. A technical compatibility rule imposes that CG subscribers
45 can only be served by CG technology while NG subscribers can be served by both technologies.
46 Furthermore, the load-balancing rule decided by the operator states that NG subscribers have to
47 be served by NG technology if NG is deployed. The capacity of each site and each technology has
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4 to be sufficient for handling traffic 95% of the time. We also assume that the upgrade mechanism
5 modeling subscribers reaction for shifting to *NG* technology depends on two parameters:

- 6 • The value of the subsidy denoted by σ .
- 7
- 8 • A discrete indicator of the level of coverage of *NG* technology denoted by c .
- 9

10 The set of possible subsidy values is denoted by \mathcal{K} . We model the level of coverage by partition-
11 ing the interval $[0, 1]$ into C smaller intervals indexed by set $\mathcal{C} = \{1, \dots, C\}$. Hence $c \in \mathcal{C}$ indicates
12 the interval (coverage range) to which belongs the sites coverage (percentage of sites equipped with
13 *NG* at the beginning of the time period).

14 We further consider two strategical guidelines that can be fixed by a telecommunication reg-
15 ulator or the operator: the quality of experience (percentage of *NG* subscribers served by *NG*
16 technology) and the sites coverage, both taken at the end of the time-horizon. All in all, the Mobile
17 Master Plan (MMP) problem for these two technologies consists in finding the subsidies decisions
18 (amount of subsidy given at each period) and networks decisions (installing *NG* technology and
19 adding modules for both technologies) for each site, while satisfying load-balancing and capacity
20 constraints at each time period, and the two strategical guidelines at the end of the time horizon.

21 Parameters and variables used throughout this paper are provided respectively in Tables 1
22 and 2.
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Parameters:

CA_{NG}	cost of adding NG technology on each site
CM_g	cost of adding a module of technology $g \in \mathcal{G}$ on each site
$M_{s,g}^0$	initial number of modules of technology $g \in \mathcal{G}$ on site $s \in \mathcal{S}$
\overline{M}_g	technical upper bound on the number of modules of technology $g \in \mathcal{G}$
$Z_{s,NG}^0$	initial presence (yes/no) of NG technology on site $s \in \mathcal{S}$
$U_{s,g}^0$	initial number of subscribers on site $s \in \mathcal{S}$ to technology $g \in \mathcal{G}$
D_g^t	unitary demand of a subscriber served by technology $g \in \mathcal{G}$ at time period $t \in \mathcal{T}$
CAP_g	capacity of each additional module of technology $g \in \mathcal{G}$
$f_{\sigma,c}$	reaction to the subsidy offered $\sigma \in \mathcal{K}$ under range of coverage interval $c \in \mathcal{C}$
L_c	lower bound of coverage range indexed by $c \in \mathcal{C}$
U_c	upper bound of coverage range indexed by $c \in \mathcal{C}$
$\overline{U}_{s,g}^t$	upper bound on the total number of subscribers to technology $g \in \mathcal{G}$ on site $s \in \mathcal{S}$ at the end of time period $t \in \mathcal{T}$
α^0	sites coverage at the beginning of the time horizon
$\underline{\alpha}$	threshold for the coverage (strategic guideline)
\underline{QoE}	threshold for the quality of service (strategic guideline)

Table 1: Model parameters

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Variables:

$z_{s,NG}^t$	binary variable indicating whether <i>NG</i> technology is deployed on site $s \in \mathcal{S}$ at the end of time-period $t \in \mathcal{T} \cup \{0\}$
$m_{s,g}^t$	total number of modules of technology $g \in \mathcal{G}$ deployed on site $s \in \mathcal{S}$ at the end of time period $t \in \mathcal{T} \cup \{0\}$
$u_{s,g}^t$	total number of subscribers to technology $g \in \mathcal{G}$ on site $s \in \mathcal{S}$ at the end of time period $t \in \mathcal{T} \cup \{0\}$
$u_{s,o,g}^t$	total number of subscribers to technology $o \in \mathcal{G}$ on site $s \in \mathcal{S}$ served by technology $g \in \mathcal{G}$ at the end of time period $t \in \mathcal{T} \cup \{0\}$
α^t	redundant variable that denotes the <i>NG</i> sites coverage (fraction of sites where <i>NG</i> technology is deployed) at the end of the time period $t \in \mathcal{T}$, which is equal to $\frac{\sum_{s \in \mathcal{S}} z_{s,NG}^t}{N_S}$
γ_c^t	binary variable indicating whether α^{t-1} belongs to range $c \in \mathcal{C}$
δ_σ^t	binary variable indicating whether subsidy offered at time period $t \in \mathcal{T}$ is $\sigma \in \mathcal{K}$

Table 2: Model variables

The MMP problem is modeled in Cambier et al. (2019) as follows:

$$\begin{aligned} \min \quad & \sum_{t \in \mathcal{T}} \sum_{\sigma \in \mathcal{K}} \sum_{c \in \mathcal{C}} \sum_{s \in \mathcal{S}} \sigma f_{\sigma,c} \delta_{\sigma}^t \gamma_c^t u_{s,CG}^{t-1} + \sum_{s \in \mathcal{S}} \sum_{g \in \mathcal{G}} CM_g(m_{s,g}^2 - M_{s,g}^0) \\ & + \sum_{s \in \mathcal{S}} CA_{NG}(z_{s,NG}^2 - Z_{s,NG}^0) \end{aligned} \quad (1)$$

$$\text{s.t. } m_{s,CG}^t \leq \overline{M}_{CG} \quad \forall s \in \mathcal{S}, \forall t \in \mathcal{T}, \quad (2)$$

$$m_{s,NG}^t \leq \overline{M}_{NG} z_{s,NG}^t \quad \forall s \in \mathcal{S}, \forall t \in \mathcal{T}, \quad (3)$$

$$m_{s,g}^{t-1} \leq m_{s,g}^t \quad \forall s \in \mathcal{S}, \forall t \in \mathcal{T}, \forall g \in \mathcal{G}, \quad (4)$$

$$u_{s,NG}^t = u_{s,NG,CG}^t + u_{s,NG,NG}^t \quad \forall s \in \mathcal{S}, \forall t \in \mathcal{T}, \quad (5)$$

$$u_{s,NG,CG}^t \leq \overline{U}_{s,NG}^t (1 - z_{s,NG}^t) \quad \forall s \in \mathcal{S}, \forall t \in \mathcal{T}, \quad (6)$$

$$D_{CG}^t (u_{s,CG}^t + u_{s,NG,CG}^t) \leq CAP_{CG} m_{s,CG}^t \quad \forall s \in \mathcal{S}, \forall t \in \mathcal{T}, \quad (7)$$

$$D_{NG}^t u_{s,NG,NG}^t \leq CAP_{NG} m_{s,NG}^t \quad \forall s \in \mathcal{S}, \forall t \in \mathcal{T}, \quad (8)$$

$$u_{s,CG}^t = u_{s,CG}^{t-1} - \sum_{\sigma \in \mathcal{K}} \sum_{c \in \mathcal{C}} f_{\sigma,c} \delta_{\sigma}^t \gamma_c^t u_{s,CG}^{t-1} \quad \forall s \in \mathcal{S}, \forall t \in \mathcal{T}, \quad (9)$$

$$u_{s,NG}^t = u_{s,NG}^{t-1} + \sum_{\sigma \in \mathcal{K}} \sum_{c \in \mathcal{C}} f_{\sigma,c} \delta_{\sigma}^t \gamma_c^t u_{s,CG}^{t-1} \quad \forall s \in \mathcal{S}, \forall t \in \mathcal{T}, \quad (10)$$

$$\sum_{s \in \mathcal{S}} u_{s,NG,NG}^2 \geq \underline{QoE} \left(\sum_{s \in \mathcal{S}} U_{s,NG}^0 + U_{s,CG}^0 \right), \quad (11)$$

$$\alpha^2 \geq \underline{\alpha}, \quad (12)$$

$$\sum_{\sigma \in \mathcal{K}} \delta_{\sigma}^t = 1 \quad \forall t \in \mathcal{T}, \quad (13)$$

$$\sum_{c \in \mathcal{C}} \gamma_c^t = 1 \quad \forall t \in \mathcal{T}, \quad (14)$$

$$\gamma_c^t \leq 1 + U_c - \alpha^{t-1} \quad \forall t \in \mathcal{T}, \forall c \in \mathcal{C}, \quad (15)$$

$$\gamma_c^t \leq 1 + \alpha^{t-1} - L_c \quad \forall t \in \mathcal{T}, \forall c \in \mathcal{C}, \quad (16)$$

$$u_{s,g}^0 = U_{s,g}^0 \quad \forall s \in \mathcal{S}, \forall g \in \mathcal{G}, \quad (17)$$

$$m_{s,g}^0 = M_{s,g}^0 \quad \forall s \in \mathcal{S}, \forall g \in \mathcal{G}, \quad (18)$$

$$z_{s,NG}^0 = Z_{s,NG}^0 \quad \forall s \in \mathcal{S}, \quad (19)$$

$$\alpha^t N_S = \sum_{s \in \mathcal{S}} z_{s,NG}^t \quad \forall t \in \mathcal{T} \cup \{0\}, \quad (20)$$

$$m_{s,g}^t \in \mathbb{N} \quad \forall s \in \mathcal{S}, \forall t \in \mathcal{T} \cup \{0\}, \forall g \in \mathcal{G}, \quad (21)$$

$$z_{s,NG}^t \in \{0, 1\} \quad \forall s \in \mathcal{S}, \forall t \in \mathcal{T} \cup \{0\}, \quad (22)$$

$$u_{s,o}^t \geq 0 \quad \forall s \in \mathcal{S}, \forall t \in \mathcal{T} \cup \{0\}, \forall o \in \mathcal{G}, \quad (23)$$

$$u_{s,o,g}^t \geq 0 \quad \forall s \in \mathcal{S}, \forall t \in \mathcal{T}, \forall o, g \in \mathcal{G}^2, \quad (24)$$

$$\gamma_c^t \in \{0, 1\} \quad \forall t \in \mathcal{T}, \forall c \in \mathcal{C}, \quad (25)$$

$$\delta_\sigma^t \in \{0, 1\} \quad \forall t \in \mathcal{T}, \forall \sigma \in \mathcal{K}. \quad (26)$$

We denote this formulation by \mathcal{M}^{det} . The objective function (1) minimizes both subscribers' migration costs and network investments. The first term stands for the cost of offering subsidies to the subscriber, the second term for the network modules cost, and the third term for the deployment of the newest technology NG .

Constraints (2)–(4) enforce the network dynamic rules ensuring the number of modules to be non-decreasing and imposing technical upper bounds on numbers of modules. Constraints (5)–(8) refer to the network dimensioning constraints, ensuring capacity constraints and load-balancing rules, making hence the link between the network dynamic and the subscriber dynamic. Constraints (9) and (10) stand for the subscriber dynamic constraints. They compute the total number of subscribers to CG and NG technologies at each site and each time period in terms of the number of subscribers to both technologies in the previous period and of the percentage of former CG subscribers who decide to shift to NG technology. Constraints (11) and (12) ensure the model strategic guidelines and refer to the end of the time horizon. Constraints (13) ensure that one and only one subsidy from the set \mathcal{K} is offered at each time period. Constraints (14)–(16) set the range of coverage for each time period according to the coverage. Constraints (17)–(19) refer to the initial conditions and constraints while constraints (20)–(26) define the domain of all variables in the formulation.

To linearize this formulation, we introduce variables:

$$\pi_{\sigma,c,s}^t = \delta_\sigma^t \gamma_c^t u_{s,CG}^{t-1}, \quad \forall t \in \mathcal{T}, \forall s \in \mathcal{S}, \forall \sigma \in \mathcal{K}, \forall c \in \mathcal{C}$$

and use the classical linearization of the product of a binary variable by a continuous one. Details on the linearization are given in Cambier et al. (2019).

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4 *2.2. Reformulation*
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6 In this section, we will consider a reformulation of \mathcal{M}^{det} that avoids the recursive structure
7 defined by constraints (9) and (10) and expresses variables u directly in term of variables δ , γ
8 and function f . This reformulation will be more amenable to obtain a robust counterpart in our
9 two-periods framework.
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14 **Proposition 2.1.** *The following non-linear equalities are valid expressions for the number of sub-*
15 *scribers on each site $s \in \mathcal{S}$ and each time-period $t \in \mathcal{T}$ in \mathcal{M}^{det}*
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$$17 \quad u_{s,CG}^t = U_{s,CG}^0 \prod_{i=1}^t \left(1 - \sum_{\sigma \in \mathcal{K}} \sum_{c \in \mathcal{C}} f_{\sigma,c} \delta_{\sigma}^i \gamma_c^i \right)$$

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21 and

$$22 \quad u_{s,NG}^t = U_{s,CG}^0 + U_{s,NG}^0 - u_{s,CG}^t = U_{s,NG}^0 + U_{s,CG}^0 \left[1 - \prod_{i=1}^t \left(1 - \sum_{\sigma \in \mathcal{K}} \sum_{c \in \mathcal{C}} f_{\sigma,c} \delta_{\sigma}^i \gamma_c^i \right) \right]$$

23
24 *Proof.* The expression for $u_{s,CG}^t$ is obtained by applying recursively over the time-horizon con-
25 straints (9) from \mathcal{M}^{det} . The expression for $u_{s,NG}^t$ is obtained by summing constraints (9) and (10)
26 which gives $u_{s,CG}^t + u_{s,NG}^t = u_{s,CG}^{t-1} + u_{s,NG}^{t-1}$ and hence recursively $u_{s,CG}^t + u_{s,NG}^t = U_{s,CG}^0 + U_{s,NG}^0$. \square
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32 These expressions allow to replace continuous variables $u_{s,NG}^t$ and $u_{s,CG}^t$ by expressions that
33 depend only on variables δ and γ . When considering $|\mathcal{T}| > 2$ periods, these new expressions contain
34 a high number of non-linearities (products of δ) which require each a linearization variable: hence
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$$38 \quad NV(|\mathcal{T}|) = \sum_{l=2}^{|\mathcal{T}|} \frac{|\mathcal{T}|!}{(|\mathcal{T}| - l)! l!} |\mathcal{K}|^l = (|\mathcal{K}| + 1)^{|\mathcal{T}|} - 1 - |\mathcal{T}| |\mathcal{K}|$$

39 additional variables are required. When focusing on a two periods framework ($T = 2$), the non-
40 linear terms are direct products of variables δ^1 and δ^2 . We can hence linearize our formulation by
41 adding only $|K|^2$ variables and $3|K|^2$ constraints (see (Fortet, 1960)). We observe that variables γ^1
42 are not needed as the coverage range in the first period depends on the initial percentage of NG
43 sites, and is hence already known.
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53 **Proposition 2.2.** *Constraints (5) and (6) can be replaced with the following non-linear equations*
54 *in formulation \mathcal{M}^{det} , for each site $s \in \mathcal{S}$ and each time-period $t \in \mathcal{T}$:*
55

$$56 \quad u_{s,NG,NG}^t = u_{s,NG}^t z_{s,NG}^t \tag{27}$$

$$u_{s,NG,CG}^t = u_{s,NG}^t(1 - z_{s,NG}^t) \quad (28)$$

Proof. First, let us assume that equations (27) and (28) are satisfied. For each site $s \in \mathcal{S}$, by summing (27) and (28), we obtain that (5) is satisfied. Clearly, equation (6) is implied by (28).

Reciprocally, let us assume that constraints (5) and (6) are both satisfied. Two cases can happen. If *NG* technology is installed ($z_{s,NG}^t = 1$): constraints (6) induce $u_{s,NG,CG}^t = 0$ and then $u_{s,NG}^t = u_{s,NG,NG}^t$ so that equations (27) and (28) are satisfied. If *NG* technology is not installed ($z_{s,NG}^t = 0$), we have $m_{s,NG}^t = 0$ (see constraints (3)) and $u_{s,NG,NG}^t = 0$ (see constraints (8)) so that equations (27) and (28) are satisfied. \square

From Propositions 2.1 and 2.2, we obtain that continuous variables u can be expressed with the following equations depending on δ , γ and f , for each site $s \in \mathcal{S}$ and each time-period $t \in \mathcal{T}$:

$$\begin{aligned} u_{s,NG,NG}^t &= \left(U_{s,NG}^0 + U_{s,CG}^0 \left[1 - \prod_{i=1}^t \left(1 - \sum_{\sigma \in \mathcal{K}} \sum_{c \in \mathcal{C}} f_{\sigma,c} \delta_{\sigma}^i \gamma_c^i \right) \right] \right) z_{s,NG}^t \\ u_{s,CG}^t + u_{s,NG,CG}^t &= u_{s,NG}^t + u_{s,CG}^t = U_{s,NG}^0 + U_{s,CG}^0 && \text{if } z_{s,NG}^t = 0, \\ u_{s,CG}^t + u_{s,NG,CG}^t &= u_{s,CG}^t = U_{s,CG}^0 \prod_{i=1}^t \left(1 - \sum_{\sigma \in \mathcal{K}} \sum_{c \in \mathcal{C}} f_{\sigma,c} \delta_{\sigma}^i \gamma_c^i \right) && \text{if } z_{s,NG}^t = 1. \end{aligned}$$

For simplifying the subsequent notations, we denote the total number of subscribers ($\sum_{s \in \mathcal{S}} U_{s,CG}^0 + U_{s,NG}^0$) by UTOT.

Consequently, formulation \mathcal{M}^{det} with two generations and two periods can be reformulated without continuous variables, as follows:

$$\min \quad \text{subcost} + \sum_{s \in \mathcal{S}} \sum_{g \in \mathcal{G}} CM_g (m_{s,g}^2 - M_{s,g}^0) + \sum_{s \in \mathcal{S}} CA_{NG} (z_{s,NG}^2 - Z_{s,NG}^0) \quad (29)$$

$$\text{s.t.} \quad \sum_{\sigma^1 \in \mathcal{K}} \sigma^1 f_{\sigma^1, c^1} \delta_{\sigma^1}^1 + \sum_{\sigma^2 \in \mathcal{K}} \sum_{c^2 \in \mathcal{C}} \sigma^2 f_{\sigma^2, c^2} \delta_{\sigma^2}^2 \gamma_{c^2}^2 (1 - \sum_{\sigma^1 \in \mathcal{K}} f_{\sigma^1, c^1} \delta_{\sigma^1}^1) \leq \frac{\text{subcost}}{\sum_{s \in \mathcal{S}} U_{s,CG}^0}, \quad (30)$$

$$(2) - (4)$$

$$D_{CG}^1 CAP_{CG} (U_{s,CG}^0 + U_{s,NG}^0) (1 - z_{s,NG}^1) \leq m_{s,CG}^1 \quad \forall s \in \mathcal{S}, \quad (31)$$

$$\frac{D_{CG}^1}{CAP_{CG}} U_{s,CG}^0 (1 - \sum_{\sigma^1 \in \mathcal{K}} f_{\sigma^1, c^1} \delta_{\sigma^1}^1) \leq m_{s,CG}^1 \quad \forall s \in \mathcal{S}, \quad (32)$$

$$\frac{D_{NG}^1}{CAP_{NG}}(U_{s,NG}^0 + U_{s,CG}^0 \sum_{\sigma^1 \in \mathcal{K}} f_{\sigma^1, c^1} \delta_{\sigma^1}^1) z_{s,NG}^1 \leq m_{s,NG}^1 \quad \forall s \in \mathcal{S}, \quad (33)$$

$$\frac{D_{CG}^2}{CAP_{CG}}(U_{s,CG}^0 + U_{s,NG}^0)(1 - z_{s,NG}^2) \leq m_{s,CG}^2 \quad \forall s \in \mathcal{S}, \quad (34)$$

$$\frac{D_{CG}^2 U_{s,CG}^0}{CAP_{CG}} \left[1 - \left(\sum_{\sigma^1 \in \mathcal{K}} f_{\sigma^1, c^1} \delta_{\sigma^1}^1 + \sum_{\sigma^2 \in \mathcal{K}} \sum_{c^2 \in \mathcal{C}} f_{\sigma^2, c^2} \delta_{\sigma^2}^2 \gamma_{c^2}^2 (1 - \sum_{\sigma^1 \in \mathcal{K}} f_{\sigma^1, c^1} \delta_{\sigma^1}^1) \right) \right] \leq m_{s,CG}^2 \quad \forall s \in \mathcal{S}, \quad (35)$$

$$\frac{D_{NG}^2}{CAP_{NG}} \left[U_{s,NG}^0 + U_{s,CG}^0 \left(\sum_{\sigma^1 \in \mathcal{K}} f_{\sigma^1, c^1} \delta_{\sigma^1}^1 + \sum_{\sigma^2 \in \mathcal{K}} \sum_{c^2 \in \mathcal{C}} f_{\sigma^2, c^2} \delta_{\sigma^2}^2 \gamma_{c^2}^2 (1 - \sum_{\sigma^1 \in \mathcal{K}} f_{\sigma^1, c^1} \delta_{\sigma^1}^1) \right) \right] z_{s,NG}^2 \leq m_{s,NG}^2 \quad \forall s \in \mathcal{S}, \quad (36)$$

$$\sum_{s \in \mathcal{S}} \left[U_{s,NG}^0 + U_{s,CG}^0 \left(\sum_{\sigma^1 \in \mathcal{K}} f_{\sigma^1, c^1} \delta_{\sigma^1}^1 + \sum_{\sigma^2 \in \mathcal{K}} \sum_{c^2 \in \mathcal{C}} f_{\sigma^2, c^2} \delta_{\sigma^2}^2 \gamma_{c^2}^2 (1 - \sum_{\sigma^1 \in \mathcal{K}} f_{\sigma^1, c^1} \delta_{\sigma^1}^1) \right) \right] z_{s,NG}^2 \geq \underline{QoE}, \quad UTOT \quad (37)$$

$$(12) - (26)$$

$$subcost \geq 0. \quad (38)$$

Constraint (30) enables to compute the subsidies cost *subcost* by using the expression from Proposition 2.1 for the number of *CG* subscribers (note that variable *subcost* will thanks to the minimization exactly be equal to the subsidies cost, but formulating with a \leq sign is more amenable for writing the robust counterpart). Load-balancing constraints (5) and (6), capacity constraints (7) and (8), subscriber dynamic constraints (9) and (10) and *QoE* threshold constraint (11) from formulation \mathcal{M}^{det} are replaced by set of constraints (31)–(37). Constraints (31) and (34) impose that all subscribers on a site (which is a constant) have to be served by *CG* technology when *NG* is not installed. Constraints (32) and (35) state that *CG* subscribers have to be served by *CG*, which is dominated by previous constraints when *NG* is not installed. Constraints (33) and (36) impose that when *NG* is installed *NG* subscribers have to be served by *NG*. The same formulas for obtaining the number of *NG* subscribers are used in the *QoE* constraint (37).

2.3. Linearization and RLT cuts

We discuss next how the model presented in Section 2.2 is linearized. Notice that all products appearing in this formulation are products of binary variables. We linearize the model presented

in Section 2.2 by applying Fortet Linearizations (see Fortet (1960)). For each $\sigma^1, \sigma^2 \in \mathcal{K}^2$, $s \in \mathcal{S}$, $t \in \mathcal{T}$, we introduce the following new binary variables:

- $\chi_{\sigma,s}$ for linearizing $\delta_{\sigma}^1 z_{s,NG}^1$ in constraints (33),
- $\pi_{\sigma,s}^t$ for linearizing $\delta_{\sigma}^t z_{s,NG}^2$ in constraints (36) and (37),
- $\eta_{\sigma^1, \sigma^2}$ for linearizing $\delta_{\sigma^1}^1 \delta_{\sigma^2}^2$ in constraints (30) and (35)–(37),
- $\zeta_{\sigma^1, \sigma^2, s}$ for linearizing $\delta_{\sigma^1}^1 \delta_{\sigma^2}^2 z_{s,NG}^2 = \pi_{\sigma^1, s}^1 \delta_{\sigma^2}^2 = \pi_{\sigma^2, s}^2 \delta_{\sigma^1}^1 = z_{s,NG}^2 \eta_{\sigma^1, \sigma^2}$ in constraints (36) and (37).

Remark 2.3. The variable δ^2 and hence some of the variables above are also multiplied by γ^2 . As in formulation \mathcal{M}^{det} , these products are not handled with Fortet linearizations, but by introducing variables $\delta_{\sigma,c}^2 = \gamma_c^2 \delta_{\sigma}^2$, for each $\sigma \in \mathcal{K}$ and each $c \in \mathcal{C}$, and by replacing γ_c^2 with $\sum_{\sigma \in \mathcal{K}} \delta_{\sigma,c}^2$ (term equal to 1 if coverage range is c and 0 otherwise). We choose to keep $\gamma_c^2 \delta_{\sigma}^2$ in the following for writing simplification.

Now, we show how our formulation can be strengthened by applying Reformulation Linearization Techniques (RLT). Multiplying constraints (13) for $t = 2$ by variable $z_{s,NG}^1$ for each site $s \in \mathcal{S}$,

$$\sum_{\sigma \in \mathcal{K}} \chi_{\sigma,s} = z_{s,NG}^1 \quad \forall s \in \mathcal{S}. \quad (39)$$

Multiplying constraints (13) by variable $z_{s,NG}^2$ for each site $s \in \mathcal{S}$, we obtain:

$$\sum_{\sigma \in \mathcal{K}} \pi_{\sigma,s}^t = z_{s,NG}^2 \quad \forall t \in \mathcal{T}, \forall s \in \mathcal{S}. \quad (40)$$

Multiplying constraints (13) for $t = 2$ by variable $\delta_{\sigma^1}^1$ for each subsidy $\sigma^1 \in \mathcal{K}$, we obtain:

$$\sum_{\sigma^2 \in \mathcal{K}} \eta_{\sigma^1, \sigma^2} = \delta_{\sigma^1}^1 \quad \forall \sigma^1 \in \mathcal{K}. \quad (41)$$

Multiplying constraints (13) for $t = 1$ by variable $\delta_{\sigma^2}^2$ for each subsidy $\sigma^2 \in \mathcal{K}$, we obtain:

$$\sum_{\sigma^1 \in \mathcal{K}} \eta_{\sigma^1, \sigma^2} = \delta_{\sigma^2}^2 \quad \forall \sigma^2 \in \mathcal{K}. \quad (42)$$

Summing constraints (41) (or (42)) on set \mathcal{K} , and applying constraints (13), we obtain:

$$\sum_{\sigma^1 \in \mathcal{K}} \sum_{\sigma^2 \in \mathcal{K}} \eta_{\sigma^1, \sigma^2} = 1. \quad (43)$$

Multiplying constraints (41) by variable $z_{s,NG}^2$ for each site $s \in \mathcal{S}$, we obtain:

$$\sum_{\sigma^2 \in \mathcal{K}} \zeta_{\sigma^1, \sigma^2, s} = \pi_{\sigma^1, s}^1 \quad \forall s \in \mathcal{S}, \quad \forall \sigma^1 \in \mathcal{K}. \quad (44)$$

Multiplying constraints (42) by variable $z_{s,NG}^2$ for each site $s \in \mathcal{S}$, we obtain:

$$\sum_{\sigma^1 \in \mathcal{K}} \zeta_{\sigma^1, \sigma^2, s} = \pi_{\sigma^2, s}^2 \quad \forall s \in \mathcal{S}, \quad \forall \sigma^2 \in \mathcal{K}. \quad (45)$$

Finally, multiplying constraints (43) by variable $z_{s,NG}^2$ for each site $s \in \mathcal{S}$, we obtain:

$$\sum_{\sigma^1 \in \mathcal{K}} \sum_{\sigma^2 \in \mathcal{K}} \zeta_{\sigma^1, \sigma^2} = z_{s,NG}^2 \quad \forall s \in \mathcal{S}. \quad (46)$$

Summarizing, the formulation presented in section (2.2) can be reinforced by adding constraints (40)–(46). We denote the corresponding linearized and reinforced formulation by $\mathcal{M}^{det, 2period}$.

3. Robust formulation

In this section, we detail the static and adjustable robust counterparts for formulation $\mathcal{M}^{det, 2period}$. The static counterpart is presented in Section 3.1. We show in Section 3.2 that this formulation can be formulated as a MILP that contains at most twice as many constraints as the nominal problem. Section 3.3 presents the adjustable counterpart.

3.1. Static robust counterpart

We now consider that the shifting function is uncertain and belongs to the uncertainty polytope \mathcal{F} . Let us recall that the shifting function is a discrete function associating to each couple $(\sigma, c) \in \mathcal{K} \times \mathcal{C}$ a reaction $f_{\sigma, c} \in [0, 1]$. Therefore, the polytope \mathcal{F} is a subset of the finite dimension space $[0, 1]^{|\mathcal{K}| \cdot |\mathcal{C}|}$. The realized shifting functions may be different in each period. We thus denote by $f^1 \in \mathcal{F}$ the shifting function realization in the first period and by $f^2 \in \mathcal{F}$ the shifting function realization in the second period.

In a static framework, the decisions cannot be adapted to mitigate the effects of the uncertainty. Consequently, the static robust counterpart of formulation $\mathcal{M}^{det, 2period}$ can be formulated as follows.

$$\min \quad subcost + \sum_{s \in \mathcal{S}} \sum_{g \in \mathcal{G}} CM_g(m_{s,g}^2 - M_{s,g}^0) + \sum_{s \in \mathcal{S}} CANG(z_{s,NG}^2 - Z_{s,NG}^0) \quad (47)$$

s.t. (2) – (4), (31), (34), (13) – (26)

$$\sum_{\sigma^1 \in \mathcal{K}} \sigma^1 f_{\sigma^1, c^1}^1 \delta_{\sigma^1}^1 + \sum_{\sigma^2 \in \mathcal{K}} \sum_{c^2 \in \mathcal{C}} \sigma^2 f_{\sigma^2, c^2}^2 \delta_{\sigma^2}^2 \gamma_{c^2}^2 (1 - \sum_{\sigma^1 \in \mathcal{K}} f_{\sigma^1, c^1}^1 \delta_{\sigma^1}^1) \leq \frac{\text{subcost}}{\sum_{s \in \mathcal{S}} U_{s, CG}^0} \quad \forall f^1 \in \mathcal{F}, \forall f^2 \in \mathcal{F}, \quad (48)$$

$$D_{CG}^1 U_{s, CG}^0 (1 - \sum_{\sigma \in \mathcal{K}} f_{\sigma, c^1}^1 \delta_{\sigma}^1) \leq CAP_{CG} m_{s, CG}^1 \quad \forall s \in \mathcal{S}, \forall f^1 \in \mathcal{F}, \quad (49)$$

$$D_{NG}^1 (U_{s, NG}^0 + U_{s, CG}^0 \sum_{\sigma \in \mathcal{K}} f_{\sigma, c^1}^1 \delta_{\sigma}^1) z_{s, NG}^1 \leq CAP_{NG} m_{s, NG}^1 \quad \forall s \in \mathcal{S}, \forall f^1 \in \mathcal{F}, \quad (50)$$

$$\frac{D_{CG}^2 U_{s, CG}^0}{CAP_{CG}} \left[1 - \left(\sum_{\sigma^1 \in \mathcal{K}} f_{\sigma^1, c^1}^1 \delta_{\sigma^1}^1 + \sum_{\sigma^2 \in \mathcal{K}} \sum_{c^2 \in \mathcal{C}} f_{\sigma^2, c^2}^2 \delta_{\sigma^2}^2 \gamma_{c^2}^2 (1 - \sum_{\sigma^1 \in \mathcal{K}} f_{\sigma^1, c^1}^1 \delta_{\sigma^1}^1) \right) \right] \leq m_{s, CG}^2 \quad \forall s \in \mathcal{S}, \forall f^1 \in \mathcal{F}, \forall f^2 \in \mathcal{F}, \quad (51)$$

$$\frac{D_{NG}^2}{CAP_{NG}} \left[U_{s, NG}^0 + U_{s, CG}^0 \left(\sum_{\sigma^1 \in \mathcal{K}} f_{\sigma^1, c^1}^1 \delta_{\sigma^1}^1 + \sum_{\sigma^2 \in \mathcal{K}} \sum_{c^2 \in \mathcal{C}} f_{\sigma^2, c^2}^2 \delta_{\sigma^2}^2 \gamma_{c^2}^2 (1 - \sum_{\sigma^1 \in \mathcal{K}} f_{\sigma^1, c^1}^1 \delta_{\sigma^1}^1) \right) \right] z_{s, NG}^2 \leq m_{s, NG}^2 \quad \forall s \in \mathcal{S}, \forall f^1 \in \mathcal{F}, \forall f^2 \in \mathcal{F}, \quad (52)$$

$$\sum_{s \in \mathcal{S}} \left[U_{s, NG}^0 + U_{s, CG}^0 \left(\sum_{\sigma^1 \in \mathcal{K}} f_{\sigma^1, c^1}^1 \delta_{\sigma^1}^1 + \sum_{\sigma^2 \in \mathcal{K}} \sum_{c^2 \in \mathcal{C}} f_{\sigma^2, c^2}^2 \delta_{\sigma^2}^2 \gamma_{c^2}^2 (1 - \sum_{\sigma^1 \in \mathcal{K}} f_{\sigma^1, c^1}^1 \delta_{\sigma^1}^1) \right) \right] z_{s, NG}^2 \geq \underline{QoE} \text{ UTOT}, \quad \forall f^1 \in \mathcal{F}, \forall f^2 \in \mathcal{F}. \quad (53)$$

Let us denote this formulation by \mathcal{M}^{stat} . Being a robust mixed-integer linear programming with polyhedral uncertainty, \mathcal{M}^{stat} has an infinite number of constraints.

Remark 3.1. As for the deterministic model, formulation \mathcal{M}^{stat} can be linearized and reinforced by constraints (40)–(46).

3.2. Constraints dominance

We observe that the previous formulation contains constraints involving quadratic dependencies on f . This is the case for each constraint that involves the numbers of subscribers to each offer in the second period, i.e. constraints (48), (51), (52) and (53). Handling constraints with non-linear dependencies of the uncertain parameters may not be easy in general. Fortunately, we show in this section that the specific structure of our constraints is simple enough to lead to a direct reformulation based on the dominance of set \mathcal{F} by two vectors denoted by \bar{f} and \underline{f} .

In what follows, let $\tilde{\delta}$ represent the value of δ in a feasible solution of \mathcal{M}^{stat} and let $\tilde{\sigma}$ denote the subsidy offered in that solution. Similarly, let $\tilde{\gamma}^2$ represent the value of γ^2 in a feasible solution of \mathcal{M}^{stat} and let \tilde{c}^2 denote the range of coverage in the second period in that solution. Our

reformulation is based on the following properties satisfied by any feasible solution of formulation \mathcal{M}^{stat} :

Proposition 3.2. *The term $\sum_{\sigma \in \mathcal{K}} f_{\sigma, c^1}^1 \tilde{\delta}_\sigma^1$ is equal to $f_{\tilde{\sigma}^1, c^1}^1$.*

Proof. This comes directly from constraints (13), $\tilde{\sigma}^1$ being the index of the only non-zero component of $\tilde{\delta}^1$. \square

Proposition 3.3. *The term $\sum_{\sigma \in \mathcal{K}} \sum_{c \in \mathcal{C}} f_{\sigma, c}^2 \tilde{\delta}_\sigma^2 \tilde{\gamma}_c^2$ is equal to $f_{\tilde{\sigma}^2, \tilde{c}^2}^2$.*

Proof. This result comes from constraints (13) and (14), $\tilde{\sigma}^2$ being the index of the unique non-zero component of $\tilde{\delta}^2$ and \tilde{c}^2 being the index of the unique non-zero component of $\tilde{\gamma}^2$. \square

These two simple results enable us to replace all robust constraints by equivalent sets of constraints, involving at most the two aforementioned vectors \underline{f} and \bar{f} , which are defined as follows for each subsidy offered $\sigma \in \mathcal{K}$ and each coverage range $c \in \mathcal{C}$:

- the weakest possible reaction is $\underline{f}_{\sigma, c} = \min_{f \in \mathcal{F}} f_{\sigma, c}$,
- the strongest possible reaction is $\bar{f}_{\sigma, c} = \max_{f \in \mathcal{F}} f_{\sigma, c}$.

Next, we present reformulations for the constraints that are linear in the uncertainty f .

Proposition 3.4. *Constraints (49) are satisfied if and only if the following constraints are satisfied:*

$$D_{CG}^1 U_{s, CG}^0 (1 - \sum_{\sigma \in \mathcal{K}} \underline{f}_{\sigma, c^1} \delta_\sigma^1) \leq CAP_{CG} m_{s, CG}^1, \quad \forall s \in \mathcal{S}. \quad (54)$$

Proof. We see that constraints (49) are equivalent to

$$\begin{aligned} \max_{f^1 \in \mathcal{F}} D_{CG}^1 U_{s, CG}^0 (1 - \sum_{\sigma \in \mathcal{K}} f_{\sigma^1, c^1}^1 \delta_\sigma^1) &\leq CAP_{CG} m_{s, CG}^1 && \forall s \in \mathcal{S}, \\ \Leftrightarrow D_{CG}^1 U_{s, CG}^0 (1 - \min_{f^1 \in \mathcal{F}} \sum_{\sigma \in \mathcal{K}} f_{\sigma, c^1}^1 \delta_\sigma^1) &\leq CAP_{CG} m_{s, CG}^1 && \forall s \in \mathcal{S}. \end{aligned}$$

From Proposition 3.2 and the definition of \underline{f} , we have

$$\sum_{\sigma \in \mathcal{K}} \underline{f}_{\sigma, c^1} \tilde{\delta}_\sigma^1 = \underline{f}_{\tilde{\sigma}^1, c^1} = \min_{f^1 \in \mathcal{F}} f_{\tilde{\sigma}^1, c^1}^1 = \min_{f^1 \in \mathcal{F}} \sum_{\sigma \in \mathcal{K}} f_{\sigma, c^1}^1 \tilde{\delta}_\sigma^1.$$

which means constraints (49) are equivalent to constraints (54). \square

Proposition 3.5. *Constraints (50) are satisfied if and only if the following constraints are satisfied:*

$$D_{NG}^1(U_{s,NG}^0 + U_{s,CG}^0 \sum_{\sigma \in \mathcal{K}} \bar{f}_{\sigma,c^1} \delta_{\sigma}^1) \leq CAP_{CG} m_{s,CG}^1 \quad \forall s \in \mathcal{S}. \quad (55)$$

Proof. The proof relies on the definition of \bar{f} and on arguments similar to those used in the proof of Proposition 3.4. \square

Next, we focus on the constraints of \mathcal{M}^{stat} that involve quadratic dependencies on f .

Proposition 3.6. *Constraints (51) are satisfied if and only if the following constraints are satisfied:*

$$\frac{D_{CG}^2 U_{s,CG}^0}{CAP_{CG}} \left[1 - \left(\sum_{\sigma^1 \in \mathcal{K}} \underline{f}_{\sigma^1, c^1} \delta_{\sigma^1}^1 + \sum_{\sigma^2 \in \mathcal{K}} \sum_{c^2 \in \mathcal{C}} \underline{f}_{\sigma^2, c^2} \tilde{\delta}_{\sigma^2}^2 \tilde{\gamma}_{c^2}^2 (1 - \sum_{\sigma^1 \in \mathcal{K}} \underline{f}_{\sigma^1, c^1} \delta_{\sigma^1}^1) \right) \right] \leq m_{s,CG}^2 \quad \forall s \in \mathcal{S}. \quad (56)$$

Proof. First, we see that if constraints (51) are satisfied, then constraints (56) are satisfied since $\underline{f} \in \mathcal{F}$.

Reciprocally, let us assume that constraints (56) are satisfied. Due to Propositions 3.2 and 3.3, we know that:

$$\sum_{\sigma^1 \in \mathcal{K}} f_{\sigma^1, c^1}^1 \tilde{\delta}_{\sigma^1}^1 + \sum_{\sigma^2 \in \mathcal{K}} \sum_{c^2 \in \mathcal{C}} f_{\sigma^2, c^2}^2 \tilde{\delta}_{\sigma^2}^2 \tilde{\gamma}_{c^2}^2 (1 - \sum_{\sigma^1 \in \mathcal{K}} f_{\sigma^1, c^1}^1 \tilde{\delta}_{\sigma^1}^1) = f_{\bar{\sigma}^1, c^1}^1 + f_{\bar{\sigma}^2, \bar{c}^2}^2 (1 - f_{\bar{\sigma}^1, c^1}^1).$$

To simplify notations, let us denote $f_{\bar{\sigma}^1, c^1}^1$ and $f_{\bar{\sigma}^2, \bar{c}^2}^2$ by x and y respectively. We also denote $\underline{f}_{\bar{\sigma}^1, c^1}$ by \underline{x} , $\bar{f}_{\bar{\sigma}^1, c^1}$ by \bar{x} , $\underline{f}_{\bar{\sigma}^2, \bar{c}^2}$ by \underline{y} and $\bar{f}_{\bar{\sigma}^2, \bar{c}^2}$ by \bar{y} .

Recalling that $(x, y) \in [0, 1]^2$, we wish to find out where function $g(x, y) = x + y(1 - x) = x + y - xy$ defined on $(x, y) \in [\underline{x}, \bar{x}] \times [\underline{y}, \bar{y}] \subseteq [0, 1]^2$ reaches its minimal value. First notice that

$$g(x, y) = x + (1 - x)y \geq x + (1 - x)\underline{y} = g(x, \underline{y}) \quad \forall x \in [\underline{x}, \bar{x}], \forall y \in [\underline{y}, \bar{y}],$$

and symmetrically $g(x, y) \geq g(\underline{x}, y), \forall x \in [\underline{x}, \bar{x}], \forall y \in [\underline{y}, \bar{y}]$. Consequently, $g(x, y) \geq g(\underline{x}, \underline{y}), \forall x \in [\underline{x}, \bar{x}], \forall y \in [\underline{y}, \bar{y}]$, and we thus have:

$$\begin{aligned} & \max_{f^1 \in \mathcal{F}, f^2 \in \mathcal{F}} \frac{D_{CG}^2 U_{s,CG}^0}{CAP_{CG}} \left[1 - \left(\sum_{\sigma^1 \in \mathcal{K}} f_{\sigma^1, c^1}^1 \delta_{\sigma^1}^1 + \sum_{\sigma^2 \in \mathcal{K}} \sum_{c^2 \in \mathcal{C}} f_{\sigma^2, c^2}^2 \delta_{\sigma^2}^2 \gamma_{c^2}^2 (1 - \sum_{\sigma^1 \in \mathcal{K}} f_{\sigma^1, c^1}^1 \delta_{\sigma^1}^1) \right) \right] \\ &= \frac{D_{CG}^2 U_{s,CG}^0}{CAP_{CG}} \left[1 - \min_{f^1 \in \mathcal{F}, f^2 \in \mathcal{F}} \left(\sum_{\sigma^1 \in \mathcal{K}} f_{\sigma^1, c^1}^1 \delta_{\sigma^1}^1 + \sum_{\sigma^2 \in \mathcal{K}} \sum_{c^2 \in \mathcal{C}} f_{\sigma^2, c^2}^2 \delta_{\sigma^2}^2 \gamma_{c^2}^2 (1 - \sum_{\sigma^1 \in \mathcal{K}} f_{\sigma^1, c^1}^1 \delta_{\sigma^1}^1) \right) \right] \\ &\leq \frac{D_{CG}^2 U_{s,CG}^0}{CAP_{CG}} \left[1 - \left(\sum_{\sigma^1 \in \mathcal{K}} \underline{f}_{\sigma^1, c^1} \delta_{\sigma^1}^1 + \sum_{\sigma^2 \in \mathcal{K}} \sum_{c^2 \in \mathcal{C}} \underline{f}_{\sigma^2, c^2} \delta_{\sigma^2}^2 \gamma_{c^2}^2 (1 - \sum_{\sigma^1 \in \mathcal{K}} \underline{f}_{\sigma^1, c^1} \delta_{\sigma^1}^1) \right) \right] \leq m_{s,CG}^2, \end{aligned}$$

which means that constraints (51) are satisfied. □

Proposition 3.7. *Constraints (53) are satisfied if and only if the following constraints are satisfied:*

$$\sum_{s \in \mathcal{S}} \left[U_{s,NG}^0 + U_{s,CG}^0 \left(\sum_{\sigma^1 \in \mathcal{K}} \underline{f}_{\sigma^1,c^1} \delta_{\sigma^1}^1 + \sum_{\sigma^2 \in \mathcal{K}} \sum_{c^2 \in \mathcal{C}} \underline{f}_{\sigma^2,c^2} \delta_{\sigma^2}^2 \gamma_{c^2}^2 (1 - \sum_{\sigma^1 \in \mathcal{K}} \underline{f}_{\sigma^1,c^1} \delta_{\sigma^1}^1) \right) \right] \geq \underline{QoE} \text{ UTOT}. \quad (57)$$

Proof. The proof relies on arguments similar to those used in the proof of Proposition 3.6. □

Proposition 3.8. *Constraints (52) are satisfied if and only if the following constraints are satisfied:*

$$\frac{D_{NG}^2}{CAP_{NG}} \left[U_{s,NG}^0 + U_{s,CG}^0 \left(\sum_{\sigma^1 \in \mathcal{K}} \bar{f}_{\sigma^1,c^1} \delta_{\sigma^1}^1 + \sum_{\sigma^2 \in \mathcal{K}} \sum_{c^2 \in \mathcal{C}} \bar{f}_{\sigma^2,c^2} \delta_{\sigma^2}^2 \gamma_{c^2}^2 (1 - \sum_{\sigma^1 \in \mathcal{K}} \bar{f}_{\sigma^1,c^1} \delta_{\sigma^1}^1) \right) \right] \leq m_{s,NG}^2 \quad \forall s \in \mathcal{S} \quad (58)$$

Proof. The proof relies on studying the maximum of function g and on arguments similar to those used in the proof of Proposition 3.6. □

We consider next constraints (48) which require an argument slightly more involved.

Proposition 3.9. *Constraints (48) are satisfied if and only if the following constraints are satisfied:*

$$\sum_{\sigma^1 \in \mathcal{K}} \sigma^1 \bar{f}_{\sigma^1,c^1} \delta_{\sigma^1}^1 + \sum_{\sigma^2 \in \mathcal{K}} \sum_{c^2 \in \mathcal{C}} \sigma^2 \bar{f}_{\sigma^2,c^2} \delta_{\sigma^2}^2 \gamma_{c^2}^2 (1 - \sum_{\sigma^1 \in \mathcal{K}} \bar{f}_{\sigma^1,c^1} \delta_{\sigma^1}^1) \leq \frac{\text{subcost}}{\sum_{s \in \mathcal{S}} U_{s,CG}^0} \quad (59)$$

$$\sum_{\sigma^1 \in \mathcal{K}} \sigma^1 \underline{f}_{\sigma^1,c^1} \delta_{\sigma^1}^1 + \sum_{\sigma^2 \in \mathcal{K}} \sum_{c^2 \in \mathcal{C}} \sigma^2 \underline{f}_{\sigma^2,c^2} \delta_{\sigma^2}^2 \gamma_{c^2}^2 (1 - \sum_{\sigma^1 \in \mathcal{K}} \underline{f}_{\sigma^1,c^1} \delta_{\sigma^1}^1) \leq \frac{\text{subcost}}{\sum_{s \in \mathcal{S}} U_{s,CG}^0} \quad (60)$$

Proof. First, we see that if constraints (48) are satisfied, constraints (59) and (60) are satisfied since $\underline{f} \in \mathcal{F}$ and $\bar{f} \in \mathcal{F}$.

Reciprocally, let us assume that constraints (59) and (60) are both satisfied: due to Propositions 3.2 and 3.3, we know that

$$\sum_{\sigma^1 \in \mathcal{K}} \sigma^1 f_{\sigma^1,c^1}^1 \tilde{\delta}_{\sigma^1}^1 + \sum_{\sigma^2 \in \mathcal{K}} \sum_{c^2 \in \mathcal{C}} \sigma^2 f_{\sigma^2,c^2}^2 \tilde{\delta}_{\sigma^2}^2 \tilde{\gamma}_{c^2}^2 (1 - \sum_{\sigma^1 \in \mathcal{K}} f_{\sigma^1,c^1}^1 \tilde{\delta}_{\sigma^1}^1) = \tilde{\sigma}^1 f_{\tilde{\sigma}^1,c^1}^1 + \tilde{\sigma}^2 f_{\tilde{\sigma}^2,c^2}^2 (1 - f_{\tilde{\sigma}^1,c^1}^1).$$

We use the same notations as those introduced in the proof of Proposition 3.6. In addition, we denote $\tilde{\sigma}^1$ by a and $\tilde{\sigma}^2$ by b . We wish to find out where function $h(x, y) = ax + by(1 - x)$ defined on $x \in [\underline{x}, \bar{x}] \times [y, \bar{y}] \subseteq [0, 1]^2$ reaches its maximal value with a and b positive reals. First notice that we have:

$$h(x, y) = ax + by(1 - x) \leq ax + b\bar{y}(1 - x) = h(x, \bar{y}) \quad \forall x \in [\underline{x}, \bar{x}], \forall y \in [y, \bar{y}].$$

Hence, our function is maximal for $y = \bar{y}$. With y fixed to \bar{y} , $h(x, \bar{y})$ becomes a linear function of x defined on $[\underline{x}, \bar{x}]$. Therefore,

$$h(x, \bar{y}) \leq \max \{h(\bar{x}, \bar{y}), h(\underline{x}, \bar{y})\} \quad \forall x \in [\underline{x}, \bar{x}]$$

and

$$h(x, y) \leq \max \{h(\bar{x}, \bar{y}), h(\underline{x}, \bar{y})\} \quad \forall x \in [\underline{x}, \bar{x}], \forall y \in [\underline{y}, \bar{y}].$$

We thus have for each $f^1 \in \mathcal{F}$, $f^2 \in \mathcal{F}$:

$$\begin{aligned} & \sum_{\sigma^1 \in \mathcal{K}} \sigma^1 f_{\sigma^1, c^1}^1 \delta_{\sigma^1}^1 + \sum_{\sigma^2 \in \mathcal{K}} \sum_{c^2 \in \mathcal{C}} \sigma^2 f_{\sigma^2, c^2}^2 \delta_{\sigma^2}^2 \gamma_{c^2}^2 (1 - \sum_{\sigma^1 \in \mathcal{K}} f_{\sigma^1, c^1}^1 \delta_{\sigma^1}^1) \\ & \leq \max \left\{ \sum_{\sigma^1 \in \mathcal{K}} \sum_{c^1 \in \mathcal{C}} \sigma^1 \bar{f}_{\sigma^1, c^1} \delta_{\sigma^1}^1 + \sum_{\sigma^2 \in \mathcal{K}} \sum_{c^2 \in \mathcal{C}} \sigma^2 \bar{f}_{\sigma^2, c^2} \delta_{\sigma^2}^2 \gamma_{c^2}^2 (1 - \sum_{\sigma^1 \in \mathcal{K}} \sum_{c^1 \in \mathcal{C}} \bar{f}_{\sigma^1, c^1} \delta_{\sigma^1}^1), \right. \\ & \quad \left. \sum_{\sigma^1 \in \mathcal{K}} \sum_{c^1 \in \mathcal{C}} \sigma^1 \underline{f}_{\sigma^1, c^1} \delta_{\sigma^1}^1 + \sum_{\sigma^2 \in \mathcal{K}} \sum_{c^2 \in \mathcal{C}} \sigma^2 \bar{f}_{\sigma^2, c^2} \delta_{\sigma^2}^2 \gamma_{c^2}^2 (1 - \sum_{\sigma^1 \in \mathcal{K}} \sum_{c^1 \in \mathcal{C}} \underline{f}_{\sigma^1, c^1} \delta_{\sigma^1}^1) \right\} \leq \frac{\text{subcost}}{\sum_{s \in \mathcal{S}} U_{s, CG}^0}, \end{aligned}$$

which means that constraints (48) are satisfied. \square

Summarizing the above results, we have shown that formulation \mathcal{M}^{stat} is equivalent to the following mixed-integer linear programming, with a finite number of constraints:

$$\min \quad \{(47) \text{ s.t. } (2) - (4), (31), (34), (54) - (60), (13) - (26)\}$$

3.3. Adjustable robust counterpart

In this section, we consider the framework where the operator can take benefit from the knowledge of the uncertainty realization in the first period when deciding the subsidy offered in the second period. We obtain a two-stage model where the operator can adapt to the uncertainty through a second-stage decision: the amount of subsidy offered in the second period. We model this decision by variables $\delta_{\sigma}^2(f^1)$ defined for each $f^1 \in \mathcal{F}$. Let us recall that all decisions concerning network investments are taken as planning decisions and considered to be taken in the first-stage, which means that $\delta_{\sigma}^2(f^1)$ is the only second-stage decision. This decision is often called the recourse in the robust optimization literature.

Plugging the recourse variables into the robust model leads to a mixed-integer formulation with an infinite number of variables since one variable $\delta(f')$ arises for each $f' \in \mathcal{F}$. Therefore, we propose

an adjustable robust partition method in the line of Bertsimas et al. (2015), splitting our uncertainty set \mathcal{F} into a finite number L of parts. Notice that, unlike Bertsimas et al. (2015), we partition the uncertainty set from the start. We denote the index set of the partition by $\mathcal{L} = \{1, \dots, L\}$. We hence have $\mathcal{F} = \cup_{l=0}^L \mathcal{F}(l)$ with $\forall l \in \mathcal{L}, l' \in \mathcal{L} \setminus \{l\}, \mathcal{F}(l) \cap \mathcal{F}(l') = \emptyset$ and consider the piecewise constant recourse defined by:

$$\delta^2(f) = \delta^2(f') \quad \forall f, f' \in \mathcal{F}(l), \forall l \in \mathcal{L}$$

Thus, we replace $\delta_\sigma^2(f)$ by $\delta_{\sigma,l}^2$ which indicates the recourse decision taken when $f \in \mathcal{F}(l)$ for each $l \in \mathcal{L}$. Consequently, the robust adjustable counterpart of formulation $\mathcal{M}^{det,2period}$ can be written as follows:

$$\min \quad subcost + \sum_{s \in \mathcal{S}} \sum_{g \in \mathcal{G}} CM_g(m_{s,g}^2 - M_{s,g}^0) + \sum_{s \in \mathcal{S}} CANG(z_{s,NG}^2 - Z_{s,NG}^0) \quad (61)$$

$$s.t. \quad (2) - (4), (31), (34), (49) - (50), (14) - (25)$$

$$\sum_{\sigma^1 \in \mathcal{K}} \sigma^1 f_{\sigma^1, c^1}^1 \delta_\sigma^1 + \sum_{\sigma^2 \in \mathcal{K}} \sum_{c^2 \in \mathcal{C}} \sigma^2 f_{\sigma^2, c^2}^2 \delta_{\sigma^2, l}^2 \gamma_{c^2}^2 (1 - \sum_{\sigma^1 \in \mathcal{K}} f_{\sigma^1, c^1}^1 \delta_\sigma^1) \leq \frac{subcost}{\sum_{s \in \mathcal{S}} U_{s,CG}^0} \quad \forall l \in \mathcal{L}, \forall f^1 \in \mathcal{F}(l), \forall f^2 \in \mathcal{F}, \quad (62)$$

$$\frac{D_{CG}^2 U_{s,CG}^0}{CAP_{CG}} \left[1 - \left(\sum_{\sigma^1 \in \mathcal{K}} f_{\sigma^1, c^1}^1 \delta_\sigma^1 + \sum_{\sigma^2 \in \mathcal{K}} \sum_{c^2 \in \mathcal{C}} f_{\sigma^2, c^2}^2 \delta_{\sigma^2, l}^2 \gamma_{c^2}^2 (1 - \sum_{\sigma^1 \in \mathcal{K}} f_{\sigma^1, c^1}^1 \delta_\sigma^1) \right) \right] \leq m_{s,CG}^2 \quad \forall s \in \mathcal{S}, \forall l \in \mathcal{L}, \forall f^1 \in \mathcal{F}(l), \forall f^2 \in \mathcal{F}, \quad (63)$$

$$\frac{D_{NG}^2}{CAP_{NG}} \left[U_{s,NG}^0 + U_{s,CG}^0 \left(\sum_{\sigma^1 \in \mathcal{K}} f_{\sigma^1, c^1}^1 \delta_\sigma^1 + \sum_{\sigma^2 \in \mathcal{K}} \sum_{c^2 \in \mathcal{C}} f_{\sigma^2, c^2}^2 \delta_{\sigma^2, l}^2 \gamma_{c^2}^2 (1 - \sum_{\sigma^1 \in \mathcal{K}} f_{\sigma^1, c^1}^1 \delta_\sigma^1) \right) \right] z_{s,NG}^2 \leq m_{s,NG}^2 \quad \forall s \in \mathcal{S}, \forall l \in \mathcal{L}, \forall f^1 \in \mathcal{F}(l), \forall f^2 \in \mathcal{F}, \quad (64)$$

$$\sum_{s \in \mathcal{S}} \left[U_{s,NG}^0 + U_{s,CG}^0 \left(\sum_{\sigma^1 \in \mathcal{K}} f_{\sigma^1, c^1}^1 \delta_\sigma^1 + \sum_{\sigma^2 \in \mathcal{K}} \sum_{c^2 \in \mathcal{C}} f_{\sigma^2, c^2}^2 \delta_{\sigma^2, l}^2 \gamma_{c^2}^2 (1 - \sum_{\sigma^1 \in \mathcal{K}} f_{\sigma^1, c^1}^1 \delta_\sigma^1) \right) \right] z_{s,NG}^2 \geq \underline{QoE} \quad UTOT, \quad \forall l \in \mathcal{L}, \forall f^1 \in \mathcal{F}(l), \forall f^2 \in \mathcal{F}, \quad (65)$$

$$\sum_{\sigma \in \mathcal{K}} \delta_\sigma^1 = 1, \quad (66)$$

$$\sum_{\sigma \in \mathcal{K}} \delta_{\sigma, l}^2 = 1 \quad \forall l \in \mathcal{L}, \quad (67)$$

$$\delta_\sigma^1 \in \{0, 1\} \quad \forall \sigma \in \mathcal{K}, \quad (68)$$

$$\delta_{\sigma, l}^2 \in \{0, 1\} \quad \forall \sigma \in \mathcal{K}, \forall l \in \mathcal{L}. \quad (69)$$

Let us denote this formulation by $\mathcal{M}^{adj,L}$ where L is the number of parts. Formulation $\mathcal{M}^{adj,L}$ is obtained by replacing each set of constraints involving the second period in formulation \mathcal{M}^{stat} (constraints (48),(51)–(53),(13)) with L sets of constraints - one for each part. We observe that in each part $\mathcal{F}(l), l \in \mathcal{L}$, constraints (62)–(65) can be reformulated in the same way as for the static model, with in the first period \underline{f} and \bar{f} replaced by $\underline{f}(l)$ and $\bar{f}(l)$ defined by $\underline{f}_{\sigma,c}(l) = \min_{f \in \mathcal{F}(l)} f_{\sigma,c}$ and $\bar{f}_{\sigma,c}(l) = \max_{f \in \mathcal{F}(l)} f_{\sigma,c}$, for each $l \in \mathcal{L}$. We notice that formulation $\mathcal{M}^{adj,1}$ is equivalent to formulation \mathcal{M}^{stat} .

Remark 3.10. As for the deterministic and static models, formulations $\mathcal{M}^{adj,L}$ can be linearized and reinforced by RLT equalities. The only slight difference is that for each linearization or equality implying variables δ_σ^2 in the deterministic model, we have now L constraints corresponding to each part.

4. Numerical experiments

In this section, we assess numerically our static and adjustable models. We first describe the uncertainty set used in the experiments in Section 4.1. The instances used and computational settings are described in Section 4.2. Then, scalability and economical results are presented in Section 4.3.

In particular, we will assess the relevance of the adjustable models compared to the deterministic and static ones. For the adjustable models, we consider three different values for L , $\{2, 4, 8\}$, to assess the scalability and economic relevance of refining the number of parts in the partition. For each model, we assess both model solution cost and simulated cost of the first-stage. The simulated cost of the first-stage is obtained by discretizing the uncertainty set in 200 values and solving 200 optimization problem, each of which considers the first-stage decisions fixed, the shifting function realization of the first period equal to the corresponding value and using the static model for finding the best second-stage decision.

4.1. Design of the uncertainty set

Let $\tilde{f}_{\sigma,c}$ denote the nominal value of $f_{\sigma,c}$. We define an uncertainty set that allows for a controlled variation around this value. Letting Γ denote the amplitude of the variation, we define

$$\mathcal{F} = \left\{ f \in [0, 1]^{|C| \cdot |K|} \mid (1 - \Gamma)\tilde{f}_{\sigma,c} \leq f_{\sigma,c} \leq (1 - \Gamma)\tilde{f}_{\sigma,c} + \Gamma, c \in \mathcal{C}, \sigma \in \mathcal{K} \right\}.$$

An illustration of \mathcal{F} for a nominal function corresponding to the reference example from Cambier et al. (2019) and $\Gamma = 0.25$ is provided in Figure 1. We see that the uncertainty set is not symmetric around the nominal value $\tilde{f}_{\sigma,c}$ as the downward deviation from $\tilde{f}_{\sigma,c}$ can be as large as $\Gamma\tilde{f}_{\sigma,c}$ while the upward deviation is bounded by $\Gamma(1 - \tilde{f}_{\sigma,c})$ for $\sigma \in \mathcal{K}$ and $c \in \mathcal{C}$, with these deviations depending on the nominal value. This definition of the uncertainty set enables us to control that each possible reaction belongs to $[0, 1]$.

For the adjustable model, we need to split \mathcal{F} into L parts. We choose to define for each l in $\{1, \dots, L\}$

$$\mathcal{F}(l) = \left\{ f \in [0, 1]^{|C| \cdot |K|} \mid (1 - \Gamma)\tilde{f}_{\sigma,c} + \frac{l-1}{L}\Gamma \leq f_{\sigma,c} \leq (1 - \Gamma)\tilde{f}_{\sigma,c} + \frac{l}{L}\Gamma, c \in \mathcal{C}, \sigma \in \mathcal{K} \right\} \subseteq \mathcal{F}.$$

An illustration of the partition for $L = 3$ and the example from Figure 1 is provided in Figure 2.

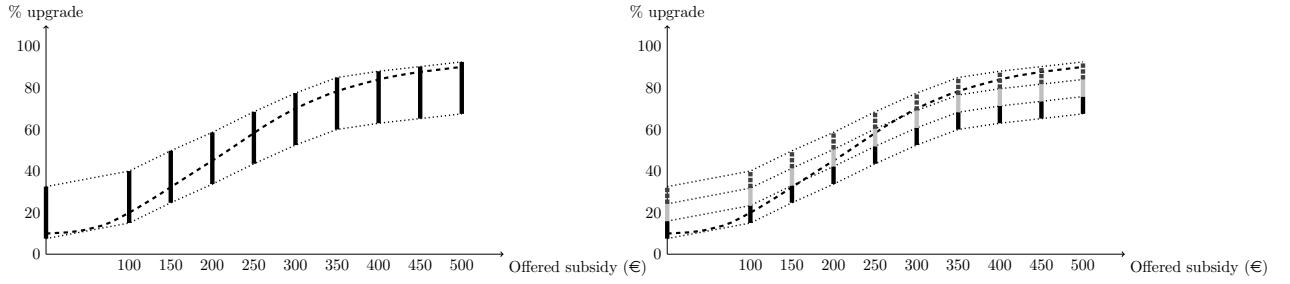


Figure 1: Example of uncertainty set with reference function from Cambier et al. (2019) and $\Gamma = 0.25$ (projected on a given range $c = [0.75, 1]$).

Figure 2: Example of partitioned ($L = 3$) uncertainty set with reference function from Cambier et al. (2019) and $\Gamma = 0.25$ (projected on a given range $c = [0.75, 1]$).

4.2. Instances parameters

We aim to assess the different models presented in Section 2 on a realistic instance for three different amplitude values Γ . We use the 3G/4G suburban instance of 100 sites from Cambier et al. (2019), adapted for a two period framework. At the beginning of the time-horizon, 34 sites are equipped with both 3G and 4G technologies while 66 are 3G-only sites. The network parameter values are realistic values taken from telecommunication equipment sellers. The strategic guideline for quality of experience (QoE) is fixed to 0.8. Regarding the subscriber dynamic, values for $\tilde{f}_{\sigma,c}$ are given in Table 3. We use the uncertainty set for f defined by these nominal values and an amplitude $\Gamma \in \{0.25, 0.30, 0.35\}$.

The computations have been made on a server of 32 processors Intel Xeon of CPU 5110 clocked at 1.6 GHz each. The code is written in Julia 1.1.0, with the package JuMP developed for discrete

Coverage level \ Subsidies (in €)	0	100	150	200	250	300	350	400	450	500
<i>low</i>	0	0	0	0	0	0	0	0	0	0
<i>medium low</i>	0.5	5	12	21	30	40	42	43	44	45
<i>medium high</i>	5	10	20	30	40	50	60	62	64	65
<i>high</i>	10	20	33	45	58	70	80	83	87	90

Table 3: Reaction of the subscribers (in %) on reference markets for given subsidies and coverage levels.

optimization (see Dunning et al. (2017) for details) and the solver used is CPLEX 12.8 (default branch-and-bound algorithm). The time limit for MILP solving is set to 7200 seconds.

4.3. Results

Table 4: Solutions for 0.25, 0.3 and 0.35 deviations

Γ	model	sol	time	first-stage (σ^1)	second-stage (σ^2)
	deterministic	6999	19	200	250
0.25	static	9691	18	250	350
	adj (2 parts)	9643	78	350	300 250
	adj(4 parts)	9643	347	350	300 250 250 200
	adj(8 parts)	9580	7200	350	300 300 300 250 250 250 200 200
0.30	static	10097	10	300	350
	adj (2 parts)	10066	36	350	300 250
	adj(4 parts)	10066	274	350	300 300 250 150
	adj(8 parts)	9965	3095	350	300 300 300 250 250 250 150 250
0.35	static	10957	8	350	350
	adj (2 parts)	10767	28	350	350 250
	adj(4 parts)	10671	130	350	350 300 250 200
	adj(8 parts)	10671	1105	350	350 350 300 300 300 200 150 100

Results obtained with each formulation are presented in Table 4. The value of the best solution found and the solving time in seconds are provided respectively in columns “sol” and “time” of Table 4. Column “first-stage” stands for the value of the subsidy offered in the first period and multi-column “second-stage” for the value of the subsidy offered in the second period. For the adjustable models, each line presents second-stage values ordered from lowest to highest reaction case.

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4 On a scalability viewpoint, we observe that the convergence of the branch-and-bound procedure
5 is obtained in less than 2 minutes for the deterministic, static and adjustable (2 parts) models for
6 each amplitude value. For each model, the larger the amplitude is, the faster the convergence is.
7
8 We also observe that whatever the amplitude is, the static model is not harder to solve than the
9 deterministic model. Refining the number of parts considerably increases the model size (in terms
10 of constraints and variables). Our results enlighten the lack of scalability of such refinements. For
11 $\Gamma = 0.25$ and 8 parts, the solver does not converge within the two-hours time limit.
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17 Next, we observe that the deterministic decision for the first-level (200 €) is never taken by any
18 of the robust frameworks. We see that for an amplitude $\Gamma = 0.25$, the first-stage decision for the
19 subsidy proposal is different between the static and adjustable models: static (250 €) and adjustable
20 (350 € for all values of L). For $\Gamma = 0.30$ deviation, the first-stage decisions for the subsidy proposal
21 are 300 € for the static model and 350 € for all adjustable models. These differences are explained
22 by an impact on network installations (fewer installations) in the adjustable cases. For $\Gamma = 0.35$
23 deviation, the decision for the static and all adjustable models is the same: 350 €. From our results,
24 we conclude that using the 2 parts model is sufficient since it converges a lot faster than using 4 or
25 8 parts and provides the same first-stage solution in each case.
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33 To estimate the costs, we use the first-stage decisions given by the three different models,
34 *deterministic*, *static*, and *adjustable* (using a 2-partition), and generate 200 scenarios for the cor-
35 responding uncertainty realization by dividing the uncertainty set into intervals of equal length.
36 We then solve one optimization problem for each of the 200 scenarios to compute the best second-
37 period decision. Each of these problems thus amounts to solve a nominal variant of the problem
38 where the first-stage decisions are fixed to the values provided by the model and f^1 depends on the
39 considered scenario.
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45 The simulations costs (curve formed by all generated scenarios) are plotted for each value of Γ
46 on Figures 3, 4 and 5. The worst-cases cost are given under column “worst-case cost” in Table 5.
47 The curves show that these worst-case costs are obtained for the scenarios where the uncertainty
48 is the lower, as these functions are mostly decreasing. We see that for each amplitude value, in the
49 worst-case, the first-stage decision of the adjustable model is the best one.
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54 On an economic viewpoint, we first notice from Table 4 that the optimal cost from the deter-
55 ministic framework (6999 k€) is around 30% lower than the best solution found (obtained with
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the adjustable model) in the worst-case (for $\Gamma = 0.25$). This can be explained by computing the worst-case for the QoE threshold when $\Gamma = 0.25$ and the decisions taken by the deterministic model are imposed. This value is equal to 0.71 while the demanded threshold is 0.8. Deterministic decisions lead hence to infeasibility in the robust context, which requires higher subsidies for robust decisions. The impact on the cost in a robust context is three-fold. It increases,

- since the subsidy proposal is higher,
- since these subsidies have a higher nominal effect (more reactions due to decisions),
- if the subscribers react better than expected (more reactions due to uncertainty).

Furthermore, we observe that for $\Gamma = 0.35$, the deterministic first-stage decision is infeasible (see Table 5 and Figure 5) when the uncertainty realization is lower than 20% (first 40 scenarios). Indeed, the maximal nominal reaction for the second period is 90% which means the worst-case is 58.5%. With an initial 4G subscribers percentage of 40%, this is not enough for reaching 80% of subscribers being 4G and hence the QoE threshold of 80% of subscribers being 4G served by 4G is infeasible (even when all sites are equipped).

Table 5: Simulated cost for 0.25, 0.3 and 0.35 deviations

Γ	model for first-stage	first-stage (σ^1)	worst-case cost
0.25	deterministic	200	10671
0.25	static	250	9659
0.25	adj	350	9270
0.30	deterministic	200	13074
0.30	static	300	10081
0.30	adj	350	9636
0.35	deterministic	200	infeasible
0.35	static	350	10565
0.35	adj	350	10565

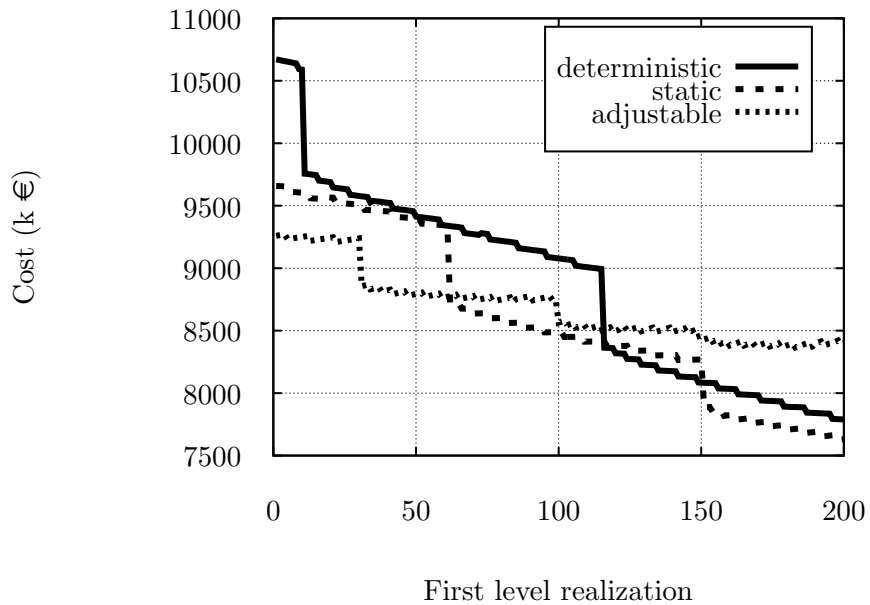


Figure 3: Simulations with first-stages imposed for $\Gamma = 0.25$ deviation

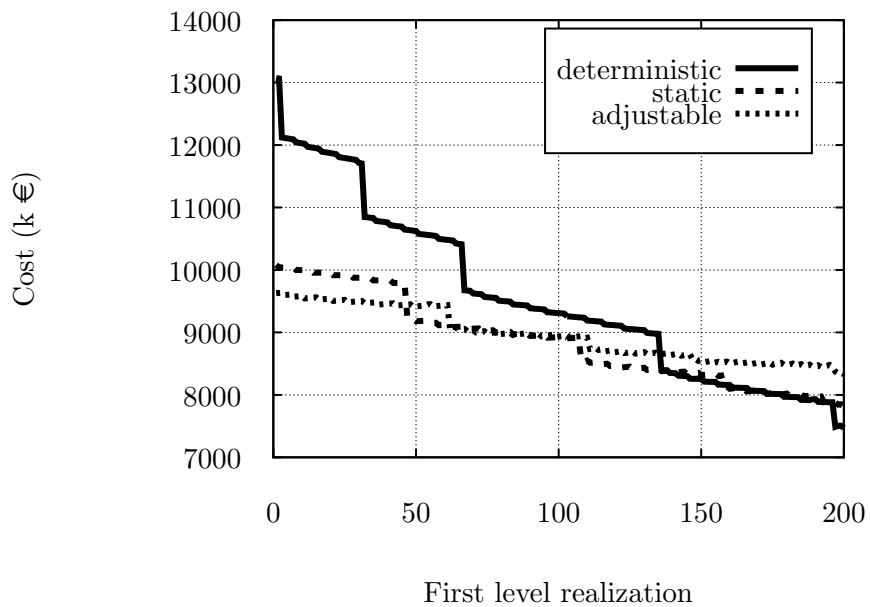


Figure 4: Simulations with first-stages imposed for $\Gamma = 0.30$

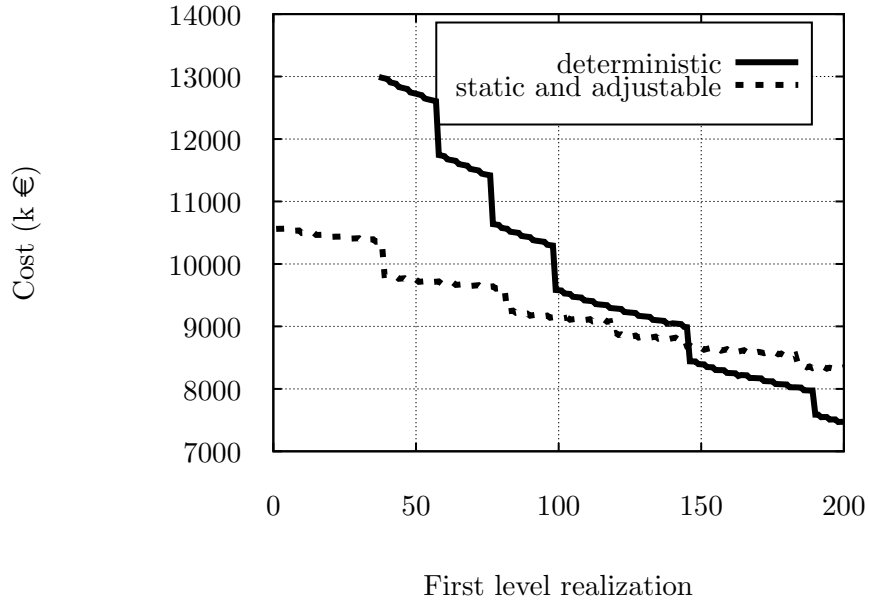


Figure 5: Simulations with first-stages imposed for $\Gamma = 0.35$

5. Conclusion

In this paper, we have introduced and strengthened a new formulation for the two periods and two generations mobile master plan problem using only integer variables. Assuming that the discrete shifting function can take any value in a given polytope, and optimizing against the worst-case outcome, we have proposed a robust counterpart for the problem. We have handled integer recourse variables by partitioning the uncertainty set while the non-linear dependencies on the uncertain parameters have been tackled by a careful analysis of dominating scenarios. Numerical experiments have been performed for static and adjustable robust frameworks on a 100 sites instance, with an uncertainty set controlling variation around the nominal value and parametrized by an amplitude Γ . Our results have illustrated the scalability of the different robust models for 100 sites as well as the economical relevance of the static and adjustable first-stage decisions over the deterministic one. These decisions can lead to saving costs as high as 30% of the total costs in the case $\Gamma = 0.3$, while the deterministic solution can become infeasible for $\Gamma = 0.35$. Our results have also underlined the impact of the quality of experience threshold, which is responsible for the higher subsidies decisions and the higher costs it involves.

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