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# Interval Hypergraphs and Catalan Numbers\*

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## Abstract

An *interval hypergraph* is a hypergraph  $H = (V, E)$  such that there exists a linear order  $\sigma$  (which is said to be *compatible*) on  $V$  such that, when  $V$  is sorted along  $\sigma$ , every hyperedge is an interval of  $V$ . A *interval cover* of a set  $V$  is an interval hypergraph  $H = (V, E)$  such that  $\bigcup_{e \in E} e = V$  and  $\forall e, e' \in E, e \not\subseteq e'$ . In this note, we show that:

- Nearly all interval hypergraphs have only two compatible orders.
- The number of interval covers of  $\{1, \dots, n\}$  is the Catalan number  $C_n = \frac{1}{n+1} \binom{2n}{n}$ .

**keywords** : Interval hypergraphs, Catalan numbers, PQ-trees, Robinson dissimilarities.

## 1 Definitions and notations

A *hypergraph* is a couple  $H = (V, E)$  where  $V$  is a finite set whose elements are *vertices* and  $E \subseteq 2^V$  is the *hyperedge* set. A hypergraph  $H = (V, E)$  is an *interval hypergraph* if there exists a *compatible (linear) order*  $\sigma$  of  $V$  such that, when  $V$  is sorted along  $\sigma$ , all hyperedges are intervals. A interval hypergraph  $H = (V, E)$  is a *interval cover* of  $V$  if  $\bigcup_{e \in E} e = V$  and  $\forall e, e' \in E, e \not\subseteq e'$ . For  $V' \subset V$ , the *restriction* of  $H = (V, E)$  to  $V'$  is the hypergraph  $H|_{V'} = (V', E')$  where  $E' = \{e \cap V' : e \in E\}$ .

A *PQ-tree*  $T$  on a set  $V$  is a tree that represents a set of permutations on  $V$ , and if  $H = (V, E)$  is an interval hypergraph, then the set of all compatible orders of  $H$  can be represented by a PQ-tree [1, 7]. The leaves of  $T$  are the elements of  $V$ , and the nodes of  $T$  are of two types: the *P-nodes* and the *Q-nodes*. We use the general convention of writing that is to represent P-nodes by circles, and Q-nodes by rectangles. On a P-node, one can apply any permutation of its children (equivalently, its children are not ordered). The children of a Q-node are ordered, and the only permutation we can apply on them is to reverse the order. For instance, the PQ-tree of figure 1 represents the set of permutations  $\{(1,2,3,4,5), (1,3,2,4,5), (2,1,3,4,5), (2,3,1,4,5), (3,1,2,4,5), (3,2,1,4,5), (5,4,1,2,3), (5,4,1,3,2), (5,4,2,1,3), (5,4,2,3,1), (5,4,3,1,2), (5,4,3,2,1)\}$ .

We say that a PQ-tree is *flat* if it has only one internal node, which is a Q-node. A flat PQ-tree represents only two permutations. Similarly, an interval hypergraph is *flat* if it has only two compatible orders, *i.e.* if its PQ-tree is flat. Conversely, a PQ-tree with one internal node, which is a P-node, is said to be *universal*. A universal PQ-tree represents all the permutations on  $S$ .

A *dissimilarity* on a finite set  $V$  is a symmetric function  $d : V \times V \rightarrow \mathbb{R}^+$  such that  $d(i, j) = 0$  if and only if  $i = j$ . Dissimilarities can be seen as a generalization of distances and can be represented by symmetric matrices on  $\mathbb{R}^+$  with a null diagonal. For a dissimilarity  $d$  on a set  $V$  and  $\delta \in \mathbb{R}$ , the graph  $G_d^\delta = (V, E_d^\delta)$  with  $xy \in E_d^\delta \iff d(x, y) \leq \delta$  is a *threshold graph* of  $d$ . The set  $\mathcal{C}_d$  of all maximal cliques of all threshold

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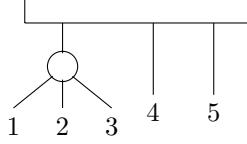


Figure 1: A PQ-tree

graphs of  $d$  is called the *class system* of  $d$ . Note that  $H_d = (V, \mathcal{C}_d)$  is a hypergraph such that  $V$  and all singletons are hyperedges.

In the rest of this paper, we will only consider hypergraphs with vertex set  $V = \{1, \dots, n\}$  and we will suppose that, if  $H$  is an interval hypergraph, then  $(1, \dots, n)$  is one of its compatible orders. We will denote the number of interval hypergraphs on  $\{1, \dots, n\}$  by  $I_n$ , the number of flat interval hypergraphs by  $F_n$  and the Catalan number  $\frac{1}{n+1} \binom{2n}{n}$  by  $C_n$ .

## 2 Results on interval hypergraphs

**Claim 1.**  $I_n = 2^{n(n+1)/2}$ .

*Proof.* Since an interval on  $\{1, \dots, n\}$  is either a pair of elements (its extremities) or a singleton, there are  $n(n+1)/2$  interval in  $\{1, \dots, n\}$ .  $\square$

**Theorem 1.**  $\lim_{n \rightarrow \infty} F_n/I_n = 1$ .

*Proof.* Let  $H = (V, E)$  be an interval hypergraph. Suppose that its PQ-tree  $T_H$  has an internal node  $\alpha$ , different from the root of  $T_H$ , and that the leaves under  $\alpha$  are  $i, \dots, i+k$ . We have  $1 < k < n-i$  and we set  $V_\alpha := \{i, \dots, i+k\}$ .

For every permutation  $\sigma$  represented by  $T_H$ ,  $V_\alpha$  is an interval of  $V$  and, if for some  $e \in E$ , there exists  $x \in e \cap V_\alpha$  and  $y \in e \setminus V_\alpha$ , then  $V_\alpha \subset e$ . So  $H$  is entirely determined by  $H|_{V_\alpha}$  and  $H|(V \setminus V_\alpha) \cup \{i\}$  and thus the number of interval hypergraphs with a node like  $\alpha$  (i.e. such that the leaves under it are  $i, \dots, i+k$ ) as internal node of their PQ-tree is  $I_n^{[k]} := I_k \cdot I_{n-k+1}$ . This value is maximum for  $k = 2$  and we have  $I_n^{[2]} = 2^{n(n-1)/2+3}$ . So the proportion of interval hypergraphs having a PQ-tree with a node like  $\alpha$  as leftmost internal node is  $\leq \frac{1}{2^{n-3}}$ . There are less than  $(n-1)^2$  possible choices for the type of  $\alpha$  (at most  $n-1$  choices for  $i$  and  $n-1$  choices for  $k$ ), so the proportion of interval hypergraphs on  $\{1, \dots, n\}$  which are compatible with a PQ-tree having internal nodes tends to 0 when  $n$  tends to  $\infty$ . If the PQ-tree of an interval hypergraph  $H$  is universal, the hyperedges of  $H$  are either singletons or  $V$ . So there are only  $2^{n+1}$  interval hypergraphs whose PQ-tree is universal, and the result follows.  $\square$

Note that, even if we do not impose  $(1, \dots, n)$  to be a compatible order, nearly all interval hypergraphs have only two compatible orders.

**Theorem 2.** *The number of interval covers of  $\{1, \dots, n\}$  is  $C_n$ .*

*Proof.* There are  $C_n$  lattice paths in  $\mathbb{Z} \times \mathbb{Z}$  from  $(1, 1)$  to  $(n, n)$  with steps  $(0, 1)$ ,  $(1, 0)$  and  $(1, 1)$  never going below the line  $y = x$  and such that steps  $(1, 1)$  only appear on the line  $y = x$  (Exercice 52 of [9]). In such a path, a *summit* is a point with neither a step  $(1, 0)$  just before it nor a step  $(0, 1)$  just after.

If we associate with each summit  $(i, j)$  the interval  $[i, j]$  (see Figure 2), we build a one-to-one correspondence between these paths and interval covers of  $\{1, \dots, n\}$ .  $\square$

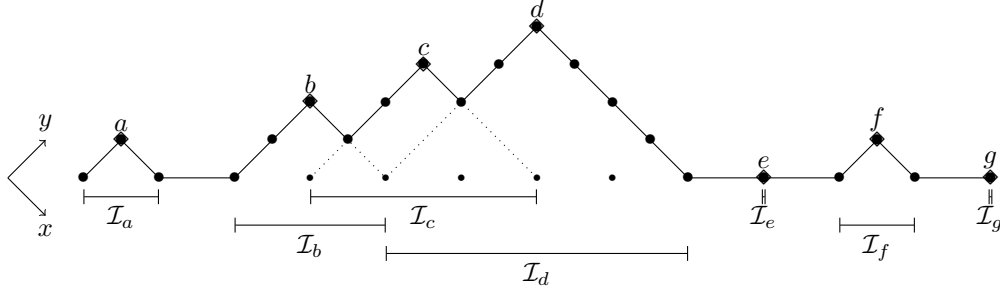


Figure 2: A lattice path on  $\mathbb{Z} \times \mathbb{Z}$  from  $(1, 1)$  to  $(13, 13)$  with summits  $a, b, \dots, g$  and the corresponding intervals  $\mathcal{I}_a, \mathcal{I}_b, \dots, \mathcal{I}_g$ , which form an interval cover of the points  $(1, 1), (2, 2), \dots, (13, 13)$ .

### 3 Robinson dissimilarities

A dissimilarity  $d$  on a finite set  $V$  is *Robinson* [8] if there exists a linear order  $\sigma$  on  $V$  (which is said to be *compatible*) such that, when  $V$  is ordered along  $\sigma$ ,  $i < j < k \implies d(i, j) \leq d(i, k)$  and  $d(j, k) \leq d(i, k)$ . A dissimilarity  $d$  on  $V$  is Robinson if and only if  $(V, \mathcal{C}_d)$  is an interval hypergraph [3, 4]; in addition,  $d$  and  $(V, \mathcal{C}_d)$  have the same set of compatible orders [7]. We will identify two dissimilarities  $d$  and  $d'$  if  $\mathcal{C}_d = \mathcal{C}_{d'}$ ; and, as for interval hypergraphs, we will suppose that  $V = \{1, \dots, n\}$  and  $(1, \dots, n)$  is a compatible order.

**Claim 2.** *The number of Robinson dissimilarities on  $V$  is  $2^{n(n-1)/2-1}$*

*Proof.*  $V \cup \{V\}$  is always a subset of  $\mathcal{C}_d$ , so the number of intervals that can be, or not, in  $\mathcal{C}_d$  is  $n(n-1)/2 - 1$ .  $\square$

**Theorem 3.** *The number of Robinson dissimilarities on  $\{1, \dots, n\}$  with values in  $\{0, 1, 2\}$  is  $C_n - 1$ .*

*Proof.* When  $d$  takes its values in  $\{0, 1, 2\}$ , the relation  $R$  on  $V$  defined by  $iRj \iff d(i, j) \leq 1$  is reflexive, symmetric and such that if  $1 \leq i < j < k \leq n$ , and  $iRk$ , then  $iRj$  and  $jRk$ . There are  $C_n$  such relations on  $V$  (Exercice 187 of [9]). Since the dissimilarities defined by  $i \neq j \implies d(i, j) = 1$  and by  $i \neq j \implies d(i, j) = 2$  are equivalent, the result follows.  $\square$

If we represent a Robinson dissimilarity  $d$  by a matrix with a horizontal diagonal (see Figure 3), we get “stacks” of lattice paths (like the one in Figure 2) with a “peak” in foreground. If  $d$  takes its values in  $\{0, \dots, k\}$ , the stack is made of  $k$  lattice paths (including the peak in foreground); equivalently,  $\mathcal{C}_d$  does not contain  $e_1, e_2, \dots, e_k$  with  $\forall 1 \leq i \leq k, 1 < |e_i| < n$  and  $e_1 \subset e_2 \subset \dots \subset e_k$ . Table 1 shows the number of

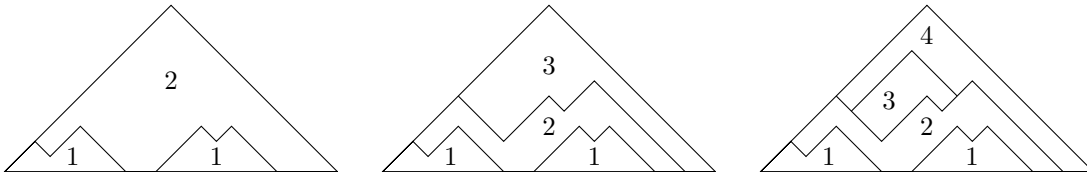


Figure 3: Robinson dissimilarities represented by matrices with a horizontal diagonal.

Robinson dissimilarities on  $\{1, \dots, n\}$  taking their values in  $\{0, \dots, k\}$  for small values of  $k$  and  $n$ . Sequences for  $k = 3, 4, 5$  match no sequence in OEIS [6] and we know no formula for them.

Table 1: Number of Robinson dissimilarities on  $\{1, \dots, n\}$  that take their values in  $\{1, \dots, k\}$  for small values of  $n$  and  $k \in \{2, 3, 4, 5, \infty\}$ . The boxes with // needed too much computation time to be filled.

$n \setminus k$	2	3	4	5	$\infty$
2	1	1	1	1	1
3	4	4	4	4	4
4	13	32	32	32	32
5	41	266	512	512	512
6	131	2254	9938	16384	16384
7	428	19934	208717	708314	1048576
8	1429	185383	//	//	134217728
9	4861	//	//	//	34359738368
$n$	$C_n - 1$	?	?	?	$2^{n(n-1)/2-1}$

## 4 Conclusion

Our original goal was to prove that nearly all Robinson dissimilarities have a flat PQ-tree (a direct consequence of Claim 2 and of the proof of Theorem 1), which was intended to be a first step toward the determination of the average complexity of algorithms like those of [2] or [5]. This led us to lattice paths, and thus to a combinatorial interpretation of Catalan numbers (Theorem 2). This interpretation, although very simple, natural and easy to prove, seems curiously to have been unknown until now.

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