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Abstract

We prove a discrete version of the first Weber inequality on three-dimensional hybrid spaces spanned by vectors of polynomials attached to the elements and faces of a polyhedral mesh. We then introduce two Hybrid High-Order methods for the approximation of the magnetostatic model, in both its (first-order) field and (second-order) vector potential formulations. These methods are applicable on general polyhedral meshes with star-shaped elements, and allow for arbitrary orders of approximation. Leveraging the previously established discrete Weber inequality, we perform a comprehensive analysis of the two methods. We finally validate them on a set of test-cases.

Keywords: Weber inequalities; Hybrid spaces; Polyhedral meshes; Hybrid High-Order methods; Magnetostatics

MSC2010 classification: 65N08, 65N12, 65N30

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ denote an open, bounded, and connected polyhedral domain. In the study of problems in electromagnetism, Weber inequalities [29] constitute a very powerful tool. They can be viewed as a generalization of the celebrated Poincaré inequality to the case of vector fields belonging to $\boldsymbol{H}(\mathbf{curl};\Omega)\cap\boldsymbol{H}(\mathrm{div};\Omega)$, and featuring either vanishing tangential component (first Weber inequality), or vanishing normal component (second Weber inequality) on the boundary $\partial\Omega$ of the domain. We refer the reader to [2, Theorems 3.4.3 and 3.5.3] for a general (from a topological viewpoint) statement of Weber inequalities.

Let us denote by $\boldsymbol{n}$ the unit normal vector field on $\partial\Omega$ pointing out of Ω. From now on, we assume that Ω is topologically trivial (a sufficient condition is that it be simply-connected), and that $\partial\Omega$ is connected. Under these assumptions, the first and second Betti numbers of Ω are both zero, i.e., Ω does not have tunnels and does not enclose any void. For a deeper insight into the role of the different topological assumptions we make on the domain, we refer to Remark 7. For any $X \subset \overline{\Omega}$, we denote by $(\cdot,\cdot)_X$ and $||\cdot||_X$ the usual inner product and norm on $L^2(X;\mathbb{R}^l)$, $l \in \{1,2,3\}$. We also let $\boldsymbol{H_0}(\mathbf{curl};\Omega) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega) \ : \ \boldsymbol{n}\times(\boldsymbol{v}\times\boldsymbol{n}) = \boldsymbol{0} \text{ on } \partial\Omega\}$, and

$$\begin{aligned}\boldsymbol{H}(\mathrm{div}^0;\Omega) &:= \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\Omega) \ : \ \mathrm{div}\,\boldsymbol{v} = 0 \text{ in } \Omega\}\\ &= \left\{\boldsymbol{v} \in L^2(\Omega;\mathbb{R}^3) \ : \ (\boldsymbol{v}, \mathbf{grad}\,\varphi)_\Omega = 0 \quad \forall \varphi \in H^1_0(\Omega)\right\}.\end{aligned}$$

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The following $L^2(\Omega;\mathbb{R}^3)$-orthogonal decomposition holds (cf. [2, Proposition 3.7.2]):

$$\boldsymbol{H_0}(\mathbf{curl};\Omega) = \mathbf{grad}\big(H_0^1(\Omega)\big) \overset{\perp}{\oplus} \Big(\boldsymbol{H_0}(\mathbf{curl};\Omega) \cap \boldsymbol{H}(\mathrm{div}^0;\Omega)\Big). \tag{2}$$

With the assumptions we have made on the topology of the domain Ω, the first Weber inequality reads: For any $\boldsymbol{v} \in \boldsymbol{H_0}(\mathbf{curl};\Omega) \cap \boldsymbol{H}(\mathrm{div}^0;\Omega)$,

$$\|\boldsymbol{v}\|_\Omega \le C_{\mathrm{W}} \,\|\mathbf{curl}\,\boldsymbol{v}\|_\Omega, \tag{3}$$

for some constant $C_{\mathrm{W}} > 0$ only depending on the domain Ω. In this work, we derive a discrete version of the first Weber inequality (3) on (three-dimensional) hybrid spaces spanned by vectors of polynomials attached to the elements and faces of a (polyhedral) mesh, as they can be encountered in Hybridizable Discontinuous Galerkin (HDG) [12] and related [28] methods, or in Hybrid High-Order (HHO) [18, 17] methods; see [11] for a discussion highlighting the analogies and differences between HDG and HHO methods in the context of scalar variable diffusion. The corresponding result is stated in Theorem 3. The proof extends the general ideas used in [15, Lemma 2.15] to derive a discrete Poincaré inequality on hybrid spaces.

In the second part of this paper, we tackle the HHO approximation of magnetostatics, in both its (first-order) field formulation and (second-order, generalized) vector potential formulation. Various discretization methods have been studied in the literature to approximate the magnetostatic equations (or, more generally, Maxwell equations). Conforming finite element discretizations were originally proposed (on tetrahedra, essentially) in the seminal work of Nédélec [24, 25]; see also [22], as well as [1], in which a unified presentation of conforming finite element methods based on notions from algebraic topology is provided. Nonconforming discretizations include the Discontinuous Galerkin method of [27] as well as the HDG method of [26, 8] and, on general polyhedral meshes, the variant [23] of [26] and the HDG method of [10]; see also [19]. Methods that support general polyhedral meshes and are built upon discrete spaces that mimick the continuity properties of the spaces appearing in the continuous weak formulation include the Virtual Element methods of [4, 3, 5], and the fully discrete method of [14] based on the discrete de Rham sequence of [16]. All the HDG methods cited above deal with the approximation of magnetostatics under its (generalized) vector potential formulation. In this paper, we first study an HHO method (which has been briefly introduced in [7]) for magnetostatics under its field formulation. We take advantage of the fact that the corresponding problem is first-order to avoid locally reconstructing a discrete curl operator as it is done for second-order problems (cf. Remark 10). Doing so, we propose a computationally inexpensive and easy-to-implement method. Second, we study an HHO method for magnetostatics under its (generalized) vector potential formulation, that can be seen as a computationally cheaper variant of the method introduced in [10] (cf. Remark 17). Our two HHO methods are applicable on general polyhedral meshes with star-shaped elements, and allow for an arbitrary order of approximation $k \ge 0$ with proven energy-error of order $k+1$ (cf. Theorems 13 and 22). Leveraging the previously established discrete Weber inequality, we carry out a comprehensive analysis of the methods, and validate them on a set of test-cases.

The article is organized as follows. In Section 2 we prove the discrete Weber inequality. Then, in Sections 3.1 and 3.2 we tackle the HHO approximation of magnetostatics, under its field and vector potential formulations, respectively.

2 A discrete Weber inequality on hybrid spaces

2.1 Discrete setting

We consider a polyhedral mesh $\mathcal{M}_h = (\mathcal{T}_h, \mathcal{F}_h)$, that is assumed to belong to a regular mesh sequence in the sense of [15, Definition 1.9]. The set $\mathcal{T}_h$ is a finite collection of nonempty, disjoint, open polyhedra T (called elements) that are star-shaped with respect to some interior point $\boldsymbol{x}_T$, and such that $\overline{\Omega} = \bigcup_{T\in\mathcal{T}_h} \overline{T}$. Concerning the role of the star-shapedness assumption, see Remark 8. The subscript h refers to the meshsize, defined by $h := \max_{T\in\mathcal{T}_h} h_T$, where h_T denotes the diameter of the element T. The set $\mathcal{F}_h$ collects the planar mesh faces and, for all $T \in \mathcal{T}_h$, we denote by $\mathcal{F}_T$ the set of faces that lie on the boundary of T. Boundary faces lying on $\partial\Omega$ are collected in the set $\mathcal{F}_h^{\mathrm{b}}$, and we denote by $\mathcal{F}_h^{\mathrm{i}} := \mathcal{F}_h \setminus \mathcal{F}_h^{\mathrm{b}}$ the set of interfaces. For all $F \in \mathcal{F}_h$, we denote by h_F its diameter and, for all $T \in \mathcal{T}_h$ and all $F \in \mathcal{F}_T$, $\boldsymbol{n}_{T,F}$ is the unit normal vector to F pointing out of T. We recall that, since $\mathcal{M}_h$ belongs to a regular mesh sequence, for all $T \in \mathcal{T}_h$, the quantity $\mathrm{card}(\mathcal{F}_T)$ is bounded from above uniformly in h and, for all $T \in \mathcal{T}_h$ and all $F \in \mathcal{F}_T$, h_F is uniformly comparable to h_T (cf. [15, Lemma 1.12]). In what follows, we will use the notation $\lesssim$ to indicate that an estimate is valid up to a multiplicative constant that may depend on the mesh regularity parameter, the ambient dimension, and (if need be) the polynomial degree, but that is independent of h.

2.2 Hybrid spaces

Let an integer polynomial degree $k \geq 0$ be given. For $X \in \mathcal{F}_h \cup \mathcal{T}_h$ and, respectively, $d \in \{2, 3\}$, we denote by $\mathbb{P}^q(X;\mathbb{R}^l)$, $q \in \mathbb{N}$, $l \in \{1, d\}$, the vector space of d-variate, l-valued polynomial functions on X of total degree at most q. When $l = 1$, we may simply write $\mathbb{P}^q(X)$. For future use, for any $T \in \mathcal{T}_h$, we let $\boldsymbol{\mathcal{G}}_T^q := \mathbf{grad}\big(\mathbb{P}^{q+1}(T)\big)$ and $\boldsymbol{\mathcal{C}}_T^q := \mathbf{curl}\big(\mathbb{P}^{q+1}(T;\mathbb{R}^3)\big)$, and we recall that the following (nonorthogonal) decomposition holds:

$$\mathbb{P}^q(T;\mathbb{R}^3) = \boldsymbol{\mathcal{G}}_T^q \oplus (\boldsymbol{x} - \boldsymbol{x}_T)\times\boldsymbol{\mathcal{C}}_T^{q-1}, \tag{4}$$

with the convention that $\boldsymbol{\mathcal{C}}_T^{-1} := \{\boldsymbol{0}\}$. For any $F \in \mathcal{F}_h$, we also let

$$\boldsymbol{\mathcal{G}}_F^{q+1} := \mathbf{grad}_\tau\big(\mathbb{P}^{q+2}(F)\big) \subset \mathbb{P}^{q+1}(F;\mathbb{R}^2)$$

denote the space of (tangential) gradients of polynomials of degree up to $q+2$ on F. Finally, for $l \in \{1, 3\}$, we define the broken space

$$\mathbb{P}^q(\mathcal{T}_h;\mathbb{R}^l) := \big\{\boldsymbol{v} \in L^2(\Omega;\mathbb{R}^l) \ : \ \boldsymbol{v}_{|T} \in \mathbb{P}^q(T;\mathbb{R}^l) \quad \forall T \in \mathcal{T}_h\big\}$$

that may be abbreviated into $\mathbb{P}^q(\mathcal{T}_h)$ whenever $l = 1$, as well as the broken subspaces $\boldsymbol{\mathcal{G}}_{\mathcal{T}_h}^q := \mathbf{grad}_h\big(\mathbb{P}^{q+1}(\mathcal{T}_h)\big)$ and $\boldsymbol{\mathcal{C}}_{\mathcal{T}_h}^q := \mathbf{curl}_h\big(\mathbb{P}^{q+1}(\mathcal{T}_h;\mathbb{R}^3)\big)$, where $\mathbf{grad}_h$ (resp. $\mathbf{curl}_h$) denotes the usual broken $\mathbf{grad}$ (resp. $\mathbf{curl}$) operator on $H^1(\mathcal{T}_h)$ (resp. $\boldsymbol{H}(\mathbf{curl};\mathcal{T}_h)$).

We introduce the following (global) hybrid spaces:

$$\underline{\mathbf{X}}_h^{k+1} := \left\{\underline{\mathbf{v}}_h = \big((\mathbf{v}_T)_{T\in\mathcal{T}_h}, (\mathbf{v}_F)_{F\in\mathcal{F}_h}\big) \ : \ \begin{array}{ll} \mathbf{v}_T \in \mathbb{P}^{k+1}(T;\mathbb{R}^3) & \forall T \in \mathcal{T}_h \\ \mathbf{v}_F \in \boldsymbol{\mathcal{G}}_F^{k+1} & \forall F \in \mathcal{F}_h \end{array}\right\}, \tag{5a}$$

$$\underline{\mathrm{Y}}_h^{k+1} := \left\{\underline{\mathrm{q}}_h = \big((\mathrm{q}_T)_{T\in\mathcal{T}_h}, (\mathrm{q}_F)_{F\in\mathcal{F}_h}\big) \ : \ \begin{array}{ll} \mathrm{q}_T \in \mathbb{P}^{k}(T) & \forall T \in \mathcal{T}_h \\ \mathrm{q}_F \in \mathbb{P}^{k+1}(F) & \forall F \in \mathcal{F}_h \end{array}\right\}, \tag{5b}$$

as well as their subspaces incorporating homogeneous essential boundary conditions:

$$\begin{aligned} \underline{\mathbf{X}}_{h,\boldsymbol{0}}^{k+1} &:= \big\{\underline{\mathbf{v}}_h \in \underline{\mathbf{X}}_h^{k+1} \ : \ \mathbf{v}_F = \boldsymbol{0} \quad \forall F \in \mathcal{F}_h^{\mathrm{b}}\big\}, \\ \underline{\mathrm{Y}}_{h,0}^{k+1} &:= \big\{\underline{\mathrm{q}}_h \in \underline{\mathrm{Y}}_h^{k+1} \ : \ \mathrm{q}_F = 0 \quad \forall F \in \mathcal{F}_h^{\mathrm{b}}\big\}. \end{aligned} \tag{6}$$

Given a mesh element $T \in \mathcal{T}_h$, we respectively denote by $\underline{\mathbf{X}}_T^{k+1}$ and $\underline{\mathrm{Y}}_T^{k+1}$ the restrictions of $\underline{\mathbf{X}}_h^{k+1}$ and $\underline{\mathrm{Y}}_h^{k+1}$ to T, and by $\underline{\mathbf{v}}_T := \big(\mathbf{v}_T, (\mathbf{v}_F)_{F\in\mathcal{F}_T}\big) \in \underline{\mathbf{X}}_T^{k+1}$ and $\underline{\mathrm{q}}_T := \big(\mathrm{q}_T, (\mathrm{q}_F)_{F\in\mathcal{F}_T}\big) \in \underline{\mathrm{Y}}_T^{k+1}$ the respective restrictions of generic vectors of polynomials $\underline{\mathbf{v}}_h \in \underline{\mathbf{X}}_h^{k+1}$ and $\underline{\mathrm{q}}_h \in \underline{\mathrm{Y}}_h^{k+1}$. Also, we let $\mathbf{v}_h$ and q_h (not underlined) be the broken polynomial functions in $\mathbb{P}^{k+1}(\mathcal{T}_h;\mathbb{R}^3)$ and in $\mathbb{P}^k(\mathcal{T}_h)$ such that

$$(\mathbf{v}_h)_{|T} := \mathbf{v}_T \quad \text{and} \quad (\mathrm{q}_h)_{|T} := \mathrm{q}_T \quad \text{for all } T \in \mathcal{T}_h.$$

Finally, we define the interpolators $\underline{\mathbf{I}}_{\mathbf{X},h}^{k+1} : H^1(\Omega;\mathbb{R}^3) \to \underline{\mathbf{X}}_h^{k+1}$ and $\underline{\mathrm{I}}_{\mathrm{Y},h}^{k+1} : H^1(\Omega) \to \underline{\mathrm{Y}}_h^{k+1}$ such that, for all $\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^3)$ and all $q \in H^1(\Omega)$,

$$\underline{\mathbf{I}}_{\mathbf{X},h}^{k+1}\boldsymbol{v} := \Big(\big(\boldsymbol{\pi}_T^{k+1}(\boldsymbol{v}_{|T})\big)_{T\in\mathcal{T}_h}, \big(\boldsymbol{\pi}_{\mathcal{G},F}^{k+1}(\boldsymbol{\gamma}_{\tau,F}(\boldsymbol{v}))\big)_{F\in\mathcal{F}_h}\Big), \tag{7a}$$

$$\underline{\mathrm{I}}_{\mathrm{Y},h}^{k+1} q := \Big(\big(\pi_T^k(q_{|T})\big)_{T\in\mathcal{T}_h}, \big(\pi_F^{k+1}(q_{|F})\big)_{F\in\mathcal{F}_h}\Big), \tag{7b}$$

where (i) for any $X \subseteq \Omega$, and any $F \in \mathcal{F}_h$ such that $F \subset \overline{X}$, $\boldsymbol{\gamma}_{\tau,F}(\boldsymbol{v}) \in L^2(F;\mathbb{R}^2)$ denotes the tangential trace on F of $\boldsymbol{v} \in H^1(X;\mathbb{R}^3)$, (ii) for $X \in \mathcal{F}_h \cup \mathcal{T}_h$ (and, respectively, $d \in \{2,3\}$), π_X^q (resp. $\boldsymbol{\pi}_X^q$), $q \in \mathbb{N}$, denotes the L^2-orthogonal projector onto $\mathbb{P}^q(X)$ (resp. $\mathbb{P}^q(X;\mathbb{R}^d)$), and (iii) $\boldsymbol{\pi}_{\mathcal{G},F}^{q+1}$ stands for the $L^2(F;\mathbb{R}^2)$-orthogonal projector onto $\boldsymbol{\mathcal{G}}_F^{q+1}$. We also introduce here, for $\mathcal{S} \in \{\mathcal{G}, \mathcal{C}\}$, the notation $\boldsymbol{\pi}_{\mathcal{S},T}^q$ for the $L^2(T;\mathbb{R}^3)$-orthogonal projector onto $\boldsymbol{\mathcal{S}}_T^q$. We will also make use of the global L^2- orthogonal projectors π_h^q, $\boldsymbol{\pi}_h^q$ and $\boldsymbol{\pi}_{\mathcal{S},h}^q$ onto, respectively, $\mathbb{P}^q(\mathcal{T}_h)$, $\mathbb{P}^q(\mathcal{T}_h;\mathbb{R}^3)$ and $\boldsymbol{\mathcal{S}}_{\mathcal{T}_h}^q$.

2.3 Gradient reconstruction in $\underline{\mathrm{Y}}_h^{k+1}$

We define the global discrete gradient reconstruction operator $\boldsymbol{G}_h^{k+1} : \underline{\mathrm{Y}}_h^{k+1} \to \mathbb{P}^{k+1}(\mathcal{T}_h;\mathbb{R}^3)$ such that its local restriction $\boldsymbol{G}_T^{k+1} : \underline{\mathrm{Y}}_T^{k+1} \to \mathbb{P}^{k+1}(T;\mathbb{R}^3)$ to any $T \in \mathcal{T}_h$ solves the following well-posed problem: For all $\underline{\mathrm{q}}_T \in \underline{\mathrm{Y}}_T^{k+1}$,

$$(\boldsymbol{G}_T^{k+1}\underline{\mathrm{q}}_T, \boldsymbol{w})_T = -(\mathrm{q}_T, \operatorname{div}\boldsymbol{w})_T + \sum_{F\in\mathcal{F}_T} (\mathrm{q}_F, \boldsymbol{w}_{|F}\cdot\boldsymbol{n}_{T,F})_F \quad \forall \boldsymbol{w} \in \mathbb{P}^{k+1}(T;\mathbb{R}^3). \tag{8}$$

We note the following commutation property (see, e.g., [15, Section 4.2.1]): For all $q \in H^1(\Omega)$, it holds

$$\boldsymbol{G}_h^{k+1}\underline{\mathrm{I}}_{\mathrm{Y},h}^{k+1} q = \boldsymbol{\pi}_h^{k+1}(\mathbf{grad}\, q). \tag{9}$$

We also have the following result in the tetrahedral case.

Lemma 1 (Norm $\|\boldsymbol{G}_h^{k+1}\cdot\|_\Omega$)**.** *Let $\mathcal{T}_h$ be a matching tetrahedral mesh. Then, the map $\|\boldsymbol{G}_h^{k+1}\cdot\|_\Omega$ defines a norm on $\underline{\mathrm{Y}}_{h,0}^{k+1}$.*

Proof. Let $\mathcal{T}_h$ be a matching tetrahedral mesh, and let $\underline{\mathrm{q}}_h \in \underline{\mathrm{Y}}_{h,0}^{k+1}$ be such that $\|\boldsymbol{G}_h^{k+1}\underline{\mathrm{q}}_h\|_\Omega = 0$. Then, for all $T \in \mathcal{T}_h$, enforcing $\boldsymbol{G}_T^{k+1}\underline{\mathrm{q}}_T = \mathbf{0}$ in the definition (8) of this quantity and integrating by parts, we obtain

$$\big(\mathbf{grad}\,\mathrm{q}_T, \boldsymbol{w}\big)_T + \sum_{F\in\mathcal{F}_T} \big(\mathrm{q}_F - \mathrm{q}_{T|F}, \boldsymbol{w}_{|F}\cdot\boldsymbol{n}_{T,F}\big)_F = 0 \qquad \forall \boldsymbol{w} \in \mathbb{P}^{k+1}(T;\mathbb{R}^3). \tag{10}$$

Let $\boldsymbol{\mathcal{N}}_T^k := \boldsymbol{\mathcal{G}}_T^{k-1} \oplus (\boldsymbol{x} - \boldsymbol{x}_T)\times\boldsymbol{\mathcal{C}}_T^{k-1}$ denote the Nédélec space of the first kind of degree k on T (cf. [24]), and let $\overline{\boldsymbol{w}} \in \mathbb{P}^{k+1}(T;\mathbb{R}^3)$ be s.t.

$$(\overline{\boldsymbol{w}}, \boldsymbol{p})_T = (\mathbf{grad}\,\mathrm{q}_T, \boldsymbol{p})_T \qquad \forall \boldsymbol{p} \in \boldsymbol{\mathcal{N}}_T^k, \tag{11a}$$

$$(\overline{\boldsymbol{w}}_{|F}\cdot\boldsymbol{n}_{T,F}, r)_F = (\mathrm{q}_F - \mathrm{q}_{T|F}, r)_F \qquad \forall r \in \mathbb{P}^{k+1}(F), \quad \forall F \in \mathcal{F}_T. \tag{11b}$$

The system (11) uniquely defines $\overline{\boldsymbol{w}}$ as a function of the Nédélec space of the second kind of degree $k+1$ on T, that is $\mathbb{P}^{k+1}(T;\mathbb{R}^3)$ (cf. [25, 6]). Testing (10) with $\overline{\boldsymbol{w}}$, and using the fact that $\mathbf{grad}\,\mathrm{q}_T \in \boldsymbol{\mathcal{G}}_T^{k-1} \subset \boldsymbol{\mathcal{N}}_T^k$ and that $\mathrm{q}_F - \mathrm{q}_{T|F} \in \mathbb{P}^{k+1}(F)$ for all $F \in \mathcal{F}_T$, we infer from (11) that $\|\mathbf{grad}\,\mathrm{q}_T\|_T^2 + \sum_{F\in\mathcal{F}_T}\|\mathrm{q}_F - \mathrm{q}_{T|F}\|_F^2 = 0$. Reproducing the same reasoning on all $T \in \mathcal{T}_h$, and using the fact that $\underline{\mathrm{q}}_h$ belongs to the space $\underline{\mathrm{Y}}_{h,0}^{k+1}$ with strongly enforced boundary conditions, finally yields $\underline{\mathrm{q}}_h = \underline{0}_h$. □

2.4 Discrete Weber inequality

We begin with a preliminary technical result.

Lemma 2. *Let $T \in \mathcal{T}_h$. For all $\boldsymbol{p} \in \mathbb{P}^q(T;\mathbb{R}^3)$ such that $\boldsymbol{p} = \mathbf{grad}\,g + (\boldsymbol{x}-\boldsymbol{x}_T)\times\mathbf{curl}\,\boldsymbol{c}$ with $g \in \mathbb{P}^{q+1}(T)$ and $\boldsymbol{c} \in \mathbb{P}^q(T;\mathbb{R}^3)$, it holds*

$$\|\boldsymbol{p} - \mathbf{grad}\,g\|_T \le 2h_T\|\mathbf{curl}\,\boldsymbol{p}\|_T. \tag{12}$$

Proof. The decomposition $\boldsymbol{p} = \mathbf{grad}\,g + (\boldsymbol{x}-\boldsymbol{x}_T)\times\mathbf{curl}\,\boldsymbol{c}$ follows from (4). Using the fact that $|(\boldsymbol{x}-\boldsymbol{x}_T)\times\mathbf{curl}\,\boldsymbol{c}| \le |\boldsymbol{x}-\boldsymbol{x}_T|\,|\mathbf{curl}\,\boldsymbol{c}| \le h_T|\mathbf{curl}\,\boldsymbol{c}|$, we infer

$$\|\boldsymbol{p} - \mathbf{grad}\,g\|_T = \|(\boldsymbol{x}-\boldsymbol{x}_T)\times\mathbf{curl}\,\boldsymbol{c}\|_T \le h_T\|\mathbf{curl}\,\boldsymbol{c}\|_T. \tag{13}$$

We now focus on the right-hand side of (13). Since $\mathbf{curl}\big(\mathbf{grad}\,g\big) = \mathbf{0}$, it holds

$$\begin{aligned}\mathbf{curl}\,\boldsymbol{p} &= \mathbf{curl}\,\big((\boldsymbol{x}-\boldsymbol{x}_T)\times\mathbf{curl}\,\boldsymbol{c}\big)\\ &= \cancel{(\boldsymbol{x}-\boldsymbol{x}_T)\,\mathrm{div}(\mathbf{curl}\,\boldsymbol{c})} - [(\boldsymbol{x}-\boldsymbol{x}_T)\cdot\mathbf{grad}](\mathbf{curl}\,\boldsymbol{c})\\ &\quad - 3\,\mathbf{curl}\,\boldsymbol{c} + [\mathbf{curl}\,\boldsymbol{c}\cdot\mathbf{grad}](\boldsymbol{x}-\boldsymbol{x}_T)\\ &= -2\,\mathbf{curl}\,\boldsymbol{c} - [(\boldsymbol{x}-\boldsymbol{x}_T)\cdot\mathbf{grad}](\mathbf{curl}\,\boldsymbol{c}),\end{aligned} \tag{14}$$

where we have (i) used with $\boldsymbol{A} = \boldsymbol{x}-\boldsymbol{x}_T$ and $\boldsymbol{B} = \mathbf{curl}\,\boldsymbol{c}$ the vector calculus identity $\mathbf{curl}(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{A}(\mathrm{div}\,\boldsymbol{B}) - [\boldsymbol{A}\cdot\mathbf{grad}]\boldsymbol{B} - (\mathrm{div}\,\boldsymbol{A})\boldsymbol{B} + [\boldsymbol{B}\cdot\mathbf{grad}]\boldsymbol{A}$, (ii) used the fact that the divergence of the curl is zero in the cancellation, and (iii) observed that $[\mathbf{curl}\,\boldsymbol{c}\cdot\mathbf{grad}](\boldsymbol{x}-\boldsymbol{x}_T) = \mathbf{curl}\,\boldsymbol{c}$ to conclude. Multiplying (14) by $-\mathbf{curl}\,\boldsymbol{c}$ and integrating over T, we get

$$\begin{aligned}-\big(\mathbf{curl}\,\boldsymbol{p},\mathbf{curl}\,\boldsymbol{c}\big)_T &= 2\|\mathbf{curl}\,\boldsymbol{c}\|_T^2 + \Big(\boldsymbol{x}-\boldsymbol{x}_T,\mathbf{grad}\left(\frac{|\mathbf{curl}\,\boldsymbol{c}|^2}{2}\right)\Big)_T\\ &= \frac{1}{2}\|\mathbf{curl}\,\boldsymbol{c}\|_T^2 + \sum_{F\in\mathcal{F}_T}\frac{1}{2}\big(|\mathbf{curl}\,\boldsymbol{c}|_{|F}^2,(\boldsymbol{x}-\boldsymbol{x}_T)_{|F}\cdot\boldsymbol{n}_{T,F}\big)_F\\ &\ge \frac{1}{2}\|\mathbf{curl}\,\boldsymbol{c}\|_T^2 \ge 0,\end{aligned} \tag{15}$$

where we have used an integration by parts formula to pass to the second line, and the fact that T is star-shaped with respect to $\boldsymbol{x}_T$ to conclude. From (15) and a Cauchy–Schwarz inequality, we infer

$$\|\mathbf{curl}\,\boldsymbol{c}\|_T \le 2\|\mathbf{curl}\,\boldsymbol{p}\|_T. \tag{16}$$

Plugging (16) into (13), (12) follows. □

We equip the space $\underline{\mathbf{X}}_h^{k+1}$ with the seminorm $\|\cdot\|_{\mathbf{X},h}$ defined by

$$\|\underline{\mathbf{v}}_h\|_{\mathbf{X},h}^2 := \|\mathbf{curl}_h\mathbf{v}_h\|_\Omega^2 + \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T} h_F^{-1}\|\boldsymbol{\pi}_{\mathcal{G},F}^{k+1}\big(\boldsymbol{\gamma}_{\tau,F}(\mathbf{v}_T) - \mathbf{v}_F\big)\|_F^2, \tag{17}$$

and we define the following discrete counterpart of $\boldsymbol{H}_{\mathbf{0}}(\mathbf{curl};\Omega)\cap\boldsymbol{H}(\mathrm{div}^0;\Omega)$:

$$\underline{\mathbb{X}}_{h,\mathbf{0}}^{k+1}:=\left\{\underline{\mathbf{w}}_h\in\underline{\mathbf{X}}_{h,\mathbf{0}}^{k+1}\ :\ \left(\mathbf{w}_h,\boldsymbol{G}_h^{k+1}\underline{\mathrm{q}}_h\right)_\Omega=0\quad\forall\underline{\mathrm{q}}_h\in\underline{\mathrm{Y}}_{h,0}^{k+1}\right\}. \tag{18}$$

Then, the following discrete Weber inequality holds.

Theorem 3 (Discrete Weber inequality)**.** *There exists a constant $c_{\mathrm{W}}>0$ independent of h such that, for all $\underline{\mathbf{v}}_h\in\underline{\mathbb{X}}_{h,\mathbf{0}}^{k+1}$, one has*

$$\|\mathbf{v}_h\|_\Omega\le c_{\mathrm{W}}\|\underline{\mathbf{v}}_h\|_{\mathbf{X},h}. \tag{19}$$

Proof. Let $\underline{\mathbf{v}}_h\in\underline{\mathbb{X}}_{h,\mathbf{0}}^{k+1}$. Since $\mathbf{v}_h\in\mathbb{P}^{k+1}(\mathcal{T}_h;\mathbb{R}^3)\subset L^2(\Omega;\mathbb{R}^3)$ and $\partial\Omega$ is connected, we can write

$$\mathbf{v}_h=\mathbf{grad}\,\varphi+\mathbf{curl}\,\boldsymbol{\psi}, \tag{20}$$

for some $\varphi\in H_0^1(\Omega)$, and some $\boldsymbol{\psi}\in H^1(\Omega;\mathbb{R}^3)$ such that $\int_\Omega\boldsymbol{\psi}=\mathbf{0}$ and $\mathrm{div}\,\boldsymbol{\psi}=0$; see, e.g., [2, Proposition 3.7.1 and Theorem 3.4.1]. Furthermore,

$$\|\boldsymbol{\psi}\|_{H^1(\Omega;\mathbb{R}^3)}\lesssim\|\mathbf{curl}\,\boldsymbol{\psi}\|_\Omega. \tag{21}$$

Since the decomposition (20) of $\mathbf{v}_h$ is $L^2(\Omega;\mathbb{R}^3)$-orthogonal, it holds

$$\|\mathbf{v}_h\|_\Omega^2=\|\mathbf{grad}\,\varphi\|_\Omega^2+\|\mathbf{curl}\,\boldsymbol{\psi}\|_\Omega^2=:\mathcal{I}_1+\mathcal{I}_2. \tag{22}$$

For the first term in (22), setting $\underline{\varphi}_h:=\underline{\mathrm{I}}_{\mathrm{Y},h}^{k+1}\varphi\in\underline{\mathrm{Y}}_{h,0}^{k+1}$, we have

$$\mathcal{I}_1=(\mathbf{v}_h,\mathbf{grad}\,\varphi)_\Omega=(\mathbf{v}_h,\boldsymbol{\pi}_h^{k+1}(\mathbf{grad}\,\varphi))_\Omega=\left(\mathbf{v}_h,\boldsymbol{G}_h^{k+1}\underline{\varphi}_h\right)_\Omega=0, \tag{23}$$

where we have used the fact that $(\mathbf{curl}\,\boldsymbol{\psi},\mathbf{grad}\,\varphi)_\Omega=0$, that $\mathbf{v}_h\in\mathbb{P}^{k+1}(\mathcal{T}_h;\mathbb{R}^3)$, the commutation property (9) of $\boldsymbol{G}_h^{k+1}$, and the fact that $\underline{\mathbf{v}}_h\in\underline{\mathbb{X}}_{h,\mathbf{0}}^{k+1}$.

Let us now estimate the second term in (22). It holds

$$\begin{aligned}\mathcal{I}_2&=\left(\mathbf{v}_h,\mathbf{curl}\,\boldsymbol{\psi}\right)_\Omega=\sum_{T\in\mathcal{T}_h}\left(\mathbf{v}_T,\mathbf{curl}\,\boldsymbol{\psi}_{|T}\right)_T\\&=\sum_{T\in\mathcal{T}_h}\left(\left(\mathbf{curl}\,\mathbf{v}_T,\boldsymbol{\psi}_{|T}\right)_T-\sum_{F\in\mathcal{F}_T}\left(\boldsymbol{\psi}_{|F}\times\boldsymbol{n}_{T,F},\mathbf{v}_{T|F}\right)_F\right)\\&=\sum_{T\in\mathcal{T}_h}\left(\left(\mathbf{curl}\,\mathbf{v}_T,\boldsymbol{\psi}_{|T}\right)_T-\sum_{F\in\mathcal{F}_T}\left(\boldsymbol{\gamma}_{\tau,F}(\boldsymbol{\psi}\times\boldsymbol{n}_{T,F}),\boldsymbol{\gamma}_{\tau,F}(\mathbf{v}_T)-\mathbf{v}_F\right)_F\right),\end{aligned}$$

where we have used the fact that $\left(\mathbf{grad}\,\varphi,\mathbf{curl}\,\boldsymbol{\psi}\right)_\Omega=0$ in the first line, an integration by parts formula on each mesh element $T\in\mathcal{T}_h$ in the second line, and the fact that the jumps of $\boldsymbol{\psi}\in H^1(\Omega;\mathbb{R}^3)$ vanish on interfaces along with $\mathbf{v}_F=\mathbf{0}$ for all $F\in\mathcal{F}_h^{\mathrm{b}}$ to insert $\mathbf{v}_F$ into the second term in the third line. Applying Cauchy–Schwarz inequalities to the right-hand side, we obtain

$$\begin{aligned}\mathcal{I}_2\le&\left(\sum_{T\in\mathcal{T}_h}\left(\|\mathbf{curl}\,\mathbf{v}_T\|_T^2+\sum_{F\in\mathcal{F}_T}h_F^{-1}\|\boldsymbol{\gamma}_{\tau,F}(\mathbf{v}_T)-\mathbf{v}_F\|_F^2\right)\right)^{1/2}\\&\times\left(\sum_{T\in\mathcal{T}_h}\left(\|\boldsymbol{\psi}_{|T}\|_T^2+\sum_{F\in\mathcal{F}_T}h_F\|\boldsymbol{\psi}_{|F}\times\boldsymbol{n}_{T,F}\|_F^2\right)\right)^{1/2}.\end{aligned} \tag{24}$$

We focus on the first factor on the right-hand side of (24). For $T \in \mathcal{T}_h$ and $F \in \mathcal{F}_T$, decomposing $\mathbf{v}_T \in \mathbb{P}^{k+1}(T;\mathbb{R}^3)$ along (4) as $\mathbf{v}_T = \mathbf{grad}\, g + (\boldsymbol{x} - \boldsymbol{x}_T) \times \mathbf{curl}\, \boldsymbol{c}$ with $g \in \mathbb{P}^{k+2}(T)$ and $\boldsymbol{c} \in \mathbb{P}^{k+1}(T;\mathbb{R}^3)$, inserting into the norm $\pm\big[\boldsymbol{\gamma}_{\tau,F}(\mathbf{grad}\, g) + \boldsymbol{\pi}^{k+1}_{\mathcal{G},F}\big(\boldsymbol{\gamma}_{\tau,F}(\mathbf{v}_T)\big)\big]$, and using the triangle inequality, we infer, since $\mathbf{v}_F \in \mathcal{G}^{k+1}_F$ and $\boldsymbol{\gamma}_{\tau,F}(\mathbf{grad}\, g) = \mathbf{grad}_\tau(g_{|F}) \in \mathcal{G}^{k+1}_F$,

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T} h_F^{-1}\|\boldsymbol{\gamma}_{\tau,F}(\mathbf{v}_T) - \mathbf{v}_F\|_F^2 \lesssim{}& \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T} h_F^{-1}\|\boldsymbol{\pi}^{k+1}_{\mathcal{G},F}\big(\boldsymbol{\gamma}_{\tau,F}(\mathbf{v}_T) - \mathbf{v}_F\big)\|_F^2 \\ &+ \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T} h_F^{-1}\|\boldsymbol{\gamma}_{\tau,F}\big(\mathbf{v}_T - \mathbf{grad}\, g\big)\|_F^2 \\ &+ \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T} h_F^{-1}\|\boldsymbol{\pi}^{k+1}_{\mathcal{G},F}\big(\boldsymbol{\gamma}_{\tau,F}(\mathbf{v}_T - \mathbf{grad}\, g)\big)\|_F^2. \end{aligned} \tag{25}$$

Using the $L^2(F;\mathbb{R}^2)$-boundedness of $\boldsymbol{\pi}^{k+1}_{\mathcal{G},F}$, a discrete trace inequality (cf., e.g., [15, Lemma 1.32]), and Lemma 2 with $\boldsymbol{p} = \mathbf{v}_T$, we infer

$$\begin{aligned}&\sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T} h_F^{-1}\|\boldsymbol{\gamma}_{\tau,F}(\mathbf{v}_T) - \mathbf{v}_F\|_F^2 \\ &\qquad\lesssim \sum_{T\in\mathcal{T}_h}\Big(\|\mathbf{curl}\,\mathbf{v}_T\|_T^2 + \sum_{F\in\mathcal{F}_T} h_F^{-1}\|\boldsymbol{\pi}^{k+1}_{\mathcal{G},F}\big(\boldsymbol{\gamma}_{\tau,F}(\mathbf{v}_T) - \mathbf{v}_F\big)\|_F^2\Big). \end{aligned} \tag{26}$$

Now, for the second factor on the right-hand side of (24), using the fact that $|\boldsymbol{\psi}_{|F} \times \boldsymbol{n}_{T,F}| \le |\boldsymbol{\psi}_{|F}|$, a continuous trace inequality (cf., e.g., [15, Lemma 1.31]), the fact that $h_T \le \mathrm{diam}(\Omega)$ for all $T \in \mathcal{T}_h$, and concluding with (21), one has

$$\Big(\sum_{T\in\mathcal{T}_h}\Big(\|\boldsymbol{\psi}_{|T}\|_T^2 + \sum_{F\in\mathcal{F}_T} h_F\|\boldsymbol{\psi}_{|F}\times\boldsymbol{n}_{T,F}\|_F^2\Big)\Big)^{1/2} \lesssim \|\boldsymbol{\psi}\|_{H^1(\Omega;\mathbb{R}^3)} \lesssim \|\mathbf{curl}\,\boldsymbol{\psi}\|_\Omega. \tag{27}$$

Plugging (26) and (27) into (24), and recalling the definition (17) of the $\|\cdot\|_{\mathbf{X},h}$-seminorm yields

$$\mathcal{I}_2 \lesssim \|\underline{\mathbf{v}}_h\|_{\mathbf{X},h}\|\mathbf{curl}\,\boldsymbol{\psi}\|_\Omega \le \|\underline{\mathbf{v}}_h\|_{\mathbf{X},h}\|\mathbf{v}_h\|_\Omega,$$

where we have used the $L^2(\Omega;\mathbb{R}^3)$-orthogonal decomposition (20) in the last bound. We conclude by combining (22), (23), and this last estimate. □

Remark 4 (Control of element unknowns)**.** *A direct proof of the fact that, for all $\underline{\mathbf{v}}_h \in \underline{\mathbb{X}}^{k+1}_{h,\mathbf{0}}$, $\|\underline{\mathbf{v}}_h\|_{\mathbf{X},h} = 0$ implies $\|\mathbf{v}_h\|_\Omega = 0$ can be obtained as follows. The volumetric term in (17) first yields that, for any $T \in \mathcal{T}_h$, $\mathbf{curl}(\mathbf{v}_{h|T}) = \mathbf{0}$ in T, meaning that $\mathbf{v}_{h|T} = \mathbf{grad}\, g$ for some $g \in \mathbb{P}^{k+2}(T)$ by Lemma 2. The boundary term in (17) then yields the continuity of the tangential component of $\mathbf{v}_h$ at interfaces, as well as $\boldsymbol{n}\times(\mathbf{v}_h\times\boldsymbol{n}) = \mathbf{0}$ on $\partial\Omega$. Hence, $\mathbf{v}_h \in \boldsymbol{H}_{\mathbf{0}}(\mathbf{curl};\Omega)$ and $\mathbf{curl}\,\mathbf{v}_h = \mathbf{0}$ in Ω. Since $\underline{\mathbf{v}}_h \in \underline{\mathbb{X}}^{k+1}_{h,\mathbf{0}}$, we also have $\mathbf{v}_h \in \boldsymbol{H}(\mathrm{div}^0;\Omega)$ by the commutation property (9). Finally, by the continuous first Weber inequality (3), $\|\mathbf{v}_h\|_\Omega = 0$. Note that this result, as opposed to the quantitative result of Theorem 3, is insensitive to the regularity of the mesh. As such, it could be stated under the sole assumption that $\mathcal{M}_h$ is a polyhedral mesh in the sense of [15, Definition 1.4] with star-shaped elements. Note also that imposing $\mathbf{v}_h \in \boldsymbol{H}(\mathrm{div}^0;\Omega)$ as we do is actually not necessary. In view of the above analysis, it is sufficient to impose that $\mathbf{v}_h$ be orthogonal to the gradient of any function in $\mathbb{P}^{k+2}(\mathcal{T}_h) \cap C^0_0(\Omega)$. This is the approach pursued in [9] on tetrahedral meshes.*

Corollary 5 (Norm $\|\cdot\|_{\mathbf{X},h}$)**.** *The map $\|\cdot\|_{\mathbf{X},h}$ defines a norm on $\underline{\mathbb{X}}^{k+1}_{h,\mathbf{0}}$.*

Proof. This is a direct consequence of Theorem 3 and of the definition (17). For $\underline{\mathbf{v}}_h \in \underline{\mathbb{X}}_{h,\mathbf{0}}^{k+1}$, if $\|\underline{\mathbf{v}}_h\|_{\mathbf{X},h} = 0$, then $\mathbf{v}_h = \mathbf{0}$, i.e. $\mathbf{v}_T = \mathbf{0}$ for all $T \in \mathcal{T}_h$. Then, for all $F \in \mathcal{F}_h$, $\|\boldsymbol{\pi}_{\mathcal{G},F}^{k+1}\mathbf{v}_F\|_F = \|\mathbf{v}_F\|_F = 0$, i.e. $\mathbf{v}_F = \mathbf{0}$, whence $\underline{\mathbf{v}}_h = \underline{\mathbf{0}}_h$. □

Corollary 6 (Generalized discrete Weber inequality)**.** *Let* $\mathrm{d}_h : \underline{\mathrm{Y}}_h^{k+1} \times \underline{\mathrm{Y}}_h^{k+1} \to \mathbb{R}$ *be a symmetric positive semi-definite bilinear form such that, for all* $\varphi \in H_0^1(\Omega)$*, letting* $\underline{\varphi}_h := \underline{\mathrm{I}}_{\mathrm{Y},h}^{k+1}\varphi \in \underline{\mathrm{Y}}_{h,0}^{k+1}$*,*

$$\mathrm{d}_h(\underline{\varphi}_h, \underline{\varphi}_h)^{1/2} \lesssim \|\,\mathbf{grad}\,\varphi\|_\Omega. \tag{28}$$

Then, there is $c_{\mathrm{W}} > 0$ *independent of* h *s.t., for all* $(\underline{\mathbf{v}}_h, \underline{\mathrm{r}}_h) \in \underline{\mathbf{X}}_{h,\mathbf{0}}^{k+1} \times \underline{\mathrm{Y}}_{h,0}^{k+1}$ *satisfying*

$$-\left(\mathbf{v}_h, \boldsymbol{G}_h^{k+1}\underline{\mathrm{q}}_h\right)_\Omega + \mathrm{d}_h(\underline{\mathrm{r}}_h, \underline{\mathrm{q}}_h) = 0 \qquad \forall \underline{\mathrm{q}}_h \in \underline{\mathrm{Y}}_{h,0}^{k+1}, \tag{29}$$

one has

$$\|\mathbf{v}_h\|_\Omega \le c_{\mathrm{W}}\left(\|\underline{\mathbf{v}}_h\|_{\mathbf{X},h}^2 + \mathrm{d}_h(\underline{\mathrm{r}}_h, \underline{\mathrm{r}}_h)\right)^{1/2}. \tag{30}$$

Proof. We follow the steps of the proof of Theorem 3. If $(\underline{\mathbf{v}}_h, \underline{\mathrm{r}}_h) \in \underline{\mathbf{X}}_{h,\mathbf{0}}^{k+1} \times \underline{\mathrm{Y}}_{h,0}^{k+1}$ satisfies (29), then (23) becomes

$$\mathcal{I}_1 = (\mathbf{v}_h, \mathbf{grad}\,\varphi)_\Omega = \left(\mathbf{v}_h, \boldsymbol{G}_h^{k+1}\underline{\varphi}_h\right)_\Omega = \mathrm{d}_h(\underline{\mathrm{r}}_h, \underline{\varphi}_h).$$

By the Cauchy–Schwarz inequality and (28), we infer

$$\mathcal{I}_1 \lesssim \mathrm{d}_h(\underline{\mathrm{r}}_h, \underline{\mathrm{r}}_h)^{1/2}\|\,\mathbf{grad}\,\varphi\|_\Omega \le \mathrm{d}_h(\underline{\mathrm{r}}_h, \underline{\mathrm{r}}_h)^{1/2}\|\mathbf{v}_h\|_\Omega, \tag{31}$$

where we have used the $L^2(\Omega;\mathbb{R}^3)$-orthogonal decomposition (20) in the last bound. The rest of the proof is unchanged provided we substitute (31) to (23). □

Two additional remarks are in order.

Remark 7 (Topological assumptions on the domain)**.** *The first Weber inequality* (3) *is actually valid under the sole topological assumption that the boundary of* Ω *is connected, so that its second Betti number is zero. The same holds for the discrete Weber inequalities of Theorem 3 and Corollary 6 (and, incidentally, this is also the case for discrete Weber inequalities on spaces with conforming unknowns, see [14, Theorem 19]). In other words, one does not need to assume, as we do, that* Ω *is topologically trivial to prove these results. This last assumption is however necessary in the applicative Section 3 to have equivalence (i) between Problems* (32) *and* (33) *in field formulation, and (ii) between Problems* (54) *and* (55) *(in the class of potentials satisfying the Coulomb gauge) in vector potential formulation when* $\operatorname{div}\boldsymbol{f} \equiv 0$*.*

Remark 8 (Star-shapedness assumption)**.** *The star-shapedness assumption of the mesh elements is crucial to prove Theorem 3 (and Corollary 6) through Lemma 2 (where it is used to infer a sign for the rightmost term in the second line of* (15)*). However, when the face unknowns* $\mathbf{v}_F$ *are rather taken in the full polynomial space* $\mathbb{P}^{k+1}(F;\mathbb{R}^2)$ *(and, correspondingly,* $\boldsymbol{\pi}_{\mathcal{G},F}^{k+1}$ *is replaced by* $\boldsymbol{\pi}_F^{k+1}$*), this assumption can be relaxed. Indeed, the discrete Weber inequality reads in this case: For all* $\underline{\mathbf{v}}_h \in \underline{\mathbb{X}}_{h,\mathbf{0}}^{k+1}$*, one has (note that* $\boldsymbol{\gamma}_{\tau,F}(\mathbf{v}_T) \in \mathbb{P}^{k+1}(F;\mathbb{R}^2)$*)*

$$\|\mathbf{v}_h\|_\Omega \le c_{\mathrm{W}}\left(\|\,\mathbf{curl}_h\mathbf{v}_h\|_\Omega^2 + \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T} h_F^{-1}\|\boldsymbol{\gamma}_{\tau,F}(\mathbf{v}_T) - \mathbf{v}_F\|_F^2\right)^{1/2},$$

which can be proven without resorting to Lemma 2, just using (24)*.*

3 Application to magnetostatics

In this section, we design and analyze HHO methods for the discretization of the magnetostatic equations. Their analysis leverages the discrete Weber inequality of Theorem 3 and its generalization pointed out in Corollary 6. We work on regular (polyhedral) mesh sequences $(\mathcal{M}_h)_{h>0}$ in the sense of [15, Definition 1.9], which are characterized by the fact that the sequence of mesh regularity parameters is bounded from below by a strictly positive real number.

3.1 Field formulation

3.1.1 The model

The (first-order) field formulation of the magnetostatic pro-blem consists in finding the magnetic field $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ such that

$$\begin{aligned} \mathbf{curl}\,\boldsymbol{u} &= \boldsymbol{f} && \text{in } \Omega, && (32a)\\ \operatorname{div}\boldsymbol{u} &= 0 && \text{in } \Omega, && (32b)\\ \boldsymbol{n}\times(\boldsymbol{u}\times\boldsymbol{n}) &= \boldsymbol{0} && \text{on } \partial\Omega, && (32c) \end{aligned}$$

where the current density $\boldsymbol{f} : \Omega \to \mathbb{R}^3$ is such that $\operatorname{div}\boldsymbol{f} = 0$ in Ω and $\boldsymbol{f}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$. We consider the following equivalent (cf. Remark 7) weak formulation of Problem (32), originally introduced in [21, Eq. (58)] (see also [20]): Find $(\boldsymbol{u}, p) \in \boldsymbol{H_0}(\mathbf{curl};\Omega)\times H^1_0(\Omega)$ s.t.

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) &= (\boldsymbol{f},\mathbf{curl}\,\boldsymbol{v})_\Omega && \forall \boldsymbol{v}\in\boldsymbol{H_0}(\mathbf{curl};\Omega), && (33a)\\ -b(\boldsymbol{u},q) + c(p,q) &= 0 && \forall q\in H^1_0(\Omega), && (33b) \end{aligned}$$

where the bilinear forms $a : \boldsymbol{H}(\mathbf{curl};\Omega)\times\boldsymbol{H}(\mathbf{curl};\Omega)\to\mathbb{R}$, $b : \boldsymbol{H}(\mathbf{curl};\Omega)\times H^1(\Omega)\to\mathbb{R}$, and $c : H^1(\Omega)\times H^1(\Omega)\to\mathbb{R}$ are given by

$$a(\boldsymbol{w},\boldsymbol{v}) := (\mathbf{curl}\,\boldsymbol{w},\mathbf{curl}\,\boldsymbol{v})_\Omega,\qquad b(\boldsymbol{w},q) := (\boldsymbol{w},\mathbf{grad}\,q)_\Omega,\qquad c(r,q) := (r,q)_\Omega. \tag{34}$$

The pressure $p : \Omega\to\mathbb{R}$ acts as the Lagrange multiplier of the divergence-free constraint on the magnetic induction. Testing (33a) with $\boldsymbol{v} = \mathbf{grad}\,p \in \boldsymbol{H_0}(\mathbf{curl};\Omega)$, it is inferred that $p = 0$ in Ω. Using the decomposition (2), Problem (33) can then be equivalently rewritten: Find $\boldsymbol{u}\in\boldsymbol{H_0}(\mathbf{curl};\Omega)\cap\boldsymbol{H}(\operatorname{div}^0;\Omega)$ s.t.

$$a(\boldsymbol{u},\boldsymbol{\eta}) = (\boldsymbol{f},\mathbf{curl}\,\boldsymbol{\eta})_\Omega \qquad \forall\boldsymbol{\eta}\in\boldsymbol{H_0}(\mathbf{curl};\Omega)\cap\boldsymbol{H}(\operatorname{div}^0;\Omega),$$

whose well-posedness is a direct consequence of the first Weber inequality (3) and of the Lax–Milgram lemma.

Remark 9 (Improved stability)**.** *Here, we take advantage of the fact that the pressure p is identically zero as a consequence of (33a) to consider a weak formulation of Problem (32) that features the bilinear contribution c defined in (34), which actually defines a norm on $H^1_0(\Omega)$. Mirroring this strategy at the discrete level enables to improve the stability of the method (cf. Lemma 12) without jeopardizing its convergence properties. At the opposite, in the model of Section 3.2 below, the pressure may be nonzero and one cannot add the same contribution without modifying the model under consideration. At the discrete level, one can then only prove a weaker stability result (cf. Lemma 21).*

3.1.2 The HHO method

We analyze in this section the HHO method for Problem (33) we have briefly introduced in [7]. This HHO method is based on the hybrid spaces introduced in Section 2.2. The discrete counterparts of the bilinear forms (34) are the bilinear forms $\mathrm{a}_h : \underline{\mathbf{X}}_h^{k+1} \times \underline{\mathbf{X}}_h^{k+1} \to \mathbb{R}$, $\mathrm{b}_h : \underline{\mathbf{X}}_h^{k+1} \times \underline{\mathrm{Y}}_h^{k+1} \to \mathbb{R}$, and $\mathrm{c}_h : \underline{\mathrm{Y}}_h^{k+1} \times \underline{\mathrm{Y}}_h^{k+1} \to \mathbb{R}$ such that

$$\mathrm{a}_h(\underline{\mathbf{w}}_h, \underline{\mathbf{v}}_h) := \big(\mathbf{curl}_h \mathbf{w}_h, \mathbf{curl}_h \mathbf{v}_h\big)_\Omega + \mathrm{s}_h(\underline{\mathbf{w}}_h, \underline{\mathbf{v}}_h), \tag{35a}$$

$$\mathrm{b}_h(\underline{\mathbf{w}}_h, \underline{\mathrm{q}}_h) := \big(\mathbf{w}_h, \boldsymbol{G}_h^{k+1} \underline{\mathrm{q}}_h\big)_\Omega, \tag{35b}$$

$$\mathrm{c}_h(\underline{\mathrm{r}}_h, \underline{\mathrm{q}}_h) := (\mathrm{r}_h, \mathrm{q}_h)_\Omega + \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} h_F (\mathrm{r}_F, \mathrm{q}_F)_F, \tag{35c}$$

where $\boldsymbol{G}_h^{k+1} : \underline{\mathrm{Y}}_h^{k+1} \to \mathbb{P}^{k+1}(\mathcal{T}_h; \mathbb{R}^3)$ is the gradient reconstruction operator introduced in Section 2.3, and $\mathrm{s}_h : \underline{\mathbf{X}}_h^{k+1} \times \underline{\mathbf{X}}_h^{k+1} \to \mathbb{R}$ is the stabilization bilinear form such that

$$\mathrm{s}_h(\underline{\mathbf{w}}_h, \underline{\mathbf{v}}_h) := \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} h_F^{-1} \big(\boldsymbol{\pi}_{\mathcal{G},F}^{k+1}\big(\boldsymbol{\gamma}_{\tau,F}(\mathbf{w}_T) - \mathbf{w}_F\big), \boldsymbol{\pi}_{\mathcal{G},F}^{k+1}\big(\boldsymbol{\gamma}_{\tau,F}(\mathbf{v}_T) - \mathbf{v}_F\big)\big)_F. \tag{36}$$

The HHO method for Problem (33) then reads: Find $(\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h) \in \underline{\mathbf{X}}_{h,\mathbf{0}}^{k+1} \times \underline{\mathrm{Y}}_{h,0}^{k+1}$ s.t.

$$\mathrm{a}_h(\underline{\mathbf{u}}_h, \underline{\mathbf{v}}_h) + \mathrm{b}_h(\underline{\mathbf{v}}_h, \underline{\mathrm{p}}_h) = (\boldsymbol{f}, \mathbf{curl}_h \mathbf{v}_h)_\Omega \qquad \forall \underline{\mathbf{v}}_h \in \underline{\mathbf{X}}_{h,\mathbf{0}}^{k+1}, \tag{37a}$$

$$-\mathrm{b}_h(\underline{\mathbf{u}}_h, \underline{\mathrm{q}}_h) + \mathrm{c}_h(\underline{\mathrm{p}}_h, \underline{\mathrm{q}}_h) = 0 \qquad \forall \underline{\mathrm{q}}_h \in \underline{\mathrm{Y}}_{h,0}^{k+1}. \tag{37b}$$

Note that, contrary to p, the discrete pressure $\underline{\mathrm{p}}_h$ is in general nonzero, as a consequence of the fact that the global discrete gradient $\boldsymbol{G}_h^{k+1} \underline{\mathrm{p}}_h$ is not irrotational.

Remark 10 (Curl reconstruction)**.** *As opposed to what is done in HHO or HDG methods for second-order problems (see Section 3.2 below), we here take advantage of the fact that the problem is first-order to avoid (locally) reconstructing a discrete curl operator. Doing so, (i) it is possible to consider a smaller local space of face unknowns (that does not need to contain $\mathbb{P}^k(F;\mathbb{R}^2)$) for $k \geq 1$ (cf. [7, Table 1]), and (ii) there is no need to solve a local problem on each mesh element (which may become, for a sequential implementation, rather costly in 3D for large polynomial degrees).*

Letting $\underline{\mathbf{Z}}_h^{k+1} := \underline{\mathbf{X}}_h^{k+1} \times \underline{\mathrm{Y}}_h^{k+1}$ and $\underline{\mathbf{Z}}_{h,\mathbf{0}}^{k+1} := \underline{\mathbf{X}}_{h,\mathbf{0}}^{k+1} \times \underline{\mathrm{Y}}_{h,0}^{k+1}$, Problem (37) can be equivalently rewritten: Find $(\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h) \in \underline{\mathbf{Z}}_{h,\mathbf{0}}^{k+1}$ such that

$$\mathrm{A}_h\big((\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h), (\underline{\mathbf{v}}_h, \underline{\mathrm{q}}_h)\big) = (\boldsymbol{f}, \mathbf{curl}_h \mathbf{v}_h)_\Omega \qquad \forall (\underline{\mathbf{v}}_h, \underline{\mathrm{q}}_h) \in \underline{\mathbf{Z}}_{h,\mathbf{0}}^{k+1}, \tag{38}$$

where the bilinear form $\mathrm{A}_h : \underline{\mathbf{Z}}_h^{k+1} \times \underline{\mathbf{Z}}_h^{k+1} \to \mathbb{R}$ is defined by

$$\mathrm{A}_h\big((\underline{\mathbf{w}}_h, \underline{\mathrm{r}}_h), (\underline{\mathbf{v}}_h, \underline{\mathrm{q}}_h)\big) := \mathrm{a}_h(\underline{\mathbf{w}}_h, \underline{\mathbf{v}}_h) + \mathrm{b}_h(\underline{\mathbf{v}}_h, \underline{\mathrm{r}}_h) - \mathrm{b}_h(\underline{\mathbf{w}}_h, \underline{\mathrm{q}}_h) + \mathrm{c}_h(\underline{\mathrm{r}}_h, \underline{\mathrm{q}}_h). \tag{39}$$

For future use, we also let

$$\begin{aligned} \underline{\mathbb{Z}}_{h,\mathbf{0}}^{k+1} &:= \Big\{(\underline{\mathbf{w}}_h, \underline{\mathrm{r}}_h) \in \underline{\mathbf{Z}}_{h,\mathbf{0}}^{k+1} \;:\; -\mathrm{b}_h(\underline{\mathbf{w}}_h, \underline{\mathrm{q}}_h) + \mathrm{c}_h(\underline{\mathrm{r}}_h, \underline{\mathrm{q}}_h) = 0 \quad \forall \underline{\mathrm{q}}_h \in \underline{\mathrm{Y}}_{h,0}^{k+1}\Big\} \\ &= \Big\{(\underline{\mathbf{w}}_h, \underline{\mathrm{r}}_h) \in \underline{\mathbf{Z}}_{h,\mathbf{0}}^{k+1} \;:\; \mathrm{A}_h\big((\underline{\mathbf{w}}_h, \underline{\mathrm{r}}_h), (\underline{\mathbf{0}}_h, \underline{\mathrm{q}}_h)\big) = 0 \quad \forall \underline{\mathrm{q}}_h \in \underline{\mathrm{Y}}_{h,0}^{k+1}\Big\}. \end{aligned} \tag{40}$$

3.1.3 Stability analysis

We recall that $\underline{\mathbf{X}}_h^{k+1}$ is equipped with the seminorm $\|\cdot\|_{\mathbf{X},h}$ defined by (17), which is such that $\|\cdot\|_{\mathbf{X},h} = \mathrm{a}_h(\cdot,\cdot)^{1/2}$. We equip $\underline{\mathrm{Y}}_h^{k+1}$ with the norm $\|\cdot\|_{\mathrm{Y},h} := \mathrm{c}_h(\cdot,\cdot)^{1/2}$ and $\underline{\mathbf{Z}}_h^{k+1}$ with the seminorm

$$\|(\underline{\mathbf{w}}_h, \underline{\mathrm{r}}_h)\|_{\mathbf{Z},h} := \left(\|\underline{\mathbf{w}}_h\|_{\mathbf{X},h}^2 + \|\underline{\mathrm{r}}_h\|_{\mathrm{Y},h}^2\right)^{1/2}. \tag{41}$$

Lemma 11 (Norm $\|\cdot\|_{\mathbf{Z},h}$)**.** *The map $\|\cdot\|_{\mathbf{Z},h}$ defines a norm on $\underline{\mathbb{Z}}_{h,\mathbf{0}}^{k+1}$.*

Proof. The seminorm property being evident, it suffices to prove that, for all $(\underline{\mathbf{w}}_h, \underline{\mathrm{r}}_h) \in \underline{\mathbb{Z}}_{h,\mathbf{0}}^{k+1}$, $\|(\underline{\mathbf{w}}_h, \underline{\mathrm{r}}_h)\|_{\mathbf{Z},h} = 0$ implies $(\underline{\mathbf{w}}_h, \underline{\mathrm{r}}_h) = (\underline{\mathbf{0}}_h, \underline{0}_h)$. Let $(\underline{\mathbf{w}}_h, \underline{\mathrm{r}}_h) \in \underline{\mathbb{Z}}_{h,\mathbf{0}}^{k+1}$ be such that $\|(\underline{\mathbf{w}}_h, \underline{\mathrm{r}}_h)\|_{\mathbf{Z},h} = 0$. Then, $\|\underline{\mathbf{w}}_h\|_{\mathbf{X},h} = 0$ and $\|\underline{\mathrm{r}}_h\|_{\mathrm{Y},h} = 0$. Since $\|\cdot\|_{\mathrm{Y},h}$ is a norm on $\underline{\mathrm{Y}}_h^{k+1}$, the second relation implies $\underline{\mathrm{r}}_h = \underline{0}_h$. Now, owing to definitions (40), (35b), and (18), since $(\underline{\mathbf{w}}_h, \underline{\mathrm{r}}_h) \in \underline{\mathbb{Z}}_{h,\mathbf{0}}^{k+1}$ and $\underline{\mathrm{r}}_h = \underline{0}_h$, we have $\underline{\mathbf{w}}_h \in \underline{\mathbb{X}}_{h,\mathbf{0}}^{k+1}$. By Corollary 5, this implies $\underline{\mathbf{w}}_h = \underline{\mathbf{0}}_h$. □

Lemma 12 (Well-posedness)**.** *For all $\underline{\mathbf{z}}_h \in \underline{\mathbf{Z}}_h^{k+1}$,*

$$\|\underline{\mathbf{z}}_h\|_{\mathbf{Z},h}^2 = \mathrm{A}_h(\underline{\mathbf{z}}_h, \underline{\mathbf{z}}_h). \tag{42}$$

Hence, Problem (37) is well-posed, and the following a priori bound holds:

$$\|(\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h)\|_{\mathbf{Z},h} \le \|\boldsymbol{f}\|_\Omega. \tag{43}$$

Proof. The identity (42) is a direct consequence of (39) and (41) along with the definitions of $\|\cdot\|_{\mathbf{X},h}$ and $\|\cdot\|_{\mathrm{Y},h}$. To prove well-posedness, since the system associated to Problem (37) is square, it is sufficient to prove injectivity. Assume that $\mathrm{A}_h\big((\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h), (\underline{\mathbf{v}}_h, \underline{\mathrm{q}}_h)\big) = 0$ for all $(\underline{\mathbf{v}}_h, \underline{\mathrm{q}}_h) \in \underline{\mathbf{Z}}_{h,\mathbf{0}}^{k+1}$. Taking $(\underline{\mathbf{v}}_h, \underline{\mathrm{q}}_h) = (\underline{\mathbf{0}}_h, \underline{\mathrm{q}}_h)$ and using (40), we first infer that $(\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h) \in \underline{\mathbb{Z}}_{h,\mathbf{0}}^{k+1}$. Taking $(\underline{\mathbf{v}}_h, \underline{\mathrm{q}}_h) = (\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h)$ and using (42), we then get

$$\|(\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h)\|_{\mathbf{Z},h}^2 = \mathrm{A}_h\big((\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h), (\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h)\big) = 0,$$

which, by Lemma 11, eventually yields $(\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h) = (\underline{\mathbf{0}}_h, \underline{0}_h)$. The a priori bound (43) directly follows from (42) with $\underline{\mathbf{z}}_h = (\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h)$, (38), the Cauchy–Schwarz inequality, and the fact that $\|\underline{\mathbf{u}}_h\|_{\mathbf{X},h} \le \|(\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h)\|_{\mathbf{Z},h}$. □

3.1.4 Error analysis

We recall that $(\boldsymbol{u}, p) \in \boldsymbol{H}_{\mathbf{0}}(\mathbf{curl};\Omega) \times H_0^1(\Omega)$ denotes the unique solution to Problem (33). We assume that $\boldsymbol{u}$ possesses the additional regularity $\boldsymbol{u} \in H^1(\Omega;\mathbb{R}^3)$, and we let $\widehat{\underline{\mathbf{u}}}_h := \underline{\mathbf{I}}_{\mathbf{X},h}^{k+1}\boldsymbol{u} \in \underline{\mathbf{X}}_{h,\mathbf{0}}^{k+1}$ and $\widehat{\underline{\mathrm{p}}}_h := \underline{\mathrm{I}}_{\mathrm{Y},h}^{k+1}p \in \underline{\mathrm{Y}}_{h,0}^{k+1}$. In the spirit of [13] (see also [15, Appendix A]), we estimate the errors

$$\underline{\mathbf{X}}_{h,\mathbf{0}}^{k+1} \ni \underline{\mathbf{e}}_h := \underline{\mathbf{u}}_h - \widehat{\underline{\mathbf{u}}}_h, \qquad \underline{\mathrm{Y}}_{h,0}^{k+1} \ni \underline{\epsilon}_h := \underline{\mathrm{p}}_h - \widehat{\underline{\mathrm{p}}}_h, \tag{44}$$

where $(\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h) \in \underline{\mathbf{X}}_{h,\mathbf{0}}^{k+1} \times \underline{\mathrm{Y}}_{h,0}^{k+1}$ is the unique solution to Problem (37). Remark that, since $p = 0$ in Ω, we actually have $\widehat{\underline{\mathrm{p}}}_h = \underline{0}_h$ and $\underline{\epsilon}_h = \underline{\mathrm{p}}_h$. Recalling (38) and (39), the errors $(\underline{\mathbf{e}}_h, \underline{\epsilon}_h) \in \underline{\mathbf{Z}}_{h,\mathbf{0}}^{k+1}$ solve

$$\mathrm{A}_h\big((\underline{\mathbf{e}}_h, \underline{\epsilon}_h), (\underline{\mathbf{v}}_h, \underline{\mathrm{q}}_h)\big) = \mathrm{l}_h(\underline{\mathbf{v}}_h) + \mathrm{m}_h(\underline{\mathrm{q}}_h) \qquad \forall (\underline{\mathbf{v}}_h, \underline{\mathrm{q}}_h) \in \underline{\mathbf{Z}}_{h,\mathbf{0}}^{k+1}, \tag{45}$$

where we have defined the consistency errors

$$\mathrm{l}_h(\underline{\mathbf{v}}_h) := (\boldsymbol{f}, \mathbf{curl}_h \mathbf{v}_h)_\Omega - \mathrm{a}_h(\widehat{\underline{\mathbf{u}}}_h, \underline{\mathbf{v}}_h), \tag{46a}$$

$$\mathrm{m}_h(\underline{\mathrm{q}}_h) := \mathrm{b}_h(\widehat{\underline{\mathbf{u}}}_h, \underline{\mathrm{q}}_h). \tag{46b}$$

Theorem 13 (Energy-error estimate)**.** *Assume that*

$$\boldsymbol{u} \in \boldsymbol{H}_{\boldsymbol{0}}(\mathbf{curl};\Omega) \cap H^1(\Omega;\mathbb{R}^3) \cap H^{k+2}(\mathcal{T}_h;\mathbb{R}^3).$$

Then, there holds, with $(\underline{\mathbf{e}}_h, \underline{\epsilon}_h) \in \underline{\mathbf{Z}}_{h,\mathbf{0}}^{k+1}$ *defined by* (44)*,*

$$\|(\underline{\mathbf{e}}_h, \underline{\epsilon}_h)\|_{\mathbf{Z},h} \lesssim \left(\sum_{T\in\mathcal{T}_h} h_T^{2(k+1)} |\boldsymbol{u}_{|T}|^2_{H^{k+2}(T;\mathbb{R}^3)}\right)^{1/2}. \tag{47}$$

Proof. Since $(\underline{\mathbf{e}}_h, \underline{\epsilon}_h) \in \underline{\mathbf{Z}}_{h,\mathbf{0}}^{k+1}$, by (42) with $\underline{\mathbf{z}}_h = (\underline{\mathbf{e}}_h, \underline{\epsilon}_h)$ and (45), we infer

$$\|(\underline{\mathbf{e}}_h, \underline{\epsilon}_h)\|_{\mathbf{Z},h} \le \max_{(\underline{\mathbf{v}}_h,\underline{\mathrm{q}}_h)\in\underline{\mathbf{Z}}_{h,\mathbf{0}}^{k+1},\, \|(\underline{\mathbf{v}}_h,\underline{\mathrm{q}}_h)\|_{\mathbf{Z},h}=1} \big(\mathrm{l}_h(\underline{\mathbf{v}}_h) + \mathrm{m}_h(\underline{\mathrm{q}}_h)\big). \tag{48}$$

Let us first focus on $\mathrm{l}_h(\underline{\mathbf{v}}_h)$ for $\underline{\mathbf{v}}_h \in \underline{\mathbf{X}}_{h,\mathbf{0}}^{k+1}$. By the definition (46a) of l_h, the fact that $\boldsymbol{f} = \mathbf{curl}\,\boldsymbol{u}$ in Ω, and the definition (7a) of $\underline{\mathbf{I}}_{\mathbf{X},h}^{k+1}\boldsymbol{u}$, there holds

$$\begin{aligned}
|\mathrm{l}_h(\underline{\mathbf{v}}_h)| &= \left|\big(\mathbf{curl}\,\boldsymbol{u} - \mathbf{curl}_h(\boldsymbol{\pi}_h^{k+1}\boldsymbol{u}), \mathbf{curl}_h\mathbf{v}_h\big)_\Omega - \mathrm{s}_h(\widehat{\underline{\mathbf{u}}}_h, \underline{\mathbf{v}}_h)\right| \\
&\le \Big(\|\mathbf{curl}\,\boldsymbol{u} - \mathbf{curl}_h(\boldsymbol{\pi}_h^{k+1}\boldsymbol{u})\|_\Omega^2 + \mathrm{s}_h(\widehat{\underline{\mathbf{u}}}_h, \widehat{\underline{\mathbf{u}}}_h)\Big)^{1/2} \|\underline{\mathbf{v}}_h\|_{\mathbf{X},h},
\end{aligned}$$

where we have used the triangle/Cauchy–Schwarz inequalities and the definition (17) of $\|\underline{\mathbf{v}}_h\|_{\mathbf{X},h}$ to pass to the second line. The quantity $\|\mathbf{curl}\,\boldsymbol{u} - \mathbf{curl}_h(\boldsymbol{\pi}_h^{k+1}\boldsymbol{u})\|_\Omega^2$ is estimated using the approximation properties of $\boldsymbol{\pi}_h^{k+1}$ (see, e.g., [15, Theorem 1.45]). For the quantity $\mathrm{s}_h(\widehat{\underline{\mathbf{u}}}_h, \widehat{\underline{\mathbf{u}}}_h)$, recalling the definition (36) of s_h and using the $L^2(F;\mathbb{R}^2)$-boundedness of $\boldsymbol{\pi}_{\mathcal{G},F}^{k+1}$, we have

$$\begin{aligned}
\mathrm{s}_h(\widehat{\underline{\mathbf{u}}}_h, \widehat{\underline{\mathbf{u}}}_h) &= \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T} h_F^{-1} \|\boldsymbol{\pi}_{\mathcal{G},F}^{k+1}\big(\boldsymbol{\gamma}_{\tau,F}(\boldsymbol{\pi}_T^{k+1}(\boldsymbol{u}_{|T}) - \boldsymbol{u})\big)\|_F^2 \\
&\le \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T} h_F^{-1} \|(\boldsymbol{\pi}_T^{k+1}(\boldsymbol{u}_{|T}) - \boldsymbol{u})_{|F}\|_F^2 \lesssim \sum_{T\in\mathcal{T}_h} h_T^{2(k+1)} |\boldsymbol{u}_{|T}|^2_{H^{k+2}(T;\mathbb{R}^3)},
\end{aligned} \tag{49}$$

where, for all $T \in \mathcal{T}_h$, we have used the approximation properties of $\boldsymbol{\pi}_T^{k+1}$ on the faces of T. Gathering the different estimates, we get

$$|\mathrm{l}_h(\underline{\mathbf{v}}_h)| \lesssim \left(\sum_{T\in\mathcal{T}_h} h_T^{2(k+1)} |\boldsymbol{u}_{|T}|^2_{H^{k+2}(T;\mathbb{R}^3)}\right)^{1/2} \|\underline{\mathbf{v}}_h\|_{\mathbf{X},h}. \tag{50}$$

Let us now focus on $\mathrm{m}_h(\underline{\mathrm{q}}_h)$ for $\underline{\mathrm{q}}_h \in \underline{\mathrm{Y}}_{h,0}^{k+1}$. Starting from (46b), performing an element-by-element integration by parts in (8), and using the fact that $\mathbf{grad}\,\mathrm{q}_T \in \boldsymbol{\mathcal{G}}_T^{k-1} \subset \mathbb{P}^{k+1}(T;\mathbb{R}^3)$, we infer

$$\begin{aligned}
\mathrm{m}_h(\underline{\mathrm{q}}_h) &= \sum_{T\in\mathcal{T}_h}\left((\mathbf{grad}\,\mathrm{q}_T, \boldsymbol{u}_{|T})_T + \sum_{F\in\mathcal{F}_T} \big(\boldsymbol{\pi}_T^{k+1}(\boldsymbol{u}_{|T})_{|F}\cdot\boldsymbol{n}_{T,F}, \mathrm{q}_F - \mathrm{q}_{T|F}\big)_F\right) \\
&= \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T} \big((\boldsymbol{\pi}_T^{k+1}(\boldsymbol{u}_{|T}) - \boldsymbol{u})_{|F}\cdot\boldsymbol{n}_{T,F}, \mathrm{q}_F - \mathrm{q}_{T|F}\big)_F,
\end{aligned}$$

where the last identity follows from another element-by-element integration by parts, and from the fact that $\operatorname{div}\boldsymbol{u} = 0$ in Ω, and that $\boldsymbol{u} \in H^1(\Omega;\mathbb{R}^3)$ along with $\mathrm{q}_F = 0$ for all $F \in \mathcal{F}_h^{\mathrm{b}}$. By the triangle

and Cauchy–Schwarz inequalities, one then gets

$$|\mathrm{m}_h(\underline{\mathrm{q}}_h)| \le \left(\sum_{T\in\mathcal{T}_h} \sum_{F\in\mathcal{F}_T} h_F^{-1} \|(\boldsymbol{\pi}_T^{k+1}(\boldsymbol{u}_{|T}) - \boldsymbol{u})_{|F}\|_F^2 \right)^{1/2} \times \left(\sum_{T\in\mathcal{T}_h} \sum_{F\in\mathcal{F}_T} h_F \|\mathrm{q}_F - \mathrm{q}_{T|F}\|_F^2 \right)^{1/2}. \quad (51)$$

Using, for all $T \in \mathcal{T}_h$, the approximation properties of $\boldsymbol{\pi}_T^{k+1}$ on the faces of T for the first factor on the right-hand side, and the triangle inequality along with a discrete trace inequality (see, e.g., [15, Lemma 1.32]) for the second factor, we infer

$$|\mathrm{m}_h(\underline{\mathrm{q}}_h)| \lesssim \left(\sum_{T\in\mathcal{T}_h} h_T^{2(k+1)} |\boldsymbol{u}_{|T}|^2_{H^{k+2}(T;\mathbb{R}^3)} \right)^{1/2} \|\underline{\mathrm{q}}_h\|_{\mathrm{Y},h}. \quad (52)$$

Plugging (50) and (52) into (48) for $(\underline{\mathbf{v}}_h, \underline{\mathrm{q}}_h)$ such that $\|(\underline{\mathbf{v}}_h, \underline{\mathrm{q}}_h)\|_{\mathbf{Z},h} = 1$, finally yields (47). □

3.1.5 Numerical results

Let the domain Ω be the unit cube $(0,1)^3$. We consider Problem (32) with exact solution

$$\boldsymbol{u}(x_1,x_2,x_3) := \begin{pmatrix} \cos(\pi x_2)\cos(\pi x_3) \\ \cos(\pi x_1)\cos(\pi x_3) \\ \cos(\pi x_1)\cos(\pi x_2) \end{pmatrix}. \quad (53)$$

Clearly, the magnetic field $\boldsymbol{u}$ satisfies (32b). The current density $\boldsymbol{f}$ is set according to (32a), and the zero tangential boundary condition (32c) is replaced by the nonhomogeneous boundary condition stemming from (53).

We solve the discrete Problem (37) with amended right-hand side accounting for the nonhomogeneous boundary condition on two refined mesh sequences, of respectively cubic and regular tetrahedral meshes. For each problem, the element unknowns for both the magnetic field and the pressure are locally eliminated using a Schur complement technique. This step is fully parallelizable. The resulting (condensed) global linear system is solved using the SparseLU direct solver of the Eigen library, on an Intel Xeon E-2176M 2.70GHz×12 with 16Go of RAM (and up to 150Go of swap). For $k \in \{0,1,2\}$, we depict on Figures 1 and 2, repectively for the cubic and (regular) tetrahedral mesh families, the relative energy-error $\|\underline{\mathbf{u}}_h - \underline{\mathbf{I}}_{\mathbf{X},h}^{k+1}\boldsymbol{u}\|_{\mathbf{X},h}/\|\underline{\mathbf{I}}_{\mathbf{X},h}^{k+1}\boldsymbol{u}\|_{\mathbf{X},h}$ (first line) and L^2-error $\|\mathbf{u}_h - \boldsymbol{\pi}_h^{k+1}\boldsymbol{u}\|_\Omega/\|\boldsymbol{\pi}_h^{k+1}\boldsymbol{u}\|_\Omega$ (second line) as functions of (i) the meshsize (left column), (ii) the solution time in seconds, i.e. the time needed to solve the (condensed) global linear system (middle column), and (iii) the number of (interface) degrees of freedom (DoF) (right column). For the two mesh families, we obtain, as predicted by Theorem 13, an energy-error convergence rate of order $k+1$. We also observe a convergence rate of order $k+2$ for the L^2-error. We remark that, whenever the solution is smooth enough (at least locally), raising the polynomial degree is computationally much more efficient than refining the mesh to increase the accuracy.

3.2 Vector potential formulation

3.2.1 The model

The (second-order) vector potential formulation of the magnetostatic problem consists, in its generalized form, in finding the magnetic vector potential $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ and the pressure $p : \Omega \to \mathbb{R}$ such

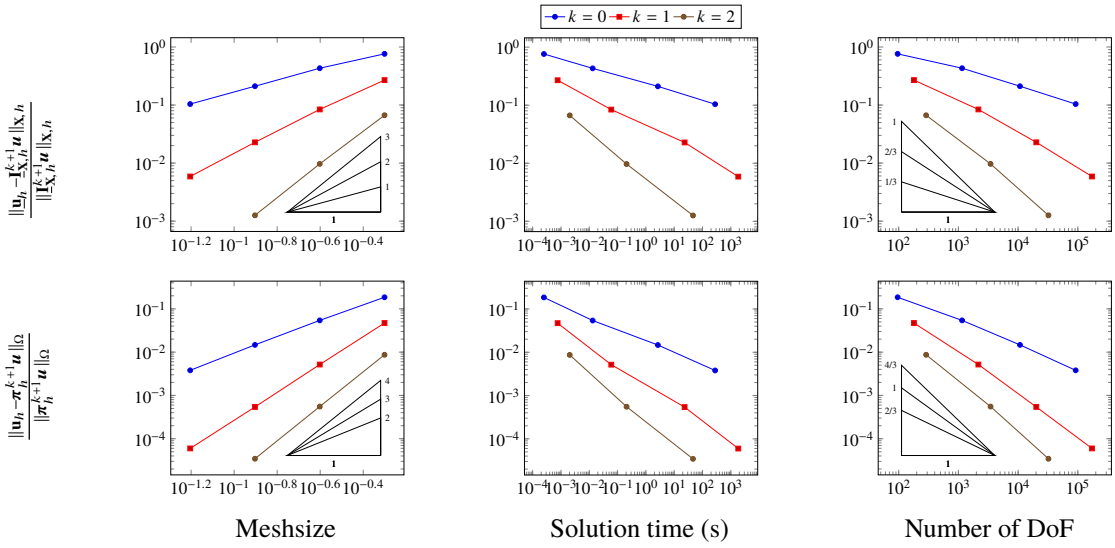

Figure 1: Errors vs. h, solution time and number of DoF on cubic meshes.

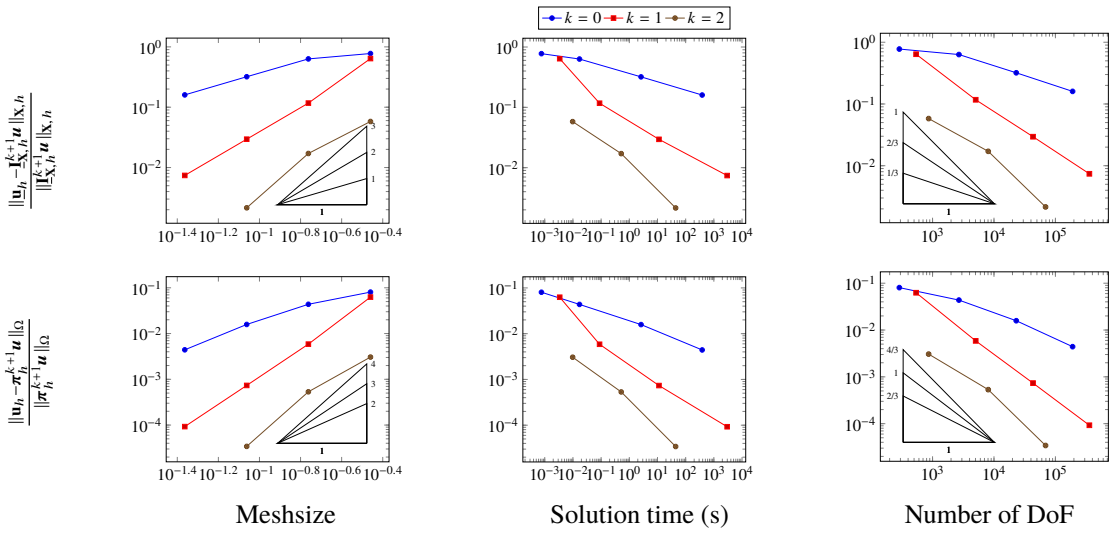

Figure 2: Errors vs. h, solution time and number of DoF on tetrahedral meshes.

that

$$\begin{aligned}
\mathbf{curl}\big(\mathbf{curl}\,\boldsymbol{u}\big) + \mathbf{grad}\,p &= \boldsymbol{f} && \text{in } \Omega, && \text{(54a)}\\
\operatorname{div}\boldsymbol{u} &= 0 && \text{in } \Omega, && \text{(54b)}\\
\boldsymbol{n}\times(\boldsymbol{u}\times\boldsymbol{n}) &= \boldsymbol{0} && \text{on } \partial\Omega, && \text{(54c)}\\
p &= 0 && \text{on } \partial\Omega, && \text{(54d)}
\end{aligned}$$

where the current density $\boldsymbol{f} : \Omega \to \mathbb{R}^3$ is no longer assumed to be divergence-free, whence the introduction of the pressure term in (54a). When $\operatorname{div}\boldsymbol{f} = 0$ in Ω, $p = 0$ in Ω and, letting $\boldsymbol{b} := \mathbf{curl}\,\boldsymbol{u}$, Problem (54) is then equivalent, in the class of vector potentials satisfying the Coulomb gauge, to the following problem (cf. Remark 7):

$$\mathbf{curl}\,\boldsymbol{b} = \boldsymbol{f} \quad \text{in } \Omega, \qquad \operatorname{div}\boldsymbol{b} = 0 \quad \text{in } \Omega, \qquad \boldsymbol{b}\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\Omega. \tag{55}$$

The field $\boldsymbol{u}$ is then the vector potential associated to the magnetic induction $\boldsymbol{b}$. Assuming $\boldsymbol{f} \in L^2(\Omega;\mathbb{R}^3)$, we consider the following equivalent weak formulation of Problem (54): Find $(\boldsymbol{u}, p) \in \boldsymbol{H_0}(\mathbf{curl};\Omega)\times H^1_0(\Omega)$ s.t.

$$\begin{aligned}
a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) &= (\boldsymbol{f},\boldsymbol{v})_\Omega && \forall \boldsymbol{v} \in \boldsymbol{H_0}(\mathbf{curl};\Omega), && \text{(56a)}\\
b(\boldsymbol{u},q) &= 0 && \forall q \in H^1_0(\Omega), && \text{(56b)}
\end{aligned}$$

where the bilinear forms $a : \boldsymbol{H}(\mathbf{curl};\Omega)\times\boldsymbol{H}(\mathbf{curl};\Omega) \to \mathbb{R}$ and $b : \boldsymbol{H}(\mathbf{curl};\Omega)\times H^1(\Omega) \to \mathbb{R}$ are defined in (34). Using the decomposition (2), Problem (56) can be equivalently rewritten under the following fully decoupled form: Find $\boldsymbol{u} \in \boldsymbol{H_0}(\mathbf{curl};\Omega)\cap\boldsymbol{H}(\operatorname{div}^0;\Omega)$ and $p \in H^1_0(\Omega)$ s.t.

$$\begin{aligned}
a(\boldsymbol{u},\boldsymbol{\eta}) &= (\boldsymbol{f},\boldsymbol{\eta})_\Omega && \forall\, \boldsymbol{\eta} \in \boldsymbol{H_0}(\mathbf{curl};\Omega)\cap\boldsymbol{H}(\operatorname{div}^0;\Omega),\\
b(\mathbf{grad}\,\xi,p) &= (\boldsymbol{f},\mathbf{grad}\,\xi)_\Omega && \forall\, \xi \in H^1_0(\Omega),
\end{aligned}$$

whose well-posedness directly follows from the first Weber inequality (3) and from the Lax–Milgram lemma.

Remark 14 (Divergence-free current density)**.** *When* $\operatorname{div}\boldsymbol{f} = 0$ *in* Ω*, since it implies* $p \equiv 0$*, one can consider adding to Problem* (56) *the bilinear contribution* c *defined in* (34)*. Mirroring this strategy at the discrete level improves the stability of the method without jeopardizing its convergence properties (cf. Remark 9).*

3.2.2 The HHO method

We consider the hybrid spaces introduced in Section 2.2, up to a slight modification of the space $\underline{\mathbf{X}}_h^{k+1}$ defined in (5a). To this end, we introduce, for any $F \in \mathcal{F}_h$ and any $q \in \mathbb{N}$, the space

$$\boldsymbol{\mathcal{P}}_F^{q+1} := \mathbb{P}^q(F;\mathbb{R}^2) \oplus \mathbf{grad}_\tau\big(\widetilde{\mathbb{P}}^{q+2}(F)\big),$$

with $\widetilde{\mathbb{P}}^{q+2}(F)$ denoting the space of homogeneous polynomials of degree $q+2$ on F. Consistently with our notation so far, we let $\boldsymbol{\pi}_{\boldsymbol{\mathcal{P}},F}^{q+1}$ denote the $L^2(F;\mathbb{R}^2)$-orthogonal projector onto $\boldsymbol{\mathcal{P}}_F^{q+1}$. With this new space at hand, we define

$$\underline{\mathbf{X}}_{\sharp,h}^{k+1} := \left\{ \underline{\mathbf{v}}_h = \big((\mathbf{v}_T)_{T\in\mathcal{T}_h},(\mathbf{v}_F)_{F\in\mathcal{F}_h}\big) \;:\; \begin{aligned} &\mathbf{v}_T \in \mathbb{P}^{k+1}(T;\mathbb{R}^3) && \forall T \in \mathcal{T}_h\\ &\mathbf{v}_F \in \boldsymbol{\mathcal{P}}_F^{k+1} && \forall F \in \mathcal{F}_h \end{aligned} \right\}, \tag{58}$$

that is from now on meant to replace the space $\underline{\mathbf{X}}_h^{k+1}$. The space $\underline{\mathrm{Y}}_h^{k+1}$ keeps the same definition (5b). We also introduce the spaces $\underline{\mathbf{X}}_{\sharp,h,\mathbf{0}}^{k+1}$ and $\underline{\mathbb{X}}_{\sharp,h,\mathbf{0}}^{k+1}$, that are respectively obtained through definitions (6) and (18), up to the replacement therein of $\underline{\mathbf{X}}_h^{k+1}$ by $\underline{\mathbf{X}}_{\sharp,h}^{k+1}$, and of $\underline{\mathbf{X}}_{h,\mathbf{0}}^{k+1}$ by $\underline{\mathbf{X}}_{\sharp,h,\mathbf{0}}^{k+1}$. Finally, we define the interpolator $\underline{\mathbf{I}}_{\mathbf{X},\sharp,h}^{k+1} : H^1(\Omega;\mathbb{R}^3) \to \underline{\mathbf{X}}_{\sharp,h}^{k+1}$ as in formula (7a), up to the replacement of the projector $\boldsymbol{\pi}_{\mathcal{G},F}^{k+1}$ by $\boldsymbol{\pi}_{\mathcal{P},F}^{k+1}$.

Remark 15 (Validity of the results of Section 2.4)**.** *For all $F \in \mathcal{F}_h$, there holds*

$$\boldsymbol{\mathcal{G}}_F^{k+1} \subset \boldsymbol{\mathcal{P}}_F^{k+1} \subset \mathbb{P}^{k+1}(F;\mathbb{R}^2).$$

It can be checked that, up to the replacement of the orthogonal projector $\boldsymbol{\pi}_{\mathcal{G},F}^{k+1}$ onto $\boldsymbol{\mathcal{G}}_F^{k+1}$ by the orthogonal projector $\boldsymbol{\pi}_{\mathcal{P},F}^{k+1}$ onto $\boldsymbol{\mathcal{P}}_F^{k+1}$, all the results in Section 2.4 remain valid when $\underline{\mathbf{X}}_h^{k+1}$ is replaced by $\underline{\mathbf{X}}_{\sharp,h}^{k+1}$ as defined in (58), including the discrete Weber inequality of Theorem 3 and its generalization of Corollary 6 (observe that the crucial estimates (25)–(26) still hold true under these changes).

We define the bilinear forms $\mathrm{a}_h : \underline{\mathbf{X}}_{\sharp,h}^{k+1} \times \underline{\mathbf{X}}_{\sharp,h}^{k+1} \to \mathbb{R}$, $\mathrm{b}_h : \underline{\mathbf{X}}_{\sharp,h}^{k+1} \times \underline{\mathrm{Y}}_h^{k+1} \to \mathbb{R}$, and $\mathrm{d}_h : \underline{\mathrm{Y}}_h^{k+1} \times \underline{\mathrm{Y}}_h^{k+1} \to \mathbb{R}$ such that

$$\mathrm{a}_h(\underline{\mathbf{w}}_h, \underline{\mathbf{v}}_h) := \big(\boldsymbol{C}_h^k \underline{\mathbf{w}}_h, \boldsymbol{C}_h^k \underline{\mathbf{v}}_h\big)_\Omega + \mathrm{s}_h(\underline{\mathbf{w}}_h, \underline{\mathbf{v}}_h), \tag{59a}$$

$$\mathrm{b}_h(\underline{\mathbf{w}}_h, \underline{\mathrm{q}}_h) := \big(\mathbf{w}_h, \boldsymbol{G}_h^{k+1} \underline{\mathrm{q}}_h\big)_\Omega, \tag{59b}$$

$$\mathrm{d}_h(\underline{\mathrm{r}}_h, \underline{\mathrm{q}}_h) := \sum_{T\in\mathcal{T}_h} \sum_{F\in\mathcal{F}_T} h_F \big(\mathrm{r}_F - \mathrm{r}_{T|F}, \mathrm{q}_F - \mathrm{q}_{T|F}\big)_F, \tag{59c}$$

where $\boldsymbol{G}_h^{k+1} : \underline{\mathrm{Y}}_h^{k+1} \to \mathbb{P}^{k+1}(\mathcal{T}_h;\mathbb{R}^3)$ is the gradient reconstruction operator introduced in Section 2.3 and $\mathrm{s}_h : \underline{\mathbf{X}}_{\sharp,h}^{k+1} \times \underline{\mathbf{X}}_{\sharp,h}^{k+1} \to \mathbb{R}$ is the stabilization bilinear form such that

$$\mathrm{s}_h(\underline{\mathbf{w}}_h, \underline{\mathbf{v}}_h) := \sum_{T\in\mathcal{T}_h} \sum_{F\in\mathcal{F}_T} h_F^{-1} \big(\boldsymbol{\pi}_{\mathcal{P},F}^{k+1}\big(\boldsymbol{\gamma}_{\tau,F}(\mathbf{w}_T) - \mathbf{w}_F\big), \boldsymbol{\pi}_{\mathcal{P},F}^{k+1}\big(\boldsymbol{\gamma}_{\tau,F}(\mathbf{v}_T) - \mathbf{v}_F\big)\big)_F. \tag{60}$$

In (59a), $\boldsymbol{C}_h^k : \underline{\mathbf{X}}_{\sharp,h}^{k+1} \to \boldsymbol{C}_{\mathcal{T}_h}^k$ (with $\boldsymbol{C}_{\mathcal{T}_h}^k$ defined in Section 2.2) is the global discrete curl reconstruction operator such that its local restriction $\boldsymbol{C}_T^k : \underline{\mathbf{X}}_{\sharp,T}^{k+1} \to \boldsymbol{C}_T^k$ to any $T \in \mathcal{T}_h$ solves the following well-posed problem: For all $\underline{\mathbf{v}}_T \in \underline{\mathbf{X}}_{\sharp,T}^{k+1}$,

$$\big(\boldsymbol{C}_T^k \underline{\mathbf{v}}_T, \boldsymbol{w}\big)_T = (\mathbf{v}_T, \mathbf{curl}\, \boldsymbol{w})_T + \sum_{F\in\mathcal{F}_T} \big(\mathbf{v}_F, \boldsymbol{\gamma}_{\tau,F}(\boldsymbol{w}\times\boldsymbol{n}_{T,F})\big)_F \quad \forall \boldsymbol{w} \in \boldsymbol{C}_T^k. \tag{61}$$

With this definition at hand, one can prove the following commutation property.

Lemma 16 (Commutation property)**.** *For all $\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^3)$, it holds*

$$\boldsymbol{C}_h^k \underline{\mathbf{I}}_{\mathbf{X},\sharp,h}^{k+1} \boldsymbol{v} = \boldsymbol{\pi}_{\boldsymbol{C},h}^k(\mathbf{curl}\, \boldsymbol{v}), \tag{62}$$

where we remind the reader that $\boldsymbol{\pi}_{\boldsymbol{C},h}^k$ is the $L^2(\Omega;\mathbb{R}^3)$-orthogonal projector onto $\boldsymbol{C}_{\mathcal{T}_h}^k$.

Proof. For $\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^3)$, let $\underline{\mathbf{v}}_h := \underline{\mathbf{I}}_{\mathbf{X},\sharp,h}^{k+1} \boldsymbol{v}$. Then, for any $T \in \mathcal{T}_h$,

$$\underline{\mathbf{v}}_T = \Big(\boldsymbol{\pi}_T^{k+1}(\boldsymbol{v}_{|T}), \big(\boldsymbol{\pi}_{\mathcal{P},F}^{k+1}\big(\boldsymbol{\gamma}_{\tau,F}(\boldsymbol{v})\big)\big)_{F\in\mathcal{F}_T}\Big).$$

Plugging $\underline{\mathbf{v}}_T$ into (61) and using the fact that $\mathbf{curl}\, \boldsymbol{w} \in \mathbb{P}^{k-1}(T;\mathbb{R}^3) \subset \mathbb{P}^{k+1}(T;\mathbb{R}^3)$ and $\boldsymbol{\gamma}_{\tau,F}(\boldsymbol{w}\times\boldsymbol{n}_{T,F}) \in \mathbb{P}^k(F;\mathbb{R}^2) \subset \boldsymbol{\mathcal{P}}_F^{k+1}$ for all $F \in \mathcal{F}_T$ to remove the projectors, one gets, for all $\boldsymbol{w} \in \boldsymbol{C}_T^k$,

$$\big(\boldsymbol{C}_T^k \underline{\mathbf{v}}_T, \boldsymbol{w}\big)_T = (\boldsymbol{v}_{|T}, \mathbf{curl}\, \boldsymbol{w})_T + \sum_{F\in\mathcal{F}_T} \big(\boldsymbol{v}_{|F}, \boldsymbol{w}_{|F}\times\boldsymbol{n}_{T,F}\big)_F. \tag{63}$$

Integrating by parts the right-hand side of (63), we readily infer the result. □

Remark 17 (Variant on $\boldsymbol{C}_h^k$)**.** *An alternative choice consists in reconstructing the discrete curl in* $\mathbb{P}^k(\mathcal{T}_h;\mathbb{R}^3)$*, which requires to solve larger local problems for* $k \geq 1$ *(for example,* $\boldsymbol{C}_T^2$ *has dimension* 26*, whereas* $\mathbb{P}^2(T;\mathbb{R}^3)$ *has dimension* 30*). In this case, the commutation property* (62) *reads* $\boldsymbol{C}_h^k \underline{\mathbf{I}}_{\mathbf{X},\sharp,h}^{k+1} \boldsymbol{v} = \boldsymbol{\pi}_h^k(\mathbf{curl}\, \boldsymbol{v})$*. This is the approach pursued in the HDG literature [26, 10]. The numerical tests we have performed (not reported here) indicate that, interestingly, reconstructing the discrete curl in* $\mathbb{P}^k(\mathcal{T}_h;\mathbb{R}^3)$ *instead of* $\boldsymbol{C}_{\mathcal{T}_h}^k$*, besides being computationally more expensive, in addition often deteriorates the accuracy of the approximation.*

The HHO method for Problem (56) reads: Find $(\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h) \in \underline{\mathbf{X}}_{\sharp,h,\mathbf{0}}^{k+1} \times \underline{\mathrm{Y}}_{h,0}^{k+1}$ s.t.

$$\mathrm{a}_h(\underline{\mathbf{u}}_h, \underline{\mathbf{v}}_h) + \mathrm{b}_h(\underline{\mathbf{v}}_h, \underline{\mathrm{p}}_h) = (\boldsymbol{f}, \mathbf{v}_h)_\Omega \qquad \forall \underline{\mathbf{v}}_h \in \underline{\mathbf{X}}_{\sharp,h,\mathbf{0}}^{k+1}, \tag{64a}$$

$$-\mathrm{b}_h(\underline{\mathbf{u}}_h, \underline{\mathrm{q}}_h) + \mathrm{d}_h(\underline{\mathrm{p}}_h, \underline{\mathrm{q}}_h) = 0 \qquad \forall \underline{\mathrm{q}}_h \in \underline{\mathrm{Y}}_{h,0}^{k+1}. \tag{64b}$$

Note that, as opposed to $\boldsymbol{u}$ and p in Problem (56), one cannot efficiently solve for $\underline{\mathbf{u}}_h$ and $\underline{\mathrm{p}}_h$ independently in Problem (64) as the **curl-grad** orthogonality is lost at the discrete level.

Problem (64) can be equivalently rewritten: Find $(\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h) \in \underline{\mathbf{Z}}_{\sharp,h,\mathbf{0}}^{k+1} := \underline{\mathbf{X}}_{\sharp,h,\mathbf{0}}^{k+1} \times \underline{\mathrm{Y}}_{h,0}^{k+1}$ s.t.

$$\mathrm{A}_h\big((\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h), (\underline{\mathbf{v}}_h, \underline{\mathrm{q}}_h)\big) = (\boldsymbol{f}, \mathbf{v}_h)_\Omega \qquad \forall\, (\underline{\mathbf{v}}_h, \underline{\mathrm{q}}_h) \in \underline{\mathbf{Z}}_{\sharp,h,\mathbf{0}}^{k+1}, \tag{65}$$

where the bilinear form $\mathrm{A}_h : \underline{\mathbf{Z}}_{\sharp,h}^{k+1} \times \underline{\mathbf{Z}}_{\sharp,h}^{k+1} \to \mathbb{R}$ is defined by

$$\mathrm{A}_h\big((\underline{\mathbf{w}}_h, \underline{\mathrm{r}}_h), (\underline{\mathbf{v}}_h, \underline{\mathrm{q}}_h)\big) := \mathrm{a}_h(\underline{\mathbf{w}}_h, \underline{\mathbf{v}}_h) + \mathrm{b}_h(\underline{\mathbf{v}}_h, \underline{\mathrm{r}}_h) - \mathrm{b}_h(\underline{\mathbf{w}}_h, \underline{\mathrm{q}}_h) + \mathrm{d}_h(\underline{\mathrm{r}}_h, \underline{\mathrm{q}}_h). \tag{66}$$

We also let, in analogy with (40),

$$\begin{aligned}\mathbb{Z}_{\sharp,h,\mathbf{0}}^{k+1} &:= \Big\{(\underline{\mathbf{w}}_h, \underline{\mathrm{r}}_h) \in \underline{\mathbf{Z}}_{\sharp,h,\mathbf{0}}^{k+1} \;:\; -\mathrm{b}_h(\underline{\mathbf{w}}_h, \underline{\mathrm{q}}_h) + \mathrm{d}_h(\underline{\mathrm{r}}_h, \underline{\mathrm{q}}_h) = 0 \;\forall \underline{\mathrm{q}}_h \in \underline{\mathrm{Y}}_{h,0}^{k+1}\Big\} \\ &= \Big\{(\underline{\mathbf{w}}_h, \underline{\mathrm{r}}_h) \in \underline{\mathbf{Z}}_{\sharp,h,\mathbf{0}}^{k+1} \;:\; \mathrm{A}_h\big((\underline{\mathbf{w}}_h, \underline{\mathrm{r}}_h), (\underline{\mathbf{0}}_h, \underline{\mathrm{q}}_h)\big) = 0 \quad \forall \underline{\mathrm{q}}_h \in \underline{\mathrm{Y}}_{h,0}^{k+1}\Big\}.\end{aligned} \tag{67}$$

3.2.3 Stability analysis

We equip $\underline{\mathbf{X}}_{\sharp,h}^{k+1}$ and $\underline{\mathrm{Y}}_h^{k+1}$ with the seminorms

$$\|\underline{\mathbf{w}}_h\|_{\mathbf{X},\sharp,h} := \Big(\|\,\mathbf{curl}_h \mathbf{w}_h\|_\Omega^2 + \mathrm{s}_h(\underline{\mathbf{w}}_h, \underline{\mathbf{w}}_h)\Big)^{1/2}, \tag{68a}$$

$$\|\underline{\mathrm{r}}_h\|_{\mathrm{Y},\sharp,h} := \left(\sum_{T\in\mathcal{T}_h} h_T^2 \|\,\mathbf{grad}\, \mathrm{r}_T\|_T^2 + \mathrm{d}_h(\underline{\mathrm{r}}_h, \underline{\mathrm{r}}_h)\right)^{1/2}. \tag{68b}$$

One can easily verify that $\|\cdot\|_{\mathrm{Y},\sharp,h}$ defines a norm on $\underline{\mathrm{Y}}_{h,0}^{k+1}$. We now equip $\underline{\mathbf{Z}}_{\sharp,h}^{k+1}$ with the seminorm

$$\|(\underline{\mathbf{w}}_h, \underline{\mathrm{r}}_h)\|_{\mathbf{Z},\sharp,h} := \Big(\|\underline{\mathbf{w}}_h\|_{\mathbf{X},\sharp,h}^2 + \|\underline{\mathrm{r}}_h\|_{\mathrm{Y},\sharp,h}^2\Big)^{1/2}. \tag{69}$$

Lemma 18 (Norm $\|\cdot\|_{\mathbf{Z},\sharp,h}$)**.** *The map* $\|\cdot\|_{\mathbf{Z},\sharp,h}$ *defines a norm on* $\underline{\mathbb{Z}}^{k+1}_{\sharp,h,\mathbf{0}}$.

Proof. The seminorm property is evident, so we only need to prove that, for all $(\underline{\mathbf{w}}_h, \underline{\mathrm{r}}_h) \in \underline{\mathbb{Z}}^{k+1}_{\sharp,h,\mathbf{0}}$, $\|(\underline{\mathbf{w}}_h, \underline{\mathrm{r}}_h)\|_{\mathbf{Z},\sharp,h} = 0$ implies $(\underline{\mathbf{w}}_h, \underline{\mathrm{r}}_h) = (\underline{\mathbf{0}}_h, \underline{0}_h)$. Let $(\underline{\mathbf{w}}_h, \underline{\mathrm{r}}_h) \in \underline{\mathbb{Z}}^{k+1}_{\sharp,h,\mathbf{0}}$ be such that $\|(\underline{\mathbf{w}}_h, \underline{\mathrm{r}}_h)\|_{\mathbf{Z},\sharp,h} = 0$. Then, $\|\underline{\mathbf{w}}_h\|_{\mathbf{X},\sharp,h} = 0$ and $\|\underline{\mathrm{r}}_h\|_{\mathrm{Y},\sharp,h} = 0$. Since $\|\cdot\|_{\mathrm{Y},\sharp,h}$ is a norm on $\underline{\mathrm{Y}}^{k+1}_{h,0}$, we directly get from the second relation that $\underline{\mathrm{r}}_h = \underline{0}_h$. Now, owing to the definitions (67) of $\underline{\mathbb{Z}}^{k+1}_{\sharp,h,\mathbf{0}}$, (59b) of b_h, and to the fact that $\underline{\mathbb{X}}^{k+1}_{\sharp,h,\mathbf{0}}$ is defined as in (18) with $\underline{\mathbf{X}}^{k+1}_{h,\mathbf{0}}$ replaced by $\underline{\mathbf{X}}^{k+1}_{\sharp,h,\mathbf{0}}$, we infer from $(\underline{\mathbf{w}}_h, \underline{\mathrm{r}}_h) \in \underline{\mathbb{Z}}^{k+1}_{\sharp,h,\mathbf{0}}$ and $\underline{\mathrm{r}}_h = \underline{0}_h$ that $\underline{\mathbf{w}}_h \in \underline{\mathbb{X}}^{k+1}_{\sharp,h,\mathbf{0}}$. By Corollary 5 and Remark 15, $\|\cdot\|_{\mathbf{X},\sharp,h}$ defines a norm on $\underline{\mathbb{X}}^{k+1}_{\sharp,h,\mathbf{0}}$, hence $\underline{\mathbf{w}}_h = \underline{\mathbf{0}}_h$. □

We now state some preliminary results for the stability analysis of Problem (64).

Lemma 19 (Equivalences of seminorms)**.** *It holds*

$$\|\underline{\mathbf{w}}_h\|^2_{\mathbf{X},\sharp,h} \lesssim \mathrm{a}_h(\underline{\mathbf{w}}_h, \underline{\mathbf{w}}_h) \lesssim \|\underline{\mathbf{w}}_h\|^2_{\mathbf{X},\sharp,h} \qquad \forall \underline{\mathbf{w}}_h \in \underline{\mathbf{X}}^{k+1}_{\sharp,h}, \tag{70a}$$

$$\|\underline{\mathrm{r}}_h\|^2_{\mathrm{Y},\sharp,h} \lesssim \sum_{T\in\mathcal{T}_h} h_T^2 \|\boldsymbol{G}_T^{k+1}\underline{\mathrm{r}}_T\|^2_T + \mathrm{d}_h(\underline{\mathrm{r}}_h, \underline{\mathrm{r}}_h) \lesssim \|\underline{\mathrm{r}}_h\|^2_{\mathrm{Y},\sharp,h} \qquad \forall \underline{\mathrm{r}}_h \in \underline{\mathrm{Y}}^{k+1}_h. \tag{70b}$$

Proof. Let us prove (70a). Let $\underline{\mathbf{w}}_h \in \underline{\mathbf{X}}^{k+1}_{\sharp,h}$, and $T \in \mathcal{T}_h$. By the definition (61) of $\boldsymbol{C}^k_T$, testing with $\boldsymbol{w} = \mathbf{curl}\,\mathbf{w}_T$, integrating by parts, and using the fact that $\boldsymbol{\gamma}_{\tau,F}(\mathbf{curl}\,\mathbf{w}_T\times\boldsymbol{n}_{T,F}) \in \mathbb{P}^k(F;\mathbb{R}^2) \subset \boldsymbol{\mathcal{P}}^{k+1}_F$, we infer

$$\|\mathbf{curl}\,\mathbf{w}_T\|^2_T = \big(\boldsymbol{C}^k_T\underline{\mathbf{w}}_T, \mathbf{curl}\,\mathbf{w}_T\big)_T + \sum_{F\in\mathcal{F}_T} \big(\boldsymbol{\pi}^{k+1}_{\mathcal{P},F}\big(\boldsymbol{\gamma}_{\tau,F}(\mathbf{w}_T) - \mathbf{w}_F\big), \boldsymbol{\gamma}_{\tau,F}(\mathbf{curl}\,\mathbf{w}_T\times\boldsymbol{n}_{T,F})\big)_F.$$

By the Cauchy–Schwarz inequality, a discrete trace inequality (see, e.g., [15, Lemma 1.32]), and recalling the definition (59a) of a_h, we get $\|\mathbf{curl}_h\mathbf{w}_h\|^2_\Omega \lesssim \mathrm{a}_h(\underline{\mathbf{w}}_h, \underline{\mathbf{w}}_h)$, and the first inequality in (70a) follows by adding $\mathrm{s}_h(\underline{\mathbf{w}}_h, \underline{\mathbf{w}}_h)$ to both sides. To prove the second inequality, we test (61) with $\boldsymbol{w} = \boldsymbol{C}^k_T\underline{\mathbf{w}}_T$ to write

$$\|\boldsymbol{C}^k_T\underline{\mathbf{w}}_T\|^2_T = \big(\mathbf{curl}\,\mathbf{w}_T, \boldsymbol{C}^k_T\underline{\mathbf{w}}_T\big)_T - \sum_{F\in\mathcal{F}_T} \big(\boldsymbol{\pi}^{k+1}_{\mathcal{P},F}\big(\boldsymbol{\gamma}_{\tau,F}(\mathbf{w}_T) - \mathbf{w}_F\big), \boldsymbol{\gamma}_{\tau,F}(\boldsymbol{C}^k_T\underline{\mathbf{w}}_T\times\boldsymbol{n}_{T,F})\big)_F,$$

and we conclude by the same kind of arguments.

The proof of (70b) is similar and is omitted for the sake of brevity. □

Lemma 20 (Control of $\boldsymbol{G}^{k+1}_h$)**.** *For all* $(\underline{\mathbf{w}}_h, \underline{\mathrm{r}}_h) \in \underline{\mathbf{Z}}^{k+1}_{\sharp,h}$*, there exists* $\underline{\mathbf{v}}^\star_h \in \underline{\mathbf{X}}^{k+1}_{\sharp,h,\mathbf{0}}$ *satisfying*

$$\|\mathbf{v}^\star_h\|^2_\Omega + \|\underline{\mathbf{v}}^\star_h\|^2_{\mathbf{X},\sharp,h} \lesssim \sum_{T\in\mathcal{T}_h} h_T^2\|\boldsymbol{G}_T^{k+1}\underline{\mathrm{r}}_T\|^2_T,$$

such that it holds

$$\sum_{T\in\mathcal{T}_h} h_T^2\|\boldsymbol{G}_T^{k+1}\underline{\mathrm{r}}_T\|^2_T \lesssim \mathrm{A}_h\big((\underline{\mathbf{w}}_h, \underline{\mathrm{r}}_h), (\underline{\mathbf{v}}^\star_h, \underline{0}_h)\big) + \|\underline{\mathbf{w}}_h\|^2_{\mathbf{X},\sharp,h}. \tag{71}$$

Proof. Let $(\underline{\mathbf{w}}_h, \underline{\mathrm{r}}_h) \in \underline{\mathbf{Z}}_{\sharp,h}^{k+1}$. We define $\underline{\mathbf{v}}_h^\star \in \underline{\mathbf{X}}_{\sharp,h,\mathbf{0}}^{k+1}$ such that

$$\mathbf{v}_T^\star := h_T^2 \boldsymbol{G}_T^{k+1} \underline{\mathrm{r}}_T \text{ for all } T \in \mathcal{T}_h \text{ and } \mathbf{v}_F^\star := \mathbf{0} \text{ for all } F \in \mathcal{F}_h. \tag{72}$$

We immediately verify, since $h_T \le \operatorname{diam}(\Omega)$ for all $T \in \mathcal{T}_h$, that there holds $\|\mathbf{v}_h^\star\|_\Omega^2 \lesssim \sum_{T\in\mathcal{T}_h} h_T^2 \|\boldsymbol{G}_T^{k+1}\underline{\mathrm{r}}_T\|_T^2$. From the definitions (72) of $\underline{\mathbf{v}}_h^\star$, (59b) of b_h, and (66) of A_h, we infer

$$\sum_{T\in\mathcal{T}_h} h_T^2 \|\boldsymbol{G}_T^{k+1}\underline{\mathrm{r}}_T\|_T^2 = \mathrm{b}_h(\underline{\mathbf{v}}_h^\star, \underline{\mathrm{r}}_h) = \mathrm{A}_h\big((\underline{\mathbf{w}}_h, \underline{\mathrm{r}}_h), (\underline{\mathbf{v}}_h^\star, \underline{0}_h)\big) - \mathrm{a}_h(\underline{\mathbf{w}}_h, \underline{\mathbf{v}}_h^\star).$$

The Cauchy–Schwarz inequality followed by the second inequality in (70a) then yield

$$\sum_{T\in\mathcal{T}_h} h_T^2 \|\boldsymbol{G}_T^{k+1}\underline{\mathrm{r}}_T\|_T^2 \lesssim \mathrm{A}_h\big((\underline{\mathbf{w}}_h, \underline{\mathrm{r}}_h), (\underline{\mathbf{v}}_h^\star, \underline{0}_h)\big) + \|\underline{\mathbf{w}}_h\|_{\mathbf{X},\sharp,h}\|\underline{\mathbf{v}}_h^\star\|_{\mathbf{X},\sharp,h}. \tag{73}$$

Using the definitions (72) and (68a) of, respectively, $\underline{\mathbf{v}}_h^\star$ and $\|\cdot\|_{\mathbf{X},\sharp,h}$, it holds

$$\begin{aligned}\|\underline{\mathbf{v}}_h^\star\|_{\mathbf{X},\sharp,h}^2 &= \sum_{T\in\mathcal{T}_h}\left(\|\operatorname{\mathbf{curl}}\mathbf{v}_T^\star\|_T^2 + \sum_{F\in\mathcal{F}_T} h_F^{-1}\|\boldsymbol{\pi}_{\boldsymbol{\mathcal{P}},F}^{k+1}\big(\boldsymbol{\gamma}_{\tau,F}(\mathbf{v}_T^\star)\big)\|_F^2\right)\\ &\lesssim \sum_{T\in\mathcal{T}_h} h_T^{-2}\|\mathbf{v}_T^\star\|_T^2 = \sum_{T\in\mathcal{T}_h} h_T^2\|\boldsymbol{G}_T^{k+1}\underline{\mathrm{r}}_T\|_T^2,\end{aligned} \tag{74}$$

where we have used the $L^2(F;\mathbb{R}^2)$-boundedness of $\boldsymbol{\pi}_{\boldsymbol{\mathcal{P}},F}^{k+1}$, as well as an inverse inequality together with a discrete trace inequality (see, e.g., [15, Lemmas 1.28 and 1.32]). Starting from (73), and using (74) combined with Young's inequality for the last term in the right-hand side eventually yields the expected result. □

We are now in position to show well-posedness for Problem (64).

Lemma 21 (Well-posedness)**.** *For all $\underline{\mathbf{z}}_h \in \underline{\mathbf{Z}}_{\sharp,h}^{k+1}$, there exists $\underline{\mathbf{v}}_h^\star \in \underline{\mathbf{X}}_{\sharp,h,\mathbf{0}}^{k+1}$ satisfying $\|\mathbf{v}_h^\star\|_\Omega + \|\underline{\mathbf{v}}_h^\star\|_{\mathbf{X},\sharp,h} \lesssim \|\underline{\mathbf{z}}_h\|_{\mathbf{Z},\sharp,h}$ such that it holds*

$$\|\underline{\mathbf{z}}_h\|_{\mathbf{Z},\sharp,h}^2 \lesssim \mathrm{A}_h\big(\underline{\mathbf{z}}_h, \underline{\mathbf{z}}_h\big) + \mathrm{A}_h\big(\underline{\mathbf{z}}_h, (\underline{\mathbf{v}}_h^\star, \underline{0}_h)\big). \tag{75}$$

Hence, Problem (64) is well-posed, and the following a priori bound holds:

$$\|(\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h)\|_{\mathbf{Z},\sharp,h} \lesssim \|\boldsymbol{f}\|_\Omega. \tag{76}$$

Proof. Let $\underline{\mathbf{z}}_h = (\underline{\mathbf{w}}_h, \underline{\mathrm{r}}_h) \in \underline{\mathbf{Z}}_{\sharp,h}^{k+1}$. By the first inequality in (70a) and (66), one has

$$\|\underline{\mathbf{w}}_h\|_{\mathbf{X},\sharp,h}^2 + \mathrm{d}_h(\underline{\mathrm{r}}_h, \underline{\mathrm{r}}_h) \lesssim \mathrm{a}_h(\underline{\mathbf{w}}_h, \underline{\mathbf{w}}_h) + \mathrm{d}_h(\underline{\mathrm{r}}_h, \underline{\mathrm{r}}_h) = \mathrm{A}_h\big(\underline{\mathbf{z}}_h, \underline{\mathbf{z}}_h\big). \tag{77}$$

By Lemma 20 combined with (77), one also has

$$\sum_{T\in\mathcal{T}_h} h_T^2\|\boldsymbol{G}_T^{k+1}\underline{\mathrm{r}}_T\|_T^2 \lesssim \mathrm{A}_h\big(\underline{\mathbf{z}}_h, (\underline{\mathbf{v}}_h^\star, \underline{0}_h)\big) + \mathrm{A}_h\big(\underline{\mathbf{z}}_h, \underline{\mathbf{z}}_h\big), \tag{78}$$

for some $\underline{\mathbf{v}}_h^\star \in \underline{\mathbf{X}}_{\sharp,h,\mathbf{0}}^{k+1}$ such that $\|\mathbf{v}_h^\star\|_\Omega + \|\underline{\mathbf{v}}_h^\star\|_{\mathbf{X},\sharp,h} \lesssim \|\underline{\mathrm{r}}_h\|_{\mathrm{Y},\sharp,h} \le \|\underline{\mathbf{z}}_h\|_{\mathbf{Z},\sharp,h}$, where we have used the second inequality in (70b) and (69). Summing (77) and (78), and using the first inequality in (70b), we infer (75). To prove well-posedness, since the system associated to Problem (64) is square, it is sufficient to prove injectivity. Assume that $\mathrm{A}_h\big((\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h), (\underline{\mathbf{v}}_h, \underline{\mathrm{q}}_h)\big) = 0$ for all $(\underline{\mathbf{v}}_h, \underline{\mathrm{q}}_h) \in \underline{\mathbf{Z}}_{\sharp,h,\mathbf{0}}^{k+1}$. Taking

$(\underline{\mathbf{v}}_h, \underline{\mathrm{q}}_h) = (\underline{\mathbf{0}}_h, \underline{\mathrm{q}}_h)$ and using (67), we first infer that $(\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h) \in \underline{\mathbb{Z}}^{k+1}_{\sharp,h,\mathbf{0}}$. Taking $(\underline{\mathbf{v}}_h, \underline{\mathrm{q}}_h) = (\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h)$ and $(\underline{\mathbf{v}}_h, \underline{\mathrm{q}}_h) = (\underline{\mathbf{v}}_h^\star, \underline{0}_h)$, and using (75), we then get

$$\|(\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h)\|^2_{\mathbf{Z},\sharp,h} \lesssim \mathrm{A}_h\big((\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h), (\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h)\big) + \mathrm{A}_h\big((\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h), (\underline{\mathbf{v}}_h^\star, \underline{0}_h)\big) = 0,$$

which, by Lemma 18, eventually yields $(\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h) = (\underline{\mathbf{0}}_h, \underline{0}_h)$. To prove the a priori bound (76), we take $\underline{\mathbf{z}}_h = (\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h)$ in (75) and we use (65). We get, by Cauchy–Schwarz inequality,

$$\|(\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h)\|^2_{\mathbf{Z},\sharp,h} \lesssim (\boldsymbol{f}, \mathbf{u}_h)_\Omega + (\boldsymbol{f}, \mathbf{v}_h^\star)_\Omega \le \|\boldsymbol{f}\|_\Omega\big(\|\mathbf{u}_h\|_\Omega + \|\mathbf{v}_h^\star\|_\Omega\big).$$

The conclusion follows from the fact that $\|\mathbf{v}_h^\star\|_\Omega \lesssim \|(\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h)\|_{\mathbf{Z},\sharp,h}$, and from the combination of Remark 15 with the generalized discrete Weber inequality (30) of Corollary 6 applied to $(\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h)$ satisfying (29) (one can easily check that d_h satisfies (28)). □

3.2.4 Error analysis

We recall that $(\boldsymbol{u}, p) \in \boldsymbol{H}_{\mathbf{0}}(\mathbf{curl};\Omega) \times H^1_0(\Omega)$ denotes the unique solution to Problem (56). We assume that $\boldsymbol{u}$ possesses the additional regularity $\boldsymbol{u} \in H^1(\Omega;\mathbb{R}^3)$, and we let $\widehat{\underline{\mathbf{u}}}_h := \underline{\mathbf{I}}^{k+1}_{\mathbf{X},\sharp,h}\boldsymbol{u} \in \underline{\mathbf{X}}^{k+1}_{\sharp,h,\mathbf{0}}$ and $\widehat{\underline{\mathrm{p}}}_h := \underline{\mathrm{I}}^{k+1}_{\mathrm{Y},h} p \in \underline{\mathrm{Y}}^{k+1}_{h,0}$. We define the errors

$$\underline{\mathbf{X}}^{k+1}_{\sharp,h,\mathbf{0}} \ni \underline{\mathbf{e}}_h := \underline{\mathbf{u}}_h - \widehat{\underline{\mathbf{u}}}_h, \qquad \underline{\mathrm{Y}}^{k+1}_{h,0} \ni \underline{\epsilon}_h := \underline{\mathrm{p}}_h - \widehat{\underline{\mathrm{p}}}_h, \tag{79}$$

where $(\underline{\mathbf{u}}_h, \underline{\mathrm{p}}_h) \in \underline{\mathbf{X}}^{k+1}_{\sharp,h,\mathbf{0}} \times \underline{\mathrm{Y}}^{k+1}_{h,0}$ is the unique solution to Problem (64). Recalling (65) and (66), the errors $(\underline{\mathbf{e}}_h, \underline{\epsilon}_h) \in \underline{\mathbf{Z}}^{k+1}_{\sharp,h,\mathbf{0}}$ solve

$$\mathrm{A}_h\big((\underline{\mathbf{e}}_h, \underline{\epsilon}_h), (\underline{\mathbf{v}}_h, \underline{\mathrm{q}}_h)\big) = \mathrm{l}_h(\underline{\mathbf{v}}_h) + \mathrm{m}_h(\underline{\mathrm{q}}_h) \qquad \forall (\underline{\mathbf{v}}_h, \underline{\mathrm{q}}_h) \in \underline{\mathbf{Z}}^{k+1}_{\sharp,h,\mathbf{0}}, \tag{80}$$

where we have defined the consistency errors

$$\mathrm{l}_h(\underline{\mathbf{v}}_h) := (\boldsymbol{f}, \mathbf{v}_h)_\Omega - \mathrm{a}_h(\widehat{\underline{\mathbf{u}}}_h, \underline{\mathbf{v}}_h) - \mathrm{b}_h(\underline{\mathbf{v}}_h, \widehat{\underline{\mathrm{p}}}_h), \tag{81a}$$

$$\mathrm{m}_h(\underline{\mathrm{q}}_h) := \mathrm{b}_h(\widehat{\underline{\mathbf{u}}}_h, \underline{\mathrm{q}}_h) - \mathrm{d}_h(\widehat{\underline{\mathrm{p}}}_h, \underline{\mathrm{q}}_h). \tag{81b}$$

Theorem 22 (Energy-error estimate)**.** *Assume that*

$$\boldsymbol{u} \in \boldsymbol{H}_{\mathbf{0}}(\mathbf{curl};\Omega) \cap H^1(\Omega;\mathbb{R}^3) \cap H^{k+2}(\mathcal{T}_h;\mathbb{R}^3), \qquad p \in H^1_0(\Omega) \cap H^{k+1}(\mathcal{T}_h).$$

Then, there holds, with $(\underline{\mathbf{e}}_h, \underline{\epsilon}_h) \in \underline{\mathbf{Z}}^{k+1}_{\sharp,h,\mathbf{0}}$ *defined by* (79)*,*

$$\|(\underline{\mathbf{e}}_h, \underline{\epsilon}_h)\|_{\mathbf{Z},\sharp,h} \lesssim \left[\sum_{T\in\mathcal{T}_h} h_T^{2(k+1)}\left(|\boldsymbol{u}|^2_{H^{k+2}(T;\mathbb{R}^3)} + |p|^2_{H^{k+1}(T)}\right)\right]^{1/2}. \tag{82}$$

Proof. Since $(\underline{\mathbf{e}}_h, \underline{\epsilon}_h) \in \underline{\mathbf{Z}}^{k+1}_{\sharp,h,\mathbf{0}}$, by (75) with $\underline{\mathbf{z}}_h = (\underline{\mathbf{e}}_h, \underline{\epsilon}_h)$ (note that $(\underline{\mathbf{v}}_h^\star, \underline{0}_h) \in \underline{\mathbf{Z}}^{k+1}_{\sharp,h,\mathbf{0}}$ and $\|(\underline{\mathbf{v}}_h^\star, \underline{0}_h)\|_{\mathbf{Z},\sharp,h} = \|\underline{\mathbf{v}}_h^\star\|_{\mathbf{X},\sharp,h} \lesssim \|\underline{\mathbf{z}}_h\|_{\mathbf{Z},\sharp,h}$) and (80), we infer

$$\|(\underline{\mathbf{e}}_h, \underline{\epsilon}_h)\|_{\mathbf{Z},\sharp,h} \lesssim \max_{(\underline{\mathbf{v}}_h,\underline{\mathrm{q}}_h)\in\underline{\mathbf{Z}}^{k+1}_{\sharp,h,\mathbf{0}},\, \|(\underline{\mathbf{v}}_h,\underline{\mathrm{q}}_h)\|_{\mathbf{Z},\sharp,h}=1} \big(\mathrm{l}_h(\underline{\mathbf{v}}_h) + \mathrm{m}_h(\underline{\mathrm{q}}_h)\big). \tag{83}$$

Let us first focus on $\mathrm{l}_h(\underline{\mathbf{v}}_h)$ for $\underline{\mathbf{v}}_h \in \underline{\mathbf{X}}_{\sharp,h,\mathbf{0}}^{k+1}$. By (81a), the fact that $\boldsymbol{f} = \mathbf{curl}(\mathbf{curl}\,\boldsymbol{u}) + \mathbf{grad}\,p$ in Ω, and element-by-element integration by parts, there holds

$$\begin{aligned}\mathrm{l}_h(\underline{\mathbf{v}}_h) &= \sum_{T\in\mathcal{T}_h}\left[\big(\mathbf{curl}(\boldsymbol{u}_{|T}),\mathbf{curl}\,\mathbf{v}_T\big)_T - \sum_{F\in\mathcal{F}_T}\big(\mathbf{curl}(\boldsymbol{u}_{|T})_{|F}\times\boldsymbol{n}_{T,F},\mathbf{v}_{T|F}\big)_F\right]\\ &\quad + \big(\mathbf{grad}\,p,\mathbf{v}_h\big)_\Omega - \mathrm{a}_h(\widehat{\underline{\mathbf{u}}}_h,\underline{\mathbf{v}}_h) - \mathrm{b}_h(\underline{\mathbf{v}}_h,\widehat{\underline{\mathrm{p}}}_h)\\ &= \sum_{T\in\mathcal{T}_h}\Bigg(\big(\mathbf{curl}(\boldsymbol{u}_{|T}),\mathbf{curl}\,\mathbf{v}_T\big)_T\\ &\quad - \sum_{F\in\mathcal{F}_T}\big(\boldsymbol{\gamma}_{\tau,F}\big(\mathbf{curl}(\boldsymbol{u}_{|T})\times\boldsymbol{n}_{T,F}\big),\boldsymbol{\gamma}_{\tau,F}(\mathbf{v}_T)-\mathbf{v}_F\big)_F\Bigg) - \mathrm{a}_h(\widehat{\underline{\mathbf{u}}}_h,\underline{\mathbf{v}}_h),\end{aligned} \tag{84}$$

where we have used the fact that the tangential component of $\mathbf{curl}\,\boldsymbol{u}$ is continuous across interfaces (as a consequence of the fact that $\mathbf{curl}\,\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl};\Omega)\cap H^1(\mathcal{T}_h;\mathbb{R}^3)$) along with $\mathbf{v}_F = \mathbf{0}$ for all $F\in\mathcal{F}_h^{\mathrm{b}}$ to insert $\mathbf{v}_F$ into the boundary term, together with the fact that $\big(\mathbf{grad}\,p,\mathbf{v}_h\big)_\Omega = \mathrm{b}_h(\underline{\mathbf{v}}_h,\widehat{\underline{\mathrm{p}}}_h)$ as a consequence of the commutation property (9). Using the definitions (59a) of a_h and (61) of $\boldsymbol{C}_T^k$ for $T\in\mathcal{T}_h$ (testing with $\boldsymbol{w} = \boldsymbol{C}_T^k\widehat{\underline{\mathbf{u}}}_T$), and integrating by parts, it holds

$$\begin{aligned}\mathrm{a}_h(\widehat{\underline{\mathbf{u}}}_h,\underline{\mathbf{v}}_h) = \sum_{T\in\mathcal{T}_h}\Bigg[&\big(\boldsymbol{C}_T^k\widehat{\underline{\mathbf{u}}}_T,\mathbf{curl}\,\mathbf{v}_T\big)_T\\ &- \sum_{F\in\mathcal{F}_T}\big(\boldsymbol{\gamma}_{\tau,F}(\boldsymbol{C}_T^k\widehat{\underline{\mathbf{u}}}_T\times\boldsymbol{n}_{T,F}),\boldsymbol{\gamma}_{\tau,F}(\mathbf{v}_T)-\mathbf{v}_F\big)_F\Bigg] + \mathrm{s}_h(\widehat{\underline{\mathbf{u}}}_h,\underline{\mathbf{v}}_h).\end{aligned} \tag{85}$$

Since, by Lemma 16, $\boldsymbol{C}_T^k\widehat{\underline{\mathbf{u}}}_T = \boldsymbol{\pi}_{C,T}^k\big(\mathbf{curl}(\boldsymbol{u}_{|T})\big)$ for all $T\in\mathcal{T}_h$, a combination of (84) and (85) yields (recall that $\mathbf{v}_T\in\mathbb{P}^{k+1}(T;\mathbb{R}^3)$)

$$\begin{aligned}\mathrm{l}_h(\underline{\mathbf{v}}_h) = \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T}&\Big(\boldsymbol{\gamma}_{\tau,F}\big((\boldsymbol{\pi}_{C,T}^k(\mathbf{curl}(\boldsymbol{u}_{|T}))-\mathbf{curl}(\boldsymbol{u}_{|T}))\times\boldsymbol{n}_{T,F}\big),\boldsymbol{\gamma}_{\tau,F}(\mathbf{v}_T)-\mathbf{v}_F\Big)_F\\ &- \mathrm{s}_h(\widehat{\underline{\mathbf{u}}}_h,\underline{\mathbf{v}}_h).\end{aligned}$$

Applying the triangle and Cauchy–Schwarz inequalities, we get

$$\begin{aligned}|\mathrm{l}_h(\underline{\mathbf{v}}_h)| \le \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T}&\big\|\big(\boldsymbol{\pi}_{C,T}^k(\mathbf{curl}(\boldsymbol{u}_{|T}))-\mathbf{curl}(\boldsymbol{u}_{|T})\big)_{|F}\big\|_F\,\|\boldsymbol{\gamma}_{\tau,F}(\mathbf{v}_T)-\mathbf{v}_F\|_F\\ &+ \mathrm{s}_h(\widehat{\underline{\mathbf{u}}}_h,\widehat{\underline{\mathbf{u}}}_h)^{1/2}\mathrm{s}_h(\underline{\mathbf{v}}_h,\underline{\mathbf{v}}_h)^{1/2}.\end{aligned} \tag{86}$$

Let us focus on $\big\|\big(\boldsymbol{\pi}_{C,T}^k(\mathbf{curl}(\boldsymbol{u}_{|T}))-\mathbf{curl}(\boldsymbol{u}_{|T})\big)_{|F}\big\|_F$ for $F\in\mathcal{F}_T$. Adding/subtracting $\mathbf{curl}(\boldsymbol{\pi}_T^{k+1}(\boldsymbol{u}_{|T}))$, using a triangle inequality, a discrete trace inequality (see, e.g., [15, Lemma 1.32]) on $\boldsymbol{\pi}_{C,T}^k(\mathbf{curl}(\boldsymbol{u}_{|T}))-\mathbf{curl}(\boldsymbol{\pi}_T^{k+1}(\boldsymbol{u}_{|T}))$, and the approximation properties of $\boldsymbol{\pi}_T^{k+1}$ on mesh faces (see, e.g., [15, Theorem 1.45]) for $\mathbf{curl}(\boldsymbol{\pi}_T^{k+1}(\boldsymbol{u}_{|T}))-\mathbf{curl}(\boldsymbol{u}_{|T})$, we infer

$$\begin{aligned}h_F^{1/2}\big\|\big(\boldsymbol{\pi}_{C,T}^k(\mathbf{curl}(\boldsymbol{u}_{|T}))-\mathbf{curl}(\boldsymbol{u}_{|T})\big)_{|F}\big\|_F \lesssim\ &\|\boldsymbol{\pi}_{C,T}^k(\mathbf{curl}(\boldsymbol{u}_{|T}))-\mathbf{curl}(\boldsymbol{u}_{|T})\|_T\\ &+ \|\mathbf{curl}(\boldsymbol{u}_{|T})-\mathbf{curl}(\boldsymbol{\pi}_T^{k+1}(\boldsymbol{u}_{|T}))\|_T + h_T^{k+1}|\boldsymbol{u}_{|T}|_{H^{k+2}(T;\mathbb{R}^3)},\end{aligned}$$

where we have used yet another triangle inequality. The second term on the right-hand side is readily estimated using again the approximation properties of $\boldsymbol{\pi}_T^{k+1}$. As far as the first term is concerned, it holds

$$\|\boldsymbol{\pi}_{C,T}^k(\mathbf{curl}(\boldsymbol{u}_{|T}))-\mathbf{curl}(\boldsymbol{u}_{|T})\|_T = \min_{\boldsymbol{p}\in\mathbb{P}^{k+1}(T;\mathbb{R}^3)}\|\mathbf{curl}\,\boldsymbol{p}-\mathbf{curl}(\boldsymbol{u}_{|T})\|_T,$$

which finally yields

$$h_F^{1/2}\|\big(\boldsymbol{\pi}_{C,T}^k(\mathbf{curl}(\boldsymbol{u}_{|T})) - \mathbf{curl}(\boldsymbol{u}_{|T})\big)_{|F}\|_F \lesssim h_T^{k+1}|\boldsymbol{u}_{|T}|_{H^{k+2}(T;\mathbb{R}^3)}.$$

Plugging this last estimate into (86), applying a discrete Cauchy–Schwarz inequality, and using (26) as well as (49) (with $\boldsymbol{\pi}_{\mathcal{P},F}^{k+1}$ instead of $\boldsymbol{\pi}_{\mathcal{G},F}^{k+1}$), we infer

$$|\mathrm{l}_h(\underline{\mathbf{v}}_h)| \lesssim \left(\sum_{T\in\mathcal{T}_h} h_T^{2(k+1)}|\boldsymbol{u}_{|T}|^2_{H^{k+2}(T;\mathbb{R}^3)}\right)^{1/2} \|\underline{\mathbf{v}}_h\|_{\mathbf{X},\sharp,h}. \tag{87}$$

Let us now focus on $\mathrm{m}_h(\underline{\mathrm{q}}_h)$ for $\underline{\mathrm{q}}_h \in \underline{\mathrm{Y}}_{h,0}^{k+1}$. Recalling (51), one can readily infer

$$|\mathrm{b}_h(\widehat{\underline{\mathbf{u}}}_h, \underline{\mathrm{q}}_h)| \lesssim \left(\sum_{T\in\mathcal{T}_h} h_T^{2(k+1)}|\boldsymbol{u}_{|T}|^2_{H^{k+2}(T;\mathbb{R}^3)}\right)^{1/2} \mathrm{d}_h(\underline{\mathrm{q}}_h, \underline{\mathrm{q}}_h)^{1/2}. \tag{88}$$

Now, applying the Cauchy–Schwarz inequality, recalling the definition (59c) of d_h, noticing that $\pi_T^k(p_{|T})_{|F} = \pi_F^{k+1}(\pi_T^k(p_{|T})_{|F})$ for all $T \in \mathcal{T}_h$ and $F \in \mathcal{F}_T$, and using the $L^2(F)$-boundedness of π_F^{k+1}, one has

$$\begin{aligned}|\mathrm{d}_h(\widehat{\underline{\mathrm{p}}}_h, \underline{\mathrm{q}}_h)| &\le \mathrm{d}_h(\widehat{\underline{\mathrm{p}}}_h, \widehat{\underline{\mathrm{p}}}_h)^{1/2}\,\mathrm{d}_h(\underline{\mathrm{q}}_h, \underline{\mathrm{q}}_h)^{1/2}\\ &\le \left(\sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T} h_F\|(p - \pi_T^k(p_{|T}))_{|F}\|_F^2\right)^{1/2} \mathrm{d}_h(\underline{\mathrm{q}}_h, \underline{\mathrm{q}}_h)^{1/2}.\end{aligned}$$

By the approximation properties of π_T^k on mesh faces (see, e.g., [15, Theorem 1.45]),

$$|\mathrm{d}_h(\widehat{\underline{\mathrm{p}}}_h, \underline{\mathrm{q}}_h)| \lesssim \left(\sum_{T\in\mathcal{T}_h} h_T^{2(k+1)}|p_{|T}|^2_{H^{k+1}(T)}\right)^{1/2} \mathrm{d}_h(\underline{\mathrm{q}}_h, \underline{\mathrm{q}}_h)^{1/2}. \tag{89}$$

Gathering (88)–(89), and recalling the definition (81b) of $\mathrm{m}_h(\underline{\mathrm{q}}_h)$, finally yields

$$|\mathrm{m}_h(\underline{\mathrm{q}}_h)| \lesssim \left(\sum_{T\in\mathcal{T}_h} h_T^{2(k+1)}\left(|\boldsymbol{u}_{|T}|^2_{H^{k+2}(T;\mathbb{R}^3)} + |p_{|T}|^2_{H^{k+1}(T)}\right)\right)^{1/2} \|\underline{\mathrm{q}}_h\|_{\mathrm{Y},\sharp,h}. \tag{90}$$

Plugging (87) and (90) into (83) for $(\underline{\mathbf{v}}_h, \underline{\mathrm{q}}_h)$ such that $\|(\underline{\mathbf{v}}_h, \underline{\mathrm{q}}_h)\|_{\mathbf{Z},\sharp,h} = 1$ finally yields (82). □

Remark 23 (The tetrahedral case)**.** *On matching tetrahedral meshes, according to Lemma 1, $\|\boldsymbol{G}_h^{k+1}\cdot\|_\Omega$ defines a norm on $\underline{\mathrm{Y}}_{h,0}^{k+1}$. Hence, in this case, and as already pointed out in [10], one can consider a modified version of Problem (64) in which d_h is removed, without jeopardizing well-posedness. Doing so, it holds $\underline{\mathbf{u}}_h \in \underline{\mathbb{X}}_{\sharp,h,\mathbf{0}}^{k+1}$ and, as a by-product of the commutation property (9), $\mathbf{u}_h \in \boldsymbol{H}(\mathrm{div}^0;\Omega)$. Furthermore, a close inspection of the proof of Theorem 22 shows that, in this case, one ends up with an energy-error estimate that is free of pressure contribution. This allows one to reproduce at the discrete level the following structure of Problem (54). When the current density $\boldsymbol{f}$ is given by the gradient of some function $\psi \in H_0^1(\Omega)$, then $\boldsymbol{u} = \mathbf{0}$ and $p = \psi$. At the discrete level, when $\boldsymbol{f} = \mathbf{grad}\,\psi$, one then gets $\underline{\mathbf{u}}_h = \underline{\mathbf{0}}_h$ and $\underline{\mathrm{p}}_h = \underline{\mathrm{I}}_{\mathrm{Y},h}^{k+1}\psi$ (note that this can be observed in practice up to machine precision only if the computation of the right-hand side is also performed up to machine precision).*

3.2.5 Numerical results

Let the domain Ω be the unit cube $(0,1)^3$. We consider Problem (54) with exact solution

$$\boldsymbol{u}(x_1,x_2,x_3) := \begin{pmatrix} \sin(\pi x_2)\sin(\pi x_3) \\ \sin(\pi x_1)\sin(\pi x_3) \\ \sin(\pi x_1)\sin(\pi x_2) \end{pmatrix}, \quad p(x_1,x_2,x_3) := \sin(\pi x_1)\sin(\pi x_2)\sin(\pi x_3).$$

Clearly, the magnetic vector potential $\boldsymbol{u}$ and the pressure p satisfy (54b), (54c), and (54d). The current density $\boldsymbol{f}$ is set according to (54a).

As in Section 3.1.5, we solve the discrete Problem (64) on two refined mesh sequences, of respectively cubic and regular tetrahedral meshes. On tetrahedral meshes, following Remark 23, we actually solve a modifed version of Problem (64) in which d_h is removed. As expected, it does not jeopardize well-posedness, nor convergence. The numerical tests show that removing d_h leads to a slightly more accurate approximation of the magnetic vector potential, but a slightly less accurate approximation of the pressure. For each problem, the element unknowns for both the magnetic vector potential and the pressure are locally eliminated using a Schur complement technique. This step is fully parallelizable. The resulting (condensed) global linear system is solved using the SparseLU direct solver of the Eigen library, on an Intel Xeon E-2176M 2.70GHz×12 with 16Go of RAM (and up to 150Go of swap). For $k \in \{0,1,2\}$, we depict on Figures 3 and 4, respectively for the cubic and (regular) tetrahedral mesh families, the relative energy-error $\|\underline{\mathbf{u}}_h - \underline{\mathbf{I}}^{k+1}_{\mathbf{X},\sharp,h}\boldsymbol{u}\|_{\mathbf{X},\sharp,h} / \|\underline{\mathbf{I}}^{k+1}_{\mathbf{X},\sharp,h}\boldsymbol{u}\|_{\mathbf{X},\sharp,h}$ (first line), L^2-error $\|\mathbf{u}_h - \boldsymbol{\pi}^{k+1}_h\boldsymbol{u}\|_\Omega / \|\boldsymbol{\pi}^{k+1}_h\boldsymbol{u}\|_\Omega$ (second line), and L^2-like-error $\|\underline{\mathrm{p}}_h - \underline{\mathrm{I}}^{k+1}_{\mathrm{Y},h}p\|_{\mathrm{Y},\sharp,h} / \|\underline{\mathrm{I}}^{k+1}_{\mathrm{Y},h}p\|_{\mathrm{Y},\sharp,h}$ (third line) as functions of (i) the meshsize (left column), (ii) the solution time in seconds, i.e. the time needed to solve the (condensed) global linear system (middle column), and (iii) the number of (interface) DoF (right column). On tetrahedral meshes, since we remove the contribution d_h, we replace the L^2-like-error measure $\|\underline{\mathrm{p}}_h - \underline{\mathrm{I}}^{k+1}_{\mathrm{Y},h}p\|_{\mathrm{Y},\sharp,h}$ by the measure

$$\|\underline{\mathrm{p}}_h - \underline{\mathrm{I}}^{k+1}_{\mathrm{Y},h}p\|_{\boldsymbol{G},h} := \left(\sum_{T\in\mathcal{T}_h} h_T^2 \|\boldsymbol{G}_T^{k+1}\big(\underline{\mathrm{p}}_T - \underline{\mathrm{I}}^{k+1}_{\mathrm{Y},T}(p_{|T})\big)\|_T^2\right)^{1/2},$$

that remains meaningful for $k = 0$. For the two mesh families, we obtain, as predicted by Theorem 22, a convergence rate of the energy-error on the magnetic vector potential and of the L^2-like-error on the pressure of order $k+1$. We also observe a convergence rate of order $k+2$ for the L^2-error on the magnetic vector potential.

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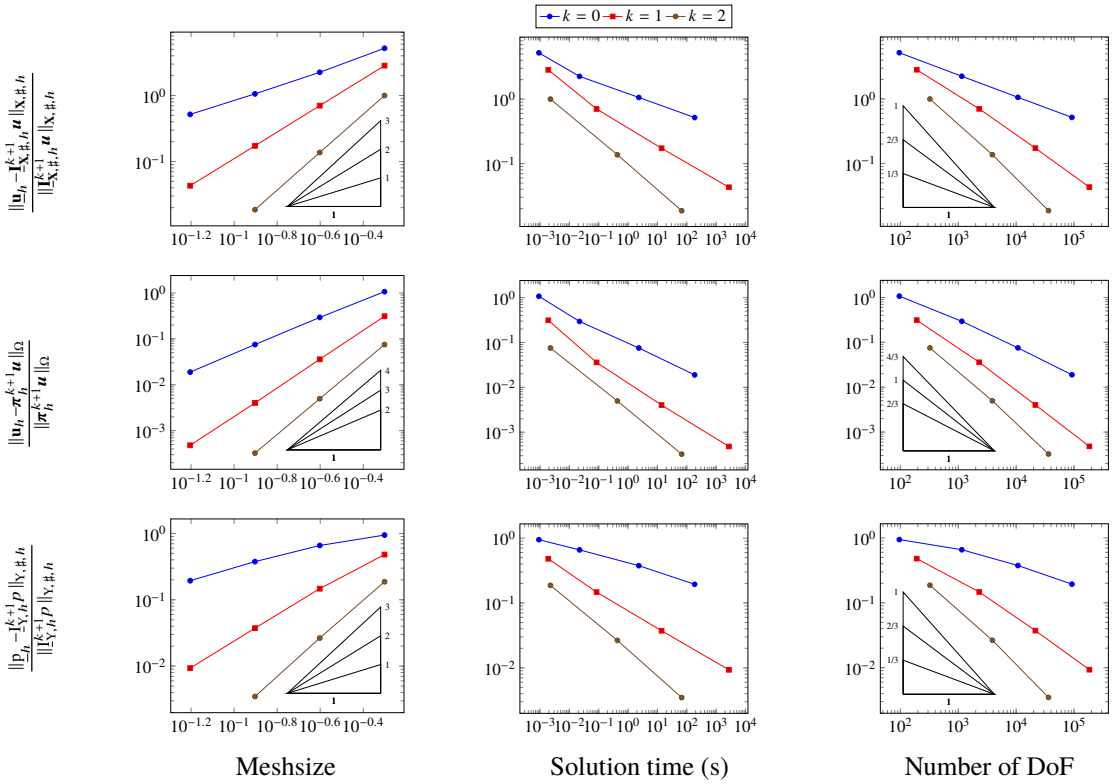

Figure 3: Errors vs. h, solution time and number of DoF on cubic meshes.

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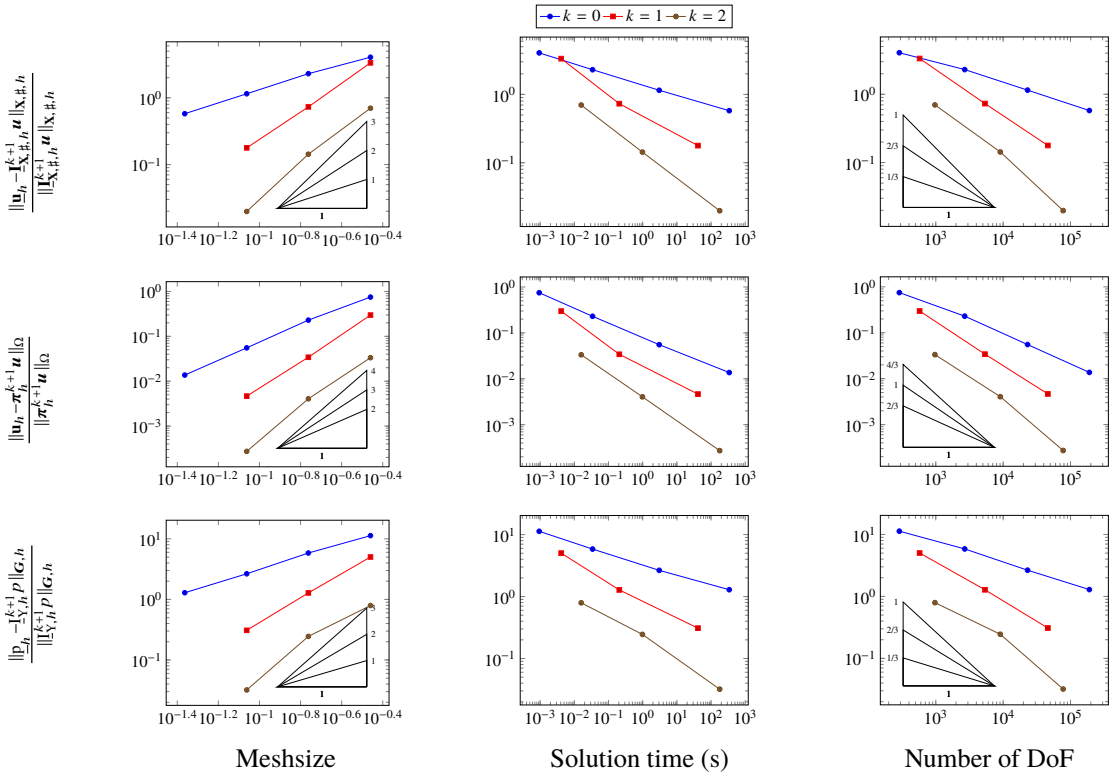

Figure 4: Errors vs. h, solution time and number of DoF on tetrahedral meshes.

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