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# A new Relaxation Method for Optimal Control of Semilinear Elliptic Variational Inequalities Obstacle Problems

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## Abstract

In this paper, we investigate optimal control problems governed by semilinear elliptic variational inequalities involving constraints on the state, and more precisely the obstacle problem. Since we adopt a numerical point of view, we first relax the feasible domain of the problem, then using both mathematical programming methods and penalization methods we get optimality conditions with smooth Lagrange multipliers. Some numerical experiments using IPOPT algorithm are presented to verify the efficiency of our approach.

*Keywords:* Optimal control, Lagrange multipliers, Variational inéqualities, mathematical programming, Smoothing methods, IPOPT.

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## 1. INTRODUCTION

In this paper, we investigate optimal control problems where the state is described by semilinear variational inequalities. These problems involve state constraints as well. We use the method of [6] to obtain a generalization of the results of the quoted paper to the semilinear case. It is known that Lagrange multipliers may not exist for such problems [7]. Nevertheless, providing qualifications conditions, one can exhibit multipliers for relaxed problems. These multipliers usually allow to get optimality conditions of Karush-Kuhn-Tucker type. Our purpose is to get optimality conditions that are useful from a numerical point of view. Indeed, we have to ensure the existence of Lagrange multipliers to prove the convergence of lagrangian methods and justify their use. These kind of problems have been extensively studied by many authors, see for instance [2, 12, 17] .

The variational inequality will be interpreted as a state equation, introducing another control function as in [6] . Then, the optimal control problem may be considered as a "standard" control problem governed by a semilinear partial differential equation, involving pure and mixed control-state constraints which are not necessarily convex. In order to derive some optimality conditions, we have to "relax" the domain; so we do not solve the original problem but this point of view will be justied and commented. Then, the use of Mathematical Programming in Banach spaces methods [19, 21] and penalization techniques provides first-order necessary optimality conditions.

The first part of this paper is devoted to the presentation of the problem: we recall some classical results on variational inequalities there. In sections 3 we give approximation formulations of the original problem. In section 4 we briefly present some Mathematical Programming results in Banach spaces. Next, we use a penalization technique and apply the tools of the previous section to the penalized problem. We obtain penalized optimality conditions, and assuming some qualification conditions we may pass to the limit to get optimality conditions for the original problem. In the last section, we present some numerical results and propose conclusion.

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## 2. PROBLEM SETTING

Let  $\Omega$  be an open, bounded subset of  $\mathbb{R}^n$  with a smooth boundary  $\partial\Omega$ . We shall denote  $\| \cdot \|_V$ , the norm in Banach space  $V$ , and more precisely  $\| \cdot \|$  the  $L^2(\Omega)$ -norm. In the same way,  $\langle \cdot, \cdot \rangle$  denotes the duality product between  $H^{-1}(\Omega)$  and  $H_0^1(\Omega)$ , we will denote similarly the  $L^2(\Omega)$ -scalar product when there is no ambiguity. Let us set

$$K = \{y \mid y \in H_0^1(\Omega), y \geq \psi \text{ a.e. in } \Omega\}, \quad (2.1)$$

where  $\psi$  is a  $H^2(\Omega) \cap H_0^1(\Omega)$  function.

In the sequel  $g$  is a non decreasing,  $C^1$  real-valued function such that  $g'$  is bounded, locally Lipschitz continuous and  $f$  belongs to  $L^2(\Omega)$ . Moreover,  $U_{ad}$  is a non empty, closed and convex subset of  $L^2(\Omega)$ .

For each  $v$  in  $U_{ad}$  we consider the following variational inequality problem : find  $y \in K$  such that

$$a(y, z) + G(y) - G(z) \geq \langle v + f, y - z \rangle \quad \forall z \in K, \quad (2.2)$$

where  $G$  is a primitive function of  $g$ , and  $a$  is a bilinear form defined on  $H_0^1(\Omega) \times H_0^1(\Omega)$  by

$$a(y, z) = \sum_{i,j=1}^n \int_{\Omega} a_{ij} \frac{\partial y}{\partial x_i} \frac{\partial z}{\partial x_j} dx + \sum_{i=1}^n \int_{\Omega} b_i \frac{\partial y}{\partial x_i} z dx + \int_{\Omega} cyz dx, \quad (2.3)$$

where  $a_{ij}, b_i, c$  belong to  $L^\infty(\Omega)$ . Moreover, we assume that  $a_{ij}$  belongs to  $C^{0,1}(\bar{\Omega})$  (the space of Lipschitz continuous functions in  $\Omega$ ) and that  $c$  is nonnegative. The bilinear form  $a(\cdot, \cdot)$  is continuous on  $H_0^1(\Omega) \cap H_0^1(\Omega)$  :

$$\exists M > 0, \forall (y, z) \in H_0^1(\Omega) \cap H_0^1(\Omega) \quad a(y, z) \leq M \|y\|_{H_0^1(\Omega)} \|z\|_{H_0^1(\Omega)} \quad (2.4)$$

and is coercive :

$$\exists \delta > 0, \forall y \in H_0^1(\Omega), \quad a(y, y) \geq \delta \|y\|_{H_0^1(\Omega)}^2. \quad (2.5)$$

We set  $A$  the elliptic differential operator from  $H_0^1(\Omega)$  to  $H^{-1}(\Omega)$  defined by

$$\forall (z, v) \in H_0^1(\Omega) \times H_0^1(\Omega) \quad \langle Ay, z \rangle = a(y, z).$$

For any  $v \in L^2(\Omega)$ , problem (2.2) has a unique solution  $y = y[v] \in H_0^1(\Omega)$ , since the coercivity of the problem in  $y$ , and  $v$ . As the obstacle function belongs to  $H^2(\Omega)$  we have an additional regularity result :  $y \in H^2(\Omega) \times H_0^1(\Omega)$  (see [3, 4]). Moreover (2.2) is equivalent to (see [17])

$$Ay + g(y) = f + v + \xi, \quad y \geq \psi, \quad \xi \geq 0, \quad \langle \xi, y - \psi \rangle = 0, \quad (2.6)$$

where " $\xi \geq 0$ " stands for " $\xi(x) \geq 0$  almost everywhere on  $\Omega$ ". The above equation can be viewed as the optimality system for problem (2.2) :  $\xi$  is the multiplier associated to the constraint  $y \geq \psi$ . It is a priori an element of  $H^{-1}(\Omega)$  but the regularity result for  $y$  shows that  $\xi \in L^2(\Omega)$ , so that  $\langle \xi, y - \psi \rangle_{H^{-1}(\Omega) \times L^2(\Omega)} = \langle \xi, y - \psi \rangle$ .

**Remark 2.1.** Applying the simple transformation  $y^* = y - \psi$ , we may assume that  $\psi = 0$ . Of course functions  $g$  and  $f$  are modified as well, but this shift preserves their generic properties (local Lipschitz-continuity, monotonicity).

In the sequel  $g$  is non decreasing,  $C^1$  real-valued function such that

$$\exists \gamma \in \mathbb{R}, \exists \beta \geq 0 \text{ such that } \forall y \in \mathbb{R} \quad |g(y)| \leq \gamma + \beta|y|. \quad (2.7)$$

We denote similarly the real valued function  $g$  and the Nemitsky operator such that  $g(y)(x) = g(y(x))$  for every  $x \in \Omega$ .

Therefore we keep the same notations. Now, let us consider the optimal control problem defined as follows :

$$\min \left\{ J(y, v) \stackrel{\text{def}}{=} \frac{1}{2} \int_{\Omega} (y - z_d)^2 dx + \frac{\nu}{2} \int_{\Omega} (v - v_d)^2 dx \mid y = y[v], v \in U_{ad}, y \in K \right\},$$

where  $z_d, v_d \in L^2(\Omega)$  and  $\nu > 0$  are given quantities.

This problem is equivalent to the problem governed by a state equation (instead of inequality) with mixed state and control constraints :

$$\min \left\{ J(y, v) = \frac{1}{2} \int_{\Omega} (y - z_d)^2 dx + \frac{\nu}{2} \int_{\Omega} (v - v_d)^2 dx \right\}, \quad (\mathcal{P})$$

$$Ay + g(y) = f + v + \xi \text{ in } \Omega, \quad y = 0 \text{ on } \partial\Omega, \quad (2.8)$$

$$(y, v, \xi) \in \mathcal{D}, \quad (2.9)$$

where

$$\mathcal{D} = \{(y, v, \xi) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \mid v \in U_{ad}, y \geq 0, \xi \geq 0, \langle y, \xi \rangle = 0\}. \quad (2.10)$$

We assume that the feasible set  $\tilde{\mathcal{D}} = \{(y, v, \xi) \in \mathcal{D} \mid \text{relation (2.8) is satisfied}\}$  is non empty. We know, then that problem  $(\mathcal{P})$  has at least an optimal solution (not necessarily unique) that we shall denote  $(\bar{y}, \bar{v}, \bar{\xi})$ , since the coercivity of the problem in  $y$ , and  $v$  see for instance [17].

Similar problems have been studied also in [9] but in the convex context ( $\mathcal{D}$  is convex). Here, the main difficulty comes from the fact that the feasible domain  $\mathcal{D}$  is non-convex and has an empty relative interior because of the bilinear constraint " $\langle y, \xi \rangle = 0$ ".

So, we cannot use generic convex analysis methods that have been used for instance in [9]. To derive optimality conditions in this case, we are going to use methods adapted to quite general mathematical programming. Unfortunately, the domain  $\mathcal{D}$  (i.e. the constraints set) does not satisfy the usual (quite weak) assumption of mathematical programming theory. This comes essentially from the fact that  $L^\infty$ -interior of  $\mathcal{D}$  is empty.

So we cannot ensure the existence of Lagrange multipliers. This problem does not satisfy classical constraint qualifications (in the usual KKT sense). One can find several counter-examples in finite and infinite dimension in [7].

### 3. A RELAXED PROBLEM

In order to "relax" the complementarity constraint " $\langle y, \xi \rangle = 0$ " we introduce a family of  $\mathcal{C}^1$  functions  $\theta_\alpha : \mathbb{R}^+ \rightarrow [0, 1[$ , ( $\alpha > 0$ ) with the following properties (see [14] for more precision on these smoothing functions):

- (i)  $\forall \alpha > 0$ ,  $\theta_\alpha$  is nondecreasing, concave and  $\theta_\alpha(1) < 1$ ,
- (ii)  $\forall \alpha > 0 \quad \theta_\alpha(0) = 0$ ,
- (iii)  $\forall x > 0 \quad \lim_{\alpha \rightarrow 0} \theta_\alpha(x) = 1 \quad \text{and} \quad \lim_{\alpha \rightarrow 0} \theta'_\alpha(0) > 0$ .

**Example 3.1.** The function below satisfy assumption (i – iii) ( see [14]):

$$\begin{aligned} \theta_\alpha^1(x) &= \frac{x}{x + \alpha}, \\ \theta_\alpha^W(x) &= 1 - e^{-\frac{x}{\alpha}}, \\ \theta_\alpha^{\log}(x) &= \frac{\log(1 + x)}{\log(1 + x + \alpha)}. \end{aligned}$$

Functions  $\theta_\alpha$  are built to approximate the complementarity constraint in the following sense :

$$\forall (x, y) \in \mathbb{R} \times \mathbb{R} \quad xy = 0 \iff \theta_\alpha(x) + \theta_\alpha(y) \leq 1 \text{ for } \alpha \text{ small enough.}$$

More precisely, we have the following proposition.

**Proposition 3.1.** *Let  $(y, v, \xi) \in \mathcal{D}$  and  $\theta_\alpha^1$  satisfying  $(i - iv)$ . Then*

$$\langle y, \xi \rangle = 0 \implies \theta_\alpha^1(y) + \theta_\alpha^1(x) \leq 1 \quad \text{a.e. in } \Omega.$$

The proof of the proposition it is based on the followings lemmas :

**Lemma 3.1.** *For any  $\varepsilon > 0$ , and  $x, y \geq 0$ , there exists  $\alpha_0 > 0$  such that*

$$\forall \alpha \leq \alpha_0, \quad (\min(x, y) = 0) \implies (\theta_\alpha(x) + \theta_\alpha(y) \leq 1) \implies (\min(x, y) \leq \varepsilon).$$

**Proof 3.1.** *The first property is obvious since  $\theta_\alpha(0) = 0$  and  $\theta_\alpha \leq 1$ .*

*Using assumption (iii) for  $x = \varepsilon$ , we have*

$$\forall r > 0, \quad \exists \alpha_0 > 0 \mid \forall \alpha \leq \alpha_0 \quad 1 - \theta_\alpha(\varepsilon) < r,$$

*so that, if we suppose that  $\min(x, y) > \varepsilon$ , assumption (i) gives*

$$\forall r > 0, \quad \theta_\alpha(x) + \theta_\alpha(y) > 2\theta_\alpha(\varepsilon) > 2(1 - r).$$

*Then if we choose  $r < \frac{1}{2}$ , we obtain that  $\theta_\alpha(x) + \theta_\alpha(y) > 1$ .*

**Lemma 3.2.** *we have*

$$(1) \quad \forall x \geq 0, \forall y \geq 0 \quad \theta_\alpha^1(x) + \theta_\alpha^1(y) \leq 1 \iff x.y \leq \alpha^2, \quad \text{and}$$

$$(2) \quad \forall x \geq 0, \forall y \geq 0 \quad x.y = 0 \implies \theta_\alpha^{\geq 1}(x) + \theta_\alpha^{\geq 1}(y) \leq 1 \implies x.y \leq \alpha^2,$$

*where  $\theta_\alpha^{\geq 1}$  verifying  $(i - iv)$  and  $\theta_\alpha^{\geq 1} \geq \theta_\alpha^1$ .*

**Proof 3.2.** (1) *We have*

$$\theta_\alpha^1(x) + \theta_\alpha^1(y) = \frac{2xy + \alpha x + \alpha y}{xy + \alpha x + \alpha y + \alpha^2},$$

*so that*

$$\begin{aligned} \theta_\alpha^1(x) + \theta_\alpha^1(y) \leq 1 &\iff 2xy + \alpha x + \alpha y \leq xy + \alpha x + \alpha y + \alpha^2 \\ &\iff x.y \leq \alpha^2. \end{aligned} \tag{3.1}$$

*The first part of (2) follows obviously from Lemma 3.1 and the second one is a direct consequence of (1) since*

$$\theta_\alpha^{\geq 1}(x) + \theta_\alpha^{\geq 1}(y) \leq 1 \implies \theta_\alpha^1(x) + \theta_\alpha^1(y) \leq 1.$$

More precisely, we consider the domain  $\mathcal{D}_\alpha$  instead of  $\mathcal{D}$ , with  $\alpha > 0$  (using the function  $\theta_\alpha^1$ ) we obtain :

$$\mathcal{D}_\alpha = \left\{ (y, v, \xi) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \mid v \in U_{ad}, y \geq 0, \xi \geq 0, \frac{y}{y + \alpha} + \frac{\xi}{\xi + \alpha} \leq 1, \text{ a.e. in } \Omega \right\}. \tag{3.2}$$

We may justify and motivate this points of view numerically, since it is usually not possible to ensure " $\langle y, \xi \rangle = 0$ " during a computation but rather " $\frac{y}{y + \alpha} + \frac{\xi}{\xi + \alpha} \leq 1$ " where  $\alpha$  is a prescribed tolerance : it may be chosen small as wanted, but strictly positive.

So the problem turns to be qualified if the bilinear constraint " $\langle y, \xi \rangle = 0$ " is relaxed to " $\frac{y}{y + \alpha} + \frac{\xi}{\xi + \alpha} \leq 1$ " a.e. in  $\Omega$ . In the sequel, we consider an optimal control problem ( $\mathcal{P}^\alpha$ ) where the feasible domain is  $\mathcal{D}_\alpha$  instead of  $\mathcal{D}$ .

Moreover, we must add a bound constraint on the control  $\xi$  to be able to ensure the existence of a solution of this relaxed problem. More precisely we consider :

$$(\mathcal{P}^\alpha) \quad \begin{cases} \min J(y, v) \\ Ay + g(y) = f + v + \xi \text{ in } \Omega, \quad y \in H_0^1(\Omega), \\ (y, v, \xi) \in \mathcal{D}_{\alpha, R} \end{cases}$$

where  $R > 0$  may be very large and

$$\mathcal{D}_{\alpha, R} = \{(y, v, \xi) \in \mathcal{D}_\alpha \mid \|\xi\|_{L^2(\Omega)} \leq R\}.$$

From now on, we omit the index  $R$  since this constant is definitely fixed, such that

$$R \geq \|\bar{\xi}\|_{L^2(\Omega)}, \quad (3.3)$$

(we recall that  $(\bar{y}, \bar{v}, \bar{\xi})$  is a solution of  $(\mathcal{P})$ ).

We will denote  $\mathcal{D}_\alpha := \mathcal{D}_{\alpha, R}$ , and  $V_{ad} = \{\xi \in L^2(\Omega) \mid \xi \geq 0, \|\xi\|_{L^2(\Omega)} \leq R\}$ .  $V_{ad}$  is obviously a closed, convex subset of  $L^2(\Omega)$

As  $(\bar{y}, \bar{v}, \bar{\xi}) \in \mathcal{D}$ , we see (with (3.3)) that  $\mathcal{D}_\alpha$  is non empty for any  $\alpha > 0$ .

### 3.1. Existence Result

In order to prove an existence result for  $(\mathcal{P}^\alpha)$ , we state first a basic but essential lemma.

**Lemma 3.3.** *Assume that  $(y_n, v_n)$  is a bounded sequence in  $H_0^1(\Omega) \times L^2(\Omega)$  such that  $\xi_n := Ay_n + g(y_n) - f - v_n$  is bounded in  $L^2(\Omega)$ . Then, one may extract subsequences (still denoted similarly) such that*

- $v_n$  converges weakly to some  $\tilde{v}$  in  $L^2(\Omega)$ ,
- $y_n$  converges strongly to some  $\tilde{y}$  in  $H_0^1(\Omega)$ ,
- $g(y_n)$  converges strongly to  $g(\tilde{y})$  in  $L^2(\Omega)$ ,
- $Ay_n + g(y_n) - f - v_n$  converges weakly to  $A\tilde{y} + g(\tilde{y}) - f - \tilde{v}$  in  $L^2(\Omega)$ .

**Proof 3.3.** *Let  $(y_n, v_n)$  be a bounded sequence in  $H_0^1(\Omega) \times L^2(\Omega)$ ; therefore  $(y_n, v_n)$  weakly converges to some  $(\tilde{y}, \tilde{v})$  in  $H_0^1(\Omega) \times L^2(\Omega)$  (up to a subsequence). Similarly,  $\xi_n$  weakly converges to some  $\tilde{\xi}$  in  $L^2(\Omega)$ . Thanks to [15] (Theorem 17.5, p174), assumption (2.7) yields that*

$$(y_n)_{n \geq 0} \text{ bounded in } L^2(\Omega) \implies (g(y_n))_{n \geq 0} \text{ bounded in } L^2(\Omega).$$

*As,  $y_n$  weakly converges to  $\tilde{y}$  in  $H_0^1(\Omega)$ , it strongly converges in  $L^2(\Omega)$  a.e in  $\Omega$ . As  $g$  is continuous,  $g(y_n)$  converges a.e. in  $\Omega$  as well (up to subsequences). We conclude then (Lebesgue theorem), that  $g(y_n)$  strongly converges to  $g(\tilde{y})$  in  $L^2(\Omega)$ .*

*Moreover when  $Ay_n = -g(y_n) + f + v_n + \xi_n$  is bounded in  $L^2(\Omega)$  it will and converge weakly to some  $\tilde{z}$  in  $L^2(\Omega)$ . As  $y_n$  weakly converges to  $\tilde{y}$  in  $H_0^1(\Omega)$ , then  $Ay_n$  converges to  $A\tilde{y}$  in  $H^{-1}(\Omega)$ , so  $\tilde{z} = A\tilde{y}$  and  $Ay_n$  weakly converges to  $A\tilde{y}$  in  $L^2(\Omega)$  as well. Therefore  $Ay_n$  strongly converges to  $A\tilde{y}$  in  $H^{-1}(\Omega)$ .*

*Finally we get the weak convergence of  $Ay_n + g(y_n) - f - v_n$  to  $A\tilde{y} + g(\tilde{y}) - f - \tilde{v}$  in  $L^2(\Omega)$  and the strong convergence of  $y_n$  to  $\tilde{y}$  in  $H_0^1(\Omega)$ .*

So that, we can consider that problem  $(\mathcal{P}^\alpha)$  is a "good" approximation of the original problem  $(\mathcal{P})$  in the following sense :

**Theorem 3.1.** *For any  $\alpha > 0$ ,  $(\mathcal{P}^\alpha)$  has at least one optimal solution (denoted  $(y_\alpha, v_\alpha, \xi_\alpha)$ ). Moreover, when  $\alpha$  goes to 0,  $y_\alpha$  strongly converges to  $\tilde{y}$  in  $H_0^1(\Omega)$  (up to a subsequence),  $v_\alpha$  strongly converges to  $\tilde{v}$  in  $L^2(\Omega)$  (up to a subsequence),  $\xi_\alpha$  weakly converges to  $\tilde{\xi}$  in  $L^2(\Omega)$  (up to a subsequence), where  $(\tilde{y}, \tilde{v}, \tilde{\xi})$  is a solution of  $(\mathcal{P})$ .*

**Proof 3.4.** Let  $(y_n, v_n, \xi_n)$  be a minimizing sequence such that  $J(y_n, v_n)$  converges to  $d^\alpha = \inf(\mathcal{P}^\alpha)$ . As  $J(y_n, v_n)$  is bounded, there exists a constant  $C$  such that we have :

$$\forall n \quad \|v_n\|_{L^2(\Omega)} \leq C.$$

So, we may extract a subsequence (denoted similarly) such that  $v_n$  converges to  $v_\alpha$  weakly in  $L^2(\Omega)$  and strongly in  $H^{-1}(\Omega)$ . As  $U_{ad}$  is a closed convex set, it is weakly closed and  $v_\alpha \in U_{ad}$ .

On the other hand, we have  $Ay_n + g(y_n) - f - v_n = \xi_n$  and. So

$$\langle Ay_n, y_n \rangle + \langle g(y_n), y_n \rangle = \langle f + v_n, y_n \rangle + \langle y_n, \xi_n \rangle.$$

In view of Lemma 3.2, we have :

$$\forall y_n \geq 0, \forall \xi_n \geq 0 \quad \frac{y_n}{y_n + \alpha} + \frac{\xi_n}{\xi_n + \alpha} \leq 1 \iff y_n \xi_n \leq \alpha^2,$$

the integral by the two ways, gives

$$\frac{y_n}{y_n + \alpha} + \frac{\xi_n}{\xi_n + \alpha} \leq 1 \iff y_n \xi_n \leq \alpha^2 \implies \langle y_n, \xi_n \rangle \leq \alpha^2 \text{Area}(\Omega).$$

So

$$\langle Ay_n, y_n \rangle + \langle g(y_n), y_n \rangle = \langle f + v_n, y_n \rangle + \langle y_n, \xi_n \rangle \leq \langle f + v_n, y_n \rangle + \alpha^2 \text{Area}(\Omega).$$

The monotonicity of  $g$  gives

$$\langle Ay_n, y_n \rangle \leq \langle Ay_n, y_n \rangle + \langle g(y_n) - g(0), y_n \rangle \leq \langle f + v_n - g(0), y_n \rangle + \alpha^2 \text{Area}(\Omega).$$

Using the coercivity of  $A$ , we obtain

$$\delta \|y_n\|_{H_0^1(\Omega)}^2 \leq \|f + v_n - g(0)\|_{H^{-1}(\Omega)} \|y_n\|_{H_0^1(\Omega)} + \alpha^2 \text{Area}(\Omega) \leq C \|y_n\|_{H_0^1(\Omega)} + \alpha^2 \text{Area}(\Omega).$$

This yields that  $y_n$  is bounded in  $H_0^1(\Omega)$ , since  $\Omega$  is bounded, so  $y_n$  converges to  $y_\alpha$  weakly in  $H_0^1(\Omega)$  and strongly in  $L^2(\Omega)$ . Moreover as  $y_n \in K$ , and  $K$  is a closed convex set,  $K$  is weakly closed and  $y_\alpha \in K$ . We have assumed that  $V_{ad}$  is  $L^2(\Omega)$ -bounded. So, we can apply Lemma 3.3, and obtain that  $\xi_n$  weakly converges to  $\xi_\alpha = Ay_\alpha + g(y_\alpha) - f - v_\alpha \in V_{ad}$  in  $L^2(\Omega)$ .

**Remark 3.1.**  $\xi_n = Ay_n + g(y_n) - f - v_n$ , weakly converges to  $\xi_\alpha = Ay_\alpha + g(y_\alpha) - f - v_\alpha$  in  $H^{-1}(\Omega)$ , Unfortunately the weak convergence of  $\xi_n$  to  $\xi_\alpha$  in  $H^{-1}(\Omega)$  is not sufficient to conclude. We need this sequence to converge weakly in  $L^2(\Omega)$ . That is the reason why we have bounded  $\xi_n$  in  $L^2(\Omega)$ .

At last,  $\frac{y_n}{y_n + \alpha} + \frac{\xi_n}{\xi_n + \alpha}$  converges to  $\frac{y_\alpha}{y_\alpha + \alpha} + \frac{\xi_\alpha}{\xi_\alpha + \alpha}$  because of the strong convergence of  $y_n$  in  $L^2(\Omega)$  and the weak convergence of  $\xi_n$  in  $L^2(\Omega)$  and we obtain  $\frac{y_\alpha}{y_\alpha + \alpha} + \frac{\xi_\alpha}{\xi_\alpha + \alpha} \leq 1$  : we just proved that  $(y_\alpha, v_\alpha, \xi_\alpha) \in \mathcal{D}_\alpha$ . The weak convergence and the lower semi-continuity of  $J$  give :

$$d^\alpha = \liminf_{n \rightarrow \infty} J(y_n, v_n) \geq J(y_\alpha, v_\alpha) \geq d^\alpha.$$

So  $J(y_\alpha, v_\alpha) = d^\alpha$  and  $(y_\alpha, v_\alpha, \xi_\alpha)$  is a solution of  $(\mathcal{P}^\alpha)$ .

- Now, let us prove the second part of the theorem. First we note that  $(\bar{y}, \bar{v}, \bar{\xi})$  belongs to  $\mathcal{D}^\alpha$  for any  $\alpha > 0$ . So :

$$\forall \alpha > 0 \quad J(y_\alpha, v_\alpha) \leq J(\bar{y}, \bar{v}) < +\infty. \quad (3.4)$$

and  $v_\alpha$  and  $y_\alpha$  are bounded respectively in  $L^2(\Omega)$  and  $H_0^1(\Omega)$ . Indeed, we use the previous arguments since  $v_\alpha$  is bounded in  $L^2(\Omega)$  and

$$\delta \|y_\alpha\|_{H_0^1(\Omega)}^2 \leq \|f + v_\alpha - g(0)\|_{H^{-1}(\Omega)} \|y_\alpha\|_{H_0^1(\Omega)} + \alpha^2 \text{Area}(\Omega) \leq C \|y_\alpha\|_{H_0^1(\Omega)} + \alpha^2 \text{Area}(\Omega).$$

So (extracting a subsequence)  $v_\alpha$  weakly converges to some  $\tilde{v}$  in  $L^2(\Omega)$  and  $y_\alpha$  converges to some  $\tilde{y}$  weakly in  $H_0^1(\Omega)$  and strongly in  $L^2(\Omega)$ . As above, it is easy to see that  $\xi_\alpha$  weakly converges to  $\tilde{\xi} = A\tilde{y} + g(\tilde{y}) - f - \tilde{v}$  in  $L^2(\Omega)$  (Thanks Lemma 3.3), and that  $\tilde{y} \in K$ ,  $\tilde{v} \in U_{ad}$ ,  $\tilde{\xi} \in V_{ad}$ . In the same way  $\frac{y_\alpha}{y_\alpha + \alpha} + \frac{\xi_\alpha}{\xi_\alpha + \alpha}$  converges to  $\frac{\tilde{y}}{\tilde{y} + \alpha} + \frac{\tilde{\xi}}{\tilde{\xi} + \alpha}$ . As  $0 \leq \frac{y_\alpha}{y_\alpha + \alpha} + \frac{\xi_\alpha}{\xi_\alpha + \alpha} \leq 1$ , from Lemma 3.2 we get :

$$0 \leq \frac{y_\alpha}{y_\alpha + \alpha} + \frac{\xi_\alpha}{\xi_\alpha + \alpha} \leq 1 \iff 0 \leq y_\alpha \xi_\alpha \leq \alpha^2,$$

at the limit as  $\alpha \searrow 0$  this implies that  $\tilde{y}\tilde{\xi} = 0 \iff \langle \tilde{y}, \tilde{\xi} \rangle = 0$ . So  $(\tilde{y}, \tilde{v}, \tilde{\xi}) \in \mathcal{D}$ . This yields that

$$J(\tilde{y}, \tilde{v}) \leq J(\tilde{y}, \tilde{v}). \quad (3.5)$$

Once again, we may pass to the inf-limite in (3.4) to obtain :

$$J(\tilde{y}, \tilde{v}) \leq \liminf_{\alpha \rightarrow 0} J(y_\alpha, v_\alpha) \leq J(\tilde{y}, \tilde{v}).$$

This implies that

$$J(\tilde{y}, \tilde{v}) = J(\tilde{y}, \tilde{v}),$$

therefore  $(\tilde{y}, \tilde{v}, \tilde{\xi})$  is a solution of  $(\mathcal{P})$ . Moreover, as  $\lim_{\alpha \rightarrow 0} J(y_\alpha, v_\alpha) = J(\tilde{y}, \tilde{v})$  and  $y_\alpha$  strongly converges to  $\tilde{y}$  in  $L^2(\Omega)$ , we get  $\lim_{\alpha \rightarrow 0} \|v_\alpha\|_{L^2(\Omega)} = \|\tilde{v}\|_{L^2(\Omega)}$ , so that  $v_\alpha$  strongly converges to  $\tilde{v}$  in  $L^2(\Omega)$ .

We already know that  $\xi_\alpha$  weakly converges to  $\tilde{\xi}$  in  $L^2(\Omega)$ . So  $\xi_\alpha + v_\alpha - g(y_\alpha) + f = Ay_\alpha$  converges to  $\tilde{\xi} + \tilde{v} - g(\tilde{y}) + f = A\tilde{y}$  weakly in  $L^2(\Omega)$  and strongly in  $H^{-1}(\Omega)$ . As  $A$  is an isomorphism from  $H_0^1(\Omega)$  to  $H^{-1}(\Omega)$  this yields that  $y_\alpha$  strongly converges to  $\tilde{y}$  in  $H_0^1(\Omega)$ .

We see then, that solutions of problem  $(\mathcal{P}^\alpha)$  are "good" approximations of the desired solution of problem  $(\mathcal{P})$ .

Now, we would like to derive optimality conditions for the problem  $(\mathcal{P}^\alpha)$ , for  $\alpha > 0$ .

In the sequel, we study the unconstrained control case:  $U_{ad} = L^2(\Omega)$ . We first present some Mathematical Programming tools that allow to prove the existence of Lagrange multipliers.

#### 4. THE MATHEMATICAL PROGRAMMING POINT OF VIEW

The non convexity of the feasible domain, does not allow to use convex analysis to get the existence of Lagrange multipliers. So we are going to use quite general mathematical programming methods in Banach spaces and adapt them to our framework.

The following results are mainly due to Zowe and Kurcyusz [21] and Troltsch [19] and we briefly present them in the following.

Let us consider real Banach spaces  $\mathcal{X}$ ,  $\mathcal{U}$ ,  $\mathcal{Z}_1$ ,  $\mathcal{Z}_2$  and a convex closed "admissible" set  $U_{ad} \subseteq \mathcal{U}$ . In  $\mathcal{Z}_2$  a convex closed cone  $P$  is given so that  $\mathcal{Z}_2$  is partially ordered by  $x \leq y \iff x - y \in P$ . We deal also with :

$$\begin{aligned} f : \mathcal{X} \times \mathcal{U} &\rightarrow \mathbb{R}, \text{ Fréchet-differentiable functional,} \\ T : \mathcal{X} \times \mathcal{U} &\rightarrow \mathcal{Z}_1 \text{ and } G : \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{Z}_2 \text{ continuously Fréchet-differentiable operators.} \end{aligned}$$

Now, consider the mathematical programming problem defined by :

$$\min \{f(x, u) \mid T(x, u) = 0, G(x, u) \leq 0, u \in U_{ad}\}. \quad (4.1)$$



We denote the partial Fréchet-derivative of  $f, T$ , and  $G$  with respect to  $x$  and  $u$  by a corresponding index  $x$  or  $u$ . We suppose that the problem (4.1) has an optimal solution that we call  $(x_0, u_0)$ , and we introduce the sets :

$$\begin{aligned}\mathcal{U}_{ad}(u_0) &= \{u \in \mathcal{U} \mid \exists \lambda \geq 0, \exists u^* \in \mathcal{U}_{ad}, u = \lambda(u^* - u_0)\}, \\ P(G(x_0, u_0)) &= \{z \in \mathcal{Z}_2 \mid \exists \lambda \geq 0, \exists p \in -P, z = p - \lambda G(x_0, u_0)\}, \\ P^+ &= \{y \in \mathcal{Z}_2^* \mid \langle y, p \rangle \geq 0, \forall p \in P\}.\end{aligned}$$

One may now announce the main result about the existence of optimality conditions.

**Theorem 4.1.** *Let  $u_0$  be an optimal control with corresponding optimal state  $x_0$  and suppose that the following regularity condition is fulfilled :*

$$\forall (z_1, z_2) \in \mathcal{Z}_1 \times \mathcal{Z}_2 \quad \text{the system} \quad \begin{cases} T'(x_0, u_0)(x, u) &= z_1 \\ G'(x_0, u_0)(x, u) - p &= z_2 \end{cases} \quad (4.2)$$

*is solvable with  $(x, u, p) \in \mathcal{X} \times \mathcal{U}_{ad}(u_0) \times P(G(x_0, u_0))$ .*

*Then a Lagrange multiplier  $(y_1, y_2) \in \mathcal{Z}_1^* \times \mathcal{Z}_2^*$  exists such that*

$$f'_x(x_0, u_0) + T'_x(x_0, u_0) * y_1 + G'_x(x_0, u_0) * y_2 = 0, \quad (4.3)$$

$$\langle f'_x(x_0, u_0) + T'_x(x_0, u_0) * y_1 + G'_x(x_0, u_0) * y_2, u - u_0 \rangle \geq 0, \quad \forall u \in \mathcal{U}_{ad}, \quad (4.4)$$

$$y_2 \in P^+, \quad \langle y_2, G(x_0, u_0) \rangle = 0. \quad (4.5)$$

Mathematical programming theory in Banach spaces allows to study problems where the feasible domain is not convex: this precisely our case (and we cannot use the classical convex theory and the Gâteaux differentiability to derive some optimality conditions). The Zowe and Kurcyusz condition [21] is a very weak condition to ensure the existence of Lagrange multipliers. It is natural to try to see if this condition is satisfied for the original problem ( $\mathcal{P}$ ) : unfortunately, it is impossible (see [5]) and this is another justification (from a theoretical point of view) of the fact that we have to take  $\mathcal{D}_\alpha$  instead of  $\mathcal{D}$ .

On the other hand, if we apply the previous general result "directly" to  $(\mathcal{P}^\alpha)$  we obtain a complicated qualification condition (4.2) which seems difficult to ensure. So we would rather mix these "mathematical-programming methods" with a penalization method in order to "relax" the state-equation as well and make the qualification condition weaker and simpler.

## 5. PENALIZATIN APPROACH

### 5.1. The penalized problem

One of the difficulties comes from the fact that we have a coupled system. It would be easier if we had only one condition. In order to split the different constraints and make them "independent", we penalize the state equation to obtain an optimization problem with non convex constraints. Then we apply previous method to get optimality conditions for the penalized problem. Of course, we may decide to penalize the bilinear constraint instead of the state equation : this leads to the same results.

Moreover we focus on the solution  $(y_\alpha, v_\alpha, \xi_\alpha)$ , so, following Barbu [2], we add some adapted penalization terms to the objective functional  $J$ .

From nowon,  $\alpha > 0$  is fixed , so we omit the index  $\alpha$  when no confusion is possible. For any  $\varepsilon > 0$  we define a penalized functional  $J_\varepsilon^\alpha$  on  $(H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega) \times L^2(\Omega)$  as following :

$$J_\varepsilon^\alpha(y, v, \xi) = \begin{cases} J(y, v) + \frac{1}{2\varepsilon} \| Ay + g(y) - f - v - \xi \|_{L^2(\Omega)}^2 \\ + \frac{1}{2} \| A(y - y_\alpha) \|_{L^2(\Omega)}^2 + \frac{1}{2} \| v - v_\alpha \|_{L^2(\Omega)}^2 \\ + \frac{1}{2} \| \xi - \xi_\alpha \|_{L^2(\Omega)}^2 \end{cases} \quad (5.1)$$

and we consider the penalized optimization problem

$$\min \{ J_\varepsilon^\alpha(y, v, \xi) \mid (y, v, \xi) \in \mathcal{D}_\alpha, y \in H^2(\Omega) \cap H_0^1(\Omega) \} \quad (\mathcal{P}_\alpha^\varepsilon)$$

**Theorem 5.1.** *The penalized problem  $(\mathcal{P}_\alpha^\varepsilon)$  has at least a solution  $(y_\varepsilon, v_\varepsilon, \xi_\varepsilon) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega) \times L^2(\Omega)$ .*

**Proof 5.1.** *The proof is almost the same as the one of Theoreme 3.1. The main difference is that we have no longer  $Ay_n + g(y_n) - f - v_n - \xi_n = 0$ , for any minimizing sequence.*

*Anyway,  $y_n, v_n, \xi_n, Ay_n$  and  $g(y_n)$  are bounded in  $L^2(\Omega)$ , and it is standard to see that any weak-cluster point of this minimizing sequence is feasible and is a solution to the problem,*

$$Ay_n + g(y_n) - f - v_n - \xi_n \rightharpoonup 0, \text{ weakly in } L^2(\Omega).$$

Now we may also give a result concerning the asymptotic behavior of the solutions of the penalized problems.

**Theorem 5.2.** *When  $\varepsilon$  goes to 0,  $(y_\varepsilon, v_\varepsilon, \xi_\varepsilon)$  strongly converges to  $(y_\alpha, v_\alpha, \xi_\alpha) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega) \times L^2(\Omega)$ .*

**Proof 5.2.** *The proof is quite similar to the one of Theoreme 3.1. We have :*

$$\forall \varepsilon > 0 \quad J_\varepsilon^\alpha(y_\varepsilon, v_\varepsilon, \xi_\varepsilon) \leq J_\varepsilon^\alpha(y_\alpha, v_\alpha, \xi_\alpha) = J(y_\alpha, v_\alpha) = j_\alpha < +\infty. \quad (5.2)$$

So

$$\frac{1}{\varepsilon} \| Ay_\varepsilon + g(y_\varepsilon) - f - v_\varepsilon - \xi_\varepsilon \|_{L^2(\Omega)}^2 + \| A(y_\varepsilon - y_\alpha) \|_{L^2(\Omega)}^2 + \| v_\varepsilon - v_\alpha \|_{L^2(\Omega)}^2 + \| \xi_\varepsilon - \xi_\alpha \|_{L^2(\Omega)}^2 \leq 2j_\alpha.$$

*Therefore  $v_\varepsilon, Ay_\varepsilon$  and  $\xi_\varepsilon$  are  $L^2(\Omega)$  - bounded; this yields that  $Ay_\varepsilon + g(y_\varepsilon) - f - v_\varepsilon$  is  $L^2(\Omega)$ -bounded and  $y_\varepsilon$  is  $H^2(\Omega) \cap H_0^1(\Omega)$ -bounded. So, using Lemma 3.3, we conclude that*

(i)  $v_\varepsilon$  converges to some  $\tilde{v}$  weakly in  $L^2(\Omega)$ ,

(ii)  $y_\varepsilon$  converges to some  $\tilde{y}$  strongly in  $H_0^1(\Omega)$ ,

(iii)  $\xi_\varepsilon$  converges to some  $\tilde{\xi}$  weakly in  $L^2(\Omega)$ , and

(iv)  $Ay_\varepsilon + g(y_\varepsilon) - f - v_\varepsilon - \xi_\varepsilon$  converges to  $A\tilde{y} + g(\tilde{y}) - f - \tilde{v} - \tilde{\xi}$  weakly in  $L^2(\Omega)$ .

*Moreover,  $\| Ay_\varepsilon + g(y_\varepsilon) - f - v_\varepsilon - \xi_\varepsilon \|_{L^2(\Omega)}^2 \leq 2\varepsilon j_\alpha$  implies the strong convergence of  $Ay_\varepsilon + g(y_\varepsilon) - f - v_\varepsilon - \xi_\varepsilon$  to 0 in  $L^2(\Omega)$ . Therefore  $A\tilde{y} + g(\tilde{y}) = f + \tilde{v} + \tilde{\xi}$ .*

*It is easy to see that  $\tilde{y} \in K, \tilde{v} \in U_{ad}$  and  $\tilde{\xi} \in V_{ad}$ . Moreover, as  $y_\varepsilon$  converges to  $\tilde{y}$  strongly in  $L^2(\Omega)$  and  $\xi_\varepsilon$  converges to  $\tilde{\xi}$  weakly in  $L^2(\Omega)$ , we know that  $\frac{y_\varepsilon}{y_\varepsilon + \alpha} + \frac{\xi_\varepsilon}{\xi_\varepsilon + \alpha} (\leq 1)$  converges to  $\frac{\tilde{y}}{\tilde{y} + \alpha} + \frac{\tilde{\xi}}{\tilde{\xi} + \alpha}$ . So  $\frac{\tilde{y}}{\tilde{y} + \alpha} + \frac{\tilde{\xi}}{\tilde{\xi} + \alpha} \leq 1$  and  $(\tilde{y}, \tilde{v}, \tilde{\xi})$  belongs to  $\mathcal{D}_\alpha$ .*

*Relation (5.2) implies that*

$$J(y_\varepsilon, v_\varepsilon) + \frac{1}{2} \| A(y_\varepsilon - y_\alpha) \|_{L^2(\Omega)}^2 + \frac{1}{2} \| v_\varepsilon - v_\alpha \|_{L^2(\Omega)}^2 + \frac{1}{2} \| \xi_\varepsilon - \xi_\alpha \|_{L^2(\Omega)}^2 \leq J(y_\alpha, v_\alpha). \quad (5.3)$$

Passing to the inf-limit and using the fact that  $(\tilde{y}, \tilde{v}, \tilde{\xi})$  belongs to  $\mathcal{D}_\alpha$ , we obtain

$$J(\tilde{y}, \tilde{v}) + \frac{1}{2} \|A(\tilde{y} - y_\alpha)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\tilde{v} - v_\alpha\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\tilde{\xi} - \xi_\alpha\|_{L^2(\Omega)}^2 \leq J(y_\alpha, v_\alpha) \leq J(\tilde{y}, \tilde{v}).$$

Therefore  $A(\tilde{y} - y_\alpha) = 0$  (which implies  $\tilde{y} = y_\alpha$  since  $A(\tilde{y} - y_\alpha) \in H_0^1(\Omega)$ ),  $\tilde{v} = v_\alpha$  and  $\tilde{\xi} = \xi_\alpha$ .

We just proved the weak convergence of  $(y_\varepsilon, v_\varepsilon, \xi_\varepsilon)$  to  $(y_\alpha, v_\alpha, \xi_\alpha)$  in  $H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ , and that  $\lim_{\varepsilon \rightarrow 0} J(y_\varepsilon, v_\varepsilon) = J(y_\alpha, v_\alpha)$ . Relation (5.3) gives

$$\|A(y_\varepsilon - y_\alpha)\|_{L^2(\Omega)}^2 + \|v_\varepsilon - v_\alpha\|_{L^2(\Omega)}^2 + \|\xi_\varepsilon - \xi_\alpha\|_{L^2(\Omega)}^2 \leq 2[J(y_\alpha, v_\alpha) - J(y_\varepsilon, v_\varepsilon)];$$

therefore we get the strong convergence of  $Ay_\varepsilon$  towards  $Ay_\alpha$  in  $L^2(\Omega)$ , that is the strong convergence of  $y_\varepsilon$  to  $y_\alpha$  in  $H^2(\Omega) \cap H_0^1(\Omega)$ . We get also the strong convergence of  $(v_\varepsilon, \xi_\varepsilon)$  towards  $(v_\alpha, \xi_\alpha)$  in  $L^2(\Omega) \times L^2(\Omega)$ . Let us remark, at last, that  $y_\varepsilon$  converges to  $y_\alpha$  uniformly in  $\bar{\Omega}$ , since  $H^2(\Omega) \cap H_0^1(\Omega) \subset \mathcal{C}(\bar{\Omega})$ .

**Corollary 5.1.** *If we define the penalized adjoint state  $p_\varepsilon$  as the solution of*

$$A^*p_\varepsilon + g'(y_\varepsilon)p_\varepsilon = y_\varepsilon - z_d \text{ on } \Omega, \quad p_\varepsilon \in H_0^1(\Omega), \quad (5.4)$$

then  $p_\varepsilon$  strongly converges to  $p_\alpha$  in  $H_0^1(\Omega)$ , where  $p_\alpha$  is defined by

$$A^*p_\alpha + g'(y_\alpha)p_\alpha = y_\alpha - z_d \text{ on } \Omega, \quad p_\alpha \in H_0^1(\Omega). \quad (5.5)$$

**Proof 5.3.** *we have seen that  $\|y_\varepsilon - y_\alpha\|_\infty \rightarrow 0$ . Therefore  $y_\varepsilon$  remains in a bounded set of  $\mathbb{R}^n$  (independent of  $\varepsilon$ ). As  $g$  is a  $C^1$  function, this means that  $\|g'(y_\varepsilon)\|_\infty$  is bounded by a constant  $C$  which does not depend on  $\varepsilon$ . In particular  $g'(y_\varepsilon)$  is bounded in  $L^2(\Omega)$  and Lebesgue's Theorem implies the strong convergence of  $g'(y_\varepsilon)$  to  $g'(y_\alpha)$  in  $L^2(\Omega)$ .*

Let  $p_\varepsilon$  be the solution of (5.4). This gives

$$\langle A^*p_\varepsilon, p_\varepsilon \rangle + \langle g'(y_\varepsilon)p_\varepsilon, p_\varepsilon \rangle = \langle y_\varepsilon - z_d, p_\varepsilon \rangle,$$

as  $g' \geq 0$  and  $A^*$  is coercive we get

$$\delta \|p_\varepsilon\|_{H_0^1(\Omega)}^2 \leq \|y_\varepsilon - z_d\|_{H^{-1}(\Omega)} \|p_\varepsilon\|_{H_0^1(\Omega)}.$$

So,  $p_\varepsilon$  is bounded in  $H_0^1(\Omega)$  and weakly converges to  $\tilde{p}$  in  $H_0^1(\Omega)$ . Moreover,  $p_\varepsilon$  is the solution to

$$A^*p_\varepsilon = -g'(y_\varepsilon)p_\varepsilon + y_\varepsilon - z_d \text{ on } \Omega,$$

the left-hand side (weakly) converges to  $-g'(y_\alpha)\tilde{p} + y_\alpha - z_d$  in  $L^2(\Omega)$ ; this achieves the proof.

## 5.2. Optimality conditions for the penalized problem

We apply Theorem 4.1 to the above penalized problem  $(\mathcal{P}_\alpha^\varepsilon)$ . We set

$$x = y, \quad u = (v, \xi), \quad (x_0, u_0) = (x_\varepsilon, v_\varepsilon, \xi_\varepsilon)$$

$$\mathcal{X} = H^2(\Omega) \cap H_0^1(\Omega), \quad \mathcal{Z}_2 = \mathcal{X}$$

$$\mathcal{U} = L^2(\Omega) \times L^2(\Omega)$$

$$\mathcal{U}_{ad} = U_{ad} \times V_{ad}, \quad P = \{y \in H^2(\Omega) \cap H_0^1(\Omega) \mid y \geq 0\} \times \mathbb{R}^+$$

We recall that  $\langle \cdot, \cdot \rangle$  denote the  $L^2(\Omega)$ -scalar product, and

$$G(y, v, \xi) = (-y, \langle 1, \frac{y}{y+\alpha} + \frac{\xi}{\xi+\alpha} \rangle - Area(\Omega)), \quad f(x, u) = J_\alpha^\varepsilon(y, v, \xi).$$

There is no equality constraint and  $G$  is  $\mathcal{C}^1$ ,

$$G'(y_\varepsilon, v_\varepsilon, \xi_\varepsilon)(y, v, \xi) = (-y, \langle y, \frac{\alpha}{(y_\varepsilon + \alpha)^2} \rangle + \langle \xi, \frac{\alpha}{(\xi_\varepsilon + \alpha)^2} \rangle).$$

Here

$$\mathcal{U}_{ad}(v_\varepsilon, \xi_\varepsilon) = \{(\lambda(v - v_\varepsilon), \mu(\xi - \xi_\varepsilon)) \mid \lambda \geq 0, \mu \geq 0, v \in U_{ad}, \xi \in V_{ad}\},$$

$$P(G(y_\varepsilon, v_\varepsilon, \xi_\varepsilon)) = \{(-p + \lambda y_\varepsilon, -\gamma - \lambda(\langle 1, \frac{y_\varepsilon}{y_\varepsilon + \alpha} + \frac{\xi_\varepsilon}{\xi_\varepsilon + \alpha} \rangle - Area(\Omega))) \in H^2(\Omega) \cap H_0^1(\Omega) \times \mathbb{R} \mid \gamma, \lambda \geq 0, p \geq 0\}$$

Let us write the the condition (4.2) : for any  $(z, \beta)$  in  $\mathcal{X} \times \mathbb{R}$  we must solve the system :

$$\begin{aligned} -y + p - \lambda y_\varepsilon &= z, \\ \langle y, \frac{\alpha}{(y_\varepsilon + \alpha)^2} \rangle + \langle \mu(\xi - \xi_\varepsilon), \frac{\alpha}{(\xi_\varepsilon + \alpha)^2} \rangle + \gamma + \lambda(\langle 1, \frac{y_\varepsilon}{y_\varepsilon + \alpha} + \frac{\xi_\varepsilon}{\xi_\varepsilon + \alpha} \rangle - Area(\Omega)) &= \beta, \end{aligned}$$

with  $\mu, \gamma, \lambda \geq 0$ ,  $\xi \in V_{ad}$ ,  $v \in U_{ad}$ , and  $y \in \mathcal{X}$ . Taking  $y$  from the first equation into the second we have to solve :

$$\langle p - \lambda y_\varepsilon - z, \frac{\alpha}{(y_\varepsilon + \alpha)^2} \rangle + \langle \mu(\xi - \xi_\varepsilon), \frac{\alpha}{(\xi_\varepsilon + \alpha)^2} \rangle + \gamma + \lambda(\langle 1, \frac{y_\varepsilon}{y_\varepsilon + \alpha} + \frac{\xi_\varepsilon}{\xi_\varepsilon + \alpha} \rangle - Area(\Omega)) = \beta.$$

So

$$\begin{aligned} \langle p, \frac{\alpha}{(y_\varepsilon + \alpha)^2} \rangle - \lambda \langle y_\varepsilon, \frac{\alpha}{(y_\varepsilon + \alpha)^2} \rangle + \langle \mu(\xi - \xi_\varepsilon), \frac{\alpha}{(\xi_\varepsilon + \alpha)^2} \rangle + \gamma + \lambda(\langle 1, \frac{y_\varepsilon}{y_\varepsilon + \alpha} + \frac{\xi_\varepsilon}{\xi_\varepsilon + \alpha} \rangle - Area(\Omega)) &= \beta + \\ \langle z, \frac{\alpha}{(y_\varepsilon + \alpha)^2} \rangle &= \rho \end{aligned}$$

with  $\mu, \gamma, \lambda \geq 0$ ,  $\xi \in V_{ad}$ ,  $v \in U_{ad}$ . We see that we may take :  $\mu = 1$ ,  $\xi = \xi_\varepsilon$ ,  $p = 0$ , and

- If  $\rho \geq 0$ , we choose  $\lambda = 0$ ,  $\gamma = \rho$
- If  $\rho < 0$ , we have two cases :

- If  $(\langle 1, \frac{y_\varepsilon}{y_\varepsilon + \alpha} + \frac{\xi_\varepsilon}{\xi_\varepsilon + \alpha} \rangle - Area(\Omega)) = \zeta < 0$ , then we set  $\gamma = \lambda \langle y_\varepsilon, \frac{\alpha}{(y_\varepsilon + \alpha)^2} \rangle$ ,  $\lambda = \frac{\rho}{\zeta}$ .
- If  $(\langle 1, \frac{y_\varepsilon}{y_\varepsilon + \alpha} + \frac{\xi_\varepsilon}{\xi_\varepsilon + \alpha} \rangle - Area(\Omega)) = 0$ , then we set  $\gamma = 0$ ,  $\lambda = -\frac{\rho}{\eta}$ , such that  $\eta = \lambda \langle y_\varepsilon, \frac{\alpha}{(y_\varepsilon + \alpha)^2} \rangle$ .

Indeed, we have

$$(\langle 1, \frac{y_\varepsilon}{y_\varepsilon + \alpha} + \frac{\xi_\varepsilon}{\xi_\varepsilon + \alpha} \rangle - Area(\Omega)) = 0,$$

in view of Lemma 3.2, we have

$$(\langle 1, \frac{y_\varepsilon}{y_\varepsilon + \alpha} + \frac{\xi_\varepsilon}{\xi_\varepsilon + \alpha} \rangle - Area(\Omega)) = 0 \iff y_\varepsilon \cdot \xi_\varepsilon = \alpha^2 \text{ a.e in } \Omega.$$

Therefore  $y_\varepsilon$  and  $\xi_\varepsilon$  are strictly positive. (Here  $y_\varepsilon > 0$ , and  $\xi_\varepsilon > 0$ , because  $\alpha > 0$  fixed). Hence,  $\eta > 0$  and  $\lambda > 0$ .

So condition (4.2) is always satisfied and we may apply Theorem 4.1, since  $J_\varepsilon^\alpha$  is Fréchet differentiable, and

$$J_\varepsilon^{\alpha'}(y_\varepsilon, v_\varepsilon, \xi_\varepsilon)(y, v, \xi) = \begin{pmatrix} (J_\varepsilon^\alpha)'_y(y_\varepsilon, v_\varepsilon, \xi_\varepsilon) & (J_\varepsilon^\alpha)'_v(y_\varepsilon, v_\varepsilon, \xi_\varepsilon) & (J_\varepsilon^\alpha)'_\xi(y_\varepsilon, v_\varepsilon, \xi_\varepsilon) \end{pmatrix} \cdot \begin{pmatrix} y \\ v \\ \xi \end{pmatrix}.$$

We have :

$$J_\varepsilon^\alpha(y, v, \xi) = \begin{cases} J(y, v) + \frac{1}{2\varepsilon} \|Ay + g(y) - f - v - \xi\|_{L^2(\Omega)}^2 \\ + \frac{1}{2} \|A(y - y_\alpha)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|v - v_\alpha\|_{L^2(\Omega)}^2 \\ + \frac{1}{2} \|\xi - \xi_\alpha\|_{L^2(\Omega)}^2. \end{cases} \quad (5.6)$$

So,

$$\begin{aligned} (J_\varepsilon^\alpha)'_y(y_\varepsilon, v_\varepsilon, \xi_\varepsilon) &= \langle 1, y_\varepsilon - z_d \rangle + \frac{1}{\varepsilon} \langle A + g'(y_\varepsilon), Ay_\varepsilon + g(y_\varepsilon) - f - v_\varepsilon - \xi_\varepsilon \rangle + \langle A, A(y_\varepsilon - y_\alpha) \rangle. \\ (J_\varepsilon^\alpha)'_v(y_\varepsilon, v_\varepsilon, \xi_\varepsilon) &= \langle \nu, v_\varepsilon - v_d \rangle + \langle 1, v_\varepsilon - v_\alpha \rangle - \frac{1}{\varepsilon} \langle 1, Ay_\varepsilon + g(y_\varepsilon) - f - v_\varepsilon - \xi_\varepsilon \rangle. \\ (J_\varepsilon^\alpha)'_\xi(y_\varepsilon, v_\varepsilon, \xi_\varepsilon) &= \langle 1, \xi_\varepsilon - \xi_\alpha \rangle - \frac{1}{\varepsilon} \langle 1, Ay_\varepsilon + g(y_\varepsilon) - f - v_\varepsilon - \xi_\varepsilon \rangle. \end{aligned}$$

Therefore

$$J_\varepsilon^\alpha'(y_\varepsilon, v_\varepsilon, \xi_\varepsilon)(y, v, \xi) = \langle y, y_\varepsilon - z_d \rangle + \nu \langle v, v_\varepsilon - v_d \rangle + \langle v, v_\varepsilon - v_\alpha \rangle + \langle \xi, \xi_\varepsilon - \xi_\alpha \rangle + \langle Ay, A(y_\varepsilon - y_\alpha) \rangle + \langle q_\varepsilon, A_\varepsilon y - v - \xi \rangle,$$

where

$$q_\varepsilon = \frac{Ay_\varepsilon + g(y_\varepsilon) - f - v_\varepsilon - \xi_\varepsilon}{\varepsilon} \in L^2(\Omega) \quad \text{and} \quad A_\varepsilon = A + g'(y_\varepsilon). \quad (5.7)$$

Thers exists  $s_\varepsilon \in \mathcal{X}^*$  and  $r_\varepsilon \in \mathbb{R}$  such that :

$$\forall y \in \mathcal{X} \quad \langle y, y_\varepsilon - z_d \rangle + \langle q_\varepsilon, A_\varepsilon y \rangle + \langle Ay, A(y_\varepsilon - y_\alpha) \rangle + r_\varepsilon \langle y, \frac{\alpha}{(y_\varepsilon + \alpha)^2} \rangle - \langle \langle s_\varepsilon, y \rangle \rangle = 0, \quad (5.8)$$

$$\forall v \in U_{ad} \quad \langle \nu(v_\varepsilon - v_d) + v_\varepsilon - v_\alpha - q_\varepsilon, v - v_\varepsilon \rangle \geq 0, \quad (5.9)$$

$$\forall \xi \in V_{ad} \quad \langle r_\varepsilon \frac{\alpha}{(\xi_\alpha + \alpha)^2} - q_\varepsilon + \xi_\varepsilon - \xi_\alpha, \xi - \xi_\varepsilon \rangle \geq 0, \quad (5.10)$$

$$r_\varepsilon \geq 0, \quad r_\varepsilon \left( \langle 1, \frac{y_\varepsilon}{y_\varepsilon + \alpha} + \frac{\xi_\varepsilon}{\xi_\varepsilon + \alpha} \rangle - Area(\Omega) \right) = 0, \quad (5.11)$$

$$\forall y \in \mathcal{X}, \quad y \geq 0, \quad \langle \langle s_\varepsilon, y \rangle \rangle \geq 0, \quad \langle \langle s_\varepsilon, y_\varepsilon \rangle \rangle = 0, \quad (5.12)$$

where  $\langle \langle \cdot, \cdot \rangle \rangle$  denotes the duality product between  $\mathcal{X}^*$  and  $\mathcal{X}$ .

Finally, we have optimality conditions on the penalized system, without any further assumption :

**Theorem 5.3.** *The solution  $(y_\varepsilon, v_\varepsilon, \xi_\varepsilon)$  of problem  $(\mathcal{P}_\varepsilon^\alpha)$  satisfies the following optimality system :*

$$\forall y \in \tilde{K} \quad \langle p_\varepsilon + q_\varepsilon, A_\varepsilon(y - y_\varepsilon) \rangle + \langle A(y - y_\varepsilon), A(y_\varepsilon - y_\alpha) \rangle + r_\varepsilon \langle y - y_\varepsilon, \frac{\alpha}{(y_\varepsilon + \alpha)^2} \rangle \geq 0, \quad (5.13)$$

$$\forall v \in U_{ad} \quad \langle \nu(v_\varepsilon - v_d) + v_\varepsilon - v_\alpha - q_\varepsilon, v - v_\varepsilon \rangle \geq 0, \quad (5.14)$$

$$\forall \xi \in V_{ad} \quad \langle r_\varepsilon \frac{\alpha}{(\xi_\alpha + \alpha)^2} - q_\varepsilon + \xi_\varepsilon - \xi_\alpha, \xi - \xi_\varepsilon \rangle \geq 0, \quad (5.15)$$

$$r_\varepsilon \geq 0, \quad r_\varepsilon \left( \langle 1, \frac{y_\varepsilon}{y_\varepsilon + \alpha} + \frac{\xi_\varepsilon}{\xi_\varepsilon + \alpha} \rangle - Area(\Omega) \right) = 0, \quad (5.16)$$

where  $p_\varepsilon$  is given by (5.4) and  $q_\varepsilon$  by (5.7).

**Proof 5.4.** *Relation (5.8) applied to  $y - y_\varepsilon$  gives :*

$$\forall y \in \mathcal{X} \quad \langle y - y_\varepsilon, y_\varepsilon - z_d \rangle + \langle q_\varepsilon, A_\varepsilon(y - y_\varepsilon) \rangle + \langle A(y - y_\varepsilon), A(y_\varepsilon - y_\alpha) \rangle + r_\varepsilon \langle y - y_\varepsilon, \frac{\alpha}{(y_\varepsilon + \alpha)^2} \rangle = \langle \langle s_\varepsilon, y \rangle \rangle - \langle \langle s_\varepsilon, y_\varepsilon \rangle \rangle,$$

So, with (5.12), we obtain

$$\forall y \in \tilde{K} \quad \langle p_\varepsilon + q_\varepsilon, A_\varepsilon(y - y_\varepsilon) \rangle + \langle A(y - y_\varepsilon), A(y_\varepsilon - y_\alpha) \rangle + r_\varepsilon \langle y - y_\varepsilon, \frac{\alpha}{(y_\varepsilon + \alpha)^2} \rangle \geq 0,$$

where  $p_\varepsilon$  is given by (5.4) and  $q_\varepsilon$  by (5.7), and  $\tilde{K} = K \cap (H^2(\Omega) \cap H_0^1(\Omega))$ .

## 6. OPTIMALITY CONDITIONS FOR $(\mathcal{P}^\alpha)$

### 6.1. Qualification assumption

Now we would like to study the asymptotic behaviour of the previous optimality conditions (5.13)-(5.16) when  $\varepsilon$  goes to 0 and we need some estimations on  $q_\varepsilon$  and  $r_\varepsilon$ . We have to assume some qualification conditions to pass to the limit in the penalized optimality system, we remark that

$$A_\varepsilon y_\varepsilon - v_\varepsilon - \xi_\varepsilon = Ay_\varepsilon + g(y_\varepsilon) - v_\varepsilon - \xi_\varepsilon - f + f + g'(y_\varepsilon)y_\varepsilon - g(y_\varepsilon).$$

We set

$$\omega_\varepsilon = g'(y_\varepsilon)y_\varepsilon - g(y_\varepsilon) \quad \text{and} \quad \omega_\alpha = g'(y_\alpha)y_\alpha - g(y_\alpha), \quad (6.1)$$

so that

$$A_\varepsilon y_\varepsilon - v_\varepsilon - \xi_\varepsilon = \varepsilon q_\varepsilon + f + \omega_\varepsilon.$$

Let us choose  $(y, v, \xi)$  in  $\tilde{K} \times U_{ad} \times V_{ad}$ , and add relation (5.13)-(5.15). We have :

$$\begin{aligned} & \langle p_\varepsilon, A_\varepsilon(y - y_\varepsilon) \rangle + \langle q_\varepsilon, A_\varepsilon(y - y_\varepsilon) \rangle + \langle A(y - y_\varepsilon), A(y_\varepsilon - y_\alpha) \rangle + \langle \nu(v_\varepsilon - v_d) + v_\varepsilon - v_\alpha, v - v_\varepsilon \rangle \\ & + \langle -q_\varepsilon, v - v_\varepsilon \rangle + r_\varepsilon \langle y - y_\varepsilon, \frac{\alpha}{(y_\varepsilon + \alpha)^2} \rangle + r_\varepsilon \langle \frac{\alpha}{(\xi_\varepsilon + \alpha)^2}, \xi - \xi_\varepsilon \rangle + \langle \xi_\varepsilon - \xi_\alpha, \xi - \xi_\varepsilon \rangle + \langle -q_\varepsilon, \xi - \xi_\varepsilon \rangle \geq 0. \end{aligned}$$

So that:

$$\begin{aligned} & \langle q_\varepsilon, f + \omega_\varepsilon + v + \xi - A_\varepsilon y \rangle - r_\varepsilon \left( \langle \frac{\alpha}{(y_\varepsilon + \alpha)^2}, y - y_\varepsilon \rangle + \langle \frac{\alpha}{(\xi_\varepsilon + \alpha)^2}, \xi - \xi_\varepsilon \rangle \right) \leq \\ & \langle p_\varepsilon, A_\varepsilon(y - y_\varepsilon) \rangle + \langle A(y - y_\varepsilon), A(y_\varepsilon - y_\alpha) \rangle + \langle \nu(v_\varepsilon - v_d) + v_\varepsilon - v_\alpha, v - v_\varepsilon \rangle + \langle \xi_\varepsilon - \xi_\alpha, \xi - \xi_\varepsilon \rangle - \varepsilon \|q_\varepsilon\|_2^2. \end{aligned}$$

The right hand side is uniformly bounded with respect to  $\varepsilon$  by a constant  $C$  which only depends of  $y, v, \xi$ . Here we use as well Theorem 5.2. Moreover relation (5.16) gives

$$r_\varepsilon \langle 1, \frac{y_\varepsilon}{y_\varepsilon + \alpha} + \frac{\xi_\varepsilon}{\xi_\varepsilon + \alpha} \rangle = r_\varepsilon \text{Area}(\Omega),$$

so that we finally obtain :

$$-\langle q_\varepsilon, Ay + g'(y_\varepsilon)y - f - v - \xi - \omega_\varepsilon \rangle - r_\varepsilon \left( \langle \frac{\alpha}{(y_\varepsilon + \alpha)^2}, y - y_\varepsilon \rangle + \langle \frac{\alpha}{(\xi_\varepsilon + \alpha)^2}, \xi - \xi_\varepsilon \rangle \right) \leq C_{(y,v,\xi)}, \quad (6.2)$$

where

$$q_\varepsilon = \frac{Ay_\varepsilon + g(y_\varepsilon) - f - v_\varepsilon - \xi_\varepsilon}{\varepsilon} \in L^2(\Omega) \quad \text{and} \quad A_\varepsilon = A + g'(y_\varepsilon), \quad \omega_\varepsilon = g'(y_\varepsilon)y_\varepsilon - g(y_\varepsilon).$$

We consider two cases :

(i) If

$$\langle 1, \frac{y_\alpha}{y_\alpha + \alpha} + \frac{\xi_\alpha}{\xi_\alpha + \alpha} \rangle < \text{Area}(\Omega),$$

as  $\langle 1, \frac{y_\varepsilon}{y_\varepsilon + \alpha} + \frac{\xi_\varepsilon}{\xi_\varepsilon + \alpha} \rangle \rightarrow \langle 1, \frac{y_\alpha}{y_\alpha + \alpha} + \frac{\xi_\alpha}{\xi_\alpha + \alpha} \rangle$ , there exists  $\varepsilon_0 > 0$  such that

$$\forall \varepsilon \leq \varepsilon_0 \quad \langle 1, \frac{y_\varepsilon}{y_\varepsilon + \alpha} + \frac{\xi_\varepsilon}{\xi_\varepsilon + \alpha} \rangle < \text{Area}(\Omega),$$

and relation (5.16) implies that  $r_\varepsilon = 0$ . So the limit value is  $r_\alpha = 0$ .

(ii) If

$$\left\langle 1, \frac{y_\alpha}{y_\alpha + \alpha} + \frac{\xi_\alpha}{\xi_\alpha + \alpha} \right\rangle = Area(\Omega),$$

as  $\left\langle 1, \frac{y_\varepsilon}{y_\varepsilon + \alpha} + \frac{\xi_\varepsilon}{\xi_\varepsilon + \alpha} \right\rangle \rightarrow \left\langle 1, \frac{y_\alpha}{y_\alpha + \alpha} + \frac{\xi_\alpha}{\xi_\alpha + \alpha} \right\rangle$ , there exists  $\varepsilon_0 > 0$  such that

$$\forall \varepsilon \leq \varepsilon_0 \quad \left\langle 1, \frac{y_\varepsilon}{y_\varepsilon + \alpha} + \frac{\xi_\varepsilon}{\xi_\varepsilon + \alpha} \right\rangle = Area(\Omega),$$

we cannot conclude immediately, so we assume the following condition :

$$\forall \alpha > 0 \text{ such that } \left\langle 1, \frac{y_\varepsilon}{y_\varepsilon + \alpha} + \frac{\xi_\varepsilon}{\xi_\varepsilon + \alpha} \right\rangle = Area(\Omega),$$

$$g' \text{ is locally lipschitz continuous,} \quad (\mathcal{H}_1)$$

$U_{ad}$  has a non empty  $L^\infty$ -interior (denoted  $\text{Int}_\infty(U_{ad})$ ) and that  $-(f + \omega_\alpha) \in \text{Int}_\infty(U_{ad})$ .

**Theorem 6.1.** Assume  $(\mathcal{H}_1)$ , then  $r_\varepsilon$  is bounded by a constant independent of  $\varepsilon$  and we may extract a subsequence that converges to  $r_\alpha$ .

**Proof 6.1.** We have already mentioned that  $r_\alpha = 0$  when  $\left\langle 1, \frac{y_\varepsilon}{y_\varepsilon + \alpha} + \frac{\xi_\varepsilon}{\xi_\varepsilon + \alpha} \right\rangle < Area(\Omega)$ . In the other case, as  $g'$  is locally lipschitz continuous, then  $\omega_\varepsilon$  uniformly converges to  $\omega_\alpha$  on  $\bar{\Omega}$ .

Indeed, we have proved that  $y_\varepsilon$  uniformly converges to  $y_\alpha$ . Therefore, there exists  $\varepsilon_0 > 0$  such that  $y_\varepsilon - y_\alpha$  remains in a bounded subset of  $\mathbb{R}^n$  independently of  $\varepsilon \in ]0, \varepsilon_0[$ . The local lipschitz continuity of  $g'$  yields

$$|g'(y_\varepsilon(x)) - g'(y_\alpha(x))| \leq M|y_\varepsilon(x) - y_\alpha(x)| \leq M \|y_\varepsilon - y_\alpha\|_\infty, \quad \forall x \in \Omega$$

where  $M$  is a constant that does not depend of  $\varepsilon$ . Thus  $\|g'(y_\varepsilon) - g'(y_\alpha)\|_\infty \rightarrow 0$ . As

$$|g'(y_\varepsilon)y_\varepsilon - g'(y_\alpha)y_\alpha| \leq |g'(y_\varepsilon)||y_\varepsilon - y_\alpha| + |g'(y_\varepsilon) - g'(y_\alpha)||y_\alpha|,$$

we get

$$\|g'(y_\varepsilon)y_\varepsilon - g'(y_\alpha)y_\alpha\|_\infty \leq M \|y_\varepsilon - y_\alpha\|_\infty + \|g'(y_\varepsilon) - g'(y_\alpha)\|_\infty \|y_\alpha\|_\infty \rightarrow 0.$$

Similarly  $\|g(y_\varepsilon) - g(y_\alpha)\|_\infty \rightarrow 0$ . As we supposed  $-(f + \omega_\alpha) \in \text{Int}_\infty(U_{ad})$ , then  $-(f + \omega_\varepsilon) \in U_{ad}$  for  $\varepsilon$  smaller than some  $\varepsilon_0 > 0$ .

Now, we choose  $y = 0, v = -(f + \omega_\varepsilon)$  and  $\xi = 0$  in relation (6.2). We obtain

$$\forall \varepsilon \leq \varepsilon_0 \quad r_\varepsilon \left( \left\langle \frac{\alpha}{(y_\varepsilon + \alpha)^2}, y_\varepsilon \right\rangle + \left\langle \frac{\alpha}{(\xi_\varepsilon + \alpha)^2}, \xi_\varepsilon \right\rangle \right) \leq C,$$

where  $C$  is independent of  $\varepsilon$  since  $\omega_\varepsilon$  is uniformly bounded with respect to  $\varepsilon$ , for  $\varepsilon \in ]0, \varepsilon_0[$ .

We still need to proof that :

$$\left\langle \frac{\alpha}{(y_\varepsilon + \alpha)^2}, y_\varepsilon \right\rangle + \left\langle \frac{\alpha}{(\xi_\varepsilon + \alpha)^2}, \xi_\varepsilon \right\rangle \neq 0, \quad \forall \alpha > 0 \text{ and } \varepsilon \rightarrow 0.$$

As

$$\left\langle 1, \frac{y_\varepsilon}{y_\varepsilon + \alpha} + \frac{\xi_\varepsilon}{\xi_\varepsilon + \alpha} \right\rangle = Area(\Omega),$$

So we have :

$$\frac{y_\varepsilon}{y_\varepsilon + \alpha} + \frac{\xi_\varepsilon}{\xi_\varepsilon + \alpha} = 1, \text{ a.e. in } \Omega,$$

in view of section 3 we obtain

$$\frac{y_\varepsilon}{y_\varepsilon + \alpha} + \frac{\xi_\varepsilon}{\xi_\varepsilon + \alpha} = 1 \iff y_\varepsilon \xi_\varepsilon = \alpha^2 \implies \langle y_\varepsilon, \xi_\varepsilon \rangle = \alpha^2 \text{Area}(\Omega) \quad \text{a.e. in } \Omega.$$

Therefore, the set  $\{x \in \Omega, /y_\varepsilon(x) \neq 0, \xi_\varepsilon(x) \neq 0\}$  is not empty, and the set  $\{x \in \Omega, /y_\varepsilon(x) = 0, \xi_\varepsilon(x) = 0\}$  is empty when  $\varepsilon$  goes to 0, since  $\alpha$  is fixed. Hence we obtain :

$$\left\langle \frac{\alpha}{(y_\varepsilon + \alpha)^2}, y_\varepsilon \right\rangle + \left\langle \frac{\alpha}{(\xi_\varepsilon + \alpha)^2}, \xi_\varepsilon \right\rangle \neq 0, \forall \alpha > 0.$$

Finally, the passage to limit as  $\varepsilon \rightarrow 0$  gives :

$$\left\langle \frac{\alpha}{(y_\alpha + \alpha)^2}, y_\alpha \right\rangle + \left\langle \frac{\alpha}{(\xi_\alpha + \alpha)^2}, \xi_\alpha \right\rangle \neq 0, \forall \alpha > 0.$$

Once we have the previous estimate, relation (6.2) becomes :

$$\forall (y, v, \xi) \in \tilde{K} \times U_{ad} \times V_{ad} \quad - \langle q_\varepsilon, Ay + g'(y_\varepsilon)y - f - v - \xi - \omega_\varepsilon \rangle \leq C_{(y,v,\xi)}. \quad (6.3)$$

Then we have to do another assumption to get the estimation of  $q_\varepsilon$  :

$$\begin{aligned} & \exists p \in [1, +\infty], \quad \exists \varepsilon_0 > 0, \quad \exists \rho > 0 \\ & \forall \varepsilon \in ]0, \varepsilon_0[, \quad \forall \chi \in L^p(\Omega) \text{ such that } \|\chi\|_{L^p(\Omega)} \leq 1, \\ & \exists (y_\chi^\varepsilon, v_\chi^\varepsilon, \xi_\chi^\varepsilon) \text{ bounded in } \tilde{K} \times U_{ad} \times V_{ad} \text{ (uniformly with respect to } \chi \text{ and } \varepsilon), \\ & \text{such that } Ay_\chi^\varepsilon + g'(y_\chi^\varepsilon)y_\chi^\varepsilon = f + \omega_\varepsilon + v_\chi^\varepsilon + \xi_\chi^\varepsilon - \rho\chi \text{ in } \Omega. \end{aligned} \quad (\mathcal{H}_2)$$

Then we may conclude :

**Theorem 6.2.** Assume  $(\mathcal{H}_1)$  and  $(\mathcal{H}_2)$ , then  $q_\varepsilon$  is bounded in  $L^{p'}(\Omega)$  by a constant independent of  $\varepsilon$  (here  $\frac{1}{p} + \frac{1}{p'} = 1$ ).

**Proof 6.2.**  $(\mathcal{H}_2)$  and relation (6.3) when applied with  $(y_\chi^\varepsilon, v_\chi^\varepsilon, \xi_\chi^\varepsilon)$  give :

$$\forall \chi \in L^p(\Omega), \quad \|\chi\|_{L^p(\Omega)} \leq 1, \quad \rho \langle q_\varepsilon, \chi \rangle \leq C_{\chi, \varepsilon} \leq C.$$

Then we may pass to the limit in the penalized optimality system and obtain the following result.

**Theorem 6.3.** Assume  $(\mathcal{H}_1)$  and  $(\mathcal{H}_2)$ , if  $(y_\alpha, v_\alpha, \xi_\alpha)$  is a solution of  $(P^\alpha)$ , then Lagrange multipliers  $(q_\alpha, r_\alpha) \in L^{p'}(\Omega) \times \mathbb{R}^+$  exist, such that

$$\forall y \in \tilde{K}, \quad [A + g'(y_\alpha)](y - y_\alpha) \in L^p(\Omega) \quad \langle p_\alpha + q_\alpha, [A + g'(y_\alpha)](y - y_\alpha) \rangle + r_\alpha \left\langle \frac{\alpha}{(y_\alpha + \alpha)^2}, y - y_\alpha \right\rangle \geq 0, \quad (6.4)$$

$$\forall v \in U_{ad}, \quad v - v_\alpha \in L^p(\Omega) \quad \langle \nu(v_\alpha - v_d) - q_\alpha, v - v_\alpha \rangle \geq 0, \quad (6.5)$$

$$\forall \xi \in V_{ad}, \quad \xi - \xi_\alpha \in L^p(\Omega) \quad \left\langle \frac{r_\alpha \alpha}{(\xi_\alpha + \alpha)^2} - q_\alpha, \xi - \xi_\alpha \right\rangle \geq 0, \quad (6.6)$$

$$r_\alpha \left( \left\langle 1, \frac{y_\alpha}{y_\alpha + \alpha} + \frac{\xi_\alpha}{\xi_\alpha + \alpha} \right\rangle - \text{Area}(\Omega) \right) = 0, \quad (6.7)$$

where  $p_\alpha$  is given by (5.5).



6.2. Sufficient condition for  $(\mathcal{H}_2)$  with  $p=2$ .

In this subsection we give an assumption dealing with  $(y_\alpha, v_\alpha, \xi_\alpha)$  where  $\varepsilon$  does not appear. We choose  $p = 2$  because it is the most useful case. We always assume that  $g'$  is locally lipschitz continuous (for example  $g$  is  $\mathcal{C}^2$ ), and we set the following

$$\begin{aligned} & \exists \rho > 0, \exists v_0 \in \text{Int}_\infty(U_{ad}), \forall \mathcal{X} \in L^2(\Omega) \text{ such that } \|\mathcal{X}\|_{L^2(\Omega)} \leq 1, \\ & \exists (y_{\mathcal{X}}, \xi_{\mathcal{X}}) \in \tilde{K} \times V_{ad} \text{ (uniformly bounded by a constant } M \text{ independent of } \mathcal{X} \text{),} \quad (\mathcal{H}_3) \\ & \text{such that } Ay_{\mathcal{X}} + g'(y_\alpha)y_{\mathcal{X}} = f + \omega_\alpha + v_0 + \xi_{\mathcal{X}} - \rho\mathcal{X} \text{ in } \Omega. \end{aligned}$$

**Proposition 6.1.** *If  $g'$  is locally lipschitz continuous then  $(\mathcal{H}_3) \implies (\mathcal{H}_2)$ .*

**Proof 6.3.** *We have seen that  $\|y_\varepsilon - y_\alpha\|_\infty \rightarrow 0$ ,  $\|g'(y_\varepsilon) - g'(y_\alpha)\|_\infty \rightarrow 0$  and  $\|\omega_\varepsilon - \omega_\alpha\|_\infty \rightarrow 0$ .*

*Let be  $\mathcal{X} \in L^2(\Omega)$  such that  $\|\mathcal{X}\|_2 \leq 1$  and  $(y_{\mathcal{X}}, v_0, \xi_{\mathcal{X}}) \in \tilde{K} \times \text{Int}_\infty(U_{ad}) \times V_{ad}$  given by  $(\mathcal{H}_3)$ . As  $v_0 \in \text{Int}_\infty(U_{ad})$ , there exists  $\rho_0 > 0$  such that  $\mathcal{B}_\infty(v_0, \rho) \subset U_{ad}$ . As  $y_{\mathcal{X}}$  is bounded by  $M$ , then for  $\varepsilon$  small enough (less than some  $\varepsilon_0 > 0$ ), we get*

$$\|\omega_\alpha - \omega_\varepsilon + (g'(y_\varepsilon) - g'(y_\alpha))y_{\mathcal{X}}\|_\infty \leq \|\omega_\alpha - \omega_\varepsilon\|_\infty + \|g'(y_\varepsilon) - g'(y_\alpha)\|_\infty \|y_{\mathcal{X}}\|_\infty \leq \rho_0,$$

*therefore  $v_{\mathcal{X}}^\varepsilon = v_0 + (g'(y_\varepsilon) - g'(y_\alpha))y_{\mathcal{X}} + \omega_\alpha - \omega_\varepsilon$  belongs to  $U_{ad}$  and*

$$\|v_{\mathcal{X}}^\varepsilon\|_2 \leq \|v_0\|_2 + \|\omega_\alpha - \omega_\varepsilon\|_2 + \|(g'(y_\varepsilon) - g'(y_\alpha))y_{\mathcal{X}}\|_2 \leq C,$$

*$v_{\mathcal{X}}^\varepsilon$  is  $L^2$ -bounded independently of  $\mathcal{X}$  and  $\varepsilon$ . Now, we set  $y_{\mathcal{X}}^\varepsilon = y_{\mathcal{X}} \in \tilde{K}$  and  $\xi_{\mathcal{X}}^\varepsilon = \xi_{\mathcal{X}} \in V_{ad}$  to obtain*

$$\begin{aligned} Ay_{\mathcal{X}}^\varepsilon + g'(y_\varepsilon)y_{\mathcal{X}}^\varepsilon &= Ay_{\mathcal{X}} + g'(y_\alpha)y_{\mathcal{X}} + (g'(y_\varepsilon) - g'(y_\alpha))y_{\mathcal{X}} \\ &= f + \omega_\alpha + v_0 + \xi_{\mathcal{X}} - \rho\mathcal{X} + (g'(y_\varepsilon) - g'(y_\alpha))y_{\mathcal{X}} \\ &= f + \omega_\varepsilon + v_0 + (g'(y_\varepsilon) - g'(y_\alpha))y_{\mathcal{X}} + \omega_\alpha - \omega_\varepsilon + \xi_{\mathcal{X}} - \rho\mathcal{X} \\ &= f + \omega_\varepsilon + v_{\mathcal{X}}^\varepsilon + \xi_{\mathcal{X}}^\varepsilon - \rho\mathcal{X}. \end{aligned} \quad (6.8)$$

*We can see that that  $(\mathcal{H}_2)$  is satisfied.*

An immediate consequence is the following Theorem: we get the existence of Lagrange multipliers:

**Theorem 6.4.** *Let  $(y_\alpha, v_\alpha, \xi_\alpha)$  be a solution of  $(P^\alpha)$  and assume  $(\mathcal{H}_1)$  and  $(\mathcal{H}_3)$ ; then Lagrange multipliers  $(q_\alpha, r_\alpha) \in L^2(\Omega) \times \mathbb{R}^+$  exist, such that*

$$\forall y \in \tilde{K}, \quad \langle p_\alpha + q_\alpha, [A + g'(y_\alpha)](y - y_\alpha) \rangle + r_\alpha \langle \frac{\alpha}{(y_\alpha + \alpha)^2}, y - y_\alpha \rangle \geq 0, \quad (6.9)$$

$$\forall v \in U_{ad}, \quad \langle \nu(v_\alpha - v_d) - q_\alpha, v - v_\alpha \rangle \geq 0, \quad (6.10)$$

$$\forall \xi \in V_{ad}, \quad \langle \frac{r_\alpha \alpha}{(\xi_\alpha + \alpha)^2} - q_\alpha, \xi - \xi_\alpha \rangle \geq 0, \quad (6.11)$$

$$r_\alpha \left( \langle 1, \frac{y_\alpha}{y_\alpha + \alpha} + \frac{\xi_\alpha}{\xi_\alpha + \alpha} \rangle - \text{Area}(\Omega) \right) = 0, \quad (6.12)$$

*where  $p_\alpha$  is given by (5.5).*

**Proof 6.4.** *We take  $v_0 = -(f + \omega_\alpha)$  to ensure  $(\mathcal{H}_3)$ . Let  $\mathcal{X} \in L^2(\Omega)$  such that  $\|\mathcal{X}\|_{L^2(\Omega)} \leq 1$ .*

*We set  $\xi_{\mathcal{X}} = \mathcal{X}^+ + \mathcal{X}^- = |\mathcal{X}| \geq 0$ , where  $\mathcal{X}^+ = \max(0, \mathcal{X})$  and  $\mathcal{X}^- = \max(0, -\mathcal{X})$ . As  $\|\mathcal{X}\|_{L^2(\Omega)} \leq 1$ , it is clear that  $\xi_{\mathcal{X}} \in V_{ad}$ .*

Let  $y_{\mathcal{X}}$  be the solution of

$$[A + g'(y_{\alpha})]y_{\mathcal{X}} = \xi_{\mathcal{X}} - \mathcal{X} = 2\mathcal{X}^- \geq 0 \quad (\text{a.e.}), \quad y \in H_0^1(\Omega),$$

thanks to the properties of  $[A + g'(y_{\alpha})]$  and the maximum principle, then  $y_{\mathcal{X}} \geq 0$  a.e. in  $\Omega$ . Therefore  $y_{\mathcal{X}} \in \tilde{K}$  and  $(\mathcal{H}_3)$  is satisfied (with  $\rho = 1$ ). The optimality system follows and we have proved that the multiplier  $q_{\alpha}$  is a  $L^2(\Omega)$ -function.

**Corollary 6.1.** *If  $g$  is linear and  $-f \in U_{ad}$ , the conclusions of Theorem are valid.*

*Proof.* If  $g$  is linear, we use the same proof as the one of Theorem 6.4 to bound  $q_{\varepsilon}$  in  $L^2(\Omega)$ . It is sufficient that  $-f \in U_{ad}$ .  $\square$

**Remark 6.1.** *We may choose for example  $U_{ad} = [a, b]$  with  $a+3+\alpha \leq b-\alpha$ ,  $\alpha > 0$ ,  $-b+\alpha \leq -a-3-\alpha$  and  $g(x) = -\frac{1}{1+x^2}$ . In this cas  $0 \leq \omega_{\alpha} \leq 3$  so that  $-(f + \omega_{\alpha}) \in [a + \alpha, b - \alpha] \subset \text{Int}_{L^{\infty}}(U_{ad})$ .*

Next we describe numerical experiments that were carried out by means of AMPL languages [1], with the IPOPT solver [20] ("Interior Point OPTimizer"), KNITRO solver [18] ("Nonlinear Interior point Trust Region Optimization") and SNOPT solver [13] ("Sparse Nonlinear OPTimizer").

## 7. NUMERICAL RESULTS

In this section, we report on some experiments considering a 2D-example. For two different smoothing functions, we present some numerical results using the IPOPT nonlinear programming algorithm on AMPL [1] optimization platform. Our aim is just to verify the qualitative numerical efficiency of our approach. The discretization process was based on finite difference schemes with a  $N \times N$  grid and the size of the grid is given by  $h = \frac{1}{N}$  on each side of the domain.

We take  $\Omega = ]0, 1[ \times ]0, 1[ \subset \mathbb{R}^2$ ,  $A := \Delta$  the Laplacian operator ( $\Delta y = \frac{\partial^2 y}{\partial x_1^2} + \frac{\partial^2 y}{\partial x_2^2}$ ). We fixe the tolerance to  $\text{tol} = 10^{-3}$  and the smoothing parameter to  $\alpha = 10^{-3}$ . In our experiments, we use the two following functions

$$\theta_{\alpha}^1(x) = \frac{x}{x + \alpha},$$

$$\theta_{\alpha}^{\log}(x) = \frac{\log(1 + x)}{\log(1 + x + \alpha)}.$$

### 7.1. Description of the example:

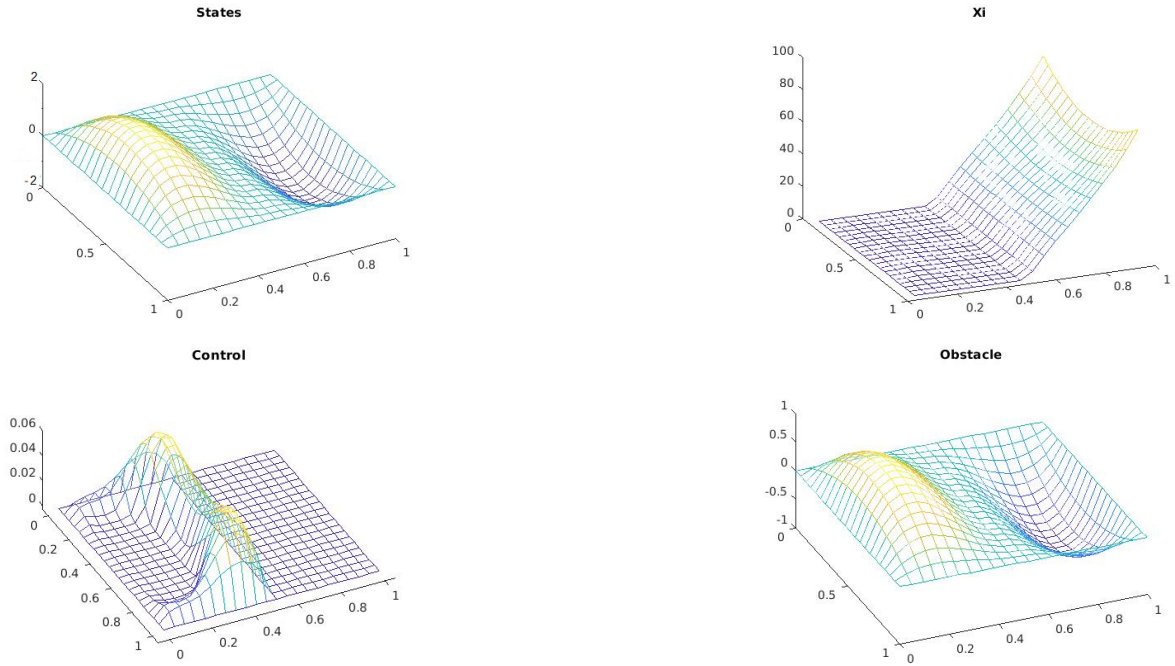
We set  $U_{ad} = L^2(\Omega)$ ,  $\nu = 0.1$ ,  $z_d = 1$ ,  $v_d = 0$ ,  $g(y) = y^3$ .

$$f(x_1, x_2) = \begin{cases} 200[2x_1(x_1 - 0.5)^2 - x_2(1 - x_2)(6x_1 - 2)] & \text{if } x_1 \leq 0.5, \\ 200(0.5 - x_1) & \text{else,} \end{cases}$$

$$\psi(x_1, x_2) = \begin{cases} 200[x_1 x_2 (x_1 - 0.5)^2 (1 - x_2)] & \text{if } x_1 \leq 0.5, \\ 200[(x_1 - 1)x_2 (x_1 - 0.5)^2 (1 - x_2)] & \text{else.} \end{cases}$$



Figure 1: Data of the considered example

Figure 2: Optimal solution using the  $\theta_\alpha^1$ ,  $N=20$  and  $\alpha = 10^{-3}$ 

## 7.2. Details of the numerical tests

### 7.2.1. Numerical simulation results using IPOPT solver

In our experiments we made a logarithmic scaling for these two functions to bound their gradients. Each constraint

$$\theta_\alpha((y - \psi)_{i,j}) + \theta_\alpha(\xi_{i,j}) \leq 1$$

is in fact replaced by the following inequality

$$\alpha^2 \ln \left( \frac{\alpha}{(y - \psi)_{i,j} + \alpha} + \frac{\alpha}{\xi_{i,j} + \alpha} \right) \geq 0, \quad 0 \leq i, j \leq N + 1$$

in the case of the  $\theta_\alpha^1$  function and

$$\alpha \ln \left( 2 - \left( \frac{\log(1 + (y - \psi)_{i,j})}{\log(1 + (y - \psi)_{i,j} + \alpha)} + \frac{\log(1 + \xi_{i,j})}{\log(1 + \xi_{i,j} + \alpha)} \right) \right) \geq 0, \quad 0 \leq i, j \leq N + 1$$

in the case of the  $\theta_\alpha^{\log}$ .

This scaling technique was proposed and used in [10] to avoid numerical issues. The two following tables give in view of exemple 7.1 and for different values of the parameter  $\alpha$ , the complementarity error, the state equation error and the solution obtained when using each of the two smoothing functions.

$\alpha$	$\ Ay - g(y) - f - v - \xi\ _2$	$\langle y - \psi, \xi \rangle / N^2$	Obj
0.1	$9.7453e - 14$	$9.911e - 3$	$2.8420e + 02$
$10^{-2}$	$8.2725e - 14$	$9.998e - 5$	$2.8564e + 02$
$10^{-3}$	$8.73843e - 14$	$9.675e - 7$	$2.8572e + 02$
$10^{-4}$	$1.75849e - 06$	$5.243e - 09$	$2.8573e + 02$

Table 1: Using the  $\theta_\alpha^1$  smoothing function -Example 7.1- N=20

$\alpha$	$\ Ay - g(y) - f - v - \xi\ _2$	$\langle y - \psi, \xi \rangle / N^2$	Obj
0.1	$9.00735e - 14$	$4.943e - 3$	$2.8455e + 02$
$10^{-2}$	$8.89491e - 14$	$6.013e - 5$	$2.8564e + 02$
$10^{-3}$	$9.22557e - 14$	$5.892e - 7$	$2.8572e + 02$
$10^{-4}$	$3.47932e - 06$	$8.124e - 8$	$2.8573e + 02$

Table 2: Using the  $\theta_\alpha^{\log}$  smoothing function -Example 7.1- N=20

## 7.2.2. Numerical comparisons using different solvers : IPOPT [20] , KNITRO [18] and SNOPT [13]

Solver	SNOPT	KNITRO	IPOPT
$\ Ay - g(y) - f - v - \xi\ _2$	$1.06926e - 12$	$6.00605e - 14$	$6.71515e - 14$
$\langle y - \psi, \xi \rangle / N^2$	$8.7111e - 5$	$8.7277e - 5$	$8.70844e - 5$
Obj	$1.502476e + 02$	$1.502476e + 02$	$1.502476e + 02$
Iter	26236	200	198

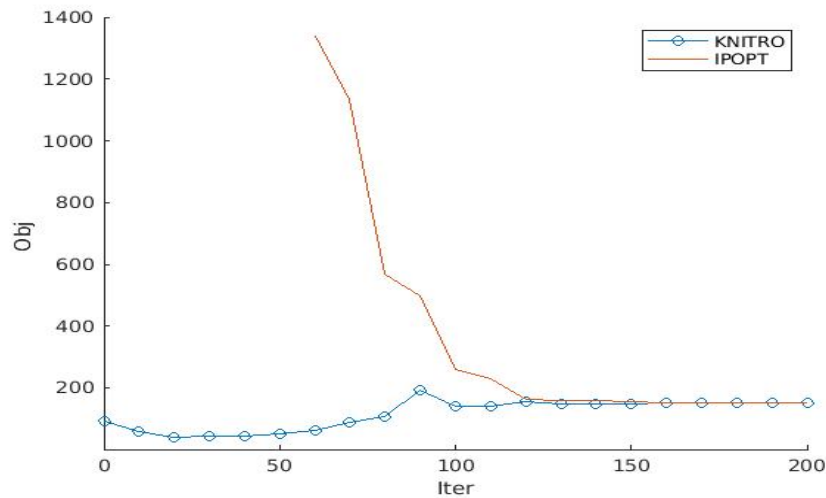
Table 3: Using the  $\theta_\alpha^1$  smoothing function -Example 7.1- N=15 and  $\alpha = 10^{-2}$ 

Figure 3: Numerical comparison of the different numerical solver IPOPT and KNITRO

We remark that :

The 3 algorithms obtain the same solution and almost the same objective value. This suggests that our approach can be implemented using any standard NLP solver.

## 8. Conclusions

In this work, we introduced a new regularization schema for optimal control of semilinear elliptic variational inequalities with complementarity constraints. We proved that Lagrange multipliers exist. The existence of Lagrange multipliers is an important tool to describe and study algorithms to compute the solutions(s) of  $(\mathcal{P}^\alpha)$  (that are "good approximations" of the original problem  $(\mathcal{P})$ ). In our numerical experiments, we used several standard NLP solvers and obtain promising results. The next step will be to develop an approach based on our optimality conditions.

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