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Distributed event-triggered formation control for multi-agent systems in presence of packet losses

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Abstract

This paper considers a distributed formation control problem for a multi-agent system. The dynamic of agents is assumed to be Euler-Lagrange. An event-triggered approach is considered, leading to intermittent communications between agents. This paper addresses the situation where communications are prone to random packet losses. To evaluate its control input, each agent maintains multi-hypothesis estimators of its own state and of the states of its neighbors accounting for the potentially lost packets. Each agent is then able to compute the expected estimation error of its own state as evaluated by its neighbors. The condition to trigger communication (CTC) involves this expected error. An analysis of the behavior of the system with such CTC is performed using stochastic Lyapunov functions. Simulations confirm the effectiveness of the proposed approach.

Key words: Communication constraints, event-triggered control, packet losses, formation stabilization, multi-agent system.

1 Introduction

Distributed control with event-triggered communication is an efficient method to coordinate Multi-agent System (MAS) with a reduced amount of communication between agents. Contrary to periodical communication [5] or intermittent communication [22], messages are transmitted only when needed. This method reduces the need for communications, and so reduces the risk of packet collisions. The main difficulty of these approaches consists in determining the communication triggering condition (CTC) that will ensure the completion of the task assigned to the MAS (see [16,19,6,27,23,15,2]). Nevertheless, since event-triggered approaches are based on the idea that a message is transmitted only when required, a loss of information may have a critical impact on the performance, and even stability of the MAS.

Packet losses may be due to collisions between packets transmitted from different agents, presence of obstacles, or interference with other communications systems. The packet losses issue is usually addressed by integrating an acknowledgement mechanism in the communication protocol to detect and possibly re-transmit lost messages. In practice, acknowledgement or re-transmitted message can also be lost. This induces communication delays if the same content is re-transmitted, and may lead to desynchronization between agents.

In [1,7,8,26,24] packet losses is addressed by combining H_∞ control and event-triggered communications. Sufficient conditions are established to ensure the global exponential stability of the system. In [7], communication delays and packet losses are considered simultaneously. In [8], two different models (Bernoulli and non-Bernoulli) are proposed to describe the packet losses in the networks. In [8] external perturbations are considered and it is shown that event-triggered control schemes are more vulnerable to packet losses than time-triggered control strategies. The work in [1] addresses the problem of a MAS following several leaders to reach its objectives. The observers of the state of other agents run by each agent account for the last send message, agent dynamics and measurement perturbations. However, agents with linear dynamics are considered in all these works.

Non-linear dynamic models are studied in [3,4]. In [3], the influence of packet losses is taken into account in the estimator models and not in the CTC. New distributed estimators are designed to guarantee the exponential stability of the estimation errors even in presence of packet losses. To update its estimate of other agents' states, each agent uses its own innovation and

the innovation of its neighbor's. This improves the accuracy of the estimates at the cost of being more sensitive to the loss of information expected from other agents.

In [4], the problem of control of a single agent in presence of potential loss of information in the measurement communication process is considered. An event-triggered strategy is proposed along with two communication protocols, with and without acknowledgment scheme. With acknowledgment scheme, the number of successive packet losses can be perfectly known, and the most recently received measurement value can be identified. Without acknowledgment scheme, this information is no longer available and a set of estimators is used to perform an estimation of this measurement, each one using a different hypothesis of the last packet received. As previously stated, the case of a MAS is not studied here.

In this paper, the problem of distributed formation control of MAS with nonlinear Euler-Lagrange dynamics with state perturbations is addressed. The reduction of communications is performed by a distributed event-triggered strategy. It extends the method proposed in [20] to the case of packet losses by modifying the CTC evaluation to account for the potentially lost information during communication in the MAS. In a similar way as [4], each agent maintains multi-hypotheses estimators of its own state, each corresponding to estimates performed by its neighbors based on the potential received and lost information. As this may lead to heavy computational burden, a practical implementation is proposed to limit the number of estimators to maintain. The convergence of the global system to the target formation is proved.

Assumptions and the formation parameterization are introduced in Section 2 and 3. The distributed control law is described in Section 3. State estimators to replace missing information in control law and evaluate the CTC are proposed in Section 4.1. Influence of packet losses on estimator is presented in Section 4.2, to evaluate an expected value of the estimation error. Knowledge of this error is improved using a feedback information from other agents, as described in Section 4.4. The distributed CTC is presented in Section 5. A simulation example is presented in Section 6 to illustrate the reduction of the number of communications obtained by the proposed approach. Finally, Section 7 presents conclusions and perspectives for future work.

2 Notations and hypotheses

Consider a vector $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$. The notation $x \geq 0$ indicates that each component x_i of x is non-negative, *i.e.*, $x_i \geq 0, \forall i \in \{1, \dots, n\}$. The absolute value of the i -th component of x is $|x_i|$ and $|x| = (|x_1|, \dots, |x_n|)^T$.

2.1 Multi-agent system

Consider a MAS consisting of N communicating agents with indexes in the set $\mathcal{N} = \{1, \dots, N\}$. In a global fixed reference frame \mathcal{R} , let $q_i \in \mathbb{R}^n$ be the vector of *coordinates* of agent i and $q = [q_1^T, \dots, q_N^T]^T \in \mathbb{R}^{Nn}$ be the *configuration* of the MAS. The relative coordinate vector between two agents i and j is $r_{ij} = q_i - q_j$.

The evolution of the state $x_i = [q_i^T, \dot{q}_i^T]^T$ of agent i is assumed to be described by the Euler-Lagrange model

$$M_i(q_i) \ddot{q}_i + C_i(q_i, \dot{q}_i) \dot{q}_i + G = u_i + d_i, \quad (1)$$

where $u_i \in \mathbb{R}^n$ is some control input, $M_i(q_i) \in \mathbb{R}^{n \times n}$ is the inertia matrix of agent i , $C_i(q_i, \dot{q}_i) \in \mathbb{R}^{n \times n}$ is the matrix of the Coriolis and centripetal terms for agent i , G accounts for gravitational acceleration supposed to be known and constant, and d_i is a time-varying state perturbation satisfying $\|d_i(t)\| \leq D_{\max}$.

One assume that the MAS satisfies is such that for each agent i :

A1) $M_i(q_i)$ is symmetric positive and there exists $k_M > 0$ satisfying $\forall x, x^T M_i(q_i) x \leq k_M x^T x$.

A2) $\dot{M}_i(q_i) - 2C_i(q_i, \dot{q}_i)$ is skew symmetric or negative definite and there exists $k_C > 0$ satisfying $\forall x, x^T C_i(q_i, \dot{q}_i) x \leq k_C \|\dot{q}_i\| x^T x$.

A3) the left side of (1) can be linearly parametrized as

$$M_i(q_i) \xi_1 + C_i(q_i, \dot{q}_i) \xi_2 = Y_i(q_i, \dot{q}_i, \xi_1, \xi_2) \theta_i \quad (2)$$

for all vectors $\xi_1, \xi_2 \in \mathbb{R}^n$, where $Y_i(q_i, \dot{q}_i, \xi_1, \xi_2)$ is a regressor matrix with known structure identical for all agents, and $\theta_i \in \mathbb{R}^p$ is a vector of constant parameters known by agent i .

A4) x_i can be measured without error.

A5) its initial value of its state $x_i(0)$ is known by of all its neighbors.

Assumptions A1, A2, and A3 have been previously considered, *e.g.*, in [11–13,18]. In what follows, the notations M_i and C_i are used in place of $M_i(q_i)$ and $C_i(q_i, \dot{q}_i)$.

2.2 Communication model

Each agent is only able to communicate with a subset of agents in the MAS. This communication topology is described by a fixed undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, where $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$ is the set of edges of the graph. The set of neighbors of agent i is $\mathcal{N}_i = \{j \in \mathcal{N} \mid (i, j) \in \mathcal{E}, i \neq j\}$ with cardinal number N_i .

One neglects communication delays between agents. Due to occlusions, fading, and packet collisions, the communication link between agents i and j may be unreliable and messages may be lost. When agent i broadcasts its k_i -th message at time t_{i,k_i} , agent $j \in \mathcal{N}_i$ either receives this message without error at time $t_{i,k_i}^j = t_{i,k_i}$ or does not receive it. To limit the amount of communications, one assumes further that there is no acknowledgement protocol and thus no possible retransmission in case of losses. Let $\{\delta_{i,k_i}^j\}_{k_i \geq 1}$ be a sequence of binary variables such that $\delta_{i,k_i}^j = 1$ if the k_i -th message sent by agent i has been received by agent j and $\delta_{i,k_i}^j = 0$ else.

Here, the δ_{i,k_i}^j s are modeled as realizations of a timing-invariant Markov processes with characteristic identical for all agents, as described in Assumption A6.

A6) *There exists $\kappa > 0$ such that for all pairs of neighbouring agents (i, j) , one has*

$$\Pr \left(\delta_{i,k_i}^j = 1 \mid \sum_{\ell=1}^{\kappa} \delta_{i,k_i-\ell}^j > 0 \right) = 1 - \pi \quad (3)$$

$$\Pr \left(\delta_{i,k_i}^j = 0 \mid \sum_{\ell=1}^{\kappa} \delta_{i,k_i-\ell}^j > 0 \right) = \pi \quad (4)$$

and

$$\Pr \left(\delta_{i,k_i}^j = 1 \mid \sum_{\ell=1}^{\kappa} \delta_{i,k_i-\ell}^j = 0 \right) = 1 \quad (5)$$

$$\Pr \left(\delta_{i,k_i}^j = 0 \mid \sum_{\ell=1}^{\kappa} \delta_{i,k_i-\ell}^j = 0 \right) = 0 \quad (6)$$

with $0 \leq \pi < 1$.

Assumption A6 implies that at least one of the last κ messages broadcast by Agent i has been received by each of its neighbours Agent j .

Let $k_i^j \leq k_i$ be the index of the last message agent j has received from its neighbor i . When a communication is triggered at time t_{i,k_i} , agent i broadcasts a message containing k_i , t_{i,k_i} , $q_i(t_{i,k_i})$, $\dot{q}_i(t_{i,k_i})$, θ_i , and $\{k_j^i\} \in j \in \mathcal{N}_i$. By sending $k_j^i \leq k_j$ for all $j \in \mathcal{N}_i$, agent i indicates the index of the last message received from each of its neighbors.

When agent j receives a message from agent i , it updates k_i^j to k_i . Moreover, $q_i(t_{i,k_i})$, $\dot{q}_i(t_{i,k_i})$, and θ_i are used to update its estimator of the state of agent i , as detailed in Section 4.1. Finally, agent j keeps track in the variables $k_j^{i,j}$ of the value of k_j^i which represents the index of the last message sent by agent j and which has been actually received by agent i . $k_j^{i,j}$ is used by agent j to evaluate the knowledge agent i has on x_j (see example Figure 1).

2.3 Target formation

A potentially time-varying target formation is defined by the set $\mathcal{R} = \{r_{ij}^*(t), (i, j) \in \mathcal{N} \times \mathcal{N}\}$, where $r_{ij}^*(t)$ is the target relative coordinate vector between agents i and j . Without loss of generality, the first agent is considered as the reference agent. Any target relative coordinate vector r_{ij}^* can be expressed as $r_{ij}^*(t) = r_{i1}^*(t) - r_{j1}^*(t)$. The target relative *configuration* vector is $r^*(t) = [r_{11}^{*T}(t) \dots r_{1N}^{*T}(t)]^T$. Each agent i knows only the relative coordinate vector between its own neighbors $r_{ij}^*(t)$, $j \in \mathcal{N}_i$. Additionally, a constant target reference velocity \dot{q}_1^* known by all agents is imposed to the MAS, where \dot{q}_1^* .

Our aim is to evaluate, in a distributed way, the control input for each agent so that the MAS converges to \mathcal{R} , while limiting the number of communications between agents and accounting for losses in the communication between agents. For that purpose, the control inputs will have to provide an asymptotic convergence of the MAS to the target configuration vector with a bounded mean-square error. Due to the packet losses, this convergence will only be achievable in the mean-square sense.

Definition 1 *The MAS asymptotically mean-square converges to the target formation with a bounded mean-square error (bounded average asymptotic convergence) iff there exists some $\varepsilon_1 > 0$ such that*

$$\forall (i, j) \in \mathcal{N}^2, \lim_{t \rightarrow \infty} \mathbb{E} \left(\|r_{ij}(t) - r_{ij}^*(t)\|^2 \right) \leq \varepsilon_1, \quad (7)$$

where the expectation is evaluated considering the packet loss events.

2.4 Overview of the proposed approach

A distributed control law is introduced in Section 3 to drive the MAS to its target formation and reference speed. This requires the knowledge by each agent of the state vector of its neighbors. Since the state vector of a neighbor j is only available at agent i when agent j broadcasts its state, agent i has to maintain an estimator of the state of each of its neighbors. This estimator is described in Section 4.

Moreover, to determine the quality of the estimate of x_i evaluated by its neighbors, agent i has also to estimate its own state x_i with the information it has transmitted to these neighbors. As soon as a function of the error between this estimate and x_i reaches some threshold, agent i triggers a communication to allow its neighbors to refresh their estimate of x_i . The main difficulty, compared to [18,20], lies in the fact that estimators have to account for packet losses. In the solution proposed here, each agent maintains several estimates of its own state accounting for different packet loss hypotheses, and an estimate of the state of its neighbors with the last information received. As will be seen in Section 4.4, the number of hypotheses can be limited to a manageable amount determined by the last received packet from agent i .

The CTC relies on the error between the values of the states of agents and of the estimates made by neighboring agents, see Section 5. Since this error cannot be exactly evaluated due to packet losses, only its expected value is used in the CTC. This paper proposes different methods to evaluate or upper-bound this expected error, which is then used to analyze the convergence and the stability of the MAS.

3 Distributed control inputs

Section 3.1 introduces the potential energy $P(q, t)$ of the MAS to quantify the discrepancy between the current and target formations. A control input accounting for agent state estimators is defined in Section 3.2.

3.1 Potential energy of the formation

In [14,25], the *potential energy* of the formation

$$P(q, t) = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N m_{ij} \|r_{ij} - r_{ij}^*\|^2 \quad (8)$$

is introduced, where the coefficients $m_{ij} = m_{ji}$ are some positive or null coefficients. $P(q, t)$ quantifies the discrepancy between the actual and target relative coordinate vectors. We take $m_{ii} = 0$, $m_{ij} = 0$ if $(i, j) \notin \mathcal{E}$, and $m_{ij} > 0$ if $(i, j) \in \mathcal{E}$. Since \mathcal{G} is connected, the minimum number of non-zero coefficients m_{ij} to properly define a target formation is $N - 1$.

Proposition 2 *The MAS asymptotically converges to the target formation with a bounded error iff there exists some $\varepsilon_2 > 0$ such that*

$$\lim_{t \rightarrow \infty} \mathbb{E} (P(q, t)) \leq \varepsilon_2, \quad (9)$$

where the expectation is evaluated considering the packet loss events.

The proof of Proposition 2 is provided in Appendix A.1.

3.2 Control input with agent state estimators

In what follows, a distributed control law is designed so that the MAS asymptotically converges with a bounded mean-square error. The control law has to make $P(q, t)$ decrease. For that purpose, one introduces, as in [25],

$$g_i = \frac{\partial P(q, t)}{\partial q_i} = \sum_{j \in \mathcal{N}_i} m_{ij} (r_{ij} - r_{ij}^*), \quad (10)$$

$$\dot{g}_i = \sum_{j \in \mathcal{N}_i} m_{ij} (\dot{r}_{ij} - \dot{r}_{ij}^*), \quad (11)$$

$$s_i = \dot{q}_i - \dot{q}_i^* + k_p g_i, \quad (12)$$

where $\dot{q}_i^* = \dot{q}_1^* - \dot{r}_{1i}^*$ is the reference velocity of agent i . The vectors g_i and \dot{g}_i characterize the evolution with q_i and \dot{q}_i of the discrepancy between the actual and target relative coordinate vectors. In (12), $k_p > 0$ is a scalar design parameter.

According to (10), to make $P(q, t)$ decrease, the control input of agent i requires r_{ij} , and thus q_j , $j \in \mathcal{N}_i$. Nevertheless, $q_j, j \in \mathcal{N}_i$ is only available to agent i when it receives a packet from agent j containing q_j , see Section 2.2. Between the reception of two packets from Agent j , an estimates \hat{q}_j^i of $q_j, j \in \mathcal{N}_i$ needs to be evaluated. This one has to account for potentially lost packets, see Section 4.1.

Using estimates \hat{q}_j^i and $\dot{\hat{q}}_j^i$ of q_j and \dot{q}_j for all $j \in \mathcal{N}_i$, agent i is able to evaluate $\bar{r}_{ij} = q_i - \hat{q}_j^i$, $\dot{\bar{r}}_{ij} = \dot{q}_i - \dot{\hat{q}}_j^i$, as well as

$$\bar{g}_i = \sum_{j \in \mathcal{N}_i} m_{ij} (\bar{r}_{ij} - r_{ij}^*) \quad (13)$$

$$\bar{s}_i = \dot{q}_i - \dot{q}_i^* + k_p \bar{g}_i. \quad (14)$$

Then, the following control input can be evaluated in a distributed way by agent i and used in (1)

$$u_i = -k_s \bar{s}_i - k_g \bar{g}_i + G - Y_i (q_i, \dot{q}_i, \bar{p}_i, \dot{\bar{p}}_i) \theta_i, \quad (15)$$

where $\bar{p}_i = k_p \bar{g}_i - \dot{q}_i^*$ and $\dot{\bar{p}}_i = k_p \dot{\bar{g}}_i - \dot{q}_i^*$ with the additional design parameters $k_g > 0$ and $k_s \geq 1 + k_p (k_M + 1)$.

The convergence properties of the MAS when each agent i applies the control input (15) will be analysed and ensured in Section 5.

4 State estimators and packet losses

This section describes the estimators involved in the control input (15) of each agent. These estimators are introduced in Section 4.1. Section 4.2 describes the way agent i estimates its own state x_i , with the information transmitted to its neighbors, to determine the quality of their estimates of x_i . In Section 4.3, the expected value of the estimation error between the current and the estimated state is evaluated. This estimator accounts for packet losses. In Section 4.4, an implicit feedback, based on packets received from other agents, is described and exploited to improve the evaluation of the state estimation error.

4.1 Estimation of the state of other agents

To evaluate (15), agent i has to maintain an estimate \hat{x}_j^i of the state x_j of all its neighbors $j \in \mathcal{N}_i$. Assume that agent j broadcasts its k -th message at time $t_{j,k}$. Then, since communication delays are neglected, depending on whether this message has been received by agent i , \hat{x}_j^i is updated as follows, see [3]

$$\hat{x}_j^i (t_{j,k}^+) = \delta_{j,k}^i x_j (t_{j,k}) + (1 - \delta_{j,k}^i) \hat{x}_j^i (t_{j,k}), \quad (16)$$

where $x_j(t_{j,k})$ is obtained from the received packet. For all $t \geq t_{j,k}$ and up to the time instant of reception of the next packet sent by agent j , the components \hat{q}_j^i and $\dot{\hat{q}}_j^i$ evolve according to

$$M_j \left(\hat{q}_j^i \right) \ddot{\hat{q}}_j^i + C_j \left(\hat{q}_j^i, \dot{\hat{q}}_j^i \right) \dot{\hat{q}}_j^i + G = \hat{u}_j^i. \quad (17)$$

where M_j and C_j are evaluated using (2) with Y_j and $\hat{\theta}_j^i = \theta_j$, where the structure of Y_j and θ_j are initially known by Agent i or have been transmitted by Agent j at time $t = 0$. The estimator (17) maintained by agent i requires itself an estimate \hat{u}_j^i of the control input u_j evaluated by agent j . This estimate \hat{u}_j^i , used by agent i , is

$$\hat{u}_j^i = -k_s \hat{\varepsilon}_j^i + G - Y_j \left(\hat{q}_j^i, \dot{\hat{q}}_j^i, -\ddot{q}_j^*, -\dot{q}_j^* \right) \hat{\theta}_j^i, \quad (18)$$

with $\hat{\varepsilon}_j^i = \hat{q}_j^i - \dot{q}_j^*$. The control input (18) thus only depends on information available at agent i .

4.2 Multi hypothesis state estimates

The estimate \hat{q}_i^j of the state of agent i , evaluated by agent j , only depends on the information provided by agent i . The estimate \hat{q}_i^j is reset to q_i as soon as a message sent by agent i is received by agent j , see (16). Consequently, when agent i has sent k_i messages, and wants to evaluate the estimate of its own state as computed by one of its neighbors, $\kappa + 2$ different hypotheses have to be considered, each of which is associated to a different estimator of q_i at time $t \in [t_{i,k_i}, t_{i,k_i+1}]$

- the k_i -th packet has been received,
- the k_i -th packet has been lost, but the $k_i - 1$ -th packet has been received,
- ...
- all packets have been lost, except the $k_i - \kappa$ -th if $k_i \geq \kappa$,
- no packet has been received if $k_i < \kappa$, but the initial state is known.

At time $t \in [t_{i,k_i}, t_{i,k_i+1}[$, the state estimates corresponding to these hypotheses are denoted $\hat{q}_i^{i,\ell}(t)$, $\ell = \max\{0, k_i - \kappa\}, \dots, k_i$ and $\hat{q}_i^{i,k_i} = \hat{q}_i^i$, introduced in Section 4.1.

Since there are at most $\kappa - 1$ consecutive losses, agent i has only to maintain κ estimates of q_i , denoted $\hat{q}_i^{(1)}(t), \dots, \hat{q}_i^{(\kappa)}(t)$. For all $t \in [t_{i,k_i}, t_{i,k_i+1}[$, one has $\hat{q}_i^{(1)}(t) = \hat{q}_i^{i,k_i}(t), \dots, \hat{q}_i^{(\kappa)}(t) = \hat{q}_i^{i,k_i-\kappa+1}(t)$. These estimates evolve according to the dynamic (17)-(18) introduced in Section 4.1. When a new packet is sent at time t_{i,k_i+1} by agent i , the estimates are updated as follows

$$\hat{q}_i^{(1)}(t_{i,k_i+1}) = q_i(t_{i,k_i+1}). \quad (19)$$

$$\hat{q}_i^{(\ell+1)}(t_{i,k_i+1}) = \hat{q}_i^{(\ell)}(t_{i,k_i+1}^-), \quad \ell = 1, \dots, \kappa - 1, \quad (20)$$

where $t_{i,k_i+1}^- = t_{i,k_i+1} - \varepsilon$ for some very small $\varepsilon > 0$.

4.3 Expected value of the estimation error of $q_i(t)$

At time $t \in [t_{i,k_i}, t_{i,k_i+1}[$, agent i has sent k_i packets. Let

$$p_{k_i,\ell}^j = \Pr\left(\delta_{i,\ell}^j = 1, \delta_{i,\ell+1}^j = 0, \dots, \delta_{i,k_i}^j = 0\right)$$

be the probability that the ℓ -th packet has been received by a given neighbor j and that all packets from the $\ell + 1$ -th to the k_i -th have been lost. Note that $p_{k_i,\ell}^j$ only depends on the considered packet loss model, and does not depend on the neighbor index j , which is omitted in what follows. When $\ell = k_i$, one has simply $p_{k_i,k_i} = 1 - \pi$.

At time $t \in [t_{i,k_i}, t_{i,k_i+1}[$, the estimation error of the coordinates of agent i , as evaluated by agent j , is

$$e_i^j(t) = \hat{q}_i^j(t) - q_i(t) \quad (21)$$

and its mean-square value is

$$\mathbb{E}\left(\|e_i^j(t)\|^2\right) = \sum_{\ell=\max\{k_i-\kappa+1,0\}}^{k_i} p_{k_i,\ell} \|\hat{q}_i^{i,\ell}(t) - q_i(t)\|^2. \quad (22)$$

$\mathbb{E}\left(\|e_i^j(t)\|^2\right)$ can be determined by agent i using $\hat{q}_i^{i,\ell}(t)$ and $p_{k_i,\ell}$, $\ell = \max\{k_i - \kappa, 0\}, \dots, k_i$. Consequently, from (22), agent i is able to determine the quality of the estimate of q_i evaluated by its neighbors. Agent i has thus to maintain κ estimates $\hat{q}_i^{i,\ell}(t)$ of $q_i(t)$. Note the expectation of e_i^j can be obtained in the same way.

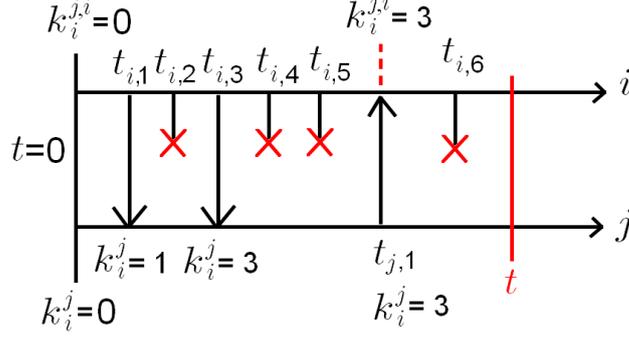


Fig. 1. Communication instants and index of last message received k_i^j and $k_i^{j,i}$.

4.4 Estimates accounting for the feedback

When agent i receives a message at time $t_{j,k_j} \in [t_{i,k_i}, t_{i,k_i+1}[$ from one of its neighbors j , this message contains the index k_i^j of the last message received by agent j and sent by agent i , see Figure 1. This index is kept by agent i in $k_i^{j,i}$, see Section 2.2. This information can significantly improve the evaluation of the mean-square value of $e_i^j(t)$. Using $k_i^{j,i}$, agent i can evaluate the probability $p_{k_i, \ell | k_i^{j,i}}^j$ that agent j has received its ℓ -th message and lost all the following message up to the k_i -th message, knowing that agent j has received its $k_i^{j,i}$ -th message. Here again, $p_{k_i, \ell | k_i^{j,i}}^j$ only depends on the values of k_i , ℓ , and $k_i^{j,i}$, and on the packet loss probability model introduced in Section 4.1. The superscript j is thus again omitted in what follows.

Consider the probability

$$p_{k_i, \ell | k_i^{j,i}} = \Pr \left(\delta_{i, \ell}^j = 1, \sum_{m=\ell+1}^{k_i} \delta_{i, m}^j = 0 \mid \delta_{i, k_i^{j,i}}^j = 1 \right) \quad (23)$$

that the ℓ -th message sent by agent i (with $\ell \leq k_i$) has been received by agent j and that all following messages, including the k_i -th have been lost, knowing that the $k_i^{j,i}$ -th message has been received. Our aim in what follows is to get an expression of $p_{k_i, \ell | k_i^{j,i}}$, for all $\ell = k_i^{j,i}, \dots, k_i$, to be used in the evaluation of mean-square value of $e_i^j(t)$.

Assume that in the time interval $[t_{i,k_i}, t_{i,k_i+1}[$, agent i has received a single message from agent j at time t_{j,k_j} . Then, upon reception of this message, agent i updates $k_i^{j,i}$ to $k_i^j \leq k_i$: By receiving this message from agent j , agent i is aware that agent j has received the $k_i^{j,i}$ -th message sent by agent i , and has not received the following ones if $k_i^j < k_i$, see Figure 1, where $k_i^j = k_i - 2$. According to Assumption A6, one has necessarily $k_i^j \geq \max\{0, k_i - \kappa\}$. Consequently, one has

$$p_{k_i, k_i^{j,i} | k_i^{j,i}} = \Pr \left(\sum_{m=k_i^{j,i}+1}^{k_i} \delta_{i, m}^j = 0 \mid \delta_{i, k_i^{j,i}}^j = 1 \right) = 1 \quad (24)$$

and for all $\ell = k_i^{j,i} + 1, \dots, k_i$,

$$p_{k_i, \ell | k_i^{j,i}} = \Pr \left(\delta_{i, \ell}^j = 1, \sum_{m=\ell+1}^{k_i} \delta_{i, m}^j = 0 \mid \delta_{i, k_i^{j,i}}^j = 1 \right) = 0. \quad (25)$$

Consider now $t \in [t_{i,k_i+1}, t_{i,k_i+2}[$ and assume that the last message received by agent i from agent j was at time $t_{j,k_j} \in [t_{i,k_i}, t_{i,k_i+1}[$. One has still $k_i^{j,i} = k_i^j \leq k_i$. Consequently, agent i knows that agent j has received the $k_i^{j,i}$ -th message sent by agent i , and has not received the following ones with indexes between $k_i^{j,i}$ and k_i . Agent i has no information about the fact the agent j has received the $k_i + 1$ -th message, except if $k_i^{j,i} = k_i - \kappa$. In the latter case, according to Assumption A6, agent i is sure that agent j has received the $k_i + 1$ -th message sent by agent i . Consequently, one has

$$p_{k_i+1, \ell | k_i^{j,i} = k_i - \kappa}(t) = \begin{cases} 1 & \ell = k_i + 1 \\ 0 & \text{else,} \end{cases} \quad (26)$$

$k_i + n$								
		5	6	7	8	9	10	
ℓ	3	1	π	0	0	0	0	
	4	0	0	0	0	0	0	
	5	0	0	0	0	0	0	
	6	*	$1 - \pi$	$(1 - \pi) \pi$	$(1 - \pi) \pi^2$	$(1 - \pi) \pi^3$	0	
	7	*	*	1	π	π^2	π^3	
	8	*	*	$-(1 - \pi) \pi$	$-(1 - \pi) \pi^2$	$-(1 - \pi) \pi^3$	$-(1 - \pi) \pi^4$	
	9	*	*	*	$1 - \pi$	$(1 - \pi) \pi$	$(1 - \pi) \pi^2$	
	10	*	*	*	*	$1 - \pi$	$(1 - \pi) \pi$	
							$1 - \pi$	$+(1 - \pi) \pi^4$

Table 1
Example of probability calculation

and

$$p_{k_i+1, \ell | k_i^{j,i} > k_i - \kappa}(t) = \begin{cases} \pi & \ell = k_i^{j,i} \\ 1 - \pi & \ell = k_i + 1 \\ 0 & \ell \in [\max\{0, k_i - \kappa\}, k_i^{j,i} - 1] \\ & \cup [k_i^{j,i} + 1, k_i]. \end{cases} \quad (27)$$

The following proposition allows one to evaluate $p_{k_i, \ell | k_i^{j,i}}$ recursively, starting from (24)-(25).

Proposition 3 Consider $t \in [t_{i, k_i+n}, t_{i, k_i+n+1}[$ with $n \geq 0$ and assume that the last message received by agent i from agent j was at time $t_{j, k_j} \in [t_{i, k_i}, t_{i, k_i+1}[$. Thus $k_i^{j,i} = k_i^j \leq k_i$. Then one may write for all $m > k_i^{j,i}$

$$p_{m, m | k_i^{j,i}} = \begin{cases} 1 - \pi + \pi p_{m-1, m-\kappa | k_i^{j,i}} & \text{if } m > \kappa \\ 1 - \pi & \text{else} \end{cases} \quad (28)$$

$$p_{m+n, m | k_i^{j,i}} = \begin{cases} \pi^n p_{m, m | k_i^{j,i}} & \text{if } n \leq \kappa \\ 0 & \text{else} \end{cases} \quad (29)$$

$$p_{m, \ell | k_i^{j,i}} = 0 \quad \text{if } \ell \leq k_i \text{ and } \ell \neq k_i^{j,i}. \quad (30)$$

The proof of Proposition 3 is in Appendix A8.

Example: Consider an agent i and a message received at time $t_{k_j} \in [t_{k_i}, t_{k_i+1}[$ from its neighbor agent j . Table 1 illustrates the different values of $p_{k_i+n, \ell | k_i^j}$, for $n \in [0, \dots, 4]$ when $\kappa = 4$, $k_i = 5$, and $k_i^j = 3$.

Then Proposition 3 can be used with Assumption **A6** to evaluate $\mathbb{E}(\|e_i^j(t)\|^2)$, taking into account the feedback information provided by neighbors as follows.

Consider some agent i and $k_i > 0$. Assume that agent i knows the index k_i^j of the last message sent by agent i and received by some neighbor agent j . At time $t \in [t_{i, k_i}, t_{i, k_i+1}[$, the mean-square value of the estimation error (21) is

$$\mathbb{E}\left(\|e_i^j(t)\|^2 | k_i^{j,i}\right) = \sum_{\ell=\max\{k_i-\kappa+1, 0\}}^{k_i} p_{k_i, \ell | k_i^{j,i}} \|\hat{q}_i^{i, \ell}(t) - q_i(t)\|^2. \quad (31)$$

In what follows, the notation $\mathbb{E}(\|e_i^j(t)\|^2)$ is used in place of $\mathbb{E}(\|e_i^j(t)\|^2 | k_i^{j,i})$. Contrary to (22), (31) depends now on the index of the neighbor agent j via $k_i^{j,i}$, and so is updated each time agent i receives a message from its neighbor, in addition to the update made each time agent i broadcast a message as in (22).

5 Event-triggered communications accounting for packet losses

This section presents a CTC which may involve one of the state estimators introduced in Section 4.

Let $m_{\min} = \min_{i,j=1,\dots,N} \{m_{ij} \neq 0\}$, $m_{\max} = \max_{i,j=1,\dots,N} \{m_{ij}\}$, $N_{\min} = \min_{i=1\dots N} (N_i)$, $\alpha_i = \sum_{j=1}^N m_{ij}$, and $\alpha_M = \max_{i=1,\dots,N} \alpha_i$. The distributed CTC (32) presented in Theorem 4 is designed to ensure an asymptotic convergence of the MAS to the target formation with a bounded mean-square error.

Theorem 4 *Consider a MAS with agent dynamics given by (1), the communication protocol defined in Section 2.2, the control law (15). Consider also the packet losses model satisfying by Assumption A6. Assuming absence of communication delays, if the communications are triggered by each agent i of the MAS when the following condition is satisfied*

$$\begin{aligned} & \alpha_M \left[\sum_{j=1}^N m_{ij} \left(k_e \mathbb{E} \left(\|e_i^j\|^2 \right) + k_p k_M \mathbb{E} \left(\|\dot{e}_i^j\|^2 \right) \right) \right. \\ & \left. + k_p k_C^2 \sum_{j=1}^N m_{ij} \left(2\mathbb{E} \left(\|e_i^j\|^2 \right) \|\dot{q}_j^i\|^2 + \mathbb{E} \left(\|e_i^j\|^4 \right) + \mathbb{E} \left(\|\dot{e}_i^j\|^4 \right) \right) \right] \\ & + k_g b_i \|\dot{q}_i - \dot{q}_i^*\|^2 \geq k_s \bar{s}_i^T \bar{s}_i + k_p k_g \bar{g}_i^T \bar{g}_i + \eta \end{aligned} \quad (32)$$

where $k_e = k_s k_p^2 + k_g k_p + \frac{k_g}{b_i}$, $\eta \geq 0$, and $0 < b_i < \frac{k_s}{k_s k_p + k_g}$ are design parameters, then

(a) The MAS with agent dynamics (1) asymptotically converges to the target formation with a bounded error such that

$$\lim_{t \rightarrow \infty} \mathbb{E} \left(\frac{1}{2} P(q, t) \right) \leq \xi \quad (33)$$

where $\xi = \frac{N}{k_g c_3} [D_{\max}^2 + \eta]$,

$$c_3 = \frac{\min \{k_1, k_p\} \min \left(1, \frac{N_{\min} m_{\min}}{m_{\max}} \right)}{\max \{1, k_M\}} \quad (34)$$

and $k_1 = k_s - (1 + k_p (k_M + 1))$.

(b) One has $t_{i,k_i} < t_{i,k_i+1}$.

The proof of (a) in Theorem 4 is given in Appendix A.3, and proof of (b) in Appendix A.4. Each agent i has to evaluate the expected values of $\|e_i^j(t)\|^2$ and $\|\dot{e}_i^j(t)\|^2$ for all $j \in \mathcal{N}_i$. This can be done by evaluating the expectation (22) or (31) detailed in Section 4.3 and 4.4.

The CTC (32) is triggered by agent i mainly when $\mathbb{E}(\|e_i^j(t)\|^2)$ and $\mathbb{E}(\|\dot{e}_i^j(t)\|^2)$ become large. Thus, it is preferable to use the knowledge of $k_i^{j,i}$ allowed from the proposed feedback mechanism to calculate (31) rather than using (22).

An analysis of the impact of the values of the parameters on the reduction of communications has been presented in [20] in absence of packet losses. These results can be extended to the case with packet losses. The choice of the parameters α_M , k_g , k_p and b_i also determines the number of messages broadcast. Choosing the coefficients m_{ij} such that $\alpha_i = \sum_{j=1}^N m_{ij}$ is small, leads to a reduction in the number of communications triggered resulting from the satisfaction of (32), at the cost of a less precise formation.

6 Example

Consider the dynamical model of N identical surface ships with coordinate vectors $q_i = [x_i \ y_i \ \psi_i]^T \in \mathbb{R}^3$, $i = 1 \dots N$, in a local Earth-fixed frame. For agent i , (x_i, y_i) represents its position and ψ_i its heading angle. The agent dynamics are expressed in the body frame (see [10]) as

$$M_{b,i} \dot{v}_i + C_{b,i}(v_i) v_i + D_{b,i} v_i = \tau_{b,i} + d_{b,i}, \quad (35)$$

where v_i is the velocity vector in the body frame. The values of $M_{b,i}$, $D_{b,i}$, and $C_{b,i}(v_i)$ are taken from [10].

One takes $N = 6$. The model (35) may be expressed as (1) with $G = 0$ using an appropriate change of variables detailed in [10]. The parameters of (15) are $k_M = \|M_i\| = 33.8$, $k_C = \|C_i(1_N)\| = 43.96$, $k_p = 6$, $k_g = 20$, $k_s = 1 + k_p (k_M + 1)$, $b_i = \frac{1}{k_g}$.

6.1 Parameters

The initial value of the configuration vector is $q(0) = [x(0)^T, y(0)^T, \psi(0)^T]^T$, $\dot{q}(0) = 0_{3N \times 1}$, with $x(0) = [-0.35, 4.59, 4.72, 0.64, 3.53, -1.26]$, $y(0) = [-1.11, -4.59, 2.42, 1.36, 1.56, 3.36]$ and $\psi(0) = 0_N$. An hexagonal target formation is considered with $r^*(0) = [r_{(1)}^*(0)^T, r_{(2)}^*(0)^T, r_{(3)}^*(0)^T]^T$ where $r_{(1)}^*(0) = [0, 2, 3, 2, 0, -1]$, $r_{(2)}^*(0) = [0, 0, \sqrt{3}, 2\sqrt{3}, 2\sqrt{3}, \sqrt{3}]$, and $r_{(3)}^*(0) = 0_N$. Moreover, the target MAS velocity is $\dot{q}_1^* = [1, 1, 0]^T$. Each agent communicates with $N/2 = 3$ other agents. From [25], one obtains the coefficients matrix $S = [m_{ij}]_{i,j=1\dots N}$ such

$$S = 0.1 \begin{bmatrix} 0 & 1.85 & 0 & 0.926 & 0 & 1.85 \\ 1.85 & 0 & 1.85 & 0 & 0.926 & 0 \\ 0 & 1.85 & 0 & 1.85 & 0 & 0.926 \\ 0.926 & 0 & 1.85 & 0 & 1.85 & 0 \\ 0 & 0.926 & 0 & 1.85 & 0 & 1.85 \\ 1.85 & 0 & 0.926 & 0 & 1.85 & 0 \end{bmatrix}.$$

One has $\alpha_i = \sum_{j=1}^N m_{ij} = 0.463$, for all $i = 1, \dots, N$ and $\alpha_M = 0.463$.

The simulation duration is $T = 4$ s, taken sufficiently large to reach a steady-state behavior, with an integration step size $\Delta t = 0.01$ s. Since time has been discretized, the minimum delay between the transmission of two messages by the same agent is set to Δt . The perturbation $d_i(t)$ is assumed constant over each interval $[k\Delta t, (k+1)\Delta t]$. The components of $d_i(t)$ are independent realizations of zero-mean uniformly distributed noise $U(-D_{\max}/\sqrt{3}, D_{\max}/\sqrt{3})$ and are thus such that $\|d_i(t)\| \leq D_{\max}$. Let N_m be the total number of messages transmitted during a simulation. The performance of the proposed approach is evaluated with $R_{\text{com}} = N_m/\bar{N}_m$, where $\bar{N}_m = NT/\Delta t \geq N_m$. One takes $\kappa = 6$.

6.2 Simulations results

Figure 2 shows the performance of the proposed approach with the CTC (32) for different values of the packet loss probability π and disturbance bound D_{\max} . Results are averaged over 50 independent realizations of the noise and of the packet losses. As expected, the number of communications required for the MAS to converge increases with π and D_{\max} .

The influence of η on the number of communication is detailed in [21]. Increasing η leads to a reduction of R_{com} but increases the potential energy $P(q, T)$, and thus the discrepancy with respect to the target formation.

Figure 3 compares results of the proposed approach obtained without (a) and with (b) the exploitation of the index $k_i^{j,i}$ of the last message sent by agent i and received by some neighbor agent j . Using the implicit feedback from neighbors, and thus $\mathbb{E}(\|e_i^j(t)\|^2 | k_j^i)$ instead of $\mathbb{E}(\|e_i^j(t)\|^2)$ in the CTC, convergence is obtained with 75% less communications.

7 Conclusion

This paper addresses the problem of communication reduction in distributed formation control of a MAS with Euler-Lagrange dynamics in presence of packet losses and perturbations. To evaluate its control input, each agent maintains estimators of the states of the other agents. Each agent also maintains a multi-hypothesis estimator of its own state accounting for potentially lost packets in the communications with its neighbors. Using these estimators, each agent is then able to compute an expected value of the estimation error of its own state as evaluated by its neighbors. An feedback from other agents may be used to get a reduced estimation error. A distributed CTC is then proposed, involving these estimation errors, to reduce the number of communications and the behavior of the MAS is analyzed using stochastic Lyapunov functions. Convergence to the target formation has been proven and the time between consecutive communications has been proven to be strictly positive. Simulations illustrate the effectiveness of the proposed approach.

In future work, communication delays will also be considered along with packet losses.

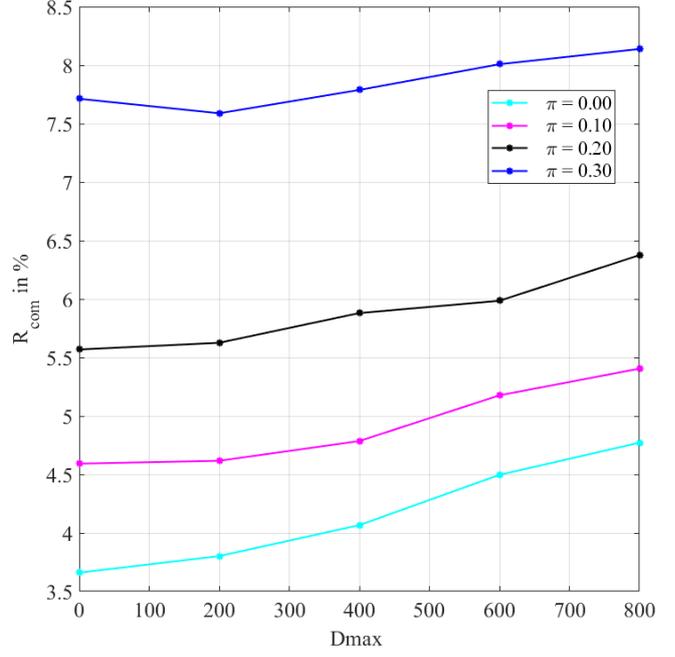
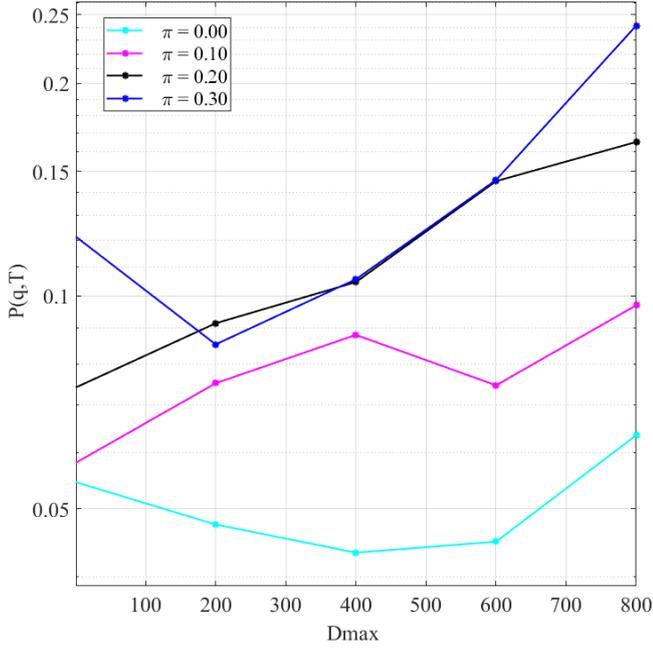
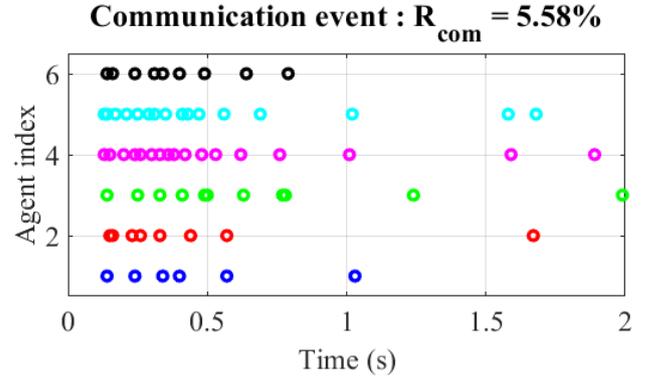
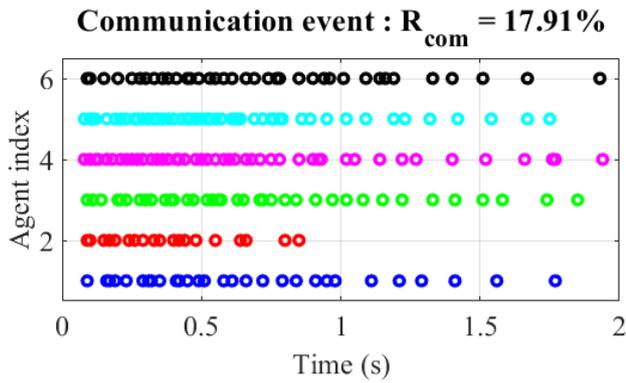
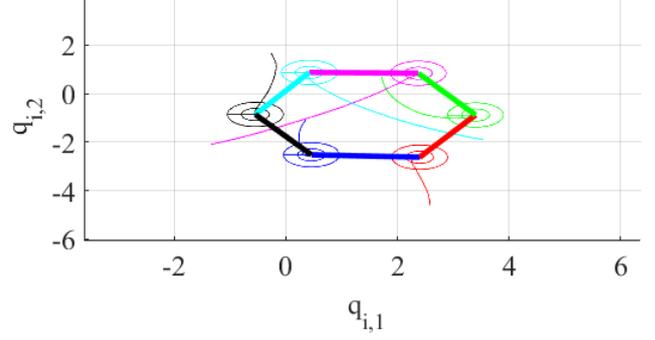
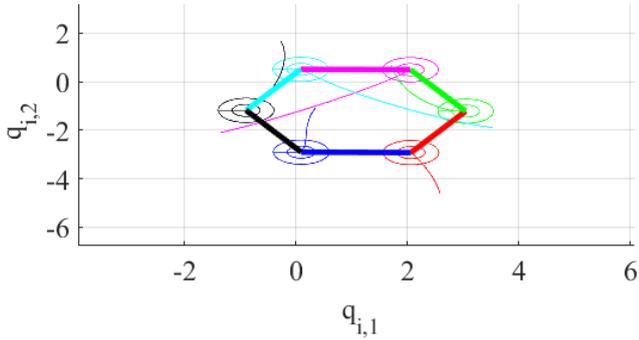


Fig. 2. Evolution of $P(q,T)$ and R_{com} for different values of D_{\max} , $\eta = 100$. The estimator (17) is considered, as well as $\mathbb{E} \left(\|e_i^j(t)\|^2 |k_j^{i,j} \right)$ from (31).



(a) without feedback

(b) with feedback

Fig. 3. Results of the method using (a) no implicit feedback (22), (b) an implicit feedback (31). $D_{\max} = 200$, $\pi = 0.2$

References

- [1] W. Chen, D. Ding, G. Wei, S. Zhang, and Y. Li. Event-based containment control for multi-agent systems with packet dropouts. *Journal of Systems Science*, 49(12):2658–2669, 2018.
- [2] B. Cheng and Z. Li. Fully distributed event-triggered protocols for linear multiagent networks. *IEEE Trans. Automatic Control*, 64(4):1655–1662, 2018.
- [3] D. Ding, Z. Wang, B. Shen, and H. Dong. Event-triggered distributed h state estimation with packet dropouts through sensor networks. *IET Control Theory & Applications*, 9(13):1948–1955, 2015.
- [4] V.S. Dolk, D. P. Borgers, and W Heemels. Output-based and decentralized dynamic event-triggered control with guaranteed l_p -gain performance and zeno-freeness. *IEEE Trans. on Automatic Control*, 62(1):34–49, 2017.
- [5] E Garcia, Y Cao, X Wang, and D. W. Casbeer. Cooperative control with general linear dynamics and limited communication: Periodic updates. In *Proc. IEEE ACC*, pages 3195–3200, 2014.
- [6] E. Garcia, Y. Cao, X. Wang, and D. W. Casbeer. Event-triggered cooperative control with general linear dynamics and communication delays. In *Proc IEEE CDC*, pages 2914–2919, 2014.
- [7] W. Hu and L. Liu. Cooperative output regulation of heterogeneous linear multi-agent systems by event-triggered control. *IEEE trans. on cybernetics*, 47(1):105–116, 2017.
- [8] Z. Hu, P. Shi, J. Zhang, and F. Deng. Control of discrete-time stochastic systems with packet loss by event-triggered approach. *IEEE Trans Systems, Man, and Cybernetics: Systems*, 1:1–10, 2018.
- [9] H. K. Khalil. *Nonlinear Systems, 3rd ed.* Upper Saddle River, NJ: Prentice-Hall, 2002.
- [10] E Kyrkjebø, K.Y. Pettersen, M. Wøndergem, and H. Nijmeijer. Output synchronization control of ship replenishment operations: Theory and experiments. *Control Engineering Practice*, 15(6):741–755, 2007.
- [11] Z Liu, W Chen, J Lu, H Wang, and J Wang. Formation control of mobile robots using distributed controller with sampled-data and communication delays. *IEEE Trans. Control Systems Technology*, 24(6):2125–2132, 2016.
- [12] C. Makkar, G. Hu, W. G. Sawyer, and W.E. Dixon. Lyapunov-based tracking control in the presence of uncertain nonlinear parameterizable friction. *IEEE Trans. Automatic Control*, 52(10):1988–1994, 2007.
- [13] J. Mei, W. Ren, and G. Ma. Distributed coordinated tracking with a dynamic leader for multiple euler-lagrange systems. *IEEE Trans. Automatic Control*, 56(6):1415–1421, 2011.
- [14] B. Nabet and N.-E. Leonard. Tensegrity models and shape control of vehicle formations. In *arXiv:0902.3710*, pages 1–31, 2009.
- [15] C. Nowzari, E. Garcia, and J. Cortés. Event-triggered communication and control of networked systems for multi-agent consensus. *Automatica*, 105:1–27, 2019.
- [16] G. S. Seyboth, D. V. Dimarogonas, and K. H. Johansson. Event-based broadcasting for multi-agent average consensus. 49(1):245–252, 2013.
- [17] D. Shi, T. Chen, and L. Shi. Event-triggered maximum likelihood state estimation. *Automatica*, 50(1):247–254, 2014.
- [18] C. Viel, S. Bertrand, M. Kieffer, and H. Piet-Lahanier. Distributed event-triggered control for multi-agent formation stabilization. *Proc IFAC World Congress*, 50(1):8025–8030, 2017.
- [19] C. Viel, S. Bertrand, M. Kieffer, and H. Piet-Lahanier. New state estimators and communication protocol for distributed event-triggered consensus of linear multi-agent systems with bounded perturbations. *IET Control Theory & Applications*, 11(11):1736–1748, 2017.
- [20] C. Viel, S. Bertrand, M. Kieffer, and H. Piet-Lahanier. Distributed event-triggered control strategies for multi-agent formation stabilization and tracking. *Automatica*, 106:110–116, 2019.
- [21] C. Viel, S. Bertrand, H. Piet-Lahanier, and M Kieffer. Distributed event-triggered control for multi-agent formation stabilization and tracking. *arXiv:1709.06652*, 2017.
- [22] G. Wen, Z. Duan, W. Yu, and G. Chen. Consensus of second-order multi-agent systems with delayed nonlinear dynamics and intermittent communications. *International Journal of Control*, 86(2):322–331, 2013.
- [23] Z.-G. Wu, Y. Xu, Y.-J. Pan, H. Su, and Y. Tang. Event-triggered control for consensus problem in multi-agent systems with quantized relative state measurements and external disturbance. *IEEE Trans. Circuits and Systems I: Regular Papers*, 65(7):2232–2242, 2018.
- [24] X. Xiao, J. H. Park, and L. Zhou. Event-triggered control of discrete-time switched linear systems with packet losses. *Applied Mathematics and Computation*, 333:344–352, 2018.

- [25] Q. Yang, M. Cao, H. Fan, J. Chen, and J. Huang. Distributed formation stabilization for mobile agents using virtual tensegrity structures. In *Proc. Chinese Control Conference (CCC)*, pages 447–452, 2015.
- [26] F. Zhou, Z. Huang, W. Liu, L. Li, and K. Gao. Periodic event-triggered condition design for the consensus of multiagent systems with communication delays. *Mathematical Problems in Engineering*, 1:1–9, 2016.
- [27] W. Zhu, Z.P. Jiang, and G. Feng. Event-based consensus of multi-agent systems with general linear models. *Automatica*, 50(2):552–558, 2014.

A Appendix

A.1 Proof of Proposition 2

Assume that there exists $\varepsilon_2 > 0$ such that

$$\lim_{t \rightarrow \infty} \mathbb{E}(P(q, t)) \leq \varepsilon_2. \quad (\text{A.1})$$

Then

$$\lim_{t \rightarrow \infty} \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N m_{ij} \mathbb{E} \left(\|r_{ij}(t) - r_{ij}^*(t)\|^2 \right) \leq \varepsilon_2. \quad (\text{A.2})$$

Since

$$m_{ij} \mathbb{E} \left(\|r_{ij}(t) - r_{ij}^*(t)\|^2 \right) \leq \sum_{i=1}^N \sum_{j=1}^N m_{ij} \mathbb{E} \left(\|r_{ij}(t) - r_{ij}^*(t)\|^2 \right). \quad (\text{A.3})$$

and $m_{ij} \geq 0$, for all (i, j) such that $m_{ij} > 0$, this implies

$$\lim_{t \rightarrow \infty} \mathbb{E} \left(\|r_{ij}(t) - r_{ij}^*(t)\|^2 \right) \leq \frac{2\varepsilon_2}{m_{ij}}. \quad (\text{A.4})$$

Consider now a pair (i, j) such that $m_{ij} = 0$. The communication graph has been assumed connected. Consequently, one find a sequence $(i_1, \dots, i_{N_{ij}})$ of $N_{ij} \leq N$ nodes with $i_1 = i$ and $i_{N_{ij}} = j$ and such that $m_{i_k i_{k+1}} > 0$ for all $k = 1, \dots, N_{ij} - 1$. Then

$$\lim_{t \rightarrow \infty} \mathbb{E} \left(\|r_{ij}(t) - r_{ij}^*(t)\|^2 \right) \leq \lim_{t \rightarrow \infty} \sum_{k=1}^{N_{ij}-1} \mathbb{E} \left(\|r_{i_k i_{k+1}}(t) - r_{i_k i_{k+1}}^*(t)\|^2 \right).$$

Using (A.4), one gets

$$\lim_{t \rightarrow \infty} \mathbb{E} \left(\|r_{ij}(t) - r_{ij}^*(t)\|^2 \right) \leq \sum_{k=1}^{N_{ij}-1} \frac{2\varepsilon_2}{m_{i_k i_{k+1}}}. \quad (\text{A.5})$$

Then, introduce

$$\varepsilon_{11} = \max_{(i,j) \in \mathcal{N}^2, m_{ij}=0} \sum_{k=1}^{N_{ij}-1} \frac{2\varepsilon_2}{m_{i_k i_{k+1}}} \quad (\text{A.6})$$

and

$$\varepsilon_{12} = \max_{(i,j) \in \mathcal{N}^2, m_{ij}>0} \frac{2\varepsilon_2}{m_{ij}}. \quad (\text{A.7})$$

Finally, combining (A.4) and (A.5), one has for all $(i, j) \in \mathcal{N}^2$,

$$\lim_{t \rightarrow \infty} \mathbb{E} \left(\|r_{ij}(t) - r_{ij}^*(t)\|^2 \right) \leq \varepsilon_1, \quad (\text{A.8})$$

with $\varepsilon_1 = \max \{ \varepsilon_{11}, \varepsilon_{12} \}$.

The converse is immediate: if there exists $\varepsilon_1 > 0$ such that (A.8) is satisfied for all $(i, j) \in \mathcal{N}^2$, then

$$\begin{aligned} \lim_{t \rightarrow \infty} \mathbb{E}(P(q, t)) &= \lim_{t \rightarrow \infty} \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N m_{ij} \mathbb{E} \left(\|r_{ij}(t) - r_{ij}^*(t)\|^2 \right) \\ &\leq \varepsilon_2, \end{aligned}$$

with $\varepsilon_2 = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N m_{ij} \varepsilon_1$.

A.2 Evaluation of $p_{m,m|k_i^{j,i}}$

Consider $t \in [t_{i,k_i+n}, t_{i,k_i+n+1}[$ with $n \geq 0$ and assume that the last message received by agent i from agent j was at time $t_{j,k_j} \in [t_{i,k_i}, t_{i,k_i+1}[$. Thus $k_i^{j,i} = k_i^j \leq k_i$ and agent i knows that agent j has received the $k_i^{j,i}$ -th message sent by agent i , and has not received the following ones with index between $k_i^{j,i}$ and k_i . Agent i has no information about the reception by agent j of the $k_i + 1, \dots, k_i + n$ -th messages, except if $k_i^{j,i} = k_i - \kappa$.

Consider some $t \geq t_{i,k_i-1}$. For all $m > k_i^{j,i}$, there exists β_m such that

$$p_{m,m|k_i^{j,i}} = \Pr\left(\delta_{i,m}^j = 1 | \delta_{i,k_i^{j,i}}^j = 1\right) = \beta_m \quad (\text{A.9})$$

and

$$\Pr\left(\delta_{i,m}^j = 0 | \delta_{i,k_i^{j,i}}^j = 1\right) = 1 - \beta_m. \quad (\text{A.10})$$

A) First, evaluate $p_{m+1,m|k_i^{j,i}}$ for some $m > k_i^{j,i}$

$$\begin{aligned} p_{m+1,m|k_i^{j,i}} &= \Pr\left(\delta_{i,m+1}^j = 0, \delta_{i,m}^j = 1 | \delta_{i,k_i^{j,i}}^j = 1\right) \\ &= \Pr\left(\delta_{i,m+1}^j = 0 | \delta_{i,m}^j = 1, \delta_{i,k_i^{j,i}}^j = 1\right) \Pr\left(\delta_{i,m}^j = 1 | \delta_{i,k_i^{j,i}}^j = 1\right) \\ &= \Pr\left(\delta_{i,m+1}^j = 0 | \delta_{i,m}^j = 1, \delta_{i,k_i^{j,i}}^j = 1\right) \beta_m \\ &= \pi \beta_m. \end{aligned}$$

Consider now $p_{m+n,m|k_i^{j,i}}$ for some $m > k_i^{j,i}$ and $n \geq 0$

$$\begin{aligned} p_{m+n,m|k_i^{j,i}} &= \Pr\left(\sum_{\ell=1}^n \delta_{i,m+\ell}^j = 0, \delta_{i,m}^j = 1 | \delta_{i,k_i^{j,i}}^j = 1\right) \\ &= \Pr\left(\sum_{\ell=1}^n \delta_{i,m+\ell}^j = 0 | \delta_{i,m}^j = 1, \delta_{i,k_i^{j,i}}^j = 1\right) \Pr\left(\delta_{i,m}^j = 1 | \delta_{i,k_i^{j,i}}^j = 1\right) \\ &= \Pr\left(\sum_{\ell=1}^n \delta_{i,m+\ell}^j = 0 | \delta_{i,m}^j = 1\right) \beta_m. \end{aligned} \quad (\text{A.11})$$

Form (A.11), two cases have to be considered

1) If $n \geq \kappa + 1$, since, according to Assumption A6,

$$\Pr\left(\sum_{\ell=1}^n \delta_{i,m+\ell}^j = 0 | \delta_{i,m}^j = 1\right) = 0 \quad (\text{A.12})$$

$$p_{m+n,m|k_i^{j,i}} = 0. \quad (\text{A.13})$$

2) If $n \leq \kappa$, then

$$\begin{aligned}
p_{m+n,m|k_i^{j,i}} &= \Pr \left(\sum_{\ell=2}^n \delta_{i,m+\ell}^j = 0 \mid \delta_{i,m+1}^j = 0, \delta_{i,m}^j = 1 \right) \\
&\quad \times \Pr \left(\delta_{i,m+1}^j = 0 \mid \delta_{i,m}^j = 1 \right) \beta_m \\
&= \Pr \left(\sum_{\ell=2}^n \delta_{i,m+\ell}^j = 0 \mid \delta_{i,m+1}^j = 0, \delta_{i,m}^j = 1 \right) \pi \beta_m \\
&= \Pr \left(\sum_{\ell=3}^n \delta_{i,m+\ell}^j = 0 \mid \delta_{i,m+2}^j = 0, \delta_{i,m+1}^j = 0, \delta_{i,m}^j = 1 \right) \\
&\quad \times \Pr \left(\delta_{i,m+2}^j = 0 \mid \delta_{i,m+1}^j = 0, \delta_{i,m}^j = 1 \right) \pi \beta_m \\
&= \Pr \left(\sum_{\ell=3}^n \delta_{i,m+z}^j = 0 \mid \delta_{i,m+2}^j = 0, \delta_{i,m+1}^j = 0, \delta_{i,m}^j = 1 \right) \pi^2 \beta_m \\
&\quad \dots \\
&= \pi^n \beta_m
\end{aligned}$$

So

$$p_{m+n,m|k_i^{j,i}} = \pi^n p_{m,m|k_i^{j,i}} \quad \text{if } n \leq \kappa. \quad (\text{A.14})$$

B) Now, evaluate $p_{m,m|k_i^{j,i}}$ with $m > k_i^{j,i}$

$$\begin{aligned}
p_{m,m|k_i^{j,i}} &= \Pr \left(\delta_{i,m}^j = 1 \mid \delta_{i,k_i^{j,i}}^j = 1 \right) \\
&= 1 - \Pr \left(\delta_{i,m}^j = 0, \mid \delta_{i,k_i^{j,i}}^j = 1 \right) \\
&= 1 - \Pr \left(\delta_{i,m}^j = 0, \delta_{i,m-1}^j = 1 \mid \delta_{i,k_i^{j,i}}^j = 1 \right) \\
&\quad - \Pr \left(\delta_{i,m}^j = 0, \delta_{i,m-1}^j = 0 \mid \delta_{i,k_i^{j,i}}^j = 1 \right) \\
&= 1 - p_{m,m-1|k_i^{j,i}} \\
&\quad - \Pr \left(\delta_{i,m}^j = 0, \delta_{i,m-1}^j = 0, \delta_{i,m-2}^j = 0 \mid \delta_{i,k_i^{j,i}}^j = 1 \right) \\
&\quad - \Pr \left(\delta_{i,m}^j = 0, \delta_{i,m-1}^j = 0, \delta_{i,m-2}^j = 1 \mid \delta_{i,k_i^{j,i}}^j = 1 \right) \\
&= 1 - p_{m,m-1|k_i^{j,i}} - p_{m,m-2|k_i^{j,i}} \\
&\quad - \Pr \left(\delta_{i,m}^j = 0, \delta_{i,m-1}^j = 0, \delta_{i,m-2}^j = 0 \mid \delta_{i,k_i^{j,i}}^j = 1 \right) \\
&= \dots \\
&= 1 - \sum_{\ell=m-\kappa+1}^{m-1} p_{m,\ell|k_i^{j,i}} \\
&\quad - \Pr \left(\sum_{\ell=m-\kappa+1}^{m-1} \delta_{i,\ell}^j = 0 \mid \delta_{i,k_i^{j,i}}^j = 1 \right)
\end{aligned}$$

and according to Assumption A6, $\Pr \left(\sum_{\ell=m-\kappa+1}^{m-1} \delta_{i,\ell}^j = 0 \mid \delta_{i,k_i^{j,i}}^j = 1 \right) = 0$. Thus

$$p_{m,m|k_i^{j,i}} = 1 - \sum_{\ell=m-\kappa+1}^{m-1} p_{m,\ell|k_i^{j,i}} \quad (\text{A.15})$$

Using (A.15) and (A.14), one gets

$$\begin{aligned} p_{m,m|k_i^{j,i}} &= 1 - \pi \sum_{\ell=m-\kappa+1}^{m-1} p_{m-1,\ell|k_i^{j,i}} \\ &= 1 - \pi \left(\sum_{\ell=(m-1)-\kappa+1}^{m-1} p_{m-1,\ell|k_i^{j,i}} - p_{m-1,m-1-\kappa+1|k_i^{j,i}} \right). \end{aligned} \quad (\text{A.16})$$

Moreover, since (A.15), one has

$$\begin{aligned} p_{m,m|k_i^{j,i}} + \sum_{\ell=m-\kappa+1}^{m-1} p_{m,\ell|k_i^{j,i}} &= 1 \\ \sum_{\ell=m-\kappa+1}^m p_{m,\ell|k_i^{j,i}} &= 1 \end{aligned}$$

and so

$$\sum_{\ell=m-\kappa}^{m-1} p_{m-1,\ell|k_i^{j,i}} = 1. \quad (\text{A.17})$$

Combining (A.17) and (A.16) one gets

$$p_{m,m|k_i^{j,i}} = 1 - \pi + \pi p_{m-1,m-\kappa|k_i^{j,i}}. \quad (\text{A.18})$$

A.3 Proof of convergence with packet losses

To prove Theorem 4 a) one shows first that the MAS is converging with a bounded mean-square error. For that purpose, one will introduce a candidate Lyapunov function and show that it satisfies the conditions introduced in the Definition 1.

Consider some $D_{\max} \geq 0$, $\eta \geq 0$, and realizations $d_i(t)$, $i = 1, \dots, N$ of the state perturbations.

Inspired by the proof developed in [17,3], consider the continuous positive-definite candidate Lyapunov function

$$V(t) = \mathbb{E} \left(\frac{1}{2} \sum_{i=1}^N \left(s_i(q(t, \delta))^T M_i s_i(q(t, \delta)) \right) + \frac{k_g}{4} P(q(t, \delta), t) \right) \quad (\text{A.19})$$

where the expectation is evaluated considering the random losses described by δ .

A.3.1 Continuity of the Lyapunov function

Assume that the first message is transmitted at time t_1 , without loss of generality, by agent 1 to N_1 neighbors. Consider some $t \in [t_1, t_2[$, where t_2 is the time at which the second message is transmitted, whatever the agent. There are 2^{N_1} possible reception scenarios, from no reception by all agents to a reception by all agents. Let σ represent the index of the σ -th scenario, $0 \leq \sigma \leq 2^{N_1}$ and $p_{\sigma,1}$ be the associated probability for the first communication. One may write

$$\begin{aligned} V(t) &= \mathbb{E} \left(\frac{1}{2} \sum_{i=1}^N \left(s_i(q(t, \delta))^T M_i s_i(q(t, \delta)) \right) + \frac{k_g}{4} P(q(t, \delta), t) \right) \\ &= \frac{1}{2} \sum_{\sigma=1}^{2^{N_1}} p_{\sigma,1} \left(\sum_{i=1}^N s_i(q(t, \delta_\sigma))^T M_i s_i(q(t, \delta_\sigma)) + \frac{k_g}{4} P(q(t, \delta_\sigma), t) \right) \end{aligned} \quad (\text{A.20})$$

where $\sum_{\sigma=1}^{2^{N_1}} p_{\sigma,1} = 1$.

For a given reception scenario σ of the first message, the time instant $t_{\sigma,2}$ of transmission of the second message and the index $i_{\sigma,2}$ of the transmitting agent both depend on σ . More generally, at time t , S_t different transmission and reception scenarios have to be considered. For a given scenario σ , let n_σ be the number of communications that have occurred. The associated

loss vector is $\delta_\sigma = (\delta_{\sigma,1}, \dots, \delta_{\sigma,n_\sigma})$, where $\delta_{\sigma,k}$ is the loss vector for the k -th communication. The probability associated to δ_σ is p_σ . The next communication time instant is $t_{\sigma,n_\sigma+1} > \underline{t}$ and the communicating agent is $i_{\sigma,n_\sigma+1}$. Let

$$\bar{t} = \min_{\sigma=1,\dots,S_{\underline{t}}} t_{\sigma,n_\sigma+1}$$

$$\bar{\sigma} = \arg \min_{\sigma=1,\dots,S_{\underline{t}}} t_{\sigma,n_\sigma+1}$$

and \bar{i} denote the index of the associated communicating agent.

For all $t \in [\underline{t}, \bar{t}]$, one has

$$V(t) = \frac{1}{2} \sum_{\sigma=1}^{S_{\underline{t}}} p_\sigma \left(\sum_{i=1}^N s_i(q(t, \delta_\sigma))^T M_i s_i(q(t, \delta_\sigma)) + \frac{k_g}{4} P(q(t, \delta_\sigma), t) \right).$$

In the scenario $\bar{\sigma}$, at time \bar{t} , agent \bar{i} is communicating. Consequently

$$\begin{aligned} V(\bar{t}) &= \frac{1}{2} \sum_{\sigma=1,\dots,S_{\underline{t}}, \sigma \neq \bar{\sigma}} p_\sigma \left(\sum_{i=1}^N s_i(q(\bar{t}, \delta_\sigma))^T M_i s_i(q(\bar{t}, \delta_\sigma)) + \frac{k_g}{4} P(q(\bar{t}, \delta_\sigma), \bar{t}) \right) \\ &\quad + \frac{1}{2} \sum_{\mu=1}^{2^{N_{\bar{i}}}} p_{(\bar{\sigma}, \mu)} \left(\sum_{i=1}^N s_i(q(\bar{t}, \delta_{(\bar{\sigma}, \mu)}))^T M_i s_i(q(\bar{t}, \delta_{(\bar{\sigma}, \mu)})) + \frac{k_g}{4} P(q(\bar{t}, \delta_{(\bar{\sigma}, \mu)}), \bar{t}) \right) \end{aligned} \quad (\text{A.21})$$

where $p_{(\bar{\sigma}, \mu)}$ denotes the probability of the μ -th loss scenario associated to the $n_{\bar{\sigma}} + 1$ communication performed by agent \bar{i} at time \bar{t} , when the previous loss scenario is $\bar{\sigma}$. One has

$$\sum_{\mu=1}^{2^{N_{\bar{i}}}} p_{(\bar{\sigma}, \mu)} = p_{\bar{\sigma}}. \quad (\text{A.22})$$

Upon reception at time \bar{t}^+ of a message sent at time \bar{t} , only the estimators are updated according to (16). The state of agents receiving a message at time \bar{t}^+ from a neighbor is continuous, *i.e.*, $q_i(\bar{t}^+, \delta_{(\bar{\sigma}, \mu)}) = q_i(\bar{t}^-, \delta_{\bar{\sigma}})$, where \bar{t}^- is a time instant immediately before transmission. This is also true for agents which do not receive the message sent at time \bar{t} . Thus, one gets $g_i(\bar{t}^+, \delta_{(\bar{\sigma}, \mu)}) = g_i(\bar{t}^-, \delta_{\bar{\sigma}})$, $s_i(q(\bar{t}^+, \delta_{(\bar{\sigma}, \mu)})) = s_i(q(\bar{t}^-, \delta_{\bar{\sigma}}))$ for $i = 1, \dots, N$, and consequently, $P(q(\bar{t}^+, \delta_{(\bar{\sigma}, \mu)}), \bar{t}^+) = P(q(\bar{t}^-, \delta_{\bar{\sigma}}), \bar{t}^-)$ for all μ . Thus, at time \bar{t}^+ , (A.21) becomes

$$\begin{aligned} V(\bar{t}^+) &= \frac{1}{2} \sum_{\sigma=1,\dots,S_{\underline{t}}, \sigma \neq \bar{\sigma}} p_\sigma \left(\sum_{i=1}^N s_i(q(\bar{t}^+, \delta_\sigma))^T M_i s_i(q(\bar{t}^+, \delta_\sigma)) + \frac{k_g}{4} P(q(\bar{t}^+, \delta_\sigma), \bar{t}^+) \right) \\ &\quad + \frac{1}{2} \sum_{\mu=1}^{2^{N_{\bar{i}}}} p_{(\bar{\sigma}, \mu)} \left(\sum_{i=1}^N s_i(q(\bar{t}^+, \delta_{(\bar{\sigma}, \mu)}))^T M_i s_i(q(\bar{t}^+, \delta_{(\bar{\sigma}, \mu)})) + \frac{k_g}{4} P(q(\bar{t}^+, \delta_{(\bar{\sigma}, \mu)}), \bar{t}^+) \right) \\ &= \frac{1}{2} \sum_{\sigma=1,\dots,S_{\underline{t}}, \sigma \neq \bar{\sigma}} p_\sigma \left(\sum_{i=1}^N s_i(q(\bar{t}^-, \delta_\sigma))^T M_i s_i(q(\bar{t}^-, \delta_\sigma)) + \frac{k_g}{4} P(q(\bar{t}^-, \delta_\sigma), \bar{t}^-) \right) \\ &\quad + \frac{1}{2} \sum_{\mu=1}^{2^{N_{\bar{i}}}} p_{(\bar{\sigma}, \mu)} \left(\sum_{i=1}^N s_i(q(\bar{t}^-, \delta_{\bar{\sigma}}))^T M_i s_i(q(\bar{t}^-, \delta_{\bar{\sigma}})) + \frac{k_g}{4} P(q(\bar{t}^-, \delta_{\bar{\sigma}}), \bar{t}^-) \right) \end{aligned}$$

and using (A.22), one gets

$$\begin{aligned} V(\bar{t}^+) &= \frac{1}{2} \sum_{\sigma=1,\dots,S_{\underline{t}}, \sigma \neq \bar{\sigma}} p_\sigma \left(\sum_{i=1}^N s_i(q(\bar{t}^-, \delta_\sigma))^T M_i s_i(q(\bar{t}^-, \delta_\sigma)) + \frac{k_g}{4} P(q(\bar{t}^-, \delta_\sigma), \bar{t}^-) \right) \\ &\quad + \frac{1}{2} p_{\bar{\sigma}} \left(\sum_{i=1}^N s_i(q(\bar{t}^-, \delta_{\bar{\sigma}}))^T M_i s_i(q(\bar{t}^-, \delta_{\bar{\sigma}})) + \frac{k_g}{4} P(q(\bar{t}^-, \delta_{\bar{\sigma}}), \bar{t}^-) \right) \\ &= V(\bar{t}^-). \end{aligned}$$

Consequently, $V(t)$ is continuous at \bar{t} .

A.3.2 Differential inequality satisfied by the Lyapunov function

Using (A.21) from the previous section, the time derivative of V exists and can be evaluated for each $t \in [t, \bar{t}]$ as follows

$$\begin{aligned} \dot{V}(t) = & \sum_{\sigma=1}^{S_t} p_{\sigma} \left(\sum_{i=1}^N \left(\frac{1}{2} s_i^T (q(t, \delta_{\sigma})) \dot{M}_i s_i (q(t, \delta_{\sigma})) + s_i^T (q(t, \delta_{\sigma})) M_i \dot{s}_i (q(t, \delta_{\sigma})) \right) \right. \\ & \left. + \frac{k_g}{4} \frac{d}{dt} P(q(t, \delta_{\sigma}), t) \right). \end{aligned} \quad (\text{A.23})$$

which may be written more concisely as

$$\dot{V} = \mathbb{E} \left(\sum_{i=1}^N \left[\frac{1}{2} s_i^T \dot{M}_i s_i + s_i^T M_i \dot{s}_i \right] + \frac{k_g}{4} \frac{d}{dt} P(q, t) \right), \quad (\text{A.24})$$

where the expectation is to be taken over all possible transmission loss events.

Our aim, in what follows is to obtain a differential inequality satisfied by V . One starts considering the two terms in the right hand side of (A.24).

In (A.23), one has

$$\begin{aligned} & \frac{1}{4} \frac{d}{dt} P(q, t) \\ = & \frac{1}{4} \frac{d}{dt} \sum_{i=1}^N \sum_{j=1}^N m_{ij} \|r_{ij} - r_{ij}^*\|^2 \\ = & \sum_{i=1}^N \left[\frac{1}{2} \sum_{j=1}^N m_{ij} (\dot{r}_{ij} - \dot{r}_{ij}^*)^T (r_{ij} - r_{ij}^*) \right] \\ = & \sum_{i=1}^N \frac{1}{2} \sum_{j=1}^N m_{ij} \left[(\dot{q}_i - \dot{q}_i^*)^T (r_{ij} - r_{ij}^*) - (\dot{q}_j - \dot{q}_j^*)^T (r_{ij} - r_{ij}^*) \right] \\ = & \sum_{i=1}^N \frac{1}{2} \sum_{j=1}^N m_{ij} \left[(\dot{q}_i - \dot{q}_i^*)^T (r_{ij} - r_{ij}^*) - (\dot{q}_i - \dot{q}_i^*)^T (r_{ji} - r_{ji}^*) \right] \end{aligned} \quad (\text{A.25})$$

Since $r_{ji} = -r_{ij}$, one gets

$$\begin{aligned} \frac{1}{4} \frac{d}{dt} P(q, t) = & \sum_{i=1}^N (\dot{q}_i - \dot{q}_i^*)^T \sum_{j=1}^N m_{ij} (r_{ij} - r_{ij}^*) \\ = & \sum_{i=1}^N (\dot{q}_i - \dot{q}_i^*)^T g_i. \end{aligned} \quad (\text{A.26})$$

Combining (A.23) and (A.26), one obtains

$$\dot{V} = \mathbb{E} \left(\sum_{i=1}^N \left[\frac{1}{2} s_i^T \dot{M}_i s_i + s_i^T M_i \dot{s}_i + k_g (\dot{q}_i - \dot{q}_i^*)^T g_i \right] \right) \quad (\text{A.27})$$

One focuses now on the term $M_i \dot{s}_i$. Using (12), one may write

$$M_i \dot{s}_i + C_i s_i = M_i (\ddot{q}_i - \ddot{q}_i^* + k_p \dot{q}_i) + C_i (\dot{q}_i - \dot{q}_i^* + k_p g_i)$$

and using (1), one gets

$$M_i \dot{s}_i + C_i s_i = u_i + d_i - G + M_i (k_p \dot{q}_i - \ddot{q}_i^*) + C_i (k_p g_i - \dot{q}_i^*). \quad (\text{A.28})$$

Now, introducing (15), one gets

$$\begin{aligned} M_i \dot{s}_i + C_i s_i &= -k_s \bar{s}_i - k_g \bar{g}_i - Y_i(q_i, \dot{q}_i, k_p \dot{g}_i - \dot{q}_i^*, k_p \bar{g}_i - \dot{q}_i^*) \theta_i \\ &\quad + M_i(k_p \dot{g}_i - \dot{q}_i^*) + C_i(k_p g_i - \dot{q}_i^*) + d_i \end{aligned} \quad (\text{A.29})$$

In what follows, one uses Y_i to represent $Y_i(q_i, \dot{q}_i, k_p \dot{g}_i - \dot{q}_i^*, k_p \bar{g}_i - \dot{q}_i^*)$. Assumption A3 leads to

$$-s_i^T Y_i \theta_i = -s_i^T (M_i(k_p \dot{g}_i - \dot{q}_i^*) + C_i(k_p \bar{g}_i - \dot{q}_i^*)). \quad (\text{A.30})$$

Considering (2) and (A.29) in (A.27), one gets

$$\begin{aligned} \dot{V} &= \mathbb{E} \left(\sum_{i=1}^N \left[\frac{1}{2} s_i^T \dot{M}_i s_i - k_s s_i^T \bar{s}_i - k_g s_i^T \bar{g}_i - s_i^T C_i s_i + s_i^T (M_i(k_p \dot{g}_i - \dot{q}_i^*) + C_i(k_p g_i - \dot{q}_i^*)) \right. \right. \\ &\quad \left. \left. - s_i^T (M_i(k_p \dot{g}_i - \dot{q}_i^*) + C_i(k_p \bar{g}_i - \dot{q}_i^*)) + k_g (\dot{q}_i - \dot{q}_i^*)^T g_i + s_i^T d_i \right] \right). \end{aligned} \quad (\text{A.31})$$

Now, introduce (10) in (12) to get

$$s_i = \dot{q}_i - \dot{q}_i^* + k_p \sum_{i=1}^N m_{ij} (q_i - q_j - r_{ij}^*). \quad (\text{A.32})$$

Since $e_j^i = \hat{q}_j^i - q_j$, one gets

$$\begin{aligned} s_i &= \dot{q}_i - \dot{q}_i^* + k_p \sum_{i=1}^N m_{ij} (q_i - \hat{q}_j^i + e_j^i - r_{ij}^*) \\ &= \dot{q}_i - \dot{q}_i^* + k_p \sum_{i=1}^N m_{ij} (\bar{r}_{ij} - r_{ij}^*) + k_p \sum_{\substack{j=1 \\ j \neq i}}^N m_{ij} e_j^i \\ &= \bar{s}_i + k_p E^i \end{aligned} \quad (\text{A.33})$$

with

$$E^i = \sum_{i=1}^N m_{ij} e_j^i, \quad (\text{A.34})$$

since $m_{ii} = 0$. Using similar derivations, one may show that

$$g_i = \bar{g}_i + E^i. \quad (\text{A.35})$$

Replacing (A.33) and (A.35) in (A.31), one gets

$$\begin{aligned} \dot{V} &= \mathbb{E} \left(\sum_{i=1}^N \left[s_i^T \left[\frac{1}{2} \dot{M}_i - C_i \right] s_i - k_s s_i^T \bar{s}_i - k_g (\dot{q}_i - \dot{q}_i^* + k_p g_i)^T \bar{g}_i \right. \right. \\ &\quad \left. \left. + k_p s_i^T (M_i \dot{E}^i + C_i E^i) + k_g (\dot{q}_i - \dot{q}_i^*)^T g_i + s_i^T d_i \right] \right). \end{aligned} \quad (\text{A.36})$$

Let $\dot{V}_1 = \sum_{i=1}^N 2k_p s_i^T (M_i \dot{E}^i + C_i E^i)$. Using Assumption A2, $\frac{1}{2} \dot{M}_i - C_i$ is skew symmetric or definite negative thus $s_i^T \left[\frac{1}{2} \dot{M}_i - C_i \right] s_i \leq 0$. For all $b > 0$ and all vectors x and y of similar size, one has

$$x^T y \leq \frac{1}{2} \left(b x^T x + \frac{1}{b} y^T y \right). \quad (\text{A.37})$$

Using (A.37) with $b = 1$, and the fact that $d_i^T d_i \leq D_{\max}^2$, one deduces that $d_i^T s_i \leq \frac{1}{2} (D_{\max}^2 + s_i^T s_i)$ and that

$$\begin{aligned} \dot{V} \leq \mathbb{E} \left(\sum_{i=1}^N \left[-k_s s_i^T \bar{s}_i - k_g k_p g_i^T \bar{g}_i + \frac{1}{2} s_i^T s_i + \frac{1}{2} D_{\max}^2 \right. \right. \\ \left. \left. + k_g (\dot{q}_i - \dot{q}_i^*)^T (g_i - \bar{g}_i) \right] + \frac{1}{2} \dot{V}_1 \right). \end{aligned} \quad (\text{A.38})$$

One notices that $r_{ij} = q_i - q_j = q_i - \hat{q}_j^i + e_j^i = \bar{r}_{ij} + e_j^i$, thus

$$\begin{aligned} \|s_i - \bar{s}_i\|^2 &= s_i^T s_i - 2s_i^T \bar{s}_i + \bar{s}_i^T \bar{s}_i \\ \|k_p E^i\|^2 &= s_i^T s_i - 2s_i^T \bar{s}_i + \bar{s}_i^T \bar{s}_i \\ s_i^T \bar{s}_i &= -\frac{1}{2} \|k_p E^i\|^2 + \frac{1}{2} s_i^T s_i + \frac{1}{2} \bar{s}_i^T \bar{s}_i \end{aligned} \quad (\text{A.39})$$

Using similar derivations, from (A.39), one shows that $g_i^T \bar{g}_i = -\frac{1}{2} \|E^i\|^2 + \frac{1}{2} g_i^T g_i + \frac{1}{2} \bar{g}_i^T \bar{g}_i$. Injecting the latter expression in (A.38), one gets

$$\begin{aligned} \dot{V} \leq \mathbb{E} \left(\sum_{i=1}^N \left[\frac{k_s}{2} \left(k_p^2 \|E^i\|^2 - s_i^T s_i - \bar{s}_i^T \bar{s}_i \right) + k_p k_g \frac{1}{2} \left(\|E^i\|^2 - g_i^T g_i - \bar{g}_i^T \bar{g}_i \right) + \frac{1}{2} s_i^T s_i + \frac{1}{2} D_{\max}^2 \right. \right. \\ \left. \left. + k_g (\dot{q}_i - \dot{q}_i^*)^T (g_i - \bar{g}_i) \right] + \frac{1}{2} \dot{V}_1 \right) \\ \leq \mathbb{E} \left(\sum_{i=1}^N \left[-\frac{(k_s - 1)}{2} s_i^T s_i - \frac{k_s}{2} \bar{s}_i^T \bar{s}_i + \frac{k_s k_p^2 + k_g k_p}{2} \|E^i\|^2 - \frac{1}{2} k_p k_g \left(g_i^T g_i + \bar{g}_i^T \bar{g}_i \right) + \frac{1}{2} D_{\max}^2 \right. \right. \\ \left. \left. + k_g (\dot{q}_i - \dot{q}_i^*)^T (g_i - \bar{g}_i) \right] + \frac{1}{2} \dot{V}_1 \right). \end{aligned} \quad (\text{A.40})$$

Using (A.37) with $b = b_i > 0$, one shows that $2\dot{q}_i^T (g_i - \bar{g}_i) \leq (b_i \|\dot{q}_i\|^2 + \frac{1}{b_i} \|E^i\|^2)$. Using this result in (A.40), one gets

$$\begin{aligned} \dot{V} \leq \frac{1}{2} \sum_{i=1}^N \left[-(k_s - 1) \mathbb{E} (s_i^T s_i) - k_s \mathbb{E} (\bar{s}_i^T \bar{s}_i) + \left(k_s k_p^2 + k_g k_p + \frac{k_g}{b_i} \right) \mathbb{E} \left(\|E^i\|^2 \right) + b_i k_g \mathbb{E} \left(\|\dot{q}_i - \dot{q}_i^*\|^2 \right) \right. \\ \left. - k_p k_g \mathbb{E} \left(g_i^T g_i + \bar{g}_i^T \bar{g}_i \right) + D_{\max}^2 \right] + \frac{1}{2} \mathbb{E} (\dot{V}_1) \end{aligned} \quad (\text{A.41})$$

Consider now \dot{V}_1 . Using (A.37) with $b = 1$ and Assumption A1, one obtains

$$\begin{aligned} \sum_{i=1}^N 2k_p s_i^T (M_i \dot{E}^i + C_i E^i) &\leq \sum_{i=1}^N k_p \left(s_i^T M_i s_i + s_i^T s_i + [\dot{E}^{iT} M_i \dot{E}^i + E^{iT} C_i^T C_i E^i] \right) \\ &\leq \sum_{i=1}^N k_p \left((k_M + 1) s_i^T s_i + [k_M \dot{E}^{iT} \dot{E}^i + E^{iT} C_i^T C_i E^i] \right) \end{aligned} \quad (\text{A.42})$$

Focus now on the terms $E^{iT} C_i^T C_i E^i$. Using Assumption A2, one has

$$\begin{aligned} \sum_{i=1}^N E^{iT} C_i^T C_i E^i &= \sum_{i=1}^N \left(\sum_{j=1}^N m_{ij} e_j^i \right)^T C_i^T C_i \left(\sum_{\ell=1}^N m_{i\ell} e_\ell^i \right) \\ &\leq \sum_{i=1}^N \sum_{j=1}^N \sum_{\ell=1}^N m_{i\ell} m_{ij} \|C_i\|^2 e_j^{iT} e_\ell^i. \end{aligned} \quad (\text{A.43})$$

Using again (A.37) with $b = 1$, one gets

$$\begin{aligned}
\sum_{i=1}^N E^{iT} C_i^T C_i E^i &\leq \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \sum_{\ell=1}^N m_{i\ell} m_{ij} \|C_i\|^2 \left(e_j^{iT} e_j^i + e_\ell^{iT} e_\ell^i \right) \\
&\leq \sum_{i=1}^N \sum_{j=1}^N \sum_{\ell=1}^N m_{i\ell} m_{ij} \|C_i\|^2 \left(e_j^{iT} e_j^i \right) \\
&\leq \sum_{i=1}^N \alpha_i \sum_{j=1}^N m_{ij} \|C_i\|^2 \left(e_j^{iT} e_j^i \right).
\end{aligned} \tag{A.44}$$

Since $m_{ij} = m_{ji}$ and $m_{ij} = 0$ if $(i, j) \notin \mathcal{N}_j$, one may write

$$\sum_{i=1}^N \sum_{j=1}^N m_{ij} \|e_j^i\|^2 = \sum_{i=1}^N \sum_{j=1}^N m_{ji} \|e_j^i\|^2 = \sum_{i=1}^N \sum_{j=1}^N m_{ij} \|e_i^j\|^2 \tag{A.45}$$

and using Assumption A2, one gets

$$\begin{aligned}
\sum_{i=1}^N E^{iT} C_i^T C_i E^i &\leq \sum_{i=1}^N \left(\alpha_M \sum_{j=1}^N \left[m_{ij} \|e_j^i\|^2 \|C_j\|^2 \right] \right) \\
&\leq \sum_{i=1}^N \left(\alpha_M \sum_{j=1}^N \left[m_{ij} \|e_j^i\|^2 k_C^2 \|\hat{q}_j\|^2 \right] \right).
\end{aligned} \tag{A.46}$$

Observing that

$$\begin{aligned}
\|\hat{q}_j\|^2 &= \|\dot{\hat{q}}_j^i + \dot{e}_j^i\|^2 \\
&= \|\dot{\hat{q}}_j^i\|^2 + \|\dot{e}_j^i\|^2 + 2\dot{\hat{q}}_j^{iT} \dot{e}_j^i \\
&\leq 2\|\dot{\hat{q}}_j^i\|^2 + 2\|\dot{e}_j^i\|^2,
\end{aligned}$$

(A.45) can be rewritten as

$$\begin{aligned}
\sum_{i=1}^N E^{iT} C_i^T C_i E^i &\leq 2\alpha_M k_C^2 \sum_{i=1}^N \sum_{j=1}^N m_{ij} \|e_i^j\|^2 \left(\|\dot{\hat{q}}_j^i\|^2 + \|\dot{e}_j^i\|^2 \right) \\
&\leq 2\alpha_M k_C^2 \sum_{i=1}^N \sum_{j=1}^N m_{ij} \left(\|e_i^j\|^2 \|\dot{\hat{q}}_j^i\|^2 + \|e_i^j\|^2 \|\dot{e}_j^i\|^2 \right).
\end{aligned}$$

Using (A.37) with $b = 1$ and $m_{ij} = m_{ji}$, one gets

$$\begin{aligned}
\sum_{i=1}^N E^{iT} C_i^T C_i E^i &\leq 2\alpha_M k_C^2 \sum_{i=1}^N \sum_{j=1}^N m_{ij} \left(\|e_i^j\|^2 \|\dot{\hat{q}}_j^i\|^2 + \frac{1}{2} \|e_i^j\|^4 + \frac{1}{2} \|\dot{e}_j^i\|^4 \right) \\
&\leq 2\alpha_M k_C^2 \sum_{i=1}^N \sum_{j=1}^N m_{ij} \left(\|e_i^j\|^2 \|\dot{\hat{q}}_j^i\|^2 + \frac{1}{2} \|e_i^j\|^4 + \frac{1}{2} \|\dot{e}_j^i\|^4 \right).
\end{aligned} \tag{A.47}$$

Similarly, one shows that $\sum_{i=1}^N E^{iT} E^i \leq \sum_{i=1}^N \alpha_M \sum_{j=1}^N m_{ij} \|e_i^j\|^2$ and $\sum_{i=1}^N \dot{E}^{iT} \dot{E}^i \leq \sum_{i=1}^N \alpha_M \sum_{j=1}^N m_{ij} \|\dot{e}_i^j\|^2$.

Injecting (A.42) and (A.47) in (A.41), one gets

$$\begin{aligned}
\dot{V} &\leq \frac{1}{2} \sum_{i=1}^N \left[- (k_s - 1 - k_p (k_M + 1)) \mathbb{E} \left(s_i^T s_i \right) - k_s \mathbb{E} \left(\bar{s}_i^T \bar{s}_i \right) + D_{\max}^2 \right. \\
&\quad - k_p k_g \mathbb{E} \left(g_i^T g_i \right) - k_p k_g \mathbb{E} \left(\bar{g}_i^T \bar{g}_i \right) + k_g b_i \mathbb{E} \left(\| \dot{q}_i - \dot{q}_i^* \|^2 \right) + k_p k_M \sum_{j=1}^N \alpha_M m_{ij} \mathbb{E} \left(\| \dot{e}_i^j \|^2 \right) \\
&\quad + \left(k_s k_p^2 + k_g k_p + \frac{k_g}{b_i} \right) \sum_{j=1}^N \alpha_M m_{ij} \mathbb{E} \left(\| e_i^j \|^2 \right) \\
&\quad \left. + 2 \alpha_M k_p k_C^2 \sum_{j=1}^N m_{ij} \left(\mathbb{E} \left(\| e_i^j \|^2 \right) \| \dot{q}_j^i \|^2 + \frac{1}{2} \mathbb{E} \left(\| e_i^j \|^4 \right) + \frac{1}{2} \mathbb{E} \left(\| \dot{e}_i^j \|^4 \right) \right) \right] \tag{A.48}
\end{aligned}$$

The CTC (32) leads to

$$\begin{aligned}
\dot{V} &\leq \frac{1}{2} \sum_{i=1}^N \mathbb{E} \left[- (k_s - 1 - k_p (k_M + 1)) s_i^T s_i - k_g k_p g_i^T g_i + D_{\max}^2 + \eta \right] \\
\dot{V} &\leq \frac{1}{2} \sum_{i=1}^N \mathbb{E} \left[- k_1 s_i^T s_i - k_g k_p g_i^T g_i + D_{\max}^2 + \eta \right] \tag{A.49}
\end{aligned}$$

with $k_1 = k_s - 1 - k_p (k_M + 1)$.

Introducing $k_m = \min \{ k_1, k_p \}$, from (A.49), one gets

$$\dot{V} \leq \frac{1}{2} \sum_{i=1}^N \mathbb{E} \left[- k_m \left(s_i^T s_i + k_g g_i^T g_i \right) + D_{\max}^2 + \eta \right]. \tag{A.50}$$

A lower bound of $\sum_{i=1}^N g_i^T g_i$ has now to be introduced using the following lemma, which proof is given in Appendix A.5.1.

Lemma 5 *For all t , one has*

$$\sum_{i=1}^N g_i^T g_i \geq \frac{N_{\min} m_{\min}}{m_{\max}} P(q, t), \tag{A.51}$$

where $m_{\min} = \min_{i,j=1 \dots N} (m_{ij} \neq 0)$, $m_{\max} = \max_{i,\ell=1 \dots N} (m_{ij})$ and $N_{\min} = \min_{i=1 \dots N} (N_i)$.

Using Lemma 5 and introducing $k_3 = \frac{N_{\min} m_{\min}}{m_{\max}}$, one may write

$$\begin{aligned}
\dot{V} &\leq \mathbb{E} \left(- \frac{k_m}{2} \left[\sum_{i=1}^N s_i^T s_i + \frac{k_3 k_g}{4} P(q, t) \right] + \frac{N}{2} (D_{\max}^2 + \eta) \right) \\
&\leq \mathbb{E} \left(- \frac{k_m}{k_M^*} \left[\frac{1}{2} \sum_{i=1}^N (k_M s_i^T s_i) + \frac{k_3 k_g}{4} P(q, t) \right] + \frac{N}{2} (D_{\max}^2 + \eta) \right) \\
&\leq \mathbb{E} \left(- \frac{k_4}{k_M^*} \left[\frac{1}{2} \sum_{i=1}^N (k_M s_i^T s_i) + \frac{k_g}{4} P(q, t) \right] + \frac{N}{2} (D_{\max}^2 + \eta) \right) \tag{A.52}
\end{aligned}$$

with $k_M^* = 1$ if $k_M < 1$ and $k_M^* = k_M$ else, and $k_4 = k_m \min(1, k_3)$. Introducing $c_3 = \frac{k_4}{k_M^*}$, one gets

$$\begin{aligned}
\dot{V} &\leq \mathbb{E} \left(- c_3 \left[\frac{1}{2} \sum_{i=1}^N (s_i^T M_i s_i) + \frac{k_g}{4} P(q, t) \right] + \frac{N}{2} [D_{\max}^2 + \eta] \right) \\
\dot{V} &\leq - c_3 V + \frac{N}{2} [D_{\max}^2 + \eta]. \tag{A.53}
\end{aligned}$$

A.3.3 Upper bound of the Lyapunov function

Consider $t \in [\underline{t}, \bar{t}]$ and the function W satisfying

$$\dot{W} = -c_3 W + \frac{N}{2} [D_{\max}^2 + \eta]. \quad (\text{A.54})$$

The solution of (A.54) with initial condition $W(\underline{t}) = V(\underline{t})$ is

$$W(t) = \exp(-c_3(t - \underline{t})) V(\underline{t}) + (1 - \exp(-c_3(t - \underline{t}))) \frac{N}{2c_3} [D_{\max}^2 + \eta]. \quad (\text{A.55})$$

Then, using [9, Lemma 3.4] (Comparison lemma), one has $V(t) \leq W(t) \forall t \in [\underline{t}, \bar{t}]$, so

$$V(t) \leq \exp(-c_3(t - \underline{t})) V(\underline{t}) + (1 - \exp(-c_3(t - \underline{t}))) \frac{N}{2c_3} [D_{\max}^2 + \eta] \quad (\text{A.56})$$

$$\leq \exp(-c_3(t - \underline{t})) \left[V(\underline{t}) - \frac{N}{2c_3} [D_{\max}^2 + \eta] \right] + \frac{N}{2c_3} [D_{\max}^2 + \eta] \quad (\text{A.57})$$

Then, since $V(\underline{t}) > \frac{N}{2c_3} [D_{\max}^2 + \eta]$, $V(t)$ is decreasing over the interval $[\underline{t}, \bar{t}]$.

Using (A.56), one may write $\forall t > 0$

$$V(t) \leq \exp(-c_3 t) V(0) + (1 - \exp(-c_3 t)) \frac{N}{2c_3} [D_{\max}^2 + \eta] \quad (\text{A.58})$$

and from (A.58), one has

$$\begin{aligned} \lim_{t \rightarrow \infty} V(t) &\leq \frac{N}{2c_3} [D_{\max}^2 + \eta] \\ \lim_{t \rightarrow \infty} \mathbb{E} \left(\frac{1}{2} \sum_{i=1}^N (s_i^T M_i s_i) + \frac{k_g}{4} P(q, t) \right) &\leq \frac{N}{2c_3} [D_{\max}^2 + \eta] \\ \lim_{t \rightarrow \infty} \mathbb{E} \left(\frac{1}{2} P(q, t) \right) &\leq \frac{N}{k_g c_3} [D_{\max}^2 + \eta]. \end{aligned} \quad (\text{A.59})$$

Asymptotically, the formation error is bounded and according to Definition 1, the system is asymptotically converging to the target formation with a bounded mean-square error.

A.4 Proof that $t_{i,k} < t_{i,k+1}$

Consider

$$C_L(t) = k_s \bar{s}_i^T(t) \bar{s}_i(t) + k_p k_g \bar{g}_i^T(t) \bar{g}_i(t) + \eta$$

and

$$\begin{aligned} C_R(t) &= \alpha_M \left[\sum_{j=1}^N m_{ij} \left(k_e \mathbb{E} \left(\|e_i^j(t)\|^2 \right) + k_p k_M \mathbb{E} \left(\|\dot{e}_i^j(t)\|^2 \right) \right) \right. \\ &\quad \left. + 2k_p k_C^2 \sum_{i=1}^N \sum_{j=1}^N m_{ij} \left(\mathbb{E} \left(\|e_i^j(t)\|^2 \right) \|\dot{q}_j^i(t)\|^2 + \frac{1}{2} \mathbb{E} \left(\|e_i^j(t)\|^4 \right) + \frac{1}{2} \mathbb{E} \left(\|\dot{e}_i^j(t)\|^4 \right) \right) \right] \\ &\quad + k_g b_i \|\dot{q}_i(t) - \dot{q}_i^*(t)\|^2, \end{aligned}$$

with $k_e = \left(k_s k_p^2 + k_g k_p + \frac{k_g}{b_i} \right)$.

According to (32), no communication is triggered as long as $C_L(t) > C_R(t)$. A communication is triggered at $t = t_{i,k}$ when

$$C_L(t_{i,k}) = C_R(t_{i,k}). \quad (\text{A.60})$$

The message sent at time $t = t_{i,k}$ by agent i implies an update of the estimates \hat{q}_i^j , $j \in \mathcal{N}_i$ of that state q_i run by the neighbors of agent i . Consequently, the expected state estimation error will be such that

$$\mathbb{E} \left(\left\| e_i^j(t_{i,k_i}^+) \right\|^2 \right) = \pi \mathbb{E} \left(\left\| e_i^j(t_{i,k_i}^-) \right\|^2 \right),$$

see in Appendix A.5.3. Nevertheless, $k_s \bar{s}_i^T \bar{s}_i + k_p k_g \bar{g}_i^T \bar{g}_i + \eta$ and $\|\dot{q}_i - \dot{q}_i^*\|^2$, which are not updated by the communication, stay unchanged. Consequently

$$C_L(t_{i,k_i}^+) = C_L(t_{i,k_i}^-). \quad (\text{A.61})$$

To prove that $t_{i,k} < t_{i,k+1}$, one has to show that $C_L(t_{i,k_i}^+) > C_R(t_{i,k_i}^+)$.

Using Appendix A.5.3 and the continuity of $\dot{q}_i(t)$ and \dot{q}_i^* , one may write

$$\begin{aligned} C_R(t_{i,k_i}^+) &= k_g b_i \|\dot{q}_i(t_{i,k_i}^-) - \dot{q}_i^*(t_{i,k_i}^-)\|^2 + \pi \alpha_M \left[\sum_{j=1}^N m_{ij} \left(k_e \mathbb{E} \left(\left\| e_i^j(t_{i,k_i}^-) \right\|^2 \right) + k_p k_M \mathbb{E} \left(\left\| \dot{e}_i^j(t_{i,k_i}^-) \right\|^2 \right) \right) \right. \\ &\quad \left. + 2k_p k_C^2 \sum_{i=1}^N \sum_{j=1}^N m_{ij} \left(\mathbb{E} \left(\left\| e_i^j(t_{i,k_i}^-) \right\|^2 \right) \left\| \dot{q}_i^j \right\|^2 + \frac{1}{2} \mathbb{E} \left(\left\| e_i^j(t_{i,k_i}^-) \right\|^4 \right) + \frac{1}{2} \mathbb{E} \left(\left\| \dot{e}_i^j(t_{i,k_i}^-) \right\|^4 \right) \right) \right] \end{aligned}$$

Using (A.60) and (A.61), one gets

$$\begin{aligned} C_R(t_{i,k_i}^+) &= k_g b_i \|\dot{q}_i(t_{i,k_i}^-) - \dot{q}_i^*(t_{i,k_i}^-)\|^2 + \pi \left[k_s \bar{s}_i^T \bar{s}_i(t_{i,k_i}^-) + k_p k_g \bar{g}_i^T \bar{g}_i(t_{i,k_i}^-) + \eta - k_g b_i \|\dot{q}_i(t_{i,k_i}^-) - \dot{q}_i^*(t_{i,k_i}^-)\|^2 \right] \\ &= (1 - \pi) k_g b_i \|\dot{q}_i(t_{i,k_i}^-) - \dot{q}_i^*(t_{i,k_i}^-)\|^2 + \pi \left[k_s \bar{s}_i^T \bar{s}_i(t_{i,k_i}^-) + k_p k_g \bar{g}_i^T \bar{g}_i(t_{i,k_i}^-) + \eta \right]. \end{aligned}$$

The CTC is not satisfied if

$$\begin{aligned} C_L(t_{i,k_i}^+) &> C_R(t_{i,k_i}^+) \\ k_s \bar{s}_i^T \bar{s}_i(t_{i,k_i}^-) + k_p k_g \bar{g}_i^T \bar{g}_i(t_{i,k_i}^-) + \eta &> (1 - \pi) k_g b_i \|\dot{q}_i(t_{i,k_i}^-) - \dot{q}_i^*(t_{i,k_i}^-)\|^2 \\ &\quad + \pi \left[k_s \bar{s}_i^T \bar{s}_i(t_{i,k_i}^-) + k_p k_g \bar{g}_i^T \bar{g}_i(t_{i,k_i}^-) + \eta \right] \end{aligned} \quad (\text{A.62})$$

$$\begin{aligned} (1 - \pi) \left(k_s \bar{s}_i^T \bar{s}_i(t_{i,k_i}^-) + k_p k_g \bar{g}_i^T \bar{g}_i(t_{i,k_i}^-) + \eta \right) &> (1 - \pi) k_g b_i \|\dot{q}_i(t_{i,k_i}^-) - \dot{q}_i^*(t_{i,k_i}^-)\|^2 \\ k_s \bar{s}_i^T \bar{s}_i(t_{i,k_i}^-) + k_p k_g \bar{g}_i^T \bar{g}_i(t_{i,k_i}^-) + \eta &> k_g b_i \|\dot{q}_i(t_{i,k_i}^-) - \dot{q}_i^*(t_{i,k_i}^-)\|^2 \end{aligned} \quad (\text{A.63})$$

and let show now that (A.63) is always satisfied.

Using the property $x^T y \geq -\frac{1}{2} \left(b_{i2} x^T x + \frac{1}{b_{i2}} y^T y \right)$ for some $b_{i2} > 0$, one deduces that

$$\begin{aligned} \bar{s}_i^T \bar{s}_i &= k_p^2 \bar{g}_i^T \bar{g}_i + \|\dot{q}_i - \dot{q}_i^*\|^2 + 2k_p \bar{g}_i^T (\dot{q}_i - \dot{q}_i^*) \\ &\geq (k_p^2 - k_p b_{i2}) \bar{g}_i^T \bar{g}_i + \left(1 - \frac{k_p}{b_{i2}} \right) \|\dot{q}_i - \dot{q}_i^*\|^2. \end{aligned} \quad (\text{A.64})$$

Using (A.64), a sufficient condition for (A.63) to be satisfied is

$$\begin{aligned} k_s (k_p^2 - k_p b_{i2}) \bar{g}_i^T \bar{g}_i + k_s \left(1 - \frac{k_p}{b_{i2}} \right) \|\dot{q}_i - \dot{q}_i^*\|^2 + k_p k_g \bar{g}_i^T \bar{g}_i + \eta &> k_g b_i \|\dot{q}_i - \dot{q}_i^*\|^2 \\ k_s \left(1 - \frac{k_p}{b_{i2}} \right) \|\dot{q}_i - \dot{q}_i^*\|^2 + [k_p k_g + k_s (k_p^2 - k_p b_{i2})] \bar{g}_i^T \bar{g}_i + \eta &> k_g b_i \|\dot{q}_i - \dot{q}_i^*\|^2 \\ k_1 \bar{g}_i^T \bar{g}_i + \eta &> k_2 \|\dot{q}_i - \dot{q}_i^*\|^2 \end{aligned} \quad (\text{A.65})$$

where $k_1 = [k_p k_g + k_s (k_p^2 - k_p b_{i2})]$ and $k_2 = [k_g b_i - k_s (1 - \frac{k_p}{b_{i2}})]$. To ensure that the inequality (A.65) is satisfied independently of the values of \bar{g}_i and \dot{q}_i , it is sufficient to find b_i and b_{i2} such that $k_1 > 0$ and $k_2 < 0$. Consider first k_1 .

$$\begin{aligned} k_p k_g + k_s (k_p^2 - k_p b_{i2}) &> 0 \\ \frac{k_g}{k_s} &> (-k_p + b_{i2}) \\ \frac{k_s k_p + k_g}{k_s} &> b_{i2}. \end{aligned} \tag{A.66}$$

Focus now on k_2

$$\begin{aligned} k_g b_i - k_s \left(1 - \frac{k_p}{b_{i2}}\right) &< 0 \\ \frac{k_g b_i}{k_s} &< 1 - \frac{k_p}{b_{i2}} \end{aligned} \tag{A.67}$$

Since $b_{i2} > 0$, one has $\frac{k_g b_i}{k_s} < 1$ and so $b_i < \frac{k_s}{k_g}$. Then

$$\frac{k_s k_p}{k_s - k_g b_i} < b_{i2}. \tag{A.68}$$

Finally, one has to find a condition on b_i such that (A.66) and (A.67) can be satisfied simultaneously

$$\frac{k_s k_p + k_g}{k_s} > b_{i2} > \frac{k_s k_p}{k_s - k_g b_i}. \tag{A.69}$$

One may find b_{i2} if

$$\begin{aligned} k_s - k_g b_i &> \frac{k_s^2 k_p}{k_s k_p + k_g} \\ \frac{1}{k_g} \left(k_s - \frac{k_s^2 k_p}{k_s k_p + k_g}\right) &> b_i \\ b_i &< \frac{k_s}{k_s k_p + k_g}. \end{aligned} \tag{A.70}$$

which also ensures that $b_i < \frac{k_s}{k_g}$. Thus, once $b_i < \frac{k_s}{k_s k_p + k_g}$, there exists some b_{i2} such that (A.69) is satisfied. As a consequence, (32) stops to be satisfied when $t = t_{i,k}^+$.

A.5 Additional proof elements

A.5.1 Upper-bound on $\sum_{i=1}^N g_i^T g_i$

From (10), one may write

$$\begin{aligned} \sum_{i=1}^N g_i^T g_i &= \sum_{i=1}^N \left(\sum_{j=1}^N m_{ij} (r_{ij} - r_{ij}^*) \right)^T \left(\sum_{\ell=1}^N m_{i\ell} (r_{i\ell} - r_{i\ell}^*) \right) \\ &= \sum_{i=1}^N \sum_{\ell=1}^N \sum_{j=1}^N m_{i\ell} m_{ij} (r_{ij} - r_{ij}^*)^T (r_{i\ell} - r_{i\ell}^*) \end{aligned} \tag{A.71}$$

Using the fact that

$$(a - b)^T (a - b) = a^T a + b^T b - 2a^T b, \tag{A.72}$$

one gets

$$\sum_{i=1}^N g_i^T g_i = \sum_{i=1}^N \left[\frac{1}{2} \sum_{\ell=1}^N \sum_{j=1}^N m_{i\ell} m_{ij} \left[\|r_{ij} - r_{ij}^*\|^2 + \|r_{i\ell} - r_{i\ell}^*\|^2 - \|r_{ij} - r_{ij}^* - (r_{i\ell} - r_{i\ell}^*)\|^2 \right] \right]. \quad (\text{A.73})$$

One has

$$\begin{aligned} (r_{ij} - r_{ij}^*) - (r_{i\ell} - r_{i\ell}^*) &= (r_{ij} - r_{i\ell}) - (r_{ij}^* - r_{i\ell}^*) \\ &= r_{\ell j} - r_{\ell j}^* \end{aligned}$$

Injecting this result in (A.73) leads to

$$\sum_{i=1}^N g_i^T g_i = \sum_{i=1}^N \left[\frac{1}{2} \sum_{\ell=1}^N \sum_{j=1}^N m_{i\ell} m_{ij} \left[\|r_{ij} - r_{ij}^*\|^2 + \|r_{i\ell} - r_{i\ell}^*\|^2 - \|r_{\ell j} - r_{\ell j}^*\|^2 \right] \right] \quad (\text{A.74})$$

with $m_{\max} = \max_{i,j=1 \dots N} (m_{ij})$

$$\begin{aligned} m_{\max} \sum_{i=1}^N g_i^T g_i &\geq \sum_{i=1}^N \left[\frac{1}{2} \sum_{\ell=1}^N \sum_{j=1}^N m_{i\ell} m_{ij} m_{\ell j} \left[\|r_{ij} - r_{ij}^*\|^2 + \|r_{i\ell} - r_{i\ell}^*\|^2 - \|r_{\ell j} - r_{\ell j}^*\|^2 \right] \right] \\ m_{\max} \sum_{i=1}^N g_i^T g_i &\geq \frac{1}{2} \sum_{i=1}^N \sum_{\ell=1}^N \sum_{j=1}^N m_{i\ell} m_{ij} m_{\ell j} \|r_{ij} - r_{ij}^*\|^2 + \frac{1}{2} \sum_{i=1}^N \sum_{\ell=1}^N \sum_{j=1}^N m_{i\ell} m_{ij} m_{\ell j} \|r_{i\ell} - r_{i\ell}^*\|^2 \\ &\quad - \frac{1}{2} \sum_{i=1}^N \sum_{\ell=1}^N \sum_{j=1}^N m_{i\ell} m_{ij} m_{\ell j} \|r_{\ell j} - r_{\ell j}^*\|^2 \\ m_{\max} \sum_{i=1}^N g_i^T g_i &\geq \frac{1}{2} \sum_{i=1}^N \sum_{\ell=1}^N \sum_{j=1}^N m_{i\ell} m_{ij} m_{\ell j} \|r_{ij} - r_{ij}^*\|^2 + \frac{1}{2} \sum_{i=1}^N \sum_{\ell=1}^N \sum_{j=1}^N m_{i\ell} m_{ij} m_{\ell j} \|r_{ij} - r_{ij}^*\|^2 \\ &\quad - \frac{1}{2} \sum_{i=1}^N \sum_{\ell=1}^N \sum_{j=1}^N m_{i\ell} m_{ij} m_{\ell j} \|r_{ij} - r_{ij}^*\|^2 \\ m_{\max} \sum_{i=1}^N g_i^T g_i &\geq \frac{1}{2} \sum_{i=1}^N \sum_{\ell=1}^N \sum_{j=1}^N m_{i\ell} m_{ij} m_{\ell j} \|r_{ij} - r_{ij}^*\|^2. \end{aligned} \quad (\text{A.75})$$

According $m_{i\ell} = 0$ if $\ell \notin \mathcal{N}_i$, one gets

$$\sum_{i=1}^N \sum_{\ell=1}^N \sum_{j=1}^N m_{i\ell} m_{ij} m_{\ell j} \|r_{ij} - r_{ij}^*\|^2 = \sum_{i=1}^N \sum_{\ell \in \mathcal{N}_i} m_{i\ell} \sum_{j=1}^N m_{ij} m_{\ell j} \|r_{ij} - r_{ij}^*\|^2$$

Let $m_{\min} = \min_{i,j=1 \dots N} (m_{ij} \neq 0)$ and $N_{\min} = \min_{i=1 \dots N} (N_i)$. One may write

$$\begin{aligned} \sum_{i=1}^N \sum_{\ell=1}^N \sum_{j=1}^N m_{i\ell} m_{ij} m_{\ell j} \|r_{ij} - r_{ij}^*\|^2 &= \sum_{i=1}^N \sum_{\ell \in \mathcal{N}_i} m_{\min} \sum_{j=1}^N m_{ij} \|r_{ij} - r_{ij}^*\|^2 \\ \sum_{i=1}^N \sum_{\ell=1}^N \sum_{j=1}^N m_{i\ell} m_{ij} m_{\ell j} \|r_{ij} - r_{ij}^*\|^2 &\geq N_{\min} m_{\min} \sum_{i=1}^N \sum_{j=1}^N m_{ij} \|r_{ij} - r_{ij}^*\|^2 \end{aligned}$$

and so

$$\begin{aligned} m_{\max} \sum_{i=1}^N g_i^T g_i &\geq N_{\min} m_{\min} \sum_{i=1}^N \sum_{j=1}^N m_{ij} \|r_{ij} - r_{ij}^*\|^2 \\ \sum_{i=1}^N g_i^T g_i &\geq \frac{N_{\min} m_{\min}}{m_{\max}} P(q, t) \end{aligned} \quad (\text{A.76})$$

A.5.2 Evaluation of c_3

$$\begin{aligned}
c_3 &= \frac{k_4}{k_M^*} \\
&= \frac{k_m \min(1, k_3)}{\max\{1, k_M\}} \\
&= \frac{\min\{k_1, k_p\} \min\left(1, \frac{N_{\min} m_{\min}}{m_{\max}}\right)}{\max\{1, k_M\}}.
\end{aligned} \tag{A.77}$$

where $k_1 = k_s - 1 - k_p (k_M + 1)$.

A.5.3 Evaluation of $\mathbb{E}\left(\|e_i^j(t_{i,k_i+1}^+)\|^2\right)$ and $\mathbb{E}\left(\|\dot{e}_i^j(t_{i,k_i+1}^+)\|^2\right)$

$\mathbb{E}\left(\|e_i^j(t_{i,k_i+1}^+)\|^2\right)$ and $\mathbb{E}\left(\|\dot{e}_i^j(t_{i,k_i+1}^+)\|^2\right)$ are evaluated assuming that the implicit feedback is not employed. A similar evaluation may be performed considering this information. Using results of (22), one may write immediately after the transmission of the $k_i + 1$ message by agent i

$$\begin{aligned}
\mathbb{E}\left(\|e_i^j(t_{i,k_i+1}^+)\|^2\right) &= \sum_{\ell=k_i-\kappa+2}^{k_i+1} p_{k_i+1,\ell} \left\| \hat{q}_i^{i,\ell}(t_{i,k_i+1}^+) - q_i(t_{i,k_i+1}^+) \right\|^2 \\
&= \sum_{\ell=k_i-\kappa+1}^{k_i} \pi p_{k_i,\ell} \left\| \hat{q}_i^{i,\ell}(t_{i,k_i+1}^+) - q_i(t_{i,k_i+1}^+) \right\|^2 \\
&\quad + p_{k_i+1,k_i+1} \left\| \hat{q}_i^{i,k_i+1}(t_{i,k_i+1}^+) - q_i(t_{i,k_i+1}^+) \right\|^2 \\
&\quad - p_{k_i+1,k_i-\kappa} \left\| \hat{q}_i^{i,k_i-\kappa+1}(t_{i,k_i+1}^+) - q_i(t_{i,k_i+1}^+) \right\|^2,
\end{aligned}$$

where $p_{k_i+1,\ell} = \pi p_{k_i,\ell}$ has been shown in (29). Since $\hat{q}_i^{i,\ell}(t_{i,k_i+1}^+) = \hat{q}_i^{i,\ell}(t_{i,k_i+1}^-)$ for all $\ell = k_i - \kappa + 1, \dots, k_i$ and $q_i(t_{i,k_i+1}^+) = q_i(t_{i,k_i+1}^-)$, one deduces

$$\begin{aligned}
\mathbb{E}\left(\|e_i^j(t_{i,k_i+1}^+)\|^2\right) &= \pi \mathbb{E}\left(\|e_i^j(t_{i,k_i+1}^-)\|^2\right) \\
&\quad + p_{k_i+1,\ell} \left\| \hat{q}_i^{i,k_i+1}(t_{i,k_i+1}^+) - q_i(t_{i,k_i+1}^+) \right\|^2 \\
&\quad - p_{k_i+1,k_i-\kappa} \left\| \hat{q}_i^{i,k_i-\kappa+1}(t_{i,k_i+1}^+) - q_i(t_{i,k_i+1}^+) \right\|^2.
\end{aligned}$$

According to Proposition 3, $p_{k_i+1,k_i-\kappa} = 0$. Moreover, at $t = t_{k_i+1}^+$, $\hat{q}_i^{i,k_i+1}(t_{i,k_i+1}^+) = q_i(t_{i,k_i+1}^+)$. Consequently,

$$\mathbb{E}\left(\|e_i^j(t_{i,k_i+1}^+)\|^2\right) = \pi \mathbb{E}\left(\|e_i^j(t_{i,k_i+1}^-)\|^2\right).$$

Similarly, one has

$$\mathbb{E}\left(\|\dot{e}_i^j(t_{i,k_i+1}^+)\|^2\right) = \pi \mathbb{E}\left(\|\dot{e}_i^j(t_{i,k_i+1}^-)\|^2\right)$$

since $\hat{q}_i^{i,\ell}(t_{i,k_i+1}^+) = \hat{q}_i^{i,\ell}(t_{i,k_i+1}^-)$ for all $\ell = k_i - \kappa + 1, \dots, k_i$ and $\dot{q}_i(t_{i,k_i+1}^+) = \dot{q}_i(t_{i,k_i+1}^-)$.