



HAL
open science

Power partitions and saddle-point method

Gérald Tenenbaum, Jie Wu, Ya-Li Li

► **To cite this version:**

Gérald Tenenbaum, Jie Wu, Ya-Li Li. Power partitions and saddle-point method. *Journal of Number Theory*, 2019, 204, pp.435-445. 10.1016/j.jnt.2019.04.013 . hal-02891953

HAL Id: hal-02891953

<https://hal.science/hal-02891953>

Submitted on 7 Jul 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

POWER PARTITIONS AND SADDLE-POINT METHOD*

GÉRALD TENENBAUM, JIE WU AND YALI LI

ABSTRACT. For $k \geq 1$, denote by $p_k(n)$ the number of partitions of an integer n into k -th powers. In this note, we apply the saddle-point method to provide a new proof for the well-known asymptotic expansion of $p_k(n)$. This approach turns out to significantly simplify those of Wright (1934), Vaughan (2015) and Gafni (2016).

1. INTRODUCTION

Let $p(n)$ denote, as usual, the number of unrestricted partitions of an integer n , i.e. the number of solutions to the equation

$$n = a_1 + a_2 + \cdots + a_d,$$

where $d \geq 1$ and the a_j are positive integers such that $a_1 \geq a_2 \geq \cdots \geq a_d \geq 1$. In 1918, Hardy and Ramanujan [4] proved the asymptotic formula

$$(1.1) \quad p(n) \sim \frac{\exp(\pi\sqrt{2n/3})}{4\sqrt{3}n} \quad (n \rightarrow \infty).$$

by using modular properties of Jacobi's Δ -function.

More generally, given an integer $k \geq 1$, let $p_k(n)$ denote the number of partitions of the integer n into k -th powers, i.e. the number of solutions to the equation

$$n = a_1^k + \cdots + a_d^k$$

where, as before, $d \geq 1$ and $a_1 \geq a_2 \geq \cdots \geq a_d \geq 1$. Thus, $p_1(n) = p(n)$.

In 1918 too, Hardy and Ramanujan [4] stated without proof the asymptotic formula

$$(1.2) \quad p_k(n) \sim \frac{\mathfrak{b}_k \exp\{\mathfrak{c}_k n^{1/(k+1)}\}}{n^{(3k+1)/(2k+2)}} \quad (n \rightarrow \infty),$$

where the constants \mathfrak{b}_k and \mathfrak{c}_k are defined by

$$(1.3) \quad \mathfrak{a}_k := \{k^{-1}\zeta(1+k^{-1})\Gamma(1+k^{-1})\}^{k/(k+1)},$$

$$(1.4) \quad \mathfrak{b}_k := \frac{\mathfrak{a}_k}{(2\pi)^{(k+1)/2}\sqrt{(1+1/k)}},$$

$$(1.5) \quad \mathfrak{c}_k := (k+1)\mathfrak{a}_k,$$

and ζ is the Riemann zeta-function. In 1934, introducing a number of complicated objects including generalised Bessel functions, Wright [9] obtained an asymptotic expansion of $p_k(n)$: for any integer $k \geq 1$, there is a real sequence $\{\alpha_{kj}\}_{j \geq 1}$ such that, for any $J \geq 1$, we have

$$(1.6) \quad p_k(n) = \frac{\mathfrak{b}_k \exp(\mathfrak{c}_k(n+h_k)^{1/(k+1)})}{(n+h_k)^{(3k+1)/(2k+2)}} \left\{ 1 + \sum_{1 \leq j < J} \frac{(-1)^j \alpha_{kj}}{(n+h_k)^{j/(k+1)}} + O\left(\frac{1}{n^{J/(k+1)}}\right) \right\},$$

where

$$h_k := \begin{cases} 0 & \text{if } k \text{ is even,} \\ (-1)^{(k+1)/2} (2\pi)^{-(k+1)} k! \zeta(k+1) & \text{if } k \text{ is odd,} \end{cases}$$

Date: October 5, 2019.

2010 Mathematics Subject Classification. 05A17, 11N37, 11P82.

Key words and phrases. Asymptotic estimates, partitions, partitions into powers, saddle-point method.

* We include here some corrections with respect to the published version.

and the implied constant depends at most on J and k . Apart from an explicit formula for α_{k1} , no further information was given about the α_{kj} beyond the statement that they depend only on k and j and that they “may be calculated with sufficient labour for any given values of k, j ”.

Of course, taking $J = 1$ in (1.6) yields an effective form of (1.2).

More recently, appealing to a relatively simple implementation of the Hardy-Littlewood circle method, Vaughan [7] obtained an explicit version of (1.6) in the case $k = 2$ and Gafni [3] generalised the argument to arbitrary, fixed k .

Gafni states her result in the following way. Let $X = X_k(n)$ denote the real solution to the equation

$$(1.7) \quad n = (\mathfrak{a}_k X)^{1+1/k} - \frac{1}{2}X - \frac{1}{2}\zeta(-k),$$

and write

$$(1.8) \quad Y = Y_k(n) := (1 + 1/k)\mathfrak{a}_k^{1+1/k} X^{1/k} - \frac{1}{4}.$$

Then, given any $k \geq 1$, there is a real sequence $\{\beta_{kj}\}_{j \geq 1}$ such that for any fixed $J \geq 1$, we have

$$(1.9) \quad p_k(n) = \frac{\exp\left\{(k+1)\mathfrak{a}_k^{1+1/k} X^{1/k} - \frac{1}{2}\right\}}{(2\pi)^{(k+1)/2} X^{3/2} Y^{1/2}} \left\{1 + \sum_{1 \leq j < J} \frac{\beta_{kj}}{Y^j} + O\left(\frac{1}{Y^J}\right)\right\}.$$

It may be checked that the asymptotic formulae (1.6) and (1.9) match each other.

In this note, our aim is to provide a new proof of (1.2), and indeed also of (1.6) and (1.9), by applying the saddle-point method along lines very similar to those employed in [5] in the case of $p(n)$.

Our approach appears to be significantly simpler than those of the quoted previous works.

Indeed, as mentioned above, Wright’s method, which provides a sharper error term, rests on the introduction of generalised Bessel functions and so may be regarded as far more difficult. Note that Vaughan [7] does motivate his study by the search of a ‘relatively simple argument’.

Our method simplifies the matter further. Actually, the saddle-point method may be seen as a crude version of the circle method in which the major arcs are reduced to a single neighbourhood of one point. In the present context, it actually provides the same accuracy.

We thank the referee for pointing out to us that the Vaughan-Gafni method has been very recently generalised to enumerate other classes of partitions—see [1] and [2]. We believe that the saddle-point method could still be used in these contexts, with expected substantial simplifications in the analysis.

The constants \mathfrak{b}_k and \mathfrak{c}_k being defined as in (1.4) and (1.5), we can state the following.

Theorem 1. *Let $k \geq 1$ be a fixed integer. There is a real sequence $\{\gamma_{kj}\}_{j \geq 1}$ such that, for any given integer $J \geq 1$, we have*

$$(1.10) \quad p_k(n) = \frac{\mathfrak{b}_k \exp(\mathfrak{c}_k n^{1/(k+1)})}{n^{(3k+1)/(2k+2)}} \left\{1 + \sum_{1 \leq j < J} \frac{\gamma_{kj}}{n^{j/(k+1)}} + O\left(\frac{1}{n^{J/(k+1)}}\right)\right\}$$

uniformly for $n \geq 1$. The implied constant depends at most on J and k .

The coefficients γ_{kj} can be made explicit directly from the computations in our proof. For instance, we find that $\gamma_{k1} = -(11k^2 + 11k + 2)/(24k\mathfrak{c}_k)$ when $k \geq 2$, in accordance with the expression given by Wright. (It can be checked, after some computations, that it matches Gafni’s formula too.) We also have

$$\gamma_{11} = -\frac{1}{48}\mathfrak{c}_1 - 1/\mathfrak{c}_1 = -\sqrt{\frac{2}{3}}\left(\frac{\pi}{48} + \frac{3}{2\pi}\right).$$

It may be seen that $|\gamma_{kj}|$ grows like $\Gamma(j)e^{O(j)}$ and thus that the series $\sum_{j \geq 1} \gamma_{kj} z^j$ has radius of convergence 0.

2. TECHNICAL PREPARATION

Define

$$(2.1) \quad F_k(s) := \sum_{n \geq 0} p_k(n) e^{-ns} \quad (\Re s > 0),$$

so that

$$(2.2) \quad p_k(n) = \frac{1}{2\pi i} \int_{\sigma-i\pi}^{\sigma+i\pi} F_k(s) e^{ns} ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(\sigma + i\tau) e^{n\sigma + in\tau} d\tau.$$

According to the principles of the saddle-point method, we aim at selecting the integration abscissa σ as a solution σ_n of $-F'_k(\sigma)/F_k(\sigma) = n$. We plainly have

$$(2.3) \quad F_k(s) = \prod_{m \geq 1} (1 - e^{-m^k s})^{-1} \quad (\Re s > 0).$$

Thus, in the same half-plane, we may define a determination of $\log F_k(s)$ by the formula

$$\Phi_k(s) := \sum_{m \geq 1} \log \left(\frac{1}{1 - e^{-m^k s}} \right)$$

where the complex logarithms are taken in principal branch. Expanding throughout and inverting summations, we get

$$(2.4) \quad \Phi_k(s) = \sum_{m \geq 1} \sum_{n \geq 1} \frac{e^{-m^k ns}}{n} = \sum_{r \geq 1} \frac{w_k(r)}{r} e^{-rs}, \quad -\Phi'_k(s) = \sum_{r \geq 1} w_k(r) e^{-rs} \quad (\Re s > 0),$$

where

$$w_k(r) := \sum_{m^k | r} m^k \quad (r \geq 1).$$

Hence $-\Phi'_k(\sigma)$ decreases from $+\infty$ to $0+$ on $(0, \infty)$, and so the equation $-\Phi'_k(\sigma) = n$ has for each integer $n \geq 1$ a unique real solution $\sigma_n = \sigma_n(k)$. Moreover, the sequence $\{\sigma_n\}_{n \geq 1}$ is decreasing and the trivial estimates $1 \leq w_k(r) \leq r^2$ yield $1/n \ll \sigma_n \ll 1/\sqrt[3]{n}$.

We start with an asymptotic expansion for the derivatives $\Phi_k^{(m)}(\sigma_n)$ in terms of powers of σ_n . It turns out that all coefficients but a finite number vanish.

Lemma 2.1. *Let $J \geq 1$, $k \geq 1$. As $n \rightarrow \infty$, we have*

$$(2.5) \quad \Phi_k(\sigma_n) = \frac{k \mathfrak{a}_k^{1+1/k}}{\sigma_n^{1/k}} + \frac{1}{2} \log \left(\frac{\sigma_n}{(2\pi)^k} \right) + \frac{1}{2} \zeta(-k) \sigma_n + O(\sigma_n^J),$$

Moreover, for fixed $m \geq 1$,

$$(2.6) \quad (-1)^m \Phi_k^{(m)}(\sigma_n) = \frac{\mathfrak{a}_k^{1+1/k}}{\sigma_n^{m+1/k}} \prod_{1 \leq \ell < m} \left(\ell + \frac{1}{k} \right) - \frac{(m-1)!}{2\sigma_n^m} - \frac{1}{2} \delta_{1m} \zeta(-k) + O(\sigma_n^J),$$

where δ_{1m} is Kronecker's symbol.

Proof. Considering Mellin's inversion formula

$$e^{-s} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(z) s^{-z} dz \quad (\Re s > 0)$$

and the convolution identity

$$(2.7) \quad \sum_{r \geq 1} \frac{w_k(r)}{r^{1+z}} = \zeta(z+1) \zeta(kz) \quad (\Re z > 1/k),$$

we derive from the series representation (2.4) the integral formula

$$(2.8) \quad \Phi_k(s) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta(z+1) \zeta(kz) \Gamma(z) \frac{dz}{s^z},$$

and in turn

$$(2.9) \quad (-1)^m \Phi_k^{(m)}(\sigma_n) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta(z+1) \zeta(kz) \Gamma(z+m) \frac{dz}{\sigma_n^{z+m}} \quad (m \geq 0).$$

Using the classical fact that $\zeta(z)$ has finite order in any vertical strip $a \leq \Re z \leq b$ ($a, b \in \mathbb{R}$ with $a < b$), or, in other words, satisfies

$$\zeta(x + iy) \ll_{a,b} 1 + |y|^A \quad (a \leq x \leq b, |y| \geq 1),$$

for suitable $A = A(a, b)$, and invoking Stirling's formula in the form

$$|\Gamma(x + iy)| = \sqrt{2\pi} |y|^{x-1/2} e^{-\pi|y|/2} \{1 + O_{a,b}(1/y)\} \quad (a \leq x \leq b, |y| \geq 1)$$

we may move the line of integration to $\Re z = -J - m - \frac{1}{2}$.

The shifted integral is clearly $\ll \sigma_n^J$.

Let us first consider the case $m = 0$ in (2.9). Then the crossed singularities are a pole of order 2 at $z = 0$, and two simple poles at $z = 1/k$ and $z = -1$. Indeed, $\zeta(z+1)\zeta(kz)$ vanishes at all negative integers ≤ -2 , so the corresponding zeros compensate the poles of $\Gamma(z)$ at negative integers ≤ -2 .

The residue at $z = 1/k$ is equal to

$$k^{-1}\zeta(1+k^{-1})\Gamma(k^{-1})\sigma_n^{-1/k} = k\mathfrak{a}_k^{1+1/k}\sigma_n^{-1/k}.$$

The residue at $z = 0$ is the coefficient of z in the Taylor expansion of

$$\begin{aligned} z^2\zeta(z+1)\zeta(kz)\Gamma(z)\sigma_n^{-z} &= z\zeta(z+1)\zeta(kz)\Gamma(z+1)\sigma_n^{-z} \\ &= (1-\gamma z)\{\zeta(0) + k\zeta'(0)z\}(1+\gamma z)(1-z\log\sigma_n) + O(z^2) \\ &= \zeta(0) + \{-\zeta(0)\log\sigma_n + k\zeta'(0)\}z + O(z^2). \end{aligned}$$

Since $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2}\log(2\pi)$, this residue equals $\frac{1}{2}\log\{\sigma_n/(2\pi)^k\}$.

The residue at $z = -1$ equals $\frac{1}{2}\zeta(-k)\sigma_n$.

This completes the proof of (2.5).

When $m = 1$, the three crossed singularities are simple poles. The residues at $z = 1/k$, $z = 0$ and $z = -1$ are respectively $(1/k)\Gamma(1+1/k)\zeta(1+1/k)\sigma_n^{-1-1/k}$, $-1/2\sigma_n$ and $-\frac{1}{2}\zeta(-k)$.

When $m \geq 2$, the only crossed singularities are two simple poles, at $z = 1/k$ and $z = 0$, with respective residues $(1/k)\Gamma(m+1/k)\zeta(1+1/k)\sigma_n^{-m-1/k}$ and $-\frac{1}{2}(m-1)!\sigma_n^{-m}$. This proves (2.6). \square

Lemma 2.2. *Let $J \geq 1$, $k \geq 1$, $m \geq 1$.*

(i) *There is a real sequence $\{a_{kj}\}_{j \geq 1}$ with $a_{k1} = -k/(2\mathfrak{c}_k)$, $a_{k2} = k/(8\mathfrak{c}_k^2)$, such that*

$$(2.10) \quad \sigma_n = \frac{\mathfrak{a}_k}{n^{k/(k+1)}} \left\{ 1 + \sum_{1 \leq j < J} \frac{a_{kj}}{n^{j/(k+1)}} + O\left(\frac{1}{n^{J/(k+1)}}\right) \right\} \quad (n \rightarrow \infty).$$

(ii) *There is a real sequence $\{b_{kj}\}_{j \geq 1}$ with $b_{k1} = -a_{k1}/k$ such that, as $n \rightarrow \infty$, we have*

$$(2.11) \quad \Phi_k(\sigma_n) = k\mathfrak{a}_k n^{1/(k+1)} \left\{ 1 + \sum_{1 \leq j < J} \frac{b_{kj}}{n^{j/(k+1)}} + O\left(\frac{1}{n^{J/(k+1)}}\right) \right\} + \frac{1}{2} \log\left(\frac{\sigma_n}{(2\pi)^k}\right).$$

(iii) *There is a real sequence $\{b_{kmj}\}_{j \geq 1}$ such that, as $n \rightarrow \infty$, we have*

$$(2.12) \quad \begin{aligned} &(-1)^m \Phi_k^{(m)}(\sigma_n) + \frac{1}{2} \delta_{1m} \zeta(-k) \\ &= \frac{n^{(mk+1)/(k+1)}}{\mathfrak{a}_k^{m-1}} \prod_{1 \leq \ell < m} \left(\ell + \frac{1}{k} \right) \left\{ 1 + \sum_{1 \leq j < J} \frac{b_{kmj}}{n^{j/(k+1)}} + O\left(\frac{1}{n^{J/(k+1)}}\right) \right\}. \end{aligned}$$

Proof. We infer from (2.6) that

$$(2.13) \quad n = \frac{\mathfrak{a}_k^{1+1/k}}{\sigma_n^{1+1/k}} - \frac{1}{2\sigma_n} - \frac{1}{2}\zeta(-k) + O(\sigma_n^J).$$

This immediately implies (2.10) by Lagrange's inversion formula — see, e.g. [8, §7.32]. We may obtain an explicit expression for the a_{kj} from the formula

$$(2.14) \quad \sigma_n = \frac{\mathfrak{a}_k}{2\pi i n^{k/(k+1)}} \oint_{|z-1|=\varrho} \frac{zG'(z)}{G(z)} dz + O\left(\frac{1}{n^{(k+J)/(k+1)}}\right)$$

where ϱ is a fixed, small positive constant and

$$G(z) := z^{-1-1/k} - 1 - \frac{1}{2\mathbf{a}_k z n^{1/(k+1)}} - \frac{\zeta(-k)}{2n}.$$

This is classically derived from Rouché's theorem and we omit the details. The values of a_{k1} and a_{k2} may be retrieved from the above or by formally inserting (2.10) into (2.13).

Inserting (2.14) back into (2.5) and (2.6) immediately yields (2.11) and (2.12). \square

With the aim of applying Laplace's method to evaluate the integral on the right-hand side of (2.2), we need to show that it is dominated by a small neighbourhood of the saddle-point σ_n . The next result meets this requirement. Here and in the sequel, all constants c_j ($j \geq 0$) are assumed, unless otherwise stated, to depend at most upon k .

Lemma 2.3. *We have*

$$(2.15) \quad \frac{|F_k(\sigma_n + i\tau)|}{|F_k(\sigma_n)|} \leq \begin{cases} e^{-c_1 \tau^2 \sigma_n^{-(2+1/k)}} & \text{if } |\tau| \leq 2\pi\sigma_n, \\ e^{-c_2 \sigma_n^{-1/k}} & \text{if } 2\pi\sigma_n < |\tau| \leq \pi. \end{cases}$$

Proof. Noticing that

$$\begin{aligned} |1 - e^{-m^k(\sigma_n + i\tau)}|^2 &= |1 - e^{-m^k\sigma_n}|^2 + 4e^{-m^k\sigma_n} \sin^2\left(\frac{1}{2}m^k\tau\right) \\ &\geq |1 - e^{-m^k\sigma_n}|^2 + 16e^{-m^k\sigma_n} \|m^k\tau/(2\pi)\|^2, \end{aligned}$$

we can write

$$\begin{aligned} \frac{|F_k(\sigma_n + i\tau)|^2}{|F_k(\sigma_n)|^2} &\leq \prod_{m \geq 1} \left(1 + \frac{16\|m^k\tau/(2\pi)\|^2}{e^{m^k\sigma_n}(1 - e^{-m^k\sigma_n})^2}\right)^{-1} \\ &\leq \prod_{(4\sigma_n)^{-1/k} < m \leq (2\sigma_n)^{-1/k}} \left(1 + \frac{16\|m^k\tau/(2\pi)\|^2}{e^{m^k\sigma_n}(1 - e^{-m^k\sigma_n})^2}\right)^{-1}. \end{aligned}$$

Thus, there is an absolute positive constant c_3 such that

$$(2.16) \quad \frac{|F_k(\sigma_n + i\tau)|}{|F_k(\sigma_n)|} \leq e^{-c_3 S(\tau; \sigma_n)}$$

with

$$S(\tau; \sigma_n) := \sum_{m \in I} \|m^k\tau/(2\pi)\|^2.$$

where we have put $I :=](4\sigma_n)^{-1/k}, (2\sigma_n)^{-1/k}[$.

If $|\tau| \leq 2\pi\sigma_n$ and $m \in I$, we have $|m^k\tau/(2\pi)| \leq \frac{1}{2}$. Thus

$$(2.17) \quad S(\tau; \sigma_n) = \sum_{(4\sigma_n)^{-1/k} < m \leq (2\sigma_n)^{-1/k}} m^{2k}\tau^2/4\pi^2 \asymp \tau^2\sigma_n^{-(2+1/k)}.$$

When $2\pi\sigma_n < |\tau| \leq 2\pi\sigma_n^{1-1/3k}$, we proceed similarly, noting that for any integer h with $|\tau|/4\sigma_n < 2\pi h \leq |\tau|/2\sigma_n$ there are $\gg (1/\sigma_n)^{1/k}/h$ integers m in I such that

$$\frac{1}{4} \leq |m^k\tau/(2\pi) - h| \leq \frac{1}{2}.$$

This yields the the required estimate

$$(2.18) \quad S(\tau; \sigma_n) \gg \sigma_n^{-1/k}.$$

When $2\pi(\sigma_n)^{1-1/3k} < |\tau| \leq \pi$, we note that (2.18) follows, via the Cauchy-Schwarz inequality, from the inequalities

$$\sum_{m \in I} \|\tau m^k/(2\pi)\| \geq \sum_{m \in I} |1 - e^{i\tau m^k}| \geq |I| - \left| \sum_{m \in I} e^{i\tau m^k} \right|$$

provided we can show the modulus of the last exponential sum is, say, $\leq (1-c)|I|$ for some positive constant c depending at most upon k . Now Dirichlet's approximation lemma guarantees that there exist integers $a \in \mathbb{Z}^*$ and q , with $1 < q \leq (1/\sigma_n)^{1-1/3k}$, $(a, q) = 1$,

$$|\tau/(2\pi) - a/q| \leq (\sigma_n)^{1-1/3k}/q.$$

If $q \leq (1/\sigma_n)^{1/3k}$, we readily deduce the required estimate from [6, Lemma 2.7 & Theorem 4.2]. If $(1/\sigma_n)^{1/3k} < q \leq (1/\sigma_n)^{1-1/3k}$, we may apply Weyl's inequality, as stated for instance in [6, Lemma 2.4], to get that the exponential sum under consideration is $\ll |I|^{1-\varepsilon_k}$ for some positive ε_k depending only on k . Hence, (2.18) holds in all circumstances. \square

3. COMPLETION OF THE PROOF

Proposition 3.1. *Let $k \geq 1$, $J \geq 1$. Then there is a real sequence $\{e_{kj}\}_{j \geq 1}$ such that for any integer $J \geq 1$ we have*

$$(3.1) \quad p_k(n) = \frac{\exp(n\sigma_n + \Phi_k(\sigma_n))}{\sqrt{2\pi\Phi_k''(\sigma_n)}} \left\{ 1 + \sum_{2 \leq j < J} \frac{e_{kj}}{n^{j/(k+1)}} + O\left(\frac{1}{n^{J/(k+1)}}\right) \right\} \quad (n \rightarrow \infty).$$

Proof. By (2.2), we have

$$(3.2) \quad p_k(n) = \frac{e^{n\sigma_n}}{2\pi} \int_{-\pi}^{\pi} e^{\Phi_k(\sigma_n + i\tau) + in\tau} d\tau.$$

From (2.15), we deduce that

$$(3.3) \quad \int_{2\pi\sigma_n < |\tau| \leq \pi} e^{\Phi_k(\sigma_n + i\tau) + in\tau} d\tau \ll e^{\Phi_k(\sigma_n) - c_4\sigma_n^{-1/k}}$$

$$\int_{\sigma_n^{1+1/3k} < |\tau| \leq 2\pi\sigma_n} e^{\Phi_k(\sigma_n + i\tau) + in\tau} d\tau \ll e^{\Phi_k(\sigma_n) - c_4\sigma_n^{-1/3k}}.$$

Since these bounds are exponentially small with respect to the expected main term, it only remains to estimate the contribution of the interval $\mathfrak{J} :=]-\sigma_n^{1+1/(3k)}, \sigma_n^{1+1/(3k}[$, corresponding to a small neighbourhood of the saddle-point.

In this range, we have

$$\Phi_k(\sigma_n + i\tau) = \sum_{0 \leq m \leq 2J+1} \frac{\Phi_k^{(m)}(\sigma_n)}{m!} (i\tau)^m + O\left(\frac{\tau^{2J+2}}{\sigma_n^{1/k+2J+2}}\right),$$

where the estimate for the error term follows from (2.6). The same formula ensures that $|\Phi_k^{(m)}(\sigma_n)\tau^m| \ll 1$ for $m \geq 3$. Thus for $\tau \in \mathfrak{J}$, we can write

$$e^{\Phi_k(\sigma_n + i\tau) + in\tau}$$

$$= e^{\Phi_k(\sigma_n) - \frac{1}{2}\Phi_k''(\sigma_n)\tau^2} \left\{ 1 + \sum_{1 \leq \ell \leq 2J} \frac{1}{\ell!} \left(\sum_{3 \leq m \leq 2J+1} \frac{\Phi_k^{(m)}(\sigma_n)}{m!} (i\tau)^m \right)^\ell + O\left(\frac{\tau^{2J+2}}{\sigma_n^{1/k+2J+2}}\right) \right\}$$

$$= e^{\Phi_k(\sigma_n) - \frac{1}{2}\Phi_k''(\sigma_n)\tau^2} \left\{ 1 + \sum_{1 \leq \ell \leq 2J} \frac{1}{\ell!} \sum_{3\ell \leq m \leq (2J+1)\ell} \lambda_{k,\ell,m}(n) \tau^m + O\left(\frac{\tau^{2J+2}}{\sigma_n^{1/k+2J+2}}\right) \right\},$$

where

$$(3.4) \quad \lambda_{k,\ell,m}(n) := i^m \sum_{\substack{3 \leq m_1, \dots, m_\ell \leq 2J+1 \\ m_1 + \dots + m_\ell = m}} \prod_{1 \leq r \leq \ell} \frac{\Phi_k^{(m_r)}(\sigma_n)}{m_r!}.$$

Since the contributions from odd powers of τ vanish, we get

$$(3.5) \quad \int_{\mathfrak{J}} e^{\Phi_k(\sigma_n + i\tau) + in\tau} d\tau = e^{\Phi_k(\sigma_n)} \left\{ I_0 + \sum_{1 \leq \ell \leq 2J} \frac{1}{\ell!} \sum_{3\ell \leq 2m \leq (2J+1)\ell} \lambda_{k,\ell,2m}(n) I_m + O(R) \right\},$$

with

$$I_m := \int_{\mathcal{J}} e^{-\frac{1}{2}\Phi_k''(\sigma_n)\tau^2} \tau^{2m} d\tau, \quad R := \sigma_n^{-1/k-2J-2} \int_{\mathcal{J}} e^{-\frac{1}{2}\Phi_k''(\sigma_n)\tau^2} \tau^{2J+2} d\tau.$$

Extending the range of integration in I_m involves an exponentially small error, so we get from the classical formula for Laplace integrals

$$I_m = \frac{\sqrt{2\pi}(2m)!}{m!2^m\Phi_k''(\sigma_n)^{m+1/2}} + O\left(e^{-c_5n^{1/(3k+3)}}\right), \quad R \asymp \sigma_n^{1+(J+1/2)/k} \asymp \frac{1}{\sqrt{\Phi_k''(\sigma_n)n^{J/(k+1)}}}.$$

Inserting these estimates back into (3.5) and expanding all arising factors $\Phi_k^{(m)}(\sigma_n)$ by (2.12), we obtain (3.1). \square

Remark. From (3.3) and (3.5) we see that, when $k \geq 2$,

$$(3.6) \quad p_k(n) = \frac{e^{n\sigma_n + \Phi_k(\sigma_n)}}{\sqrt{2\pi\Phi_k''(\sigma_n)}} \left\{ 1 - \frac{2k^2 + 5k + 2}{24k\mathbf{c}_k} \left(\frac{\sigma_n}{\mathbf{a}_k}\right)^{1/k} + O\left(\sigma_n^{2/k}\right) \right\}$$

where, in view of (2.6), the quantity inside curly brackets may be replaced by an asymptotic series in powers of $\sigma_n^{1/k}$. Inserting (2.5) and (2.10) in the main term, we thus get a formula which is very close to, but simpler than (1.9), since it follows from (2.13) that X and $1/\sigma_n$ agree to any power of σ_n .

We are now in a position to complete the proof of Theorem 1.

We infer from (2.10) and (2.11) that

$$n\sigma_n + \Phi_k(\sigma_n) = \mathbf{c}_k n^{1/(k+1)} + \sum_{1 \leq j < J} \frac{a_{kj}^*}{n^{j/(k+1)}} + O\left(\frac{1}{n^{J/(k+1)}}\right) + \frac{1}{2} \log\left(\frac{\sigma_n}{(2\pi)^k}\right)$$

with $a_{kj}^* := \mathbf{a}_k(ka_{k,j+1} + b_{k,j+1})$. Exponentiating and expanding, we get

$$(3.7) \quad \begin{aligned} & \exp(n\sigma_n + \Phi(\sigma_n)) \\ &= \frac{\sqrt{\sigma_n}}{(2\pi)^{k/2}} \exp(\mathbf{c}_k n^{1/(k+1)}) \left\{ 1 + \sum_{1 \leq \ell < J} \frac{1}{\ell!} \left(\sum_{1 \leq j < J} \frac{a_{kj}^*}{n^{j/(k+1)}} \right)^\ell + O\left(\frac{1}{n^{J/(k+1)}}\right) \right\} \\ &= \frac{\sqrt{\sigma_n}}{(2\pi)^{k/2}} \exp(\mathbf{c}_k n^{1/(k+1)}) \left\{ 1 + \sum_{1 \leq j < J} \frac{f_{kj}}{n^{j/(k+1)}} + O\left(\frac{1}{n^{J/(k+1)}}\right) \right\} \end{aligned}$$

with

$$f_{kj} := \sum_{1 \leq \ell < J} \frac{1}{\ell!} \sum_{\substack{1 \leq j_1, \dots, j_\ell < J \\ j_1 + \dots + j_\ell = j}} a_{kj_1}^* \cdots a_{kj_\ell}^*.$$

It remains to insert back into (3.1) and expand $\sqrt{\sigma_n/\Phi_k''(\sigma_n)}$ according to (2.10) and (2.12) with $m = 2$ to obtain the required asymptotic formula.

REFERENCES

- [1] B.C. Berndt, A. Malik, and A. Zaharescu, Partitions into k th-powers of terms in an arithmetic progression, *Math. Z.* **290** (2018), 1277–1307.
- [2] A. Dunn and N. Robles, Polynomial partition asymptotics, *J. Math. Anal. App.* **459** (2018), 359–384.
- [3] A. Gafni, *Power partitions*, *J. Number Theory* **163** (2016), 19–42.
- [4] G. Hardy and S. Ramanujan, *Asymptotic formulae in combinatory analysis*, *Proc. London Math. Soc. (2)* **17** (1918), 75–115.
- [5] G. Tenenbaum, *Applications de la méthode du col*, Cours M2 (2015/2016), Méthodes analytiques, Institut Élie Cartan de Lorraine, Université de Lorraine.
- [6] R. C. Vaughan, *The Hardy-Littlewood method*, Cambridge tracts in mathematics, no. 125, second edition, Cambridge University Press, 1997.
- [7] R. C. Vaughan, *Squares: additive questions and partitions*, *Int. J. Number Theory* **11**, no. 5 (2015), 1367–1409.
- [8] E.T. Whittaker & G.N. Watson, *A course of modern analysis* (4-ième éd.), Cambridge University Press, 1927.
- [9] E. M. Wright, *Asymptotic partition formulae, III. Partitions into k -th powers*, *Acta Math.* **63** (1) (1934), 143–191.

INSTITUT ÉLIE CARTAN DE LORRAINE, UNIVERSITÉ DE LORRAINE,
BP 70239, 54506 VANDŒUVRE-LÈS-NANCY CEDEX, FRANCE
Email address: gerald.tenenbaum@univ-lorraine.fr

CNRS LAMA 8050, LABORATOIRE D'ANALYSE ET DE MATHÉMATIQUES APPLIQUÉES,
UNIVERSITÉ PARIS-EST CRÉTEIL, 94010 CRÉTEIL CEDEX, FRANCE
Email address: jie.wu@math.cnrs.fr

YALI LI, SCHOOL OF MATHEMATICS AND STATISTICS, HENAN UNIVERSITY, KAIFENG, HENAN 475004,
P. R. CHINA