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► **To cite this version:**

Gérald Tenenbaum, Jie Wu, Ya-Li Li. Power partitions and saddle-point method. *Journal of Number Theory*, 2019, 204, pp.435-445. 10.1016/j.jnt.2019.04.013 . hal-02891953

**HAL Id: hal-02891953**

**<https://hal.science/hal-02891953v1>**

Submitted on 7 Jul 2020

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# POWER PARTITIONS AND SADDLE-POINT METHOD\*

GÉRALD TENENBAUM, JIE WU AND YALI LI

ABSTRACT. For  $k \geq 1$ , denote by  $p_k(n)$  the number of partitions of an integer  $n$  into  $k$ -th powers. In this note, we apply the saddle-point method to provide a new proof for the well-known asymptotic expansion of  $p_k(n)$ . This approach turns out to significantly simplify those of Wright (1934), Vaughan (2015) and Gafni (2016).

## 1. INTRODUCTION

Let  $p(n)$  denote, as usual, the number of unrestricted partitions of an integer  $n$ , i.e. the number of solutions to the equation

$$n = a_1 + a_2 + \cdots + a_d,$$

where  $d \geq 1$  and the  $a_j$  are positive integers such that  $a_1 \geq a_2 \geq \cdots \geq a_d \geq 1$ . In 1918, Hardy and Ramanujan [4] proved the asymptotic formula

$$(1.1) \quad p(n) \sim \frac{\exp(\pi\sqrt{2n/3})}{4\sqrt{3}n} \quad (n \rightarrow \infty).$$

by using modular properties of Jacobi's  $\Delta$ -function.

More generally, given an integer  $k \geq 1$ , let  $p_k(n)$  denote the number of partitions of the integer  $n$  into  $k$ -th powers, i.e. the number of solutions to the equation

$$n = a_1^k + \cdots + a_d^k$$

where, as before,  $d \geq 1$  and  $a_1 \geq a_2 \geq \cdots \geq a_d \geq 1$ . Thus,  $p_1(n) = p(n)$ .

In 1918 too, Hardy and Ramanujan [4] stated without proof the asymptotic formula

$$(1.2) \quad p_k(n) \sim \frac{\mathfrak{b}_k \exp\{\mathfrak{c}_k n^{1/(k+1)}\}}{n^{(3k+1)/(2k+2)}} \quad (n \rightarrow \infty),$$

where the constants  $\mathfrak{b}_k$  and  $\mathfrak{c}_k$  are defined by

$$(1.3) \quad \mathfrak{a}_k := \{k^{-1}\zeta(1+k^{-1})\Gamma(1+k^{-1})\}^{k/(k+1)},$$

$$(1.4) \quad \mathfrak{b}_k := \frac{\mathfrak{a}_k}{(2\pi)^{(k+1)/2}\sqrt{(1+1/k)}},$$

$$(1.5) \quad \mathfrak{c}_k := (k+1)\mathfrak{a}_k,$$

and  $\zeta$  is the Riemann zeta-function. In 1934, introducing a number of complicated objects including generalised Bessel functions, Wright [9] obtained an asymptotic expansion of  $p_k(n)$ : for any integer  $k \geq 1$ , there is a real sequence  $\{\alpha_{kj}\}_{j \geq 1}$  such that, for any  $J \geq 1$ , we have

$$(1.6) \quad p_k(n) = \frac{\mathfrak{b}_k \exp(\mathfrak{c}_k(n+h_k)^{1/(k+1)})}{(n+h_k)^{(3k+1)/(2k+2)}} \left\{ 1 + \sum_{1 \leq j < J} \frac{(-1)^j \alpha_{kj}}{(n+h_k)^{j/(k+1)}} + O\left(\frac{1}{n^{J/(k+1)}}\right) \right\},$$

where

$$h_k := \begin{cases} 0 & \text{if } k \text{ is even,} \\ (-1)^{(k+1)/2} (2\pi)^{-(k+1)} k! \zeta(k+1) & \text{if } k \text{ is odd,} \end{cases}$$

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*Date:* October 5, 2019.

*2010 Mathematics Subject Classification.* 05A17, 11N37, 11P82.

*Key words and phrases.* Asymptotic estimates, partitions, partitions into powers, saddle-point method.

\* We include here some corrections with respect to the published version.

and the implied constant depends at most on  $J$  and  $k$ . Apart from an explicit formula for  $\alpha_{k1}$ , no further information was given about the  $\alpha_{kj}$  beyond the statement that they depend only on  $k$  and  $j$  and that they “may be calculated with sufficient labour for any given values of  $k, j$ ”.

Of course, taking  $J = 1$  in (1.6) yields an effective form of (1.2).

More recently, appealing to a relatively simple implementation of the Hardy-Littlewood circle method, Vaughan [7] obtained an explicit version of (1.6) in the case  $k = 2$  and Gafni [3] generalised the argument to arbitrary, fixed  $k$ .

Gafni states her result in the following way. Let  $X = X_k(n)$  denote the real solution to the equation

$$(1.7) \quad n = (\mathfrak{a}_k X)^{1+1/k} - \frac{1}{2}X - \frac{1}{2}\zeta(-k),$$

and write

$$(1.8) \quad Y = Y_k(n) := (1 + 1/k)\mathfrak{a}_k^{1+1/k} X^{1/k} - \frac{1}{4}.$$

Then, given any  $k \geq 1$ , there is a real sequence  $\{\beta_{kj}\}_{j \geq 1}$  such that for any fixed  $J \geq 1$ , we have

$$(1.9) \quad p_k(n) = \frac{\exp\left\{(k+1)\mathfrak{a}_k^{1+1/k} X^{1/k} - \frac{1}{2}\right\}}{(2\pi)^{(k+1)/2} X^{3/2} Y^{1/2}} \left\{1 + \sum_{1 \leq j < J} \frac{\beta_{kj}}{Y^j} + O\left(\frac{1}{Y^J}\right)\right\}.$$

It may be checked that the asymptotic formulae (1.6) and (1.9) match each other.

In this note, our aim is to provide a new proof of (1.2), and indeed also of (1.6) and (1.9), by applying the saddle-point method along lines very similar to those employed in [5] in the case of  $p(n)$ .

Our approach appears to be significantly simpler than those of the quoted previous works.

Indeed, as mentioned above, Wright’s method, which provides a sharper error term, rests on the introduction of generalised Bessel functions and so may be regarded as far more difficult. Note that Vaughan [7] does motivate his study by the search of a ‘relatively simple argument’.

Our method simplifies the matter further. Actually, the saddle-point method may be seen as a crude version of the circle method in which the major arcs are reduced to a single neighbourhood of one point. In the present context, it actually provides the same accuracy.

We thank the referee for pointing out to us that the Vaughan-Gafni method has been very recently generalised to enumerate other classes of partitions—see [1] and [2]. We believe that the saddle-point method could still be used in these contexts, with expected substantial simplifications in the analysis.

The constants  $\mathfrak{b}_k$  and  $\mathfrak{c}_k$  being defined as in (1.4) and (1.5), we can state the following.

**Theorem 1.** *Let  $k \geq 1$  be a fixed integer. There is a real sequence  $\{\gamma_{kj}\}_{j \geq 1}$  such that, for any given integer  $J \geq 1$ , we have*

$$(1.10) \quad p_k(n) = \frac{\mathfrak{b}_k \exp(\mathfrak{c}_k n^{1/(k+1)})}{n^{(3k+1)/(2k+2)}} \left\{1 + \sum_{1 \leq j < J} \frac{\gamma_{kj}}{n^{j/(k+1)}} + O\left(\frac{1}{n^{J/(k+1)}}\right)\right\}$$

uniformly for  $n \geq 1$ . The implied constant depends at most on  $J$  and  $k$ .

The coefficients  $\gamma_{kj}$  can be made explicit directly from the computations in our proof. For instance, we find that  $\gamma_{k1} = -(11k^2 + 11k + 2)/(24k\mathfrak{c}_k)$  when  $k \geq 2$ , in accordance with the expression given by Wright. (It can be checked, after some computations, that it matches Gafni’s formula too.) We also have

$$\gamma_{11} = -\frac{1}{48}\mathfrak{c}_1 - 1/\mathfrak{c}_1 = -\sqrt{\frac{2}{3}}\left(\frac{\pi}{48} + \frac{3}{2\pi}\right).$$

It may be seen that  $|\gamma_{kj}|$  grows like  $\Gamma(j)e^{O(j)}$  and thus that the series  $\sum_{j \geq 1} \gamma_{kj} z^j$  has radius of convergence 0.

## 2. TECHNICAL PREPARATION

Define

$$(2.1) \quad F_k(s) := \sum_{n \geq 0} p_k(n) e^{-ns} \quad (\Re s > 0),$$

so that

$$(2.2) \quad p_k(n) = \frac{1}{2\pi i} \int_{\sigma-i\pi}^{\sigma+i\pi} F_k(s) e^{ns} ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(\sigma + i\tau) e^{n\sigma + in\tau} d\tau.$$

According to the principles of the saddle-point method, we aim at selecting the integration abscissa  $\sigma$  as a solution  $\sigma_n$  of  $-F'_k(\sigma)/F_k(\sigma) = n$ . We plainly have

$$(2.3) \quad F_k(s) = \prod_{m \geq 1} (1 - e^{-m^k s})^{-1} \quad (\Re s > 0).$$

Thus, in the same half-plane, we may define a determination of  $\log F_k(s)$  by the formula

$$\Phi_k(s) := \sum_{m \geq 1} \log \left( \frac{1}{1 - e^{-m^k s}} \right)$$

where the complex logarithms are taken in principal branch. Expanding throughout and inverting summations, we get

$$(2.4) \quad \Phi_k(s) = \sum_{m \geq 1} \sum_{n \geq 1} \frac{e^{-m^k n s}}{n} = \sum_{r \geq 1} \frac{w_k(r)}{r} e^{-rs}, \quad -\Phi'_k(s) = \sum_{r \geq 1} w_k(r) e^{-rs} \quad (\Re s > 0),$$

where

$$w_k(r) := \sum_{m^k | r} m^k \quad (r \geq 1).$$

Hence  $-\Phi'_k(\sigma)$  decreases from  $+\infty$  to  $0+$  on  $(0, \infty)$ , and so the equation  $-\Phi'_k(\sigma) = n$  has for each integer  $n \geq 1$  a unique real solution  $\sigma_n = \sigma_n(k)$ . Moreover, the sequence  $\{\sigma_n\}_{n \geq 1}$  is decreasing and the trivial estimates  $1 \leq w_k(r) \leq r^2$  yield  $1/n \ll \sigma_n \ll 1/\sqrt[3]{n}$ .

We start with an asymptotic expansion for the derivatives  $\Phi_k^{(m)}(\sigma_n)$  in terms of powers of  $\sigma_n$ . It turns out that all coefficients but a finite number vanish.

**Lemma 2.1.** *Let  $J \geq 1$ ,  $k \geq 1$ . As  $n \rightarrow \infty$ , we have*

$$(2.5) \quad \Phi_k(\sigma_n) = \frac{k \mathfrak{a}_k^{1+1/k}}{\sigma_n^{1/k}} + \frac{1}{2} \log \left( \frac{\sigma_n}{(2\pi)^k} \right) + \frac{1}{2} \zeta(-k) \sigma_n + O(\sigma_n^J),$$

Moreover, for fixed  $m \geq 1$ ,

$$(2.6) \quad (-1)^m \Phi_k^{(m)}(\sigma_n) = \frac{\mathfrak{a}_k^{1+1/k}}{\sigma_n^{m+1/k}} \prod_{1 \leq \ell < m} \left( \ell + \frac{1}{k} \right) - \frac{(m-1)!}{2\sigma_n^m} - \frac{1}{2} \delta_{1m} \zeta(-k) + O(\sigma_n^J),$$

where  $\delta_{1m}$  is Kronecker's symbol.

*Proof.* Considering Mellin's inversion formula

$$e^{-s} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(z) s^{-z} dz \quad (\Re s > 0)$$

and the convolution identity

$$(2.7) \quad \sum_{r \geq 1} \frac{w_k(r)}{r^{1+z}} = \zeta(z+1) \zeta(kz) \quad (\Re z > 1/k),$$

we derive from the series representation (2.4) the integral formula

$$(2.8) \quad \Phi_k(s) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta(z+1) \zeta(kz) \Gamma(z) \frac{dz}{s^z},$$

and in turn

$$(2.9) \quad (-1)^m \Phi_k^{(m)}(\sigma_n) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta(z+1) \zeta(kz) \Gamma(z+m) \frac{dz}{\sigma_n^{z+m}} \quad (m \geq 0).$$

Using the classical fact that  $\zeta(z)$  has finite order in any vertical strip  $a \leq \Re z \leq b$  ( $a, b \in \mathbb{R}$  with  $a < b$ ), or, in other words, satisfies

$$\zeta(x + iy) \ll_{a,b} 1 + |y|^A \quad (a \leq x \leq b, |y| \geq 1),$$

for suitable  $A = A(a, b)$ , and invoking Stirling's formula in the form

$$|\Gamma(x + iy)| = \sqrt{2\pi} |y|^{x-1/2} e^{-\pi|y|/2} \{1 + O_{a,b}(1/y)\} \quad (a \leq x \leq b, |y| \geq 1)$$

we may move the line of integration to  $\Re z = -J - m - \frac{1}{2}$ .

The shifted integral is clearly  $\ll \sigma_n^J$ .

Let us first consider the case  $m = 0$  in (2.9). Then the crossed singularities are a pole of order 2 at  $z = 0$ , and two simple poles at  $z = 1/k$  and  $z = -1$ . Indeed,  $\zeta(z+1)\zeta(kz)$  vanishes at all negative integers  $\leq -2$ , so the corresponding zeros compensate the poles of  $\Gamma(z)$  at negative integers  $\leq -2$ .

The residue at  $z = 1/k$  is equal to

$$k^{-1}\zeta(1+k^{-1})\Gamma(k^{-1})\sigma_n^{-1/k} = k\mathfrak{a}_k^{1+1/k}\sigma_n^{-1/k}.$$

The residue at  $z = 0$  is the coefficient of  $z$  in the Taylor expansion of

$$\begin{aligned} z^2\zeta(z+1)\zeta(kz)\Gamma(z)\sigma_n^{-z} &= z\zeta(z+1)\zeta(kz)\Gamma(z+1)\sigma_n^{-z} \\ &= (1-\gamma z)\{\zeta(0) + k\zeta'(0)z\}(1+\gamma z)(1-z\log\sigma_n) + O(z^2) \\ &= \zeta(0) + \{-\zeta(0)\log\sigma_n + k\zeta'(0)\}z + O(z^2). \end{aligned}$$

Since  $\zeta(0) = -\frac{1}{2}$  and  $\zeta'(0) = -\frac{1}{2}\log(2\pi)$ , this residue equals  $\frac{1}{2}\log\{\sigma_n/(2\pi)^k\}$ .

The residue at  $z = -1$  equals  $\frac{1}{2}\zeta(-k)\sigma_n$ .

This completes the proof of (2.5).

When  $m = 1$ , the three crossed singularities are simple poles. The residues at  $z = 1/k$ ,  $z = 0$  and  $z = -1$  are respectively  $(1/k)\Gamma(1+1/k)\zeta(1+1/k)\sigma_n^{-1-1/k}$ ,  $-1/2\sigma_n$  and  $-\frac{1}{2}\zeta(-k)$ .

When  $m \geq 2$ , the only crossed singularities are two simple poles, at  $z = 1/k$  and  $z = 0$ , with respective residues  $(1/k)\Gamma(m+1/k)\zeta(1+1/k)\sigma_n^{-m-1/k}$  and  $-\frac{1}{2}(m-1)!\sigma_n^{-m}$ . This proves (2.6).  $\square$

**Lemma 2.2.** *Let  $J \geq 1$ ,  $k \geq 1$ ,  $m \geq 1$ .*

(i) *There is a real sequence  $\{a_{kj}\}_{j \geq 1}$  with  $a_{k1} = -k/(2\mathfrak{c}_k)$ ,  $a_{k2} = k/(8\mathfrak{c}_k^2)$ , such that*

$$(2.10) \quad \sigma_n = \frac{\mathfrak{a}_k}{n^{k/(k+1)}} \left\{ 1 + \sum_{1 \leq j < J} \frac{a_{kj}}{n^{j/(k+1)}} + O\left(\frac{1}{n^{J/(k+1)}}\right) \right\} \quad (n \rightarrow \infty).$$

(ii) *There is a real sequence  $\{b_{kj}\}_{j \geq 1}$  with  $b_{k1} = -a_{k1}/k$  such that, as  $n \rightarrow \infty$ , we have*

$$(2.11) \quad \Phi_k(\sigma_n) = k\mathfrak{a}_k n^{1/(k+1)} \left\{ 1 + \sum_{1 \leq j < J} \frac{b_{kj}}{n^{j/(k+1)}} + O\left(\frac{1}{n^{J/(k+1)}}\right) \right\} + \frac{1}{2} \log\left(\frac{\sigma_n}{(2\pi)^k}\right).$$

(iii) *There is a real sequence  $\{b_{kmj}\}_{j \geq 1}$  such that, as  $n \rightarrow \infty$ , we have*

$$(2.12) \quad \begin{aligned} &(-1)^m \Phi_k^{(m)}(\sigma_n) + \frac{1}{2} \delta_{1m} \zeta(-k) \\ &= \frac{n^{(mk+1)/(k+1)}}{\mathfrak{a}_k^{m-1}} \prod_{1 \leq \ell < m} \left( \ell + \frac{1}{k} \right) \left\{ 1 + \sum_{1 \leq j < J} \frac{b_{kmj}}{n^{j/(k+1)}} + O\left(\frac{1}{n^{J/(k+1)}}\right) \right\}. \end{aligned}$$

*Proof.* We infer from (2.6) that

$$(2.13) \quad n = \frac{\mathfrak{a}_k^{1+1/k}}{\sigma_n^{1+1/k}} - \frac{1}{2\sigma_n} - \frac{1}{2}\zeta(-k) + O(\sigma_n^J).$$

This immediately implies (2.10) by Lagrange's inversion formula — see, e.g. [8, §7.32]. We may obtain an explicit expression for the  $a_{kj}$  from the formula

$$(2.14) \quad \sigma_n = \frac{\mathfrak{a}_k}{2\pi i n^{k/(k+1)}} \oint_{|z-1|=\varrho} \frac{zG'(z)}{G(z)} dz + O\left(\frac{1}{n^{(k+J)/(k+1)}}\right)$$

where  $\varrho$  is a fixed, small positive constant and

$$G(z) := z^{-1-1/k} - 1 - \frac{1}{2\mathbf{a}_k z n^{1/(k+1)}} - \frac{\zeta(-k)}{2n}.$$

This is classically derived from Rouché's theorem and we omit the details. The values of  $a_{k1}$  and  $a_{k2}$  may be retrieved from the above or by formally inserting (2.10) into (2.13).

Inserting (2.14) back into (2.5) and (2.6) immediately yields (2.11) and (2.12).  $\square$

With the aim of applying Laplace's method to evaluate the integral on the right-hand side of (2.2), we need to show that it is dominated by a small neighbourhood of the saddle-point  $\sigma_n$ . The next result meets this requirement. Here and in the sequel, all constants  $c_j$  ( $j \geq 0$ ) are assumed, unless otherwise stated, to depend at most upon  $k$ .

**Lemma 2.3.** *We have*

$$(2.15) \quad \frac{|F_k(\sigma_n + i\tau)|}{|F_k(\sigma_n)|} \leq \begin{cases} e^{-c_1 \tau^2 \sigma_n^{-(2+1/k)}} & \text{if } |\tau| \leq 2\pi\sigma_n, \\ e^{-c_2 \sigma_n^{-1/k}} & \text{if } 2\pi\sigma_n < |\tau| \leq \pi. \end{cases}$$

*Proof.* Noticing that

$$\begin{aligned} |1 - e^{-m^k(\sigma_n + i\tau)}|^2 &= |1 - e^{-m^k\sigma_n}|^2 + 4e^{-m^k\sigma_n} \sin^2\left(\frac{1}{2}m^k\tau\right) \\ &\geq |1 - e^{-m^k\sigma_n}|^2 + 16e^{-m^k\sigma_n} \|m^k\tau/(2\pi)\|^2, \end{aligned}$$

we can write

$$\begin{aligned} \frac{|F_k(\sigma_n + i\tau)|^2}{|F_k(\sigma_n)|^2} &\leq \prod_{m \geq 1} \left(1 + \frac{16\|m^k\tau/(2\pi)\|^2}{e^{m^k\sigma_n}(1 - e^{-m^k\sigma_n})^2}\right)^{-1} \\ &\leq \prod_{(4\sigma_n)^{-1/k} < m \leq (2\sigma_n)^{-1/k}} \left(1 + \frac{16\|m^k\tau/(2\pi)\|^2}{e^{m^k\sigma_n}(1 - e^{-m^k\sigma_n})^2}\right)^{-1}. \end{aligned}$$

Thus, there is an absolute positive constant  $c_3$  such that

$$(2.16) \quad \frac{|F_k(\sigma_n + i\tau)|}{|F_k(\sigma_n)|} \leq e^{-c_3 S(\tau; \sigma_n)}$$

with

$$S(\tau; \sigma_n) := \sum_{m \in I} \|m^k\tau/(2\pi)\|^2.$$

where we have put  $I := ](4\sigma_n)^{-1/k}, (2\sigma_n)^{-1/k}[$ .

If  $|\tau| \leq 2\pi\sigma_n$  and  $m \in I$ , we have  $|m^k\tau/(2\pi)| \leq \frac{1}{2}$ . Thus

$$(2.17) \quad S(\tau; \sigma_n) = \sum_{(4\sigma_n)^{-1/k} < m \leq (2\sigma_n)^{-1/k}} m^{2k}\tau^2/4\pi^2 \asymp \tau^2\sigma_n^{-(2+1/k)}.$$

When  $2\pi\sigma_n < |\tau| \leq 2\pi\sigma_n^{1-1/3k}$ , we proceed similarly, noting that for any integer  $h$  with  $|\tau|/4\sigma_n < 2\pi h \leq |\tau|/2\sigma_n$  there are  $\gg (1/\sigma_n)^{1/k}/h$  integers  $m$  in  $I$  such that

$$\frac{1}{4} \leq |m^k\tau/(2\pi) - h| \leq \frac{1}{2}.$$

This yields the the required estimate

$$(2.18) \quad S(\tau; \sigma_n) \gg \sigma_n^{-1/k}.$$

When  $2\pi(\sigma_n)^{1-1/3k} < |\tau| \leq \pi$ , we note that (2.18) follows, via the Cauchy-Schwarz inequality, from the inequalities

$$\sum_{m \in I} \|\tau m^k/(2\pi)\| \geq \sum_{m \in I} |1 - e^{i\tau m^k}| \geq |I| - \left| \sum_{m \in I} e^{i\tau m^k} \right|$$

provided we can show the modulus of the last exponential sum is, say,  $\leq (1-c)|I|$  for some positive constant  $c$  depending at most upon  $k$ . Now Dirichlet's approximation lemma guarantees that there exist integers  $a \in \mathbb{Z}^*$  and  $q$ , with  $1 < q \leq (1/\sigma_n)^{1-1/3k}$ ,  $(a, q) = 1$ ,

$$|\tau/(2\pi) - a/q| \leq (\sigma_n)^{1-1/3k}/q.$$

If  $q \leq (1/\sigma_n)^{1/3k}$ , we readily deduce the required estimate from [6, Lemma 2.7 & Theorem 4.2]. If  $(1/\sigma_n)^{1/3k} < q \leq (1/\sigma_n)^{1-1/3k}$ , we may apply Weyl's inequality, as stated for instance in [6, Lemma 2.4], to get that the exponential sum under consideration is  $\ll |I|^{1-\varepsilon_k}$  for some positive  $\varepsilon_k$  depending only on  $k$ . Hence, (2.18) holds in all circumstances.  $\square$

### 3. COMPLETION OF THE PROOF

**Proposition 3.1.** *Let  $k \geq 1$ ,  $J \geq 1$ . Then there is a real sequence  $\{e_{kj}\}_{j \geq 1}$  such that for any integer  $J \geq 1$  we have*

$$(3.1) \quad p_k(n) = \frac{\exp(n\sigma_n + \Phi_k(\sigma_n))}{\sqrt{2\pi\Phi_k''(\sigma_n)}} \left\{ 1 + \sum_{2 \leq j < J} \frac{e_{kj}}{n^{j/(k+1)}} + O\left(\frac{1}{n^{J/(k+1)}}\right) \right\} \quad (n \rightarrow \infty).$$

*Proof.* By (2.2), we have

$$(3.2) \quad p_k(n) = \frac{e^{n\sigma_n}}{2\pi} \int_{-\pi}^{\pi} e^{\Phi_k(\sigma_n + i\tau) + in\tau} d\tau.$$

From (2.15), we deduce that

$$(3.3) \quad \int_{2\pi\sigma_n < |\tau| \leq \pi} e^{\Phi_k(\sigma_n + i\tau) + in\tau} d\tau \ll e^{\Phi_k(\sigma_n) - c_4\sigma_n^{-1/k}}$$

$$\int_{\sigma_n^{1+1/3k} < |\tau| \leq 2\pi\sigma_n} e^{\Phi_k(\sigma_n + i\tau) + in\tau} d\tau \ll e^{\Phi_k(\sigma_n) - c_4\sigma_n^{-1/3k}}.$$

Since these bounds are exponentially small with respect to the expected main term, it only remains to estimate the contribution of the interval  $\mathfrak{J} := ]-\sigma_n^{1+1/(3k)}, \sigma_n^{1+1/(3k}[$ , corresponding to a small neighbourhood of the saddle-point.

In this range, we have

$$\Phi_k(\sigma_n + i\tau) = \sum_{0 \leq m \leq 2J+1} \frac{\Phi_k^{(m)}(\sigma_n)}{m!} (i\tau)^m + O\left(\frac{\tau^{2J+2}}{\sigma_n^{1/k+2J+2}}\right),$$

where the estimate for the error term follows from (2.6). The same formula ensures that  $|\Phi_k^{(m)}(\sigma_n)\tau^m| \ll 1$  for  $m \geq 3$ . Thus for  $\tau \in \mathfrak{J}$ , we can write

$$e^{\Phi_k(\sigma_n + i\tau) + in\tau}$$

$$= e^{\Phi_k(\sigma_n) - \frac{1}{2}\Phi_k''(\sigma_n)\tau^2} \left\{ 1 + \sum_{1 \leq \ell \leq 2J} \frac{1}{\ell!} \left( \sum_{3 \leq m \leq 2J+1} \frac{\Phi_k^{(m)}(\sigma_n)}{m!} (i\tau)^m \right)^\ell + O\left(\frac{\tau^{2J+2}}{\sigma_n^{1/k+2J+2}}\right) \right\}$$

$$= e^{\Phi_k(\sigma_n) - \frac{1}{2}\Phi_k''(\sigma_n)\tau^2} \left\{ 1 + \sum_{1 \leq \ell \leq 2J} \frac{1}{\ell!} \sum_{3\ell \leq m \leq (2J+1)\ell} \lambda_{k,\ell,m}(n) \tau^m + O\left(\frac{\tau^{2J+2}}{\sigma_n^{1/k+2J+2}}\right) \right\},$$

where

$$(3.4) \quad \lambda_{k,\ell,m}(n) := i^m \sum_{\substack{3 \leq m_1, \dots, m_\ell \leq 2J+1 \\ m_1 + \dots + m_\ell = m}} \prod_{1 \leq r \leq \ell} \frac{\Phi_k^{(m_r)}(\sigma_n)}{m_r!}.$$

Since the contributions from odd powers of  $\tau$  vanish, we get

$$(3.5) \quad \int_{\mathfrak{J}} e^{\Phi_k(\sigma_n + i\tau) + in\tau} d\tau = e^{\Phi_k(\sigma_n)} \left\{ I_0 + \sum_{1 \leq \ell \leq 2J} \frac{1}{\ell!} \sum_{3\ell \leq 2m \leq (2J+1)\ell} \lambda_{k,\ell,2m}(n) I_m + O(R) \right\},$$

with

$$I_m := \int_{\mathcal{J}} e^{-\frac{1}{2}\Phi_k''(\sigma_n)\tau^2} \tau^{2m} d\tau, \quad R := \sigma_n^{-1/k-2J-2} \int_{\mathcal{J}} e^{-\frac{1}{2}\Phi_k''(\sigma_n)\tau^2} \tau^{2J+2} d\tau.$$

Extending the range of integration in  $I_m$  involves an exponentially small error, so we get from the classical formula for Laplace integrals

$$I_m = \frac{\sqrt{2\pi}(2m)!}{m!2^m\Phi_k''(\sigma_n)^{m+1/2}} + O\left(e^{-c_5 n^{1/(3k+3)}}\right), \quad R \asymp \sigma_n^{1+(J+1/2)/k} \asymp \frac{1}{\sqrt{\Phi_k''(\sigma_n)n^{J/(k+1)}}}.$$

Inserting these estimates back into (3.5) and expanding all arising factors  $\Phi_k^{(m)}(\sigma_n)$  by (2.12), we obtain (3.1).  $\square$

*Remark.* From (3.3) and (3.5) we see that, when  $k \geq 2$ ,

$$(3.6) \quad p_k(n) = \frac{e^{n\sigma_n + \Phi_k(\sigma_n)}}{\sqrt{2\pi\Phi_k''(\sigma_n)}} \left\{ 1 - \frac{2k^2 + 5k + 2}{24k\mathbf{c}_k} \left(\frac{\sigma_n}{\mathbf{a}_k}\right)^{1/k} + O\left(\sigma_n^{2/k}\right) \right\}$$

where, in view of (2.6), the quantity inside curly brackets may be replaced by an asymptotic series in powers of  $\sigma_n^{1/k}$ . Inserting (2.5) and (2.10) in the main term, we thus get a formula which is very close to, but simpler than (1.9), since it follows from (2.13) that  $X$  and  $1/\sigma_n$  agree to any power of  $\sigma_n$ .

We are now in a position to complete the proof of Theorem 1.

We infer from (2.10) and (2.11) that

$$n\sigma_n + \Phi_k(\sigma_n) = \mathbf{c}_k n^{1/(k+1)} + \sum_{1 \leq j < J} \frac{a_{kj}^*}{n^{j/(k+1)}} + O\left(\frac{1}{n^{J/(k+1)}}\right) + \frac{1}{2} \log\left(\frac{\sigma_n}{(2\pi)^k}\right)$$

with  $a_{kj}^* := \mathbf{a}_k(ka_{k,j+1} + b_{k,j+1})$ . Exponentiating and expanding, we get

$$(3.7) \quad \begin{aligned} & \exp(n\sigma_n + \Phi(\sigma_n)) \\ &= \frac{\sqrt{\sigma_n}}{(2\pi)^{k/2}} \exp(\mathbf{c}_k n^{1/(k+1)}) \left\{ 1 + \sum_{1 \leq \ell < J} \frac{1}{\ell!} \left( \sum_{1 \leq j < J} \frac{a_{kj}^*}{n^{j/(k+1)}} \right)^\ell + O\left(\frac{1}{n^{J/(k+1)}}\right) \right\} \\ &= \frac{\sqrt{\sigma_n}}{(2\pi)^{k/2}} \exp(\mathbf{c}_k n^{1/(k+1)}) \left\{ 1 + \sum_{1 \leq j < J} \frac{f_{kj}}{n^{j/(k+1)}} + O\left(\frac{1}{n^{J/(k+1)}}\right) \right\} \end{aligned}$$

with

$$f_{kj} := \sum_{1 \leq \ell < J} \frac{1}{\ell!} \sum_{\substack{1 \leq j_1, \dots, j_\ell < J \\ j_1 + \dots + j_\ell = j}} a_{kj_1}^* \cdots a_{kj_\ell}^*.$$

It remains to insert back into (3.1) and expand  $\sqrt{\sigma_n/\Phi_k''(\sigma_n)}$  according to (2.10) and (2.12) with  $m = 2$  to obtain the required asymptotic formula.

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