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Serdar Şahin[⋆†], Antonio Maria Cipriano[⋆], Charly Poulliat[†] and Marie-Laure Boucheret[†]
[⋆] THALES, Gennevilliers, 92230, France, Email: name.surname@thalesgroup.com
[†] University of Toulouse, IRIT-ENSEEIHT, Toulouse, 31000, France, Email: name.surname@enseeiht.fr

***Abstract*—This paper investigates the dynamic behaviour of doubly iterative bit-interleaved coded modulation (BICM) receivers based on expectation propagation (EP). When implemented in the frequency domain, for single-carrier (SC) systems, such receivers achieve attractive performance-complexity trade-offs in quasi-static wideband channels. With this category of receivers, conventional binary extrinsic information transfer (EXIT) functions are subject to a great number of parameters, including channel realizations, constellation and inner iteration parameters. Hence, this paper proposes a novel extrinsic information evolution analysis method which simplifies the receiver's EXIT function into independent inner transfer functions. The core idea is to track state-evolution dynamics of EP through numerically stable extrinsic variance/information transfer (EXVIT) functions. Numerical results attest to the accuracy of this method for tracking the asymptotic receiver behaviour.**

***Index Terms*—Bit-interleaved coded modulation, inter-symbol interference, expectation propagation, EXIT, evolution analysis.**

I. Introduction

Iterative detection and decoding techniques, involving soft-input soft-output (SISO) detectors and decoders, have been widely investigated to push achievable communication rate boundaries closer to their theoretical limits. Recent advances on probabilistic message passing algorithms, such as expectation propagation (EP), has led to novel iterative architectures for enabling close-to-optimal detection in a variety of interference-limited applications. The signal processing dynamics of such algorithms are often computationally intensive, and thus, predicting their behaviour is a topic of interest.

The predictability of the iterative detection dynamics is of significant importance both for facilitating algorithm design, or for physical layer link abstraction. For instance, in channel coding, density evolution (DE) was proposed for tracking the dynamics of probability density distributions (PDFs) of exchanged bit log-likelihood ratios (LLRs) between SISO modules [1], [2]. Extrinsic information transfer (EXIT) analysis [3] simplified DE to an asymptotic single-parameter tracking problem, and it has lead to great achievements on the design and analysis of bit-interleaved coded modulation (BICM) systems with iterative detection.

When considering communications systems, where a codeword is subject to P independent channel realizations (due to fading, pseudo-random precoding or frequency-hopping, for instance) [4], the receiver's EXIT function becomes an at least P-dimensional function. Analyzing receiver behaviour under such circumstances requires numerical Monte Carlo integration over each dimension, during the EXIT function synthesis. Moreover, when the receiver has configurable parameters, the dimensionality of the EXIT function further increases.

This issue has been addressed for soft-input soft-output (SISO) minimum mean squared error (MMSE) linear equalizers with interference cancellation (IC) [5], by using analytical approximations and average-mutual information based EXIT models of their sub-blocks [6]–[8]. This paper generalizes this approach to non-linear turbo equalizers involving EP.

A recently proposed doubly-iterative FD equalizer, based on EP, called FD self-iterated linear equalizer with EP-based IC (FD SILE-EPIC), has been shown to achieve attractive performance-complexity trade-offs [9]. This receiver performs self-iterations, through the exchange of extrinsic Gaussian messages between the equalizer and the demapper modules, prior to decoding. It was shown to improve the performance upon alternative self-iterated receivers that uses hard decision or soft a posteriori probability (APP) feedback, which have been often discarded in the past to their unpredictability and due to high error propagation. These disadvantages are not a concern with FD SILE-EPIC. EP message-passing is shown to be asymptotically predictable through asymptotic mean-squared error (MSE) state-evolution functions [10], [11], thanks to its divergence-free "decision device" [12]. However these proofs are limited to Gaussian i.i.d. channels, and predictability of EP in general is only an experimental observation [9].

This paper investigates the detection dynamics of EP-based BICM receivers through extrinsic message evolution analysis. First, the conventional binary EXIT analysis is carried out in order to outline its dependency on receiver parameters and channel realizations, which makes it costly to compute. Next, a novel evolution analysis method is proposed in our context, which replaces this receiver's EXIT function with analytical receiver models and symbolwise extrinsic information/variance transfer (EXVIT) functions. The synthesis method for this approach is given, and to further illustrate the implications of our proposal, analytical expressions are given for the BPSK and Gray-mapped QPSK constellations. Finally, the asymptotic prediction capabilities of this method is illustrated with numerical results.

The remainder of this paper is organized as follows. The system model and the receiver are given in section II, and the conventional analysis method is discussed in section III. The proposed evolution analysis is discussed in section IV and its implications are assessed in section V with numerical results.

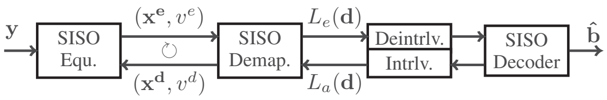

Fig. 1. Double-loop turbo equalization structure.

II. System Model

A. Single Carrier Multi-Block Transmission with BICM

Single-carrier transmissions of P blocks of K symbols is carried out with a BICM scheme[1]. A K_b-bit information block $\mathbf{b}$ is encoded by a rate-R_c forward-error-correction code and then interleaved into a K_d-bit $\mathbf{d}$. This codeword is then split into P binary sequences $\mathbf{d}_p$, and a memoryless modulator φ maps each sequence to $\mathbf{x}_p \in \mathcal{X}^K$, with $|\mathcal{X}| = M$ and $Q = \log_2 M$. This symbolwise operation maps the vector $\mathbf{d}_{p,k} \triangleq [d_{p,Qk}, \ldots, d_{p,Q(k+1)-1}]$ to the symbol $x_{p,k}$, and we use $\varphi_q^{-1}(x_{p,k})$ or $d_{p,k,q}$ to refer to $d_{p,kQ+q}$.

Each transmitted block $\mathbf{x}_p$ goes to through a possibly independent ISI channel, and the received baseband observations are $\mathbf{y}_p = \mathbf{H}_p\mathbf{x}_p + \mathbf{w}_p$, for $p = 0, \ldots, P-1$, with $\mathbf{H}_p$ the channel matrix and $\mathbf{w}_p \sim \mathcal{CN}(\mathbf{0}_K, \sigma^2_{w_p}\mathbf{I}_K)$ the additive white Gaussian noise (AWGN). Assuming $\mathbf{H}_p$ to be a circulant matrix[2], whose first line is the impulse response $\mathbf{h}_p = [h_{p,0}, \ldots, h_{p,L-1}, \mathbf{0}_{1,K-L}]$, $L < K$ being the channel spread, the frequency domain observations $\underline{\mathbf{y}}_p = \mathcal{F}_K\mathbf{y}_p$ follow

$$\underline{\mathbf{y}}_p = \underline{\mathbf{H}}_p\underline{\mathbf{x}}_p + \underline{\mathbf{w}}_p, \tag{1}$$

where $\underline{\mathbf{x}}_p = \mathcal{F}_K\mathbf{x}_p$, $\underline{\mathbf{w}}_p = \mathcal{F}_K\mathbf{w}_p$ and $\underline{\mathbf{H}}_p = \mathcal{F}_K\mathbf{H}_p\mathcal{F}_K^H$, with $\mathcal{F}_K$ is the normalized K-DFT matrix with its elements given by $[\mathcal{F}_K]_{k,l} = \exp(-2j\pi kl/K)/\sqrt{K}$, and such that $\mathcal{F}_K\mathcal{F}_K^H = \mathbf{I}_K$. Thanks to DFT properties, $\underline{\mathbf{H}}_p = \mathbf{Diag}(\underline{\mathbf{h}}_p)$ with $\underline{\mathbf{h}}_p = \sqrt{K}\mathcal{F}_K\mathbf{h}_p$, and the FD noise is $\underline{\mathbf{w}} \sim \mathcal{CN}(\mathbf{0}_K, \sigma^2_{w_p}\mathbf{I}_K)$.

B. Double-Loop EP Receiver: FD SILE-EPIC

EP is an approximate Bayesian inference algorithm which can be seen as an extension of loopy belief propagation which assign a PDF family to each variable node [13], and computes messages by the means of the reverse-information projection.

This concept has been used to derive a doubly iterative frequency domain equalizer for BICM in [9], the FD SILE-EPIC. In addition to the conventional turbo loop for exchanging extrinsic LLRs between the demapper and the decoder, there is a loop for exchanging extrinsic symbols between the equalizer and the demapper. The receiver structure and exchanged quantities are illustrated in Fig 1. At the SISO demapper, a priori and extrinsic bit LLRs are respectively denoted by $L_a(d_{p,k,q})$ and $L_e(d_{p,k,q})$, and a priori and extrinsic symbol estimates are respectively denoted $x^e_{p,k}$ and $x^d_{p,k}$ with respective residual noise variances v^e and v^d. Each turbo-iteration (TI) $\tau = 0, \ldots, \mathcal{T}$ consists of $\mathcal{S}_\tau$ self-iterations (SIs) (may depend on τ), for $s = 0, \ldots, \mathcal{S}_\tau$, and then the decoder is updated with the extrinsic receiver LLRs.

At the initialization of a TI τ, equalizer's messages are reset, with $x^{e(\tau,s=-1)}_{p,k} = 0$, $v^{e(\tau,s=-1)}_p = +\infty$, and at the initial TI, the decoder message is set to $L_a^{(\tau=0)}(d_{p,k,q}) = 0, \forall k, j$.

The SISO demodulator computes prior probability mass function (PMF) on transmitted symbols with, $\forall\alpha \in \mathcal{X}$,

$$\mathcal{P}^{(\tau)}_{p,k}(\alpha) \propto \prod_{q=0}^{Q-1} \exp(-\varphi_q^{-1}(\alpha)L_a^{(\tau)}(d_{p,k,q})), \tag{2}$$

and then deduces the posterior PMF, $\forall\alpha \in \mathcal{X}$, with

$$\mathcal{D}^{(\tau,s)}_{p,k}(\alpha) \propto \phi(x^{e(\tau,s)}_{p,k}, \alpha, v^{e(\tau,s)}_p)\mathcal{P}^{(\tau)}_{p,k}(\alpha). \tag{3}$$

The posterior PMF $\mathcal{D}^{(\tau,s)}_{p,k}$ is projected on $\mathcal{CN}(\mu^d_{p,k}, \gamma^d_p)$ through the EP-based reverse-information projection, with

$$\begin{aligned}
\mu^{d(\tau,s)}_{p,k} &\triangleq \mathbb{E}_\mathcal{D}[x_{p,k}] = \textstyle\sum_{\alpha\in\mathcal{X}} \alpha\mathcal{D}^{(\tau,s)}_{p,k}(\alpha),\\
\gamma^{d(\tau,s)}_{p,k} &\triangleq \text{Var}_\mathcal{D}[x_{p,k}] = \textstyle\sum_{\alpha\in\mathcal{X}} |\alpha - \mu^{d(\tau,s)}_{p,k}|^2\mathcal{D}^{(\tau,s)}_{p,k}(\alpha),\\
\gamma^{d(\tau,s)}_p &\triangleq K^{-1}\textstyle\sum_k \gamma^{d(\tau,s)}_{p,k}.
\end{aligned} \tag{4}$$

Then the extrinsic symbol PDF $\mathcal{CN}(x^\star_{p,k}, v^\star_p)$ is computed by dividing $\mathcal{CN}(\mu^d_{p,k}, \gamma^d_p)$ by the prior symbol PDF $\mathcal{CN}(x^e_{p,k}, v^e_p)$

$$x^{\star(\tau,s+1)}_{p,k}/v^{\star(\tau,s+1)}_p = \mu^{d(\tau,s)}_{p,k}/\gamma^{d(\tau,s)}_p - x^{e(\tau,s)}_{p,k}/v^{e(\tau,s)}_p, \tag{5}$$

$$1/v^{\star(\tau,s+1)}_p = 1/\gamma^{d(\tau,s)}_p - 1/v^{e(\tau,s)}_p. \tag{6}$$

However, as EP is an approximate inference algorithm, damping is used to avoid local fixed points [13]. Thus, for $s = 0$,

$$x^{d(\tau,0)}_{p,k} = x^{\star(\tau,0)}_{p,k} \triangleq \mathbb{E}_\mathcal{P}[x_{p,k}], \tag{7}$$

$$v^{d(\tau,0)}_p = v^{\star(\tau,0)}_p \triangleq K^{-1}\textstyle\sum_k \text{Var}_\mathcal{P}[x_{p,k}], \tag{8}$$

which is equivalent to the conventional decoder's extrinsic feedback [5], and for $s > 0$, exponential smoothing is used with parameters $0 \le \beta^{(\tau,s)} \le 1$ on the extrinsic estimates

$$x^{d(\tau,s)}_{p,k} = (1-\beta^{(\tau,s)})x^{\star(\tau,s)}_{p,k} + \beta^{(\tau,s)}x^{d(\tau,s-1)}_{p,k}, \tag{9}$$

$$v^{d(\tau,s)}_p = (1-\beta^{(\tau,s)})v^{\star(\tau,s)}_p + \beta^{(\tau,s)}v^{d(\tau,s-1)}_p. \tag{10}$$

At the SISO equalizer, applying DFT on the demapper feedback, the extrinsic outputs are given by FD MMSE filtering

$$\underline{x}^{e(\tau,s)}_{p,k} = \underline{x}^{d(\tau,s)}_{p,k} + \underline{f}^{(\tau,s)*}_{p,k}(\underline{y}_{p,k} - \underline{h}_{p,k}\underline{x}^{d(\tau,s)}_{p,k}), \tag{11}$$

$$v^{e(\tau,s)}_p = 1/\xi^{(\tau,s)}_p - v^{d(\tau,s)}_p, \tag{12}$$

$$\underline{f}^{(\tau,s)}_{p,k} = \underline{h}_{p,k}/[\xi^{(\tau,s)}_p(k_w\sigma^2_{w_p} + v^{d(\tau,s)}_p|\underline{h}_{p,k}|^2)], \tag{13}$$

$$\xi^{(\tau,s)}_p = K^{-1}\textstyle\sum_k |\underline{h}_{p,k}|^2/(k_w\sigma^2_{w_p} + v^{d(\tau,s)}_p|\underline{h}_{p,k}|^2), \tag{14}$$

where $k_w = 1/2$ for real constellations (BPSK), else $k_w = 1$.

[1] *Notations:* Bold lowercase letters denote vectors: let $\mathbf{u}$ be a $N \times 1$ vector, then $u_n, n = 0, \ldots, N-1$ are its entries. Capital bold letters denote matrices: for a $N \times M$ matrix $\mathbf{A}$, $[\mathbf{A}]_{n,:}$ and $[\mathbf{A}]_{:,m}$ respectively denote its n^{th} row and m^{th} column, and $a_{n,m} = [\mathbf{A}]_{n,m}$ is the entry (n, m). $\mathbf{I}_N$ is the $N \times N$ identity matrix, $\mathbf{0}_{N,M}$ and $\mathbf{1}_{N,M}$ are respectively all zeros and all ones.

Let x and y be two random variables, then $\mu_x = \mathbb{E}[x]$ is the expected value, $\sigma_x^2 = \text{Var}[x]$ is the variance and $\sigma_{x,y} = \text{Cov}[x, y]$ is the covariance. The probability of x taking a value α is $\mathbb{P}[x = \alpha]$, and probability density functions (PDF) are denoted as $p(\cdot)$. $\mathcal{CN}(\mu_x, \sigma_x^2)$ denotes the circularly-symmetric complex Gaussian distribution of mean μ_x and variance σ_x^2. We denote $\phi(x; \mu, \sigma^2) \triangleq \exp(-k_w|x-\mu|^2/\sigma^2)/\sqrt{\pi\sigma^2/k_w}$ where $k_w = 1/2$ for $x \in \mathbb{R}$ and $k_w = 1$ for $x \in \mathbb{C}$. The mutual information between two random variables x and y is defined in bits as $I(x; y) \triangleq \mathbb{E}_{p(x,y)}[\log_2(p(y|x)/p(y))]$.

[2] The circulant channel matrix assumption can be satisfied in practice with signalling schemes such as cyclic prefix insertion. This model can be extended to SC frequency-division-multiple-access (FDMA) and its variants, by incorporating involved subband masks and pulse-shaping into $\mathbf{H}_p$.

At the final self iteration, extrinsic bit LLRs are computed

$$L_e^{(\tau)}(d_{p,k,q}) = \ln \frac{\sum_{\alpha \in \mathcal{X}_q^0} \mathcal{D}_{p,k}^{(\tau,\mathcal{S}_\tau)}(\alpha)}{\sum_{\alpha \in \mathcal{X}_q^1} \mathcal{D}_{p,k}^{(\tau,\mathcal{S}_\tau)}(\alpha)} - L_a^{(\tau)}(d_{p,k,q}), \quad (15)$$

with $\mathcal{X}_q^b = \{\alpha \in \mathcal{X} : \varphi_q^{-1}(x) = b\}$, $b \in \mathbb{F}_2$. These LLRs are the processed by the SISO decoder to produce $L_a^{(\tau+1)}(d_{p,k,q})$.

III. Conventional Evolution Analysis

A. Binary Extrinsic Information Evolution: EXIT

Extrinsic information transfer (EXIT) functions track the mutual information (MI) carried by extrinsic messages [3], to study the behaviour of SISO modules within a BICM system.

The SISO receiver is described by the transfer function $\mathcal{T}_{\text{REC}}$, which depends on the channel realizations with

$$I_{E,p}^{(\tau)} = \mathcal{T}_{\text{REC}}(I_A^{(\tau)}; \mathbf{h}_p, \sigma_{w_p}^2, \{\beta^{(\tau,s)}\}_{s=0}^{\mathcal{S}_\tau}), \quad (16)$$

where the prior information $I_A^{(\tau)}$ and the extrinsic information $I_{E,p}^{(\tau)}$ are the average MI between coded bits and respectively the a priori and extrinsic LLRs of the module. Indeed $I_A^{(\tau)} \triangleq (PKQ)^{-1} \sum_{p,k,q} I(d_{p,k,q}; L_a^{(\tau)}(d_{p,k,q}))$ and $I_{E,p}^{(\tau)} \triangleq (KQ)^{-1} \sum_{k,q} I(d_{p,k,q}; L_e^{(\tau)}(d_{p,k,q}))$. The SISO decoder's EXIT function $\mathcal{T}_C$ is similarly given by

$$I_A^{(\tau+1)} = \mathcal{T}_{\text{DEC}}(\{I_E^{(\tau)}\}_{p=0}^{P-1}). \quad (17)$$

This is a multi-dimensional EXIT that accounts for different channel realizations over the codeword, but which provides a scalar output due to decoder's averaging effect.

Receiver EXIT functions are synthesized by isolating the concerned SISO module and feeding it prior LLRs matching the desired I_A, and then computing an histogram on extrinsic LLRs, through equations (2)-(3) and (15), to estimate I_E. Hence, a prior LLR generator is needed, and the following assumption and property provide a solution to this end [3].

Assumption 1. *A priori LLRs of a SISO module are i.i.d. with* $L_a^{(\tau)}(d_{p,k,q}) \sim \mathcal{N}(\bar{d}_{p,k,q}\mu_a^{(\tau)}, \sigma_a^{2(\tau)})$ *, with* $\bar{d}_{p,k,q} = 1 - 2d_{p,k,q}$.

Property 1. *If Assumption 1 holds, as LLRs are symmetrically distributed,* $\sigma_a^{2(\tau)} = 2\mu_a^{(\tau)}$*, and* $I(d_{p,k,q}; L_a^{(\tau)}(d_{p,k,q})) = J(\mu_a^{(\tau)})$ *with* $J(\mu) \triangleq 1 - \int_L \log_2(1 + e^{-L})\phi(L; \mu, 2\mu)\,\mathrm{d}L$.

Property 1 states that μ_a needed for input LLR generation at I_A can be obtained by the binary MI function $J(\cdot)$.

EXIT analysis has partially been carried out on FD SILE-EPIC, in order to deduce achievable rates in [9], and Fig. 2 illustrates its operating principle. However as $\mathcal{T}_{\text{REC}}$ depends on inner iterations and its damping parameters, computing an EXIT for each possible configuration is impractical.

B. On the Asymptotic Predictability of FD SILE-EPIC

The consistent Gaussian model (Assumption 1) accurately characterizes EXIT functions for many cases, and in particular for the MAP detector, but some sub-optimal receivers' EXIT yield erroneous predictions when $I_A \neq 0$. Indeed, as one SISO module's prior inputs are provided from another SISO module's extrinsic outputs, the prediction accuracy depends on whether the following assumption is true.

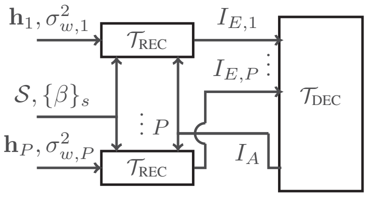

Fig. 2. EXIT evolution analysis model for the receiver.

Assumption 2. *There exists* $\mu_e^{(\tau)} > 0$*, such that extrinsic LLRs of a SISO module are approximately i.i.d. with* $L_e^{(\tau)}(d_{p,k,q}) \sim \mathcal{N}(\bar{d}_{p,k,q}\mu_e^{(\tau)}, 2\mu_e^{(\tau)})$*, and* $\mu_e^{(\tau)} \approx J^{-1}(I_E^{(\tau)})$.

The i.i.d. assumption is often true, in the large system limit thanks to interleaving, and Gaussian model holds for demapper output with BPSK and Gray-mapped QPSK constellations. However, the consistent Gaussian approximation of LLRs at the decoder output is lost across turbo iterations due to non-linear dynamics of channel decoding [14]. Hence, EXIT remains accurate for a few iterations, but its accuracy needs to be experimentally evaluated. A study has been carried out for FD SILE-EPIC in [9], where finite-length trajectories of average MI were in accordance with the EXIT functions.

IV. Proposed Evolution Analysis

A. Symbolwise Extrinsic Information Evolution: EXVIT

Synthesis of EXIT functions for evolution analysis of SISO receivers can be impractical as numerical integrations need to be carried out for each possible channel realization. For FD SILE-EPIC, the number of self-iterations and the damping add additional static parameters which further increase the dimensionality of the EXIT generation procedure. To alleviate this issue, a symbolwise evolution analysis method is proposed here, removing the dependence of the synthesis process on channel and on inner loop parameters.

The asymptotic behaviour of some EP-based algorithms has previously been investigated through state evolution [10], [11], where MSE on extrinsic estimates are tracked. Here, the BICM context with the SISO demodulator with prior information, and damping brings additional modelling complexity.

In this section, a symbolwise extrinsic parameter transfer model is proposed, which consists in characterizing

- the equalizer output (demodulator input) with modulation constrained normalized mutual information I_L,
- the equalizer input (demodulator output) with the covariance v^d of the soft symbol feedback.

The choice of these measures aims to ensure evolution analysis' ability to track evolution dynamics with sufficient accuracy, and to keep them numerically stable with values on finite intervals. Various candidate measures were evaluated for extrinsic evolution dynamics in [15]. Mutual information and second-order statistics of the extrinsic PDFs (called "fidelity" therein) are measures that maintain accuracy, by avoiding restrictive assumptions on the measured extrinsic PDFs. Regarding the numerical stability, we have $I_L \in [0, 1]$, and $v^d \in [0, \sigma_x^2]$, where $\sigma_x^2 = 1$ is the average symbol power.

B. Transfer Function Synthesis for EXVIT Analysis

As equalizer's extrinsic estimates are unbiased and decorrelated [12], and EP is asymptotically Bayes optimal [10], the following assumption is considered.

Assumption 3. *The extrinsic symbol outputs of the equalizer are i.i.d. and* $x_{p,k}^e = x_{p,k} + w_{p,k}^e$ *with* $w_{p,k}^e \sim \mathcal{CN}(0, v_p^e)$.

Property 2. *If Assumption 3 holds, then the constrained AWGN capacity is* $\psi_{\mathcal{X}}(v_p^e) \triangleq I(x_{p,k}^e; x_{p,k}) = Q - r(v_p^e)$ *with*

$$r(v^e) = \int_{x^e} \log_2\left(\frac{\sum_{x\in\mathcal{X}} \phi(x_k^e; x, v^e)}{\phi(x_k^e; x_k, v^e)}\right) \phi(x_k^e; x_k, v^e)\, \mathrm{d}x_k^e.$$

$\psi_{\mathcal{X}}(\cdot)$ *depends on* $\mathcal{X}$*, and it does not have a closed form in general. It can be obtained by using Monte Carlo integration.*

Hence, the transfer function of the equalizer is defined as

$$I_{L,p}^{(\tau,s)} = \mathcal{T}_{\text{EQU}}(v_p^{d(\tau,s)}; \mathbf{h}_p, \sigma_{w_p}^2), \tag{18}$$

under Assumption 3, with the analytical transfer function

$$\mathcal{T}_{\text{EQU}}(v_p^{d(\tau,s)}; \mathbf{h}_p, \sigma_{w_p}^2) \triangleq \psi_{\mathcal{X}}(v_p^{e(\tau,s)})/Q, \tag{19}$$

where $v_p^{e(\tau,s)}$ is given by equations (12)-(14).

To capture the non-linear demapping and EP-feedback behaviour with an extrinsic mutual information information to variance transfer function, the dynamics of the extrinsic symbol PDF $\mathcal{CN}(x_{p,k}^{d(\tau,s)}, v_p^{d(\tau,s)})$ is tracked, given the prior information $I_A^{(\tau)}$, and the extrinsic equalizer output, $I_{L,p}^{(\tau,s)}$.

The EP-based feedback from the demapper is modelled with

$$v_p^{d(\tau,s+1)} = \mathcal{T}_{\text{EP}}(I_{L,p}^{(\tau,s)}, I_A^{(\tau)}, v_p^{d(\tau,s)}; \beta^{(\tau,s)}), \tag{20}$$

with $\mathcal{T}_{\text{EP}} \triangleq \mathcal{T}_{\text{Damp}} \circ \mathcal{T}_{\text{EP*}}$ and where $\mathcal{T}_{\text{Damp}}$ analytically models damping and $\mathcal{T}_{\text{EP*}}$ characterizes the behaviour of $\mathcal{CN}(x_{p,k}^{\star(\tau,s)}, v_p^{\star(\tau,s)})$. In detail, eq. (10) applies damping with

$$\mathcal{T}_{\text{Damp}}(v_p^\star, v_p^{d(\tau,s)}; \beta^{(\tau,s)}) \triangleq (1-\beta^{(\tau,s)}) v_p^{\star(\tau,s)} + \beta v_p^{d(\tau,s)},$$

and the transfer of the pure extrinsic symbols is obtained, following Assumptions 1 and 3, given I_L and I_A, with

$$\mathcal{T}_{\text{EP*}}(I_L, I_A) \triangleq [1/\gamma^d(I_L, I_A) - 1/v^e]^{-1}, \tag{21}$$

$$\gamma^d(I_L, I_A) \triangleq \frac{1}{KM} \sum_{k=0}^{K-1} \sum_{x_k \in \mathcal{X}} \int f_\gamma(x_k, x_k^e, \mathbf{L}_{\mathbf{a},k})\, \mathrm{d}\mathbf{L}_{\mathbf{a},k}\, \mathrm{d}x_k^e,$$

$$f_\gamma(x_k, x_k^e, \mathbf{L}_{\mathbf{a},k}) \triangleq \gamma_k^d(x_k, x_k^e, \mathbf{L}_{\mathbf{a},k}) \phi(x_k^e; x_k, v^e) \prod_{q=0}^{Q-1} \phi(L_{a,k,q}; (1-2\varphi_q^{-1}(x_k))\mu_a, 2\mu_a), \tag{22}$$

$$\gamma_k^d(x_k, x_k^e, \mathbf{L}_{\mathbf{a},k}) \triangleq \sum_{\alpha\in\mathcal{X}} |\alpha - \sum_{\alpha'\in\mathcal{X}} \alpha' \mathcal{D}_k(\alpha')|^2 \mathcal{D}_k(\alpha), \tag{23}$$

$$\mathcal{D}_k(\alpha) = Z_k^{-1} \phi(x_k^e; \alpha, v^e) \prod_{q=0}^{Q-1} \exp(-\varphi_q^{-1}(\alpha) L_{a,k,q}), \tag{24}$$

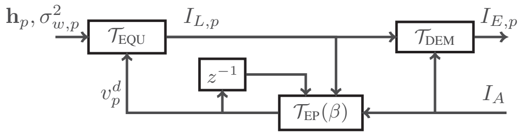

Fig. 3. Proposed evolution analysis model for $\mathcal{T}_{\text{REC}}$.

with Z_k is such that $\sum_{\alpha\in\mathcal{X}} \mathcal{D}_k(\alpha) = 1$, $\mu_a = J^{-1}(I_A)$, $v^e = \psi_{\mathcal{X}}^{-1}(QI_L)$ and $\mathbf{L}_{\mathbf{a},k} = [L_{a,k,0}, \ldots, L_{a,k,Q-1}]$. While $\mathcal{T}_{\text{EP*}}$ has no closed form in general, it only depends on $\mathcal{X}$ and on K.

Finally, to complete the characterization of FD SILE-EPIC, SISO demapper's extrinsic LLR output needs to be characterized with respect to the final self-iteration estimates. Following Assumption 3, the EXIT function of a SISO demapper is used

$$I_{E,p}^{(\tau)} = \mathcal{T}_{\text{DEM}}(I_A^{(\tau)}, I_{L,p}^{(\tau,\mathcal{S}_\tau)}). \tag{25}$$

In conclusion, symbolwise EXVIT analysis consists in applying transfer functions $\mathcal{T}_{\text{EP}}$ and $\mathcal{T}_{\text{EQU}}$ successively, on each block p to obtain $\{v_p^{d(\tau,\mathcal{S}_\tau)}\}_{p=0}^{P-1}$ and then $\{I_{L,p}^{(\tau,\mathcal{S}_\tau)}\}_{p=0}^{P-1}$, starting with $v_p^{d(\tau,-1)} = 1, \forall p$. Then, the extrinsic output of the overall receiver is $\{I_{E,p}^{(\tau)}\}_{p=0}^{P-1}$, computed with $\mathcal{T}_{\text{DEM}}$.

Unlike EXIT analysis, EXVIT has analytical dependence on channel realizations $\{\mathbf{h}_p, \sigma_{w_p}^2\}_{p=0}^{P-1}$ and on $\{\beta^{(\tau,s)}\}_{s=0}^{\mathcal{S}_\tau}$. Numerical integrations are only needed for the two-dimensional $\mathcal{T}_{\text{EP*}}$ and $\mathcal{T}_{\text{DEM}}$ functions, which only depend on $\mathcal{X}$ and on K.

C. Simplifying the decoder's EXIT function

The decoder's EXIT function in eq. (17) requires as many inputs as P, the number of channel realizations over the codeword, which elevates synthesis complexity. A similar issue is noted for evolution analysis of BPSK MIMO systems in [7]. The problem is identical from the MI point of view, hence the following assumption is made to address it.

Assumption 4. *The decoder behaviour is identical for inputs* $\{L_e(\mathbf{d}_{p,k})\}_{k=0}^{K-1}$, $p = 0, \ldots, P-1$ *and for* $\{\tilde{L}_e(\mathbf{d}_{k'})\}_{k'=0}^{KP-1}$, *if they have the same average MI with respect to* $\mathbf{d}$.

Thus the SISO decoder's EXIT function is reduced to single dimensional mapping with the effective mutual information

$$I_A^{(\tau+1)} = \mathcal{T}_{\text{DEC}}\left(P^{-1} \textstyle\sum_{p=0}^{P-1} I_E^{(\tau)}\right). \tag{26}$$

D. Analytic evolution analysis for BPSK/QPSK

This section provides analytical derivation of $\mathcal{T}_{\text{EP*}}$ for BPSK constellation (extension to Gray-mapped QPSK is straightforward). Due the symmetry of the constellation, and i.i.d. LLRs, $\gamma_k^d(x, x_k^e, L_{a,k}) = \gamma_k^d(1, x^e, L_a)$, $x^e = 1 + w^e$, $w^e \sim \mathcal{N}(0, v^e)$, and $L_a = \mu_a + w^a$, with $w^a \sim \mathcal{N}(0, 2\mu^a)$. Then

$$\gamma^d(w^e, w^a) \triangleq \gamma_k^d(1, x^e, L_a) = 1 - |\mu^d(w^e, w^a)|^2, \tag{27}$$

$$\mu^d(w^e, w^a) \triangleq \textstyle\sum_\alpha \alpha \mathcal{D}(\alpha) = \tanh\left(\frac{1}{2}(\mu_d + w^d)\right), \tag{28}$$

$$\mu_d \triangleq \mu_a + 2/v^e, \qquad w^d \triangleq w^a + 2w^e/v^e, \tag{29}$$

with $\mu_a = J^{-1}(I_A)$, $v^e = \psi_{\mathcal{X}}^{-1}(I_L) = 2/J^{-1}(I_L)$. APP covariance is $\gamma^d(I_L, I_A) = 1 - \mathbb{E}_{w_e,w_a}[|\mu^d(w^e, w^a)|^2]$, and the

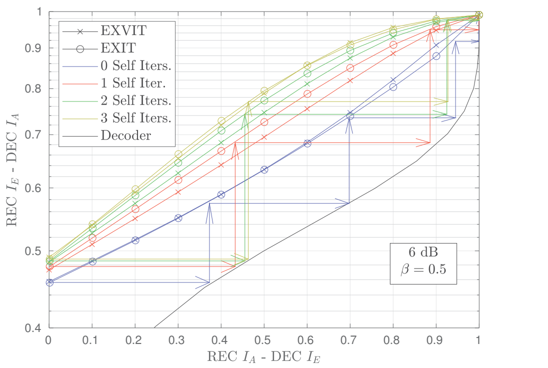

Fig. 4. EXIT & EXVIT functions with $[1, 5/7]_8$ recursive convolutional code.

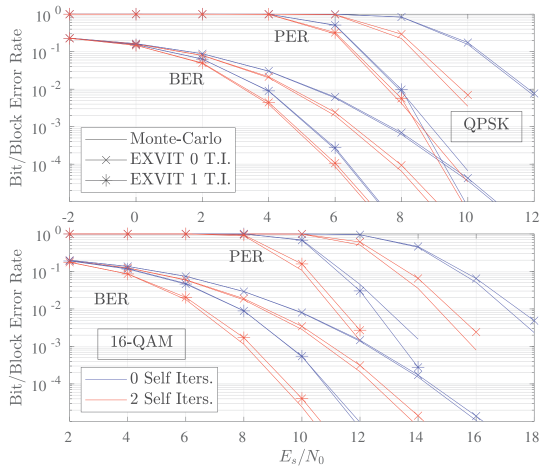

Fig. 5. BER and BLER prediction with analytical EXVIT.

Gaussian integrals therein can be approximated analytically with an n_q^{th} order Gauss-Hermite quadrature [16].

V. Results and Conclusion

In the following, BPSK and QPSK EXVIT functions are evaluated analytically, with $n_q = 15$, and 16-QAM EXVIT, and all EXIT functions are given by Monte-Carlo evaluations.

First, the accuracy of the proposed EXVIT model is evaluated by comparison with the measured MI trajectories of a coded BPSK system with $K_b = 16384$. The closeness of transfer functions to the measured MI trajectories indicates the accuracy of an analysis method. Equivalent EXIT functions in Proakis C channel, $[0.23, 0.46, 0.69, 0.46, 0.23]$, are plotted in Fig. 4, for a signal-to-noise ratio (SNR) of 5 dB. Both EXVIT and EXIT transfer functions appear to be close the MI trajectories for the first few turbo-iterations, hence both methods manage to predict initial turbo-iterations fairly well. Besides, there is a relatively slight difference between both curves, hence they would predict similar decoding thresholds or achievable rates.

Next, to assess the accuracy of EXVIT for predicting more quantitatively, transmissions of $K_b = 16384$ bits over a 10-tap Rayleigh fading channel with uniform power profile is considered with $\beta = 0.3$. The codeword, obtained with the recursive convolutional code $[1, 5/7]_8$ is subject to $P = 8$ channel realizations, and the decoder's bit error rate (BER) and packet error rate (PER) are quantified with regards to its input prior information,. Finite-length simulations are drawn with solid lines on Fig. 5, and EXVIT is used to track the evolution of the extrinsic information of the equalizer, which is fed into the decoder, and to predict the corresponding error rate performance. Predictions appear to be close to the actual Monte-Carlo simulations, for QPSK and 16-QAM constellations, at the zeroeth and the first turbo-iterations. These results attest to the accuracy of the proposed method.

In conclusion, this paper proposes a method for the symbolwise extrinsic analysis of an EP-based, highly non-linear receiver's behaviour. This approach replaces the EXIT function of the receiver, with inner EXVIT functions having analytical dependence on the channel and the receiver parameters. The receiver is shown to be predicted with adequate accuracy over multi-block channels only with a few numerical integrations and independently of the modulation format.

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