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A GENERAL DECAY AND OPTIMAL DECAY RESULT IN A HEAT SYSTEM WITH A VISCOELASTIC TERM*

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Abstract We consider a quasilinear heat system in the presence of an integral term and establish a general and optimal decay result from which improves and generalizes several stability results in the literature.

Key words

2010 MR Subject Classification

1 Introduction

In this work, we consider the following problem:

$$\begin{cases} A(t)|u_t|^{m-2}u_t - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds = 0 & \Omega \times (0, +\infty), \\ u(x, t) = 0 & \partial\Omega \times \mathbb{R}^+, \\ u(x, 0) = u_0(x) & \Omega, \end{cases} \quad (1.1)$$

where $m \geq 2$, Ω is a bounded domain of \mathbb{R}^n , $n \in \mathbb{N}^* := \{1, 2, \dots\}$, with a smooth boundary $\partial\Omega$, $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a positive nonincreasing function, and

$$A : \mathbb{R}^+ \rightarrow M_n(\mathbb{R})$$

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is a bounded square matrix satisfying $A \in C(\mathbb{R}^+)$ and, for some positive constant c_0 ,

$$(A(t)v, v) \geq c_0|v|^2, \quad \forall t \in \mathbb{R}^+, \forall v \in \mathbb{R}^n, \quad (1.2)$$

where (\cdot, \cdot) and $|\cdot|$ are the inner product and the norm, respectively, in \mathbb{R}^n . The equation in consideration arises from various mathematical models in engineering and physics. For instance, in the study of heat conduction in materials with memory, the classical Fourier law is replaced by the following form (cf. [9]):

$$q = -d\nabla u - \int_{-\infty}^t \nabla (k(x, t)u(x, \tau)) d\tau,$$

where u is the temperature, d the diffusion coefficient and the integral term represents the memory effect in the material. This type of problems has considered by a number of researchers; see [2, 9, 11] and the references therein. From a mathematical point of view, we expect that the integral term would be dominated by the leading term in the equation, so that the theory of parabolic equation can be applied. In fact, this has been confirmed by the work of Yin [11], in which he considered a general equation of the form

$$u_t = \operatorname{div} A(x, t, u, u_x) + a(x, t, u, u_x) + \int_0^t \operatorname{div} B(x, t, \tau, u, u_x) d\tau$$

and proved the existence of a unique weak solution under suitable conditions on A , B and a . See more results concerning global existence and asymptotic behavior in Nakao and Ohara [7], Nakao and Chen [8], and Engler *et al.* [3]. Pucci and Serrin [10] discussed the following system:

$$A(t)|u_t|^{m-2}u_t = \Delta u - f(x, u),$$

for $m > 1$ and f satisfying

$$(f(x, u), u) \geq 0$$

and showed that strong solutions tend to the rest state as $t \rightarrow +\infty$, however, no rate of decay has been given. Berrimi and Messaoudi [1] showed that, if A satisfies (1.2), then solutions with small initial energy decay exponentially for $m = 2$ and polynomially if $m > 2$. Messaoudi and Tellab [5] considered (1.1), under condition (1.2) and for relaxation function g satisfying a general decay condition of the form

$$g'(t) \leq -\xi(t)g(t), \quad \forall t \in \mathbb{R}^+,$$

for some nonincreasing differentiable function $\xi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, and established a general decay result, from which the exponential and polynomial decay rates of [1] are only special cases. Recently, Liu and Chen [4] investigated (1.1), with a nonlinear source term, and established a general decay result under suitable conditions on g and the nonlinear source term. They also proved a blow-up result for the solution with both positive and negative initial energy.

In this work, we discuss (1.1) when g is of a more general decay, and establish a general and optimal decay result, which improves those of Berrimi and Messaoudi [1], Liu and Chen [4], and Messaoudi and Tellab [5].

2 Preliminaries

In this section, we present some material needed in the proof of our result. For the relaxation function g we assume that (G1) The function $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a differentiable function satisfying

$$g(0) > 0 \quad \text{and} \quad 1 - \int_0^{+\infty} g(s)ds = l > 0.$$

(G2) There exist a constant $p \in [1, 3/2)$ and a nonincreasing differentiable function $\xi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying

$$g'(t) \leq -\xi(t)g^p(t), \quad \forall t \in \mathbb{R}^+.$$

(G3) We also assume that

$$\begin{aligned} 2 \leq m \leq \frac{2n}{n-2} & \quad \text{if } n \geq 3, \\ m \geq 2 & \quad \text{if } n = 1, 2. \end{aligned}$$

Remark 2.1 There are many functions satisfying (G1) and (G2). Examples of such functions are, for $b > 0$, $\alpha > 0$, $\nu > 1$, and $a > 0$ small enough,

$$g_1(t) = ae^{-b(t+1)^\alpha} \quad \text{and} \quad g_2(t) = \frac{a}{(1+t)^\nu}.$$

We will also be using the embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$, $L^r(\Omega) \hookrightarrow L^q(\Omega)$, for $2 \leq q \leq r < +\infty$, and Poincaré's inequality. The same embedding constant C_* will be used, and C denotes a generic positive constant.

We introduce the following:

$$E(t) = \frac{1}{2}(g \circ \nabla u)(t) + \frac{1}{2} \left(1 - \int_0^t g(s)ds \right) \|\nabla u(t)\|_2^2, \quad \forall t \in \mathbb{R}^+, \quad (2.1)$$

where $\|\cdot\|_q = \|\cdot\|_{(L^q(\Omega))^n}$, for $1 \leq q < +\infty$, and

$$(g \circ \nabla u)(t) = \int_0^t g(t-\tau) \|\nabla u(\cdot, t) - \nabla u(\cdot, \tau)\|_2^2 d\tau, \quad \forall t \in \mathbb{R}^+. \quad (2.2)$$

Similarly to [10], we give the definition of a strong solution of (1.1).

Definition 2.2 A strong solution of (1.1) on $[0, T]$ is a function

$$u \in C([0, T]; (H_0^1(\Omega))^n) \cap C^1((0, T); (L^m(\Omega))^n)$$

which satisfies

$$\begin{aligned} & \int_0^t \int_\Omega \left(\nabla u(x, s) - \int_0^s g(s-\tau) \nabla u(x, \tau) d\tau \right) \cdot \nabla \phi(x, s) dx ds \\ & + \int_0^t A(s) |u_t|^{m-2} u_t(x, s) \cdot \phi(x, s) dx ds = 0, \end{aligned}$$

for all t in $[0, T]$ and all ϕ in $C([0, T]; (H_0^1(\Omega))^n)$.

Remark 2.3 Similarly to [10], we assume the existence of a solution. For the linear case ($m = 2$), one can easily establish the existence of a weak solution by the Galerkin method. In the one-dimensional case ($n = 1$), the existence is established in a more general setting by Yin [11].

Finally, we state an important lemma [6].

Lemma 2.4 Assume that g satisfies (G1) and (G2) and u is the solution of (1.1), then there exists a positive constant k_0 such that

$$\xi(t)(g \circ \nabla u)(t) \leq k_0 (-E'(t))^{\frac{1}{2p-1}}, \quad \forall t \in \mathbb{R}^+, \quad (2.3)$$

where E' is given in (3.1) below.

We also recall the following particular case of the well-known Jensen inequality which will be of essential use in obtaining our result: let $f : \Omega \rightarrow \mathbb{R}^+$ and $h : \Omega \rightarrow \mathbb{R}^+$ be integrable functions on Ω such that

$$\int_{\Omega} h(x) dx = k > 0.$$

Then, for any $p > 1$, we have

$$\frac{1}{k} \int_{\Omega} (f(x))^{\frac{1}{p}} h(x) dx \leq \left(\frac{1}{k} \int_{\Omega} f(x) h(x) dx \right)^{\frac{1}{p}}. \quad (2.4)$$

3 Decay Result

In this section, we state and prove our main result. We start with a lemma.

Lemma 3.1 Let u be the solution of (1.1). Then the energy satisfies

$$E'(t) = - \int_{\Omega} A(t) |u_t|^m dx - \frac{1}{2} g(t) \|\nabla u(t)\|_2^2 + \frac{1}{2} (g' \circ \nabla u)(t) \leq 0, \quad \forall t \in \mathbb{R}^+. \quad (3.1)$$

Proof By multiplying the first equation in (1.1) by u_t , integrating over Ω we get (3.1), after routine manipulations. \square

Lemma 3.2 Let u be a solution of problem (1.1). Then, for any $\delta > 0$, we have

$$\|\nabla u(t)\|_2^2 \leq c_4 \delta E(t) - \frac{C_\delta}{c_0} E'(t) + c_5 (g \circ \nabla u)(t), \quad \forall t \in \mathbb{R}^+, \quad (3.2)$$

where c_0 is introduced in (1.2), c_4 and c_5 are two positive constants, and C_δ is a positive constant depending on δ .

Proof Multiplying the first equation in (1.1) by u and integrating over Ω , we get

$$\|\nabla u(t)\|_2^2 = - \int_{\Omega} A(t) |u_t|^{m-2} u_t u(x, t) dx + \int_{\Omega} \int_0^t g(t-s) \nabla u(x, s) \cdot \nabla u(x, t) ds dx. \quad (3.3)$$

Now, we estimate the right-hand side of (3.3). By using Young's and Poincaré's inequalities, the boundedness of A , conditions (G1) and (G3), and the fact that

$$E(t) \leq E(0),$$

we find, for any $\delta > 0$,

$$\begin{aligned} - \int_{\Omega} A(t) |u_t|^{m-2} u_t u dx &\leq \delta \|u(\cdot, t)\|_m^m + C_\delta \|u_t(\cdot, t)\|_m^m \\ &\leq \delta C_*^m \|\nabla u(\cdot, t)\|_2^m + C_\delta \|u_t(\cdot, t)\|_m^m \\ &\leq \delta C_*^m \left(\frac{2E(0)}{l} \right)^{\frac{m-2}{2}} \left(\frac{2}{l} E(t) \right) + C_\delta \|u_t(\cdot, t)\|_m^m. \end{aligned}$$

On the other hand, using condition (1.2) and identity (3.1), we have

$$\|u_t(\cdot, t)\|_m^m = \int_{\Omega} (|u_t|^{\frac{m}{2}})^2 dx \leq \frac{1}{c_0} \int_{\Omega} A(t) |u_t|^m dx \leq \frac{-1}{c_0} E'(t).$$

By combining the above two inequalities, we conclude that

$$-\int_{\Omega} A(t)|u_t|^{m-2}u_t u dx \leq c_1 \delta E(t) - \frac{C_\delta}{c_0} E'(t). \quad (3.4)$$

Next, we estimate the second term of the right-hand side of (3.3) carefully. By Young's inequality, we easily see that

$$\begin{aligned} & \int_{\Omega} \nabla u(x, t) \int_0^t g(t-s) \nabla u(x, s) ds dx \\ & \leq \frac{1}{2} \|\nabla u(\cdot, t)\|_2^2 + \frac{1}{2} \int_{\Omega} \left(\int_0^t g(t-s) (|\nabla u(x, s) - \nabla u(x, t)| + |\nabla u(x, t)|) ds \right)^2 dx. \end{aligned} \quad (3.5)$$

Using the fact that

$$\int_0^t g(s) ds \leq 1 - l$$

and Young's and Hölder's inequalities, we obtain, for any $\eta > 0$,

$$\begin{aligned} & \int_{\Omega} \left(\int_0^t g(t-s) (|\nabla u(x, s) - \nabla u(x, t)| + |\nabla u(x, t)|) ds \right)^2 dx \\ & = \int_{\Omega} \left(\int_0^t g(t-s) (|\nabla u(s) - \nabla u(t)|) \right)^2 dx + \int_{\Omega} \left(\int_0^t g(t-s) |\nabla u(t)| ds \right)^2 dx \\ & \quad + 2 \int_{\Omega} \left(\int_0^t g(t-s) (|\nabla u(s) - \nabla u(t)|) ds \right) \left(\int_0^t g(t-s) |\nabla u(t)| ds \right) dx \\ & \leq (1 + \eta) \int_{\Omega} \left(\int_0^t g(t-s) |\nabla u(t)| ds \right)^2 dx + \left(1 + \frac{1}{\eta} \right) \int_{\Omega} \left(\int_0^t g(t-s) |\nabla u(s) - \nabla u(t)| ds \right)^2 dx \\ & \leq (1 + \eta)(1 - l)^2 \|\nabla u(\cdot, t)\|_2^2 + \left(1 + \frac{1}{\eta} \right) (1 - l)(g \circ \nabla u)(t). \end{aligned} \quad (3.6)$$

Substituting (3.6) in (3.5), we get

$$\begin{aligned} & \int_{\Omega} \nabla u(x, t) \int_0^t g(t-s) \nabla u(x, s) ds dx \\ & \leq \frac{1}{2} (1 + (1 + \eta)(1 - l)^2) \|\nabla u(\cdot, t)\|_2^2 + \frac{1}{2} \left(1 + \frac{1}{\eta} \right) (1 - l)(g \circ \nabla u)(t). \end{aligned} \quad (3.7)$$

Combining (3.3), (3.4) and (3.7), we find

$$\begin{aligned} \|\nabla u(\cdot, t)\|_2^2 & \leq c_1 \delta E(t) - \frac{C_\delta}{c_0} E'(t) + \frac{1}{2} (1 + (1 + \eta)(1 - l)^2) \|\nabla u(\cdot, t)\|_2^2 \\ & \quad + \frac{1}{2} \left(1 + \frac{1}{\eta} \right) (1 - l)(g \circ \nabla u)(t). \end{aligned} \quad (3.8)$$

We then choose $0 < \eta < l(2 - l)/(1 - l)^2$, which makes $c_2 = \frac{1}{2} (1 + (1 + \eta)(1 - l)^2) < 1$, and, therefore, (3.8) takes the form

$$\|\nabla u(\cdot, t)\|_2^2 \leq c_1 \delta E(t) - \frac{C_\delta}{c_0} E'(t) + c_2 \|\nabla u(\cdot, t)\|_2^2 + c_3 (g \circ \nabla u)(t),$$

where $c_3 = \frac{1}{2} (1 + \frac{1}{\eta})(1 - l)$. This yields (3.2) with $c_4 = \frac{c_1}{1 - c_2}$ and $c_5 = \frac{c_3}{1 - c_2}$. \square

Theorem 3.3 Let u be the solution of (1.1). Then, there exist strictly two positive constants λ_0 and λ_1 such that the energy satisfies, for all $t \in \mathbb{R}^+$,

$$E(t) \leq \lambda_0 e^{-\lambda_1 \int_0^t \xi(s) ds} \quad \text{if } p = 1, \quad (3.9)$$

$$E(t) \leq \lambda_0 \left(1 + \int_0^t \xi^{2p-1}(s) ds \right)^{\frac{-1}{2p-2}} \quad \text{if } p > 1. \quad (3.10)$$

Moreover, if ξ and p in (G_2) satisfy

$$\int_0^{+\infty} \left(1 + \int_0^t \xi^{2p-1}(s) ds \right)^{\frac{-1}{2p-2}} dt < +\infty, \quad (3.11)$$

then, for all $t \in \mathbb{R}^+$,

$$E(t) \leq \lambda_0 \left(1 + \int_0^t \xi^p(s) ds \right)^{\frac{-1}{p-1}} \quad \text{if } p > 1. \quad (3.12)$$

Remark 3.4 Estimates (3.10) and (3.11) yield

$$\int_0^{+\infty} E(t) dt < +\infty. \quad (3.13)$$

Proof From (3.1) and for any $\kappa > 0$, we have

$$\begin{aligned} E'(t) &\leq 0 = -\kappa E(t) + \kappa E(t) \\ &\leq -\kappa E(t) + \kappa \left(\frac{1}{2} (g \circ \nabla u)(t) + \frac{1}{2} \left(1 - \int_0^t g(s) ds \right) \|\nabla u(\cdot, t)\|_2^2 \right) \\ &\leq -\kappa E(t) + \frac{\kappa}{2} (g \circ \nabla u)(t) + \frac{\kappa}{2} \|\nabla u(\cdot, t)\|_2^2. \end{aligned}$$

Recalling Lemma 3.2, we get

$$\begin{aligned} E'(t) &\leq -\kappa E(t) + \frac{\kappa}{2} (g \circ \nabla u)(t) + \frac{\kappa}{2} \left(c_4 \delta E(t) - \frac{C_\delta}{c_0} E'(t) + c_5 (g \circ \nabla u)(t) \right) \\ &\leq -\kappa \left(1 - \frac{c_4}{2} \delta \right) E(t) - \frac{\kappa C_\delta}{2c_0} E'(t) + \frac{\kappa(1+c_5)}{2} (g \circ \nabla u)(t). \end{aligned}$$

Then we have

$$\left(1 + \frac{\kappa C_\delta}{2c_0} \right) E'(t) \leq -\kappa \left(1 - \frac{c_4}{2} \delta \right) E(t) + \frac{\kappa(1+c_5)}{2} (g \circ \nabla u)(t).$$

By choosing δ small enough, we obtain, for two positive constants λ and γ ,

$$E'(t) \leq -\lambda E(t) + \gamma (g \circ \nabla u)(t). \quad (3.14)$$

Case of $p = 1$ Multiplying (3.14) by $\xi(t)$ and exploiting (G_2) , we get

$$\begin{aligned} \xi(t) E'(t) &\leq -\lambda \xi(t) E(t) + \gamma (\xi g \circ \nabla u)(t) \\ &\leq -\lambda \xi(t) E(t) - \gamma (g' \circ \nabla u)(t) \\ &\leq -\lambda \xi(t) E(t) - \gamma E'(t). \end{aligned} \quad (3.15)$$

We then set $L = (\xi + \gamma)E \sim E$ to obtain, from (3.15),

$$L'(t) \leq -\lambda \xi(t) E(t) \leq -\lambda_1 \xi(t) L(t). \quad (3.16)$$

A simple integration of (3.16) leads to

$$L(t) \leq C e^{-\lambda_1 \int_0^t \xi(s) ds}.$$

This gives (3.9), by virtue of $L \sim E$.

Case of $p > 1$ To establish (3.10), we again consider (3.14) and use Lemma 2.1 to get

$$\xi(t) E'(t) \leq -\lambda \xi(t) E(t) + C (-E'(t))^{\frac{1}{2p-1}}.$$

Multiplication of the last inequality by $\xi^\alpha E^\alpha(t)$, where $\alpha = 2p - 2 > 0$, gives

$$\frac{1}{\alpha+1} \xi^{\alpha+1} \frac{d}{dt} E^{\alpha+1}(t) \leq -\lambda \xi^{\alpha+1}(t) E^{\alpha+1}(t) + c (\xi E)^\alpha(t) (-E'(t))^{\frac{1}{\alpha+1}}.$$

Use of Young's inequality, with $q = \alpha + 1$ and $q^* = \frac{\alpha+1}{\alpha}$, yields, for any $\varepsilon > 0$,

$$\begin{aligned} \frac{1}{\alpha+1} \xi^{\alpha+1} \frac{d}{dt} E^{\alpha+1}(t) &\leq -\lambda \xi^{\alpha+1}(t) E^{\alpha+1}(t) + C (\varepsilon \xi^{\alpha+1}(t) E^{\alpha+1}(t) - C_\varepsilon E'(t)) \\ &= -(\lambda - \varepsilon C) \xi^{\alpha+1}(t) E^{\alpha+1}(t) - C_\varepsilon E'(t). \end{aligned}$$

We then choose $0 < \varepsilon < \frac{\lambda}{C}$ and recall that $\xi' \leq 0$, to obtain, for $c_6 > 0$,

$$(\xi^{\alpha+1} E^{\alpha+1}(t))'(t) \leq \xi^{\alpha+1} \frac{d}{dt} E^{\alpha+1}(t) \leq -c_6 \xi^{\alpha+1}(t) E^{\alpha+1}(t) - C E'(t);$$

which implies

$$(\xi^{\alpha+1} E^{\alpha+1} + C E)'(t) \leq -c_6 \xi^{\alpha+1}(t) E^{\alpha+1}(t).$$

Let $W = \xi^{\alpha+1} E^{\alpha+1} + C E \sim E$. Then

$$W'(t) \leq -C \xi^{\alpha+1}(t) W^{\alpha+1}(t) = -C \xi^{2p-1}(t) W^{2p-1}(t).$$

Integrating over $(0, t)$ and using the fact that $W \sim E$, we obtain, for some $\lambda_0 > 0$,

$$E(t) \leq \lambda_0 \left(\int_0^t \xi^{2p-1}(s) ds + 1 \right)^{\frac{-1}{2p-2}};$$

so (3.10) holds

To establish (3.12), we put

$$\eta(t) = \int_0^t \|\nabla u(t) - \nabla u(t-s)\|_2^2 ds.$$

Using Remark 3.1, we have

$$\begin{aligned} \eta(t) &\leq 2 \int_0^t (\|\nabla u(t)\|_2^2 + \|\nabla u(t-s)\|_2^2) ds \\ &\leq \frac{4}{1-l} \int_0^t (E(t) + E(t-s)) ds \\ &= \frac{8}{1-l} \int_0^t E(s) ds < \frac{8}{1-l} \int_0^{+\infty} E(s) ds < +\infty. \end{aligned}$$

This implies that

$$\sup_{t \in \mathbb{R}^+} \eta^{1-\frac{1}{p}}(t) < +\infty. \quad (3.17)$$

Assume that $\eta(t) > 0$. Then, from (3.14), we find

$$\begin{aligned} \xi(t) E'(t) &\leq -\lambda \xi(t) E(t) + \gamma \xi(t) (g \circ \nabla u)(t) \\ &= -\lambda \xi(t) E(t) + \gamma \frac{\eta(t)}{\eta(t)} \int_0^t (\xi^p(s) g^p(s))^{\frac{1}{p}} \|\nabla u(t) - \nabla u(t-s)\|_2^2 ds. \end{aligned} \quad (3.18)$$

Applying Jensen's inequality (2.4) for the second term of the right-hand side of (3.18), with

$$\Omega = [0, t], \quad f(s) = \xi^p(s) g^p(s) \quad \text{and} \quad h(s) = \|\nabla u(t) - \nabla u(t-s)\|_2^2,$$

to get

$$\xi(t) E'(t) \leq -\lambda \xi(t) E(t) + \gamma \eta(t) \left(\frac{1}{\eta(t)} \int_0^t \xi^p(s) g^p(s) \|\nabla u(t) - \nabla u(t-s)\|_2^2 ds \right)^{\frac{1}{p}}.$$

Therefore, using (3.17) we obtain

$$\begin{aligned}\xi(t)E'(t) &\leq -\lambda\xi(t)E(t) + \gamma\eta^{1-\frac{1}{p}}(t) \left(\xi^{p-1}(0) \int_0^t \xi(s)g^p(s) \|\nabla u(t) - \nabla u(t-s)\|_2^2 ds \right)^{\frac{1}{p}} \\ &\leq -\lambda\xi(t)E(t) + C(-g' \circ \nabla u)^{\frac{1}{p}}(t),\end{aligned}$$

and then

$$\xi(t)E'(t) \leq -\lambda\xi(t)E(t) + C(-E'(t))^{\frac{1}{p}}. \quad (3.19)$$

If $\eta(t) = 0$, then $s \rightarrow \nabla u(s)$ is a constant function on $[0, t]$. Therefore

$$(g \circ \nabla u)(t) = 0,$$

and hence we have, from (3.14),

$$E'(t) \leq -\lambda E(t),$$

which implies (3.19).

Now, multiplying (3.19) by $\xi^\alpha(t)E^\alpha(t)$, for $\alpha = p-1$, and repeating the same computations as in above, we arrive at, for some $\lambda_0 > 0$,

$$E(t) \leq \lambda_0 \left(\int_0^t \xi^p(s) ds + 1 \right)^{\frac{-1}{p-1}}.$$

This completes the proof of our main result. \square

The following examples illustrate our result and show the optimal decay rate in the polynomial case:

Example 3.5 Let $g(t) = a(1+t)^{-\nu}$, where $\nu > 2$, and $a > 0$ so that

$$\int_0^{+\infty} g(t) dt < 1. \quad (3.20)$$

We have

$$g'(t) = -a\nu(1+t)^{-\nu-1} = -b \left(a(1+t)^{-\nu} \right)^{\frac{\nu+1}{\nu}},$$

where $b = \nu a^{-\frac{1}{\nu}}$. Then (G2) holds with $\xi(t) = b$ and $p = \frac{\nu+1}{\nu} \in (1, \frac{3}{2})$. Therefore (3.11) yields

$$\int_0^{+\infty} (b^{2p-1}t + 1)^{\frac{-1}{2p-2}} dt < +\infty,$$

and hence, by (3.12), we get

$$E(t) \leq C(1+t)^{\frac{-1}{p-1}} = C(1+t)^{-\nu},$$

which is the optimal decay.

Example 3.6 Let $g(t) = ae^{-(1+t)^\nu}$, where $0 < \nu \leq 1$, and $a > 0$ is chosen so that (3.20) holds. Then

$$g'(t) = -a\nu(1+t)^{\nu-1}e^{-(1+t)^\nu}.$$

Therefore (G2) holds with $p = 1$ and $\xi(t) = \nu(1+t)^{\nu-1}$. Consequently, we can use (3.9) to deduce

$$E(t) \leq Ce^{-\lambda(1+t)^\nu}.$$

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