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Abstract	In this paper, we consider a vibrating system of Timoshenko-type in a bounded one-dimensional domain with discrete time delay and complementary frictional damping and infinite memory controls all acting on the transversal displacement. We show that the system is well-posed in the sense of semigroup and that, under appropriate assumptions on the weights of the delay and the history data, the stability of the system holds in case of the equal-speed propagation as well as in the opposite case in spite of the presence of a discrete time delay, where the decay rate of solutions is given in terms of the smoothness of the initial data and the growth of the relaxation kernel at infinity. The results of this paper extend the ones obtained by the present author and Messaoudi in (Acta Math Sci 36:1–33, 2016) to the case of presence of discrete delay.	
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Keywords (separated by '-')	Well-posedness - General decay - Time delay - Infinite memory - Frictional damping - Viscoelastic - Timoshenko-type - Semigroup theory - Energy method	
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# Well-posedness and energy decay for Timoshenko systems with discrete time delay under frictional damping and/or infinite memory in the displacement

Aissa Guesmia<sup>1</sup>

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**Abstract** In this paper, we consider a vibrating system of Timoshenko-type in a bounded one-dimensional domain with discrete time delay and complementary frictional damping and infinite memory controls all acting on the transversal displacement. We show that the system is well-posed in the sens of semigroup and that, under appropriate assumptions on the weights of the delay and the history data, the stability of the system holds in case of the equal-speed propagation as well as in the opposite case in spite of the presence of a discrete time delay, where the decay rate of solutions is given in terms of the smoothness of the initial data and the growth of the relaxation kernel at infinity. The results of this paper extend the ones obtained by the present author and Messaoudi in (Acta Math Sci 36:1–33, 2016) to the case of presence of discrete delay.

**Keywords** Well-posedness · General decay · Time delay · Infinite memory · Frictional damping · Viscoelastic · Timoshenko-type · Semigroup theory · Energy method

**Mathematics Subject Classification** 35B37 · 35L55 · 74D05 · 93D15 · 93D20

## 1 Introduction

In this paper, we are concerned with the well-posedness and the long-time behavior of the solution of the following Timoshenko system:

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$$\begin{cases}
 \rho_1 \varphi_{tt}(x, t) - k_1(\varphi_x(x, t) + \psi(x, t))_x + d(x)\varphi_t(x, t - \tau) + b(x)\varphi_t(x, t) \\
 \quad + \int_0^{+\infty} g(s)(a(x)\varphi_x(x, t - s))_x ds = 0, \\
 \rho_2 \psi_{tt}(x, t) - k_2 \psi_{xx}(x, t) + k_1(\varphi_x(x, t) + \psi(x, t)) = 0, \\
 \varphi(0, t) = \psi_x(0, t) = \varphi(L, t) = \psi_x(L, t) = 0, \\
 \varphi(x, -t) = \varphi_0(x, t), \quad \varphi_t(x, 0) = \varphi_1(x), \quad \varphi_t(x, -\tau p) = f_0(x, -\tau p), \\
 \psi(x, 0) = \psi_0(x), \quad \psi_t(x, 0) = \psi_1(x),
 \end{cases} \quad (1.1)$$

for  $(x, t, p) \in ]0, L[ \times ]0, +\infty[ \times ]0, 1[$ ,  $d : [0, L] \rightarrow \mathbb{R}$ ,  $a, b : [0, L] \rightarrow \mathbb{R}_+$  and  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  are given functions (to be specified later), where  $\mathbb{R}_+ = [0, +\infty[$ ,  $L$ ,  $\tau$ ,  $\rho_i$ ,  $k_i$  ( $i = 1, 2$ ) are positive constants,

$\varphi_0 : ]0, L[ \rightarrow \mathbb{R}$ ,  $\varphi_1, \psi_0, \psi_1 : ]0, L[ \rightarrow \mathbb{R}$  and  $f_0 : ]0, L[ \times ]-\tau, 0[ \rightarrow \mathbb{R}$

are given initial data, and

$$(\varphi, \psi) : ]0, L[ \times ]0, +\infty[ \rightarrow \mathbb{R}^2$$

is the state of (1.1). A subscript  $y$  and the notation  $\partial_y$  denote the derivative with respect to  $y$ . We also use the prime notation to denote the derivative when the function has only one variable. The infinite integral in (1.1),  $b(x)\varphi_t(x, t)$  and  $d(x)\varphi_t(x, t - \tau)$  represent, respectively, the infinite memory, the frictional damping and the discrete time delay. For simplicity of notation, the space and time variables are used only when it is necessary to avoid ambiguity.

Our aim is the study of the well-posedness and asymptotic behavior of the solutions of (1.1) in case of the equal-speed propagation

$$\frac{k_1}{\rho_1} = \frac{k_2}{\rho_2} \quad (1.2)$$

as well as in the opposite case. The equality (1.2) means that the first two equations in (1.1) have the same speeds of wave propagation  $\sqrt{\frac{k_1}{\rho_1}}$  and  $\sqrt{\frac{k_2}{\rho_2}}$ , respectively.

Timoshenko [69], in 1921, introduced the following model to describe the transverse vibration of a beam:

$$\begin{cases}
 \rho u_{tt} = (K(u_x - \varphi))_x, & \text{in } ]0, L[ \times ]0, +\infty[, \\
 I_\rho \varphi_{tt} = (EI\varphi_x)_x + K(u_x - \varphi), & \text{in } ]0, L[ \times ]0, +\infty[,
 \end{cases} \quad (1.3)$$

where  $t$  denotes the time variable and  $x$  is the space variable along the beam of length  $L$ , in its equilibrium configuration,  $u$  is the transverse displacement of the beam and  $\varphi$  is the rotation angle of the filament of the beam. The coefficients  $\rho$ ,  $I_\rho$ ,  $E$ ,  $I$  and  $K$  are, respectively, the density (the mass per unit length), the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus. Since then, this model has attracted the attention of many researchers and an important amount of work has been devoted to the issue of the stabilization and the search for the minimum dissipation by which the solutions decay uniformly to the stable state as time goes to infinity. To achieve this goal, diverse types of dissipative mechanisms have been used and several stability results have been obtained. We mention some of these results (for more results, we refer the reader to the list of references of this paper, which is not exhaustive, and the references therein).

**Absence of delay:**  $d \equiv 0$ . In the case of presence of controls on both the rotation angle and the transverse displacement, investigations showed that the Timoshenko systems are stable without any restriction on the constants  $\rho_1$ ,  $\rho_2$ ,  $k_1$  and  $k_2$ . In this regards, many decay

estimates were obtained; see [26,31,39,40,56]. However, in the case of only one control on the rotation angle, the rate of decay depends heavily on the constants  $\rho_1$ ,  $\rho_2$ ,  $k_1$  and  $k_2$  and the regularity of the initial data. Precisely, if (1.2) holds, the results obtained are similar to those established for the case of the presence of controls in both equations. We quote in this regard [2,7,14,21–24,26,41,42,45–47,63]. But, if (1.2) does not hold, a situation which is more interesting from the physics point of view, then it has been shown that the Timoshenko system is not exponentially stable even for exponentially decaying relaxation functions or linear frictional damping, and only weak decay estimates can be obtained for regular solutions in the presence of dissipation. This has been demonstrated in [2,14,23,24,26,43], for the case of finite or infinite memory, and in [17,22], for complementary frictional damping and finite or infinite memory acting on the rotation angle equation. We also refer the reader to [55] (and its references) concerning the stability of Timoshenko-type systems in  $\mathbb{R}$  (instead of  $]0, L[$ ) with controls acting on the rotation angle.

For the stability of Timoshenko systems via heat effect, we mention the pioneer work [44] devoted to the study of the following system:

$$\begin{cases} \rho_1 \varphi_{tt} - \sigma(\varphi_x, \psi)_x = 0, & \text{in } ]0, L[ \times ]0, +\infty[, \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma\theta_x = 0, & \text{in } ]0, L[ \times ]0, +\infty[, \\ \rho_3 \theta_t - k\theta_{xx} + \gamma\psi_{tx} = 0, & \text{in } ]0, L[ \times ]0, +\infty[, \end{cases} \quad (1.4)$$

where  $\theta$  denotes the temperature difference. In their work, Rivera and Racke [44] established, under appropriate conditions on the function  $\sigma$  and the positive constants  $\rho_i$ ,  $b$ ,  $k$  and  $\gamma$ , several exponential decay results for the linearized system with various boundary conditions. They also proved a non-exponential stability result for the case of non-equal speed of propagation, and proved an exponential decay result for the nonlinear case. Guesmia et al. [27] discussed a linear version of (1.4) and completed the work of [44] by establishing some polynomial decay results in the case of non-equal speed of propagation.

In (1.4), the heat flux is given by Fourier's law. As a result, this theory predicts an infinite speed of heat propagation; that is, any thermal disturbance at one point has an instantaneous effect elsewhere in the body. Experiments showed that heat conduction in some dielectric crystals at low temperatures is free of this paradox and disturbances, which are almost entirely thermal, propagate in a finite speed. This phenomenon in dielectric crystals is called second sound. To overcome this physical paradox, many theories have merged. One of which suggests that we should replace Fourier's law by Cattaneo's law. In line with this theory, (1.4), in its linear form, becomes

$$\begin{cases} \rho_1 \varphi_{tt} - k_1(\varphi_x + \psi)_x = 0, & \text{in } ]0, L[ \times ]0, +\infty[, \\ \rho_2 \psi_{tt} - k_2 \psi_{xx} + k_1(\varphi_x + \psi) + \delta\theta_x = 0, & \text{in } ]0, L[ \times ]0, +\infty[, \\ \rho_3 \theta_t + \gamma q_x + \delta\psi_{tx} = 0, & \text{in } ]0, L[ \times ]0, +\infty[, \\ \tau q_t + q + k\theta_x = 0, & \text{in } ]0, L[ \times ]0, +\infty[, \end{cases} \quad (1.5)$$

where  $q$  denotes the heat flux. Fernández Sare and Racke [15] studied (1.5) and proved that (1.2) is no longer sufficient to obtain exponential stability even in the presence of an extra viscoelastic dissipation of the form  $\int_0^{+\infty} g(s)\psi_{xx}(t-s)ds$  in the second equation. Very recently, Santos et al. [62] considered (1.5), introduced a new stability number

$$\chi = \left( \tau - \frac{\rho_1}{k_1 \rho_3} \right) \left( \rho_2 - \frac{k_2 \rho_1}{k_1} \right) - \frac{\tau \rho_1 \delta^2}{k_1 \rho_3} \quad (1.6)$$

and used the semigroup method to obtain an exponential decay result, for  $\chi = 0$ , and a polynomial decay, for  $\chi \neq 0$ . See, also [26, 29, 30, 39, 54, 58, 59]. Notice that, when  $\tau = 0$  (Fourier's law),  $\chi = 0$  if and only if (1.2) holds.

In all above mentioned works, the stability was either via both equation control or the angular rotation equation control. Recently, Almeida Júnior et al. [4] considered the situation when the control is only on the transverse displacement equation, which is more realistic from the physics point of view. Precisely, they looked into the following system:

$$\begin{cases} \rho_1 \varphi_{tt} - k_1(\varphi_x + \psi)_x + \mu \varphi_t = 0, & \text{in } ]0, L[ \times ]0, +\infty[, \\ \rho_2 \psi_{tt} - k_2 \psi_{xx} + k_1(\varphi_x + \psi) = 0, & \text{in } ]0, L[ \times ]0, +\infty[, \end{cases} \quad (1.7)$$

where  $\mu$  is a positive constant, and showed that the linear frictional damping  $\mu \varphi_t$  is strong enough to obtain exponential stability of (1.7) provided that (1.2) holds. They, also, proved some non-exponential and polynomial decay results in the case of non-equal speed situation. The results of [4] were, very recently, extended in [25] to the case where the linear frictional damping  $\mu \varphi_t$  is replaced by a nonlinear one and/or an infinite memory. The same authors of [4] considered in [5]

$$\begin{cases} \rho_1 \varphi_{tt} - \kappa(\varphi_x + \psi)_x + \sigma \theta_x = 0, & \text{in } ]0, L[ \times ]0, +\infty[, \\ \rho_2 \psi_{tt} - b \psi_{xx} + \kappa(\varphi_x + \psi) - \sigma \theta = 0, & \text{in } ]0, L[ \times ]0, +\infty[, \\ \rho_3 \theta_t - \gamma \theta_{xx} + \sigma(\varphi_x + \psi)_t = 0, & \text{in } ]0, L[ \times ]0, +\infty[, \end{cases} \quad (1.8)$$

with various boundary conditions, and established the exponential stability of (1.8) for equal-speed case, and non-exponential stability for the opposite case. In the case of lack of exponential stability, they proved some algebraic (polynomial) stability for strong solutions.

**Presence of delay:**  $d \neq 0$ . The questions related to well-posedness and stability/instability of Timoshenko-type systems as well as evolution equations with time delay have attracted considerable attention in recent years and many researchers have shown that the time delay can destabilize a system that was asymptotically stable in the absence of time delay.

When the delay and controls are present on the rotation angle equation, we mention the following Timoshenko system:

$$\begin{cases} \rho_1 \varphi_{tt} - k_1(\varphi_x + \psi)_x = 0, \\ \rho_2 \psi_{tt} - k_2 \psi_{xx} + k_1(\varphi_x + \psi) + \int_0^t g(s) \psi_{xx}(t-s) ds + \mu_1 \psi_t + \mu_2 \psi_t(t-\tau) = 0, \end{cases} \quad (1.9)$$

in  $]0, L[ \times ]0, +\infty[$ , studied in [57], where  $\mu_1, \mu_2$  and  $\tau$  are fixed non-negative constants. The author of [57] proved the stability of (1.9) under the assumptions (1.2) and  $0 < \mu_2 \leq \mu_1$ , where the decay rate of solutions depends on the one of  $g$ . The obtained stability results in [57] generalize the ones of [60] concerning (1.9) in the case  $g \equiv 0$  and  $0 < \mu_2 < \mu_1$ , and they were generalized in [32] to the case  $g \equiv 0$  and variable time delay  $\tau(t)$ . In [61], the stability of Timoshenko systems with two internal time delays and two boundary linear feedbacks was proved under some smallness conditions on  $L$  and the weights of the delays.

When no frictional damping is present, the stability of this Timoshenko system

$$\begin{cases} \rho_1 \varphi_{tt} - k_1(\varphi_x + \psi)_x = 0, \\ \rho_2 \psi_{tt} - k_2 \psi_{xx} + k_1(\varphi_x + \psi) + \int_0^{+\infty} g(s) \psi_{xx}(t-s) ds + D(\psi) = 0, \end{cases} \quad (1.10)$$

in  $]0, L[ \times ]0, +\infty[$ , was proved in [20], in both discrete time delay case

$$D(\psi) = \mu_2 \psi_t(t - \tau)$$

and distributed one

$$D(\psi) = \int_0^{+\infty} f(s) \psi_t(t - s) ds,$$

where  $\mu_2 \in \mathbb{R}^*$  and  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$  is a given function. In contrast to the situation of absence of delay and/or presence of frictional damping, (1.10) is not necessarily dissipative with respect to its classical energy. To overcome subsequently the difficulties generated by the non-dissipativeness character of (1.10), some new functionals were introduced in [20] to get crucial estimates on some terms generated by the time delay and the infinite memory. The results of [20] generalizes the ones of [18] concerning the particular case  $D(\psi) = \mu_2 \psi_t(t - \tau)$  and  $g$  converges exponentially to zero at infinity.

Similar stability results for various hyperbolic evolution equations with frictional damping and/or memory and/or time delay exist in the literature, in this regard, we refer the reader to [1, 3, 6, 8–10, 12, 13, 16, 19, 28, 34–38, 48–52, 64–68].

As far as we know, the problem of stability of Timoshenko system with a time delay under infinite memory and/or frictional damping all acting on the transversal displacement has never been treated in the literature. Our goal in this paper is to investigate the effect of each control on the asymptotic behavior of the solutions of (1.1) in the presence of a time delay, and on the decay rate of its energy, when both controls are acting cooperatively, allowing each control to vanish on the whole domain. To our best knowledge, this situation has never been considered before in the literature. Under appropriate assumptions on the history data  $\varphi_0$ , we give an explicit characterization of the decay rate of solutions depending on the growth of  $g$  at infinity and the following relations between the weights  $b$  and  $d$  of, respectively, the frictional damping and time delay:

$$\inf_{[0, L]} (b - |d|) > 0 \quad (1.11)$$

and

$$\inf_{[0, L]} (b - |d|) \leq 0. \quad (1.12)$$

Contrarily to the case (1.11), system (1.1) is not necessarily dissipative with respect to its classical energy when (1.12) holds (see (4.1) and (4.2) below). This creates some difficulties and, so, we prove the exponential stability of (1.1) provided that (1.2) holds,  $g$  converges exponentially to zero at infinity and  $\|d\|_\infty$  is small enough. In the case when (1.11) holds, we give two general decay estimates (corresponding to the case (1.2) and the opposite one) depending on the smoothness of initial data and growth of  $g$  at infinity characterized by the condition (2.9) below introduced in [16]. These results give a generalization of the ones proved by the present author and Messaoudi in [25] concerning the case  $d \equiv 0$ .

The proof of the well-posedness is based on the maximal monotone operators and semi-group approach (see, for example [33, 53]). However, the proof of stability estimates is based on the multiplier method combined with some integral or differential inequalities (see, for example [1, 3, 10, 33–37]) and an approach introduced in [16, 19], for a class of abstract hyperbolic systems of single or coupled equations with one infinite memory. In the case when (1.2) does not hold, we use also some ideas given in [3, 14, 17] to get the decay rate of solutions in terms of the regularity of initial data and the general growth of  $g$  at infinity.



The paper is organized as follows. In Sect. 2, we set up the hypotheses and present our well-posedness and stability results. In Sect. 3, we prove the well-posedness result. In Sect. 4, we establish some lemmas needed for the proof of the stability results which will be completed in Sect. 5 when (1.2) and (1.11) hold, in Sect. 6 when (1.2) and (1.12) hold, and in Sect. 7 when (1.2) does not hold and (1.11) holds. Finally, some general comments and issues will be given in Sect. 8.

## 2 Preliminaries and obtained results

### 2.1 Hypotheses

We consider the following hypotheses:

(H1) The functions  $a, b : [0, L] \rightarrow \mathbb{R}_+$  and  $d : [0, L] \rightarrow \mathbb{R}$  are such that

$$a \in C^1([0, L]), \quad b, d \in C([0, L]), \quad (2.1)$$

$$\inf_{[0, L]} (a + b) > 0, \quad (2.2)$$

$$a \equiv 0 \quad \text{or} \quad \inf_{[0, L]} a > 0. \quad (2.3)$$

(H2) The function  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a non-increasing of class  $C^1(\mathbb{R}_+)$  such that  $g(0) > 0$  and

$$g_0 \|a\|_\infty < \frac{k_1 k_2}{k_0 k_1 + k_2}, \quad (2.4)$$

where  $g_0 = \int_0^{+\infty} g(s) ds$  and  $k_0$  is the smallest constant depending only on  $L$  and satisfying (Poincaré's inequality)

$$\int_0^L v^2 dx \leq k_0 \int_0^L v_x^2 dx, \quad \forall v \in H_*^1(]0, L[) \quad (2.5)$$

with

$$H_*^1(]0, L[) = \left\{ v \in H^1(]0, L[), \int_0^L v dx = 0 \right\}. \quad (2.6)$$

(H3) There exist a positive constant  $\alpha$  and an increasing strictly convex function  $G : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  of class  $C^1(\mathbb{R}_+) \cap C^2(]0, +\infty[)$  satisfying

$$G(0) = G'(0) = 0 \quad \text{and} \quad \lim_{t \rightarrow +\infty} G'(t) = +\infty \quad (2.7)$$

such that

$$g'(t) \leq -\alpha g(t), \quad \forall t \geq 0 \quad (2.8)$$

or

$$\int_0^{+\infty} \frac{g(t)}{G^{-1}(-g'(t))} dt + \sup_{t \in \mathbb{R}_+} \frac{g(t)}{G^{-1}(-g'(t))} < +\infty. \quad (2.9)$$

*Remark 2.1* 1. The hypothesis (2.9) was introduced in [16] and it allows a wider class of relaxation functions than the ones considered in [14, 43] (see examples given in [16, 26]).

197 2. As in [25], using the second equation and boundary conditions in (1.1), we easily verify  
198 that

$$199 \quad \partial_{tt} \left( \int_0^L \psi \, dx \right) + \frac{k_1}{\rho_2} \int_0^L \psi \, dx = 0.$$

200 By solving this ordinary differential equation and using the initial data of  $\psi$ , we find

$$201 \quad \int_0^L \psi \, dx = \left( \int_0^L \psi_0 \, dx \right) \cos \left( \sqrt{\frac{k_1}{\rho_2}} t \right) + \sqrt{\frac{\rho_2}{k_1}} \left( \int_0^L \psi_1 \, dx \right) \sin \left( \sqrt{\frac{k_1}{\rho_2}} t \right). \quad (2.10)$$

202 Let

$$203 \quad \tilde{\psi} = \psi - \frac{1}{L} \left( \int_0^L \psi_0 \, dx \right) \cos \left( \sqrt{\frac{k_1}{\rho_2}} t \right) - \frac{1}{L} \sqrt{\frac{\rho_2}{k_1}} \left( \int_0^L \psi_1 \, dx \right) \sin \left( \sqrt{\frac{k_1}{\rho_2}} t \right). \quad (2.11)$$

204 Then, one can easily check that

$$205 \quad \int_0^L \tilde{\psi} \, dx = 0, \quad (2.12)$$

206 and, hence, Poincaré's inequality (2.5) is applicable for  $\tilde{\psi}$  provided that  $\tilde{\psi} \in H^1(]0, L[)$ .  
207 In addition,  $(\varphi, \tilde{\psi})$  satisfies (1.1) with initial data

$$208 \quad \tilde{\psi}_0 = \psi_0 - \frac{1}{L} \int_0^L \psi_0 \, dx \quad \text{and} \quad \tilde{\psi}_1 = \psi_1 - \frac{1}{L} \int_0^L \psi_1 \, dx$$

209 instead of  $\psi_0$  and  $\psi_1$ , respectively. In the sequel, we work with  $\tilde{\psi}$  instead of  $\psi$ , but, for  
210 simplicity of notation, we use  $\psi$  instead of  $\tilde{\psi}$ .

211 3. Thanks to Poincaré's inequality (2.5) (applied for  $\psi \in H_*^1(]0, L[)$ ), we have

$$212 \quad k_1 \int_0^L (\varphi_x + \psi)^2 \, dx \geq k_1(1 - \hat{\epsilon}) \int_0^L \varphi_x^2 \, dx + k_0 k_1 \left( 1 - \frac{1}{\hat{\epsilon}} \right) \int_0^L \psi_x^2 \, dx, \quad (2.13)$$

213 for any  $0 < \hat{\epsilon} < 1$ . Then, according to (2.4), we can choose  $\hat{\epsilon} > 0$  such that

$$214 \quad \frac{k_0 k_1}{k_0 k_1 + k_2} < \hat{\epsilon} < \frac{1}{k_1} (k_1 - g_0 \|a\|_\infty)$$

215 and deduce from (2.13) that

$$216 \quad \hat{k} \int_0^L (\varphi_x^2 + \psi_x^2) \, dx \leq \int_0^L (-g_0 \|a\|_\infty \varphi_x^2 + k_2 \psi_x^2 + k_1 (\varphi_x + \psi)^2) \, dx, \quad (2.14)$$

217 where  $\hat{k} = \min \{ k_1(1 - \hat{\epsilon}) - g_0 \|a\|_\infty, k_2 + k_0 k_1(1 - \frac{1}{\hat{\epsilon}}) \} > 0$ .

218 Because  $\int_0^L \varphi_x^2 \, dx$  and  $\int_0^L \psi_x^2 \, dx$  define norms, for  $\varphi$  and  $\psi$  on  $H_0^1(]0, L[)$  and  
219  $H_*^1(]0, L[)$ , respectively, then

$$220 \quad \int_0^L (-g_0 \|a\|_\infty \varphi_x^2 + k_2 \psi_x^2 + k_1 (\varphi_x + \psi)^2) \, dx$$

221 defines a norm on  $H_0^1(]0, L[) \times H_*^1(]0, L[)$ , for  $(\varphi, \psi)$ , equivalent to the one induced  
222 by  $(H^1(]0, L[))^2$ .

## 2.2 Well-posedness

We give here a brief idea about the formulation of (1.1) into an abstract first order system and the related existence, uniqueness and smoothness of solution. Following the ideas of [11, 48], let

$$\eta(x, t, s) = \varphi(x, t) - \varphi(x, t - s), \quad \text{for } (x, t, s) \in ]0, L[ \times ]0, +\infty[ \times ]0, +\infty[ \quad (2.15)$$

and

$$z(x, t, p) = \varphi_t(x, t - \tau p), \quad \text{for } (x, t, p) \in ]0, L[ \times ]0, +\infty[ \times ]0, 1[. \quad (2.16)$$

Then

$$\begin{cases} \eta_t + \eta_s - \varphi_t = 0, & \text{in } ]0, L[ \times ]0, +\infty[ \times ]0, +\infty[, \\ \eta(0, t, s) = \eta(L, t, s) = 0, & \text{in } ]0, +\infty[ \times ]0, +\infty[, \\ \eta(x, t, 0) = 0, & \text{in } ]0, L[ \times ]0, +\infty[, \end{cases} \quad (2.17)$$

$$\begin{cases} \tau z_t + z_p = 0, & \text{in } ]0, L[ \times ]0, +\infty[ \times ]0, 1[, \\ z(x, t, 0) = \varphi_t(x, t), & \text{in } ]0, L[ \times ]0, +\infty[, \\ z(x, t, 1) = \varphi_t(x, t - \tau), & \text{in } ]0, L[ \times ]0, +\infty[ \end{cases} \quad (2.18)$$

and

$$\begin{cases} \eta_0(x, s) := \eta(x, 0, s) = \varphi_0(x, 0) - \varphi_0(x, s), & \text{in } ]0, L[ \times ]0, +\infty[, \\ z_0(x, p) := z(x, 0, p) = f_0(x, -\tau p), & \text{in } ]0, L[ \times ]0, 1[. \end{cases}$$

Let

$$U = (\varphi, \psi, \varphi_t, \psi_t, z, \eta)^T, \quad (2.19)$$

$$U_0 = (\varphi_0(\cdot, 0), \psi_0, \varphi_1, \psi_1, z_0, \eta_0)^T \quad (2.20)$$

and

$$\mathcal{H} = H_0^1(]0, L]) \times H_*^1(]0, L]) \times L^2(]0, L]) \times L_*^2(]0, L]) \times L_\xi \times L_g, \quad (2.21)$$

where

$$L_*^2(]0, L]) = \left\{ v \in L^2(]0, L]), \int_0^L v \, dx = 0 \right\}, \quad (2.22)$$

$$L_g = \left\{ v : \mathbb{R}_+ \rightarrow H_0^1(]0, L]), \int_0^L a \int_0^{+\infty} g(s) v_x^2(s) \, ds \, dx < +\infty \right\}, \quad (2.23)$$

$$L_\xi = \left\{ v : ]0, 1[ \rightarrow L^2(]0, L]), \int_0^L \xi \int_0^1 v^2(p) \, dp \, dx < +\infty \right\} \quad (2.24)$$

and  $\xi : [0, L] \rightarrow \mathbb{R}_+$  defined by

$$\xi = \begin{cases} \tau b & \text{if (1.11) holds and } d \neq 0, \\ \tau \|d\|_\infty & \text{if (1.12) holds or } d \equiv 0. \end{cases} \quad (2.25)$$

The spaces  $L_g$  and  $L_\xi$  endowed with the inner products

$$\langle v, w \rangle_{L_g} = \int_0^L a \int_0^{+\infty} g(s) v_x(s) w_x(s) \, ds \, dx$$

Author Proof

250 and

$$251 \quad \langle v, w \rangle_{L_\xi} = \int_0^L \xi \int_0^1 v(p)w(p) dp dx$$

252 are Hilbert spaces by virtue of the following Poincaré’s inequality:

$$253 \quad \exists \tilde{k}_0 > 0 : \int_0^L v^2 dx \leq \tilde{k}_0 \int_0^L v_x^2 dx, \quad \forall v \in H_0^1(]0, L[) \quad (2.26)$$

254 and the fact that  $a > 0$  if  $a \neq 0$  (according to (2.3)), and  $\xi > 0$  if  $d \neq 0$  (by virtue of (2.25)).  
 255 The space  $\mathcal{H}$  is equipped with the inner product defined by

$$256 \quad \langle V, W \rangle_{\mathcal{H}} = \langle v_6, w_6 \rangle_{L_g} + \langle v_5, w_5 \rangle_{L_\xi} + k_1 \int_0^L (\partial_x v_1 + v_2)(\partial_x w_1 + w_2) dx$$

$$257 \quad + \int_0^L (-g_0 a \partial_x v_1 \partial_x w_1 + k_2 \partial_x v_2 \partial_x w_2 + \rho_1 v_3 w_3 + \rho_2 v_4 w_4) dx,$$

258 for any  $V = (v_1, \dots, v_6)^T \in \mathcal{H}$  and  $W = (w_1, \dots, w_6)^T \in \mathcal{H}$ . Because  $L_g$  and  $L_\xi$  are  
 259 Hilbert spaces, then also  $\mathcal{H}$  is a Hilbert space according to (2.14).

260 Now, we define the linear operators  $B$  and  $A$  by

$$261 \quad B(v_1, \dots, v_6)^T = -\frac{\xi_0}{\rho_1} (0, 0, v_3, 0, 0, 0)^T, \quad (2.27)$$

262 where

$$263 \quad \xi_0 = \begin{cases} 0 & \text{if (1.11) holds,} \\ \|d\|_\infty & \text{if (1.12) holds} \end{cases} \quad (2.28)$$

264 and

$$265 \quad A \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix} = \begin{pmatrix} -v_3 \\ -v_4 \\ -\frac{k_1}{\rho_1} \partial_x (\partial_x v_1 + v_2) + \frac{g_0}{\rho_1} \partial_x (a \partial_x v_1) + \frac{b+\xi_0}{\rho_1} v_3 + \frac{d}{\rho_1} v_5(1) - \frac{1}{\rho_1} \int_0^{+\infty} g(s) \partial_x (a \partial_x v_6(s)) ds \\ -\frac{k_2}{\rho_2} \partial_{xx} v_2 + \frac{k_1}{\rho_2} (\partial_x v_1 + v_2) \\ \frac{1}{\tau} \partial_p v_5 \\ -v_3 + \partial_s v_6 \end{pmatrix}.$$

266 The system (1.1) is equivalent to

$$267 \quad \begin{cases} U'(t) + (A + B)U(t) = 0 & \text{on } ]0, +\infty[, \\ U(0) = U_0. \end{cases} \quad (2.29)$$

268 The domain of  $B$  is given by  $D(B) = \mathcal{H}$ . However, the domain of  $A$  is defined by

$$269 \quad D(A) = \{V = (v_1, \dots, v_6)^T \in \mathcal{H}, AV \in \mathcal{H}, \partial_x v_2(0) = \partial_x v_2(L) = 0, v_5(0) = v_3, v_6(0) = 0\}$$

270 and it can be characterized by

$$271 \quad D(A) = \left\{ (v_1, \dots, v_6)^T \in H_0^1(]0, L[) \times \left( H^2(]0, L[) \cap H_*^1(]0, L[) \right) \times H_0^1(]0, L[) \right.$$

$$272 \quad \left. \times H_*^1(]0, L[) \times L_\xi \times L_g, k_1 \partial_{xx} v_1 - g_0 \partial_x (a \partial_x v_1) + \int_0^{+\infty} g(s) \partial_x (a \partial_x v_6(s)) ds \in L^2(]0, L[), \right.$$

$$273 \quad \left. \partial_p v_5 \in L_\xi, \partial_s v_6 \in L_g, \partial_x v_2(0) = \partial_x v_2(L) = 0, v_5(0) = v_3, v_6(0) = 0 \right\}.$$

274 We use the classical notation  $D(A^0) = \mathcal{H}$ ,  $D(A^1) = D(A)$  and

275 
$$D(A^n) = \{V \in D(A^{n-1}), AV \in D(A^{n-1})\}, \text{ for } n = 2, 3, \dots,$$

276 endowed with the graph norm  $\|V\|_{D(A^n)} = \sum_{k=0}^n \|A^k V\|_{\mathcal{H}}$ .

277 *Remark 2.2* If  $a \equiv 0$  (resp.  $d \equiv 0$ ), the variable  $\eta$  (resp.  $z$ ) is not considered, and therefore,  
 278 the corresponding components in the definition of  $U$ ,  $U_0$ ,  $\mathcal{H}$ ,  $B$ ,  $A$  and  $D(A)$  will not appear.

279 Our well-posedness result reads as follows:

280 **Theorem 2.3** Assume that (H1)–(H3) are satisfied. For any  $n \in \mathbb{N}$  and  $U_0 \in D(A^n)$ , the  
 281 system (2.29) has a unique solution

282 
$$U \in \cap_{k=0}^n C^{n-k}(\mathbb{R}_+; D(A^k)). \tag{2.30}$$

283 **2.3 Stability**

284 The energy functional associated with (1.1) is defined by

285 
$$E(t) := \frac{1}{2} \|U(t)\|_{\mathcal{H}}^2$$

286 
$$= \frac{1}{2} (g \circ \varphi_x)(t) + \frac{1}{2} \int_0^L \xi \int_0^1 \varphi_t^2(t - \tau p) dp dx$$

287 
$$+ \frac{1}{2} \int_0^L (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + k_1 (\varphi_x + \psi)^2 + k_2 \psi_x^2 - g_0 a \varphi_x^2) dx, \tag{2.31}$$

288 where

289 
$$(\phi \circ v)(t) = \int_0^L a \int_0^{+\infty} \phi(s) (v(t) - v(t-s))^2 ds dx, \tag{2.32}$$

290 for any  $v : \mathbb{R} \rightarrow L^2(]0, L[)$  and  $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ .

291 Now, we give our first stability result which concerns the case when (1.2) and (1.11) hold.

292 **Theorem 2.4** Assume that (1.2), (1.11) and (H1)–(H3) are satisfied and let  $U_0 \in \mathcal{H}$  such  
 293 that

294 
$$\sup_{t \in \mathbb{R}_+} \int_t^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))} \int_0^L \varphi_{0x}^2(s-t) dx ds < +\infty. \tag{2.33}$$

295 Then there exist positive constants  $\epsilon_0$ ,  $\alpha_1$  and  $\alpha_2$ , for which  $E$  satisfies

296 
$$E(t) \leq \alpha_1 \tilde{G}^{-1}(\alpha_2 t), \quad \forall t \in \mathbb{R}_+, \tag{2.34}$$

297 where  $\tilde{G}(t) = \int_t^1 \frac{1}{G_0(s)} ds$  and

298 
$$G_0(s) = \begin{cases} s & \text{if (2.8) holds,} \\ sG'(\epsilon_0 s) & \text{if (2.9) holds.} \end{cases} \tag{2.35}$$

299 *Remark 2.5* 1. Because  $\lim_{t \rightarrow 0^+} \tilde{G}(t) = +\infty$  (by virtue of (H3)), then (2.34) implies that

300 
$$\lim_{t \rightarrow +\infty} E(t) = 0. \tag{2.36}$$

Author Proof

2. In case (2.8),  $\tilde{G}(s) = -ln s$  and (2.34) is reduced to

$$E(t) \leq \alpha_1 e^{-\alpha_2 t}, \quad \forall t \in \mathbb{R}_+, \quad (2.37)$$

which is the best decay rate given by (2.34). For specific examples of decay rates given by (2.34), see [17].

Our second stability result concerns the case when (1.2) and (1.12) hold.

**Theorem 2.6** Assume that (1.2), (1.12), (H1) and (H2) are satisfied and

$$\inf_{[0, L]} a > 0 \quad \text{and} \quad (2.8) \quad \text{holds.} \quad (2.38)$$

Then there exists a positive constant  $d_0$  independent of  $d$  such that, if

$$\|d\|_\infty^2 + \|d\|_\infty < d_0, \quad (2.39)$$

then, for any  $U_0 \in \mathcal{H}$ , there exist positive constants  $\alpha_1$  and  $\alpha_2$ , for which  $E$  satisfies (2.37).

When (1.2) does not hold and (1.11) holds, we prove the following stability result:

**Theorem 2.7** Assume that (1.11) and (H1)–(H3) are satisfied. Let  $n \in \mathbb{N}^*$  and  $U_0 \in D(A^n)$  such that

$$\sup_{t \in \mathbb{R}_+} \max_{k=0, \dots, n} \int_t^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))} \int_0^L \left( \frac{\partial^k \varphi_{0x}(s-t)}{\partial s^k} \right)^2 dx ds < +\infty. \quad (2.40)$$

Then there exist positive constant  $\epsilon_0$  and  $c_n$  such that  $E$  satisfies

$$E(t) \leq G_n \left( \frac{c_n}{t} \right), \quad \forall t > 0, \quad (2.41)$$

where  $G_m(s) = G_1(sG_{m-1}(s))$ , for  $m = 2, \dots, n$  and  $s \in \mathbb{R}_+$ ,  $G_1 = G_0^{-1}$  and  $G_0$  is defined in (2.35).

**Remark 2.8** When (2.8) holds,  $G_n(s) = s^n$  and (2.41) becomes

$$E(t) \leq \frac{c_n}{t^n}, \quad \forall t > 0, \quad (2.42)$$

which is the best decay rate given by (2.41). For specific examples of decay rates given by (2.41), see [19].

### 3 Well-posedness

The proof of Theorem 2.3 is based on the semigroup approach by proving that  $A+B$  generates a  $C_0$ -semigroup in  $\mathcal{H}$ . We consider the case  $\inf_{[0, L]} a > 0$  and  $d \neq 0$ ; the proof in cases  $a \equiv 0$  and/or  $d \equiv 0$  is similar and simpler.

First, we prove that  $-A$  is dissipative. Let  $V = (v_1, \dots, v_6)^T \in D(A)$ . Exploiting the definition of  $D(A)$  and integrating by parts, we find

$$\begin{aligned} \langle -AV, V \rangle_{\mathcal{H}} &= -\frac{1}{2} \int_0^L a \int_0^{+\infty} g(s) \partial_s (\partial_x v_6(s))^2 ds dx - \frac{1}{2\tau} \int_0^L \xi \int_0^1 \partial_p (v_5(p))^2 dp dx \\ &\quad - \int_0^L (b + \xi_0) v_3^2 dx - \int_0^L dv_3 v_5(1) dx. \end{aligned}$$

331 Integrating by parts for the first two terms of the above equality, using Young’s inequality

$$332 \quad \lambda_1 \lambda_2 \leq \frac{\lambda}{2} \lambda_1^2 + \frac{1}{2\lambda} \lambda_2^2, \quad \forall \lambda_1, \lambda_2 \in \mathbb{R}, \quad \forall \lambda > 0 \quad (3.1)$$

333 (with  $\lambda_1 = |v_3|$ ,  $\lambda_2 = |v_5(1)|$  and  $\lambda = 1$ ) and noting that  $v_5(0) = v_3$  and  $v_6(0) = 0$  (from  
334 the definition of  $D(A)$ ), we get

$$335 \quad \langle -AV, V \rangle_{\mathcal{H}} \leq \frac{1}{2} \int_0^L a \int_0^{+\infty} g'(s) (\partial_x v_6(s))^2 ds dx$$

$$336 \quad + \int_0^L \left( -b - \xi_0 + \frac{\xi}{2\tau} + \frac{|d|}{2} \right) v_3^2 dx + \int_0^L \left( \frac{|d|}{2} - \frac{\xi}{2\tau} \right) v_5^2(1) dx. \quad (3.2)$$

337 The definitions (2.25) and (2.28) of  $\xi$  and  $\xi_0$  imply that, if (1.11) holds and  $d \neq 0$ ,

$$338 \quad -b - \xi_0 + \frac{\xi}{2\tau} + \frac{|d|}{2} = \frac{|d|}{2} - \frac{\xi}{2\tau} = \frac{|d| - b}{2} \leq 0,$$

339 and, if (1.12) holds or  $d = 0$ ,

$$340 \quad -b - \xi_0 + \frac{\xi}{2\tau} + \frac{|d|}{2} = -b + \frac{|d| - \|d\|_{\infty}}{2} \leq 0 \quad \text{and} \quad \frac{|d|}{2} - \frac{\xi}{2\tau} = \frac{|d| - \|d\|_{\infty}}{2} \leq 0.$$

341 Consequently, the last two integrals in (3.2) are non-positive. Therefore

$$342 \quad \langle -AV, V \rangle_{\mathcal{H}} \leq \frac{1}{2} \int_0^L a \int_0^{+\infty} g'(s) (\partial_x v_6(s))^2 ds dx \leq 0, \quad (3.3)$$

343 since  $g$  is non-increasing. Then  $-A$  is dissipative.

344 Second, we show that  $Id + A$  is surjective. For this purpose, let  $F = (f_1, \dots, f_6)^T \in \mathcal{H}$ ,  
345 we seek  $V = (v_1, \dots, v_6)^T \in D(A)$  satisfying

$$346 \quad (Id + A)V = F. \quad (3.4)$$

347 The first two equations in (3.4) are equivalent to

$$348 \quad \begin{cases} v_3 = v_1 - f_1, \\ v_4 = v_2 - f_2. \end{cases} \quad (3.5)$$

349 Using the first equation in (3.5), the last two equations in (3.4) are equivalent to

$$350 \quad \begin{cases} v_5 + \frac{1}{\tau} \partial_p v_5 = f_5, \\ v_6 + \partial_s v_6 = v_1 - f_1 + f_6, \end{cases} \quad (3.6)$$

351 then, by solving the ordinary differential equations (3.6) and noting that  $v_5(0) = v_3 = v_1 - f_1$   
352 and  $v_6(0) = 0$  (see definition of  $D(A)$ ), we get

$$353 \quad v_5 = \left( v_1 - f_1 + \tau \int_0^p f_5(y) e^{\tau y} dy \right) e^{-\tau p} = e^{-\tau p} v_1 - \left( f_1 - \tau \int_0^p f_5(y) e^{\tau y} dy \right) e^{-\tau p}$$

$$354 \quad (3.7)$$

355 and

$$356 \quad v_6 = \left( \int_0^s e^y (v_1 - f_1 + f_6(y)) dy \right) e^{-s} = (1 - e^{-s}) v_1 - \left( \int_0^s e^y (f_1 - f_6(y)) dy \right) e^{-s}.$$

$$357 \quad (3.8)$$

358 We see that, if

$$359 \quad (v_1, v_2) \in H_0^1(]0, L[) \times (H^2(]0, L[) \cap H_*^1(]0, L[)), \quad (3.9)$$

360 then, from (3.5) to (3.8), we have  $(v_3, v_4) \in H_0^1(]0, L[) \times H_*^1(]0, L[)$ ,  $(v_5, v_6) \in L_\xi \times L_g$ ,  
 361  $(\partial_p v_5, \partial_s v_6) \in L_\xi \times L_g$ ,  $v_5(0) = v_3$  and  $v_6(0) = 0$ .

362 Next, plugging (3.5) and (3.7) into the third and fourth equations in (3.4), we get

$$363 \quad \begin{cases} \frac{1}{\rho_1} (\rho_1 + b + \xi_0 + de^{-\tau}) v_1 - \frac{k_1}{\rho_1} (\partial_x v_1 + v_2)_x \\ + \frac{g_0}{\rho_1} (a \partial_x v_1)_x - \frac{1}{\rho_1} \int_0^{+\infty} g(s) (a \partial_x v_6(s))_x ds = f_7, \\ v_2 - \frac{k_2}{\rho_2} \partial_{xx} v_2 + \frac{k_1}{\rho_2} (\partial_x v_1 + v_2) = f_2 + f_4, \end{cases} \quad (3.10)$$

364 where

$$365 \quad f_7 = \frac{1}{\rho_1} (\rho_1 + b + \xi_0 + de^{-\tau}) f_1 + f_3 - \frac{\tau de^{-\tau}}{\rho_1} \int_0^1 e^{\tau y} f_5(y) dy.$$

366 So, it is sufficient to prove that (3.10), with  $v_6$  given in (3.8), has a solution  $(v_1, v_2)$  satisfying  
 367 (3.9),

$$368 \quad \partial_x v_2(0) = \partial_x v_2(L) = 0 \quad (3.11)$$

369 and

$$370 \quad k_1 \partial_{xx} v_1 - g_0 \partial_x (a \partial_x v_1) + \int_0^{+\infty} g(s) \partial_x (a \partial_x v_6(s)) ds \in L^2(]0, L[), \quad (3.12)$$

371 and then, we replace  $v_1$  and  $v_2$  in (3.5), (3.7) and (3.8) to get  $V \in D(A)$  satisfying (3.4). Let  
 372  $(v_1, v_2)$  satisfying (3.9)–(3.11). By multiplying the equations in (3.10) by  $\rho_1 w_1$  and  $\rho_2 w_2$ ,  
 373 respectively, integrating their sum by parts on  $]0, L[$  and exploiting (3.8) and (3.11), we find  
 374 that  $(v_1, v_2)$  is a solution of the system

$$375 \quad L_1((v_1, v_2), (w_1, w_2)) = L_2(w_1, w_2), \quad \forall (w_1, w_2) \in H_0^1(]0, L[) \times H_*^1(]0, L[), \quad (3.13)$$

376 where

$$377 \quad L_1((v_1, v_2), (w_1, w_2)) = \int_0^L (k_1 (\partial_x v_1 + v_2) (\partial_x w_1 + w_2) + k_2 \partial_x v_2 \partial_x w_2) dx,$$

$$378 \quad + \int_0^L (-a g_1 \partial_x v_1 \partial_x w_1 + (\rho_1 + b + \xi_0 + de^{-\tau}) v_1 w_1 + \rho_2 v_2 w_2) dx,$$

$$379 \quad L_2((w_1, w_2)) = \int_0^L (\rho_1 f_7 w_1 + \partial_x f_8 \partial_x w_1 + \rho_2 (f_2 + f_4) w_2) dx,$$

$$380 \quad g_1 = \int_0^{+\infty} e^{-s} g(s) ds \quad \text{and} \quad f_8 = a \int_0^{+\infty} e^{-s} g(s) \int_0^s e^y (f_1 - f_6(y)) dy ds.$$

381 Since, it is easy to prove that  $L_1$  is a bilinear, continuous and coercive form and  $L_2$  is a linear  
 382 and continuous form on, respectively,

$$383 \quad (H_0^1(]0, L[) \times H_*^1(]0, L[))^2 \quad \text{and} \quad H_0^1(]0, L[) \times H_*^1(]0, L[)$$

384 (noting that  $g_1 < g_0$  and using (2.14)), so, applying the Lax-Milgram theorem, we deduce  
 385 that (3.13) admits a unique solution

$$386 \quad (v_1, v_2) \in H_0^1(]0, L[) \times H_*^1(]0, L[).$$



387 Applying the classical elliptic regularity, it follows that  $(v_1, v_2)$  satisfies (3.9)–(3.12). There-  
 388 fore, the operator  $Id + A$  is surjective.

389 Third, we see that the linear operator  $B$  is Lipschitz continuous.

390 Because  $-A$  is dissipative and  $Id + A$  is surjective, then  $A$  is a maximal monotone operator.  
 391 Therefore, using Lummer–Phillips theorem (see [53]), we deduce that  $A$  is an infinitesimal  
 392 generator of a linear  $C_0$ -semigroup on  $\mathcal{H}$ . Finally, also  $A + B$  is an infinitesimal generator  
 393 of a linear  $C_0$ -semigroup on  $\mathcal{H}$  (see [53]: Ch. 3-Theorem 1.1). Consequently, Theorem 2.3  
 394 holds from the Hille–Yosida theorem (see [33, 53]).

## 395 4 Some needed lemmas

396 We will use  $c$  (sometimes  $c_y, c_{y,y_1}, \dots$ , which depends on some parameters  $y, y_1, \dots$ ),  
 397 throughout the rest of this paper, to denote a generic positive constant which depends con-  
 398 tinuously on the initial data  $U_0$  and can be different from step to step, but it does not depend  
 399 neither on  $b$  nor on  $d$ .

400 To get our stability results, we prove first some needed lemmas, for all  $U_0 \in D(A)$ ; so all  
 401 the calculations are justified. By a simple density arguments ( $D(A)$  is dense in  $\mathcal{H}$ ), (2.34) and  
 402 (2.37) remain valid for any  $U_0 \in \mathcal{H}$ . The first next seven lemmas, used in [25], are adapted  
 403 in the present paper to (1.1) by considering the needed modifications related to the presence  
 404 of delay.

405 We start by giving the following estimates for the derivative of  $E$ :

406 **Lemma 4.1** *The energy functional satisfies, if (1.11) holds and  $d \neq 0$ ,*

$$407 \quad E'(t) \leq \frac{1}{2} g' \circ \varphi_x - \frac{1}{2} \inf_{[0,L]} (b - |d|) \int_0^L \varphi_t^2 dx, \quad (4.1)$$

408 *and, if (1.12) holds or  $d \equiv 0$ ,*

$$409 \quad E'(t) \leq \frac{1}{2} g' \circ \varphi_x + \int_0^L (-b + \|d\|_\infty) \varphi_t^2 dx. \quad (4.2)$$

410 *Proof* By exploiting (2.29), (3.2) and the definition (2.27) of  $B$ , we obtain

$$411 \quad E'(t) \leq \frac{1}{2} g' \circ \varphi_x + \int_0^L \left( -b + \frac{\xi}{2\tau} + \frac{|d|}{2} \right) \varphi_t^2 dx + \int_0^L \left( \frac{|d|}{2} - \frac{\xi}{2\tau} \right) \varphi_t^2 (t - \tau) dx. \quad (4.3)$$

413 So, from (2.25), we see that, if (1.11) holds and  $d \neq 0$ , then

$$414 \quad -b + \frac{\xi}{2\tau} + \frac{|d|}{2} = \frac{|d|}{2} - \frac{\xi}{2\tau} = -\frac{1}{2} (b - |d|) \leq -\frac{1}{2} \inf_{[0,L]} (b - |d|) \leq 0.$$

415 However, if (1.12) holds or  $d \equiv 0$ , we have

$$416 \quad -b + \frac{\xi}{2\tau} + \frac{|d|}{2} = -b + \frac{\|d\|_\infty + |d|}{2} \leq -b + \|d\|_\infty \quad \text{and} \quad \frac{|d|}{2} - \frac{\xi}{2\tau} = \frac{|d| - \|d\|_\infty}{2} \leq 0.$$

417 Hence, (4.3) yields (4.1) and (4.2).  $\square$

418 **Remark 4.2** 1. When (1.11) holds,  $E' \leq 0$ , since  $g$  is non-increasing, and then (1.1) is  
 419 dissipative. However, when (1.12) holds, we are unable to determine the sign of  $E'$  from  
 420 (4.2), and therefore, (1.1) is not necessarily dissipative with respect to  $E$  at this stage.

2. Using the definition of  $E$ , (4.1) and (4.2), we see that, for some non-negative constant  $\alpha_0$ ,  $E' \leq \alpha_0 E$ . Then, by integrating,

$$E(t) \leq e^{\alpha_0(t-t_0)} E(t_0), \quad \forall t \geq t_0 \geq 0.$$

So, if  $E(t_0) = 0$ , for some  $t_0 \in \mathbb{R}_+$ , then  $E(t) = 0$ , for all  $t \geq t_0$ , and therefore, (2.34), (2.37) and (2.41) hold. Consequently, without loss of generality, we can assume that  $E(t) > 0$ , for all  $t \in \mathbb{R}_+$ .

**Lemma 4.3** *The following inequalities hold:*

$$\exists d_1 > 0 : \left( \int_0^L a \int_0^{+\infty} g(s)(\varphi(t) - \varphi(t-s)) ds dx \right)^2 \leq d_1 g \circ \varphi_x, \tag{4.4}$$

$$\exists d_2 > 0 : \left( \int_0^L a \int_0^{+\infty} g'(s)(\varphi(t) - \varphi(t-s)) ds dx \right)^2 \leq -d_2 g' \circ \varphi_x, \tag{4.5}$$

$$\exists d_3 > 0 : \left( \int_0^L a' \int_0^{+\infty} g(s)(\varphi(t) - \varphi(t-s)) ds dx \right)^2 \leq d_3 g \circ \varphi_x. \tag{4.6}$$

$$\left( \int_0^{+\infty} g(s)(\varphi_x(t) - \varphi_x(t-s)) ds \right)^2 \leq g_0 \int_0^{+\infty} g(s)(\varphi_x(t) - \varphi_x(t-s))^2 ds, \tag{4.7}$$

$$\left( \int_0^{+\infty} g'(s)(\varphi_x(t) - \varphi_x(t-s)) ds \right)^2 \leq -g(0) \int_0^{+\infty} g'(s)(\varphi_x(t) - \varphi_x(t-s))^2 ds. \tag{4.8}$$

*Proof* If  $a \equiv 0$ , (4.4)–(4.6) are trivial. If  $\inf_{[0,L]} a > 0$ , we use the fact that  $a$  and  $a'$  are bounded and apply Poincaré’s and Hölder’s inequalities (2.26) (for  $\varphi$ ) and

$$\left( \int_0^L |f_1 f_2| dx \right)^2 \leq \left( \int_0^L f_1^2 dx \right) \left( \int_0^L f_2^2 dx \right), \quad \forall f_1, f_2 \in L^2([0, L]), \tag{4.9}$$

respectively, to get (4.4)–(4.6). Using again Hölder’s inequality (4.9), (4.7) and (4.8) hold. Notice that the constants  $d_i$  do not depend neither on  $b$  nor on  $d$ .  $\square$

**Lemma 4.4** *The functional*

$$I_t(t) := -\rho_1 \int_0^L a \varphi_t \int_0^{+\infty} g(s)(\varphi(t) - \varphi(t-s)) ds dx \tag{4.10}$$

satisfies, for any  $\delta > 0$ ,

$$\begin{aligned} I_t'(t) &\leq -\rho_1 g_0 \int_0^L a \varphi_t^2 dx + \delta \int_0^L (\varphi_t^2 + \varphi_x^2 + \psi_x^2) dx \\ &\quad + \delta \int_0^L (b^2 \varphi_t^2 + d^2 \varphi_t^2(t-\tau)) dx + c \left( 1 + \frac{1}{\delta} \right) g \circ \varphi_x - \frac{c}{\delta} g' \circ \varphi_x. \end{aligned} \tag{4.11}$$

*Proof* First, note that

$$\begin{aligned} \partial_t \left( \int_0^{+\infty} g(s)(\varphi(t) - \varphi(t-s)) ds \right) &= \partial_t \left( \int_{-\infty}^t g(t-s)(\varphi(t) - \varphi(s)) ds \right) \\ &= \int_{-\infty}^t g(t-s) \varphi_t(t) ds + \int_{-\infty}^t g'(t-s)(\varphi(t) - \varphi(s)) ds \\ &= g_0 \varphi_t + \int_0^{+\infty} g'(s)(\varphi(t) - \varphi(t-s)) ds. \end{aligned} \tag{4.12}$$

448 Then, by differentiating  $I_1$ , and using the first equation and the boundary conditions in (1.1),  
449 we find

$$\begin{aligned}
 450 \quad I_1'(t) &= -\rho_1 g_0 \int_0^L a \varphi_t^2 dx - \rho_1 \int_0^L a \varphi_t \int_0^{+\infty} g'(s)(\varphi(t) - \varphi(t-s)) ds dx \\
 451 &+ k_1 \int_0^L a(\varphi_x + \psi) \int_0^{+\infty} g(s)(\varphi_x(t) - \varphi_x(t-s)) ds dx \\
 452 &+ \int_0^L a(b\varphi_t + d\varphi_t(t-\tau)) \int_0^{+\infty} g(s)(\varphi(t) - \varphi(t-s)) ds dx \\
 453 &+ \int_0^L a^2 \left( \int_0^{+\infty} g(s)(\varphi_x(t) - \varphi_x(t-s)) ds \right)^2 dx - g_0 \int_0^L a^2 \varphi_x \int_0^{+\infty} g(s)(\varphi_x(t) \\
 454 &- \varphi_x(t-s)) ds dx + \int_0^L aa' \left( \int_0^{+\infty} g(s)(\varphi(t) - \varphi(t-s)) ds \right) \\
 455 &\times \left( \int_0^{+\infty} g(s)(\varphi_x(t) - \varphi_x(t-s)) ds \right) dx \\
 456 &+ k_1 \int_0^L a'(\varphi_x + \psi) \int_0^{+\infty} g(s)(\varphi(t) - \varphi(t-s)) ds dx \\
 457 &- g_0 \int_0^L aa' \varphi_x \int_0^{+\infty} g(s)(\varphi(t) - \varphi(t-s)) ds dx.
 \end{aligned}$$

458 Therefore, applying Young's and Hölder's inequalities (3.1) and (4.9), for the last eight terms  
459 of the above equality, and using (4.4)–(4.7), Poincaré's inequality (2.26), for  $\varphi$ , and the fact  
460 that  $a$  and  $a'$  are bounded, we get (4.11).  $\square$

461 **Lemma 4.5** *The functional*

$$462 \quad I_2(t) := \int_0^L (\rho_1 \varphi \varphi_t + \rho_2 \psi \psi_t) dx \quad (4.13)$$

463 *satisfies, for any  $\delta > 0$ ,*

$$\begin{aligned}
 464 \quad I_2'(t) &\leq \int_0^L (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2) dx - k_1 \int_0^L (\varphi_x + \psi)^2 dx - k_2 \int_0^L \psi_x^2 dx \\
 465 &+ g_0 \int_0^L a \varphi_x^2 dx + \delta \int_0^L \varphi_x^2 dx + \frac{c}{\delta} \int_0^L (b^2 \varphi_t^2 + d^2 \varphi_t^2(t-\tau)) dx + \frac{c}{\delta} g \circ \varphi_x. \\
 466 & \quad (4.14)
 \end{aligned}$$

467 *Proof* By differentiating  $I_2$ , and using the first two equations and boundary conditions in  
468 (1.1), we have

$$\begin{aligned}
 469 \quad I_2'(t) &= \int_0^L (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2) dx - k_1 \int_0^L (\varphi_x + \psi)^2 dx - k_2 \int_0^L \psi_x^2 dx \\
 470 &+ g_0 \int_0^L a \varphi_x^2 dx - \int_0^L \varphi(b\varphi_t + d\varphi_t(t-\tau)) dx \\
 471 &- \int_0^L a \varphi_x \int_0^{+\infty} g(s)(\varphi_x(t) - \varphi_x(t-s)) ds dx.
 \end{aligned}$$

472 Consequently, applying Young's and Hölder's inequalities (3.1) and (4.9), for the last two  
473 terms of the above equality, and using (4.7), Poincaré's inequality (2.26), for  $\varphi$ , and the fact  
474 that  $a$  is bounded, we find (4.14).  $\square$

Author Proof

475 **Lemma 4.6** *The functional*

$$\begin{aligned}
 476 \quad I_3(t) &:= -\rho_2 \int_0^L \psi_t(\varphi_x + \psi) dx - \frac{k_2\rho_1}{k_1} \int_0^L \psi_x \varphi_t dx \\
 477 \quad &+ \frac{\rho_2}{k_1} \int_0^L a\psi_t \int_0^{+\infty} g(s)\varphi_x(t-s) ds dx \tag{4.15}
 \end{aligned}$$

478 *satisfies, for any  $\delta, \delta_1 > 0$ ,*

$$\begin{aligned}
 479 \quad I'_3(t) &\leq k_1 \int_0^L (\varphi_x + \psi)^2 dx - \rho_2 \int_0^L \psi_t^2 dx \\
 480 \quad &+ g_0 \left( \frac{\delta_1}{2} - 1 \right) \int_0^L a\varphi_x^2 dx + \frac{g_0 k_0 \|a\|_\infty}{2\delta_1} \int_0^L \psi_x^2 dx \\
 481 \quad &+ \frac{c}{\delta} \int_0^L (b^2\varphi_t^2 + d^2\varphi_t^2(t-\tau)) dx \\
 482 \quad &+ \delta \int_0^L (\psi_t^2 + \varphi_x^2 + \psi_x^2) dx + \frac{c}{\delta} (g \circ \varphi_x - g' \circ \varphi_x) \\
 483 \quad &+ \left( \frac{k_2\rho_1}{k_1} - \rho_2 \right) \int_0^L \varphi_{xt} \psi_t dx, \tag{4.16}
 \end{aligned}$$

484 *where  $k_0$  is defined in (2.5).*

485 *Proof* Similarly to (4.12) and using that  $\lim_{s \rightarrow +\infty} g(s) = 0$ , we see that

$$\begin{aligned}
 486 \quad \partial_t \left( \int_0^{+\infty} g(s)\varphi_x(t-s) ds \right) &= \partial_t \left( \int_{-\infty}^t g(t-s)\varphi_x(s) ds \right) \\
 487 \quad &= g(0)\varphi_x + \int_{-\infty}^t g'(t-s)\varphi_x(s) ds \\
 488 \quad &= g(0)\varphi_x + \int_0^{+\infty} g'(s)(\varphi_x(t-s) - \varphi_x(t) + \varphi_x(t)) ds. \\
 489 \quad &= - \int_0^{+\infty} g'(s)(\varphi_x(t) - \varphi_x(t-s)) ds.
 \end{aligned}$$

490 Therefore, exploiting the first two equations and boundary conditions in (1.1), we have

$$\begin{aligned}
 491 \quad I'_3(t) &= k_1 \int_0^L (\varphi_x + \psi)^2 dx - \rho_2 \int_0^L \psi_t^2 dx + \left( \frac{k_2\rho_1}{k_1} - \rho_2 \right) \int_0^L \varphi_{xt} \psi_t dx \\
 492 \quad &- g_0 \int_0^L a\varphi_x^2 dx - g_0 \int_0^L a\varphi_x \psi dx + \int_0^L a(\varphi_x + \psi) \int_0^{+\infty} g(s)(\varphi_x(t) - \varphi_x(t-s)) ds dx \\
 493 \quad &- \frac{\rho_2}{k_1} \int_0^L a\psi_t \int_0^{+\infty} g'(s)(\varphi_x(t) - \varphi_x(t-s)) ds dx + \frac{k_2}{k_1} \int_0^L \psi_x (b\varphi_t + d\varphi_t(t-\tau)) dx.
 \end{aligned}$$

494 By applying Young's inequality (3.1), for the last four terms of the above equality, Poincaré's  
 495 inequality (2.5), for  $\psi$ , and using (4.7), (4.8) and the fact that  $a$  is bounded, (4.16) is estab-  
 496 lished. □

497 Now, as in [7], we use a function  $w$  to get a crucial estimate.

498 **Lemma 4.7** *The function*

$$499 \quad w(x, t) = \int_0^x \psi(y, t) dy \quad (4.17)$$

500 *satisfies the estimates ( $\tilde{k}_0$  is the constant defined in (2.26))*

$$501 \quad \int_0^L w_x^2 dx = \int_0^L \psi^2 dx, \quad \forall t \geq 0, \quad (4.18)$$

$$502 \quad \int_0^L w_t^2 dx \leq \tilde{k}_0 \int_0^L \psi_t^2 dx, \quad \forall t \geq 0. \quad (4.19)$$

503 *Proof* We just have to note that  $w_x = \psi$  to get (4.18). On the other hand, using (2.12) (remind  
504 that we are working with  $\tilde{\psi}$ , but we use  $\psi$  instead of  $\tilde{\psi}$ ; see Remark 2.1-2),

$$505 \quad w_t(0, t) = 0 \quad \text{and} \quad w_t(L, t) = \partial_t \int_0^L \psi_t(y, t) dy = \partial_t \int_0^L \psi(y, t) dy = 0.$$

506 Then, applying (4.18) to  $w_t$  and using Poincaré's inequality (2.26), for  $w_t$ , we arrive at  
507 (4.19).  $\square$

508 **Lemma 4.8** *The functional*

$$509 \quad I_4(t) := \rho_1 \int_0^L (w\varphi_t + \varphi\varphi_t) dx \quad (4.20)$$

510 *satisfies, for any  $\delta, \epsilon, \epsilon_1 > 0$ ,*

$$511 \quad I_4'(t) \leq \left(\rho_1 + \frac{c_0}{\epsilon}\right) \int_0^L \varphi_t^2 dx + c_0 \epsilon \int_0^L \psi_t^2 dx \\ 512 \quad + \left(g_0 \|a\|_\infty \left(1 + \frac{\epsilon_1}{2}\right) - k_1\right) \int_0^L (\varphi_x + \psi)^2 dx + \frac{g_0 k_0 \|a\|_\infty}{2\epsilon_1} \int_0^L \psi_x^2 dx \\ 513 \quad + \delta \int_0^L (\varphi_x^2 + \psi_x^2) dx + \frac{c}{\delta} \int_0^L (b^2 \varphi_t^2 + d^2 \varphi_t^2(t - \tau)) dx + \frac{c}{\delta} g \circ \varphi_x, \quad (4.21)$$

514 *where  $k_0$  is defined in (2.5),  $c_0 = \frac{\rho_1}{2} \sqrt{\tilde{k}_0}$  and  $\tilde{k}_0$  is defined in (2.26).*

515 *Proof* Using the first two equations and boundary conditions in (1.1), and exploiting the fact  
516 that  $w(0, t) = w(L, t) = 0$  and  $w_x = \psi$ , we find

$$517 \quad I_4'(t) = \rho_1 \int_0^L \varphi_t^2 dx - k_1 \int_0^L (\varphi_x + \psi)^2 dx \\ 518 \quad + g_0 \int_0^L a(\varphi_x + \psi - \psi)(\varphi_x + \psi) dx + \rho_1 \int_0^L w_t \varphi_t dx \\ 519 \quad - \int_0^L (w + \varphi)(b\varphi_t + d\varphi_t(t - \tau)) dx - \int_0^L a(\varphi_x + \psi) \int_0^{+\infty} g(s)(\varphi_x(t) \\ 520 \quad - \varphi_x(t - s)) ds dx.$$

521 Applying Young's inequality (3.1), for the last four terms of the above equality, Poincaré's  
522 inequalities (2.5), for  $\psi$ , and (2.26), for  $\varphi$  and  $w$ , and exploiting (4.7), (4.18), (4.19) and the  
523 fact that  $a$  is bounded, we get (4.21).  $\square$

524 We use a functional introduced in [48] (in case  $\xi \equiv 1$ ) to get an estimation on the delay  
525 term.

526 **Lemma 4.9** *The functional*

$$527 \quad I_5(t) = \int_0^L \xi \int_0^1 e^{-2\tau p} \varphi_t^2(t - \tau p) dp dx \quad (4.22)$$

528 satisfies

$$529 \quad I_5'(t) \leq -2e^{-2\tau} \int_0^L \xi \int_0^1 \varphi_t^2(t - \tau p) dp dx \\ 530 \quad + \frac{1}{\tau} \int_0^L \xi \varphi_t^2 dx - \frac{e^{-2\tau}}{\tau} \int_0^L \xi \varphi_t^2(t - \tau) dx. \quad (4.23)$$

531 *Proof* Using (2.16) and the first equation in (2.18), the derivative of  $I_5$  entails

$$532 \quad I_5'(t) = 2 \int_0^L \xi \int_0^1 e^{-2\tau p} \varphi_{tt}(t - \tau p) \varphi_t(x, t - \tau p) dp dx \\ 533 \quad = -\frac{2}{\tau} \int_0^L \xi \int_0^1 e^{-2\tau p} \varphi_{tp}(t - \tau p) \varphi_t(t - \tau p) dp dx \\ 534 \quad = -\frac{1}{\tau} \int_0^L \xi \int_0^1 e^{-2\tau p} \partial_p \varphi_t^2(t - \tau p) dp dx.$$

535 Then, by using an integrating by parts, the above formula can be rewritten as

$$536 \quad I_5'(t) = -2 \int_0^L \xi \int_0^1 e^{-2\tau p} \varphi_t^2(t - \tau p) dp dx + \frac{1}{\tau} \int_0^L \xi \varphi_t^2 dx - \frac{e^{-2\tau}}{\tau} \int_0^L \xi \varphi_t^2(t - \tau) dx,$$

537 which gives (4.23), since  $-2e^{-2\tau p} \leq -2e^{-2\tau}$ , for any  $p \in ]0, 1[$ .  $\square$

538 Let  $a_0 := \inf_{[0, L]} a$ ,  $b_0 := \inf_{[0, L]} b$  and, for  $N, N_1, N_2, N_3, N_4 \geq 0$ ,

$$539 \quad I_6 := NE + N_1 I_1 + N_2 I_2 + I_3 + N_3 I_4 + N_4 I_5. \quad (4.24)$$

540 Then, by combining (4.11), (4.14), (4.16), (4.21) and (4.23), we obtain

$$541 \quad I_6'(t) \leq - \int_0^L \left( l_0 \varphi_t^2 + l_1 \psi_t^2 + l_2 (\varphi_x + \psi)^2 + l_3 \psi_x^2 \right) dx + l_4 g_0 \int_0^L a \varphi_x^2 dx + NE'(t) \\ 542 \quad - 2e^{-2\tau} N_4 \int_0^L \xi \int_0^1 \varphi_t^2(t - \tau p) dp dx + \delta(N_1 + c_{N_2, N_3}) \int_0^L \left( \varphi_t^2 + \psi_t^2 + \varphi_x^2 + \psi_x^2 \right) dx \\ 543 \quad - \int_0^L \left( \frac{e^{-2\tau} N_4}{\tau} \xi - \left( \delta N_1 + \frac{c_{N_2, N_3}}{\delta} \right) d^2 \right) \varphi_t^2(t - \tau) dx \\ 544 \quad + \int_0^L \left( \frac{N_4}{\tau} \xi + \left( \delta N_1 + \frac{c_{N_2, N_3}}{\delta} \right) b^2 \right) \varphi_t^2 dx \\ 545 \quad + \left( c_{N_1} + \frac{c_{N_1, N_2, N_3}}{\delta} \right) g \circ \varphi_x - \frac{c_{N_1}}{\delta} g' \circ \varphi_x + \left( \frac{\rho_1 k_2}{k_1} - \rho_2 \right) \int_0^L \varphi_{xt} \psi_t dx, \quad (4.25)$$

546 where

$$547 \quad l_0 = N_1 \rho_1 g_0 a_0 - (N_2 + N_3) \rho_1 - \frac{c_0 N_3}{\epsilon},$$

$$548 \quad l_1 = \rho_2 (1 - N_2) - c_0 \epsilon N_3, \quad l_2 = k_1 (N_2 + N_3 - 1) - g_0 \|a\|_\infty \left( 1 + \frac{\epsilon_1}{2} \right) N_3,$$

$$549 \quad l_3 = k_2 N_2 - \frac{g_0 k_0 \|a\|_\infty}{2} \left( \frac{N_3}{\epsilon_1} + \frac{1}{\delta_1} \right) \quad \text{and} \quad l_4 = N_2 + \frac{\delta_1}{2} - 1.$$

Now, as in [25], we choose carefully the constants  $N$ ,  $N_i$ ,  $\delta$ ,  $\delta_1$ ,  $\epsilon$  and  $\epsilon_1$  to get desired signs of  $l_i$ .

**Case 1**  $a \equiv 0$ : the second integral in (4.25) drops,  $g \circ \varphi_x = g' \circ \varphi_x = 0$  (from the definition (2.32)) and the constants  $l_0$ ,  $l_1$ ,  $l_2$  and  $l_3$  do not depend neither on  $\delta_1$  nor on  $\epsilon_1$ . On the other hand,

$$l_0 = -(N_2 + N_3)\rho_1 - \frac{c_0 N_3}{\epsilon} \geq N_1 b_0 - (N_2 + N_3)\rho_1 - \frac{c_0 N_3}{\epsilon} - N_1 b := \tilde{l}_0 - N_1 b,$$

so  $\tilde{l}_0 := N_1 b_0 - (N_2 + N_3)\rho_1 - \frac{c_0 N_3}{\epsilon}$ . Therefore, we choose

$$N_3 = 1, \quad 0 < N_2 < 1, \quad 0 < \epsilon < \frac{\rho_2}{c_0}(1 - N_2) \quad \text{and} \quad N_1 > \frac{1}{b_0}(N_2 + N_3) + \frac{c_0 N_3}{\epsilon b_0}.$$

Notice that  $N_3$ ,  $N_2$  and  $\epsilon$  do not depend neither on  $b$  nor on  $d$ . Moreover, because  $b_0 > 0$  thanks to (2.2) and  $a \equiv 0$ ,  $N_1$  exists and can be taken in the form  $N_1 = \frac{c}{b_0}$ , and then  $\tilde{l}_0$  as well as  $l_1$ ,  $l_2$  and  $l_3$  do not depend neither on  $b$  nor on  $d$ . According to these choices, we get

$$L := \min \left\{ \frac{\tilde{l}_0}{\rho_1}, \frac{l_1}{\rho_2}, \frac{l_2}{k_1}, \frac{l_3}{k_2} \right\} > 0,$$

and then, using (2.14) and (4.25),

$$\begin{aligned} I'_6(t) \leq & - \left( L - c\delta \left( 1 + \frac{1}{b_0} \right) \right) \int_0^L (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + k_1(\varphi_x + \psi)^2 \\ & + k_2 \psi_x^2) dx - 2e^{-2\tau} N_4 \int_0^L \xi \int_0^1 \varphi_t^2(t - \tau p) dp dx \\ & - \int_0^L \left( \frac{e^{-2\tau} N_4}{\tau} \xi - c \left( \frac{\delta}{b_0} + \frac{1}{\delta} \right) d^2 \right) \varphi_t^2(t - \tau) dx \\ & + NE'(t) + \int_0^L \left( \frac{N_4}{\tau} \xi + c \left( \frac{b^2 \delta}{b_0} + \frac{b^2}{\delta} + \frac{b}{b_0} \right) \right) \varphi_t^2 dx \\ & + \left( \frac{\rho_1 k_2}{k_1} - \rho_2 \right) \int_0^L \varphi_{xt} \psi_t dx. \end{aligned} \quad (4.26)$$

Next, choosing  $\delta > 0$  such that

$$L - c\delta \left( 1 + \frac{1}{b_0} \right) > 0.$$

Notice that  $L$  and  $c$  do not depend on  $\delta$ ,  $b$  and  $d$ ; so  $\delta$  exists and can be taken in the form

$$\delta = \frac{cb_0}{b_0 + 1}, \quad (4.27)$$

and consequently,  $L - c\delta \left( 1 + \frac{1}{b_0} \right)$  is a positive constant which does not depend neither on  $b$  nor on  $d$ . At the end, we choose  $N_4$  such that

$$\frac{e^{-2\tau} N_4}{\tau} \xi - c \left( \frac{\delta}{b_0} + \frac{1}{\delta} \right) d^2 \geq 0. \quad (4.28)$$

If  $d \equiv 0$ , then  $\xi \equiv 0$  (thanks to (2.25)), and therefore (4.28) is satisfied, for any  $N_4 \geq 0$ . Otherwise, if  $d \neq 0$ , then  $\xi = \tau b$  (in virtue of (2.25) and because (1.11) is assumed in this

577 case  $a \equiv 0$ ; see Theorem 2.6), consequently, the choice (4.27) and the inequality  $|d| < b$   
 578 (according to (1.11)) imply that  $N_4$  can be taken in the form

$$N_4 = \frac{c\|b\|_\infty(b_0 + 1)}{b_0}. \tag{4.29}$$

580 Thus, using (2.31), we get from (4.26)

$$\begin{aligned} I'_6(t) \leq & -cE_0(t) - \frac{c\|b\|_\infty(b_0 + 1)}{b_0}E_1(t) + NE'(t) \\ & + \frac{c(\|b\|_\infty(b_0 + 1) + 1)}{b_0} \int_0^L b\varphi_t^2 dx + \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \int_0^L \varphi_{xt} \psi_t dx, \end{aligned} \tag{4.30}$$

583 where

$$E_0(t) = E(t) - E_1(t) \quad \text{and} \quad E_1(t) = \frac{1}{2} \int_0^L \xi \int_0^1 \varphi_t^2(t - \tau p) dp dx. \tag{4.31}$$

585 **Case 2.**  $a_0 > 0$ : we choose

$$\epsilon_1 = \frac{k_1 - g_0\|a\|_\infty}{g_0\|a\|_\infty}, \quad \delta_1 = \frac{k_0 g_0\|a\|_\infty}{k_2},$$

$$\frac{k_1 \delta_1}{2k_1 - g_0\|a\|_\infty(2 + \epsilon_1)} < N_3 < \epsilon_1 \left( \frac{k_2(2 - \delta_1)}{g_0 k_0\|a\|_\infty} - \frac{1}{\delta_1} \right),$$

$$\max \left\{ 1 - N_3 \left( 1 - \frac{g_0\|a\|_\infty(2 + \epsilon_1)}{2k_1} \right), \frac{g_0 k_0\|a\|_\infty}{2k_2} \left( \frac{N_3}{\epsilon_1} + \frac{1}{\delta_1} \right) \right\} < N_2 < 1 - \frac{\delta_1}{4},$$

$$0 < \epsilon < \min \left\{ \left( 2(1 - N_2) - \frac{\delta_1}{2} \right) \frac{\rho_2}{c_0 N_3}, \frac{\rho_2(1 - N_2)}{c_0 N_3} \right\}$$

593 and

$$N_1 > \max \left\{ \frac{(N_2 + N_3)\rho_1 + \frac{c_0 N_3}{\epsilon}}{\rho_1 g_0 a_0}, \frac{\left( 2N_2 + N_3 + \frac{\delta_1}{2} - 1 \right) \rho_1 + \frac{c_0 N_3}{\epsilon}}{\rho_1 g_0 a_0} \right\}.$$

595 Notice that  $\epsilon_1$  and  $\delta_1$  exist and are positive thanks to (2.4) and the property  $g_0 > 0$  (because  
 596  $g(0) > 0$ ; see (H2)),  $N_2$  exists according to the choice of  $N_3$ ,  $\epsilon$  exists from the choice of  
 597  $N_2$ , and  $N_1$  exists because  $\rho_1 g_0 a_0 > 0$ . On the other hand, to prove the existence of  $N_3$ , we  
 598 repeat the calculations given in [25]. Using the definitions of  $\epsilon_1$  and  $\delta_1$ , we see that  $N_3$  exists  
 599 if and only if

$$k_0^2 k_1 (g_0\|a\|_\infty)^3 < k_2 (k_2 - k_0 g_0\|a\|_\infty) (k_1 - g_0\|a\|_\infty)^2.$$

601 Let  $y_0 = \frac{k_1 k_2}{k_0 k_1 + k_2}$ ,  $y = g_0\|a\|_\infty \in ]0, y_0[$  (see (2.4)) and

$$f(y) = k_0^2 k_1 y^3 - k_2 (k_2 - k_0 y) (k_1 - y)^2.$$

603 We have

$$f'(y) = 3(k_0^2 k_1 + k_0 k_2) y^2 - 2(2k_0 k_1 k_2 + k_2^2) y + k_0 k_1^2 k_2 + 2k_1 k_2^2$$

Author Proof



605 and

$$606 \quad f''(y) = 6(k_0^2 k_1 + k_0 k_2) y - 2(2k_0 k_1 k_2 + k_2^2).$$

607 Let  $y_1 = \frac{2k_0 k_1 k_2 + k_2^2}{3(k_0^2 k_1 + k_0 k_2)}$ . We notice that  $f'$  is decreasing on  $]0, y_1[$ , it is increasing on  $]y_1, +\infty[$   
608 and

$$609 \quad f'(y_0) = \frac{k_0^2 k_1^3 k_2 + 2k_0 k_1^2 k_2^2}{k_0 k_1 + k_2} > 0,$$

610 Moreover,  $y_1 \leq y_0$  if and only if  $k_2 \leq k_0 k_1$ , and, if  $k_2 \leq k_0 k_1$ ,

$$611 \quad f'(y_1) = \frac{5k_0^2 k_1^2 k_2^2 - k_2^4 + 2k_0 k_1 k_2^3 + 3k_0^3 k_1^3 k_2}{3(k_0^2 k_1 + k_0 k_2)} \geq \frac{9k_2^4}{3(k_0^2 k_1 + k_0 k_2)} > 0.$$

612 Therefore,  $f'$  is positive on  $]0, y_0[$ , and then  $f(y) < f(y_0)$ , for any  $y \in ]0, y_0[$ . But  
613  $f(y_0) = 0$ , hence  $f$  is negative on  $]0, y_0[$ . This guarantees the existence of  $N_3$ .

614 By virtue of these choices, we notice that

$$615 \quad L := \min \left\{ \frac{l_0}{\rho_1}, \frac{l_1}{\rho_2}, \frac{l_2}{k_1}, \frac{l_3}{k_2} \right\} > 0, \quad l_4 \leq L,$$

616 and  $L$  does not depend on  $\delta$ ,  $b$  and  $d$ . Then, using (2.14) and (4.25), we find

$$617 \quad I_6'(t) \leq -(L - c\delta) \int_0^L (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + k_1(\varphi_x + \psi)^2 + k_2 \psi_x^2 - a g_0 \varphi_x^2) dx + N E'(t) \\ 618 \quad - 2e^{-2\tau} N_4 \int_0^L \xi \int_0^1 \varphi_t^2(t - \tau p) dp dx - \int_0^L \left( \frac{e^{-2\tau} N_4}{\tau} \xi - c \left( \delta + \frac{1}{\delta} \right) d^2 \right) \varphi_t^2(t - \tau) dx \\ 619 \quad + \int_0^L \left( \frac{N_4}{\tau} \xi + c \left( \delta + \frac{1}{\delta} \right) b^2 \right) \varphi_t^2 dx + c \left( 1 + \frac{1}{\delta} \right) g \circ \varphi_x - \frac{c}{\delta} g' \circ \varphi_x \\ 620 \quad + \left( \frac{\rho_1 k_2}{k_1} - \rho_2 \right) \int_0^L \varphi_{xt} \psi_t dx. \quad (4.32)$$

621 Therefore, choosing  $\delta > 0$  and  $N_4 \geq 0$  such that  $L - c\delta > 0$  and

$$622 \quad \frac{e^{-2\tau} N_4}{\tau} \xi - c \left( \delta + \frac{1}{\delta} \right) d^2 \geq 0.$$

623 In virtue of (2.25),  $N_4$  can be chosen in the form  $N_4 = c \|d\|_\infty$ . Then, using (2.31), (4.30)  
624 and (4.32), we find, in both cases  $a \equiv 0$  and  $a_0 > 0$ ,

$$625 \quad I_6'(t) \leq -c E_0(t) - \tilde{c} E_1(t) + N E'(t) \\ 626 \quad + c \int_0^L \tilde{\xi} \varphi_t^2 dx + \left( \frac{\rho_1 k_2}{k_1} - \rho_2 \right) \int_0^L \varphi_{xt} \psi_t dx + c(g \circ \varphi_x - g' \circ \varphi_x), \quad (4.33)$$

627 where, thanks to the definition of  $\xi$  in case (1.11),

$$628 \quad \tilde{c} = \begin{cases} \frac{c \|b\|_\infty (b_0 + 1)}{b_0} & \text{if } a \equiv 0, \\ c \|d\|_\infty & \text{if } a_0 > 0 \end{cases} \quad (4.34)$$

629 and

$$630 \quad \tilde{\xi} = \begin{cases} \frac{\|b\|_\infty (b_0 + 1) + 1}{b_0} b & \text{if } a \equiv 0, \\ \|b\|_\infty b & \text{if } a_0 > 0 \text{ and (1.11) holds,} \\ \|d\|_\infty^2 + \|b\|_\infty b & \text{if } a_0 > 0 \text{ and (1.12) holds.} \end{cases} \quad (4.35)$$

631 Now, we estimate the term  $g \circ \varphi_x$  in (4.33).

632 **Case 1 (2.8) holds:** then

$$633 \quad g \circ \varphi_x \leq -\frac{1}{\alpha} g' \circ \varphi_x. \quad (4.36)$$

634 **Case 2 (2.9) holds:** this case does not concern Theorem 2.6 because of (2.38). For Theorem  
635 2.4 and Theorem 2.7, we apply here an inequality given in [19] (and in [16] in a less general  
636 form).

637 **Lemma 4.10** For any  $\epsilon_0 > 0$ , we have

$$638 \quad G'(\epsilon_0 E(t)) g \circ \varphi_x \leq -c g' \circ \varphi_x + c \epsilon_0 E(t) G'(\epsilon_0 E(t)). \quad (4.37)$$

639 *Proof* In Theorem 2.4 and Theorem 2.7, it is assumed that (1.11) holds. Then, thanks to (4.1),  
640  $E$  is non-increasing. Therefore, the proof is the same as in [19]-Lemma 3.6 (for  $B^{\frac{1}{2}} = \partial_x$   
641 and  $\|\cdot\| = \|\cdot\|_{L^2(0,L)}$ ).  $\square$

642 Using (4.33), (4.36) and (4.37), we see that, in both two previous cases,

$$643 \quad \frac{G_0(E(t))}{E(t)} I'_6(t) \leq -\frac{G_0(E(t))}{E(t)} ((c - \tilde{\epsilon}_0) E_0(t) + (\tilde{c} - \tilde{\epsilon}_0) E_1(t)) + N \frac{G_0(E(t))}{E(t)} E'(t) \\ 644 \quad - c (1 + G'(\epsilon_0 E(t))) g' \circ \varphi_x + c \frac{G_0(E(t))}{E(t)} \int_0^L \tilde{\xi} \varphi_t^2 dx \\ 645 \quad + \left( \frac{\rho_1 k_2}{k_1} - \rho_2 \right) \frac{G_0(E(t))}{E(t)} \int_0^L \varphi_{xt} \psi_t dx, \quad (4.38)$$

646 where  $G_0$  is defined in (2.35) and

$$647 \quad \tilde{\epsilon} = \begin{cases} 0 & \text{if (2.8) holds,} \\ c \epsilon_0 & \text{if (2.9) holds.} \end{cases} \quad (4.39)$$

648 On the other hand, by (2.14) and the definitions of the functionals  $I_i$  and  $E$ , there exists a  
649 positive constant  $\beta$  (not depending on  $N$ ,  $b$  and  $d$ ) satisfying

$$650 \quad |N_1 I_1 + N_2 I_2 + I_3 + N_3 I_4 + N_4 I_5| \leq \beta E,$$

651 which implies that

$$652 \quad (N - \beta) E \leq I_6 \leq (N + \beta) E. \quad (4.40)$$

653 Now, at this stage, we distinguish the cases of Theorems 2.4, 2.6 and 2.7.

## 654 5 General stability: (1.2) and (1.11) hold

655 Using (4.1) (in case  $d \neq 0$ ), (4.2) (in case  $d \equiv 0$ ) and the property  $g' \leq 0$ , we have

$$656 \quad N E'(t) + c \int_0^L \tilde{\xi} \varphi_t^2 dx \leq \int_0^L \left( c \tilde{\xi} - \frac{N}{2} \inf_{[0,L]} (b - |d|) \right) \varphi_t^2 dx \quad (5.1)$$

657 and

$$658 \quad -g' \circ \varphi_x \leq -2E'(t). \quad (5.2)$$

Therefore, inserting (5.1) and (5.2) into (4.38), choosing  $\epsilon_0 > 0$  such that  $\tilde{\epsilon}$  defined in (4.39) satisfies

$$\tilde{\epsilon} < \begin{cases} \min\{c, \tilde{c}\} & \text{if } d \neq 0, \\ c & \text{if } d \equiv 0 \end{cases}$$

(if  $d \equiv 0$ , then  $\xi = E_1 = 0$  and  $E = E_0$ ) and choosing  $N \geq 0$  such that

$$c\tilde{\xi} - \frac{N}{2} \inf_{[0, L]} (b - |d|) \leq 0 \quad \text{and} \quad N > \beta$$

( $N$  exists according to (1.11), (4.35) and the boundedness of  $b$ ), we deduce, from (1.2), (4.38), (4.40) and the fact that  $G'(\epsilon_0 E)$  is non-increasing, that  $I_6 \sim E$ , the last term in (4.38) vanishes and, for some positive constant  $\beta_1$ ,

$$\frac{G_0(E(t))}{E(t)} I_6'(t) + cE'(t) \leq -\beta_1 G_0(E(t)). \quad (5.3)$$

Let  $\tau_0 > 0$  and

$$F = \tau_0 \left( \frac{G_0(E)}{E} I_6 + cE \right). \quad (5.4)$$

We have  $F \sim E$  (because  $I_6 \sim E$  and  $\frac{G_0(E)}{E}$  is non-increasing) and, using (5.3),

$$F' \leq -\tau_0 \beta_1 G_0(E). \quad (5.5)$$

Then, for  $\tau_0 > 0$  such that

$$F \leq E \quad \text{and} \quad F(0) \leq 1, \quad (5.6)$$

we get, for  $\alpha_2 = \tau_0 \beta_1 > 0$  (since  $G_0$  is increasing),

$$F' \leq -\alpha_2 G_0(F). \quad (5.7)$$

Then (5.7) implies that

$$(\tilde{G}(F))' \geq \alpha_2, \quad (5.8)$$

where  $\tilde{G}(t) = \int_t^1 \frac{1}{G_0(s)} ds$ . Integrating (5.8) over  $[0, t]$  yields

$$\tilde{G}(F(t)) \geq \alpha_2 t + \tilde{G}(F(0)). \quad (5.9)$$

Because  $F(0) \leq 1$ ,  $\tilde{G}(1) = 0$  and  $\tilde{G}$  is decreasing, we obtain from (5.9) that

$$\tilde{G}(F(t)) \geq \alpha_2 t,$$

which implies that

$$F(t) \leq \tilde{G}^{-1}(\alpha_2 t).$$

The fact that  $F \sim E$  gives (2.34). This completes the proof of Theorem 2.4.

## 6 Exponential stability: (1.2) and (1.12) hold

Exploiting (2.38), (4.2), (4.35) and the property  $g' \leq 0$ , we find

$$\begin{aligned} NE'(t) + c \int_0^L \tilde{\xi} \varphi_t^2 dx &\leq \int_0^L (N(-b + \|d\|_\infty) + c(\|d\|_\infty^2 + \|b\|_\infty b)) \varphi_t^2 dx \\ &\leq \int_0^L (c\|b\|_\infty - N)b\varphi_t^2 dx + \frac{2}{\rho_1} (N\|d\|_\infty + c\|d\|_\infty^2) E_0(t) \end{aligned} \quad (6.1)$$

and

$$-g' \circ \varphi_x \leq -2E'(t) + 2\|d\|_\infty \int_0^L \varphi_t^2 dx \leq -2E'(t) + \frac{4}{\rho_1} \|d\|_\infty E_0(t). \quad (6.2)$$

Therefore, choosing  $N \geq 0$  such that

$$N \geq c\|b\|_\infty \quad \text{and} \quad N > \beta;$$

so  $c\|b\|_\infty - N \leq 0$  and  $I_6 \sim E$  by virtue of (4.40). The constant  $N$  can be chosen in the form

$$N = c(1 + \|b\|_\infty), \quad (6.3)$$

and therefore, inserting (6.1) and (6.2) into (4.38) and noting that the last term in (4.38) vanishes (thanks to (1.2)),  $G_0 = Id$  and  $\tilde{\epsilon}_0 = 0$  (according to (2.35) and (4.39)), we conclude that, for some positive constant  $\beta_2$  which does not depend neither on  $b$  nor on  $d$ ,

$$I'_6(t) + cE'(t) \leq -(c - \beta_2(1 + \|b\|_\infty)) (\|d\|_\infty^2 + \|d\|_\infty) E_0(t) - \tilde{c}E_1(t).$$

Let  $F = I_6 + cE$ . The property  $I_6 \sim E$  and condition (2.39), for

$$d_0 = \frac{c}{\beta_2(1 + \|b\|_\infty)}, \quad (6.4)$$

lead to  $F \sim E$  and

$$F' \leq -\alpha_2 F, \quad (6.5)$$

for some positive constant  $\alpha_2$ . By integrating (6.5) over  $[0, t]$  and using again the equivalence  $F \sim E$ , we find (2.37). This ends the proof of Theorem 2.6.

## 7 Weak stability: (1.2) does not hold and (1.11) holds

In this section, we treat the case when (1.2) does not hold which is more realistic from the physics point of view. We need to estimate the last term in (4.38) using the system (7.1) resulting from differentiating (1.1) with respect to time

$$\begin{cases} \rho_1 \varphi_{ttt} - k_1(\varphi_{xt} + \psi_t)_x + d\varphi_{tt}(t - \tau) + b\varphi_{tt} + \int_0^{+\infty} g(s)(a\varphi_{xt}(t - s))_x ds = 0, \\ \rho_2 \psi_{ttt} - k_2 \psi_{xxt} + k_1(\varphi_{xt} + \psi_t) = 0, \\ \varphi_t(0, t) = \psi_{xt}(0, t) = \varphi_t(L, t) = \psi_{xt}(L, t) = 0. \end{cases} \quad (7.1)$$

System (7.1) is well posed for initial data  $U_0 \in D(A)$  thanks to Theorem 2.3. Let  $E_2$  be the second-order energy (the energy of (7.1)) defined by

$$E_2(t) = \frac{1}{2} \|U_t(t)\|_{\mathcal{H}}^2. \quad (7.2)$$

714 A simple calculation (as for (4.1) and (4.2)) implies, in case (1.11), that

$$715 \quad E_2'(t) \leq \frac{1}{2} g' \circ \varphi_{xt} - \frac{1}{2} \inf_{[0,L]} (b - |d|) \int_0^L \varphi_{tt}^2 dx; \quad (7.3)$$

716 so  $E_2$  is non-increasing (according to (1.11)). Let  $\tau_0 = 1$  in (5.4). Thus, similarly to (5.5)  
 717 (with the same choices of  $\epsilon_0$  and  $N$ ), we deduce from (4.38) that

$$718 \quad F'(t) \leq -\beta_1 G_0(E(t)) + \left( \frac{\rho_1 k_2}{k_1} - \rho_2 \right) \frac{G_0(E(t))}{E(t)} \int_0^L \varphi_{xt} \psi_t dx. \quad (7.4)$$

719 Now, as in [25], we use some ideas of [17].

720 **Lemma 7.1** *For any  $\epsilon > 0$ , we have*

$$721 \quad \left( \frac{\rho_1 k_2}{k_1} - \rho_2 \right) \int_S^T \frac{G_0(E(t))}{E(t)} \int_0^L \varphi_{xt} \psi_t dx dt \leq \epsilon \int_S^T G_0(E(t)) dt$$

$$722 \quad + c_\epsilon \frac{G_0(E(0))}{E(0)} (E(S) + E_2(S)), \quad \forall T \geq S \geq 0. \quad (7.5)$$

723 *Proof* By integration with respect to  $t$ , we get

$$724 \quad \left( \frac{\rho_1 k_2}{k_1} - \rho_2 \right) \int_S^T \frac{G_0(E(t))}{E(t)} \int_0^L \varphi_{xt} \psi_t dx dt = \left( \frac{\rho_1 k_2}{k_1} - \rho_2 \right) \left[ \frac{G_0(E(t))}{E(t)} \int_0^L \varphi_{xt} \psi dx \right]_S^T$$

$$725 \quad - \left( \frac{\rho_1 k_2}{k_1} - \rho_2 \right) \int_S^T \left( \frac{G_0(E(t))}{E(t)} \right)' \int_0^L \varphi_{xt} \psi dx dt$$

$$726 \quad - \left( \frac{\rho_1 k_2}{k_1} - \rho_2 \right) \int_S^T \frac{G_0(E(t))}{E(t)} \int_0^L \varphi_{xtt} \psi dx dt. \quad (7.6)$$

727 Moreover, applying Poincaré’s inequality (2.5), for  $\psi$ , and using the definition of  $E$  and  $E_2$   
 728 and their non-increasingness, we find

$$729 \quad \left| \left( \frac{\rho_1 k_2}{k_1} - \rho_2 \right) \int_0^L \varphi_{xt} \psi dx \right| \leq c (E(t) + E_2(t))$$

$$730 \quad \leq c (E(S) + E_2(S)), \quad \forall 0 \leq S \leq t.$$

731 Thus, by integrating by parts the last integral in (7.6) with respect to  $x$  and noting that  $\frac{G_0(E)}{E}$   
 732 is non-increasing, we have

$$733 \quad \left( \frac{\rho_1 k_2}{k_1} - \rho_2 \right) \int_S^T \frac{G_0(E(t))}{E(t)} \int_0^L \varphi_{xt} \psi_t dx dt$$

$$734 \quad \leq c \frac{G_0(E(0))}{E(0)} (E(S) + E_2(S)) + c \int_S^T \frac{G_0(E(t))}{E(t)} \int_0^L |\varphi_{tt}| |\psi_x| dx dt, \quad \forall T \geq S \geq 0. \quad (7.7)$$

735 On the other hand, according to (1.11) and (7.3) (notice also that  $g$  is non-increasing), we  
 736 have

$$737 \quad \int_0^L \varphi_{tt}^2 dx \leq \frac{-2}{\inf_{[0,L]} (b - |d|)} E_2'(t).$$

738 Then, using (2.14) and Young's inequality (3.1), we estimate the last integral in (7.7) as  
 739 follows:

$$740 \quad c \int_S^T \frac{G_0(E(t))}{E(t)} \int_0^L |\varphi_{tt}| |\psi_x| dx dt \leq \frac{\epsilon \hat{k}}{2} \int_S^T \frac{G_0(E(t))}{E(t)} \psi_x^2 dx dt - c_\epsilon \frac{G_0(E(0))}{E(0)} \int_S^T E_2'(t) dt$$

$$741 \quad \leq \epsilon \int_S^T G_0(E(t)) dt + c_\epsilon \frac{G_0(E(0))}{E(0)} E_2(S), \quad \forall T \geq S \geq 0.$$

742 Inserting this inequality into (7.7), we get (7.5). □

743 Now, exploiting (7.4) and (7.5) and choosing  $\epsilon \in ]0, \beta_1[$ , we get, for  $\beta_3 = \beta_1 - \epsilon$ ,

$$744 \quad \int_S^T F'(t) dt \leq -\beta_3 \int_S^T G_0(E(t)) dt + c \frac{G_0(E(0))}{E(0)} (E(S) + E_2(S)), \quad \forall T \geq S \geq 0.$$

(7.8)

746 By combining (7.8) and the property  $F \sim E$ , we deduce that, for some positive constant  $\beta_4$ ,

$$747 \quad \int_S^T G_0(E(t)) dt \leq \beta_4 \left( 1 + \frac{G_0(E(0))}{E(0)} \right) (E(S) + E_2(S)), \quad \forall T \geq S \geq 0. \quad (7.9)$$

748 Choosing  $S = 0$  in (7.9) and using the fact that  $G_0(E)$  is non-increasing, we get

$$749 \quad G_0(E(T))T \leq \int_0^T G_0(E(t)) dt \leq \beta_4 \left( 1 + \frac{G_0(E(0))}{E(0)} \right) (E(0) + E_2(0)), \quad \forall T \geq 0,$$

750 which gives (2.41), for  $n = 1$ , with  $c_1 = \beta_4 \left( 1 + \frac{G_0(E(0))}{E(0)} \right) (E(0) + E_2(0))$ , since  $G_0^{-1}$  is  
 751 increasing.

752 By induction on  $n$ , suppose that (2.41) holds and let  $U_0 \in D(A^{n+1})$  such that  $a \equiv 0$   
 753 or (2.8) holds or (2.40) holds, for  $n + 1$  instead of  $n$ . We have  $U_t(0) \in D(A^n)$  (thanks to  
 754 Theorem 2.3) and  $U_t$  satisfies the first two equations and the boundary conditions of (1.1).  
 755 On the other hand, if  $a \neq 0$  and (2.8) does not hold, then  $U_t(0)$  satisfies (2.40) (because  $U_0$   
 756 satisfies (2.40), for  $n + 1$ ). Then the energy  $E_2$  of (7.1) (defined in (7.2)) also satisfies, for  
 757 some positive constant  $\tilde{c}_n$ ,

$$758 \quad E_2(t) \leq G_n \left( \frac{\tilde{c}_n}{t} \right), \quad \forall t > 0. \quad (7.10)$$

759 Now, choosing  $S = \frac{T}{2}$  in (7.9), combining with (2.41) and (7.10), and using the fact that  
 760  $G_0(E)$  is non-increasing, we deduce that

$$761 \quad G_0(E(T))T \leq 2 \int_{\frac{T}{2}}^T G_0(E(t)) dt \leq 2\beta_4 \left( 1 + \frac{G_0(E(0))}{E(0)} \right) \left( G_n \left( \frac{2c_n}{T} \right) + G_n \left( \frac{2\tilde{c}_n}{T} \right) \right),$$

762 this implies that, for  $c_{n+1} = \max \left\{ 4\beta_4 \left( 1 + \frac{G_0(E(0))}{E(0)} \right), 2c_n, 2\tilde{c}_n \right\}$  (notice that  $G_n$  is increas-  
 763 ing),

$$764 \quad E(T) \leq G_0^{-1} \left( \frac{c_{n+1}}{T} G_n \left( \frac{c_{n+1}}{T} \right) \right) = G_1 \left( \frac{c_{n+1}}{T} G_n \left( \frac{c_{n+1}}{T} \right) \right) = G_{n+1} \left( \frac{c_{n+1}}{T} \right).$$

765 This proves (2.41), for  $n + 1$ . The proof of Theorem 2.7 is completed.

## 8 General comments and issues

We give in this last section some general comments and issues.

*Remark 8.1* When (1.2) does not hold and (1.12) holds, proving the stability of (1.1) seems a delicate question (even under smallness condition on  $\|d\|_\infty$ ). In this case, there is a double difficulty: the presence of the last term in (4.38) which can not be absorbed by  $E$  itself and the fact that (1.1) and (7.1) are not necessarily dissipative with respect to  $E$  and  $E_2$ , respectively (see (4.2) and (7.2)).

*Remark 8.2* The regularity  $g \in C^1(\mathbb{R}_+)$  can be weakened by assuming that  $g$  is differentiable almost everywhere on  $\mathbb{R}_+$ . On the other hand, our condition (2.9) implies that the set

$$\{s \in \mathbb{R}_+, g(s) > 0 \text{ and } g'(0) = 0\} \quad (8.1)$$

is empty. Using the arguments of [64–68], our stability results can be extended to the case of convolution kernels  $g$  having flat zones up to a certain extent; that is, the set (8.1) is not negligible but small enough in some sense.

*Remark 8.3* It is interesting to determine the biggest value of  $d_0$  in (2.39) which guarantees the exponential stability (2.37) of (1.1) when (1.2) and (1.12) hold. On the other hand, is the system (1.1) unstable when (1.2) and (1.12) hold, but  $\|d\|_\infty$  is not small enough?

*Remark 8.4* Another interesting question concerns the stability of (1.1) with an additional discrete time delay  $\tilde{d}\psi_t(t - \tilde{\tau})$  considered on the second equation, where  $\tilde{\tau}$  is a positive constant and  $\tilde{d} : [0, L] \rightarrow \mathbb{R}$  is a given function.

*Remark 8.5* The arguments applied in [20] to get the stability of (1.10) can be adapted to (1.1) and a general stability estimate can be proved when (1.2), (1.12) and (2.9) hold (so  $g$  can converge to zero at infinity less faster than exponentially). The arguments of [20] are based on an approach introduced and developed in [64–68]. This approach allowed us to deal with some arbitrary decaying kernels  $g$  without assuming explicit conditions on their derivatives  $g'$  and to avoid passing by  $E'$  in objective to overcome subsequently the difficulties generated by the non-dissipativeness character of (1.10). On the other hand, the arguments of [20] can be used to obtain the stability of (1.1) in case where the discrete time delay  $d\varphi_t(t - \tau)$  is replaced by a distributed one

$$\int_0^{+\infty} f(s)\varphi_t(t - s) ds,$$

for some given function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ . Moreover, the results of the present paper remain true if we replace the linear damping  $b\varphi_t$  by a non-linear one  $bh(\varphi_t)$ , for some given function  $h : \mathbb{R} \rightarrow \mathbb{R}$ . Finally, some other Timoshenko-type systems with controls and time delays on the displacement can be considered (see [25] concerning the case where no delay is considered). To keep away this paper of being too long, we do not discuss these situations.

*Remark 8.6* When  $\inf_{[0, L]} a > 0$  and  $\|d\|_\infty$  is small enough, the stability estimates (2.34) and (2.41) hold true also in case

$$\inf_{[0, L]} (b - |d|) = 0. \quad (8.2)$$

More precisely, we have the following:

**Theorem 8.7** Assume that (H1)–(H3) and (8.2) are satisfied and  $\inf_{[0,L]} a > 0$ . Let

$$\xi = \begin{cases} \tau b & \text{if } d \neq 0, \\ 0 & \text{if } d \equiv 0 \end{cases} \quad \text{and} \quad \xi_0 = 0 \quad (8.3)$$

instead of (2.25) and (2.28). Then the well-posedness result of Theorem 2.3 holds true. Moreover, there exists a positive constant  $d_0$  independent of  $d$  such that, if

$$\|d\|_\infty < d_0, \quad (8.4)$$

then

1. Case (1.2) holds: for any  $U_0 \in \mathcal{H}$  such that (2.8) or (2.33) holds,  $E$  satisfies (2.34).
2. Case (1.2) does not hold: for any  $n \in \mathbb{N}^*$  and  $U_0 \in D(A^n)$  such that (2.8) or (2.40) holds,  $E$  satisfies (2.41).

*Proof* First, according to (8.2) and (8.3), (2.27) and (3.2) imply that  $B \equiv 0$  and (3.3), respectively. The rest of the proof of Theorem 2.3 is identical to the one given in Sect. 3.

Second, under the choice (8.3), (4.3) and (8.2) imply that

$$-g' \circ \varphi_x \leq -2E'(t) \quad (8.5)$$

and

$$E'(t) \leq -\frac{1}{2} \int_0^L b\varphi_t^2 dx + \frac{\|d\|_\infty}{2} \int_0^L \varphi_t^2 dx. \quad (8.6)$$

Similarly to (8.5), we have also

$$-g' \circ \varphi_{xt} \leq -2E_2'(t). \quad (8.7)$$

Because  $\xi \leq \tau b$ , then

$$\tilde{c} = c\|d\|_\infty \quad \text{and} \quad \tilde{\xi} = \|b\|_\infty b \quad (8.8)$$

instead of (4.34) and (4.35). Consequently, using (8.6), we have

$$NE'(t) + c \int_0^L \tilde{\xi}\varphi_t^2 dx \leq \int_0^L \left( c\tilde{\xi} - \frac{N}{2}b \right) \varphi_t^2 dx + \frac{N\|d\|_\infty}{\rho_1} E_0(t). \quad (8.9)$$

Therefore, inserting (8.5) and (8.9) into (4.38), we get

$$\begin{aligned} \frac{G_0(E(t))}{E(t)} I_6'(t) &\leq -\frac{G_0(E(t))}{E(t)} \left( \left( c - \frac{N\|d\|_\infty}{\rho_1} - \tilde{\epsilon}_0 \right) E_0(t) + (\tilde{c} - \tilde{\epsilon}_0) E_1(t) \right) \\ &\quad - c \left( 1 + G'(\epsilon_0 E(t)) \right) E'(t) + \frac{G_0(E(t))}{E(t)} \int_0^L \left( c\tilde{\xi} - \frac{N}{2}b \right) \varphi_t^2 dx \\ &\quad + \left( \frac{\rho_1 k_2}{k_1} - \rho_2 \right) \frac{G_0(E(t))}{E(t)} \int_0^L \varphi_{xt} \psi_t dx. \end{aligned} \quad (8.10)$$

Choosing  $N \geq 0$  such that

$$c\tilde{\xi} - \frac{N}{2}b \leq 0 \quad \text{and} \quad N > \beta;$$



so  $N$  can be taken as in (6.3), therefore  $I_6 \sim E$  (due to (4.40)) and, for some positive constant  $\beta_5$  which does not depend neither on  $b$  nor on  $d$  (notice that  $G'(\epsilon_0 E)$  is non-increasing),

$$\begin{aligned} \frac{G_0(E(t))}{E(t)} I_6'(t) &\leq -\frac{G_0(E(t))}{E(t)} ((c - \beta_5(1 + \|b\|_\infty)\|d\|_\infty - \tilde{\epsilon}_0) E_0(t) + (\tilde{c} - \tilde{\epsilon}_0) E_1(t)) \\ &\quad - cE'(t) + \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \frac{G_0(E(t))}{E(t)} \int_0^L \varphi_{x_t} \psi_t dx. \end{aligned} \quad (8.11)$$

Next, exploiting (8.4), for  $d_0 = \frac{c}{\beta_5(1+\|b\|_\infty)}$ , and choosing  $\epsilon > 0$  such that

$$c - \beta_5(1 + \|b\|_\infty)\|d\|_\infty - \tilde{\epsilon}_0 > 0 \quad \text{and} \quad \tilde{c} - \tilde{\epsilon}_0 > 0,$$

we deduce from (8.11) that, for some positive constant  $\beta_6$ ,

$$\frac{G_0(E(t))}{E(t)} I_6'(t) + cE'(t) \leq -\beta_6 G_0(E(t)) + \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \frac{G_0(E(t))}{E(t)} \int_0^L \varphi_{x_t} \psi_t dx. \quad (8.12)$$

If (1.2) holds, then (8.12) coincides with (5.3) and the proof of (2.34) can be finished as in Sect. 5.

If (1.2) does not hold, we consider the functional  $F$  defined in (5.4) with  $\tau_0 = 1$ , and then (8.12) becomes identical to (7.4). Consequently, the proof of (2.41) can be ended as in Sect. 7.  $\square$

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