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# ASYMPTOTIC STABILITY OF BRESSE SYSTEM WITH ONE INFINITE MEMORY IN THE LONGITUDINAL DISPLACEMENTS

AISSA GUESMIA

**Abstract.** The asymptotic stability of one-dimensional linear Bresse systems under infinite memories was proved by Guesmia and Kafini [12] under three infinite memories, Guesmia and Kirane [13] under two infinite memories, and De Lima Santos et al. [6] under one infinite memory acting on the shear angle displacements. The subject of this paper is to complete these results by proving that the asymptotic stability of Bresse systems holds also under one infinite memory acting on the longitudinal displacements.

**Keywords:** Bresse system, Infinite memory, Asymptotic behavior, Energy method, integral inequalities.

**AMS Classification:** 35B40, 35L45, 74H40, 93D20, 93D15.

## 1. INTRODUCTION

In this paper, we consider a Bresse system in one-dimensional open bounded domain under the homogeneous Dirichlet-Neumann-Neumann boundary conditions and with one infinite memory acting on the third equation (longitudinal displacements)

$$(1.1) \quad \begin{cases} \rho_1 \varphi_{tt} - k_1(\varphi_x + \psi + lw)_x - lk_3(w_x - l\varphi) = 0, \\ \rho_2 \psi_{tt} - k_2 \psi_{xx} + k_1(\varphi_x + \psi + lw) = 0, \\ \rho_1 w_{tt} - k_3(w_x - l\varphi)_x + lk_1(\varphi_x + \psi + lw) + \int_0^{+\infty} g(s)w_{xx}(x, t-s) ds = 0, \\ \varphi(0, t) = \psi_x(0, t) = w_x(0, t) = \varphi(L, t) = \psi_x(L, t) = w_x(L, t) = 0, \\ \varphi(x, 0) = \varphi_0(x), \varphi_t(x, 0) = \varphi_1(x), \\ \psi(x, 0) = \psi_0(x), \psi_t(x, 0) = \psi_1(x), \\ w(x, -t) = w_0(x, t), w_t(x, 0) = w_1(x), \end{cases}$$

where  $(x, t) \in ]0, L[ \times \mathbb{R}_+$ ,  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a given function, and  $L, l, \rho_i, k_i \in \mathbb{R}_+^*$ .

The Bresse system [3] is known as the circular arch problem, where  $\varphi$ ,  $w$  and  $\psi$  represent, respectively, the vertical, longitudinal and shear angle displacements. For more details, we refer to [15] and [16] (see also [12] and [13]).

The stability of Bresse systems with (local or global) frictional dampings was obtained by several researchers in the last few years; see [1], [7], [19] and [21] for the case of one frictional damping acting on the shear angle displacements, [2], [23] and [24] for the case of two frictional dampings, and [4], [20], [22] and [24] for the

case of three frictional dampings. When each equation is controlled by a frictional damping, the exponential stability of Bresse systems was proved regardless to the speeds of wave propagations given by

$$(1.2) \quad s_1 = \sqrt{\frac{k_1}{\rho_1}}, \quad s_2 = \sqrt{\frac{k_2}{\rho_2}} \quad \text{and} \quad s_3 = \sqrt{\frac{k_3}{\rho_1}}.$$

When at least one equation is free, the obtained stability estimate is of exponential or polynomial type depending on some relations between  $s_i$ . When only one frictional damping is considered, it was proved that the exponential stability is equivalent to

$$(1.3) \quad s_1 = s_2 = s_3.$$

Similar stability results were proved in [8], [17] and [18] in case where the Bresse system is coupled with one or two heat equations in a certain manner.

The stability of Bresse systems with memories was also recently studied. When the three equations are controlled via infinite memories of the form

$$\int_0^{+\infty} g_1(s) \varphi_{xx}(x, t-s) ds, \quad \int_0^{+\infty} g_2(s) \psi_{xx}(x, t-s) ds \quad \text{and} \quad \int_0^{+\infty} g_3(s) w_{xx}(x, t-s) ds,$$

where  $g_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  are differentiable, non-increasing and integrable functions on  $\mathbb{R}_+$ , the stability was proved in [12] regardless to  $s_i$ . The obtained decay estimate in [12] depends only on the arbitrary growth at infinity of  $s \mapsto g_i(s)$ . When only two memories are considered, the stability of Bresse systems was proved in [13], where the decay rate depends also on  $s_i$  and the smoothness of initial data.

As far as we know, the first stability result for Bresse systems with only one infinite memory is the one obtained in [6] under

$$(1.4) \quad \int_0^{+\infty} g(s) \psi_{xx}(x, t-s) ds$$

acting on the shear angle displacements (the second equation in (1.1)), where  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  converges exponentially to zero at infinity.

Our objectif in this paper is to prove that the asymptotic stability of Bresse systems holds also under one infinite memory acting on the longitudinal displacements; that is (1.1) is stable, where the decay rate of solutions depends on the arbitrary growth at infinity of the kernel  $g$ , the speeds of wave propagations (1.2) and the smoothness of initial data.

The paper is organized as follows. In section 2, we present our hypotheses and state our main results. The proof of our main results will be given in section 3.

## 2. HYPOTHESES AND MAIN RESULTS

Following the method of [5], we consider the functional

$$(2.1) \quad \eta(x, t, s) = w(x, t) - w(x, t-s) \quad \text{in } ]0, L[ \times \mathbb{R}_+ \times \mathbb{R}_+.$$

This functional satisfies

$$(2.2) \quad \begin{cases} \eta_t + \eta_s - w_t = 0 & \text{in } ]0, L[ \times \mathbb{R}_+ \times \mathbb{R}_+, \\ \eta_x(0, t, s) = \eta_x(L, t, s) = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}_+, \\ \eta(x, t, 0) = 0 & \text{in } ]0, L[ \times \mathbb{R}_+. \end{cases}$$

Let  $\eta^0(x, s) = \eta(x, 0, s)$ ,  $U^0 = (\varphi_0, \psi_0, w_0, \varphi_1, \psi_1, w_1, \eta^0)^T$ ,  $U = (\varphi, \psi, w, \varphi_t, \psi_t, w_t, \eta)^T$  and

$$(2.3) \quad g^0 = \int_0^{+\infty} g(s) ds.$$

Then the system (1.1) takes the following abstract form:

$$(2.4) \quad \begin{cases} U_t = AU, \\ U(t=0) = U^0, \end{cases}$$

where  $A$  is the linear operator defined by

$$AU = \begin{pmatrix} \varphi_t \\ \psi_t \\ w_t \\ \frac{k_1}{\rho_1} \varphi_{xx} - \frac{l^2 k_3}{\rho_1} \varphi + \frac{k_1}{\rho_1} \psi_x + \frac{l}{\rho_1} (k_1 + k_3) w_x \\ -\frac{k_1}{\rho_2} \varphi_x + \frac{k_2}{\rho_2} \psi_{xx} - \frac{k_1}{\rho_2} \psi - \frac{lk_1}{\rho_2} w \\ -\frac{l}{\rho_1} (k_1 + k_3) \varphi_x - \frac{lk_1}{\rho_1} \psi + \frac{1}{\rho_1} (k_3 - g^0) w_{xx} - \frac{l^2 k_1}{\rho_1} w + \frac{1}{\rho_1} \int_0^{+\infty} g \eta_{xx} ds \\ w_t - \eta_s \end{pmatrix}.$$

Let

$$(2.5) \quad L_0 = \left\{ v : \mathbb{R}_+ \rightarrow H_*^1(]0, L[), \int_0^L \int_0^{+\infty} g v_x^2 ds dx < +\infty \right\}$$

and

$$(2.6) \quad H = H_0^1(]0, L[) \times (H_*^1(]0, L[))^2 \times L^2(]0, L[) \times (L_*^2(]0, L[))^2 \times L_0,$$

where

$$(2.7) \quad L_*^2(]0, L[) = \left\{ v \in L^2(]0, L[), \int_0^L v dx = 0 \right\}$$

and

$$(2.8) \quad H_*^1(]0, L[) = \left\{ v \in H^1(]0, L[), \int_0^L v dx = 0 \right\}.$$

The domain  $D(A)$  of  $A$  is defined by

$$(2.9) \quad \begin{aligned} D(A) = \left\{ V = (v_1, \dots, v_7)^T \in H, AV \in H, v_7(0) = 0, \partial_x v_2(0) = \partial_x v_3(0) = 0, \right. \\ \left. \partial_x v_2(L) = \partial_x v_3(L) = 0, \partial_x v_7(\cdot, 0) = \partial_x v_7(\cdot, L) = 0 \right\}. \end{aligned}$$

*Remark 2.1.* As in [13], integrating on  $]0, L[$  the second and third equations in (1.1), and using the boundary conditions, we verify that

$$(2.10) \quad \partial_{tt} \left( \int_0^L \psi dx \right) + \frac{k_1}{\rho_2} \int_0^L \psi dx + \frac{lk_1}{\rho_2} \int_0^L w dx = 0$$

and

$$(2.11) \quad \partial_{tt} \left( \int_0^L w \, dx \right) + \frac{l^2 k_1}{\rho_1} \int_0^L w \, dx + \frac{l k_1}{\rho_1} \int_0^L \psi \, dx = 0.$$

Therefore, (2.10) implies that

$$(2.12) \quad \int_0^L w \, dx = -\frac{\rho_2}{l k_1} \partial_{tt} \left( \int_0^L \psi \, dx \right) - \frac{1}{l} \int_0^L \psi \, dx.$$

Substiting (2.12) into (2.11), we get

$$(2.13) \quad \partial_{ttt} \left( \int_0^L \psi \, dx \right) + \left( \frac{k_1}{\rho_2} + \frac{l^2 k_1}{\rho_1} \right) \partial_{tt} \left( \int_0^L \psi \, dx \right) = 0.$$

Let  $l_0 = \sqrt{\frac{k_1}{\rho_2} + \frac{l^2 k_1}{\rho_1}}$ . Then, solving (2.13), we find

$$(2.14) \quad \int_0^L \psi \, dx = \tilde{c}_1 \cos(l_0 t) + \tilde{c}_2 \sin(l_0 t) + \tilde{c}_3 t + \tilde{c}_4,$$

where  $\tilde{c}_1, \dots, \tilde{c}_4$  are real constants. By combining (2.12) and (2.14), we get

$$(2.15) \quad \int_0^L w \, dx = \tilde{c}_1 \left( \frac{\rho_2 l_0^2}{l k_1} - \frac{1}{l} \right) \cos(l_0 t) + \tilde{c}_2 \left( \frac{\rho_2 l_0^2}{l k_1} - \frac{1}{l} \right) \sin(l_0 t) - \frac{\tilde{c}_3}{l} t - \frac{\tilde{c}_4}{l}.$$

Let  $(\tilde{\psi}_0(x), \tilde{w}_0(x)) = (\psi_0(x), w_0(x, 0))$ . Using the initial data of  $\psi$  and  $w$  in (1.1), we see that

$$\begin{cases} \tilde{c}_1 = \frac{k_1}{\rho_2 l_0^2} \int_0^L \tilde{\psi}_0 \, dx + \frac{l k_1}{\rho_2 l_0^2} \int_0^L \tilde{w}_0 \, dx, \\ \tilde{c}_2 = \frac{k_1}{\rho_2 l_0^3} \int_0^L \psi_1 \, dx + \frac{l k_1}{\rho_2 l_0^3} \int_0^L w_1 \, dx, \\ \tilde{c}_3 = \left( 1 - \frac{k_1}{\rho_2 l_0^2} \right) \int_0^L \psi_1 \, dx - \frac{l k_1}{\rho_2 l_0^2} \int_0^L w_1 \, dx, \\ \tilde{c}_4 = \left( 1 - \frac{k_1}{\rho_2 l_0^2} \right) \int_0^L \tilde{\psi}_0 \, dx - \frac{l k_1}{\rho_2 l_0^2} \int_0^L \tilde{w}_0 \, dx. \end{cases}$$

Let

$$(2.16) \quad \tilde{\psi} = \psi - \frac{1}{L} (\tilde{c}_1 \cos(l_0 t) + \tilde{c}_2 \sin(l_0 t) + \tilde{c}_3 t + \tilde{c}_4)$$

and

$$(2.17) \quad \tilde{w} = w - \frac{1}{L} \left( \tilde{c}_1 \left( \frac{\rho_2 l_0^2}{l k_1} - \frac{1}{l} \right) \cos(l_0 t) + \tilde{c}_2 \left( \frac{\rho_2 l_0^2}{l k_1} - \frac{1}{l} \right) \sin(l_0 t) - \frac{\tilde{c}_3}{l} t - \frac{\tilde{c}_4}{l} \right).$$

Then, from (2.14) and (2.15), one can check that

$$(2.18) \quad \int_0^L \tilde{\psi} \, dx = \int_0^L \tilde{w} \, dx = \int_0^L \tilde{\eta} \, dx = 0,$$

where

$$\tilde{\eta}(x, t, s) = \tilde{w}(x, t) - \tilde{w}(x, t - s) \quad \text{in } ]0, L[ \times \mathbb{R}_+ \times \mathbb{R}_+.$$

Therefore, Poincaré's inequality

$$(2.19) \quad \exists c_0 > 0 : \int_0^L v^2 dx \leq c_0 \int_0^L v_x^2 dx, \quad \forall v \in H_*^1(]0, L[) \cup H_0^1(]0, L[)$$

is applicable for  $\tilde{\psi}$ ,  $\tilde{w}$  and  $\tilde{\eta}$ . In addition,  $(\varphi, \tilde{\psi}, \tilde{w})$  satisfies the boundary conditions and the first three equations in (1.1) with initial data

$$\psi_0 - \frac{1}{L}(\tilde{c}_1 + \tilde{c}_4), \quad \psi_1 - \frac{1}{L}(l_0 \tilde{c}_2 + \tilde{c}_3), \quad w_0 - \frac{1}{L} \left( \tilde{c}_1 \left( \frac{\rho_2 l_0^2}{l k_1} - \frac{1}{l} \right) - \frac{\tilde{c}_4}{l} \right) \quad \text{and} \quad w_1 - \frac{1}{L} \left( \tilde{c}_2 l_0 \left( \frac{\rho_2 l_0^2}{l k_1} - \frac{1}{l} \right) - \frac{\tilde{c}_3}{l} \right)$$

instead of  $\psi_0$ ,  $\psi_1$ ,  $w_0$  and  $w_1$ , respectively. In the sequel, we work with  $\tilde{\psi}$ ,  $\tilde{w}$  and  $\tilde{\eta}$  instead of  $\psi$ ,  $w$  and  $\eta$ , respectively, but, for simplicity of notation, we use  $\psi$ ,  $w$  and  $\eta$ .

Now, the following hypothesis guarantees the well-posedness of (2.4):

**(H1)** Assume that the function  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is differentiable, non-increasing and integrable on  $\mathbb{R}_+$ , and there exists a positive constant  $k_0$  such that, for any

$$(\varphi, \psi, w)^T \in H_0^1(]0, L[) \times (H_*^1(]0, L[))^2,$$

we have

$$(2.20) \quad k_0 \int_0^L (\varphi_x^2 + \psi_x^2 + w_x^2) dx \leq \int_0^L (k_2 \psi_x^2 + k_1 (\varphi_x + \psi + lw)^2 + k_3 (w_x - l\varphi)^2 - g^0 w_x^2) dx.$$

Moreover, assume that there exists a positive constant  $\beta$  such that

$$(2.21) \quad -\beta g(s) \leq g'(s), \quad \forall s \in \mathbb{R}_+.$$

We notice that, under the hypothesis **(H1)**, the sets  $L_0$  and  $H$  are Hilbert spaces equipped with the inner products that generate the norms, for  $v \in L_0$  and  $V = (v_1, \dots, v_7)^T \in H$ ,

$$(2.22) \quad \|v\|_{L_0}^2 = \int_0^L \int_0^{+\infty} g v_x^2 ds dx$$

and

$$(2.23) \quad \|V\|_H^2 = \int_0^L (k_2 (\partial_x v_2)^2 + k_1 (\partial_x v_1 + v_2 + l v_3)^2 + k_3 (\partial_x v_3 - l v_1)^2 - g^0 (\partial_x v_3)^2) dx \\ + \int_0^L (\rho_1 v_4^2 + \rho_2 v_5^2 + \rho_1 v_6^2) dx + \|v_7\|_{L_0}^2.$$

Exactly as in [13] one can prove that  $A$  generates a  $C_0$ -semigroup of contractions in  $H$  by proving that  $-A$  is maximal monotone (it is enough to neglect the second memory in the first system considered in [13], and the proof is the same as in [13]), and deduce the following well-posedness results of (2.4).

**Theorem 2.2.** *Assume that **(H1)** holds. Let  $n \in \mathbb{N}$  and  $U^0 \in D(A^n)$ . Then (2.4) has a unique solution*

$$(2.24) \quad U \in \cap_{k=0}^n C^{n-k}(\mathbb{R}_+; D(A^k)).$$

To get the stability of (2.4), we consider the following additional hypothesis:

**(H2)** Assume that  $g(0) > 0$ , and there exist a positive constant  $\alpha$  and an increasing strictly convex function  $G : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  of class  $C^1(\mathbb{R}_+) \cap C^2(]0, +\infty[)$  satisfying

$$G(0) = G'(0) = 0 \quad \text{and} \quad \lim_{t \rightarrow +\infty} G'(t) = +\infty$$

such that

$$(2.25) \quad g'(s) \leq -\alpha g(s), \quad \forall s \in \mathbb{R}_+$$

or

$$(2.26) \quad \int_0^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))} ds + \sup_{s \in \mathbb{R}_+} \frac{g(s)}{G^{-1}(-g'(s))} < +\infty.$$

Let us consider the energy functional  $E$  associated to (2.4) defined by

$$(2.27) \quad E(t) = \frac{1}{2} \|U(t)\|_H^2.$$

First, we consider the case (1.3).

**Theorem 2.3.** Assume that **(H1)**, **(H2)** and (1.3) are satisfied. Let  $U^0 \in H$  be such that

$$(2.28) \quad (2.25) \text{ holds} \quad \text{or} \quad \sup_{t \in \mathbb{R}_+} \int_t^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))} \int_0^L (\eta_x^0(x, s-t))^2 dx ds < +\infty.$$

Then there exist positive constants  $\tilde{l}$  and  $\tilde{g}$  (not depending neither on  $l$  nor on  $g$ ) so that, if

$$(2.29) \quad l < \tilde{l} \quad \text{and} \quad g^0 < \tilde{g},$$

then there exist positive constants  $c'$  and  $c''$  satisfying

$$(2.30) \quad E'(t) \leq c'' \tilde{G}^{-1}(c't), \quad \forall t \in \mathbb{R}_+,$$

where

$$(2.31) \quad \tilde{G}(s) = \int_s^1 \frac{1}{G_0(\tau)} d\tau \quad \text{and} \quad G_0(s) = \begin{cases} s & \text{if (2.25) holds,} \\ sG'(s) & \text{if (2.26) holds.} \end{cases}$$

When (1.3) does not hold and  $s_1 = s_2$ ; that is

$$(2.32) \quad s_1 = s_2 \quad \text{and} \quad s_1 \neq s_3,$$

we prove the following weaker stability result.

**Theorem 2.4.** Assume that **(H1)**, **(H2)** and (2.32) are satisfied. Let  $n \in \mathbb{N}^*$  and  $U^0 \in D(A^n)$  be such that

$$(2.33) \quad (2.25) \text{ holds} \quad \text{or} \quad \sup_{t \in \mathbb{R}_+} \max_{k=0, \dots, n} \int_t^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))} \int_0^L (\partial_s^k \eta_x^0(x, s-t))^2 dx ds < +\infty.$$

Then there exist positive constants  $\tilde{l}$  and  $\tilde{g}$  (not depending neither on  $l$  nor on  $g$ ) such that, if (2.29) holds, then there exists a positive constant  $c_n$  satisfying

$$(2.34) \quad E(t) \leq c_n G_n\left(\frac{c_n}{t}\right), \quad \forall t > 0,$$

where  $G_m(s) = G_1(sG_{m-1}(s))$ , for  $m = 2, \dots, n$ ,  $G_1 = G_0^{-1}$  and  $G_0$  is defined in (2.31).

*Remark 2.5.* 1. If (2.25) holds, then (2.30) and (2.34) give, respectively, for some positive constants  $d_1$  and  $d_2$ ,

$$(2.35) \quad E(t) \leq d_1 e^{-d_2 t}, \quad \forall t \in \mathbb{R}_+$$

and

$$(2.36) \quad E(t) \leq \frac{d_1}{t^n}, \quad \forall t > 0.$$

The particular estimates (2.35) and (2.36) (for  $n = 1$ ) coincide with the ones of [6] obtained in case (1.4) with  $g$  converging exponentially to zero at infinity.

2. Condition (2.26) (introduced in [10]) allows  $s \mapsto g(s)$  to have a decay rate at infinity arbitrarily close to  $\frac{1}{s}$ . For specific examples of  $g$  and  $U^0$  satisfying (2.26) and (2.33), respectively, and the corresponding decay rates (2.30) and (2.34), see [10] and [11].

### 3. PROOF OF (2.30) AND (2.34)

**3.1. Preliminaries lemmas.** We will use  $c, c_{y_1}, c_{y_1, y_2}, \dots$ , throughout the rest of this paper, to denote generic positive constants which depend continuously on the initial data  $U^0$  and some constants  $y_1, y_2, \dots$ , introduced in the proof.

First, simple computations (see [13]), we see that

$$(3.1) \quad E'(t) = \frac{1}{2} \int_0^L \int_0^{+\infty} g' \eta_x^2 ds dx.$$

Recalling that  $g$  is non-increasing, (3.1) implies that  $E$  is non-increasing.

We start our lemmas by considering the following functional:

$$(3.2) \quad I(t) = -\rho_1 \int_0^L w_t \int_0^{+\infty} g(s) \eta ds dx.$$

**Lemma 3.1.** *For any  $\delta_0 > 0$ , there exists  $c_{\delta_0, l, g^0, g(0)} > 0$  such that*

$$(3.3) \quad I'(t) \leq -\rho_1 g^0 \int_0^L w_t^2 dx + \delta_0 E(t) + c_{\delta_0, l, g^0, g(0)} \int_0^L \int_0^{+\infty} (g(s) - g'(s)) \eta_x^2 ds dx.$$

*Proof.* First, noticing that

$$\begin{aligned} \partial_t \int_0^{+\infty} g(s) \eta ds &= \partial_t \int_{-\infty}^t g(t-s)(w(t) - w(s)) ds \\ &= \int_{-\infty}^t g'(t-s)(w(t) - w(s)) ds + \left( \int_{-\infty}^t g(t-s) ds \right) w_t; \end{aligned}$$

that is

$$(3.4) \quad \partial_t \int_0^{+\infty} g(s) \eta ds = \int_0^{+\infty} g'(s) \eta ds + g^0 w_t.$$



Second, using Young's and Hölder's inequalities, we get the following inequality: for any  $\lambda > 0$ , there exists  $c_\lambda > 0$  such that, for any  $v \in L^2([0, L])$  and  $f \in \{\eta, \eta_x\}$ ,

$$(3.5) \quad \left| \int_0^L v \int_0^{+\infty} g(s) f \, ds \, dx \right| \leq \lambda \int_0^L v^2 \, dx + c_{\lambda, g^0} \int_0^L \int_0^{+\infty} g(s) f^2 \, ds \, dx.$$

Similarly,

$$(3.6) \quad \left| \int_0^L v \int_0^{+\infty} g'(s) f \, ds \, dx \right| \leq \lambda \int_0^L v^2 \, dx - c_{\lambda, g(0)} \int_0^L \int_0^{+\infty} g'(s) f^2 \, ds \, dx.$$

Now, direct computations, using the third equation in (1.1), integrating by parts and using the boundary conditions and (3.4), yield

$$\begin{aligned} I'(t) = & -\rho_1 g^0 \int_0^L w_t^2 \, dx + \int_0^L \left( \int_0^{+\infty} g(s) \eta_x \, ds \right)^2 \, dx \\ & + k_3 \int_0^L (w_x - l\varphi) \int_0^{+\infty} g(s) \eta_x \, ds \, dx + lk_1 \int_0^L (\varphi_x + \psi + lw) \int_0^{+\infty} g(s) \eta \, ds \, dx \\ & - \rho_1 \int_0^L w_t \int_0^{+\infty} g'(s) \eta \, ds \, dx - g^0 \int_0^L w_x \int_0^{+\infty} g(s) \eta_x \, ds \, dx. \end{aligned}$$

Using (3.5) and (3.6) for the last four terms in the above equality, Poincaré's inequality (2.19) for  $\eta$ , and (2.20) and Hölder's inequality to estimate

$$\int_0^L w_x^2 \, dx \quad \text{and} \quad \left( \int_0^{+\infty} g(s) \eta_x \, ds \right)^2,$$

respectively, we get (3.3).  $\square$

**Lemma 3.2.** *Let*

$$(3.7) \quad J(t) = -\rho_2 \int_0^L (\varphi_x + \psi + lw) \psi_t \, dx - \frac{k_2 \rho_1}{k_1} \int_0^L \psi_x \varphi_t \, dx.$$

*Then, for any  $\epsilon_0, \epsilon_1 > 0$ , there exists  $c_{\epsilon_0, l} > 0$  such that*

$$\begin{aligned} (3.8) \quad J'(t) \leq & k_1 \int_0^L (\varphi_x + \psi + lw)^2 \, dx + \frac{lk_2 k_3 \epsilon_1}{2k_1} \int_0^L (w_x - l\varphi)^2 \, dx + \frac{lk_2 k_3}{2k_1 \epsilon_1} \int_0^L \psi_x^2 \, dx \\ & + (-\rho_2 + \epsilon_0) \int_0^L \psi_t^2 \, dx + c_{\epsilon_0, l} \int_0^L w_t^2 \, dx + \left( \frac{k_2 \rho_1}{k_1} - \rho_2 \right) \int_0^L \psi_t \varphi_{xt} \, dx. \end{aligned}$$

*Proof.* By exploiting the first two equations in (1.1), integrating by parts and using the boundary conditions, we get

$$\begin{aligned} J'(t) = & k_1 \int_0^L (\varphi_x + \psi + lw)^2 \, dx + \left( \frac{k_2 \rho_1}{k_1} - \rho_2 \right) \int_0^L \psi_t \varphi_{xt} \, dx - \rho_2 \int_0^L \psi_t^2 \, dx \\ & - \rho_2 l \int_0^L \psi_t w_t \, dx - \frac{lk_2 k_3}{k_1} \int_0^L (w_x - l\varphi) \psi_x \, dx. \end{aligned}$$

Applying Young's inequality for the last two terms in the above equality, we find (3.8).  $\square$

**Lemma 3.3.** *Let*

(3.9)

$$K(t) = \rho_1 \int_0^L (\varphi_x + \psi + lw) w_t dx + \frac{k_3 \rho_1}{k_1} \int_0^L (w_x - l\varphi) \varphi_t dx - \frac{\rho_1}{k_1} \int_0^L \varphi_t \int_0^{+\infty} g(s) w_x(t-s) ds dx.$$

Then, for any  $\delta_0, \epsilon_0 > 0$ , there exist  $c_{\delta_0, l, g^0, g(0)}, c_{\epsilon_0, l} > 0$  such that

(3.10)

$$\begin{aligned} K'(t) \leq & -lk_1 \int_0^L (\varphi_x + \psi + lw)^2 dx + \frac{lk_3^2}{k_1} \int_0^L (w_x - l\varphi)^2 dx + \delta_0 E(t) \\ & - \frac{lk_3 g^0}{k_1} \int_0^L (w_x - l\varphi) w_x dx + \int_0^L (c_{\epsilon_0, l} w_t^2 + \epsilon_0 \psi_t^2) dx - \frac{lk_3 \rho_1}{k_1} \int_0^L \varphi_t^2 dx \\ & + \rho_1 \left( \frac{k_3}{k_1} - 1 \right) \int_0^L w_{xt} \varphi_t dx + c_{\delta_0, l, g^0, g(0)} \int_0^L \int_0^{+\infty} (g(s) - g'(s)) \eta_x^2 ds dx. \end{aligned}$$

*Proof.* First, we notice that

$$\begin{aligned} \partial_t \int_0^{+\infty} g(s) w_x(t-s) ds &= \partial_t \int_{-\infty}^t g(t-s) w_x(s) ds \\ &= g(0) w_x(t) + \int_{-\infty}^t g'(t-s) w_x(s) ds \\ &= - \int_0^{+\infty} g'(s) w_x(t) ds + \int_0^{+\infty} g'(s) w_x(t-s) ds; \end{aligned}$$

that is

$$(3.11) \quad \partial_t \int_0^{+\infty} g(s) w_x(t-s) ds = - \int_0^{+\infty} g'(s) \eta_x ds.$$

Now, using the first and third equations in (1.1), integrating by parts, recalling (3.11) and using the boundary conditions, we find

$$\begin{aligned} K'(t) = & -lk_1 \int_0^L (\varphi_x + \psi + lw)^2 dx + \frac{lk_3^2}{k_1} \int_0^L (w_x - l\varphi)^2 dx + \rho_1 \left( \frac{k_3}{k_1} - 1 \right) \int_0^L \varphi_t w_{xt} dx \\ & + l\rho_1 \int_0^L w_t^2 dx - \frac{lk_3 \rho_1}{k_1} \int_0^L \varphi_t^2 dx - \frac{lk_3 g^0}{k_1} \int_0^L (w_x - l\varphi) w_x dx + \rho_1 \int_0^L \psi_t w_t dx \\ & + \frac{\rho_1}{k_1} \int_0^L \varphi_t \int_0^{+\infty} g'(s) \eta_x ds dx + \frac{lk_3}{k_1} \int_0^L (w_x - l\varphi) \int_0^{+\infty} g(s) \eta_x ds dx. \end{aligned}$$

By applying (3.5), (3.6) and Young's inequality for the last three terms in the above equality and exploiting (2.20), we deduce (3.10).  $\square$

**Lemma 3.4.** *Let*

(3.12)

$$P(t) = -\rho_1 k_3 \int_0^L (w_x - l\varphi) \int_0^x w_t(y, t) dy dx - \rho_1 k_1 \int_0^L \varphi_t \int_0^x (\varphi_x + \psi + lw)(y, t) dy dx.$$

Then, for any  $\epsilon_0, \delta_0, \epsilon_2 > 0$ , there exist  $c_{\epsilon_0, l}, c_{\delta_0, g^0} > 0$  such that

$$(3.13) \quad \begin{aligned} P'(t) \leq & k_1^2 \int_0^L (\varphi_x + \psi + lw)^2 dx - k_3^2 \int_0^L (w_x - l\varphi)^2 dx + \delta_0 E(t) \\ & + \left( -\rho_1 k_1 + \frac{\rho_1 k_1 \epsilon_2}{2} + \epsilon_0 \right) \int_0^L \varphi_t^2 dx + c_{\epsilon_0, l} \int_0^L w_t^2 dx + \frac{c_0 \rho_1 k_1}{2\epsilon_2} \int_0^L \psi_t^2 dx \\ & + k_3 g^0 \int_0^L (w_x - l\varphi) w_x dx + c_{\delta_0, g^0} \int_0^L \int_0^{+\infty} g(s) \eta_x^2 ds dx. \end{aligned}$$

*Proof.* By exploiting the first and third equations in (1.1), integrating by parts and using (2.18) and the boundary conditions, we get

$$(3.14) \quad \begin{aligned} P'(t) = & \rho_1 k_3 \int_0^L w_t^2 dx - \rho_1 k_1 \int_0^L \varphi_t^2 dx - k_3^2 \int_0^L (w_x - l\varphi)^2 dx + k_1^2 \int_0^L (\varphi_x + \psi + lw)^2 dx \\ & + k_3 g^0 \int_0^L (w_x - l\varphi) w_x dx - \rho_1 \int_0^L \varphi_t \int_0^x (k_1 \psi_t(y, t) + l(k_1 - k_3) w_t(y, t)) dy dx \\ & - k_3 \int_0^L (w_x - l\varphi) \int_0^{+\infty} g(s) \eta_x ds dx. \end{aligned}$$

Noticing that the functions

$$x \mapsto \int_0^x \psi_t(y, t) dy \quad \text{and} \quad x \mapsto \int_0^x w_t(y, t) dy$$

vanish at 0 and  $L$  (because of (2.18)), then, applying (2.19), we have

$$(3.15) \quad \int_0^L \left( \int_0^x \psi_t(y, t) dy \right)^2 dx \leq c_0 \int_0^L \psi_t^2 dx$$

and

$$(3.16) \quad \int_0^L \left( \int_0^x w_t(y, t) dy \right)^2 dx \leq c_0 \int_0^L w_t^2 dx.$$

By applying Young's inequality and (3.5) for the last two terms in (3.14), and recalling (2.20), (3.15) and (3.16), we conclude (3.13).  $\square$

**Lemma 3.5.** *Let*

$$(3.17) \quad R(t) = \int_0^L (\rho_1 \varphi \varphi_t + \rho_2 \psi \psi_t + \rho_1 w w_t) dx.$$

Then, for any  $\delta_0 > 0$ , there exists  $c_{\delta_0, g^0} > 0$  such that

$$(3.18) \quad \begin{aligned} R'(t) \leq & \int_0^L (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \rho_1 w_t^2 - k_2 \psi_x^2 - k_1 (\varphi_x + \psi + lw)^2 - k_3 (w_x - l\varphi)^2) dx \\ & + g^0 \int_0^L w_x^2 dx + \delta_0 E(t) + c_{\delta_0, g^0} \int_0^L \int_0^{+\infty} g(s) \eta_x^2 ds dx. \end{aligned}$$

*Proof.* By exploiting the first three equations in (1.1), integrating by parts and using the boundary conditions, we get

$$\begin{aligned} R'(t) &= \int_0^L (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \rho_1 w_t^2 - k_2 \psi_x^2 - k_1 (\varphi_x + \psi + lw)^2 - k_3 (w_x - l\varphi)^2) dx \\ &\quad + g^0 \int_0^L w_x^2 dx - \int_0^L w_x \int_0^{+\infty} g(s) \eta_x ds dx. \end{aligned}$$

Applying (2.20) and (3.5) for the last term in the above equality, we arrive at (3.18).  $\square$

Let  $N, N_1, N_2, N_3, N_4 > 0$  and

$$(3.19) \quad F := NE + N_1 I + N_2 P + \frac{N_3}{l} K + N_4 R + J.$$

Then, by combining (3.3), (3.8), (3.10), (3.13) and (3.18), and exploiting (3.1) to estimate the integral of  $-g' \eta_x^2$ , we obtain

$$\begin{aligned} (3.20) \quad F'(t) &\leq \int_0^L \left( l_1 \varphi_t^2 + l_2 \psi_t^2 + l_3 w_t^2 + \left( l_4 + \frac{lk_2 k_3}{2k_1 \epsilon_1} \right) \psi_x^2 + \left( l_5 + \frac{lk_2 k_3 \epsilon_1}{2k_1} \right) (w_x - l\varphi)^2 \right) dx \\ &\quad + \int_0^L (l_6 (\varphi_x + \psi + lw)^2 + g^0 N_4 w_x^2 + g^0 l_7 (w_x - l\varphi) w_x) dx + \delta_0 c_{l, N_1, \dots, N_4} E(t) \\ &\quad + (N - c_{\delta_0, l, g^0, g(0), N_1, N_3}) E'(t) + c_{\delta_0, l, g^0, g(0), N_1, \dots, N_4} \int_0^L \int_0^{+\infty} g(s) \eta_x^2 ds dx + \left( \frac{k_2 \rho_1}{k_1} - \rho_2 \right) \int_0^L \psi_t \varphi_{xt} dx \\ &\quad + \frac{N_3 \rho_1}{l} \left( \frac{k_3}{k_1} - 1 \right) \int_0^L w_{xt} \varphi_t dx + \epsilon_0 c_{l, N_2, N_3} \int_0^L (\varphi_t^2 + \psi_t^2) dx + c_{l, \epsilon_0, N_2, N_3} \int_0^L w_t^2 dx, \end{aligned}$$

where

$$\begin{aligned} l_1 &= \rho_1 k_1 \left( \frac{\epsilon_2}{2} - 1 \right) N_2 - \frac{\rho_1 k_3 N_3}{k_1} + \rho_1 N_4, \quad l_2 = \frac{c_0 \rho_1 k_1 N_2}{2\epsilon_2} + \rho_2 N_4 - \rho_2, \\ l_3 &= -\rho_1 g_0 N_1 + \rho_1 N_4, \quad l_4 = -k_2 N_4, \quad l_5 = -k_3^2 N_2 + \frac{k_3^2 N_3}{k_1} - k_3 N_4, \\ l_6 &= k_1 + k_1^2 N_2 - k_1 N_3 - k_1 N_4 \quad \text{and} \quad l_7 = k_3 \left( N_2 - \frac{N_3}{k_1} \right). \end{aligned}$$

At this point, we choose carefully the constants  $N, N_i, \epsilon_i$  and  $\delta_0$  to get suitable values of  $l_i$ . We choose  $\epsilon_2 \in \left] 0, \frac{2(k_1 + k_3)}{k_1} \right[$ . After we pick  $N_2 > 0$  small enough so that

$$\begin{aligned} \frac{c_0 \rho_1 k_1}{2\rho_2 \epsilon_2} N_2 &< 1, \quad \left( k_1 \left( \frac{\epsilon_2}{2} - 1 \right) + \frac{c_0 \rho_1 k_1}{2\rho_2 \epsilon_2} - k_3 \right) N_2 < 1, \\ \left( \frac{k_3 \epsilon_2}{2} + \frac{k_1 + k_3}{k_1} \left( \frac{c_0 \rho_1 k_1}{2\rho_2 \epsilon_2} - k_3 \right) \right) N_2 &< 1 \quad \text{and} \quad \frac{c_0 \rho_1}{2\rho_2 \epsilon_2} (k_1 + k_3) N_2 < 1. \end{aligned}$$

Next we fix  $N_3 > 0$  such that

$$N_3 < \frac{k_1}{k_3} \left( 1 + \left( k_3 - \frac{c_0 \rho_1 k_1}{2\rho_2 \epsilon_2} \right) N_2 \right)$$

and

$$N_3 > \max \left\{ k_1 \left( 1 + \frac{c_0 \rho_1}{2\rho_2 \epsilon_2} \right) N_2, \frac{k_1^2}{k_3} \left( \frac{\epsilon_2}{2} - 1 \right) N_2, \frac{k_1}{k_1 + k_3} \left( 1 + \frac{k_1 \epsilon_2}{2} N_2 \right) \right\}.$$

Noticing that  $N_3$  exists according to the choice of  $N_2$ . Then we choose  $N_4 > 0$  so that

$$N_4 < \min \left\{ 1 - \frac{c_0 \rho_1 k_1}{2 \rho_2 \epsilon_2} N_2, \frac{k_3}{k_1} N_3 + k_1 \left( 1 - \frac{\epsilon_2}{2} \right) N_2 \right\}$$

and

$$N_4 > \max \left\{ -k_3 N_2 + \frac{k_3}{k_1} N_3, 1 + k_1 N_2 - N_3 \right\}.$$

The constant  $N_4$  exists thanks to the choice of  $N_3$ ,  $N_2$  and  $\epsilon_2$ . By virtue of the choice of  $\epsilon_2$ ,  $N_2$ ,  $N_3$  and  $N_4$ , we see that

$$(3.21) \quad \max \{l_1, l_2, l_4, l_5, l_6\} < 0.$$

At this step, we choose  $\epsilon_1 = \sqrt{\frac{l_5}{l_4}}$  and we put  $\tilde{l} = \frac{2k_1}{k_2 k_3} \sqrt{l_4 l_5}$  ( $\epsilon_1$  and  $\tilde{l}$  are well defined from (3.21)). Then, if  $l$  satisfies the first inequality in (2.29), we get

$$(3.22) \quad l_4 + \frac{lk_2 k_3}{2k_1 \epsilon_1} < 0 \quad \text{and} \quad l_5 + \frac{lk_2 k_3 \epsilon_1}{2k_1} < 0.$$

On the other hand, from (2.20) and Young's inequality, we find that

$$(3.23) \quad \begin{aligned} & \int_0^L (g^0 N_4 w_x^2 + g^0 l_7 (w_x - l\varphi) w_x) \, dx \\ & \leq g^0 \int_0^L \left( \left( N_4 + \frac{1}{2k_3} \right) w_x^2 + \frac{k_3 l_7^2}{2} (w_x - l\varphi)^2 \right) \, dx \\ & \leq g^0 \left( \frac{1}{k_0} \left( N_4 + \frac{1}{2k_3} \right) + \frac{l_7^2}{2} \right) \int_0^L (k_2 \psi_x^2 + k_1 (\varphi_x + \psi + lw)^2 + k_3 (w_x - l\varphi)^2) \, dx. \end{aligned}$$

Let

$$\tilde{g} := \frac{2k_0}{2 \left( N_4 + \frac{1}{2k_3} \right) + k_0 l_7^2} \min \left\{ -\frac{1}{k_2} \left( l_4 + \frac{lk_2 k_3}{2k_1 \epsilon_1} \right), -\frac{1}{k_3} \left( l_5 + \frac{lk_2 k_3 \epsilon_1}{2k_1} \right), -\frac{1}{k_1} l_6 \right\}$$

( $\tilde{g} > 0$  according to (3.21) and (3.22)). Because  $\epsilon_1$ ,  $l_4$ ,  $l_5$ ,  $l_6$  and  $l_7$  do not depend neither on  $l$  nor on  $g$ , then  $\tilde{l}$  and  $\tilde{g}$  do not depend neither on  $l$  nor on  $g$ . So, if  $g^0$  satisfies the second inequality in (2.29), then

$$\lambda_0 := \max \left\{ l_4 + \frac{lk_2 k_3}{2k_1 \epsilon_1} + g^0 k_2 \left( \frac{1}{k_0} \left( N_4 + \frac{1}{2k_3} \right) + \frac{l_7^2}{2} \right), l_5 + \frac{lk_2 k_3 \epsilon_1}{2k_1} + g^0 k_3 \left( \frac{1}{k_0} \left( N_4 + \frac{1}{2k_3} \right) + \frac{l_7^2}{2} \right), \right.$$

$$(3.24) \quad \left. l_6 + g^0 k_1 \left( \frac{1}{k_0} \left( N_4 + \frac{1}{2k_3} \right) + \frac{l_7^2}{2} \right) \right\} < 0,$$

and therefore, using (3.23),

$$(3.25) \quad \begin{aligned} & \int_0^L \left( \left( l_4 + \frac{lk_2 k_3}{2k_1 \epsilon_1} \right) \psi_x^2 + \left( l_5 + \frac{lk_2 k_3 \epsilon_1}{2k_1} \right) (w_x - l\varphi)^2 + l_6 (\varphi_x + \psi + lw)^2 + g^0 N_4 w_x^2 + g^0 l_7 (w_x - l\varphi) w_x \right) \, dx \\ & \leq \lambda_0 \int_0^L (\psi_x^2 + (w_x - l\varphi)^2 + (\varphi_x + \psi + lw)^2) \, dx. \end{aligned}$$

After, because  $l$ ,  $l_1$ ,  $l_2$ ,  $N_2$  and  $N_3$  do not depend on  $\epsilon_0$ , we can choose  $\epsilon_0 > 0$  small enough such that

$$(3.26) \quad l_1 + \epsilon_0 c_{l, N_2, N_3} < 0 \quad \text{and} \quad l_2 + \epsilon_0 c_{l, N_2, N_3} < 0,$$

and then we fix  $N_1$  large enough so that

$$(3.27) \quad l_3 + c_{l, \epsilon_0, N_2, N_3} < 0.$$

Therefore, from (3.24), (3.26) and (3.27) we see that

$$\max \left\{ \frac{1}{\rho_1} (l_1 + \epsilon_0 c_{l, N_2, N_3}), \frac{1}{\rho_2} (l_2 + \epsilon_0 c_{l, N_2, N_3}), \frac{1}{\rho_1} (l_3 + c_{l, N_2, N_3, \epsilon_0}), \frac{\lambda_0}{k_2}, \frac{\lambda_0}{k_3}, \frac{\lambda_0}{k_1} \right\} < 0.$$

Finally, we choose  $\delta_0 > 0$  small enough such that

$$\tilde{c}_1 := -2 \max \left\{ \frac{1}{\rho_1} (l_1 + \epsilon_0 c_{l, N_2, N_3}), \frac{1}{\rho_2} (l_2 + \epsilon_0 c_{l, N_2, N_3}), \frac{1}{\rho_1} (l_3 + c_{l, N_2, N_3, \epsilon_0}), \frac{\lambda_0}{k_2}, \frac{\lambda_0}{k_3}, \frac{\lambda_0}{k_1} \right\} - \delta_0 c_{l, N_1, \dots, N_4} > 0.$$

Then, using (2.23) and (2.27), and recalling that  $\frac{k_2 \rho_1}{k_1} - \rho_2 = 0$  (because  $s_1 = s_2$  thanks to (1.3) or (2.32)), we deduce from (3.20) and (3.25) that

$$(3.28) \quad F'(t) \leq -\tilde{c}_1 E(t) + (N - c) E'(t) + c \int_0^L \int_0^{+\infty} g(s) \eta_x^2 ds dx + \frac{N_3 \rho_1}{l} \left( \frac{k_3}{k_1} - 1 \right) \int_0^L w_{xt} \varphi_t dx.$$

Now, we estimate the integral of  $g \eta_x^2$  in (3.28). When (2.25) holds, we see that, by virtue of (3.1),

$$(3.29) \quad \int_0^L \int_0^{+\infty} g(s) \eta_x^2 ds dx \leq -\frac{2}{\alpha} E'(t).$$

When (2.26) holds, we apply Lemma 3.6 [11] (in the particular case  $B = -\partial_{xx}$  and  $\|\cdot\| = \|\cdot\|_{L^2([0, L])}$ ) to get the following inequality.

**Lemma 3.6.** *There exists a positive constant  $c$  such that, for any  $\tau_0 > 0$ , we have*

$$(3.30) \quad G'(\tau_0 E(t)) \int_0^L \int_0^{+\infty} g(s) \eta_x^2 ds dx \leq -c E'(t) + c \tau_0 E(t) G'(\tau_0 E(t)).$$

*Proof.* See Lemma 3.6 [11]. □

Using (3.29) and (3.30), we get, for the two cases (2.25) and (2.26) and for any  $\tau_0 > 0$ ,

$$(3.31) \quad \frac{G_0(\tau_0 E(t))}{\tau_0 E(t)} \int_0^L \int_0^{+\infty} g(s) \eta_x^2 ds dx \leq c G_0(\tau_0 E(t)) - c E'(t) - c \frac{G_0(\tau_0 E(t))}{\tau_0 E(t)} E'(t),$$

where  $G_0$  is defined in (2.31). By multiplying (3.28) by  $\frac{G_0(\tau_0 E(t))}{E(t)}$  and combining with (3.31), we obtain, for any  $\tau_0 > 0$ ,

$$(3.32) \quad \begin{aligned} \frac{G_0(\tau_0 E(t))}{E(t)} F'(t) &\leq -(\tilde{c}_1 - c \tau_0) G_0(\tau_0 E(t)) + \left( (N - c) \frac{G_0(\tau_0 E(t))}{E(t)} - c \tau_0 \right) E'(t) \\ &\quad + \frac{G_0(\tau_0 E(t))}{E(t)} \frac{N_3 \rho_1}{l} \left( \frac{k_3}{k_1} - 1 \right) \int_0^L w_{xt} \varphi_t dx. \end{aligned}$$

On the other hand, from (2.20), (2.23) and (2.27), we deduce that there exists a positive constant  $\gamma$  (independent of  $N$ ) satisfying

$$\left| N_1 I + N_2 P + \frac{N_3}{l} K + N_4 R + J \right| \leq \gamma E,$$

which, combined with (3.19), implies that

$$(3.33) \quad (N - \gamma)E \leq F \leq (N + \gamma)E.$$

Choosing  $N$  so that  $N > \max\{\gamma, c\}$  ( $c$  is the constant in (3.32)) and using (3.32), (3.33) and  $E' \leq 0$ , we deduce that  $F \sim E$  and

$$(3.34) \quad \begin{aligned} \frac{G_0(\tau_0 E(t))}{E(t)} F'(t) &\leq -(\tilde{c}_1 - c\tau_0)G_0(\tau_0 E(t)) - c\tau_0 E'(t) \\ &\quad + \frac{G_0(\tau_0 E(t))}{E(t)} \frac{N_3 \rho_1}{l} \left( \frac{k_3}{k_1} - 1 \right) \int_0^L w_{xt} \varphi_t dx. \end{aligned}$$

Let  $\tilde{\tau} > 0$  and

$$(3.35) \quad \tilde{F} = \tilde{\tau} \left( \frac{G_0(\tau_0 E(t))}{E(t)} F + c\tau_0 E(t) \right).$$

Because  $\frac{G_0(\tau_0 E(t))}{E(t)}$  is non-increasing, then, thanks to (3.33),

$$(3.36) \quad c\tilde{\tau}\tau_0 E \leq \tilde{F} \leq \tilde{\tau} \left( (N + \gamma) \frac{G_0(\tau_0 E(0))}{E(0)} + c\tau_0 \right) E.$$

We have, using (3.34), (3.35) and the fact that  $\frac{G_0(\tau_0 E)}{E}$  is non-increasing, (3.37)

$$\tilde{F}'(t) \leq -\tilde{\tau}(\tilde{c}_1 - c\tau_0)G_0(\tau_0 E(t)) + \tilde{\tau} \frac{G_0(\tau_0 E(t))}{E(t)} \frac{N_3 \rho_1}{l} \left( \frac{k_3}{k_1} - 1 \right) \int_0^L w_{xt} \varphi_t dx.$$

**3.2. Proof of (2.30).** Let us choose  $\tilde{\tau} > 0$  such that

$$(3.38) \quad \tilde{F} \leq \tau_0 E \quad \text{and} \quad \tilde{F}(0) \leq 1.$$

According to (1.3), the coefficient of the integral in (3.37) vanishes, and hence, by choosing  $\tau_0 > 0$  small enough such that  $\tilde{c}_1 - c\tau_0 > 0$  and using the first inequality in (3.38), we get, for  $c' = \tilde{\tau}(\tilde{c} - c\tau_0)$ ,

$$(3.39) \quad \tilde{F}' \leq -c' G_0(\tilde{F}),$$

whereupon

$$(3.40) \quad (\tilde{G}(\tilde{F}))' \geq c',$$

where  $\tilde{G}$  is defined in (2.31). Integrating (3.40) over  $[0, t]$  yields

$$(3.41) \quad \tilde{G}(\tilde{F}(t)) \geq c't + \tilde{G}(\tilde{F}(0)).$$

Because  $\tilde{F}(0) \leq 1$  (from (3.38)),  $\tilde{G}(1) = 0$  and  $\tilde{G}$  is decreasing, we obtain from (3.41) that  $\tilde{G}(\tilde{F}(t)) \geq c't$ , which implies that  $\tilde{F}(t) \leq \tilde{G}^{-1}(c't)$ . Then (3.36) gives (2.30).

**3.3. Proof of (2.34).** In this section, we treat the case when (2.32) holds. We need to estimate the last integral in (3.37) using the following systems resulting

from differentiating (1.1) with respect to time  $t$ :

$$(3.42) \quad \begin{cases} \rho_1 \varphi_{ttt} - k_1(\varphi_{xt} + \psi_t + lw_t)_x - lk_3(w_{xt} - l\varphi_t) = 0, \\ \rho_2 \psi_{ttt} - k_2 \psi_{xxt} + k_1(\varphi_{xt} + \psi_t + lw_t) = 0, \\ \rho_1 w_{ttt} - k_3(w_{xt} - l\varphi_t)_x + lk_1(\varphi_{xt} + \psi_t + lw_t) + \int_0^{+\infty} g(s)w_{xxt}(x, t-s) ds = 0, \\ \varphi_t(0, t) = \psi_{xt}(0, t) = w_{xt}(0, t) = \varphi_t(L, t) = \psi_{xt}(L, t) = w_{xt}(L, t) = 0. \end{cases}$$

Thanks to Theorem 2.2, we have, for any initial data  $U^0 \in D(A)$ , the system (3.42) has a unique solution  $U$  satisfying

$$\partial_t U \in C(\mathbb{R}_+; H).$$

Let  $U^0 \in D(A)$  and  $\tilde{E}$  be the energy of (3.42) defined by

$$(3.43) \quad \tilde{E}(t) = \frac{1}{2} \|\partial_t U(t)\|_H^2.$$

Similarly to (3.1), we have

$$(3.44) \quad \tilde{E}'(t) = \frac{1}{2} \int_0^L \int_0^{+\infty} g'(s) \eta_{xt}^2 ds dx \leq 0;$$

so  $\tilde{E}$  is non-increasing. We use an idea introduced in [9] to get this lemma.

**Lemma 3.7.** *For any  $\epsilon > 0$ , there exists  $c_\epsilon > 0$  such that*

$$(3.45) \quad \left| \frac{N_3 \rho_1}{l} \left( \frac{k_3}{k_1} - 1 \right) \int_0^L w_{xt} \varphi_t dx \right| \leq c_\epsilon \int_0^L \int_0^{+\infty} g(s) \eta_{xt}^2 ds dx + \epsilon E(t) - c_\epsilon E'(t).$$

*Proof.* We have, by the definition of  $\eta$ ,

$$(3.46) \quad \begin{aligned} \frac{N_3 \rho_1}{l} \left( \frac{k_3}{k_1} - 1 \right) \int_0^L \varphi_t w_{xt} dx &= \frac{N_3 \rho_1}{g^0 l} \left( \frac{k_3}{k_1} - 1 \right) \int_0^L \varphi_t \int_0^{+\infty} g(s) \eta_{xt} ds dx \\ &\quad + \frac{N_3 \rho_1}{g^0 l} \left( \frac{k_3}{k_1} - 1 \right) \int_0^L \varphi_t \int_0^{+\infty} g(s) w_{xt}(t-s) ds dx. \end{aligned}$$

Using (3.5) (for  $f = \eta_{xt}$  and  $v = \varphi_t$ ) and (2.27), we get, for all  $\epsilon > 0$ ,

$$(3.47) \quad \left| \frac{N_3 \rho_1}{g^0 l} \left( \frac{k_3}{k_1} - 1 \right) \int_0^L \varphi_t \int_0^{+\infty} g(s) w_{xt} \eta ds dx \right| \leq \frac{\epsilon}{2} E(t) + c_\epsilon \int_0^L \int_0^{+\infty} g(s) \eta_{xt}^2 ds dx.$$

On the other hand, by integrating with respect to  $s$  and using the definition of  $\eta$ , we obtain

$$\begin{aligned} \int_0^L \varphi_t \int_0^{+\infty} g(s) w_{xt}(t-s) ds dx &= - \int_0^L \varphi_t \int_0^{+\infty} g(s) \partial_s(w_x(t-s)) ds dx \\ &= \int_0^L \varphi_t \left( g(0) w_x(t) + \int_0^{+\infty} g'(s) w_x(t-s) ds \right) dx \\ &= - \int_0^L \varphi_t \int_0^{+\infty} g'(s) \eta_x ds dx. \end{aligned}$$



Therefore, using (3.6) (for  $f = \eta_x$  and  $v = \varphi_t$ ) and (3.1),

$$(3.48) \quad \left| \frac{N_3 \rho_1}{l} \left( \frac{k_3}{k_1} - 1 \right) \int_0^L \varphi_t \int_0^{+\infty} g(s) w_{xt}(t-s) ds dx \right| \leq \frac{\epsilon}{2} E(t) - c_\epsilon E'(t).$$

Inserting (3.47) and (3.48) into (3.46), we obtain (3.45).  $\square$

Now, by combining (3.45), (3.37) with  $\tilde{\tau} = 1$  and choosing  $\epsilon = \frac{\tilde{c}_1}{2}$ , we get

$$(3.49) \quad \tilde{F}'(t) \leq - \left( \frac{\tilde{c}_1}{2} - c\tau_0 \right) G_0(\tau_0 E(t)) - c \frac{G_0(\tau_0 E(0))}{E(0)} E' + c \frac{G_0(\tau_0 E(t))}{E(t)} \int_0^L \int_0^{+\infty} g(s) \eta_{xt}^2 ds dx.$$

Similarly to (3.29) and (3.30), using (3.44), we find

$$(3.50) \quad \int_0^L \int_0^{+\infty} g(s) \eta_{xt}^2 ds dx \leq -\frac{2}{\alpha} \tilde{E}'(t)$$

when (2.25) holds. When (2.26) holds, there exists a positive constant  $c$  such that, for any  $\tau_0 > 0$ , we have as for (3.30) (see the proof of Lemma 3.6 [11])

$$(3.51) \quad G'(\tau_0 E(t)) \int_0^L \int_0^{+\infty} g(s) \eta_{xt}^2 ds dx \leq -c \tilde{E}'(t) + c\tau_0 E(t) G'(\tau_0 E(t)).$$

From (3.50) and (3.51), we find that, in both cases (2.25) and (2.26),

$$(3.52) \quad \frac{G_0(\tau_0 E(t))}{\tau_0 E(t)} \int_0^L \int_0^{+\infty} g(s) \eta_{xt}^2 ds dx \leq c G_0(\tau_0 E(t)) - c \frac{G_0(\tau_0 E(t))}{\tau_0 E(t)} \tilde{E}'(t) - c \tilde{E}'(t).$$

Inserting (3.52) in (3.49), choosing  $\tau_0 > 0$  small enough such that  $\frac{\tilde{c}_1}{2} - c\tau_0 > 0$ , and using the fact that  $\frac{G_0(\tau_0 E)}{E}$  is non-increasing, we find, for some  $\tilde{c}_2 > 0$ ,

$$(3.53) \quad G_0(\tau_0 E(t)) \leq -\tilde{c}_2 \tilde{F}'(t) - c \left( 1 + \frac{G_0(\tau_0 E(0))}{E(0)} \right) (E'(t) + \tilde{E}'(t)).$$

By integration with respect to  $t$  and using (3.36), we get, for some  $\tilde{c}_3 > 0$ ,

$$(3.54) \quad \int_S^T G_0(\tau_0 E(t)) dt \leq \tilde{c}_3 \left( 1 + \frac{G_0(\tau_0 E(0))}{E(0)} \right) (E(S) + \tilde{E}(S)), \quad \forall T \geq S \geq 0.$$

Choosing  $S = 0$  in (3.54) and using the fact that  $G_0(\tau_0 E)$  is non-increasing, we obtain

$$(3.55) \quad G_0(\tau_0 E(T))T \leq \int_0^T G_0(E(t)) dt \leq \tilde{c}_3 \left( 1 + \frac{G_0(\tau_0 E(0))}{E(0)} \right) (E(0) + \tilde{E}(0)).$$

Because  $G_0^{-1}$  is increasing, (2.34) for  $n = 1$  is deduced from (3.55) with

$$c_1 = \max \left\{ \frac{1}{\tau_0}, \tilde{c}_3 \left( 1 + \frac{G_0(\tau_0 E(0))}{E(0)} \right) (E(0) + \tilde{E}(0)) \right\}.$$

By induction on  $n$ , one can prove that (2.34) holds, for  $n = 2, 3, \dots$ ; see [11] and [13].

*Remark 3.8.* We give in this remark some general comments and open problems.

- (1) Our stability results (2.30) and (2.34) hold under the smallness conditions (2.29) on  $l$  and  $g^0$ , and the boundedness conditions (2.28) and (2.33) on the initial data  $\eta^0$ . It is interesting to drop these conditions or determine the biggest values of  $\tilde{l}$  and  $\tilde{g}$  in (2.29) for which (2.30) and (2.34) hold.
- (2) Another interesting question concerns the stability of (1.1) when  $s_1 \neq s_2$ .
- (3) The case where only one infinity memory is considered in the vertical displacements; that is the integral in (1.1) is replaced by

$$\int_0^{+\infty} g(s) \varphi_{xx}(x, t-s) ds$$

and considered on the first equation in (1.1), seems very delicate. The particular case of Timoshenko systems ((1.1) with  $l = 0$ ) under infinity memory and/or frictional damping in the vertical displacements was studied in [14] and some stability estimates were proved.

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