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# HOMOTOPY THEORY OF CURVED OPERADS AND CURVED ALGEBRAS

GABRIEL C. DRUMMOND-COLE AND JOAN BELLIER-MILLÈS

ABSTRACT. Curved algebras are algebras endowed with a predifferential, which is an endomorphism of degree  $-1$  whose square is not necessarily  $0$ . This makes the usual definition of quasi-isomorphism meaningless and therefore the homotopical study of curved algebras cannot follow the same path as differential graded algebras.

In this article, we propose to study curved algebras by means of curved operads. We develop the theory of bar and cobar constructions adapted to this new notion as well as Koszul duality theory. To be able to provide meaningful definitions, we work in the context of objects which are filtered and complete and become differential graded after applying the associated graded functor.

This setting brings its own difficulties but it nevertheless permits us to define a combinatorial model category structure that we can transfer to the category of curved operads and to the category of algebras over a curved operad using free-forgetful adjunctions.

We address the case of curved associative algebras. We recover the notion of curved  $A_\infty$ -algebras, and we show that the homotopy categories of curved associative algebras and of curved  $A_\infty$ -algebras are Quillen equivalent.

## INTRODUCTION

**Motivation.** The primary goal of this paper is to give a framework to deal with the homological and homotopical theory of curved algebras.

The most elementary definition of an associative algebra is an underlying vector space or module endowed with algebraic structures (multiplication, unit). The study of extensions or of deformations of an associative algebra leads to the definition of the Hochschild (co)homology, which is defined by means of the Hochschild (co)chain complex. Revisiting this definition in terms of derived functors leads up to the notion of resolution and in particular of quasi-isomorphism. Going further, the theory of model categories provides powerful tools to extend the previous ideas to many other contexts including the study of other kinds of algebras (commutative algebras, Lie algebras, ...). The category of dg modules over a ring is endowed with a model category structure whose weak equivalences are quasi-isomorphisms. Hochschild (co)homology has a meaningful interpretation in this context and it opens the door for (co)homology theories of other types of algebras.

Now suppose that we want to follow this path for *curved* associative algebras. Such algebras are equipped not with a differential but rather with a predifferential. Instead of squaring to zero, the square of the predifferential is equal to the bracket

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with a closed element called the curvature. This difference means that it is not reasonable to expect the category of curved algebras to have an underlying category of dg modules. There is therefore the need to define a new category (possibly containing the category of dg modules as a subcategory) endowed with a notion of weak equivalences to replace quasi-isomorphisms. Only then we will be able to achieve our goal.

**Approach and antecedents.** At the heart of the homological and homotopical study of algebras are the notions of an operad, used to encode algebras, and Koszul duality theory for such operads. Koszul duality theory is a homological theory in its definition and in its range of applications. The approach taken in this paper is:

- (1) to define curved algebras as representations of curved operads,
- (2) to find an appropriate base category to study curved algebras, and
- (3) to extend Koszul duality theory to the curved context.

The constructions given in this paper are in some sense dual to the constructions given in [HM12]. However, the homotopy theory of curved algebras is complicated by the need to work in a kind of filtered context. In an unfiltered context, a homotopy algebra equipped with a non-zero curvature is *isomorphic* to an “algebra” which has a curvature but otherwise the zero algebra structure [Pos00, 7.3] (see also Proposition 3.3 in [DSV20] for a filtered version with the curvature in filtration degree 0).

Moving to Koszul duality poses its own problems and this is the reason why we cannot use the existing literature [Lyu13, Lyu14]. The formulas that one is led naturally to write are infinite sums whose terms eventually go to smaller and smaller submodules of the filtration. These sums then only make sense if one further refines from a filtered to a complete context. Depending on the side of the duality, there are further minor refinements to make in order to fully capture the phenomena at play; see below for the gory details. The filtered context already appears in [DDL18], and previously, in a more restricted setting in [Pos12].

In this last article, Positselski has similar motivations to ours. He proposes a framework to study the derived category of a curved associative or  $A_\infty$ -algebra. He is also able to develop a Koszul duality theory in this context. We highly recommend the reading of this article’s engaging and fruitful introduction. We cannot however use his framework since our main example doesn’t fit into it. We therefore propose a filtered framework which is different in two ways: we consider filtrations on the base ring which aren’t induced by a maximal ideal and we consider filtrations on our objects (algebras, operads, ...) which aren’t induced by the filtration on the base ring. Another example we have in mind is the curved  $A_\infty$ -algebras appearing in Floer theory in [FOOO07], which fit into the framework introduced here.

**On filtered objects.** Dealing with filtered and complete filtered objects poses multiple technical challenges. If the ground category is Abelian, the categories of (complete) filtered objects in the ground category are only quasi-Abelian [Sch99]. Colimits are more difficult to compute and the monoidal product must be redefined in order to inherit various desirable qualities.

These details can be done more or less by hand; for example, the reference [Fre17, 7.3] contains a good deal of the setup work for the case where the ground ring is a field. It is also possible to perform an  $\infty$ -category treatment as in [GP18]. But for us, a useful way to organize the necessary changes was by recognizing that the

various categories of objects of interest form a lattice of *normal reflective embeddings*. A subcategory inclusion is reflective when it admits a left adjoint (called a *reflector*) and a reflective inclusion of closed symmetric monoidal categories is a normal reflective embedding when the reflector is extended into a strong symmetric monoidal functor.

Most of these categorical details are siloed off in Appendix A.

**Our main algebraic categories.** Let us give a little more precise focus here to guide the development of the exposition. The reader is encouraged to think of most of the symmetric monoidal categories as making up a scaffolding for the two cases of actual interest. These are *curved augmented operads* in the category of *gr-dg complete filtered objects* (treated in Section 2) and *altipotent cooperads* in the category of *dg complete filtered objects* (treated in Section 3).

**Our operads.** Again, a curved associative algebra is an associative algebra  $A$  endowed with a predifferential  $d$  satisfying

$$d^2 = [\theta, -]$$

where  $\theta$  is a closed element in  $A$  of degree  $-2$ . In the filtered complete context  $A = F_0A \supset F_1A \supset \dots$ , a filtered object endowed with a predifferential is gr-dg when its associated graded is differentially graded, and we will assume that  $\theta \in F_1A$  so that our curved algebras are gr-dg. The curved algebras in this article can therefore be considered as infinitesimal deformations of flat algebras. We extend this definition to define the notion of a curved operad. A first example is given by the endomorphism operad of a gr-dg object and there is a curved operad  $\mathbf{cAs}$  whose algebras are precisely curved associative algebras.

**Our cooperads.** To tell a story about the bar-cobar adjunction, we need a notion dual to the notion of curved (augmented) operads. We use infinitesimal cooperads for this dual notion. Infinitesimal cooperads are quite similar to (not necessarily coaugmented) cooperads in a filtered context. The only difference is that the counit is constrained to lie in degree 1 of the filtration. As in the article [HM12], this can be seen as an incarnation of the fact that curvature and counit play a dual role and that asking for the curvature to be in filtration degree 1 corresponds dually to the fact that the counit also is in filtration degree 1.

**Outline of the remaining contents.** We introduce several (co)free constructions. In the operadic context, we define a free pointed gr-dg operad and a free curved operad. The first one is used to build the cobar construction of an infinitesimal cooperad, the latter is used to endow the category of curved operads with a model structure. In the cooperadic context, we define the notion of altipotency which is a variation of the notion of conilpotency adapted to the complete setting. We provide a cofree construction in this setting and we use it in the definition of the bar construction of a curved operad.

The bar and the cobar constructions, presented in Section 4, fit as usual in an adjunction and are represented by a notion of curved twisting morphisms. These are different from the constructions in [Lyu14] because of the filtered complete framework and by the fact that we consider a curvature only on one side of the adjunction. In the context of  $\mathbf{R}$ -modules (for a field  $\mathbf{R}$  of characteristic 0), the counit of the adjunction provides a graded quasi-isomorphism between curved operads. This resolution is functorial and we can hope to obtain a “smaller” resolution

when dealing with specific examples. This is the objective of the Koszul duality theory for curved operads that we develop in Section 5. The constructions in this section are a little bit more subtle than the classical constructions because of the fact that infinite sums appear. In particular, it is difficult to describe the Koszul dual (infinitesimal) cooperad associated with a quadratic curved operad. Nevertheless, we define the Koszul dual (infinitesimal) operad which is easier to compute. Moreover, in the situation where the curved operad is Koszul, a Poincaré–Birkhoff–Witt type isomorphism provides a description of the underlying  $\mathbb{S}$ -module by means of the (classical) Koszul dual cooperad of a quadratic operad which is the associated graded of the curved operad. Under the Koszul condition, we again obtain a resolution of the quadratic curved operad, smaller in a precise meaning since the generators embed to the bar construction as the 0-homology group for the “syzygy degree”.

We make explicit the case of the curved operad  $cAs$  encoding curved associative algebras in Section 6. The curved operad  $cAs$  is Koszul and we can compute its Koszul dual (infinitesimal) cooperad (and operad), as well as the Koszul resolution. The algebras over the Koszul resolution are the curved  $A_\infty$ -algebras which appear in the literature (to give only a few examples see [GJ90, CD01, Kel06, FOOO07, Nic08, Pos19, DSV20]). We finally show that the homotopy categories of curved associative algebras and of curved  $A_\infty$ -algebras are Quillen equivalent.

**Model category structure.** Speaking of resolutions and Quillen equivalence are indications that we have a model category structure in mind. We establish the existence and properties of this structure in Appendix C. More precisely, there we describe a model structure on the base category of complete gr-dg  $\mathbf{R}$ -modules. This model category enjoys several nice properties: it is a proper cofibrantly generated model structure, it is combinatorial and it is a monoidal model category structure. Classical theorems allow us to transfer this cofibrantly generated model structure along a free-forgetful adjunction. This enables us to endow the category of complete curved operads with a cofibrantly generated model structure. The bar-cobar resolution and the Koszul resolution are cofibrant in the underlying category of complete gr-dg  $\mathbb{S}$ -modules. Similarly, we describe a free functor in the context of algebras over a curved operad and endow the category of algebras over a curved operad with a cofibrantly generated model structure. We provide base change results to compare the homotopy categories of algebras over some curved operads.

In [LGL18], Le Grignou and Lejay endow the category of algebras over a curved cooperad with a model structure. For some curved cooperad, we can try to compare our curved algebras over the dual operad with their algebras and their model structure. Because of the freedom we have for our filtrations, it is not clear how to compare the two categories of algebras and the two model structures in general.

**Conventions.** The ground category is a Grothendieck category  $\mathcal{A}$  equipped with a closed (symmetric monoidal) tensor product. We assume moreover that the tensor product preserve colimits in each variable. For various parts of the exposition, weaker hypotheses suffice but this seems a reasonable place to cut things off. For example,  $\mathcal{A}$  can be

- the category of  $\mathbf{R}$ -modules for  $\mathbf{R}$  a commutative ring,
- the category of graded  $\mathbf{R}$ -modules or complexes of  $\mathbf{R}$ -modules,
- the category of sheaves of  $\mathbf{R}$ -modules on a topological space  $X$ ,

- the category of graded sheaves of  $\mathbf{R}$ -modules or complexes of sheaves of  $\mathbf{R}$ -modules on  $X$ , or
- for a (graded) ringed space  $(X, \mathcal{O}_X)$ , the category of sheaves of (graded)  $\mathcal{O}_X$ -modules.

When we deal with symmetric operads, we want to assume that  $\mathcal{A}$  is  $\mathbb{Q}$ -linear. In the example above, this is the assumption that  $\mathbf{R}$  is a  $\mathbb{Q}$ -algebra.

We use the notation  $(\mathcal{M}, \mathbb{I}, \otimes, \mathbb{1})$  to denote an additive closed monoidal category with small colimits and limits. We assume moreover that the monoidal structure preserves colimits in each variable. Examples of such categories  $\mathcal{M}$  are given by the categories  $\mathcal{A}$ ,  $\text{Filt}(\mathcal{A})$ ,  $\widehat{\text{Filt}}(\mathcal{A})$ , and  $\widehat{\text{Filt}}^{\text{gr}}(\mathcal{A})$  when  $\mathcal{A}$  is a closed symmetric monoidal Grothendieck category with small colimits and limits and such that the monoidal structure preserves colimits in each variable.

We want to work with  $\mathbb{S}$ -modules or *collections* in  $\mathcal{M}$ , which are functors from the groupoid of finite sets to  $\mathcal{M}$ . We also want to (simultaneously) work with  $\mathbb{N}$ -modules in  $\mathcal{M}$ , which are functors from the groupoid of ordered finite sets to  $\mathcal{M}$ . We sometimes implicitly pass to a skeleton of either of these categories with objects  $[n] = \{1, \dots, n\}$ .

The categories of  $\mathbb{S}$ -modules and  $\mathbb{N}$ -modules support a number of monoidal products built using the monoidal product of  $\mathcal{M}$ . The primary one we will want to use is the *composition product* that we denote by  $\circ$ . The forgetful functor from  $\mathbb{S}$ -modules to  $\mathbb{N}$ -modules does not intertwine the composition products on each of these categories, but nevertheless we will use the same symbol for both cases. In the few cases where this might cause confusion, we will be explicit about the distinctions.

In Sections 5 and 6, we restrict ourselves to categories  $\mathcal{A}$  of (unbounded)  $\mathbb{Z}$ -graded  $\mathbf{R}$ -modules or complexes of  $R$ -modules, for  $R$  a commutative ring.

When  $M$  is a  $\mathbb{Z}$ -graded object, we denote by  $sM$  the suspension of  $M$ , that is the graded object such that  $(sM)_n := M_{n-1}$ .

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## 1. THE FILTERED FRAMEWORK

In this preliminary section we establish definitions, terminology, and notation for the base world of filtered and complete objects in which we will work.

**1.1. Predifferential graded objects.** We first fix our convention for filtered objects and predifferential graded objects.

**Definition 1.2.**

- A *filtered object*  $(\vec{X}, F)$  (often expressed just as  $X$ ) in  $\mathcal{A}$  is a  $\mathbb{N}^{\text{op}}$ -indexed diagram

$$F_0X \leftarrow F_1X \leftarrow F_2X \leftarrow \cdots$$

where each map is a monomorphism. We often think of the object  $(\vec{X}, F)$  as the object  $X := F_0X$  equipped with the extra data of the family of subobjects  $\{F_pX\}$ .

Every non filtered object  $X$  gives rise to a trivially filtered object with

$$F_pX = \begin{cases} X & p = 0 \\ \emptyset & p > 0. \end{cases}$$

Morphisms of filtered modules  $f : (\vec{X}, F) \rightarrow (\vec{Y}, F')$  are morphisms of diagrams. In other words, they are  $\mathcal{A}$ -maps  $X \rightarrow Y$  which are *filtration preserving* in that  $f(F_pX) \subset F'_pY$ , for all  $p \in \mathbb{N}$ .

- To the filtered object  $(\vec{X}, F)$ , we associate the graded object (i.e., object indexed by the elements of  $\mathbb{N}^{\text{op}}$ )  $\text{Gr } X$  defined by  $(\text{Gr } X)_p := F_pX/F_{p+1}X$ .

We denote by  $\text{Filt}(\mathcal{A})$  the category of filtered objects in the category  $\mathcal{A}$ .

*Remark 1.3.* • Since we are only interested in *decreasing* filtration, we omit the adjective “decreasing” in this article.

- The indexing category  $\mathbb{N}$  is not the most general possibility. The case of  $\mathbb{Z}$ -filtered objects is also interesting but it’s technically useful for us to work with a partially ordered monoid with the identity as its lowest element. Of course, there are many examples of such monoids other than  $\mathbb{N}$ , such as  $\mathbb{Q}_+$  and  $\mathbb{R}_+$ . But in most examples which arise in practice, it seems that coarsening the filtration to a discrete countable filtration by a monoid isomorphic to  $\mathbb{N}$  does no harm in terms of the algebra that interests us. On the other hand, working with non-discrete filtrations poses a number of technical problems, most importantly the fact that the associated graded functor fails to be conservative in general. So we restrict our attention to  $\mathbb{N}$ -filtrations.

*Remark 1.4.* It is well-known that the category  $\text{Filt}(\mathcal{A})$  is not Abelian in general. However, this category is a reflective subcategory of the category of  $\mathbb{N}^{\text{op}}$ -indexed diagrams in  $\mathcal{A}$  and thus is bicomplete. See Appendix A for details.

**Notation 1.5.** Denote by  $\mathbf{g}\mathcal{A}$  the closed symmetric monoidal category of  $\mathbb{Z}$ -graded  $\mathcal{A}$ -objects and by  $\mathbf{dg}\mathcal{A}$  the closed symmetric monoidal category of  $\mathbb{Z}$ -graded complexes in  $\mathcal{A}$  (dg objects in  $\mathcal{A}$  for short). A *predifferential* on a graded object  $X$  in  $\mathbf{g}\mathcal{A}$  is a degree  $-1$  map of graded objects  $d : X \rightarrow X$ . We denote by  $\mathbf{pg}\mathcal{A}$  the closed symmetric monoidal category of predifferential graded object (pg object in  $\mathcal{A}$  for short).

*Remark 1.6.* When  $\mathcal{A}$  is a Grothendieck category, the category  $\mathbf{pg}\mathcal{A}$  of predifferential graded objects is again a Grothendieck category. Colimits are taken degreewise and filtered colimits are degreewise exact hence exact. Finally, if  $\{U_i\}_{i \in I}$  is a family of generators of  $\mathcal{A}$ , then  $\{D^{n,\infty}U_i := (\coprod_{k \leq n} U_i, d)\}_{n \in \mathbb{Z}, i \in I}$ , where the  $k$ -th copy of  $U_i$  is in degree  $k$ ,  $d$  sends  $u$  in the  $k$ -th copy to  $u$  in the  $(k-1)$ -th copy and where the filtration is given by  $F_p(D^{n,\infty}U_i) = \coprod_{k \leq n-2p} U_i$ , is a family of generators of  $\mathbf{pg}\mathcal{A}$ .

1.7. **“Gr” and associated graded.** For any property  $p$  of  $\mathbf{g}\mathcal{A}$ , we say that a filtered object  $X$  is *gr- $p$*  if the associated graded object  $\mathrm{Gr} X$  is  $p$ . We extend this terminology in the obvious way to all other contexts where we have a variant of the associated graded functor (and we make the definitions precise when the terminology is not obvious). This convention already appears in [Sjö73].

The following example illustrates this terminological choice.

**Definition 1.8.** Let  $(\vec{X}, F, d)$  be an object of  $\mathrm{Filt}(\mathbf{pg}\mathcal{A})$ . When the predifferential  $d$  induces a differential on  $\mathrm{Gr} X$ , that is to say,

$$d^2 : F_p X \rightarrow F_p X$$

factors through  $F_{p+1} X$  for all  $p$ , we call  $(\vec{X}, F, d)$  *gr-dg*. A natural way to associate a dg object to a gr-dg object  $(\vec{X}, F, d)$  is to consider the associated graded object

$$(X, d)^{\mathrm{gr}} := (\mathrm{Gr} X, \mathrm{Gr} d).$$

Accordingly, we call *gr-homology* of the gr-dg object  $(\vec{X}, F, d)$  the graded object

$$\mathbf{H}_\bullet^{\mathrm{gr}}(\vec{X}, F, d) := \mathbf{H}_\bullet((X, d)^{\mathrm{gr}}) = \mathbf{H}_\bullet(\mathrm{Gr} X, \mathrm{Gr} d).$$

We define the corresponding notion of “quasi-isomorphism” between gr-dg objects. We say that a map  $f : (\vec{X}, F, d) \rightarrow (\vec{Y}, F', d')$  is a *graded quasi-isomorphism* if it induces a quasi-isomorphism  $f^{\mathrm{gr}} : (X, d)^{\mathrm{gr}} \rightarrow (Y, d')^{\mathrm{gr}}$ , that is when  $\mathbf{H}_\bullet^{\mathrm{gr}}(f) : \mathbf{H}_\bullet^{\mathrm{gr}}(\vec{X}, F, d) \rightarrow \mathbf{H}_\bullet^{\mathrm{gr}}(\vec{Y}, F', d')$  is an isomorphism.

*Remark 1.9.* The graded quasi-isomorphisms are the weak equivalences in the model category structure on gr-dg  $\mathbf{R}$ -modules given in Appendix C.

From now on we use this kind of “gr” terminology without comment.

1.10. **Complete objects.** Given a filtered object  $X = (\vec{X}, F)$ , the filtration structure induces maps  $F_p X \xrightarrow{i_p} F_0 X$ .

**Definition 1.11.** We say that  $X$  is *complete* if the natural morphism

$$X \rightarrow \lim_p \mathrm{coker} i_p$$

is an isomorphism.

We use the notation  $\widehat{\mathrm{Filt}}(\mathcal{A})$  for the category of complete filtered objects in  $\mathcal{A}$ .



*Remark 1.12.* Complete objects are a reflective subcategory of filtered objects, with a *completion functor* as a reflector which we will write

$$V \mapsto \hat{V}, \quad V \mapsto \widehat{V}, \quad V \mapsto V^\wedge,$$

whichever seems typographically most appropriate in context. See Appendix A for details.

**1.13. Monoidal products.** The closed symmetric monoidal product of  $\mathcal{A}$  extends to the filtered (see Corollary A.30) and complete (Corollary A.35) settings. The product in the filtered setting has components:

$$F_p(V \bar{\otimes} W) := \text{im} \left( \left( \text{colim}_{a+b \geq p} F_a V \otimes F_b W \right) \rightarrow V \otimes W \right),$$

and the product in the complete setting is the completion:

$$V \hat{\otimes} W := (V \bar{\otimes} W)^\wedge.$$

In both cases the internal hom objects can be calculated in  $\mathbb{N}^{\text{op}}$ -indexed diagrams in  $\mathcal{A}$ .

**Definition 1.14.** We denote by  $\widehat{\text{Filt}}^{\text{gr}}(\mathcal{A})$  the category of complete gr-dg objects  $(\vec{X}, F, d)$ . It is a full subcategory of  $\widehat{\text{Filt}}(\text{pg}\mathcal{A})$ . Moreover, when  $\mathcal{A}$  is assumed to be a Grothendieck category, it is a reflexive subcategory of complete pg modules (see Corollary A.42) and the closed symmetric monoidal product of  $\mathcal{A}$  extends to the complete gr-dg setting (see Corollary A.47). The monoidal product is again  $\hat{\otimes}$ .

## 2. OPERADS IN THE COMPLETE AND FILTERED SETTING

Let  $(\mathcal{M}, \Pi, \otimes, \mathbb{1})$  be an additive closed monoidal category with small colimits and limits. We assume moreover that the monoidal structure preserves colimits in each variable. In this situation,  $\mathcal{M}$  is enriched over the category of sets (see [Fre09, 1.1.7]) and for a set  $K$  and an object  $M \in \mathcal{M}$ , we have a tensor product  $K \otimes M$  in  $\mathcal{M}$  given by

$$K \otimes M := \Pi_{k \in K} M.$$

Examples of such categories  $\mathcal{M}$  are given by the categories  $\mathcal{A}$ ,  $\text{Filt}(\mathcal{A})$ ,  $\widehat{\text{Filt}}(\mathcal{A})$ , and  $\widehat{\text{Filt}}^{\text{gr}}(\mathcal{A})$  when  $\mathcal{A}$  is a closed symmetric monoidal Grothendieck category with small colimits and limits and such that the monoidal structure preserves colimits in each variable.

**2.1. Complete  $\mathbb{S}$ -objects.** We present the monoidal category of symmetric objects (or  $\mathbb{S}$ -modules) in  $\mathcal{M}$  denoted by  $(\mathbb{S}\text{-Mod}(\mathcal{M}), \circ, I)$ . We refer for instance to [LV12, 5.1] in the case of modules and to [Fre09, 2.2] for more general situations.

An  $\mathbb{S}$ -module in  $\mathcal{M}$  is a collection  $M = \{M(0), M(1), \dots, M(n), \dots\}$  of right- $\mathbb{S}_n$ -objects  $M(n)$  in  $\mathcal{M}$ . An action of  $\mathbb{S}_n$  on an object  $M(n)$  is defined as a morphism of monoids  $\mathbb{S}_n \otimes \mathbb{1} \rightarrow M(n)$  in  $\mathcal{M}$ . A morphism  $f : M \rightarrow N$  in the category of  $\mathbb{S}\text{-Mod}(\mathcal{M})$  is a componentwise morphism.

We define the monoidal product  $\circ$  of  $\mathbb{S}$ -objects in  $\mathcal{M}$  by

$$M \circ N(n) := \prod_{k \geq 0} \left( M(k) \otimes_{\mathbb{S}_k} \left( \prod_{i_1 + \dots + i_k = n} \text{Ind}_{\mathbb{S}_{i_1} \times \dots \times \mathbb{S}_{i_k}}^{\mathbb{S}_n} (N(i_1) \otimes \dots \otimes N(i_k)) \right) \right),$$

where  $\text{Ind}_H^G M := G \otimes_H M$  is the *induced representation*. We emphasize that, when for example  $\mathcal{M} = \widehat{\text{Filt}}(\mathcal{A})$ , the sum  $\amalg$  stands for the completion of the sum in  $\mathcal{A}$  with respect to the filtration.

To simplify the notations, we denote by

$$\mathbb{S}\text{-Mod}(\widehat{\mathcal{A}}^{\text{gr}}) \text{ the category } \mathbb{S}\text{-Mod}(\widehat{\text{Filt}}^{\text{gr}}(\mathcal{A}))$$

and by

$$\mathbb{S}\text{-Mod}(\widehat{\text{dg}}\mathcal{A}) \text{ the full subcategory of } \mathbb{S}\text{-Mod}(\widehat{\mathcal{A}}^{\text{gr}})$$

given by complete pg- $\mathbb{S}$ -modules  $M = \{M(n)\}$  such that the  $M(n)$  are dg-objects. We remark that the category of  $\mathbb{S}$ -modules defined in [LV12] can be seen as a full subcategory of the category  $\mathbb{S}\text{-Mod}(\widehat{\text{dg}}\mathcal{A})$  by defining on the  $\mathbb{S}$ -module  $M$  the trivial filtration  $F_0 M = M$  and  $F_p M = \{0\}$  for all  $p > 0$ . For example,  $I := \{0, \mathbf{R}, 0, \dots\}$  endowed with the trivial filtration and the trivial differential is an object of  $\mathbb{S}\text{-Mod}(\widehat{\text{dg}}\mathcal{A})$ .

This gives a monoidal product on the categories  $\mathbb{S}\text{-Mod}(\widehat{\mathcal{A}}^{\text{gr}})$  and  $\mathbb{S}\text{-Mod}(\widehat{\text{dg}}\mathcal{A})$ . We are interested in the category  $(\mathbb{S}\text{-Mod}(\widehat{\mathcal{A}}^{\text{gr}}), \circ, I)$  in Section 2.5 and in the category  $(\mathbb{S}\text{-Mod}(\widehat{\text{dg}}\mathcal{A}), \circ, I)$  in Section 3.

*Remark 2.2.* Using the convention that an empty tensor product is equal to  $\mathbf{R}$ , we get that one of the components of  $(M \circ N)(0)$  is  $M(0)$ .

**Definition 2.3.** An *operad*  $(\mathcal{O}, \gamma, \eta, d)$  in the category  $\mathcal{M}$  is a monoid in the monoidal category  $(\mathbb{S}\text{-Mod}(\mathcal{M}), \circ, I)$ . The map  $\gamma : \mathcal{O} \circ \mathcal{O} \rightarrow \mathcal{O}$  is the *composition product*, the map  $\eta : I \rightarrow \mathcal{O}$  is the *unit* and the map  $d : \mathcal{O} \rightarrow \mathcal{O}$  is a derivation for the composition on  $\mathcal{O}$ . We denote by  $\text{Op}(\mathcal{M})$  the category of operads in  $\mathcal{M}$ .

Thus we have, for instance:

- a *graded operad* is an operad in the closed symmetric monoidal category  $(\mathfrak{g}\mathcal{A}, \otimes)$ ,
- a *filtered operad* is an operad in the closed symmetric monoidal category  $(\text{Filt}(\mathcal{A}), \otimes)$ , and
- a *complete operad* is an operad in the closed symmetric monoidal category  $(\widehat{\text{Filt}}(\mathcal{A}), \otimes)$ .
- a *complete gr-dg operad* is an operad in the closed symmetric monoidal category  $(\widehat{\text{Filt}}^{\text{gr}}(\mathcal{A}), \otimes)$ .

*Remark 2.4.* The phrase “complete filtered operad” is a priori ambiguous. We will always use this phrase to mean an operad in a symmetric monoidal category of complete filtered objects in some ground category. We will never use it to mean an operad equipped with a complete filtration each stage of which is itself an operad.

Let  $\mathcal{O}$  be an operad in  $\mathcal{M}$ . The total object obtained by taking the coproduct (in  $\mathcal{M}$ )

$$\coprod_{n \geq 0} \mathcal{O}(n)$$

supports a pre-Lie bracket  $\{-, -\}$ , and then a Lie bracket  $[-, -]$ , which is given on  $\mathcal{O}(p) \otimes \mathcal{O}(q)$  by

$$\{-, -\} := \sum_{i=1}^p \sum_{\mathbf{P}} (- \circ_i -)^{\sigma_{\mathbf{P}}},$$

The notation  $- \circ_i -$  stands for the partial composition product on  $\mathcal{O}$  where we plug the element in  $\mathcal{O}(q)$  in the  $i$ th entry of the element in  $\mathcal{O}(p)$ . The sum runs over some ordered partitions  $\mathbf{P}$  by means of which we can define the permutation  $\sigma_{\mathbf{P}}$ . (We refer to [LV12], Section 5.4.3 for more details.) If we fix a map  $\mu : I \rightarrow \mathcal{O}(1)$  (or an element  $\mu \in \mathcal{O}(1)$  when it is meaningful), we therefore obtain, by a slight abuse of notation, an endomorphism  $[\mu, -]$  of  $\mathcal{O}$  given by the formula

$$[\mu, -] := (\mu \circ_1 -) - (-1)^{|\mu||-|} \sum_{j=1}^q (- \circ_j \mu).$$

(Or

$$[\mu, \nu] := \mu \circ_1 \nu - (-1)^{|\mu||\nu|} \sum_{j=1}^q \nu \circ_j \mu$$

when this is meaningful.) Moreover, the associativity of the map  $\gamma$  shows that  $[\mu, -]$  is a derivation.

**2.5. Curved operads.** We give the definition of a curved operad in the category  $\widehat{\text{Filt}}^{\text{gr}}(\mathcal{A})$  and we present the curved endomorphism operad in this context. We emphasize that our operads are allowed to have 0-ary elements, that is  $\mathcal{O}(0) \neq \emptyset$  a priori. We use the notation of the book [LV12].

**Definition 2.6.** A *curved operad*  $(\mathcal{O}, \gamma, \eta, d, \theta)$  is a complete gr-dg operad  $(\mathcal{O}, \gamma, \eta, d)$  equipped with a map  $\theta : I \rightarrow (F_1\mathcal{O}(1))_{-2}$  such that

$$\begin{cases} d^2 &= [\theta, -] \quad (\text{or } [-, \theta] + d^2 = 0), \\ d(\theta) &= 0 \quad (\theta \text{ is closed}). \end{cases}$$

The map (or element)  $\theta$  is called the *curvature*.

*Remark 2.7.* When dealing with curvature, we often think about the map  $\theta$  as an element  $\theta \in F_1\mathcal{O}(1)$  of degree  $-2$ . If necessary, it is possible to replace “the element  $\theta$ ” by “the map  $\theta$ ” everywhere in this article.

**Definition 2.8.** A *morphism*  $f : (\mathcal{O}, d, \theta) \rightarrow (\mathcal{P}, d', \theta')$  of curved operads is a morphism of operads such that

$$\begin{cases} f \cdot d &= (-1)^{|f|} d' \cdot f, \\ f(\theta) &= \theta'. \end{cases}$$

We denote by  $\text{cOp}(\mathcal{A})$  the category of curved operads in  $\widehat{\text{Filt}}^{\text{gr}}(\mathcal{A})$ .

A *weak equivalence between curved operads* is a morphism of curved operads whose underlying  $\mathbb{S}$ -module map is a weak equivalence.

An important example of a curved operad is given by the endomorphism operad of a complete gr-dg object.

**Definition 2.9.** The *endomorphism operad* of a complete gr-dg object  $(X, d)$  is the curved operad

$$\text{End}_X := (\{\text{Hom}(X^{\otimes n}, X)\}_{n \geq 0}, \gamma, \partial, \theta),$$

where the composition map  $\gamma$  is given by the composition of functions and

$$\begin{cases} \partial(f) &:= [d, f], \quad \text{for } f \in \text{Hom}(X^{\otimes n}, X), \\ \theta &:= d^2. \end{cases}$$

It is straightforward to check that this really defines a curved operad.

**Definition 2.10.** A *representation* of a curved operad  $\mathcal{O}$  on the complete gr-dg object  $X$  is a map of curved operads  $\mathcal{O} \rightarrow \text{End}_X$ . We also say that this defines an  *$\mathcal{O}$ -algebra structure* on  $X$ .

*Remark 2.11.* An example of such an algebra is given by curved associative algebras (see Section 6) and curved Lie algebras in a complete setting.

An *augmentation* of a curved operad  $(\mathcal{O}, \gamma, \eta, d, \theta)$  is a map  $\mathcal{O} \rightarrow I$  of curved operads such that  $\varepsilon \cdot \eta = \text{id}_I$ .

An augmentation realizes the underlying  $\mathbb{S}$ -object of  $\mathcal{O}$  as the biproduct (i.e., both the product and coproduct) of  $I$  and  $\ker \varepsilon$ , which we denote  $\overline{\mathcal{O}}$  as usual.

**2.12. Free complete operad.** The construction of the free operad given in [BJT97, Rez96] and in [LV12], Section 5.5.1 and 5.8.6 apply in the filtered and in the complete setting by replacing the biproduct and the composition product  $\circ$  by their filtered or complete analog.

We recall the definition of the tree operad. For  $M$  an  $\mathbb{S}$ -module in  $\mathcal{M}$ , we define

$$\mathcal{T}_0 M = I$$

and recursively define

$$\mathcal{T}_n M = I \amalg (M \circ \mathcal{T}_{n-1} M).$$

There is an inclusion  $\mathcal{T}_0 M \rightarrow \mathcal{T}_1 M$ . Then given a map  $\iota_{n-1} : \mathcal{T}_{n-1} M \rightarrow \mathcal{T}_n M$ , there is a map  $\iota_n : \mathcal{T}_n M \rightarrow \mathcal{T}_{n+1} M$  which takes the  $I$  factor to the  $I$  factor and takes  $M \circ \mathcal{T}_{n-1} M \rightarrow M \circ \mathcal{T}_n M$  using  $\text{id}_M$  on the  $M$  factor and  $\iota_{n-1}$  on the other factor. We write  $\mathcal{T}M$  the colimit (in filtered  $\mathbb{S}$ -modules) over  $n$  of  $\mathcal{T}_n M$  and  $\widehat{\mathcal{T}}M$  the colimit (in complete  $\mathbb{S}$ -modules) over  $n$  of  $\mathcal{T}_n M$ .

There is an injection from  $I = \mathcal{T}_0 M \rightarrow \mathcal{T}_n M$  which passes to a map  $\eta : I \rightarrow \mathcal{T}M$  (resp.  $\eta : I \rightarrow \widehat{\mathcal{T}}M$ ). Similarly, the map  $M \rightarrow \mathcal{T}_1 M$  induces a map  $j : M \rightarrow \mathcal{T}M$  (resp.  $j : M \rightarrow \widehat{\mathcal{T}}M$ ).

The two constructions  $\mathcal{T}M$  and  $\widehat{\mathcal{T}}M$  are the free operads in the filtered and in the complete setting with the same arguments as in Theorem 5.5.1 in [LV12].

**Theorem 2.13.** *There is an operad structure  $\gamma$  (resp.  $\hat{\gamma}$ ) on  $\mathcal{T}M$  (resp.  $\widehat{\mathcal{T}}M$ ) such that  $\mathcal{T}(M) := (\mathcal{T}M, \gamma, j)$  (resp.  $\widehat{\mathcal{T}}(M) := (\widehat{\mathcal{T}}M, \hat{\gamma}, j)$ ) is the free operad on  $M$  in the category of filtered operads (resp. of complete operads).*

*Proof.* See [LV12, Theorem 5.5.1]. □

Next we verify that this standard construction commutes with completion, so that the complete free operad is the aritywise completion with respect to the filtration of the usual free operad.

A lax symmetric monoidal functor induces a functor on operads, acting aritywise (see, e.g., [Fre17, Proposition 3.1.1] or [YJ15, Theorem 12.11(1)] applied to Corollary 11.16). Therefore, given an operad  $\mathcal{O}$  in filtered objects, the aritywise completion  $\widehat{\mathcal{O}}$  is (functorially) a complete operad. Moreover, the completion functor and the inclusion of complete operads into filtered operads are still adjoint [YJ15, Corollary 12.13] (see also [Fre17, Proposition 3.1.5], where the hypotheses require  $G$  to be strong symmetric monoidal but the argument only uses the lax structure). We record the conclusion as follows.

**Proposition 2.14.** *Aritywise completion and inclusion form an adjunction between complete filtered operads and filtered operads.*

This construction is then compatible with the free operad in the following sense.

**Proposition 2.15.** *Let  $M$  be a filtered graded  $\mathbb{S}$ -object. The completion of the free filtered (graded) operad on  $M$  and the map of complete  $\mathbb{S}$ -objects  $\widehat{M} \rightarrow \widehat{\mathcal{T}M}$  induced by completion exhibits  $\widehat{\mathcal{T}M}$  as the free complete operad  $\widehat{\mathcal{T}M}$  on the complete  $\mathbb{S}$ -object  $\widehat{M}$ .*

*Proof.* By adjunction, maps of complete operads from  $\widehat{\mathcal{T}M}$  to a complete graded operad  $\mathcal{X}$  are in bijection first with operad maps from  $\mathcal{T}M$  to  $\mathcal{X}$ , then to  $\mathbb{S}$ -object maps from  $M$  to  $\mathcal{X}$ , and finally to complete  $\mathbb{S}$ -object maps from  $\widehat{M}$  to  $\mathcal{X}$ .

A priori there are two recipes for a map from  $\widehat{M}$  to  $\widehat{\mathcal{T}M}$ . To see that they agree, it suffices to compare them on  $M$ , where both are the composition  $M \rightarrow \mathcal{T}M \rightarrow \widehat{\mathcal{T}M}$ .  $\square$

*Remark 2.16.* (1) When  $(M, d_M)$  is a predifferential graded  $\mathbb{S}$ -object, the free (complete) operad on  $M$  is naturally endowed with a predifferential  $\tilde{d}_M$  induced by  $d_M$  defined as follows: it is the unique derivation which extends the map

$$M \xrightarrow{d_M} M \hookrightarrow \mathcal{T}M.$$

We denote by  $\mathcal{T}(M, d_M)$  the free pg operad  $(\mathcal{T}M, \tilde{d}_M)$ . Moreover, when  $(M, d_M)$  is a gr-dg  $\mathbb{S}$ -object, so is  $\mathcal{T}(M, d_M)$  and it is the free gr-dg operad on  $(M, d_M)$ . This is true in the filtered and in the complete settings.

- (2) To lighten the notation, when the setting is explicit, we use the notation  $\mathcal{T}$  for the free operad both in the filtered and in the complete setting.
- (3) By the previous proposition, it is possible to think of an element in the free complete operad on the complete graded  $\mathbb{S}$ -object  $M$  as a (possibly infinite) sum of trees, whose vertices of arity  $k$  are indexed by element in  $M(k)$ .

**2.17. Free curved operad on a gr-dg  $\mathbb{S}$ -module.** We now provide a curved version of the free-forgetful functors adjunction.

**Definition 2.18.** We say that a couple  $(\mathcal{O}, d_{\mathcal{O}}, \theta)$  is a *pointed complete gr-dg operad* when  $\mathcal{O}$  is a complete gr-dg operad and  $\theta$  is a *closed* element in  $F_1\mathcal{O}_{-2}$  (that is  $d_{\mathcal{O}}(\theta) = 0$ ).

In the next proposition,  $\mathcal{T}$  denote either the free filtered gr-dg operad or the free complete gr-dg operad.

**Proposition 2.19.** *We define the functor*

$$\begin{aligned} \mathcal{T}_+ &: \mathbb{S}\text{-Mod}(\widehat{\mathcal{A}}^{\text{gr}}) \rightarrow \text{Pointed filtered/compl. gr-dg operads,} \\ (M, d_M) &\mapsto \left( \mathcal{T}(M \amalg \vartheta I), \tilde{d}_M, \vartheta \right), \end{aligned}$$

where  $\vartheta$  is a formal parameter in homological degree  $-2$  and in filtration degree 1 and  $\tilde{d}_M$  is the unique derivation which extends the map

$$M \amalg \vartheta I \rightarrow M \xrightarrow{d_M} M \rightarrow \mathcal{T}(M \amalg \vartheta I).$$

There is an adjunction

$$\mathcal{T}_+ : \mathbb{S}\text{-Mod}(\widehat{\mathcal{A}}^{\text{gr}}) \rightleftarrows \text{Pointed filt./compl. gr-dg operads} : U.$$

Moreover, the pointed filtered / complete gr-dg operad  $(\mathcal{T}_+(M, d_M), \gamma, j_+)$ , where  $\gamma$  is the free composition product and  $j_+$  is the map  $M \xrightarrow{j} \mathcal{T}M \rightarrow \mathcal{T}(\vartheta I \amalg M)$  is the free filtered / complete gr-dg operad.

*Proof.* We denote by  $\mathbf{pOp}$  the category of pointed operads under consideration and we consider a pointed operad  $(\mathcal{O}, d_{\mathcal{O}}, \theta)$  in it. By Theorem 2.13, we have

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Op}}(\mathcal{T}(M \amalg \vartheta I), \mathcal{O}) &\cong \mathrm{Hom}_{\mathbb{S}\text{-Mod}(\widehat{\mathcal{A}}^{\mathrm{gr}})}(M \amalg \vartheta I, U(\mathcal{O})) \\ &\cong \mathrm{Hom}_{\mathbb{S}\text{-Mod}(\widehat{\mathcal{A}}^{\mathrm{gr}})}(M, U(\mathcal{O})) \times \mathrm{Hom}_{\mathbb{S}\text{-Mod}(\widehat{\mathcal{A}}^{\mathrm{gr}})}(\vartheta I, U(\mathcal{O})), \end{aligned}$$

where  $U(\mathcal{O})$  is the underlying filtered / complete graded  $\mathbb{S}$ -modules associated with  $\mathcal{O}$ . The fiber over the map  $\vartheta \mapsto \theta$  on the right-hand side is naturally isomorphic to  $\mathrm{Hom}_{\mathbb{S}\text{-Mod}(\widehat{\mathcal{A}}^{\mathrm{gr}})}(M, U(\mathcal{O}))$ . On the other hand, the fiber over the map  $\vartheta \mapsto \theta$  on the left-hand side consists of those operad maps  $\mathcal{T}_+M \rightarrow \mathcal{O}$  sending  $\vartheta$  to  $\theta$ . The condition on the predifferentials follows directly from the construction of  $\tilde{d}_M$ . We therefore obtain the adjunction

$$\mathrm{Hom}_{\mathbf{pOp}}(\mathcal{T}_+(M, d_M), (\mathcal{O}, d_{\mathcal{O}}, \theta)) \cong \mathrm{Hom}_{\mathbb{S}\text{-Mod}(\widehat{\mathcal{A}}^{\mathrm{gr}})}((M, d_M), U(\mathcal{O}, d_{\mathcal{O}}, \theta)),$$

where  $U(\mathcal{O}, d_{\mathcal{O}}, \theta)$  is the underlying filtered / complete gr-dg  $\mathbb{S}$ -modules associated with  $(\mathcal{O}, d_{\mathcal{O}}, \theta)$ . It is clear from the proof (and Theorem 2.13) that the tuple  $(\mathcal{T}_+(M, d_M), \gamma, j_+)$  is the free filtered / complete gr-dg operad.  $\square$

We construct the free curved operad by means of the functor  $\mathcal{T}_+$ .

**Theorem 2.20.** *We define the functor*

$$\begin{aligned} \mathbf{cT} : \mathbb{S}\text{-Mod}(\widehat{\mathcal{A}}^{\mathrm{gr}}) &\rightarrow \text{Curved operads} \\ (M, d_M) &\mapsto (\mathcal{T}(M \amalg \vartheta I) / (\mathrm{im}(d_M^2 - [\vartheta, -])), \bar{d}_M, \bar{\vartheta}), \end{aligned}$$

where  $\vartheta$  is a formal parameter in homological degree  $-2$  and in filtration degree  $1$ , the map  $\bar{d}_M$  is the derivation induced by  $\tilde{d}_M$  and  $(\mathrm{im}(d_M^2 - [\vartheta, -]))$  is the ideal generated by the image of the map  $d_M^2 - [\vartheta, -] : M \rightarrow \mathcal{T}(M \amalg \vartheta I)$ .

There is an adjunction

$$\mathbf{cT} : \mathbb{S}\text{-Mod}(\widehat{\mathcal{A}}^{\mathrm{gr}}) \rightleftarrows \text{Curved operads} : U.$$

Moreover, let  $\bar{j} : (M, d_M) \rightarrow \mathbf{cT}(M, d_M)$  denote the composition  $(M, d_M) \xrightarrow{j} \mathcal{T}(M, d_M) \rightarrow \mathbf{cT}(M, d_M)$ . The curved operad  $(\mathbf{cT}(M, d_M), \bar{\gamma}, \bar{j})$  is the free curved operad on  $(M, d_M)$  in the category of curved operads.

*Proof.* The map  $\tilde{d}_M$  induces a well-defined map  $\bar{d}_M$  on the quotient of the free operad  $\mathcal{T}(M \amalg \vartheta I) / (\mathrm{im}(d_M^2 - [\vartheta, -]))$  since  $\tilde{d}_M(\vartheta) = 0$ . From the fact that

$$\mathrm{im}(d_M^2 - [\vartheta, -]) \subset (\mathrm{im}(\tilde{d}_M^2 - [\bar{\vartheta}, -])) = (\mathrm{im}(d_M^2 - [\vartheta, -]))$$

we obtain that  $\mathbf{cT}(M, d_M)$  is a curved operad.

We denote by  $U_p$  is the forgetful functor from curved operads to pointed gr-dg operads. Morphisms of curved operads are morphisms of pointed gr-dg operads between curved operads. This means that a morphism of curved operads  $\bar{f} : \mathbf{cT}(M, d_M) \rightarrow (\mathcal{O}, d_{\mathcal{O}}, \theta)$  is the same thing as a morphism of pointed gr-dg operads  $\bar{f} : U_p \mathbf{cT}(M, d_M) \rightarrow U_p(\mathcal{O}, d_{\mathcal{O}}, \theta)$ . Morphisms  $U_p \mathbf{cT}(M, d_M) \rightarrow U_p(\mathcal{O}, d_{\mathcal{O}}, \theta)$  in pointed gr-dg operads coincide with morphisms  $\mathcal{T}_+(M, d_M) \rightarrow U_p(\mathcal{O}, d_{\mathcal{O}}, \theta)$  in pointed gr-dg operads with the condition that the ideal  $(\mathrm{im}(d_M^2 - [\vartheta, -]))$  is sent

to 0. Since  $(\text{im}(d_M^2 - [\bar{\vartheta}, -])) = (\text{im}(d_M^2 - [\vartheta, -]))$  and morphisms of pointed gr-dg operads commute with the predifferentials and send the marked point to the marked point, we get that this last condition is automatically satisfied when  $U_p(\mathcal{O}, d_{\mathcal{O}}, \theta)$  is the underlying pointed gr-dg operad of a curved operad. Hence, we have

$$\text{Hom}_{\mathfrak{COp}}(\mathfrak{cT}(M, d_M), (\mathcal{O}, d_{\mathcal{O}}, \theta)) \cong \text{Hom}_{\mathfrak{pOp}}(\mathcal{T}_+(M, d_M), U_p(\mathcal{O}, d_{\mathcal{O}}, \theta)).$$

The Theorem follows from Proposition 2.19.  $\square$

*Remark 2.21.* This adjunction is useful to define a model category structure on complete curved operads so that the bar-cobar resolution and the Koszul resolution provide  $\mathbb{S}$ -cofibrant resolutions (see Appendix C for the details on the model structure).

We now give the notion of quasi-free complete curved operad. The cobar construction (see Section 4) is a quasi-free operad and cofibrant complete curved operad (see Appendix C) are retracts of quasi-free complete curved operads.

**Definition 2.22.** We call a complete curved operad  $\mathcal{O}$  *quasi-free* if there exists a complete  $\mathbb{S}$ -module  $M$  and a predifferential  $d : \mathcal{T}_+(M) \rightarrow \mathcal{T}_+(M)$  such that

$$\mathcal{O} \cong (\mathcal{T}_+(M), d) / (\text{im}(d^2 - [\vartheta, -])).$$

It is different from a free complete curved operad by the fact that the predifferential  $d$  is not a priori induced by a map  $M \rightarrow M$ .

### 3. COMPLETE COOPERADS

We present in this section the notion of a complete cooperad and associated notions, most importantly *altipotence*, a variation of conilpotence where instead of being eventually zero, iterated powers of the decomposition map eventually arrive in arbitrarily high filtration degree. This material is fairly technical, but the raison d'être and upshot are straightforward: the goal throughout is to establish a context in which the usual formulas for the cofree conilpotent cooperad in terms of decorated trees extend to the complete context. The tension is to find a full subcategory of cooperads that is big enough to contain the (complete) tree cooperad but small enough so that the tree cooperad is the cofree object there. It is possible (in fact likely) that there are more natural alternatives to working with altipotence. Since for us the altipotent cooperads are not of intrinsic interest but merely a tool to study complete operads, in that capacity they are perfectly adequate.

**Definition 3.1.** We define a *cooperad*  $(\mathcal{C}, \Delta, \varepsilon, d)$  in the category  $\mathcal{M}$  to be a comonoid in the monoidal category  $(\mathbb{S}\text{-Mod}(\mathcal{M}), \circ, I)$ . The map  $\Delta : \mathcal{C} \rightarrow \mathcal{C} \circ \mathcal{C}$  is a *decomposition map*, the map  $\varepsilon : \mathcal{C} \rightarrow I$  is the *counit map*. Morphisms of cooperads are morphisms of comonoids in  $\mathbb{S}$ -modules. We denote by  $\text{Coop}(\mathcal{M})$  the category of cooperads in  $\mathcal{M}$ .

Thus:

- a graded cooperad (in  $\mathcal{A}$ ) is a cooperad in the category  $(\mathfrak{g}\mathcal{A}, \otimes)$ ,
- a filtered cooperad is a cooperad in the category  $(\text{Filt}(\mathcal{A}), \bar{\otimes})$ , and
- a *complete cooperad* is a cooperad in the category  $(\widehat{\text{Filt}}(\mathcal{A}), \hat{\otimes})$ ,
- a *dg complete cooperad* is a cooperad in the category  $(\widehat{\text{Filt}}(\text{dg}\mathcal{A}), \hat{\otimes})$ .

We need an other notion of “cooperad” in the next section to define a bar construction and a cobar construction.

**Definition 3.2.** When  $(M, F)$  is a (complete) filtered object in  $\mathcal{M}$ , we define the  $k$ -shifted filtrations of  $F$  by  $(M, F)[k] = (M, F[k])$  where

$$F[k]_p M = \begin{cases} F_0 M & \text{when } k + p \leq 0, \\ F_{k+p} M & \text{when } k + p \geq 1. \end{cases}$$

With a slight abuse of notation, we often simply write  $M[k]$  instead of  $(M, F)[k]$  and  $F_p M[k]$  instead of  $F[k]_p M$ .

We denote by  $I[-1]$  the (complete) filtered  $\mathbb{S}$ -module endowed with the following filtration

$$F_0 I[-1] = I, \quad F_1 I[-1] = I \text{ and } F_p I[-1] = \emptyset \text{ for all } p \geq 2.$$

**Definition 3.3.** We define an *infinitesimal cooperad*  $(\mathcal{C}, \Delta, \varepsilon, d)$  in the category  $\mathcal{M}$  to be the following data:

- a coassociative map  $\Delta : \mathcal{C} \rightarrow \mathcal{C} \circ \mathcal{C}$  called the *decomposition map*;
- a map  $\varepsilon : \mathcal{C} \rightarrow I[-1]$  called the *counit map* which fits into the following commutative diagrams

$$\begin{array}{ccccc} (\mathcal{C}(n)[-n])_n \cong \mathcal{C} \circ (I[-1]) & \xleftarrow{\text{id} \circ \varepsilon} & \mathcal{C} \circ \mathcal{C} & \xrightarrow{\varepsilon \circ \text{id}} & I[-1] \circ \mathcal{C} \cong (\mathcal{C}(n)[-1])_n \\ & \swarrow \text{incl.} & \uparrow \Delta & \searrow \text{incl.} & \\ & & \mathcal{C} & & \end{array}$$

Morphisms of infinitesimal cooperads are morphisms of  $\mathbb{S}$ -modules which preserve the decomposition map and the counit map. We denote by  $\text{infCoop}(\mathcal{M})$  the category of infinitesimal cooperads in  $\mathcal{M}$ .

The same definition is meaningful in the case where the filtration is a graduation and we can therefore provide the same definition in a graded setting. Because generically the associated graded functor is not colax, it does not take filtered (infinitesimal) cooperads to graded (infinitesimal) cooperads. For that we need a flatness condition.

**Proposition 3.4.** *There is a functor  $\text{Gr}$  from the category of gr-flat filtered (or complete, infinitesimal) cooperads to graded (infinitesimal) cooperads covering the associated graded functor.*

See Appendix B for the proof.

*Remark 3.5 (Warning).* Because the associated graded functor does not extend to arbitrary filtered or complete (infinitesimal) cooperads, any “gr” definition which uses a cooperadic structure on the associated graded is implicitly assumed to only be defined for gr-flat filtered or gr-flat complete (infinitesimal) cooperads.

**3.6. The tree cooperad.** We recall the construction of the *tree cooperad* in an additive category. This construction is taken directly from [LV12, 5.8.6] with very mild adaptations (using coinvariants instead of invariants and working in greater generality). We recall it explicitly because we will need to work with it intimately.

The following construction works in a cocomplete symmetric monoidal additive category equipped with the composition product  $\circ$  and unit  $I$ . We however work



in the category  $\mathcal{M}$  described in the beginning of Section 2 for the results arriving after.

**Construction 3.7** (Tree cooperad). For  $M$  an  $\mathbb{S}$ -module in  $\mathcal{M}$ , we write  $\mathcal{T}^c M$  as the colimit (in  $\mathbb{S}$ -modules) over  $n$  of  $\mathcal{T}_n M$ , where  $\mathcal{T}_n M$  is defined in Section 2.12.

There is a projection using the zero map from  $\mathcal{T}_n M \rightarrow I$  which passes to a map  $\varepsilon : \mathcal{T}^c M \rightarrow I$ . There is an inclusion  $\eta : I = \mathcal{T}_0 M \rightarrow \mathcal{T}^c M$ .

Finally we define a decomposition map. We proceed inductively. First define  $\Delta_0 : \mathcal{T}_0 M \rightarrow \mathcal{T}_0 M \circ \mathcal{T}_0 M$  as the canonical map  $I \rightarrow I \circ I$ . We use the same definition for  $I \subset \mathcal{T}_n M$ . For the  $M \circ \mathcal{T}_{n-1} M$  summand of  $\mathcal{T}_n M$ , we define a map to the biproduct

$$M \circ \mathcal{T}_{n-1} M \rightarrow (I \circ (M \circ \mathcal{T}_{n-1} M)) \amalg ((M \circ \mathcal{T}_{n-1} M) \circ \mathcal{T}_n M)$$

by  $I \circ \text{id}$  on the first factor and

$$\begin{aligned} M \circ \mathcal{T}_{n-1} M &\xrightarrow{\text{id}_M \circ \Delta_{n-1}} M \circ (\mathcal{T}_{n-1} M \circ \mathcal{T}_{n-1} M) \\ &\cong (M \circ \mathcal{T}_{n-1} M) \circ \mathcal{T}_{n-1} M \xrightarrow{\text{id} \circ \iota_{n-1}} (M \circ \mathcal{T}_{n-1} M) \circ \mathcal{T}_n M \end{aligned}$$

on the second factor.

This passes to the colimit to give a map  $\Delta : \mathcal{T}^c M \rightarrow \mathcal{T}^c M \circ \mathcal{T}^c M$ . Verifying left unitality is direct, right unitality is by induction, and associativity is by a combination of induction and combinatorics involving the  $I$  factor.

This defines a cooperad, called the *tree cooperad*.

The tree cooperad is often constructed (e.g., in [LV12]) as an explicit model for the cofree conilpotent cooperad. We care about this explicit model more or less because it lets us perform calculations. However, in the complete case which interest us, the cooperads is not conilpotent in general. Therefore our goal is to weaken the conilpotence condition and find a more relaxed setting in which the tree cooperad will still be cofree.

**Lemma 3.8.** *There is a natural retract of collections  $r_n$  from  $\mathcal{T}_n M$  to  $\mathcal{T}_{n-1} M$ .*

*Proof.* We can define  $r_n$  inductively. For  $n = 1$  the retract  $r_1 : I \amalg M \circ \mathcal{T}_0 M \rightarrow I$  is projection to the first factor. Suppose given  $r_j$  for  $1 \leq j < n$ . Define the  $n$  component at  $M$ ,

$$r_n : I \amalg (M \circ \mathcal{T}_n M) \rightarrow I \amalg (M \circ \mathcal{T}_{n-1} M),$$

as the identity on the first factor and  $\text{id} \circ r_{n-1}$  on the second factor. By induction  $r_n$  is a retract of the desired inclusion.  $\square$

The composite of  $r_2 \cdots r_n : \mathcal{T}_n M \rightarrow \mathcal{T}_1 M$  with the projection  $\mathcal{T}_1 M \rightarrow M \circ I \cong M$  provides a map  $\epsilon_n(M) : \mathcal{T}_n M \rightarrow M$ , which extends to a map  $\epsilon(M) : \mathcal{T}^c M \rightarrow M$  (functorial in  $M$ ).

We will use the notation  $Q_n M$  for the ‘‘graded’’ component  $\mathcal{T}_n M / \mathcal{T}_{n-1} M$ .

**Corollary 3.9.** *The underlying collection of the tree cooperad  $\mathcal{T}^c M$  is the coproduct of  $Q_n M$  over all  $n$ . Similarly, the finite stage  $\mathcal{T}_n M$  is the coproduct of these quotients through stage  $n$ .*

*Proof.* The retract of Lemma 3.8 realizes the object  $\mathcal{T}_n M$  as the coproduct of  $Q_n M$  and  $\mathcal{T}_{n-1} M$ , since  $\mathbb{S}\text{-Mod}(\mathcal{M})$  is additive with kernels. Then by induction  $\mathcal{T}_n^c M$  is isomorphic to the coproduct

$$\coprod_{j \leq n} Q_j M,$$

and the colimit of these is the infinite coproduct.  $\square$

*Remark 3.10.* Intuitively we can think of the  $n$ th graded stage  $Q_n M$  as consisting of rooted trees decorated by elements of  $M$  with maximal number of vertices encountered in a simple path from the root to a leaf precisely equal to  $n$ .

**Corollary 3.11.** *Let  $M$  be a complete collection in  $\widehat{\text{Filt}}(\mathcal{A})$ . A map of complete collections with codomain  $\mathcal{T}^c M$  is determined uniquely by its projections to  $Q_n M$  (but need not exist).*

*Proof.* This follows from Lemma A.17 which says the map from the coproduct to the product is monic.  $\square$

**Construction 3.12.** Let  $\mathcal{C}$  be a complete cooperad and  $M$  a complete collection equipped with a map of collections  $\varphi : \mathcal{C} \rightarrow M$ .

We inductively construct a map  $\Phi_n : \mathcal{C} \rightarrow Q_n M$  as follows. We define  $\Phi_0$  as  $\mathcal{C} \rightarrow I \cong \mathcal{T}_0^c M$  as the counit  $\varepsilon$ . We define  $\Phi_1$  as  $\varphi$ . Next assuming  $\Phi_0, \dots, \Phi_{n-1}$  are defined, we define a map to the biproduct

$$\sum_{i=0}^n \Phi_i : \mathcal{C} \rightarrow \mathcal{T}_n^c M = I \amalg (M \circ \mathcal{T}_{n-1}^c M)$$

as  $\Phi_0$  on the first summand  $I$  and

$$\left( \varphi \circ \sum_{i=1}^n \Phi_{i-1} \right) \cdot \Delta$$

on the second summand. Inductively this agrees with  $\sum_{i=0}^{n-1} \Phi_i$  except on the factor  $Q_n M$  of the biproduct  $\mathcal{T}_n^c M$ , and so defines  $\Phi_n$ .

This gives a map into the product of quotients  $\prod Q_n M$ .

**Lemma 3.13.** *Let  $\mathcal{C}$  be a complete cooperad and  $M$  a complete collection. Let  $\Phi : \mathcal{C} \rightarrow \mathcal{T}^c M$  be a map of collections and let  $\varphi : \mathcal{C} \rightarrow M$  be the projection*

$$\mathcal{C} \xrightarrow{\Phi} \mathcal{T}^c M \rightarrow \prod Q_n M \rightarrow M.$$

*Then  $\Phi$  is a map of cooperads if and only if the projection*

$$\mathcal{C} \xrightarrow{\Phi} \mathcal{T}^c M \rightarrow \prod_n Q_n M \rightarrow Q_N M$$

*coincides with the map  $\Phi_N$  obtained from  $\varphi$  via Construction 3.12 for all  $N$ .*

*Proof.* The map  $\Phi$  is compatible with counits if and only if  $\mathcal{C} \xrightarrow{\Phi} \mathcal{T}^c M \rightarrow Q_0 M \cong I$  is the counit  $\varepsilon$  of  $\mathcal{C}$ , which coincides with  $\Phi_0$ . Compatibility with  $\Phi_1$  is already a hypothesis of the lemma.

Let us first show necessity. Suppose shown that the projection of  $\Phi$  to  $\mathcal{T}_{n-1} M$  must be  $\sum_{j=0}^{n-1} \Phi_{j-1}$  in order that  $\Phi$  be a map of cooperads. Let  $N$  be arbitrary (a priori unrelated to  $n$ ) and consider the composition of the ‘‘comultiplication’’  $\Delta_N$  :

$\mathcal{T}_N M \rightarrow \mathcal{T}_N M \circ \mathcal{T}_N M$  used in Construction 3.7 and the map  $\mathcal{T}_N M \circ \mathcal{T}_N M \xrightarrow{\epsilon_N(M)\text{oid}} M \circ \mathcal{T}_N M$ . More or less by unitality, this composition is the inclusion of the non- $I$  summand of  $\mathcal{T}_N M$  into the non- $I$  summand of  $\mathcal{T}_{N+1} M$ . Moreover, the restriction of this composition along  $Q_N M \rightarrow \mathcal{T}_N M$  lifts as follows (again essentially by definition of  $\Delta_N$ ):

$$\begin{array}{ccccccc}
 & & & & & & (M \circ \mathcal{T}_{N-1} M) / (M \circ \mathcal{T}_{N-2} M) \\
 & & & & & & \downarrow \\
 Q_N M & \xrightarrow{\quad} & \mathcal{T}_N M & \xrightarrow{\Delta_N} & \mathcal{T}_N M \circ \mathcal{T}_N M & \xrightarrow{\epsilon_N(M)\text{oid}} & M \circ \mathcal{T}_N M.
 \end{array}$$

In particular for  $N > n$  the projection of the horizontal composition to  $M \circ \mathcal{T}_{n-1} M$  vanishes. Then the following diagram commutes:

$$\begin{array}{ccccc}
 \mathcal{C} & \xrightarrow{\Phi} & \mathcal{T}^c M & \xrightarrow{\text{projection}} & \mathcal{T}_n M \\
 \Delta \downarrow & & \Delta \downarrow & & \downarrow \\
 \mathcal{C} \circ \mathcal{C} & \xrightarrow{\Phi \circ \Phi} & \mathcal{T}^c M \circ \mathcal{T}^c M & \longrightarrow & M \circ \mathcal{T}_{n-1} M.
 \end{array}$$

The composition along the bottom of the diagram is  $\varphi \circ \sum_{j=0}^{n-1} \Phi_j$  by induction, which suffices to show that the projection of  $\Phi$  to  $\mathcal{T}_n M$  is  $\sum_{j=0}^n \Phi_j$ , extending the induction.

Now we turn to sufficiency. Suppose that  $\Phi$  has projections as in Construction 3.12. We would like to argue that  $\Phi$  is a cooperad map. We reduce the question to an inductive question about the finite stages  $\mathcal{T}_r M \circ \mathcal{T}_s M$  by the following categorical argument. First,  $\circ$  preserves filtered colimits in each variable, since  $\otimes$  does, so maps to  $\mathcal{T}^c M \circ \mathcal{T}^c M$  are the same as maps to  $\text{colim}_{r,s} \mathcal{T}_r M \circ \mathcal{T}_s M$ . All of the connecting maps in the diagram for this colimit split, which means that  $\text{colim}_{r,s} \mathcal{T}_r M \circ \mathcal{T}_s M$  splits as a coproduct. Writing the coproduct explicitly is somewhat inconvenient since  $\circ$  doesn't naively distribute over coproducts on the right, but in any event every term in this coproduct appears at the stage indexed by some finite pair  $(r, s)$ . Then by Lemma A.17, there is a monomorphism from  $\text{colim}_{r,s} \mathcal{T}_r M \circ \mathcal{T}_s M$  to  $\prod_{r,s} \mathcal{T}_r M \circ \mathcal{T}_s M$ . Moreover, the projection of  $\mathcal{T}^c M \xrightarrow{\Delta} \mathcal{T}^c M \circ \mathcal{T}^c M \rightarrow \mathcal{T}_r M \circ \mathcal{T}_s M$  factors through  $\mathcal{T}_{r+s} M$ . Therefore to check commutativity of the square

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{\Phi} & \mathcal{T}^c M \\
 \downarrow & & \downarrow \Delta \\
 \mathcal{C} \circ \mathcal{C} & \xrightarrow{\Phi \circ \Phi} & \mathcal{T}^c M \circ \mathcal{T}^c M
 \end{array}$$

it suffices to check commutativity of the square

$$\begin{array}{ccccc}
 & & \xrightarrow{\sum_j^{r+s} \Phi_j} & & \\
 \mathcal{C} & \xrightarrow{\Phi} & \mathcal{T}^c M & \longrightarrow & \mathcal{T}_{r+s} M \\
 \downarrow & & & & \downarrow \text{projection of } \Delta \\
 \mathcal{C} \circ \mathcal{C} & \xrightarrow{\Phi \circ \Phi} & \mathcal{T}^c M \circ \mathcal{T}^c M & \longrightarrow & \mathcal{T}_r M \circ \mathcal{T}_s M \\
 & & \xrightarrow{(\sum_j^r \Phi_j) \circ (\sum_j^s \Phi_j)} & & 
 \end{array}$$

for all  $r$  and  $s$ .

We proceed by induction on  $r$ . The base cases with  $r = 0$  follow from unitality considerations, so assume that  $r$  is strictly positive. Consider the following diagram:

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{\sum_j^{r+s} \Phi_j} & \mathcal{T}_{r+s}M \\
 \Delta \downarrow & & \downarrow \text{projection} \\
 \mathcal{C} \circ \mathcal{C} & \xrightarrow{\varphi \circ \sum_j^{r+s-1} \Phi_j} & M \circ \mathcal{T}_{r+s-1}M \\
 \text{id} \circ \Delta \downarrow & & \downarrow \text{projection of } \Delta \\
 \mathcal{C} \circ \mathcal{C} \circ \mathcal{C} & \xrightarrow{\varphi \circ \sum_j^{r-1} \Phi_j \circ \sum_j^s \Phi_j} & M \circ \mathcal{T}_{r-1}M \circ \mathcal{T}_sM.
 \end{array}$$

The upper square commutes by construction of  $\Phi_j$  and the lower square commutes by the inductive premise. The outer rectangle constitutes the correct commutativity for all of the summands of the next step in the induction except for the  $I \circ \mathcal{T}_sM$  term, which again follows by unitality. Note that comparing this outside cell to the next inductive step uses the fact that  $(\text{id} \circ \Delta) \cdot \Delta = (\Delta \circ \text{id}) \cdot \Delta$  in the cooperad  $\mathcal{C}$ . This completes the induction.  $\square$

This lemma constitutes a uniqueness result for extending  $\varphi$  to a cooperad map  $\Phi$ . However, it is only a partial existence result because Construction 3.12 a priori lands in the product of  $Q_nM$ , not in the subobject  $\mathcal{T}^cM$  of the product. In the next section we will provide conditions for a refined existence result, conditions under which the projections  $\Phi_n$  determine a map with codomain  $\mathcal{T}^cM$ .

**3.14. Gr-coaugmentation and gr-conilpotence.** We now move on to the definition of altpotence. To manage that, we generalize the notions of coradical filtration and of primitives for a coaugmented cooperad to the setting of complete filtered (infinitesimal) cooperads.

**Definition 3.15.** Let  $(\mathcal{C}, \Delta, \varepsilon, d)$  be a complete cooperad.

- A *coaugmentation* for this cooperad is a map  $\eta : I \rightarrow \mathcal{C}$  of cooperads (on  $I$ ,  $\Delta(1) = 1 \otimes 1$ ) such that  $\varepsilon \cdot \eta = \text{id}_I$ .
- When  $\mathcal{C}$  is gr-flat, a *gr-coaugmentation*  $\eta$  is a map of filtered complete  $\mathbb{S}$ -modules, such that applying the graded functor  $\text{Gr}$  to  $(\mathcal{C}, \Delta, \varepsilon, d, \eta)$  provides a flat coaugmented cooperad.

Let  $(\mathcal{C}, \Delta, \varepsilon, d, \eta)$  be a gr-coaugmented gr-flat cooperad.

- The *infinitesimal coideal* of  $\eta$ , notated  $\tilde{\mathcal{C}}$ , is the complete filtered  $\mathbb{S}$ -module which is the pushout (in complete filtered objects) of the diagram  $I[-1] \leftarrow I \xrightarrow{\eta} \mathcal{C}$ , where  $I \rightarrow I[-1]$  is the identity in filtration degree zero. We denote by  $\tilde{\eta}$  the map  $I[-1] \rightarrow \tilde{\mathcal{C}}$  in the pushout.

These definitions extend directly to infinitesimal cooperads. Let  $(\mathcal{C}, \Delta, \varepsilon, d)$  be a complete infinitesimal cooperad.

- A *coaugmentation* for this infinitesimal cooperad is a map  $\eta : I[-1] \rightarrow \mathcal{C}$  of infinitesimal cooperads (on  $I[-1]$ ,  $\Delta(1) = 1 \otimes 1$  defines a filtered map) such that  $\varepsilon \cdot \eta = \text{id}_{I[-1]}$ .
- When  $\mathcal{C}$  is gr-flat, a *gr-coaugmentation*  $\eta$  is a map of filtered complete  $\mathbb{S}$ -modules, such that applying the graded functor  $\text{Gr}$  to  $(\mathcal{C}, \Delta, \varepsilon, d, \eta)$  provides a flat coaugmented infinitesimal cooperad.

*Remark 3.16.* In keeping with our conventions, a gr-coaugmentation only has to be compatible with the decomposition map, the differential and the counit on the graded level. However, a gr-coaugmentation still satisfies  $\varepsilon \cdot \eta = \text{id}_I$  (resp. in the infinitesimal setting,  $\varepsilon \cdot \eta = \text{id}_{I[-1]}$ ) at the filtered level. This follows from the fact that since  $I$  has nothing in positive filtration degree, the counit  $\varepsilon$  factors through its 0th graded component. Respectively, in the infinitesimal setting, this follows from the fact that since  $I[-1]$  is characterized by its filtration degree 1 component, the counit  $\varepsilon$  is characterized by its 1st graded component.

*Remark 3.17.* The infinitesimal coideal  $\tilde{\mathcal{C}}$  has the same underlying object as  $\mathcal{C}$ , but the image of  $\eta$  is forced to live in filtration degree 1. There is a map of complete objects  $\mathcal{C} \rightarrow \tilde{\mathcal{C}}$ . Then given a map of complete collections  $\tilde{\mathcal{C}} \xrightarrow{\varphi} M$ , we can use the recipe of Construction 3.12 to extend  $\mathcal{C} \rightarrow \tilde{\mathcal{C}} \rightarrow M$  to a map  $\mathcal{C} \rightarrow \prod Q_n M$ . We use the same notation in this case (i.e., we do not record whether the map  $\varphi$  started with domain  $\mathcal{C}$  or  $\tilde{\mathcal{C}}$ ).

**Lemma 3.18.** *Let  $\mathcal{C}$  be a gr-coaugmented complete cooperad. Then the cooperad structure on  $\mathcal{C}$  provides the infinitesimal coideal  $\tilde{\mathcal{C}}$  with an infinitesimal cooperad structure. We denote the decomposition map by  $\tilde{\Delta} : \tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{C}} \circ \tilde{\mathcal{C}}$ , the counit map by  $\tilde{\varepsilon} : \tilde{\mathcal{C}} \rightarrow I[-1]$ , and the gr-coaugmentation by  $\tilde{\eta} : I[-1] \rightarrow \tilde{\mathcal{C}}$ . We also denote by  $\tilde{d} : \tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{C}}$  the differential induced by the one on  $\mathcal{C}$ .*

*For simplicity, we still denote this infinitesimal cooperad by  $\tilde{\mathcal{C}}$ .*

*Proof.* The map  $\tilde{\varepsilon} : \tilde{\mathcal{C}} \rightarrow I[-1]$  is obtained by the universal property of the pushout  $\tilde{\mathcal{C}}$  and the two maps  $\text{id}_{I[-1]}$  and  $\mathcal{C} \xrightarrow{\varepsilon} I \rightarrow I[-1]$ . The differential  $\tilde{d}$  is defined similarly. Since  $\tilde{\mathcal{C}}$  has the same underlying object as  $\mathcal{C}$  the maps  $\tilde{\Delta}$ ,  $\tilde{\varepsilon}$  and  $\tilde{\eta}$  are predetermined and the only question is whether they respect the filtration of  $\tilde{\mathcal{C}}$ . This is evident by construction for  $\tilde{\varepsilon}$  and  $\tilde{\eta}$ . It remains to treat the case of  $\tilde{\Delta}$ . Because  $\eta$  is a gr-coaugmentation, we know that

$$\Delta\eta \equiv (\eta \circ \eta)\Delta_I \pmod{F_1(\mathcal{C} \circ \mathcal{C})}$$

which immediately implies that

$$\tilde{\Delta}\tilde{\eta} \equiv 0 \pmod{F_1(\tilde{\mathcal{C}} \circ \tilde{\mathcal{C}})}. \quad \square$$

The tree cooperad is coaugmented by the inclusion of  $I$  into  $\mathcal{T}^c M$  and the tree infinitesimal cooperad  $\tilde{\mathcal{T}}^c M$  is coaugmented by the inclusion of  $I[-1]$  into  $\tilde{\mathcal{T}}^c M$ .

**Lemma 3.19.** *The categories of gr-coaugmented complete cooperads and of gr-coaugmented complete infinitesimal cooperads are equivalent through the map  $\mathcal{C} \mapsto \tilde{\mathcal{C}}$ .*

*Proof.* It is enough to describe an inverse functor to  $\mathcal{C} \mapsto \tilde{\mathcal{C}}$ . By Remark 3.16, for a gr-coaugmented complete infinitesimal cooperad  $\mathcal{C}$ , we can write  $\mathcal{C} \cong \eta(I[-1]) \amalg \ker \varepsilon$ . The inverse functor is given by  $\mathcal{C} \mapsto \eta(I[-1])[1] \amalg \ker \varepsilon$ .  $\square$

For convenience, we describe the notion of altipotent cooperad for cooperads but this notion make also sense for infinitesimal cooperads by the preceding lemma and we use it in this last setting.

Now we make some definitions related to conilpotence in the complete context. We cannot follow either [LV12] or [LGL18] without modification.

**Definition 3.20.** Let  $(\mathcal{C}, \Delta, \varepsilon, d)$  be a complete (gr-flat) cooperad endowed with a gr-coaugmentation  $\eta$ . We define the *reduced decomposition map*  $\bar{\Delta} : \mathcal{C} \rightarrow \mathcal{C} \circ \mathcal{C}$  by

$$\bar{\Delta} := \Delta - ((\eta \cdot \varepsilon) \circ \text{id}) \cdot \Delta - (\text{id} \circ (\eta \cdot \varepsilon)) \cdot \Delta + ((\eta \cdot \varepsilon) \circ (\eta \cdot \varepsilon)) \cdot \Delta.$$

**Definition 3.21** (Coradical filtration). We define a bigraded collection  $\mathfrak{c}^{p,n}\mathcal{C}$  of subcollections of  $\mathcal{C}$  recursively as follows. We define  $\mathfrak{c}^{p,0}\mathcal{C}$  as  $F_p\mathcal{C}$  and define  $\mathfrak{c}^{0,n}\mathcal{C}$  as  $F_0\mathcal{C}$ . Next, supposing that  $\mathfrak{c}^{p',n'}\mathcal{C}$  is defined for  $n' < n$  and  $p' \leq p$ , we first define a subobject  $\tilde{\mathfrak{c}}^{p,n}(\mathcal{C} \circ \mathcal{C})$  of  $\mathcal{C} \circ \mathcal{C}$ . We define it as the sub-object in  $\mathcal{C} \circ \mathcal{C}$

$$\coprod_{m \geq 0} \bigcup_{p_0 + \dots + p_m + \delta = p} F_{p_0}\mathcal{C}(m) \otimes_{\mathbb{S}_m} \bigotimes_{i=1}^m F_{p_i} \mathfrak{c}^{p_i + \delta, n-1}\mathcal{C}.$$

Then define  $\mathfrak{c}^{p,n}\mathcal{C}$  as the following pullback:

$$\begin{array}{ccc} \mathfrak{c}^{p,n}\mathcal{C} & \xrightarrow{\quad \quad \quad} & \mathcal{C} \\ \downarrow & & \downarrow \bar{\Delta} \\ \tilde{\mathfrak{c}}^{p,n}(\mathcal{C} \circ \mathcal{C}) & \xrightarrow{\quad \quad \quad} & \mathcal{C} \circ \mathcal{C}. \end{array}$$

Recursively, there are natural inclusions  $\mathfrak{c}^{p,n}\mathcal{C} \rightarrow \mathfrak{c}^{p-1,n}\mathcal{C}$  and  $\mathfrak{c}^{p,n}\mathcal{C} \rightarrow \mathfrak{c}^{p,n+1}\mathcal{C}$ .

**Definition 3.22.** Let  $\mathcal{C}$  be a complete (gr-flat) cooperad which is equipped with a gr-coaugmentation  $\eta$ . We define the *k-primitives* of  $\mathcal{C}$  to be the  $\mathbb{S}$ -module:

$$\text{Prim}_k\mathcal{C} := \ker \varepsilon \cap \bigcap_{p \geq 0} \mathfrak{c}^{p+k,p}\mathcal{C},$$

and the *primitives* to be the  $\mathbb{S}$ -module:

$$\text{Prim}\mathcal{C} := \bigcup_{k \geq 0} \text{Prim}_k\mathcal{C}.$$

We also fix the following notations

$$\text{Prim}_k^+\mathcal{C} := \bigcap_{p \geq 0} \mathfrak{c}^{p+k,p}\mathcal{C}.$$

**Definition 3.23.** Let  $(\mathcal{C}, \eta)$  be a complete gr-coaugmented (gr-flat) cooperad. If

- (1) the map

$$\text{colim}_n \mathfrak{c}^{p,n}\mathcal{C} \rightarrow \mathcal{C}$$

is an isomorphism for all  $p$ ,

- (2) the coaugmentation  $\eta(I)$  lies in  $\text{Prim}_0^+\mathcal{C}$ , and

- (3) for all  $k \geq 0$  we have

$$\bar{\Delta}(\text{Prim}_k^+\mathcal{C}) \subset \coprod_m \bigcup_{k_0 + \dots + k_m = k+1} F_{k_0}\mathcal{C}(m) \otimes (\text{Prim}_{k_1}^+\mathcal{C} \otimes \dots \otimes \text{Prim}_{k_m}^+\mathcal{C}), \quad (*)$$

then we call  $\mathcal{C}$  *altipotent*.

*Remark 3.24.* The Latin prefix ‘‘alti-’’ denotes height. This condition is a priori stronger than gr-conilpotence and weaker than conilpotence. In a gr-conilpotent cooperad, repeatedly applying the reduced decomposition map would eventually increase the filtration degree by 1. In a conilpotent cooperad, repeatedly applying the reduced decomposition map would eventually reach zero. In an altipotent cooperad, repeatedly applying the reduced decomposition map eventually increases the filtration degree past any fixed number.

**3.25. Existence of extensions in the altipotent setting.** In this section we show that the tree cooperad is the cofree altipotent cooperad.

**Lemma 3.26.** *Let  $\mathcal{C}$  be a complete altipotent cooperad and  $M$  a complete collection. Let  $\varphi : \mathcal{C} \rightarrow M$  be a map of collections which extends to a map  $\tilde{\mathcal{C}} \rightarrow M$  that we still denote by  $\varphi$ . Then for  $p \geq 0$ , we have*

$$\Phi_j(\eta(I)) \subset F_p Q_j M, \quad \text{for all } j \geq 2p, \quad (1)$$

$$\Phi_j(\text{Prim}_k \mathcal{C}) \subset F_{p+k} Q_j M, \quad \text{for all } j \geq 2p+1 \text{ and all } k \geq 0, \quad (2)$$

where the maps  $\Phi_j$  are defined in Section 3.6.

*Proof.* We will prove the statement by induction on  $p$ . The base case  $p = 0$  is true because  $F_0 Q_j M = Q_j M$  for Statement (1) and  $\varphi$  and  $\Phi_j$  are filtered for Statement (2).

Let  $j \geq 2p \geq 2$ . We need to control the terms in

$$\left( \varphi \circ \sum_{i=1}^j \Phi_{i-1} \right) \cdot \Delta(\eta(I))$$

where at least one  $\Phi_{j-1}$  appears. First, we recall that

$$\Delta(\eta) = \bar{\Delta}(\eta) + (\eta \otimes \eta) \Delta_I$$

with  $\bar{\Delta}(\eta(I))$  lying in

$$\coprod_m \coprod_{k_0 + \dots + k_m = 1} F_{k_0} \mathcal{C}(m) \otimes (\text{Prim}_{k_1}^+ \mathcal{C} \otimes \dots \otimes \text{Prim}_{k_m}^+ \mathcal{C})$$

since  $\mathcal{C}$  is altipotent. (In particular,  $\bar{\Delta}(\eta(I)) \subset F_1(\mathcal{C} \circ \mathcal{C})$ .) By means of the fact that  $\varphi$  extends to a map  $\tilde{\mathcal{C}} \rightarrow M$ , we have  $\varphi(\eta(I)) \subset F_1 \mathcal{C}$ . Therefore, the induction hypothesis gives  $(\varphi \otimes \Phi_{j-1})(\eta \otimes \eta) \Delta_I(I) \subset F_{1+(p-1)} Q_j M = F_p Q_j M$  (using  $2p-1 \geq 2(p-1)$ ). Then we consider the term

$$\left( \varphi \circ \sum_{i=1}^j \Phi_{i-1} \right) \cdot \bar{\Delta}(\eta(I))$$

where at least one  $\Phi_{j-1}$  appears. We need to control the piece

$$\varphi(F_{k_0} \mathcal{C}(m)) \otimes (\Phi_{j_1}(\text{Prim}_{k_1}^+ \mathcal{C}) \otimes \dots \otimes \Phi_{j_m}(\text{Prim}_{k_m}^+ \mathcal{C}))$$

where at least one  $j_a$  is  $j-1$ . Using the fact that  $\text{Prim}_k^+ \mathcal{C} = \eta(I) \amalg \text{Prim}_k \mathcal{C}$ , we have that it is enough to consider the two following situations:

- $\Phi_{j-1}$  is applied to  $\eta(I)$  (for simplicity we assume that  $j_m = j-1$  but the reasoning works for all  $j_l$ ) and  $F_{k_0} \mathcal{C}(m) \otimes (\text{Prim}_{k_1}^+ \mathcal{C} \otimes \dots \otimes \text{Prim}_{k_{m-1}}^+ \mathcal{C}) \subset F_1(\mathcal{C}(m) \otimes \mathcal{C}^{\otimes(m-1)})$  since  $\bar{\Delta}(\eta(I)) \subset F_1(\mathcal{C} \circ \mathcal{C})$ ,
- $\Phi_{j-1}$  is applied to  $\text{Prim}_{k_l} \mathcal{C}$  for  $k_l = 0$  or  $k_l = 1$  (for simplicity we assume that  $l = m$  but the reasoning works for all  $l$ ) and

$$F_{k_0} \mathcal{C}(m) \otimes (\text{Prim}_{k_1}^+ \mathcal{C} \otimes \dots \otimes \text{Prim}_{k_{m-1}}^+ \mathcal{C}) \subset F_{1-k}(\mathcal{C}(m) \otimes \mathcal{C}^{\otimes(m-1)}).$$

By the induction hypothesis, the application of  $\Phi_{j-1}$  to  $\eta(I)$  (we have  $j-1 \geq 2p-1 \geq 2(p-1)$ ) lies in  $F_{p-1} Q_{j-1} M$ . The piece to control is therefore in  $F_p Q_j M$ . Similarly, the application  $\Phi_{j-1}$  to  $\text{Prim}_k \mathcal{C}$  (we have  $j-1 \geq 2p-1 = 2(p-1)+1$ )

lies in  $F_{(p-1)+k}Q_{j-1}M$ . The piece to control is therefore in  $F_{(p-1+k)+(1-k)}Q_jM = F_pQ_jM$ .

Let  $j \geq 2p+1$ . We emphasize the fact that we will use the induction hypothesis for  $\eta(I)$  and  $p$  that we just proved. We have on  $(\text{Prim}_k\mathcal{C})(m)$

$$\Delta = \bar{\Delta} + ((\eta \cdot \varepsilon) \otimes \text{id}) \cdot \Delta + (\text{id} \otimes (\eta \cdot \varepsilon)^{\otimes m}) \cdot \Delta.$$

By means of the fact that  $\varphi$  extends to a map  $\tilde{\mathcal{C}} \rightarrow M$ , we have  $\varphi(\eta(I)) \subset F_1\mathcal{C}$ . Therefore, the induction hypothesis gives  $\varphi(\eta(I)) \otimes \Phi_{j-1}((\text{Prim}_k\mathcal{C})(m)) \subset F_{1+(p-1)+k}Q_jM = F_{p+k}Q_jM$ . By the induction hypothesis for  $\eta(I)$  and  $j-1 \geq 2p$ , we get that the term  $(\text{id} \otimes (\eta \cdot \varepsilon)^{\otimes m}) \cdot \Delta$  lies in  $F_{k+p}Q_jM$  (we can assume that  $m \geq 1$  otherwise this term doesn't appear in the computation of  $\Phi_j((\text{Prim}_k\mathcal{C})(m))$ ). Since

$$\bar{\Delta}((\text{Prim}_k\mathcal{C})(m)) \subset \coprod_m \coprod_{k_1+\dots+k_m=k+1} F_{k_0}\mathcal{C}(m) \otimes (\text{Prim}_{k_1}^+\mathcal{C} \otimes \dots \otimes \text{Prim}_{k_m}^+\mathcal{C}),$$

we have to consider the two following situations:

- $\Phi_{j-1}$  is applied to  $\eta(I)$  (for simplicity we assume that  $j_m = j-1$  but the reasoning works for all  $j_l$ ) and  $F_{k_0}\mathcal{C}(m) \otimes (\text{Prim}_{k_1}^+\mathcal{C} \otimes \dots \otimes \text{Prim}_{k_{m-1}}^+\mathcal{C}) \subset F_{k+1}(\mathcal{C}(m) \otimes \mathcal{C}^{\otimes(m-1)})$ ,
- $\Phi_{j-1}$  is applied to  $\text{Prim}_{k_l}\mathcal{C}$  (for simplicity we assume that  $l = m$  but the reasoning works for all  $l$ ) and  $F_{k_0}\mathcal{C}(m) \otimes (\text{Prim}_{k_1}^+\mathcal{C} \otimes \dots \otimes \text{Prim}_{k_{m-1}}^+\mathcal{C}) \subset F_{k+1-k_m}(\mathcal{C}(m) \otimes \mathcal{C}^{\otimes(m-1)})$  for  $k_m \leq k+1$ .

By induction hypothesis, the application of  $\Phi_{j-1}$  to  $\eta(I)$  (using  $j-1 \geq 2p$ ) lies in  $F_pQ_{j-1}M$ . The resulting term in  $Q_jM$  is therefore in  $F_{p+k+1}Q_jM$ . Similarly, the application  $\Phi_{j-1}$  to  $\text{Prim}_{k_l}\mathcal{C}$  lies in  $F_{p-1+k_l}Q_{j-1}M$ . The resulting term in  $Q_jM$  is therefore in

$$F_{(p-1+k_l)+(k+1-k_l)}Q_jM = F_{p+k}\mathcal{C}. \quad \square$$

**Lemma 3.27.** *Let  $\mathcal{C}$  be a complete altipotent cooperad and let  $M$  be a complete collection. Let  $\varphi : \tilde{\mathcal{C}} \rightarrow M$  be a map of collections. Then for  $p, n \geq 0$ , we have*

$$\Phi_j(\mathfrak{cr}^{p,n}\mathcal{C}) \subset F_pQ_jM, \quad \text{for all } j \geq n+2p. \quad (3)$$

*Proof.* We will prove the statement by induction on  $n$  and  $p$ . The base cases with  $n=0$  (any  $p$ ) and  $p=0$  (any  $n$ ) are true because  $\mathfrak{cr}^{p,0}\mathcal{C}$  is just  $F_p\mathcal{C}$  and  $\varphi$  is filtered and because  $F_0Q_jM = Q_jM$ .

Let  $(p, n)$  be a pair of integers greater than or equal to 1. Suppose that the statement (3) is true for all pairs  $(p', n')$  with  $n' < n$  and for all pairs  $(p', n)$  with  $p' < p$ . Let us prove the statement for  $(p, n)$ .

Let  $j$  be an integer. The map  $\Phi_j$  is defined recursively by means of the formula

$$\sum_{i=0}^j \Phi_i = \varepsilon + \left( \varphi \circ \sum_{i=1}^j \Phi_{i-1} \right) \cdot \Delta.$$

Recall that  $\mathcal{T}_j^c M = I \amalg (M \circ \mathcal{T}_{j-1}^c M)$  and  $M \circ \mathcal{T}_{j-1}^c M$  has summands which are quotients of

$$M(m) \otimes (\mathcal{T}_{j-1}^c M)^{\otimes m}.$$

It follows that  $\Phi_j$  is given by the terms in the sum  $(\varphi \circ \sum^{j-1} \Phi_*) \cdot \Delta$  which contain at least one application of  $\Phi_{j-1}$  on some factor  $\mathcal{T}_{j-1}^c M$ .



Assume  $j \geq n + 2p$ . On  $\mathbf{cr}^{p,n}\mathcal{C}(m)$ , we have

$$\Delta = \bar{\Delta} + ((\eta \cdot \varepsilon) \otimes \text{id}) \cdot \Delta + (\text{id} \otimes (\eta \cdot \varepsilon)^{\otimes m}) \cdot \Delta - ((\eta \cdot \varepsilon) \otimes (\eta \cdot \varepsilon)) \cdot \Delta.$$

The term  $\varphi(\eta(I)) \otimes \Phi_{j-1}(\mathbf{cr}^{p,n}\mathcal{C}(m))$  lies in  $F_p Q_j M$  by the induction hypothesis for the pair  $(p-1, n)$  since  $\mathbf{cr}^{p,n}\mathcal{C} \subset \mathbf{cr}^{p-1,n}\mathcal{C}$  and by the fact that  $\varphi(\eta(I)) \subset F_1\mathcal{C}$ . Because  $j-1 \geq n+2p-1 \geq 2p$ , we have by Lemma 3.26 that  $\Phi_{j-1}(\eta(I))$  and  $\Phi_{j-1}(\eta \cdot \varepsilon(\mathbf{cr}^{p,n}\mathcal{C}(m)))$  lie in  $F_p Q_{j-1} M$ . It follows that the terms  $(\text{id} \otimes (\eta \cdot \varepsilon)^{\otimes m}) \cdot \Delta(\mathbf{cr}^{p,n}\mathcal{C}(m))$  and  $-((\eta \cdot \varepsilon) \otimes (\eta \cdot \varepsilon)) \cdot \Delta(\mathbf{cr}^{p,n}\mathcal{C}(m))$  lie in  $F_p Q_j M$ . Using the fact that  $\bar{\Delta}(\mathbf{cr}^{p,n}\mathcal{C}(m)) \subset \tilde{\mathbf{cr}}^{p,n}(\mathcal{C} \circ \mathcal{C})$  and the induction hypothesis for  $\Phi_{j-1}$  and couples  $(p', n-1)$  and the fact that the  $\Phi_i$ s are filtered, we get that this last term also lies in  $F_p Q_j M$ . This finishes the proof.  $\square$

As a direct consequence of Lemma 3.27, we get the following corollary:

**Corollary 3.28.** *Let  $\mathcal{C}$  be a complete altipotent cooperad and let  $M$  be a complete collection. Let  $\varphi : \tilde{\mathcal{C}} \rightarrow M$  be a map of collections. Then the components of the composition of maps of collections*

$$\mathbf{cr}^{p,n}\mathcal{C} \rightarrow \mathcal{C} \xrightarrow{\prod \Phi_j} \prod Q_j M$$

land in filtration degree at least  $p$  in  $Q_j M$  for  $j \geq n + 2p$ .

Finally, we obtain the following lemma.

**Lemma 3.29.** *Let  $\mathcal{C}$  be a complete altipotent cooperad and  $M$  a complete collection. Let  $\varphi : \tilde{\mathcal{C}} \rightarrow M$  be a map of collections. Then the map of collections*

$$\mathcal{C} \xrightarrow{\prod \Phi_n} \prod Q_n M$$

factors as a map of collections through the tree cooperad  $\mathcal{T}^c M$ .

*Proof.* Corollary 3.28 tells us that we have a (unique) dotted factorization as in the following diagram

$$\begin{array}{ccc} & \prod_{j=0}^{n+2p-1} F_0(Q_j M) \times \prod_{j=n+2p}^{\infty} F_p(Q_j M) & \\ & \nearrow \text{dotted} & \downarrow \\ \mathbf{cr}^{p,n}\mathcal{C} & \longrightarrow & \prod_{j=0}^{\infty} F_0(Q_j M). \end{array}$$

Taking componentwise quotients by filtration degree  $p$  takes the vertical map to the left hand vertical map of the following commutative square:

$$\begin{array}{ccc} \prod_{j=0}^{n+2p-1} F_0(Q_j M)/F_p(Q_j M) & \xrightarrow{\cong} & \prod_{j=0}^{n+2p-1} F_0(Q_j M)/F_p(Q_j M) \\ \downarrow & & \downarrow \\ \prod_{j=0}^{\infty} F_0(Q_j M)/F_p(Q_j M) & \longleftarrow & \prod_{j=0}^{\infty} F_0(Q_j M)/F_p(Q_j M). \end{array}$$

Consider the bottom right entry here. The coproduct passes through the quotient to give

$$\begin{aligned} \prod_{j=0}^{\infty} F_0(Q_j M)/F_p(Q_j M) &\cong \left( \prod_{j=0}^{\infty} F_0(Q_j M) \right) / \left( \prod_{j=0}^{\infty} F_p(Q_j M) \right) \\ &\cong \mathcal{T}^c M / F_p \mathcal{T}^c M. \end{aligned}$$

This gives a lift of the map from

$$\mathfrak{c}\mathfrak{r}^{p,n}\mathcal{C} \rightarrow \prod Q_j M / F_p Q_j M$$

to a map

$$\mathfrak{c}\mathfrak{r}^{p,n}\mathcal{C} \rightarrow \mathcal{T}^c M / F_p \mathcal{T}^c M$$

which maps  $F_p \mathcal{C} \subset \mathfrak{c}\mathfrak{r}^{p,n}\mathcal{C}$  to 0.

Now (using condition (1) of Definition 3.23 for the isomorphism in the following line), we get a map from

$$F_0 \mathcal{C} / F_p \mathcal{C} \cong (\operatorname{colim}_n \mathfrak{c}\mathfrak{r}^{p,n}\mathcal{C}) / F_p \mathcal{C} \cong \operatorname{colim}_n (\mathfrak{c}\mathfrak{r}^{p,n}\mathcal{C} / F_p \mathcal{C})$$

to  $\mathcal{T}^c M / F_p \mathcal{T}^c M$ .

By examining projections, we see that these maps are compatible for different choices of  $p$  so that we can take the limit over  $p$ . This then yields a map from the limit, i.e., the complete cooperad  $\mathcal{C}$ , to  $\mathcal{T}^c M$  which by construction has projections  $\Phi_n$ .  $\square$

**3.30. Properties of the tree cooperad.** Now we show that the tree cooperad is cofree in the category of altipotent cooperads in the gr-flat setting. To do this, we prove a sequence of lemmas telling us more and more about the associated graded functor and this cooperad. First

**Lemma 3.31.** *Let  $D$  be a diagram. Suppose that the colimit functor from  $D$ -indexed diagrams in the ground category to the ground category preserves monomorphisms. Then the associated graded functor  $\operatorname{Gr}$  from (complete) filtered objects to graded objects preserves the colimits of  $D$ -indexed diagrams.*

*Proof.* The graded functor commutes with completion, so it suffices to consider the filtered case. Also, the graded functor from  $\mathbb{N}$ -diagrams to graded objects is a left adjoint so commutes with all colimits. So preservation of  $D$ -indexed colimits by the associated graded can be checked in terms of preservation of those colimits by the inclusion of filtered objects into diagrams. Colimits in filtered objects are obtained by applying the reflector to the same colimits in  $\mathbb{N}$ -diagrams. Then  $D$ -indexed colimits are preserved as soon as the reflector is an isomorphism. In  $\mathbb{N}$ -diagrams colimits are computed objectwise, so the reflector being an isomorphism is implied by  $D$ -indexed colimits in the ground category preserving monomorphisms.  $\square$

**Corollary 3.32.** *Then the associated graded functor  $\operatorname{Gr}$  from (complete) filtered objects to graded objects preserves filtered colimits, and in particular transfinite compositions and coproducts. If the ground category be  $\mathbb{Q}$ -linear, then the associated graded functor also preserves the coinvariants of a finite group action.*

*Proof.* The first statement follows directly from Lemma 3.31 and the fact that the ground category satisfies AB5. For the second statement, if the ground category is  $\mathbb{Q}$ -linear then the comparison map between coinvariants and invariants of a finite group action is a (natural) isomorphism. The  $\operatorname{Gr}$  functor is then a composition of left and right adjoints so preserves such (co)invariants.  $\square$

**Lemma 3.33.** *The tree cooperad on a gr-flat  $\mathbb{S}$ -module is gr-flat. The inclusion of  $I$  as  $\eta$  is a gr-coaugmentation.*

*Proof.* The general shape of the argument is to argue that  $\text{Gr}$  preserves all of the categorical building blocks of the tree cooperad, and then that these building blocks preserve flat objects.

The tree cooperad is built as a directed colimit of finite stages, each of which is built from the monoidal unit  $I$  and previous stages by the  $\circ$  product and a binary coproduct. So we will be done by induction as soon as we argue that each of these procedures preserves gr-flat objects.

The monoidal unit is gr-flat. For coproducts, directed colimits, by Corollary 3.32 we know that  $\text{Gr}$  preserves such colimits. Flat objects (and thus gr-flat objects) are closed under coproducts and directed colimits (the latter by AB5).

The  $\circ$  product is built as a coproduct of summands each of which either a monoidal product or a  $\mathbb{S}_n$ -quotient of a sum of monoidal products. We know by Lemma B.4 that  $\text{Gr}$  preserves the monoidal product and the coproducts, and by Corollary 3.32 that it preserves the  $\mathbb{S}_n$ -quotients under the  $\mathbb{Q}$ -linear assumption. Flat objects are closed under the monoidal product by definition, and are (again) closed under arbitrary coproducts. Moreover, the finite group coinvariants of an object  $X$  are a retract of  $X$ . Thus if the  $X$  is flat then its coinvariants are. Then the  $\circ$  product preserves gr-flatness as well and we are done.

The given  $\eta$  is already a coaugmentation before taking associated graded, which implies by functoriality that it remains one after doing so.  $\square$

**Lemma 3.34.** *The tree cooperad on a gr-flat  $\mathbb{S}$ -module, equipped with the inclusion of  $I$  as  $\eta$ , is altipotent.*

*Proof.* Next, we claim by induction that  $\mathcal{T}_n^c M$  is in  $\mathfrak{c}\mathfrak{r}^{p,n+1}\mathcal{T}^c M$  for all  $p$ . For  $n = 0$ , this is true because  $\bar{\Delta}\eta = 0$  in the tree cooperad. Then  $\bar{\Delta}\mathcal{T}_n^c M$  is represented by summands

$$(\mathcal{T}_{n_0}^c M)(m) \otimes_{\mathbb{S}_m} \bigotimes_{i=1}^m \mathcal{T}_{n_i}^c M$$

with each  $n_i$  strictly less than  $n$ , so by induction  $\bar{\Delta}\mathcal{T}_n^c M$  is in  $\tilde{\mathfrak{c}}\mathfrak{r}^{p,n}(\mathcal{T}^c M \circ \mathcal{T}^c M)$  for all  $p$  (take  $p_i = 0$  and  $\delta = p$  in the definition of  $\tilde{\mathfrak{c}}\mathfrak{r}^{p,n}$ ). This completes the induction.

Then for fixed  $p$  the inclusions  $\mathcal{T}_n^c M \rightarrow \mathfrak{c}\mathfrak{r}^{p,n+1}\mathcal{T}^c M$  pass to the colimit over  $n$  so that

$$\mathcal{T}^c M \cong \text{colim}_n \mathcal{T}_n^c M \rightarrow \text{colim}_n \mathfrak{c}\mathfrak{r}^{p,n+1}\mathcal{T}^c M \rightarrow \mathcal{T}^c M$$

is the identity. Since our ground category satisfies AB5, the colimit of monomorphisms is monic so this suffices to show that the final map here is an isomorphism. This is condition (1) of Definition 3.23.

Condition (2) is easy because  $\bar{\Delta}\eta = 0$ .

Condition (3) follows because the primitives  $\text{Prim}\mathcal{T}^c M$  are precisely  $M$ , so  $\text{Prim}^+\mathcal{T}^c M \cong I \amalg M$ , and  $\bar{\Delta}$  vanishes on this subobject.  $\square$

**Corollary 3.35.** *The tree cooperad, viewed as a functor from gr-flat complete  $\mathbb{S}$ -modules to altipotent cooperads, is right adjoint to the infinitesimal coideal. That is, the tree cooperad is cofree in the category of altipotent complete cooperads.*

*Proof.* By Lemma 3.34, the tree cooperad (equipped with  $\eta$ ) lies in the full subcategory of altipotent cooperads. By Corollary 3.11, an  $\mathbb{S}$ -module map from  $\mathcal{C}$  to the tree cooperad is uniquely determined by its projections to  $Q_n M$ . By Lemma 3.13, such a map is a cooperad map from  $\mathcal{C}$  to the tree cooperad if and only if it is

further determined by the projection  $\mathcal{C} \rightarrow \mathcal{T}^c M \rightarrow Q_1 M \cong M$  via the recipe of Construction 3.12.

Being compatible with the gr-coaugmentation means that the triangle

$$\begin{array}{ccc} I & \xrightarrow{\quad} & \mathcal{T}^c M \\ & \searrow & \nearrow \\ & \mathcal{C} & \end{array}$$

commutes after taking the associated graded. This then implies that  $I \rightarrow \mathcal{C} \rightarrow M$  should factor through strictly positive filtration degree. This means that  $\varphi : \mathcal{C} \rightarrow M$  must factor through the infinitesimal coideal  $\tilde{\mathcal{C}} \rightarrow M$ .

The procedure just outlined gives an injective map from the set of cooperad maps  $\mathcal{C} \rightarrow \mathcal{T}^c M$  to the set of complete  $\mathbb{S}$ -module maps  $\tilde{\mathcal{C}} \rightarrow M$ . In the other direction, given a map of collections  $\varphi : \mathcal{C} \rightarrow \tilde{\mathcal{C}} \rightarrow M$ , Lemma 3.29 tells us that map of collections  $\Phi : \mathcal{C} \rightarrow \prod Q_n M$  obtained via Construction 3.12 factors through  $\mathcal{T}^c M$ . Then the other direction of Lemma 3.13 ensures that this extension is in fact a map of cooperads. This establishes surjectivity.  $\square$

*Remark 3.36.* It is possible to see an element in the cofree altipotent complete cooperad on  $M$  as a (possibly infinite) sum of trees, whose vertices of arity  $k$  are indexed by element in  $M(k)$ .

**3.37. Coderivations.** To have a complete treatment, we briefly review the standard fact that coderivations valued in a free cooperad are determined by their projections to the cogenerators.

Let  $f$  and  $g$  be maps of  $\mathbb{S}$ -modules from  $P$  to  $Q$  and let  $h$  be a map of  $\mathbb{S}$ -modules from  $R$  to  $S$ . We define a map  $h \circ' (f; g)$  from  $R \circ P$  to  $S \circ Q$ . The components of  $h \circ' (f; g)$  are induced by

$$h(n) \otimes \bigotimes_{i=1}^{n-1} f(k_i) \otimes g(k_n) : R(n) \otimes \bigotimes_{i=1}^n P(k_i) \rightarrow S(n) \otimes \bigotimes_{i=1}^n Q(k_i).$$

This is closely related to the *infinitesimal composite* of  $f$  and  $g$  as defined in [LV12, Section 6.1.3]. Neither is quite a generalization of the other as defined, but the family resemblance should be clear.

Now suppose  $f : \mathcal{C} \rightarrow \mathcal{D}$  is a map of complete cooperads then a *coderivation of  $f$*  is a morphism  $d$  of complete  $\mathbb{S}$ -modules  $\mathcal{C} \rightarrow \mathcal{D}$  such that

$$\Delta_{\mathcal{D}} \cdot d = (d \circ f) \cdot \Delta + (f \circ' d) \cdot \Delta.$$

A coderivation of  $\text{id}_{\mathcal{C}}$  is also called just a coderivation of  $\mathcal{C}$ .

**Proposition 3.38.** *Let  $M$  be a complete  $\mathbb{S}$ -module and  $\mathcal{C}$  a gr-coaugmented altipotent cooperad equipped with a map  $f : \mathcal{C} \rightarrow \mathcal{T}^c M$ . Projection to cogenerators  $\mathcal{T}^c M \rightarrow M$  induces a bijection between coderivations of  $f$  and  $\mathbb{S}$ -module maps from  $\mathcal{C}$  to  $M$ .*

*Proof.* Equip the  $\mathbb{S}$ -module  $\mathcal{C}1 \amalg \mathcal{C}x$  with a decomposition map induced by that of  $\mathcal{C}$ :

$$\mathcal{C}1 \amalg \mathcal{C}x \rightarrow (\mathcal{C}1 \amalg \mathcal{C}x) \circ (\mathcal{C}1 \amalg \mathcal{C}x)$$

where,

- writing  $i$  for the canonical inclusions of  $\mathcal{C}1$  and  $\mathcal{C}x$  into  $\mathcal{C}1 \amalg \mathcal{C}x$  and
- writing  $i^\dagger$  for the inclusion of  $\mathcal{C}x$  into  $\mathcal{C}1 \amalg \mathcal{C}x$  in the  $\mathcal{C}1$  factor,

we set

$$\mathcal{C}1 \xrightarrow{\Delta} (\mathcal{C} \circ \mathcal{C})1 \cong (\mathcal{C}1 \circ \mathcal{C}1) \xrightarrow{i \circ i^\dagger} (\mathcal{C}1 \amalg \mathcal{C}x) \circ (\mathcal{C}1 \amalg \mathcal{C}x)$$

and

$$\mathcal{C}x \xrightarrow{\Delta} (\mathcal{C} \circ \mathcal{C})x \cong (\mathcal{C}x \circ \mathcal{C}x) \xrightarrow{(i \circ i^\dagger) + (i^\dagger \circ (i^\dagger; i))} (\mathcal{C}1 \amalg \mathcal{C}x) \circ (\mathcal{C}1 \amalg \mathcal{C}x).$$

Coassociativity of  $\mathcal{C}$  implies that  $\mathcal{C}1 \amalg \mathcal{C}x$  is a gr-coaugmented altipotent cooperad with this structure decomposition and structure data

$$\text{counit: } \mathcal{C}1 \amalg \mathcal{C}x \xrightarrow{\text{projection}} \mathcal{C}1 \cong \mathcal{C} \xrightarrow{\epsilon} I;$$

$$\text{gr-coaugmentation: } I \xrightarrow{\eta} \mathcal{C} \cong \mathcal{C}1 \xrightarrow{i} \mathcal{C}1 \amalg \mathcal{C}x.$$

It is also clear that this construction is functorial. Then

$$\begin{aligned} \text{Hom}_{\text{Coop}}(\mathcal{C}1 \amalg \mathcal{C}x, \mathcal{T}^c M) &\cong \text{Hom}_{\mathbb{S}}(\overline{\mathcal{C}1 \amalg \mathcal{C}x}, M) \\ &\cong \text{Hom}_{\mathbb{S}}(\overline{\mathcal{C}1}, M) \times \text{Hom}_{\mathbb{S}}(\mathcal{C}x, M) \\ &\cong \text{Hom}_{\text{Coop}}(\mathcal{C}, \mathcal{T}^c M) \times \text{Hom}_{\mathbb{S}}(\mathcal{C}, M). \end{aligned}$$

The final entry has a natural projection to  $\text{Hom}_{\text{Coop}}(\mathcal{C}, \mathcal{T}^c M)$ . The fiber over the morphism  $f$  is naturally isomorphic to  $\text{Hom}_{\mathbb{S}}(\mathcal{C}, M)$ . On the other hand, the fiber over the morphism  $f$  in  $\text{Hom}_{\text{Coop}}(\mathcal{C}1 \amalg \mathcal{C}x, \mathcal{T}^c M)$  consists of those cooperad maps of the form

$$\mathcal{C}1 \amalg \mathcal{C}x \cong \mathcal{C} \amalg \mathcal{C} \xrightarrow{f \amalg d} \mathcal{T}^c M$$

for some  $d$ .

The condition to be a cooperad map is automatically satisfied on the  $\mathcal{C}1$  factor since  $f$  is a map of cooperads. On the  $\mathcal{C}x$  factor, for  $f \amalg d$  to be a map of cooperads we get an equation. One side of the equation is

$$\Delta_{\mathcal{T}^c M} \cdot d.$$

The other side is a sum of two terms coming from the two terms defining the decomposition map on  $\mathcal{C}x$ . The first term is

$$\begin{array}{ccccc} \mathcal{C}x & \xrightarrow{\Delta} & (\mathcal{C} \circ \mathcal{C})x & \xrightarrow{\cong} & \mathcal{C}x \circ \mathcal{C}x & \xrightarrow{i \circ i^\dagger} & (\mathcal{C}1 \amalg \mathcal{C}x) \circ (\mathcal{C}1 \amalg \mathcal{C}x) \\ \downarrow \cong & & \searrow \cong & & \downarrow \cong & & \downarrow (f \amalg d) \circ (f \amalg d) \\ \mathcal{C} & \xrightarrow{\Delta} & \mathcal{C} \circ \mathcal{C} & \xrightarrow{d \circ f} & \mathcal{T}^c M \circ \mathcal{T}^c M & & \end{array}$$

and the other is

$$\begin{array}{ccccc} \mathcal{C}x & \xrightarrow{\Delta} & (\mathcal{C} \circ \mathcal{C})x & \xrightarrow{\cong} & \mathcal{C}x \circ \mathcal{C}x & \xrightarrow{i^\dagger \circ (i^\dagger; i)} & (\mathcal{C}1 \amalg \mathcal{C}x) \circ (\mathcal{C}1 \amalg \mathcal{C}x) \\ \downarrow \cong & & \searrow \cong & & \downarrow \cong & & \downarrow f \circ (f \amalg d) \\ \mathcal{C} & \xrightarrow{\Delta} & \mathcal{C} \circ \mathcal{C} & \xrightarrow{f \circ (i^\dagger; f)} & \mathcal{T}^c M \circ \mathcal{T}^c M & & \end{array}$$

The equation of the first of these three terms with the sum of the latter two is precisely the condition for  $d$  to be a coderivation of  $f$ .  $\square$

## 4. BAR AND COBAR CONSTRUCTIONS IN THE CURVED SETTING

In this section, we define a bar-cobar adjunction between the setting of complete augmented curved operads and complete altipotent infinitesimal cooperads. It doesn't strictly extend the classical constructions between augmented operads and conilpotent cooperads but the constructions are very close. More precisely, we define a functor *bar*  $\hat{B}$  which associate a complete altipotent infinitesimal cooperad to an augmented curved complete operad and we define a functor *cobar*  $\hat{\Omega}$  in the opposite direction. We obtain an adjunction between the two functors

$$\hat{B} : \text{Compl. aug. curved operads} \rightleftarrows \text{Compl. altip. inf. cooperads} : \hat{\Omega}.$$

It will result from this adjunction an  $\mathbb{S}$ -cofibrant resolution of a curved operad. This  $\mathbb{S}$ -cofibrant resolution seems uselessly big by means of the presence of a formal parameter  $\vartheta$  which is not present in the classical cobar construction. It is in fact possible to provide a definition of a "smaller" construction which does not involve the parameter  $\vartheta$ . However, we cannot use this smaller cobar functor in order to get an  $\mathbb{S}$ -cofibrant resolution in the model category structure given in Appendix C.

We consider curved operads in the category  $\mathcal{M}$  defined as the full subcategory of gr-flat object in  $\widehat{\text{Filt}}^{\text{gr}}(\mathcal{A})$  and cooperads in the category  $\mathcal{M}'$  defined as the full subcategory of gr-flat objects in  $\widehat{\text{Filt}}(\text{dg}\mathcal{A})$ . By Lemma B.4, this ensures that the functor Gr is strong monoidal.

**4.1. Bar construction.** Let  $(\mathcal{O}, \gamma, d, \varepsilon, \theta, \eta)$  be an augmented curved complete operad. The augmentation ideal  $\bar{\mathcal{O}} := \ker(\varepsilon : \mathcal{O} \rightarrow I)$  of  $\mathcal{O}$  is a complete gr-dg  $\mathbb{S}$ -module.

*Remark 4.2.* The predifferential  $d$  on  $\mathcal{O}$  induces a predifferential  $\bar{d}$  on the complete  $\mathbb{S}$ -module  $\bar{\mathcal{O}}$  and the composition product  $\gamma$  induces a map  $\bar{\gamma} : \bar{\mathcal{O}} \circ \bar{\mathcal{O}} \rightarrow \bar{\mathcal{O}}$  which gives by means of  $\eta$  a map  $\bar{\gamma}_{(1)} : \bar{\mathcal{O}} \circ_{(1)} \bar{\mathcal{O}} \rightarrow \bar{\mathcal{O}}$ .

The *bar construction* of the augmented curved complete operad  $(\mathcal{O}, \gamma, d, \varepsilon, \theta, \eta)$  is given by the altipotent complete infinitesimal cooperad

$$\hat{B}\mathcal{O} := \left( \widetilde{\mathcal{T}}^c(s\bar{\mathcal{O}}), \Delta_\beta, \varepsilon_\beta, d_\beta := d_0 + d_1 + d_2, \eta_\beta \right),$$

where  $\widetilde{\mathcal{T}}^c(s\bar{\mathcal{O}})$  is the infinitesimal cooperad associated with  $\mathcal{T}^c(s\bar{\mathcal{O}})$  (see Lemma 3.18), the map  $d_2$  is the unique coderivation of degree  $-1$  which extends the map

$$\widetilde{\mathcal{T}}^c(s\bar{\mathcal{O}}) \rightarrow s^2 (\bar{\mathcal{O}} \circ_{(1)} \bar{\mathcal{O}}) \xrightarrow{\gamma_s \otimes \bar{\gamma}_{(1)}} s\bar{\mathcal{O}},$$

where  $\gamma_s(s \otimes s) = s$ , the map  $d_1$  is the unique coderivation of degree  $-1$  which extends the map

$$\widetilde{\mathcal{T}}^c(s\bar{\mathcal{O}}) \rightarrow s\bar{\mathcal{O}} \xrightarrow{\text{id}_s \otimes \bar{d}} s\bar{\mathcal{O}},$$

and the map  $d_0$  is the unique coderivation of degree  $-1$  which extends the map

$$\widetilde{\mathcal{T}}^c(s\bar{\mathcal{O}}) \rightarrow I \xrightarrow{-s\theta} s\bar{\mathcal{O}}.$$

For instance, we have pictorially

$$d_0(\Upsilon) = - \begin{array}{c} \diagup \quad \diagdown \\ \text{---} \text{---} \\ | \\ s\theta \end{array} + \begin{array}{c} | \\ \text{---} \text{---} \\ \diagdown \quad \diagup \\ s\theta \end{array} + \begin{array}{c} \diagdown \quad \diagup \\ \text{---} \text{---} \\ | \\ s\theta \end{array} \in s^2 (\bar{\mathcal{O}} \circ_{(1)} \bar{\mathcal{O}}),$$

for  $\Upsilon \in \mathcal{O}(2)$  of degree  $-1$  and  $\theta$  is identified with  $\theta(\bar{1}) \in \mathcal{O}(1)$ . The counit  $\varepsilon_\beta$  is the usual projection onto the trivial tree and the gr-coaugmentation  $\eta_\beta$  is the inclusion of the trivial tree.

*Remark 4.3.* Since  $d_\beta(\bar{1}) = d_0(\bar{1}) = -\frac{1}{s\theta}$ , it follows that the bar construction is not coaugmented as a filtered cooperad whenever the curvature is non zero. However, when  $\mathcal{O}$  is gr-flat, it is gr-coaugmented as a complete cooperad since  $\theta$  is in  $F_1\mathcal{O}$  (see Definition 3.15).

**Lemma 4.4.** *The coderivation  $d_\beta$  is a differential and the bar construction induces a functor  $\hat{B} : \text{Comp. aug. curved op.} \rightarrow \text{Compl. altip. inf. coop.}$ .*

*Proof.* We can split the square of the coderivation  $d_\beta$  as follows

$$(d_0 + d_1 + d_2)^2 = \underbrace{d_0^2} + \underbrace{d_0d_1 + d_1d_0} + \underbrace{d_1^2 + d_0d_2 + d_2d_0} + \underbrace{d_1d_2 + d_2d_1} + \underbrace{d_2^2}.$$

We have  $d_0^2 = 0$  because of sign considerations. For the same reason and due to the fact that  $\theta$  is closed, we get  $d_0d_1 + d_1d_0 = 0$ . The bracket of two coderivations is a coderivation so  $[d_0, d_2] = d_0d_2 + d_2d_0$  and  $d_1^2$  are coderivations. Therefore, the corestriction of the equality  $d_1^2 + d_0d_2 + d_2d_0 = 0$  to  $s\bar{\mathcal{O}}$  is enough to prove the equality. The corestriction of  $d_1^2 + d_0d_2 + d_2d_0$  to  $s\bar{\mathcal{O}}$  is equal to  $\bar{d}^2 + [-\bar{\theta}, -]$ , which is zero since  $d^2 - [\theta, -]$  is and  $\mathcal{O}$  is augmented. The map  $d$  is a derivation with respect to  $\gamma$  for the augmented curved complete operad  $\mathcal{O}$ , so by sign considerations, we obtain that  $d_1d_2 + d_2d_1 = 0$ . The associativity of the composition product  $\gamma$  of the augmented curved complete operad  $\mathcal{O}$  gives the last equality  $d_2^2 = 0$ .  $\square$

**4.5. Cobar construction.** We define here a cobar construction  $\hat{\Omega}$  which associates a complete curved operad to a complete altipotent infinitesimal cooperad.

We recall that the functor  $\mathcal{T}_+$  applied on a complete gr-dg  $\mathbb{S}$ -module  $M$  is given by pointed complete gr-dg operad  $(\mathcal{T}(M \amalg \vartheta I), \vartheta)$ , where the generator  $\vartheta$  lives in arity one, weight one and degree  $-2$  (the correct arity, weight and degree for a curvature). When  $M$  is a filtered complete graded  $\mathbb{S}$ -module, we denote by  $s^{-1}M$  the desuspension of  $M$ , that is the filtered complete graded  $\mathbb{S}$ -module such that  $(s^{-1}M)_n := M_{n+1}$  and  $F_p(s^{-1}M) := s^{-1}F_pM$ . Let  $(\mathcal{C}, \Delta, d, \varepsilon, \eta)$  be a complete altipotent infinitesimal cooperad. The *cobar construction*  $\hat{\Omega}\mathcal{C}$  of  $\mathcal{C}$  is defined as the (quasi-free) complete augmented curved operad

$$\hat{\Omega}\mathcal{C} := (\mathcal{T}_+(s^{-1}\mathcal{C}), \gamma_\omega, d_\omega := d_1 + d_2 + d_\vartheta, \vartheta),$$

where:

- (1) the map  $d_1$  is the unique derivation of degree  $-1$  which extends the map

$$s^{-1}\mathcal{C} \amalg \vartheta I \rightarrow s^{-1}\mathcal{C} \xrightarrow{\text{id}_{s^{-1}} \otimes d} s^{-1}\mathcal{C} \rightarrow \mathcal{T}(s^{-1}\mathcal{C}) \rightarrow \mathcal{T}(s^{-1}\mathcal{C} \amalg \vartheta I),$$

- (2) the map  $d_2$  is the unique derivation of degree  $-1$  which extends the map

$$s^{-1}\mathcal{C} \amalg \vartheta I \rightarrow s^{-1}\mathcal{C} \xrightarrow{\Delta_{s^{-1}} \otimes \Delta_{(1)}} s^{-2}(\mathcal{C} \circ_{(1)} \mathcal{C}) \rightarrow \mathcal{T}(s^{-1}\mathcal{C}) \rightarrow \mathcal{T}(s^{-1}\mathcal{C} \amalg \vartheta I),$$

where  $\Delta_{s^{-1}}(s^{-1}) = -s^{-1} \otimes s^{-1}$ , and

- (3) the map  $d_\vartheta$  is the unique derivation of degree  $-1$  which extends the map

$$s^{-1}\mathcal{C} \amalg \vartheta I \rightarrow s^{-1}\mathcal{C} \xrightarrow{\text{id}_{s^{-1}} \otimes \varepsilon} s^{-1}I \rightarrow \vartheta I \rightarrow \mathcal{T}(s^{-1}\mathcal{C} \amalg \vartheta I).$$

(The composition is filtered but the second map isn't.)

The cobar construction is a quasi-free complete curved operad because of the following lemma which implies that

$$\hat{\Omega}\mathcal{C} = (\mathcal{T}_+(s^{-1}\mathcal{C}), d_\omega) / (\text{im}(d_\omega^2 - [\vartheta, -])).$$

**Lemma 4.6.** *The square of the derivation  $d_\omega$  is equal to  $[\vartheta, -]$ , the curvature  $\vartheta$  is closed and the cobar construction induces a functor  $\hat{\Omega} : \text{Compl. altip. inf. coop.} \rightarrow \text{Comp. aug. curved op.}$ .*

*Proof.* The free complete operad  $\mathcal{T}(s^{-1}\mathcal{C} \amalg \vartheta I)$  is augmented and the map  $d_\omega$  is zero on  $I$ . (In fact, since the category of augmented curved operads is equivalent to the category of non-unital curved operad, we could have consider the non-unital curved operad  $\overline{\mathcal{T}}(s^{-1}\mathcal{C} \amalg \vartheta I)$ .)

Because of the fact that  $d_\omega^2 = \frac{1}{2}[d_\omega, d_\omega]$  and  $[\vartheta, -]$  are derivations, it is enough to prove the equality  $d_\omega^2 = [\vartheta, -]$  on the generators  $s^{-1}\mathcal{C} \amalg \vartheta I$ . We have  $d_\omega^2 = (d_1 + d_2 + d_\vartheta)^2 = d_1^2 + (d_1d_2 + d_2d_1) + d_2^2 + (d_1d_\vartheta + d_\vartheta d_1) + d_\vartheta^2 + (d_2d_\vartheta + d_\vartheta d_2)$ . The term  $d_1^2$  (resp.  $d_2^2$ ) is zero since  $d$  is a differential (resp.  $\Delta$  is coassociative). The term  $d_\vartheta^2$  is zero since  $\varepsilon$  is zero on  $\vartheta$ . The sum  $d_1d_2 + d_2d_1$  is zero because  $d$  is a coderivation with respect to  $\Delta$  and the term  $d_1d_\vartheta + d_\vartheta d_1$  is zero because  $d(\vartheta) = 0$  and because of the Koszul sign rule. It remains to compute the term  $d_2d_\vartheta + d_\vartheta d_2$ . We have  $d_2d_\vartheta = 0$  since  $d_2$  is zero on  $\vartheta$ . Finally,  $d_\vartheta d_2 = [\vartheta, -]$  because  $\Delta$  is counital.

The curvature  $\vartheta$  is closed by definition of  $d_\omega$ . This finishes the proof.  $\square$

**4.7. The convolution curved Lie algebra.** As usual, the bar and cobar constructions represent a bifunctor. It is given by curved twisting morphisms in a curved Lie algebra.

Let  $\mathcal{O}$  be a curved complete operad and  $\mathcal{C}$  be a complete altipotent infinitesimal cooperad. We use the notation  $\text{Hom}_{\mathbb{S}}(\mathcal{C}, \mathcal{O})$  for

$$\prod_n \text{Hom}_{\mathbb{S}}(\mathcal{C}(n), \mathcal{O}(n)),$$

where  $\text{Hom}_{\mathbb{S}}$  stands for the morphisms of complete filtered  $\mathbb{S}$ -modules, and we fix the element  $\Theta := (\theta_{\mathcal{O}} \cdot \varepsilon_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{O}) \in \text{Hom}_{\mathbb{S}}(\mathcal{C}, \mathcal{O})_{-2}$ . Since  $\theta_{\mathcal{O}}$  is in filtration level 1 of  $\mathcal{O}$ , this map lies in  $F_1\text{Hom}_{\mathbb{S}}(\mathcal{C}, \mathcal{O})_{-2}$ . We define on  $\text{Hom}_{\mathbb{S}}(\mathcal{C}, \mathcal{O})$  the pre-Lie product  $\star$  given by

$$f \star g : \mathcal{C} \xrightarrow{\Delta(1)} \mathcal{C} \circ_{(1)} \mathcal{C} \xrightarrow{f \circ g} \mathcal{O} \circ_{(1)} \mathcal{O} \xrightarrow{\gamma(1)} \mathcal{O}$$

using the decomposition map of Lemma 3.18.

The product  $\star$  induces a Lie bracket  $\{f, g\} := f \star g - (-1)^{|f||g|} g \star f$  on  $\text{Hom}_{\mathbb{S}}(\mathcal{C}, \mathcal{O})$ . We denote by  $\partial$  the derivation of  $\star$  given by  $\partial(f) := d_{\mathcal{O}} \cdot f - (-1)^{|f|} f \cdot d_{\mathcal{C}}$ .

**Lemma 4.8.** *The tuple  $(\text{Hom}_{\mathbb{S}}(\mathcal{C}, \mathcal{O}), \{-, -\}, \partial, \Theta)$  forms a curved Lie algebra, called the convolution curved Lie algebra.*

*Proof.* We do the computations for the curvature:

$$\begin{aligned} \partial^2(f) &= d_{\mathcal{O}} \cdot \partial(f) - (-1)^{|\partial(f)|} \partial(f) \cdot d_{\mathcal{C}} \\ &= d_{\mathcal{O}}^2 \cdot f - (-1)^{|f|} d_{\mathcal{O}} \cdot f \cdot d_{\mathcal{C}} + (-1)^{|f|} (d_{\mathcal{O}} \cdot f \cdot d_{\mathcal{C}} - (-1)^{|f|} f \cdot d_{\mathcal{C}}^2) \\ &= d_{\mathcal{O}}^2 \cdot f = [\theta, -] \cdot f = \Theta \star f - f \star \Theta = \{\Theta, f\} \end{aligned}$$

and  $\partial(\Theta) = d_{\mathcal{O}} \cdot \theta \cdot \varepsilon - (-1)^{|\Theta|} \theta \cdot \varepsilon \cdot d_{\mathcal{C}} = 0$  since  $d_{\mathcal{O}} \cdot \theta = 0$  and  $\varepsilon \cdot d_{\mathcal{C}} = 0$ .  $\square$



Fix an augmented curved complete operad  $(\mathcal{O}, \gamma, d_{\mathcal{O}}, \varepsilon_{\mathcal{O}}, \theta, \eta_{\mathcal{O}})$  and a complete altipotent infinitesimal cooperad  $(\mathcal{C}, \Delta, d_{\mathcal{C}}, \varepsilon_{\mathcal{C}}, \eta_{\mathcal{C}})$ . An element  $\alpha : \mathcal{C} \rightarrow \mathcal{O}$  of degree  $-1$  in the curved Lie algebra  $\text{Hom}_{\mathbb{S}}(\mathcal{C}, \mathcal{O})$  such that  $\varepsilon_{\mathcal{O}} \cdot \alpha = 0$  which is called a *curved twisting morphism* if it is a solution of the *curved Maurer-Cartan equation*

$$\Theta + \partial(\alpha) + \frac{1}{2}\{\alpha, \alpha\} = 0.$$

We denote by  $\text{Tw}(\mathcal{C}, \mathcal{O})$  the set of curved twisting morphisms. By means of the fact that  $\mathcal{C}$  is an infinitesimal cooperad, a curved Maurer-Cartan satisfies  $\text{im}(\alpha \cdot \eta_{\mathcal{C}}) \subset F_1\mathcal{O}$ . We show that it is representable on the left and on the right by the bar and the cobar constructions.

**Proposition 4.9.** *For any complete altipotent infinitesimal cooperad  $\mathcal{C}$  and any complete augmented curved operad  $\mathcal{O}$ , there are natural bijections*

$$\text{Hom}_{\text{comp. aug. curv. op.}}(\hat{\Omega}\mathcal{C}, \mathcal{O}) \cong \text{Tw}(\mathcal{C}, \mathcal{O}) \cong \text{Hom}_{\text{compl. alti. inf. coop.}}(\mathcal{C}, \hat{B}\mathcal{O}).$$

Therefore the functors  $\hat{B}$  and  $\hat{\Omega}$  form a pair of adjoint functors between the categories of complete augmented curved operads and the category of complete altipotent infinitesimal cooperads.

*Proof.* We make the first bijection explicit. A morphism of augmented complete operads  $f_{\alpha} : \mathcal{T}(s^{-1}\mathcal{C} \amalg \vartheta I) \rightarrow \mathcal{O}$  sending  $\vartheta$  to  $\theta_{\mathcal{O}}$  is uniquely determined by a map  $-s\alpha : s^{-1}\mathcal{C} \rightarrow \mathcal{O}$  of degree 0 such that, since  $f_{\alpha}$  is augmented,  $\varepsilon_{\mathcal{O}} \cdot (s\alpha) = 0$ . This is equivalent to the data of a map  $\alpha : \mathcal{C} \rightarrow \mathcal{O}$  of degree  $-1$  such that  $\varepsilon_{\mathcal{O}} \cdot \alpha = 0$ . Moreover,  $f_{\alpha}$  commutes with the predifferential if and only if the following diagram commutes

$$\begin{array}{ccccc} s^{-1}\mathcal{C} & \xrightarrow{-s\alpha} & \mathcal{O} & \xrightarrow{d_{\mathcal{O}}} & \mathcal{O} \\ \downarrow d_v + d_1 + d_2 & & & & \uparrow \text{id} + \gamma_{(1)} \\ \vartheta I \amalg s^{-1}\mathcal{C} \amalg (s^{-1}\mathcal{C} \circ_{(1)} s^{-1}\mathcal{C}) & \xrightarrow{\theta_{\mathcal{O}} \iota_v + (-s\alpha) + (-s\alpha) \circ_{(1)} (-s\alpha)} & \mathcal{O} \amalg \mathcal{O} \circ_{(1)} \mathcal{O} & & \end{array}$$

where  $\iota_{\vartheta} : \vartheta I \rightarrow I$  is the projection to  $I$ . We obtain therefore  $\Theta + \partial(\alpha) + \alpha \star \alpha = 0$ .

We now make the second bijection explicit. A morphism of complete altipotent infinitesimal cooperads  $g_{\alpha} : \mathcal{C} \rightarrow \widetilde{\mathcal{T}}^c(s\overline{\mathcal{O}})$  is uniquely determined by a map  $-s\alpha : \mathcal{C} \rightarrow s\overline{\mathcal{O}}$  (see Corollary 3.35 with Lemma 3.19), that is by a map  $\alpha : \mathcal{C} \rightarrow \mathcal{O}$  of degree  $-1$  satisfying  $\varepsilon_{\mathcal{O}} \cdot \alpha = 0$ . Moreover,  $g_{\alpha}$  commutes with the differential if and only if the following diagram commutes

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\varepsilon_{\mathcal{C}} - s\alpha + ((-s\alpha) \circ_{(1)} (-s\alpha)) \cdot \Delta_{(1)}} & I \amalg s\overline{\mathcal{O}} \amalg s\overline{\mathcal{O}} \circ_{(1)} s\overline{\mathcal{O}} \\ \downarrow d_{\mathcal{C}} & & \downarrow d_{\omega}^{s\overline{\mathcal{O}}} = (d_0 + d_1 + d_2)|_{s\overline{\mathcal{O}}} \\ \mathcal{C} & \xrightarrow{s\alpha} & s\overline{\mathcal{O}} \end{array}$$

Similarly as before, this is equivalent to the equality  $\Theta + \partial(\alpha) + \alpha \star \alpha = 0$ . This concludes the proof.  $\square$

**4.10. Bar-cobar resolution.** In this section, we assume that the category  $\mathcal{A}$  is the category of  $\mathbf{R}$ -modules since we use results of [LV12] which are proved in this setting.

A curved complete operad  $\mathcal{O}$  is *weight graded* if it is endowed with a extra grading  $\mathcal{O}^{(w)}$  compatible with the composition product. It is connected for this weight grading if we have

$$\mathcal{O} = I \amalg \mathcal{O}^{(1)} \amalg \mathcal{O}^{(2)} \amalg \dots \amalg \mathcal{O}^{(w)} \amalg \dots,$$

that is  $\mathcal{O}^{(0)} = I$ .

**Proposition 4.11.** *Let  $\mathcal{O}$  complete curved operad such that  $\mathrm{Gr}\mathcal{O}$  is a connected weight graded operad. The counit of the adjunction of Proposition 4.9*

$$\hat{\Omega}\hat{B}\mathcal{O} \rightarrow \mathcal{O}$$

*is a graded quasi-isomorphism which is an  $\mathbb{S}$ -cofibrant resolution in the model category structure on complete curved operads given in Appendix C, that is to say cofibrant in the model category structure on complete gr-dg  $\mathbb{S}$ -modules.*

*Proof.* Since  $\mathrm{Gr}$  is strong monoidal, we have  $\mathrm{Gr}\hat{\Omega}\hat{B}\mathcal{O} \cong \Omega\mathrm{BGr}\mathcal{O}$ , where the construction  $\Omega$  and  $B$  are defined similarly as the constructions  $\hat{\Omega}$  and  $\hat{B}$  and applies to the graded setting. Let us describe  $\Omega\mathrm{BGr}\mathcal{O}$ . We compute the differential on  $\Omega\mathrm{BGr}\mathcal{O} \cong \mathcal{T}(\vartheta I \amalg s^{-1}\tilde{I} \amalg s^{-1}\overline{\mathcal{T}^c(s\overline{\mathrm{Gr}\mathcal{O}})})$ . The predifferential on  $\hat{\Omega}\hat{B}\mathcal{O}$  is

$$d_\omega = d_1 + d_2 + d_\vartheta = d'_0 + d'_1 + d'_2 + d_2 + d_\vartheta.$$

When we applies the functor  $\mathrm{Gr}$ , the term  $d'_0$  cancels, the term  $d'_1$  is induced by the differential on  $\mathrm{Gr}\mathcal{O}$ , the term  $d'_2$  is induced by the composition product on  $\mathrm{Gr}\mathcal{O}$ , the term  $d_2$  is induced by the cofree decomposition map on  $\mathcal{T}^c(s\overline{\mathrm{Gr}\mathcal{O}})$  since the part  $d_2(s^{-1}\tilde{I})$  cancels, and finally the term  $d_\vartheta$  is unchanged and induced by the map  $s^{-1}\tilde{I} \rightarrow \vartheta I$ ,  $s^{-1}1 \mapsto \vartheta$ .

The ideal  $(s^{-1}\tilde{I} \amalg \vartheta I)$  in  $\Omega\mathrm{BGr}\mathcal{O}$  is equal to the coproduct  $\coprod_{k \geq 0} (s^{-1}\tilde{I} \amalg \vartheta I)_{(k)}$ , where  $(s^{-1}\tilde{I} \amalg \vartheta I)_{(k)}$  is the subobject given by trees on  $\vartheta I \amalg s^{-1}\tilde{I} \amalg s^{-1}\overline{\mathcal{T}^c(s\overline{\mathrm{Gr}\mathcal{O}})}$  with  $k$  apparitions of  $\vartheta I \amalg s^{-1}\tilde{I}$ . We have

$$\Omega\mathrm{BGr}\mathcal{O} \cong \Omega_{cl}B_{cl}\mathrm{Gr}\mathcal{O} \coprod \coprod (s^{-1}\tilde{I} \amalg \vartheta I),$$

where  $\Omega_{cl}$  and  $B_{cl}$  are the classical cobar and bar constructions for the augmented graded (uncurved) operad  $\mathrm{Gr}\mathcal{O}$  (in [LV12] for example). The differential on  $\Omega\mathrm{BGr}\mathcal{O}$  splits on the two subobjects: it is equal to the one of  $\Omega_{cl}B_{cl}\mathrm{Gr}\mathcal{O}$  on  $\Omega_{cl}B_{cl}\mathrm{Gr}\mathcal{O}$  and is induced by the map which sends  $s^{-1}\tilde{I}$  to  $\vartheta I$  on  $(s^{-1}\tilde{I} \amalg \vartheta I)$ . The subobject  $(s^{-1}\tilde{I} \amalg \vartheta I)_{(k)}$  is a coproduct of several copies of the tensor product of  $k$  copies of the chain complex

$$\dots \rightarrow 0 \rightarrow I[-1] \xrightarrow{\mathrm{id}} I[-1] \rightarrow 0 \rightarrow \dots$$

and is acyclic, so is  $(s^{-1}\tilde{I} \amalg \vartheta I)$ .

Then, by Theorem 6.6.3 in [LV12] gives that  $\Omega_{cl}B_{cl}\mathrm{Gr}\mathcal{O}$  is quasi-isomorphic to  $\mathrm{Gr}\mathcal{O}$ . We therefore obtain that the morphism  $\hat{\Omega}\hat{B}\mathcal{O} \rightarrow \mathcal{O}$  is a graded quasi-isomorphism.

The counit of the adjunction is given by the composition

$$\hat{\Omega}\hat{B}\mathcal{O} \rightarrow \mathcal{T}_+(\overline{\mathcal{O}}) \rightarrow \mathcal{O}$$

where the two maps are strict surjections, so is the composition. It remains to show that  $\hat{\Omega}\hat{B}\mathcal{O}$  is  $\mathbb{S}$ -cofibrant. The curvature of  $\hat{\Omega}\hat{B}\mathcal{O}$  is  $\vartheta$ . Let  $t \in \hat{\Omega}\hat{B}\mathcal{O}$ . We have  $d_\omega^2(t) = [\vartheta, t]$  where the bracket is computed by means of the free product. It follows that the identity  $[\vartheta, t] = 0$  implies  $t \in \mathcal{T}_+(0) = \mathcal{T}(\vartheta I)$  (induction on the roots of the terms in  $t$ ). This shows that for any  $t \in s^{-1}\hat{B}\mathcal{O}$ , we have  $d_\omega^k(t) \neq 0$ . Let us write the dg  $\mathbb{S}$ -module  $s^{-1}\hat{B}\mathcal{O}$  as a direct sum of  $\mathbb{S}_m$ -modules

$$\cdots \rightarrow 0 \rightarrow M \xrightarrow{\text{id}} M \rightarrow 0 \rightarrow \cdots \text{ and } \cdots \rightarrow 0 \rightarrow M \rightarrow 0 \rightarrow \cdots$$

of rank 1. Using the fact that the differential on  $s^{-1}\hat{B}\mathcal{O}$  commutes with the predifferential  $d_\omega$ , this gives the beginning of the description of  $\hat{\Omega}\hat{B}\mathcal{O}$  as a direct sum of  $\hat{Z}_{q,n}^{0,\infty} \otimes M$  and  $\hat{Z}_{q,n}^{1,\infty} \otimes M$ . We can continue this process with a complement in the quadratic part in  $\hat{\Omega}\hat{B}\mathcal{O}$  of the terms that we have just obtained, and so on and so forth. This provides a description of  $\hat{\Omega}\hat{B}\mathcal{O}$  as a direct sum of  $\hat{Z}_{q,n}^{0,\infty} \otimes M$  and  $\hat{Z}_{q,n}^{1,\infty} \otimes M$ . It can be seen as a retract of a direct sum of  $\hat{Z}_{q,n}^{0,\infty} \otimes \mathbf{R}[\mathbb{S}_m]$  and  $\hat{Z}_{q,n}^{1,\infty} \otimes \mathbf{R}[\mathbb{S}_m]$  (see the proof of Proposition C.29).  $\square$

## 5. KOSZUL DUALITY FOR CURVED OPERADS

In this section, we describe the Koszul dual cooperad of a homogeneous quadratic curved operad  $\mathcal{O}$  and we relate it to the bar construction  $\hat{B}\mathcal{O}$ . We propose a definition for a curved operad to be Koszul and then we show that in this case, the Koszul dual cooperad is quasi-isomorphic to the bar construction  $\hat{B}\mathcal{O}$ . It follows from the proof a sort of Poincaré–Birkhoff–Witt theorem which provides a way to make the Koszul dual cooperad explicit.

Under the Koszul property, we finally obtain the  $\mathbb{S}$ -cofibrant resolution of curved complete operads

$$\hat{\Omega}\mathcal{O}^i \rightarrow \mathcal{O},$$

which has the advantage to have a domain with a smaller space of generators in comparison to the bar-cobar resolution.

The two generic examples are the following:

- a curved associative algebra whose curvature is a sum of squares (with conditions so that it is Koszul);
- the operads encoding curved associative algebras or curved Lie algebras.

In this section, we work with the category  $\mathcal{A}$  of  $\mathbf{R}$ -modules where  $\mathbf{R}$  is a field and we consider the closed monoidal category  $\mathcal{M} = \widehat{\text{Filt}}^{\text{gr}}(\mathcal{A})$  in the operadic side and the closed monoidal category  $\mathcal{M}' = \widehat{\text{Filt}}(\text{dg}\mathcal{A})$  in the cooperadic side. Section 5.1 applies to more general category  $\mathcal{A}$  (except for Proposition 5.5 and after Section 5.16).

**5.1. Subcomonoid generated by a symmetric module.** We denote by  $\mathcal{N}$  the category  $\mathbb{S}\text{-Mod}(\mathcal{M})$  and by  $\mathcal{N}'$  the category  $\mathbb{S}\text{-Mod}(\mathcal{M}')$ . Let  $\mathcal{C}$  be a comonoid in  $\mathcal{N}'$  and  $S$  be an object in  $\mathcal{N}'$ . In Appendix B of [Val08], Vallette defined the notion of the subcomonoid of  $\mathcal{C}$  generated by  $S$  in an abelian setting. We extend the definition to quasi-abelian categories. The only difference is the necessity to consider strict morphisms in order to identify certain coimage with cokernel (and

in the dual case certain image with kernel). We only emphasize the differences in the unfamiliar case of the subcomonoid generated by an object. The case of an ideal and of a monoid quotient generated by an object is dual.

**Definition 5.2.** • Let  $\mathcal{I} \twoheadrightarrow \mathcal{C} \twoheadrightarrow \mathcal{Q}$  be an exact sequence in  $\mathcal{N}'$ , where  $\mathcal{C}$  is a comonoid. The epimorphism  $\mathcal{C} \twoheadrightarrow \mathcal{Q}$  in  $\mathcal{N}'$  is a *coideal epimorphism* if  $\mathcal{I} \twoheadrightarrow \mathcal{C}$  is a monomorphism of comonoids in  $\mathcal{N}'$ .

- Let  $\xi : \mathcal{C} \twoheadrightarrow S$  be an epimorphism in  $\mathcal{N}'$ , where  $\mathcal{C}$  is a comonoid. We consider the category  $\mathcal{S}_\xi$  of sequences  $(\mathbf{S}) : \mathcal{I} \twoheadrightarrow \mathcal{C} \twoheadrightarrow \mathcal{Q}$  as in the previous item and such that the composite  $\mathcal{I} \twoheadrightarrow \mathcal{C} \twoheadrightarrow S$  is equal to 0. A morphism between  $(\mathbf{S})$  and  $(\mathbf{S}') : \mathcal{I}' \twoheadrightarrow \mathcal{C} \twoheadrightarrow \mathcal{Q}'$  is given by a pair  $(i : \mathcal{I} \rightarrow \mathcal{I}', p : \mathcal{Q} \rightarrow \mathcal{Q}')$  such that  $i$  is a morphism of comonoids and  $p$  is a morphism in  $\mathcal{N}'$ , and such that the following diagram commutes:

$$\begin{array}{ccccc}
 \mathcal{I} & & & & \\
 i \downarrow & \searrow & & & \\
 \mathcal{I}' & \longrightarrow & \mathcal{C} & \longrightarrow & \mathcal{Q} \\
 & & & \searrow & \downarrow p \\
 & & & & \mathcal{Q}'
 \end{array}$$

- We aim to consider the largest subcomonoid of  $\mathcal{C}$  vanishing on  $S$ . This notion is given by the terminal object  $(\mathbf{S}) : \mathcal{C}(S) \twoheadrightarrow \mathcal{C} \twoheadrightarrow (S)$  in  $\mathcal{S}_\xi$ , when the latter admits one.

We now make the coideal quotient  $(S)$  and the subcomonoid  $\mathcal{C}(S)$  explicit.

**Definition 5.3.** We denote by  $i_{\mathcal{C}} : \mathcal{C} \hookrightarrow S \amalg \mathcal{C}$  is the inclusion and by  $\pi_{\mathcal{C}} : S \amalg \mathcal{C} \twoheadrightarrow \mathcal{C}$  is the projection.

- The *multilinear part* in  $S$  of  $(\mathcal{C} \circ (S \amalg \mathcal{C})) \circ \mathcal{C}$  is given either by the cokernel

$$\text{coker} \left( \mathcal{C} \circ \mathcal{C} \circ \mathcal{C} \xrightarrow{\text{id} \circ i_{\mathcal{C}} \circ \text{id}} (\mathcal{C} \circ (S \amalg \mathcal{C})) \circ \mathcal{C} \right),$$

or equivalently, by the kernel

$$\ker \left( (\mathcal{C} \circ (S \amalg \mathcal{C})) \circ \mathcal{C} \xrightarrow{\text{id} \circ \pi_{\mathcal{C}} \circ \text{id}} \mathcal{C} \circ \mathcal{C} \circ \mathcal{C} \right),$$

by means of the fact that the monoidal structure commutes with colimits and  $i_{\mathcal{C}}$  is a section of  $\pi_{\mathcal{C}}$ . It is denoted by  $\mathcal{C} \circ (\underline{S} \amalg \mathcal{C}) \circ \mathcal{C}$ .

- The *coideal quotient*  $(S)$  of  $\mathcal{C}$  generated by  $\xi : \mathcal{C} \twoheadrightarrow S$  is given by the coimage

$$(S) := \text{coim} \left( \mathcal{C} \xrightarrow{\Delta^{\circ 2}} \mathcal{C}^{\circ 3} \xrightarrow{\text{proj} \cdot (\text{id}_{\mathcal{C}} \circ (\xi \amalg \text{id}_{\mathcal{C}}) \circ \text{id}_{\mathcal{C}})} \mathcal{C} \circ (\underline{S} \amalg \mathcal{C}) \circ \mathcal{C} \right),$$

where the map  $\text{proj}$  is the projection  $\mathcal{C} \circ (S \amalg \mathcal{C}) \circ \mathcal{C} \rightarrow \mathcal{C} \circ (\underline{S} \amalg \mathcal{C}) \circ \mathcal{C}$ .

**Lemma 5.4.** *The coideal quotient of  $\mathcal{C}$  generated by  $\xi : \mathcal{C} \twoheadrightarrow S$  is also given by the coimage*

$$\text{coim} \left( \mathcal{C} \xrightarrow{(\text{id} \otimes \Delta) \cdot \Delta_{(1)}} \mathcal{C} \circ_{(1)} \mathcal{C}^{\circ 2} \xrightarrow{\text{id}_{\mathcal{C}} \circ_{(1)} \xi \circ \text{id}_{\mathcal{C}}} \mathcal{C} \circ_{(1)} (S \circ \mathcal{C}) \right).$$

*Proof.* It is clear that

$$\ker \left( \mathcal{C} \xrightarrow{\Delta^{\circ 2}} \mathcal{C}^{\circ 3} \xrightarrow{\text{proj} \cdot (\text{id}_{\mathcal{C}} \circ (\xi \amalg \text{id}_{\mathcal{C}}) \circ \text{id}_{\mathcal{C}})} \mathcal{C} \circ (\underline{S} \amalg \mathcal{C}) \circ \mathcal{C} \right) \subseteq \\ \ker \left( \mathcal{C} \xrightarrow{(\text{id} \otimes \Delta) \cdot \Delta_{(1)}} \mathcal{C} \circ_{(1)} \mathcal{C}^{\circ 2} \xrightarrow{\text{id}_{\mathcal{C}} \circ_{(1)} \xi \circ \text{id}_{\mathcal{C}}} \mathcal{C} \circ_{(1)} (S \circ \mathcal{C}) \right)$$

since  $\mathcal{C} \circ_{(1)} \mathcal{C}^{\circ 2}$  is a quotient of  $\mathcal{C}^{\circ 3}$ . The coassociativity of  $\Delta$  gives the converse inclusion.  $\square$

When the epimorphism  $\xi$  is strict (see Definition A.13), so is the epimorphism  $\mathcal{C} \twoheadrightarrow (S)$ . We therefore have  $(S) \cong \text{coker}(\mathcal{C}(S) \rightarrow \mathcal{C})$ . Under this assumption, we obtain, by the universal property of the cokernel, the equivalence between the fact that the composition  $\mathcal{C}(S) \twoheadrightarrow \mathcal{C} \rightarrow I$  is non zero and the fact that the counit does not factor through the coideal quotient  $(S)$ .

**Proposition 5.5.** *We suppose that the epimorphism  $\xi : \mathcal{C} \twoheadrightarrow S$  is strict and that the counit  $\mathcal{C} \rightarrow I$  does not factor through the coideal quotient  $(S)$ . Then, the subcomonoid of  $\mathcal{C}$  generated by  $S$  is*

$$\mathcal{C}(S) := \ker(\mathcal{C} \rightarrow (S)),$$

that is

$$\mathcal{C}(S) = \ker \left( \mathcal{C} \xrightarrow{(\text{id} \otimes \Delta) \cdot \Delta_{(1)}} \mathcal{C} \circ_{(1)} \mathcal{C}^{\circ 2} \xrightarrow{\text{id}_{\mathcal{C}} \circ_{(1)} \xi \circ \text{id}_{\mathcal{C}}} \mathcal{C} \circ_{(1)} (S \circ \mathcal{C}) \right).$$

*Proof.* The counit is given by the composite  $\mathcal{C}(S) \twoheadrightarrow \mathcal{C} \rightarrow I$ . See Proposition 60 in [Val08, Appendix B] for the fact that the comultiplication restricts to  $\mathcal{C}(S)$ .  $\square$

**5.6. Koszul dual infinitesimal cooperad.** We define the notion of homogeneous quadratic curved operad and Koszul dual infinitesimal cooperad associated with it.

Following [LV12, 5.5.4], we describe a weight grading on the free operad. It is given by the number of generating operations needed in the construction of a given element in the tree operad. Let  $M$  be an  $\mathbb{S}$ -module in  $\mathcal{M}$  and let  $\mathcal{T}M$  be the tree operad on  $M$ . We define the weight  $w(\mu)$  of an element  $\mu$  recursively as follows:  $w(\text{id}) = 0$ ,  $w(\mu) = 1$  when  $\mu \in M(n)$  for some  $n$ ; and more generally  $w(\nu; \nu_1, \dots, \nu_m) = w(\nu) + w(\nu_1) + \dots + w(\nu_m)$  for some  $m$  and  $\nu \in M(m)$ . We denote by  $\mathcal{T}M^{(r)}$  the  $\mathbb{S}$ -module of elements of weight  $r$ .

**Definition 5.7.** A curved complete operad  $(\mathcal{O}, \theta)$  is called *homogeneous quadratic* if all of the following hold:

- (1) the operad  $\mathcal{O}$  admits a homogeneous quadratic presentation

$$(\mathcal{O}, \theta) \cong (\mathcal{T}E/(R), \theta) \cong \mathcal{c}\mathcal{T}(E) / \left( R \amalg (\vartheta - \tilde{\theta}(1)) \right),$$

where  $E$  is an  $\mathbb{S}$ -module in  $\mathcal{M}$  and  $(R)$  is the ideal generated by a strict  $\mathbb{S}$ -submodule  $R$  of  $\mathcal{T}E^{(2)}$  in  $\mathcal{M}$  and the curvature  $\theta$  is induced by a map  $\tilde{\theta} : I \rightarrow F_1 \mathcal{T}E^{(2)}$ ,

- (2) the sub- $\mathbb{S}$ -module  $R$  is a direct sum of homological degree and filtration degree homogeneous subspaces,
- (3) the predifferential of  $\mathcal{O}$  vanishes, and

- (4) the counit  $\mathcal{T}^c(sE) \rightarrow I$  does not factor through the coideal quotient  $(S)$ , where we fix

$$\mathcal{C} := \mathcal{T}^c(sE) \text{ and } S := \left( I \amalg \mathcal{T}^c(sE)^{(2)} \right) / \left( s^2 R \amalg (1 - s^2 \tilde{\theta}(1)) \right).$$

- Remark 5.8.*
- The condition that the differential of  $\mathcal{O}$  is 0 implies that the bracket with the curvature is 0.
  - The filtration on  $E$  induces a filtration on  $\mathcal{T}E$ . The fact that the inclusion  $R \hookrightarrow \mathcal{T}E^{(2)}$  is strict means that the filtration on  $R$  is the one induced by the filtration on  $E$ .

**Definition 5.9.** Let  $\mathcal{O} := c\mathcal{T}(E)/(R \amalg (\vartheta - \theta))$  be a curved complete operad equipped with a homogeneous quadratic presentation. We define the *Koszul dual infinitesimal cooperad*  $\mathcal{O}^i$  of  $\mathcal{O}$  to be the sub-cooperad  $\widetilde{\mathcal{C}(S)}$  of  $\widetilde{\mathcal{T}^c(sE)}$  generated by the strict epimorphism  $\mathcal{C} \twoheadrightarrow S$  (with the notations of Definition 5.7). It is an altipotent infinitesimal cooperad since it is a (strict) sub-cooperad of an altipotent infinitesimal cooperad.

We sometimes denote it by  $\mathcal{O}^i = \widetilde{\mathcal{C}(sE, s^2 R \amalg (1 - s^2 \tilde{\theta}(1))}$ . Its counit is given by the composite  $\varepsilon_{\mathcal{O}^i} : \mathcal{O}^i \twoheadrightarrow \widetilde{\mathcal{T}^c(sE)} \rightarrow I[-1]$ .

**5.10. Koszul dual operad.** It is hard to explicitly describe the elements of  $\mathcal{O}^i$  by means of its definition. Under certain assumptions, we can however get a better understanding by means of the Koszul dual operad  $\mathcal{O}^!$ .

Considering a specific case of Section 4 in [HM12], we propose the following definition.

**Definition 5.11.** We say that a complete operad  $\mathcal{O}$  is *constant-quadratic* if it admits a presentation of the form  $\mathcal{O} = \mathcal{T}E/(R)$ , where  $E$  is a complete dg  $\mathbb{S}$ -module and  $(R)$  is the ideal generated by a complete dg  $\mathbb{S}$ -module  $R \subset I \amalg \mathcal{T}E^{(2)}$ . (We assume that the filtration on  $R$  is induced by the filtration on  $E$ .) We require that  $R$  is a coproduct of homological degree and filtration degree homogeneous subobjects. Thus the complete operad  $\mathcal{O}$  is homological degree graded and filtration degree graded and has a weight filtration induced by the  $\mathbb{S}$ -module of generators  $E$ . We assume further that the following conditions hold:

- (I) The space of generators is minimal, that is  $R \cap I = \{0\}$ .
- (II) The space of relations is maximal, that is  $(R) \cap \{I \amalg E \amalg \mathcal{T}E^{(2)}\} = R$ .

In the context of the definition above, we denote by  $q : \mathcal{T}E \rightarrow \mathcal{T}E^{(2)}$  the canonical projection and by  $qR \subset \mathcal{T}E^{(2)}$  the image under  $q$  of  $R$ . Since  $R \cap I = \{0\}$ , there exists a filtered map  $\varphi : qR \rightarrow I$  such that  $R$  is the graph of  $\varphi$ :

$$R = \{X - \varphi(X), X \in qR\}.$$

We propose the following generalization of Section 2.1.9 in [GK94]. In our context, the situation isn't symmetric since we define a constant-quadratic (resp. curved) operad associated with a curved (resp. constant-quadratic) operad. We make it however as symmetric as possible so that it is easy to extend these definitions to a curved-constant-quadratic setting. Let  $E$  be a complete  $\mathbb{S}$ -module of finite dimension in each arity which splits as follows:

$$E = F_0 E \amalg F_1 E = (E_{(0)}^0 \amalg E_{(0)}^1) \amalg E_{(1)}.$$

- In the case where  $(\mathcal{O}, \theta)$  is a homogeneous quadratic curved operad, we assume moreover that

$$\mathrm{im} \theta \subset E_{(1)} \circ_{(1)} (E_{(0)}^0 \amalg E_{(1)}) + (E_{(0)}^0 \amalg E_{(1)}) \circ_{(1)} E_{(1)}. \quad (4)$$

- In the case where  $\mathcal{O}$  is a constant-quadratic complete operad, we assume moreover that we have a splitting  $R \cong \varphi^{-1}(0) \amalg R^{c^q}$  such that

$$R^{c^q} \subset E_{(0)}^1 \circ_{(1)} (E_{(0)}^0 \amalg E_{(0)}^1) + (E_{(0)}^0 \amalg E_{(0)}^1) \circ_{(1)} E_{(0)}^1.$$

We consider on  $E^*$  the filtration

$$F_0 E^* := E_{(0)}^0 \amalg E_{(1)}^* \text{ and } F_1 E^* := E_{(0)}^1 \amalg E_{(1)}^*. \quad (5)$$

To define a pairing between  $\mathcal{T}(E)^{(2)}$  and  $\mathcal{T}(E^*)^{(2)}$ , we need to choose a tree basis of the trees made of two vertices. When the trees are reduced (that is with at least one leaf), the tree basis can be provided by shuffle trees (as explained in [LV12, Section 8.2.5]). When one of the tree has 0 leaf (necessarily the one above), we choose as a tree basis the one such that the tree of arity 0 is put further left. This defines a natural pairing  $\mathcal{T}(E)^{(2)} \otimes \mathcal{T}(E^*)^{(2)}$  which can be extended in a unique way in a natural pairing  $\langle -, - \rangle : (\vartheta I \amalg \mathcal{T}(E)^{(2)}) \otimes (I \amalg \mathcal{T}(E^*)^{(2)})$  such that  $\langle \vartheta I, \mathcal{T}(E^*)^{(2)} \rangle = \{0\}$ ,  $\langle \mathcal{T}(E)^{(2)}, I \rangle = \{0\}$  and  $\langle \vartheta 1, 1 \rangle = 1$  where 1 is the generator of  $I$ .

We propose the two following definitions of Koszul dual operad in the curved / constant-quadratic context.

**Definition 5.12.** In this definition,  $E$  is a complete  $\mathbb{S}$ -module of finite dimension in each arity admitting a decomposition  $E = F_0 E \amalg F_1 E = (E_{(0)}^0 \amalg E_{(0)}^1) \amalg E_{(1)}$ .

- (1) Given a homogeneous quadratic complete curved operad

$$\mathcal{O} = c\mathcal{T}(E) / (R \amalg (\vartheta - \theta))$$

such that Inclusion (4) is satisfied, we define its *Koszul dual complete constant-quadratic operad* by

$$\mathcal{O}^! := \mathcal{T}(E^*) / ((R \amalg (\vartheta - \theta))^\perp).$$

It is a constant-quadratic infinitesimal complete operad.

- (2) Given a constant-quadratic complete operad

$$\mathcal{O} = \mathcal{T}(E) / (R)$$

such that Inclusion (5) is satisfied, we define its *Koszul dual complete curved operad* by

$$\mathcal{O}^! := c\mathcal{T}(E^*) / (R^\perp).$$

**Proposition 5.13.** *Let  $\mathcal{O}$  be a homogeneous quadratic curved complete operad (resp. a constant-quadratic complete operad) generated by a complete  $\mathbb{S}$ -module  $E$  of finite dimension in each arity admitting a splitting  $E = F_0 E \amalg F_1 E = (E_{(0)}^0 \amalg E_{(0)}^1) \amalg E_{(1)}$  such that Inclusion (4) (resp. (5)) is satisfied. We have*

$$(\mathcal{O}^!)^! \cong \mathcal{O}.$$

*Proof.* The natural map  $E \rightarrow (E^*)^*$  is a filtered isomorphism since  $E$  is finite-dimensional and by the definitions of the filtration on  $E^*$ . Then it is direct to check the two computations.  $\square$

Moreover, under the condition that the operad  $\mathcal{O}$  is Koszul, we will see in Theorem 5.20 a description of the elements in  $\mathcal{O}^i$  by means of the elements in the Koszul dual cooperad of the (uncurved) operad  $\text{Gr } \mathcal{O}$ .

**5.14. Bar construction of a curved quadratic operad.** We fix a homogeneous quadratic curved operad  $\mathcal{O} = \mathcal{c}\mathcal{T}(E)/(R \amalg (\vartheta - \theta))$ . We consider on  $\hat{\mathcal{B}}\mathcal{O}$  a second homological degree called *syzygy degree*. The operad  $\mathcal{O} \cong (\mathcal{T}E/(R), \theta)$  is weight-graded by the weight grading of  $\mathcal{T}E$  (that is the number of generators in  $E$ ) since  $R$  is homogeneous. We define the syzygy degree of an element in  $\hat{\mathcal{B}}\mathcal{O}$  recursively as follows: the syzygy degree of  $\text{id}$  is 0, the syzygy degree of an element in  $s\bar{\mathcal{O}}$  is 1 minus its weight in  $\mathcal{O}$  and the syzygy degree of an element  $(\nu; \nu_1, \dots, \nu_m)$  is the sum of the syzygy degrees of the elements  $\nu, \nu_1, \dots, \nu_m$ . The differentials  $d_0$  and  $d_2$  lower the syzygy degree by 1.

**Proposition 5.15.** *Let  $\mathcal{O} = \mathcal{c}\mathcal{T}(E)/(R \amalg (\vartheta - \theta))$  be a complete curved operad equipped with a homogeneous quadratic presentation. Let  $\mathcal{O}^i = \tilde{\mathcal{C}}(sE, s^2R \amalg (1 - s^2\tilde{\theta}(1)))$  be its Koszul dual infinitesimal cooperad. The natural inclusion*

$$i : \mathcal{O}^i \hookrightarrow \tilde{\mathcal{T}}^c(sE) \hookrightarrow \hat{\mathcal{B}}\mathcal{O}$$

*induces an isomorphism of complete cooperads*

$$i : \mathcal{O}^i \xrightarrow{\cong} \text{H}_0(\hat{\mathcal{B}}\mathcal{O}),$$

*where the homology degree is taken to be the syzygy degree.*

*Proof.* Syzygy degree 0 elements in  $\hat{\mathcal{B}}\mathcal{O}$  are given by  $\tilde{\mathcal{T}}^c(sE)$ . Syzygy degree  $-1$  elements coincide with

$$\tilde{\mathcal{T}}^c(sE) \circ_{(1)} \left( \left( \tilde{\mathcal{T}}^c(sE)^{(2)}/s^2R \right) \circ \tilde{\mathcal{T}}^c(sE) \right)$$

seen in  $\hat{\mathcal{B}}\mathcal{O}$ . The differential  $d$  provides therefore a map

$$\tilde{\mathcal{T}}^c(sE) \rightarrow \tilde{\mathcal{T}}^c(sE) \circ_{(1)} \left( \left( \tilde{\mathcal{T}}^c(sE)^{(2)}/s^2R \right) \circ \tilde{\mathcal{T}}^c(sE) \right).$$

Through the isomorphism

$$\tilde{\mathcal{T}}^c(sE)^{(2)}/s^2R \cong \left( I \amalg \tilde{\mathcal{T}}^c(sE)^{(2)} \right) / \left( s^2R \amalg (1 - s^2\tilde{\theta}(1)) \right) = S,$$

this map coincides with the map

$$\left( \tilde{\mathcal{C}} \xrightarrow{(\text{id} \otimes \Delta) \cdot \Delta_{(1)}} \tilde{\mathcal{C}} \circ_{(1)} \tilde{\mathcal{C}} \circ_{(1)} \xrightarrow{\text{id}_{\tilde{\mathcal{C}} \circ_{(1)}} \xi \circ \text{id}_{\tilde{\mathcal{C}}}} \tilde{\mathcal{C}} \circ_{(1)} (S \circ \tilde{\mathcal{C}}) \right).$$

We conclude by Proposition 5.5 (and Lemma 3.19).  $\square$

**5.16. Curved Koszul operad.** Since we are working with  $\mathcal{A} = \text{dg-}\mathbf{R}\text{-Mod}$  (with  $\mathbf{R}$  a field), the cooperad splits as  $\mathbb{S}$ -modules as  $\mathcal{O}^i \cong I[-1] \amalg \bar{\mathcal{O}}^i$ . We denote by  $\eta : I[-1] \rightarrow \mathcal{O}^i$  coming from this isomorphism.

Using the weight grading on  $\mathcal{T}^c(sE)$ , we can write

$$\mathcal{T}^c(sE) \cong \mathcal{T}^c(sE)_{\text{even}} \amalg \mathcal{T}^c(sE)_{\text{odd}},$$

where  $\mathcal{T}^c(sE)_{\text{even}} := \amalg_{k \geq 0} \mathcal{T}^c(sE)^{(2k)}$  and  $\mathcal{T}^c(sE)_{\text{odd}} := \amalg_{k \geq 0} \mathcal{T}^c(sE)^{(2k+1)}$ .



**Lemma 5.17.** *The cooperad  $\mathcal{O}^i$  splits as follows:*

$$\mathcal{O}^i \cong \mathcal{O}_{\text{even}}^i \amalg \mathcal{O}_{\text{odd}}^i,$$

where  $\mathcal{O}_{\text{even}}^i = \mathcal{O}^i \cap \widetilde{\mathcal{T}}^c(sE)_{\text{even}}$  and  $\mathcal{O}_{\text{odd}}^i = \mathcal{O}^i \cap \widetilde{\mathcal{T}}^c(sE)_{\text{odd}}$ .

*Proof.* We have

$$\mathcal{O}^i = \ker \left( \widetilde{\mathcal{T}}^c(sE) \xrightarrow{(\text{id} \otimes \Delta) \cdot \Delta_{(1)}} \widetilde{\mathcal{T}}^c(sE) \circ_{(1)} \widetilde{\mathcal{T}}^c(sE)^{\circ 2} \right. \\ \left. \xrightarrow{\text{id}_{\widetilde{\mathcal{T}}^c(sE) \circ_{(1)}} \xi \circ \text{id}_{\widetilde{\mathcal{T}}^c(sE)}} \widetilde{\mathcal{T}}^c(sE) \circ_{(1)} (S \circ \widetilde{\mathcal{T}}^c(sE)) \right)$$

where  $S = (I[-1] \amalg \mathcal{T}^c(sE)^{(2)}) / (s^2 R \amalg (1 - s^2 \tilde{\theta}(1)))$ . The map  $(\text{id} \otimes \Delta) \cdot \Delta_{(1)}$  and  $\text{id}_{\widetilde{\mathcal{T}}^c(sE) \circ_{(1)}} \xi \circ \text{id}_{\widetilde{\mathcal{T}}^c(sE)}$  stabilize the odd part, resp. the even part. It follows that the kernel splits as desired.  $\square$

As a consequence and using the definition of  $\mathcal{O}^i$  and the fact that  $\eta$  is a gr-coaugmentation, we can write

$$\eta(1) = 1 + \left( -s^2 \tilde{\theta} + \sum s^2 r \right) + \dots \in I \amalg sE \amalg \mathcal{T}^c(sE)^{(2)} \amalg \dots, \quad (6)$$

with  $\sum s^2 r \in s^2 R$ . There is a natural morphism  $\kappa : \mathcal{O}^i \rightarrow \mathcal{O}$  of degree  $-1$  defined as the composite

$$\kappa : \mathcal{O}^i \hookrightarrow \widetilde{\mathcal{T}}^c(sE) \twoheadrightarrow sE \cong E \rightarrow \mathcal{O}.$$

**Lemma 5.18.** *We have  $\Theta + \frac{1}{2}\{\kappa, \kappa\} = 0$  for  $\Theta = \theta \cdot \varepsilon_{\mathcal{O}^i}$ . Hence  $\kappa$  is a curved twisting morphism.*

*Proof.* On  $\eta(I[-1])$ , a direct computation using Equality (6) shows that the equality  $\Theta + \frac{1}{2}\{\kappa, \kappa\} = 0$  is true. Then, on  $\overline{\mathcal{O}^i} \subset \widetilde{\mathcal{T}}^c(sE)^{(\geq 1)} = sE \amalg \mathcal{T}^c(sE)^{(2)} \amalg \dots$ , using the fact that the decomposition map is the cofree one, we only have to consider elements whose projection on  $\mathcal{T}^c(sE)^{(2)}$  is non zero. By means of Lemma 5.17 and using the definition of  $\mathcal{O}^i$ , we get that these elements write

$$\sum s^2 r + \dots \in \mathcal{T}^c(sE)^{(2)} \amalg \mathcal{T}^c(sE)^{(\geq 2)}, \quad (7)$$

with  $\sum s^2 r \in s^2 R$ . Since  $\theta|_{\overline{\mathcal{O}^i}} \equiv 0$ , we obtain that  $\kappa$  is a curved twisting morphism.  $\square$

**Definition 5.19.** A curved complete operad  $(\mathcal{O}, \theta)$  is called *Koszul* if  $\mathcal{O} \cong (\widehat{\text{Gr } \mathcal{O}}, \bar{\theta})$ , where  $\text{Gr } \mathcal{O}$  is the graded uncurved operad associated with  $\mathcal{O}$  and  $\text{Gr } \mathcal{O}$  admits a homogeneous quadratic presentation  $\text{Gr } \mathcal{O} \cong \mathcal{T}E/(R)$  in  $\text{Filt}(\mathcal{A})$  for which it is a Koszul operad.

**Theorem 5.20.** *Let  $\mathcal{O}$  be a curved complete Koszul operad. Then the map*

$$i : \mathcal{O}^i \rightarrow \hat{\text{B}}\mathcal{O}$$

*is a quasi-isomorphism of cooperads. Moreover, there is a Poincaré–Birkhoff–Witt type isomorphism of  $\mathbb{S}$ -modules*

$$\mathcal{O}^i \cong (\widehat{\text{Gr } \mathcal{O}})^i.$$

As we will see in examples, this last isomorphism provides a way to understand the cooperad  $\mathcal{O}^i$ .

*Proof.* We denote by  $F_p$  the filtration on  $\hat{\mathcal{B}}\mathcal{O}$ . It is induced by the filtration on  $\mathcal{O}$  for which  $\mathcal{O}$  is complete. We consider the (increasing) filtration  $G_p := F_{-p}$  on  $\hat{\mathcal{B}}\mathcal{O}$  (note that  $G_p$  is not bounded below). We have

$$d_0 : G_p \hat{\mathcal{B}}\mathcal{O} \rightarrow G_{p-1} \hat{\mathcal{B}}\mathcal{O} \text{ and } d_2 : G_p \hat{\mathcal{B}}\mathcal{O} \rightarrow G_p \hat{\mathcal{B}}\mathcal{O}.$$

The filtration  $G_p$  is complete since  $\hat{\mathcal{B}}\mathcal{O}$  is complete for the filtration  $F_p$  and it is exhaustive since  $\hat{\mathcal{B}}\mathcal{O} = G_0 \hat{\mathcal{B}}\mathcal{O}$ . We consider the spectral sequence  $E_{p,q}^\bullet$  associated with this filtration. Using that the functor  $\text{Gr}$  is strong monoidal ( $\mathbf{R}$  is a field), we have

$$E_{p,q}^0 = G_p \hat{\mathcal{B}}\mathcal{O}_{p+q} / G_{p-1} \hat{\mathcal{B}}\mathcal{O}_{p+q} \cong (\tilde{\mathcal{B}}_{cl} \text{Gr } \mathcal{O})_{p+q}^{(p)}$$

where the upper index stands for the degree of the graduation in  $\tilde{\mathcal{B}}_{cl} \text{Gr } \mathcal{O}$  and the last isomorphism comes from the fact that  $\text{Gr}$  is strong monoidal (see Lemma B.4). The differential is given by  $d^0 = d_{\tilde{\mathcal{B}}_{cl} \text{Gr } \mathcal{O}}$  (induced by  $d_2$ ). Since  $\text{Gr } \mathcal{O}$  is a Koszul operad, we get

$$E_{p,q}^1 = (\widetilde{\text{Gr } \mathcal{O}})_{p+q}^{(p)}.$$

Now let us consider on  $\hat{\mathcal{B}}\mathcal{O}$  and on  $\tilde{\mathcal{B}}_{cl} \text{Gr } \mathcal{O}$  the syzygy degrees described in the beginning of Section 5.14. We have the following compatibility condition: for  $b \in F_p \hat{\mathcal{B}}\mathcal{O} \setminus F_{p+1} \hat{\mathcal{B}}\mathcal{O}$  with a homogeneous syzygy degree, the associated element in  $(\tilde{\mathcal{B}}_{cl} \text{Gr } \mathcal{O})^{(p)}$  has the same syzygy degree. The  $E^1$  page is concentrated in syzygy degree 0 (in  $\tilde{\mathcal{B}}_{cl} \text{Gr } \mathcal{O}$ ) as the Koszul dual cooperad is, and by means of the compatibility of the syzygy degrees, we therefore obtain that the differentials  $d^r$ ,  $r \geq 1$ , induced by  $d_0$  and  $d_2$  are 0 (they both decrease the syzygy degree). This implies that the spectral sequence is regular (see Definition 5.2.10 in [Wei94]) and by the Complete Convergence Theorem 5.5.10 in [Wei94] that the spectral sequence converges to  $\mathbf{H}_\bullet(\hat{\mathcal{B}}\mathcal{O})$  (the spectral sequence is bounded above).

We have therefore

$$(\widetilde{\text{Gr } \mathcal{O}})_{p+q}^{(p)} = \Pi_q E_{p,q}^1 = \Pi_q E_{p,q}^\infty \cong \text{Gr}_p \mathbf{H}_\bullet(\hat{\mathcal{B}}\mathcal{O})$$

and using the syzygy degree, we get  $(\widetilde{\text{Gr } \mathcal{O}})_{p+q}^{(p)} \cong \text{Gr}_p \mathbf{H}_0(\hat{\mathcal{B}}\mathcal{O}) \cong \text{Gr}_p \mathbf{H}_\bullet(\hat{\mathcal{B}}\mathcal{O})$ . By means of Proposition 5.15, we obtain moreover that  $(\widetilde{\text{Gr } \mathcal{O}})_{p+q}^{(p)} \cong \mathcal{O}^i$  (where we have dropped the  $\sim$  to lighten the notation).

Finally, the morphism  $\mathcal{O}^i \rightarrow \hat{\mathcal{B}}\mathcal{O}$  is a quasi-isomorphism and we have an isomorphism of  $\mathbb{S}$ -modules  $\mathcal{O}^i \cong (\widetilde{\text{Gr } \mathcal{O}})_{p+q}^{(p)}$ .  $\square$

**Proposition 5.21.** *Suppose that  $\mathcal{O}$  is a Koszul curved complete operad. The natural map  $p = f_\kappa : \hat{\mathcal{O}}^i \rightarrow \mathcal{O}$  is an  $\mathbb{S}$ -cofibrant resolution of curved complete operads.*

*Proof.* We have already seen that, since  $\text{Gr}$  is strong monoidal, we have

$$\text{Gr } \hat{\mathcal{O}}^i \cong \Omega \text{Gr } (\mathcal{O}^i) \cong \Omega_{cl}(\text{Gr } \mathcal{O})^i \coprod \left( s^{-1} \eta_{\mathcal{O}^i}(\tilde{I}) \amalg \vartheta I \right).$$

Since  $\text{Gr } \mathcal{O}$  is a Koszul operad, Theorem 7.4.2 in [LV12] gives that  $\Omega_{cl}(\text{Gr } \mathcal{O})^i$  is quasi-isomorphic to  $\text{Gr } \mathcal{O}$ . We therefore obtain that  $\hat{\mathcal{O}}^i \rightarrow \mathcal{O}$  is a graded quasi-isomorphism.

It is a strict surjection and the same reasoning as in the proof of Proposition 4.11 shows that it is cofibrant in the model category structure on complete gr-dg  $\mathbb{S}$ -modules.  $\square$

## 6. THE ASSOCIATIVE CASE

In this section, we make the case of the operad encoding curved associative algebras explicit. It is a curved operad that we denote cAs and we prove that it encodes curved associative algebras.

Again, we assume that  $\mathcal{A}$  is the category of  $\mathbf{R}$ -modules, with  $\mathbf{R}$  a field.

**6.1. The curved operad encoding curved associative algebras.** We recall that the operad As encoding associative algebras is the (trivially) filtered (complete) operad defined by

$$\text{As} := \mathcal{T}(\Upsilon) / \left( \searrow\Upsilon - \swarrow\Upsilon \right).$$

Its representations in  $\mathbf{R}$ -modules are associative algebras.

**Proposition 6.2.** *The curved operad encoding curved associative algebras is the complete (pointed) gr-dg operad defined by*

$$\text{cAs} := \left( \mathcal{T}(\spadesuit \amalg \Upsilon) / \left( \searrow\Upsilon - \swarrow\Upsilon \right), 0, \theta := \spadesuit\Upsilon - \swarrow\Upsilon \right),$$

where  $\spadesuit$  is of degree  $-2$ ,  $\Upsilon$  is of degree  $0$  and the predifferential is zero. We filter it by the number of  $\spadesuit$ , say

$$F_p \text{cAs} := \{ \mu \in \text{cAs} \text{ s.t. the number of } \spadesuit \text{ in } \mu \text{ is greater than or equal to } p \}.$$

It is a curved operad and its curvature belongs to  $F_1 \text{cAs}$ . We can also write

$$\text{cAs} = \text{c}\mathcal{T}(\spadesuit \amalg \Upsilon) / \left( \searrow\Upsilon - \swarrow\Upsilon, \vartheta - \theta \right),$$

where  $\theta$  is defined above.

We denote by  $\mu_n$  the element in cAs obtained as the  $(n-1)$ -iterated composition of the generator  $\Upsilon$ . (The way of composing does not matter by means of the associativity relation.)

*Proof.* The only thing to prove is that the bracket with the curvature is always zero (since the predifferential of cAs is zero). Using the Koszul sign rule, we get that  $[\theta, \spadesuit] = 0$ . And a quick calculation (using associativity) shows

$$[\theta, \mu_n] = \begin{array}{c} \mu_n \\ \theta \end{array} - \sum_j \begin{array}{c} \dot{\theta} \\ \mu_n \\ \dot{\theta} \end{array} = \sum_j \begin{array}{c} \spadesuit \\ \mu_{n+1} \end{array} - \sum_j \begin{array}{c} \spadesuit \\ \mu_{n+1} \end{array} = 0.$$

□

**Lemma 6.3.** *A cAs-algebra on a complete gr-dg module  $A$  is the same data as a curved complete associative algebra  $(A, \mu, d_A, \theta)$  with curvature  $\theta \in F_1 A$ .*

*Proof.* A map of curved operad  $\text{cAs} \rightarrow \text{End}_A$  is characterized by the image of the generators  $\spadesuit$  and  $\Upsilon$  which give respectively two maps  $\theta : \mathbf{R} \rightarrow A$  and  $\mu : A^{\otimes 2} \rightarrow A$ . The relation defining cAs ensures that  $\mu$  is associative. The fact that the curvature is sent to the curvature says that  $d_A^2 = [\theta, -]$ . □

*Remark 6.4.* • The inclusion  $\text{As} \rightarrow \text{cAs}$  is not a map of curved operads (since the curvature of As is zero) so it cannot be a quasi-isomorphism of curved complete operads.

- The projection  $\text{cAs} \rightarrow \text{As}$ , sending the 0-ary element to 0, is a map of curved operads but the map  $\text{Gr cAs} \rightarrow \text{Gr As}$  is not a quasi-isomorphism. Therefore, it is not a quasi-isomorphism of curved complete operads.

**6.5. Homotopy curved associative algebras.** By forgetting the curvature of cAs, we obtain a complete operad which is the coproduct  $\text{As} * \mathcal{T}(\spadesuit)$  of the complete operads  $\mathcal{T}(\spadesuit)$  and  $\text{As}$ . Let us explain how the coproduct  $*$  is constructed.

**Definition 6.6.** Let  $\mathcal{O}$  and  $\mathcal{P}$  be two augmented (complete) operads. Then the *coproduct* in the category of (complete) operads is defined to be

$$\mathcal{T}(\overline{\mathcal{O}} \amalg \overline{\mathcal{P}}) / (R_{\mathcal{O}} \amalg R_{\mathcal{P}}),$$

where  $R_{\mathcal{O}}$  and  $R_{\mathcal{P}}$  are the relations in  $\mathcal{O}$  and  $\mathcal{P}$  respectively.

**Proposition 6.7.** *If  $\mathcal{O}$  and  $\mathcal{P}$  are both quadratic augmented (complete) operads, then the coproduct  $\mathcal{O} * \mathcal{P}$  is a quadratic augmented operad.*

*Proof.* For any two presented operads  $\mathcal{O} = \mathcal{T}(E_1)/(R_1)$  and  $\mathcal{P} = \mathcal{T}(E_2)/(R_2)$ , the coproduct operad  $\mathcal{O} * \mathcal{P}$  is naturally presented by  $\mathcal{T}(E_1 \amalg E_2)/(R_1 \amalg R_2)$ . If  $(E_1, R_1)$  and  $(E_2, R_2)$  are both quadratic presentations, then so is  $(E_1 \amalg E_2, R_1 \amalg R_2)$ .  $\square$

**Proposition 6.8.** *If  $\mathcal{O}$  and  $\mathcal{P}$  are both quadratic augmented (complete) operads, then the Koszul dual infinitesimal cooperad of the coproduct  $\mathcal{O} * \mathcal{P}$  is the coproduct infinitesimal cooperad  $\mathcal{O}^i \oplus \mathcal{P}^i$  (whose underlying  $\mathbb{S}$ -module is  $I[-1] \amalg \overline{\mathcal{O}^i} \amalg \overline{\mathcal{P}^i}$ ).*

*Proof.* The infinitesimal cooperad  $\mathcal{O}^i$  is the sub-cooperad of  $\widetilde{\mathcal{T}}^c(sE_1)$  which is universal among the infinitesimal sub-cooperads  $\mathcal{C}$  of  $\widetilde{\mathcal{T}}^c(sE_1)$  such that the composite

$$\mathcal{C} \hookrightarrow \widetilde{\mathcal{T}}^c(sE_1) \twoheadrightarrow \widetilde{\mathcal{T}}^c(sE_1)^{(2)}/s^2R_1$$

is zero (see [LV12, 7.1.4]). The infinitesimal cooperad  $\mathcal{P}^i$  is defined similarly. It follows that the map  $\mathcal{O}^i \oplus \mathcal{P}^i \rightarrow \widetilde{\mathcal{T}}^c(sE_1 \amalg sE_2)$  satisfies that the composite

$$\begin{aligned} \mathcal{O}^i \oplus \mathcal{P}^i &\hookrightarrow \widetilde{\mathcal{T}}^c(sE_1 \amalg sE_2) \twoheadrightarrow \mathcal{T}^c(sE_1 \amalg sE_2)^{(2)}/(s^2R_1 \amalg s^2R_2) \cong \\ &\mathcal{T}^c(sE_1)^{(2)}/s^2R_1 \amalg sE_1 \circ_{(1)} sE_2 \amalg sE_2 \circ_{(1)} sE_1 \amalg \mathcal{T}^c(sE_2)^{(2)}/s^2R_2 \end{aligned}$$

is zero. By the universal property of  $(\mathcal{O} * \mathcal{P})^i$ , there exists a unique morphism of infinitesimal cooperads  $\mathcal{O}^i \oplus \mathcal{P}^i \rightarrow (\mathcal{O} * \mathcal{P})^i$  which makes the following diagram commutative:

$$\begin{array}{ccc} \mathcal{O}^i \oplus \mathcal{P}^i & \twoheadrightarrow & \widetilde{\mathcal{T}}^c(sE_1 \amalg sE_2). \\ \downarrow & \nearrow & \\ (\mathcal{O} * \mathcal{P})^i & & \end{array}$$

Moreover, using the fact that  $(\mathcal{O} * \mathcal{P})^i$  is an infinitesimal sub-cooperad of the cofree infinitesimal cooperad  $\widetilde{\mathcal{T}}^c(sE_1 \amalg sE_2)$  and that its projection onto  $sE_1 \circ_{(1)} sE_2 \amalg sE_2 \circ_{(1)} sE_1$  is zero, we get that the map  $(\mathcal{O} * \mathcal{P})^i \hookrightarrow \widetilde{\mathcal{T}}^c(sE_1 \amalg sE_2)$  factors through  $\widetilde{\mathcal{T}}^c(sE_1) \oplus \widetilde{\mathcal{T}}^c(sE_2)$ . The composite  $(\mathcal{O} * \mathcal{P})^i \hookrightarrow \widetilde{\mathcal{T}}^c(sE_1) \oplus \widetilde{\mathcal{T}}^c(sE_2) \twoheadrightarrow \widetilde{\mathcal{T}}^c(sE_1)$  is an infinitesimal cooperad morphism, hence its image  $pr_1((\mathcal{O} * \mathcal{P})^i)$  is an infinitesimal sub-cooperad of  $\widetilde{\mathcal{T}}^c(sE_1)$ . Using the map  $\mathcal{O}^i \oplus \mathcal{P}^i \rightarrow (\mathcal{O} * \mathcal{P})^i$ , we obtain that it contains  $\mathcal{O}^i$ . Moreover, the definition of  $(\mathcal{O} * \mathcal{P})^i$  ensures that the composite

$$pr_1((\mathcal{O} * \mathcal{P})^i) \hookrightarrow \widetilde{\mathcal{T}}^c(sE_1) \twoheadrightarrow \mathcal{T}^c(sE_1)^{(2)}/s^2R_1$$

is zero. The infinitesimal cooperad  $pr_1((\mathcal{O} * \mathcal{P})^i)$  is therefore an infinitesimal sub-cooperad of  $\mathcal{O}^i$  (by the universal property satisfied by  $\mathcal{O}^i$ ). Eventually,  $pr_1((\mathcal{O} * \mathcal{P})^i) = \mathcal{O}^i$ . Similarly,  $pr_2((\mathcal{O} * \mathcal{P})^i) = \mathcal{P}^i$ . Using the fact that  $\amalg$  is a biproduct

and that  $\mathcal{O}^i$  and  $\mathcal{P}^i$  are counital, it follows that there exists a unique morphism  $(\mathcal{O} * \mathcal{P})^i \rightarrow \mathcal{O}^i \oplus \mathcal{P}^i$  which commutes with the projections. This map is injective since  $(\mathcal{O} * \mathcal{P})^i \rightarrow \widetilde{\mathcal{T}}^c(sE_1) \oplus \widetilde{\mathcal{T}}^c(sE_2)$  is. The composite  $\mathcal{O}^i \oplus \mathcal{P}^i \rightarrow (\mathcal{O} * \mathcal{P})^i \rightarrow \mathcal{O}^i \oplus \mathcal{P}^i$  is the identity and this proves the proposition.  $\square$

We are now able to compute the Koszul dual infinitesimal cooperad of the curved operad cAs. We denote the element  $s\blacktriangleright$  by  $\mu_0^c$  and the set of generators of  $\text{As}^i$  (that we identify with  $\widetilde{\text{As}}^i$ ) presented in [LV12, 9.1.5] by  $\{\mu_n^c\}_{n \geq 1}$ .

**Theorem 6.9.** *The homogeneous quadratic curved complete operad cAs is Koszul. The Koszul dual infinitesimal cooperad  $\text{cAs}^i$  is equal to*

$$\text{cAs}^i = \left( \left\{ \hat{\mu}_n^c := \sum_{\substack{k \geq 0 \\ S \subseteq [n+k], |S|=k}} (-1)^{s_0 + \dots + s_k - k(n+k)} \mu_{n+k}^S \right\}_{n \geq 0}, \Delta, 0 \right),$$

where  $S = \{s_j\}_{j=0}^k$ ,  $\mu_{n+k}^S$  is the element  $\mu_{n+k}^c$  on which we have grafted the element  $s\blacktriangleright$  in the position given by the set  $S$ , and  $\Delta$  is the cofree decomposition. The terms  $\hat{\mu}_n^c$  have degree  $n - 1$ . Explicitly, for any  $n \geq 0$ , we have

$$\Delta(\hat{\mu}_n^c) = \sum_{\substack{k \geq 0 \\ i_1 + \dots + i_k = n}} (-1)^{\sum (i_j - 1)(k - j)} (\hat{\mu}_k^c; \hat{\mu}_{i_1}^c, \dots, \hat{\mu}_{i_k}^c). \quad (8)$$

It is isomorphic to

$$\text{cAs}^i \cong (s\blacktriangleright \Pi \widetilde{\text{As}}^i, \Delta^\theta, 0),$$

where for any  $n \geq 0$ ,

$$\Delta^\theta(\mu_n^c) = \sum_{\substack{k \geq 0 \\ i_1 + \dots + i_k = n}} (-1)^{\sum (i_j - 1)(k - j)} (\mu_k^c; \mu_{i_1}^c, \dots, \mu_{i_k}^c).$$

Moreover, the natural map

$$f_\kappa : \text{cA}_\infty := \hat{\Omega} \text{cAs}^i \rightarrow \text{cAs}$$

is an  $\mathbb{S}$ -cofibrant resolution (that is cofibrant in the model category structure on complete *gr-dg*  $\mathbb{S}$ -modules given in Section C.21).

*Proof.* The operad cAs is the completion of the operad Gr cAs. The operad Gr cAs admits a presentation in filtered modules similar to the one defining cAs and it is in fact the coproduct  $\text{As} * \mathcal{T}(\blacktriangleright)$ . By Proposition 6.8, we get

$$(\text{Gr cAs})^i \cong \mathcal{T}^c(s\blacktriangleright) \oplus \text{As}^i \cong s\blacktriangleright \Pi \text{As}^i.$$

Let us prove Formula (8). We recall the decomposition map on  $\text{As}^i$  from [LV12, Lemma 9.1.2]

$$\Delta(\mu_n^c) = \sum_{i_1 + \dots + i_k = n} (-1)^{\sum (i_j - 1)(k - j)} (\mu_k^c; \mu_{i_1}^c, \dots, \mu_{i_k}^c).$$

(Replacing  $i_j + 1$  as it is written in *op. cit.* by  $i_j - 1$  does not change the sign but seems more natural with the computations to come.) Then, being careful with the

Koszul sign rule, since  $\mu_n^S$  is the element  $\mu_n^c$  on which we have grafted the element  $s^\bullet$  in the position given by the set  $S$ , we get

$$\Delta(\mu_n^S) = \sum_{\substack{l_0+i_1+\dots+i_k=n \\ S=S_0 \sqcup S_1 \sqcup \dots \sqcup S_k}} (-1)^{\sum (i_j-1)(k+l_0-(j+n_j))+L_0+K} \left( \mu_{k+l_0}^{S_0}; \mu_{i_1}^{S_1}, \dots, \mu_{i_k}^{S_k} \right),$$

where

$$\begin{cases} |S_j| &= l_j, \\ m_j &= i_j - l_j, \\ n_j &= |\{s \in S_0; s < \max\{S_j\}\}|, \\ L_0 &= \sum_{s \in S_0} \sum_{\{t; \max\{S_t\} < s\}} l_t, \text{ and} \\ K &= \sum l_t \sum_{j>t} (i_j - 1). \end{cases}$$

Some terms are missing in the first sum appearing to compute the sign: they correspond to the entries labelled by  $s \in S_0$  and are equal to  $(1+1)(k+l_0-*)$ , where  $*$  is the position of  $s$  in  $\mu_{k+l_0}^c$ ; therefore they have no effect. The sum  $L_0$  is the Koszul sign due to the elements  $s^\bullet$  corresponding to the set  $S_0$  passing through the elements  $s^\bullet$  corresponding to some set  $S_j$ . The sum  $K$  is the Koszul sign due to the elements  $s^\bullet$  of the  $S_j$  passing through some  $\mu_{i_t}^c$ . Moreover, we compute the sign in front of  $(\mu_{k+l_0}^{S_0}; \mu_{i_1}^{S_1}, \dots, \mu_{i_k}^{S_k})$  to the right of Formula (8). There is a shifting sending the entries of  $\mu_{i_j}^c$  appearing in  $\Delta(\mu_n^c)$  to  $\{1, \dots, i_j\}$ . We denote by  $\tilde{S}_j = \{\tilde{s}_j\}$  the set obtained by shifting this way the elements in  $S_j$ . We obtain:

$$(-1)^{\sum (i_j-l_j-1)(k-j)} (-1)^{\sum_{\tilde{s}_0} \tilde{s}_0 - l_0(k+l_0) + \dots + \sum_{\tilde{s}_k} \tilde{s}_k - l_k i_k}.$$

We should therefore compare this sign with

$$(-1)^{\sum s - |S|n} (-1)^{\sum (i_j-1)(k+l_0-j-n_j)+L_0+\sum l_t \sum_{j>t} (i_j-1)}.$$

Since  $|S| = \sum l_j$  and  $n = l_0 + \sum i_j$ , it is enough to compare

$$-\sum l_j(k-j) + \sum_j \sum_{\tilde{S}_j} \tilde{s}_j - l_0(k+l_0) - \sum l_j i_j$$

with

$$\sum_S s - \left( \sum l_j \right) \left( l_0 + \sum i_j \right) + \sum (i_j - 1)(l_0 - n_j) + L_0 + \sum l_t \sum_{j>t} (i_j - 1).$$

We have

$$\begin{aligned} & -\sum l_t i_j + \sum_{j>t} l_t \sum (i_j - 1) = \\ & = -l_0 \sum i_j - \sum_{t \geq 1} l_t \sum_{j \neq t} i_j - \sum l_j i_j + l_0 \sum_{j>0} (i_j - 1) + \sum_{t \geq 1} l_t \sum_{j>t} i_j - \sum_{t \geq 1} l_t (k - t) \\ & = -\sum_{t \geq 1} l_t \sum_{j<t} i_j - \sum l_j i_j - l_0 k - \sum_{t \geq 1} l_t (k - t). \end{aligned}$$

It follows that we are interested in the difference

$$D := \sum_S s - \left( \sum_{j>0} l_j \right) l_0 - \sum_{t \geq 1} l_t \sum_{j<t} i_j + \sum (i_j - 1)(l_0 - n_j) + L_0 - \sum_j \sum_{\tilde{S}_j} \tilde{s}_j.$$

A combinatorial check shows that

$$\sum_S s - \sum_j \sum_{\tilde{S}_j} \tilde{s}_j = \sum_{s \in S_0} \sum_{\{t; \max\{S_t\} < s\}} (i_t - 1) + \sum_{t \geq 1} l_t \left( \sum_{j < t} i_j + |\{s \in S_0; s < \max\{S_t\}\}| \right),$$

where  $\sum_{t \geq 1} l_t |\{s \in S_0; s < \max\{S_t\}\}| = \sum_{s \in S_0} \sum_{\{t; s < \max\{S_t\}\}} l_t$ . Moreover,

$$\sum_{s \in S_0} \sum_{\{t; s < \max\{S_t\}\}} l_t + L_0 = \sum_{s \in S_0} \sum_{t > 0} l_t = l_0 \sum_{t > 0} l_t.$$

Then

$$\begin{aligned} D &= \sum_{s \in S_0} \sum_{\{t; \max\{S_t\} < s\}} (i_t - 1) + \sum (i_j - 1)(l_0 - n_j) \\ &= \sum_{s \in S_0} \sum_{\{t; \max\{S_t\} < s\}} (i_t - 1) + \sum (i_j - 1) |\{s \in S_0; s > \max\{S_j\}\}| \\ &= 2 \sum_{s \in S_0} \sum_{\{t; \max\{S_t\} < s\}} (i_t - 1). \end{aligned}$$

We conclude that Formula (8) is true.

It follows that the  $\hat{\mu}_n^c$ ,  $n \geq 0$ , span an infinitesimal sub-cooperad  $\mathcal{C}$  of  $\widetilde{\mathcal{T}}^c(s \blacktriangleright \amalg s \blacktriangleright)$ . Moreover, an explicitation of the first terms of  $\hat{\mu}_0^c$  and  $\hat{\mu}_2^c$  shows that the composite  $\mathcal{C} \hookrightarrow \widetilde{\mathcal{T}}^c(s \blacktriangleright \amalg s \blacktriangleright) \twoheadrightarrow (I[-1] \amalg \mathcal{T}^c(s \blacktriangleright \amalg s \blacktriangleright)^{(2)}) / (s^2 (\blacktriangleright \blacktriangleright - \blacktriangleright \blacktriangleright) \amalg (1 - s^2 \tilde{\theta}(1)))$  is zero. By the universal property of the infinitesimal cooperad  $\mathbf{cAs}^i$ , it follows that there exists a unique (injective) morphism  $\mathcal{C} \hookrightarrow \mathbf{cAs}^i$  such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & \widetilde{\mathcal{T}}^c(sE), \\ \downarrow & & \nearrow \\ \mathbf{cAs}^i & & \end{array}$$

where  $E = \blacktriangleright \amalg \blacktriangleright$ . A proof similar to the proof of Proposition 6.1.7 in [HM12] shows that the quadratic operad  $\mathbf{Gr cAs}$  is a Koszul operad. By the Poincaré–Birkhoff–Witt theorem of Theorem 5.20, we have  $\mathbf{cAs}^i \cong (\widetilde{\mathbf{Gr cAs}})^i$ . It suffices now to check the dimension in each arity to get  $\mathbf{cAs}^i = \{\hat{\mu}_n^c\}_{n \geq 0}$ . Going through the Poincaré–Birkhoff–Witt isomorphism, we obtain the decomposition map  $\Delta^\theta$ .  $\square$

We can also compute the Koszul dual infinitesimal operad (see Definition 5.12).

**Proposition 6.10.** *The Koszul dual complete infinitesimal operad is given by*

$$\mathbf{cAs}^! \cong \widetilde{\mathcal{T}}(\blacktriangleright \amalg \blacktriangleright) / \left( \blacktriangleright \blacktriangleright - \blacktriangleright \blacktriangleright, \blacktriangleright \blacktriangleright - |, \blacktriangleright \blacktriangleright - | \right).$$

*Algebras over  $\mathbf{cAs}^!$  are unital associative algebras.*

*Proof.* The computation of the Koszul dual operad  $\mathbf{As}^!$  is given in Section 7.6.4 in [LV12]. The computation of the extra terms is similar. We define algebras over a complete infinitesimal operad  $\mathcal{O}$  to be morphisms of complete infinitesimal operads  $\mathcal{O} \rightarrow \widetilde{\mathbf{End}}_A$ , where  $\mathbf{End}_A$  is the endomorphism operad defined in Definition 2.9.  $\square$

*Remark 6.11.* We therefore recover in an operadic context the curved Koszul duality theory of the operad encoding unital associative algebras presented in [HM12].

**Proposition 6.12.** *A  $cA_\infty$ -algebra is equivalent to a complete graded vector space  $(A, F)$  equipped with an  $n$ -ary filtered operation*

$$m_n : A^{\otimes n} \rightarrow A \text{ of degree } n - 2 \text{ for all } n \geq 0,$$

which satisfies the following relations

$$\sum_{p+q+r=n} (-1)^{p+qr} m_k \circ_{p+1} m_q = 0, \quad n \geq 0, \quad (9)$$

where  $k = p + 1 + r$ .

*This notion of algebras coincides, under different settings, with the notions of  $A_\infty$ -algebras given in [GJ90], of curved  $A_\infty$ -algebras defined in [CD01], of weak  $A_\infty$ -algebras given in [Kel06], of filtered  $A_\infty$ -algebras used in [FOOO07], of  $[0, \infty[-$ algebras studied in [Nic08] and of weakly curved  $A_\infty$ -algebras studied in [Pos12] to give only one reference for each name.*

*Proof.* A morphism of curved operad  $\gamma_A : \hat{\Omega}cAs^i \rightarrow \text{End}_{(A, d_A)}$  is characterized by a map of complete  $\mathbb{S}$ -modules  $cAs^i \rightarrow \text{End}_A$  and therefore by a collection of filtered applications

$$\tilde{m}_n : A^{\otimes n} \rightarrow A, \text{ of degree } n - 2 \text{ for all } n,$$

such that  $\tilde{m}_1$  increases the filtration degree by 1. We denote by  $m_1$  the endomorphism  $\tilde{m}_1 - d_A$  and by  $m_n$  the endomorphisms  $\tilde{m}_n$  for  $n \neq 1$ . The fact that  $\gamma_A$  commutes with the predifferentials and the fact that  $\gamma_A$  sends the curvature to the curvature ensure that Equation (9) is satisfied for  $n$ .  $\square$

*Remark 6.13.* (1) By the fact that complete curved associative algebras are examples of complete curved operads, Sections 4 and 5 apply to complete curved associative algebras and we therefore obtain functorial resolutions of complete curved associative algebras by the bar-cobar resolution and Koszul resolutions for Koszul complete curved associative algebras (cofibrant in the underlying category of complete gr-dg  $\mathbf{R}$ -modules). These results can also be seen as an example of an extension of the results in [Mil12] to the curved / infinitesimal setting: the operadic Koszul morphism  $\kappa : cAs^i \rightarrow cAs$  allows to define a bar-cobar adjunction on the level of curved associative algebras and infinitesimal coassociative coalgebras.

(2) The same yoga would work for curved Lie algebras since we can define the *curved Lie* operad  $cLie$  as the curved operad  $cAs$ . We obtain the curved operad  $cL_\infty$  whose algebras coincides with curved  $L_\infty$ -algebras defined in [Zwi93] and used or studied for example in [Cos11, Mar12, MZ12, LS12]. It is however impossible to associate in the same way a curved operad associated with the operad  $Com$ .

**6.14. Homotopy categories.** We now apply the results of Sections C.30 and C.38 to the morphism

$$f_\kappa : cA_\infty \rightarrow cAs.$$

By Theorem C.35, the categories of algebras  $\text{Alg}(cAs)$  and  $\text{Alg}(cA_\infty)$  admit a cofibrantly generated model structure and by Theorem C.40, the morphism  $f_\kappa$  produces a Quillen adjunction between the model categories of curved algebras.



**Theorem 6.15.** *The functor  $f_\kappa^*$  and  $f_{\kappa_*}$*

$$\mathbb{L}f_\kappa^* : \mathbf{Hoalg}(cA_\infty) \rightleftarrows \mathbf{Hoalg}(cAs) : \mathbb{R}f_{\kappa_*} = f_{\kappa_*}$$

*are equivalences of the homotopy categories.*

*Proof.* To apply Theorem C.41, it is enough to show that the complete curved operad  $cAs$  and  $cA_\infty$  are  $\mathbb{S}$ -split and that the morphism  $f_\kappa$  is compatible with the chosen splittings. The two operads come from non symmetric operads. In this situation, we can consider the splitting

$$\mathcal{O}(n) = \mathcal{O}_{ns}(n) \otimes \mathbf{R}[\mathbb{S}_n] \rightarrow (\mathcal{O}_{ns}(n) \otimes \mathbf{R}[\mathbb{S}_n])_{ns} \otimes \mathbf{R}[\mathbb{S}_n]$$

induced by the map  $\mathcal{O}_{ns}(n) \rightarrow (\mathcal{O}_{ns}(n) \otimes \mathbf{R}[\mathbb{S}_n])_{ns}$ . The map  $f_\kappa$  is compatible with these splittings.  $\square$

*Remark 6.16.* There are two possible notions of morphisms for  $cA_\infty$ -morphisms:

- the classical morphisms of algebras of the curved operad  $cA_\infty$ , that is to say maps  $f : A \rightarrow B$  compatible with the predifferential and the algebra structures;
- the notion, sometimes called  $\infty$ - $cA_\infty$ -morphisms, of morphisms of infinitesimal coalgebras (cooperads) between an extension of the bar constructions on  $A$  and on  $B$  to homotopy curved algebras (or operads). (See for example Section 7.1 in [Pos12].)

In the previous theorem, we consider the first notion of morphism. The study of  $\infty$ -morphisms requires extra constructions which will be given in an other work.

## APPENDIX A. CATEGORICAL FILTRATIONS AND COMPLETIONS

In this appendix, we will establish definitions of filtered and complete filtered objects for use in both our ground categories and in the categories of operads over them.

We want to establish the ability to work concretely with filtered and complete filtered objects, which we do by realizing them as reflective subcategories of diagram categories that are easier to manipulate. The strategy of studying a category by embedding it as a reflective subcategory of a better-behaved category is quite old [GZ67]. In fact the cases of filtered and complete filtered objects constitute two of the original motivating examples. Consequently, much of the content of this appendix may be well-known to experts.

The main technical point for us will be the transfer of a closed monoidal structure to a reflective subcategory in a coherent fashion, using a criterion of Day [Day72].

The executive summary of (most of) this appendix is that the following diagram of subcategory inclusions is in fact a diagram of normal reflective embeddings. This means that:

- (1) each of the solid arrows has a left adjoint reflector functor,
- (2) the counit of each adjunction is an isomorphism,
- (3) each category is closed symmetric monoidal,
- (4) the given arrows are all lax symmetric monoidal with the unit of the adjunction as the monoidal natural transformation, and
- (5) the left adjoints are all strong symmetric monoidal.

The diagram is as follows.

$$\mathcal{A}^{\text{ob}\mathbb{N}} \longrightarrow \mathcal{A}^{\mathbb{N}^{\text{op}}} \longleftarrow \text{Filt}(\mathcal{A}) \longleftarrow \widehat{\text{Filt}}(\mathcal{A}).$$

Moreover, in the case where  $\mathcal{A}$  is replaced by the category  $\text{pg}\mathcal{A}$  of predifferential graded objects in  $\mathcal{A}$ , we can complete the diagram

$$\text{pg}\mathcal{A}^{\text{ob}\mathbb{N}} \longrightarrow \text{pg}\mathcal{A}^{\mathbb{N}^{\text{op}}} \longleftarrow \text{Filt}(\text{pg}\mathcal{A}) \longleftarrow \widehat{\text{Filt}}(\text{pg}\mathcal{A}) \longleftarrow \widehat{\text{Filt}}^{\text{gr}}(\mathcal{A}).$$

Throughout this appendix, we let  $(\mathcal{A}, \otimes, \mathbf{1}, \underline{\mathcal{A}})$  be a closed symmetric monoidal Grothendieck category with initial object  $\emptyset$ .

We consider  $\mathbb{N}$  as a poset category with initial object 0. When desirable, we enrich  $\mathbb{N}$  in  $\mathcal{A}$  by saying  $\mathbb{N}(a, b)$  is  $\emptyset$  if  $a > b$  and  $\mathbf{1}$  otherwise.

In general the underlying category of this enriched category is not equivalent to  $\mathbb{N}$  itself (they are equivalent if and only if  $\mathbf{1}$  admits no automorphisms and no maps to  $\emptyset$ ).

Because of the diagrams of reflexive subcategories, the condition that  $\mathcal{A}$  is a Grothendieck category ensures in fact that the categories  $\text{Filt}(\mathcal{A})$ ,  $\widehat{\text{Filt}}(\mathcal{A})$  and  $\widehat{\text{Filt}}^{\text{gr}}(\mathcal{A})$  are quasi-abelian (see for example [Rie16, Lemma 3.3.5] and [FS10, Proposition 4.20]). This property is useful for Section 5.

Various generalizations are possible: for many of the results we can replace  $\mathcal{A}$  with a regular cocomplete closed symmetric monoidal category and  $\mathbb{N}$  with a nontrivial poset with an initial object.

### A.1. Categorical filtered objects.

**Definition A.2.** Let  $\mathcal{C}$  be a category. A  $\mathbb{N}$ -filtered object in a category  $\mathcal{C}$  is an  $\mathbb{N}^{\text{op}}$ -indexed diagram in  $\mathcal{C}$  where all maps are monomorphisms. The full subcategory spanned by  $\mathbb{N}$ -filtered objects is denoted  $\text{Filt}(\mathcal{C})$ .

**Lemma A.3.** *The full subcategory spanned by  $\mathbb{N}$ -filtered objects in  $\mathcal{A}$  inside  $\mathcal{A}^{\mathbb{N}^{\text{op}}}$  is a reflective subcategory. Moreover, the reflector takes the functor  $F : \mathbb{N}^{\text{op}} \rightarrow \mathcal{A}$  to the functor  $rF$  whose value at  $e$  is the image of  $F(e) \rightarrow F(0)$ .*

*Proof.* For a diagram  $F : \mathbb{N}^{\text{op}} \rightarrow \mathcal{A}$  there is a map  $\psi_e : F(e) \rightarrow F(0)$  for each object  $e$  of  $\mathbb{N}$ . Define  $rF(e)$  as  $\text{im}(\psi_e)$ . By the universal property of the image in a regular category, the morphisms in the diagram  $rF$  are well-defined monomorphisms.

To see the adjunction, let  $F$  be a diagram and  $G$  a filtered object. The data of a  $\mathbb{N}$ -filtered object map from  $rF$  to  $G$  consists of a map  $f_e$  from the image of  $\psi_e$  to  $G(e)$  for each  $e$ . Such data defines a map  $\tilde{f}_e : F(e) \rightarrow \text{im}(\psi_e) \rightarrow G(e)$ . On the other hand, given a map  $\tilde{f}_e : F(e) \rightarrow G(e)$  compatible with  $F(0) \rightarrow G(0)$ , there is a unique lift through the image as in the following diagram:

$$\begin{array}{ccc} F(e) & \xrightarrow{\quad} & G(e) \\ \downarrow & & \downarrow \\ rF(e) & \xrightarrow{\quad} & F(0) \longrightarrow G(0). \end{array}$$

(A dotted arrow points from  $rF(e)$  to  $G(e)$ .)

The coherence conditions for two different values of  $e$  to be an  $\mathbb{N}$ -filtered object map or a map of diagrams coincide, in that the commutativity of the outside cell

implies that the right square commutes if and only if the left square commutes.

$$\begin{array}{ccccc}
 & & \curvearrowright & & \\
 F(e) & \xrightarrow{\tilde{f}_e} & G(e) & \xleftarrow{f_e} & rF(e) \\
 \downarrow & & \downarrow & & \downarrow \\
 F(e') & \xrightarrow{\tilde{f}_{e'}} & G(e') & \xleftarrow{f_{e'}} & rF(e') \\
 & & \curvearrowleft & & 
 \end{array}$$

Then this objectwise natural bijection restricts to a natural bijection between filtered maps from  $rF$  to  $G$  and maps of diagrams from  $F$  to  $iG$ .  $\square$

**A.4. Categorical complete filtered objects.** In this section, we use the fact that  $\mathcal{A}$  is an Abelian category, that it admits  $\mathbb{N}^{\text{op}}$ -indexed limits and that it satisfies AB4. A *filtered object* means a  $\mathbb{N}$ -filtered object in  $\mathcal{A}$ . For a filtered object  $V$  we write  $i_e$  (or  $i_e^V$  if  $V$  is ambiguous) for the monomorphism  $F_e V \rightarrow F_0 V$ .

**Definition A.5.** A filtered object is *complete* if the natural map  $F_0 V \rightarrow \lim_a \text{coker } i_a$  is an isomorphism.

The *completion* of the filtered object  $V$  is the sequence  $\hat{V}$  (also denoted  $V^\wedge$ ) defined as

$$F_e \hat{V} := \ker(\lim_a \text{coker } i_a \xrightarrow{\pi_e} \text{coker } i_e)$$

with the natural maps induced by the identity on  $\lim_a \text{coker}(i_a)$ .

**Lemma A.6.** *There is a natural isomorphism from  $\lim_{a \geq e} \text{coker}(F_a V \rightarrow F_e V)$  to  $F_e \hat{V}$ .*

*Proof.* The natural map is constructed as follows. The map  $F_e V \rightarrow \text{coker}(F_a V \rightarrow F_e V)$  is epic and the composition  $F_e V \rightarrow \text{coker}(F_a V \rightarrow F_e V) \rightarrow \text{coker } i_e$  is equal to the composition  $F_e V \rightarrow F_0 V \rightarrow \text{coker } i_e = 0$ . So, passing to the limit,  $F_e V \rightarrow \lim \text{coker}(F_a V \rightarrow F_e V)$  factors through the kernel of  $\lim_a \text{coker } i_a \rightarrow \text{coker } i_e$ .

To see that this is an isomorphism, note that for  $a \geq e$ , the sequence

$$0 \rightarrow \text{coker}(F_a V \rightarrow F_e V) \rightarrow \text{coker } i_a \rightarrow \text{coker } i_e \rightarrow 0$$

is exact by the third isomorphism theorem which exists because the category satisfies AB4. Then taking limits over  $a$ , we get an exact sequence

$$0 \rightarrow \lim(\text{coker } F_a V \rightarrow F_e V) \rightarrow F_0 \hat{V} \rightarrow \text{coker } i_e$$

since the limit is left exact. But  $F_0 \hat{V} \rightarrow \text{coker } i_e$  is epic since the epimorphism  $F_0 V \rightarrow \text{coker } i_e$  factors through  $F_0 \hat{V}$ , so the sequence is in fact short exact, which is the desired statement.  $\square$

**Lemma A.7.** *Completion is a functor from filtered objects to the full subcategory of complete filtered objects.*

*Proof.* If completion is viewed as an assignment with codomain sequences, then functoriality is clear. It should be verified that the codomain can be restricted, first to filtered objects and then to complete filtered objects.

So first we verify that the induced maps  $F_{e'} \hat{V} \rightarrow F_e \hat{V}$  are monomorphisms for  $e' > e$ .

By inspection  $F_0\hat{V} \cong \lim_a \operatorname{coker} i_a$  and so the map  $F_e\hat{V} \rightarrow F_0\hat{V}$  is the inclusion of a kernel and thus a monomorphism. Since the map  $F_{e'}\hat{V} \rightarrow F_0\hat{V}$  can be factored  $F_{e'}\hat{V} \rightarrow F_e\hat{V} \rightarrow F_0\hat{V}$ , the left map  $F_{e'}\hat{V} \rightarrow F_e\hat{V}$  is also a monomorphism.

To see that completions are complete, by the construction of  $\hat{V}$  we have coherent isomorphisms  $\operatorname{coker} i_e^{\hat{V}} \cong \operatorname{coker} i_e^V$  so the composition of the natural map with the two natural isomorphisms

$$F_0\hat{V} \rightarrow \lim_a \operatorname{coker} i_a^{\hat{V}} \cong \lim_a \operatorname{coker} F_a V \cong F_0\hat{V}$$

is the identity, which implies that the natural map is an isomorphism.  $\square$

**Lemma A.8.** *There is a natural filtered map  $r$  (or  $r_V$ ) from a filtered object  $V$  to its completion  $\hat{V}$ . This natural map is an isomorphism on complete objects.*

*Proof.* We have already used a similar natural map (let us call it  $r_0$ ) from  $F_0V$  to  $F_0\hat{V} \cong \lim_a \operatorname{coker} i_a$ . The composition

$$F_e V \xrightarrow{i_e} F_0 V \xrightarrow{r_0} F_0\hat{V} \xrightarrow{\pi_e} \operatorname{coker} i_e$$

is zero so that there is a unique induced map  $r_e$  from  $F_e V$  to  $F_e\hat{V} = \ker(\pi_e)$  as in the following diagram:

$$\begin{array}{ccc} F_e V & \xrightarrow{i_e} & F_0 V \\ \downarrow & & \downarrow \\ r_e \swarrow & 0 \longrightarrow & \operatorname{coker} i_e \searrow r_0 \\ \downarrow & & \uparrow \pi_e \\ F_e\hat{V} = \ker \pi_e & \longrightarrow & F_0\hat{V}. \end{array}$$

The collection  $r_e$  is coherent so constitute the data of morphism of filtered objects.

If  $V$  is complete then the natural map  $F_0V \rightarrow F_0\hat{V}$  is an isomorphism, so that  $F_e\hat{V} \rightarrow F_0\hat{V} \rightarrow F_0V$  is naturally isomorphic to  $\ker \operatorname{coker} i_e$  which in turn is naturally isomorphic to the monomorphism  $i_e$ . The identity isomorphism  $F_e V \rightarrow F_e V$  satisfies the universal property of  $r_e$  under the identification of this natural isomorphism, so  $r_e$  itself must be an isomorphism.  $\square$

**Lemma A.9.** *For any filtered object  $V$  the two morphisms  $r^{\hat{V}}$  and  $\widehat{r^V}$  from  $\hat{V}$  to  $\widehat{\hat{V}}$  coincide.*

The proof is straightforward; we record the details in order to have a complete<sup>1</sup> account. Because  $F_e W \rightarrow F_0 W$  is a monomorphism for a filtered complex  $W$  it suffices to check the maps  $F_0\hat{V} \rightarrow F_0\widehat{\hat{V}}$  coincide.

*Proof.* These are two maps

$$F_0\hat{V} \rightarrow \lim_b \operatorname{coker} \ker(F_0\hat{V} \xrightarrow{\pi_b} \operatorname{coker} i_b).$$

To check if these maps agree it suffices to check on all projections over the limit. The  $e$  term of the limit is the coimage of  $\pi_e$ , and so it suffices to check equality after postcomposing the monomorphism  $\operatorname{coker} \ker(\pi_e) \rightarrow \operatorname{coker} i_b$ .

<sup>1</sup>In more ways than one.

The  $e$  component of  $r^{\hat{V}}$  is just the left half of the coimage factorization  $F_0\hat{V} \rightarrow \text{coker ker}(F_0\hat{V} \xrightarrow{\pi_e} \text{coker } i_e)$  so postcomposing the given monomorphism gives  $\pi_e$ .

On the other hand, the  $e$  component of  $r^{\hat{V}}$  is given by the projection  $\pi_e$  from  $F_0\hat{V}$  to  $\text{coker } i_e$  followed by the map  $\text{coker } i_e \rightarrow \text{coker ker } \pi_e$  induced by  $r_0^V$ . Post-composition then gives

$$F_0\hat{V} \xrightarrow{\pi_e} \text{coker } i_e \xrightarrow{\text{induced by } r_0^V} \text{coker ker } \pi_e \xrightarrow{\text{right half of } \pi_e \text{ coimage factorization}} \text{coker } i_e.$$

Then to show this to be equal to  $\pi_e$  it suffices to check that the composition of the right two maps in the composition is the identity of  $\text{coker } i_e$ . Then this equality can be checked after precomposition with the epimorphism  $F_0V \rightarrow \text{coker } i_e$ . We conclude by the commutativity of the following diagram, which shows these two to be equal (unlabeled morphisms are induced by cokernels and coimage factorizations).

$$\begin{array}{ccc}
 F_0V & \xrightarrow{\quad} & \text{coker } i_e \\
 \searrow^{r_0^V} & & \downarrow \text{induced by } r_0^V \\
 & F_0\hat{V} \xrightarrow{\quad} & \text{coker ker } \pi_e \\
 & \searrow^{\pi_e} & \downarrow \\
 & & \text{coker } i_e \\
 \swarrow & & \swarrow \\
 & & \text{coker } i_e
 \end{array}$$

□

**Corollary A.10.** *The inclusion of complete filtered objects inside filtered objects is reflective with reflector  $V \xrightarrow{r} \hat{V}$ .*

*Proof.* We have already defined a unit in Lemma A.7. For  $W$  a complete filtered object, the  $W$  component of the counit is supposed to be a map of filtered objects from  $\hat{W}$  to  $W$ . Since  $W$  is complete, we may take the inverse of the unit, which is an isomorphism on the complete filtered object  $W$ . So  $\epsilon_W = r_W^{-1}$ .

Then  $W \xrightarrow{r^W} \hat{W} \xrightarrow{\epsilon_W} W$  is the identity on  $W$  by definition and the composition  $\hat{V} \xrightarrow{\hat{r}^{\hat{V}}} \hat{V} \xrightarrow{\epsilon_{\hat{V}}} \hat{V}$  is the identity by Lemma A.9. □

**A.11. Coproducts and products in complete filtered objects.** Next we spend a little time showing that in complete objects, the canonical map from the coproduct to the product is a monomorphism. This allows us to detect whether two maps into a coproduct are equal by means of the product projections. We now use that  $\mathcal{A}$  moreover has all products and coproducts and is AB5.

*Remark A.12.* Biproducts are inherited by a reflective subcategory. To see this, recall that a coproduct in the subcategory is calculated as the reflector applied to the coproduct of the same objects in the ambient category. On the other hand, the product of the same objects in the reflective subcategory is calculated in the ambient category, where it coincides with a coproduct in the ambient category. This implies that the given coproduct already (essentially) lies in the reflective subcategory, so the reflector is an isomorphism on this coproduct.

We recall the definition of strict morphism from [Del71, 1.1.5].

**Definition A.13.** In a category with finite limits and colimits, we say that a morphism  $f : Y \rightarrow X$  is *strict* if the natural morphism  $\text{coim}(f) \rightarrow \text{im}(f)$  is an isomorphism.

In the context of (possibly complete) filtered objects, a morphism  $f : Y \rightarrow X$  is strict if and only if it is strictly compatible with filtration, that is to say, for each  $n$ , the following is a pullback square:

$$\begin{array}{ccc} f(F_n Y) & \longrightarrow & F_n X \\ \downarrow & & \downarrow \\ f(F_0 Y) & \longrightarrow & F_0 X. \end{array}$$

**Lemma A.14.** *Suppose  $Y \rightarrow X$  is a strict, monic morphism of filtered objects. Then the induced map  $F_0 Y / F_n Y \rightarrow F_0 X / F_n X$  is a monomorphism for all  $n$ .*

*Proof.* Using the fact that  $Y \rightarrow X$  is strict, a standard diagram chase in the (Abelian) ground category shows that the map of cokernels of the Cartesian square

$$F_n X / f(F_n Y) \rightarrow F_0 X / f(F_0 Y)$$

is monic.

Now consider the snake lemma in the (Abelian) ground category for the diagram

$$\begin{array}{ccccccc} F_n Y & \longrightarrow & F_0 Y & \longrightarrow & F_0 Y / F_n Y & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F_n X & \longrightarrow & F_0 X & \longrightarrow & F_0 X / F_n X. \end{array}$$

Since  $Y \rightarrow X$  is monic, the kernel of  $F_0 Y \rightarrow F_0 X$  is zero so the kernel of  $F_0 Y / F_n Y \rightarrow F_0 X / F_n X$  is in an exact sequence following a zero term and preceding a monomorphism.  $\square$

**Lemma A.15.** *The natural map in diagrams from a coproduct to the product over the same indexing set is monic.*

*Proof.* The map from each finite subcoproduct to the product admits a retraction so is monic. By AB5, the map in the colimit from the total coproduct to the product is also monic. <sup>2</sup>  $\square$

**Lemma A.16.** *The natural map in filtered objects from a coproduct to the product over the same indexing set is strict.*

*Proof.* Let  $X^i$  be a family of filtered objects, and let  $Y$  map to  $\coprod F_0 X^i$  and  $\prod F_n X^i$  over  $\prod F_0 X^i$ . We construct a map from  $Y$  to  $\prod F_n X^i$  over both of these. Since the map from  $\prod F_n X^i$  to  $\prod F_0 X^i$  this will show that this lift is unique, witnessing the desired universal property.

Since the ground category satisfies AB4,  $\prod F_n X^i$  is the kernel of the map  $\prod F_0 X^i \xrightarrow{q} \prod (F_0 X^i / F_n X^i)$ , so it suffices to show that the map from  $Y$  to  $\prod F_n X^i$  vanishes under  $q$ . By Lemma A.15, it suffices to show that this vanishing after composing with the monomorphism from  $\prod (F_0 X^i / F_n X^i)$  to  $\prod (F_0 X^i / F_n X^i)$ . This in turn can be checked by projection to each  $F_0 X^j / F_n X^j$  factor. But these composites factor as

$$Y \rightarrow \prod F_0 X^i \rightarrow \prod F_0 X^i \rightarrow F_0 X^j \rightarrow F_0 X^j / F_n X^j$$

<sup>2</sup>We learned this short proof from Zhen Lin Low.

and because the map from  $Y$  to  $\coprod F_0 X^i$  factors through  $\coprod F_n X^i$ , the entire composite is zero, as desired.  $\square$

**Lemma A.17.** *The natural map in complete objects from a coproduct to the product over the same indexing set is monic.*

*Proof.* Lemma A.15 shows this in diagrams. We will upgrade this first to the filtered and then to the complete context.

Both products and coproducts in the filtered context coincide with those in diagrams, the former without any hypotheses and the latter because the ground category satisfies AB4. Monomorphisms in filtered objects are created in diagrams (or by the terminal entry), so we have the same result there.

Now passing to the complete context, we need to verify that this monomorphism passes to the completion. By Lemma A.16, the monomorphism is also strict, so by Lemma A.14, the successive quotients

$$\left( \coprod_i F_0 X^i \right) / \left( \coprod_i F_n X^i \right) \rightarrow \left( \prod_i F_0 X^i \right) / \left( \prod_i F_n X^i \right)$$

are monic. The limit functor is a right adjoint so left exact, so we get a monomorphism in filtration degree zero at the level of completions. But this is enough to detect monomorphisms in complete objects.  $\square$

**A.18. Tensor products in the filtered and complete settings.** Now we give the criterion of Day for transfer of a closed symmetric monoidal structure.

**Definition A.19.** A class of objects  $G$  in a category  $\mathcal{C}$  is *strongly generating* if a morphism  $f : x \rightarrow y$  in  $\mathcal{C}$  is an isomorphism whenever the induced maps of sets  $\mathcal{C}(g, x) \rightarrow \mathcal{C}(g, y)$  is an isomorphism for all  $g \in G$ .

**Theorem A.20 (Day).** *Let  $\mathcal{D}$  be a closed symmetric monoidal category and let  $i : \mathcal{C} \rightarrow \mathcal{D}$  be a reflective subcategory, with reflector  $r$ . Let  $G_{\mathcal{D}}$  be a strongly generating class of objects of  $\mathcal{D}$ . The following are equivalent:*

- (1) *There exists:*
  - (a) *A closed symmetric monoidal structure on  $\mathcal{C}$ ,*
  - (b) *A symmetric monoidal enrichment of the inclusion  $i$  which commutes with the underlying set functor, and*
  - (c) *A strong symmetric monoidal enrichment of the reflector  $r$ .*
- (2) *(Day's condition (1)) For  $d$  in  $\mathcal{D}$  and  $c$  in  $\mathcal{C}$ , the component of the unit of the reflection adjunction  $\text{id}_{\mathcal{D}} \rightarrow ir$  is an isomorphism for the hom object  $\underline{\mathcal{D}}(d, ic)$ .*
- (3) *(Day's condition (2), simplified) For  $d$  in  $G_{\mathcal{D}}$  and  $c$  in  $\mathcal{C}$ , the component of the unit of the reflection adjunction  $\text{id}_{\mathcal{D}} \rightarrow ir$  is an isomorphism for the hom object  $\underline{\mathcal{D}}(d, ic)$ .*

In the case that Day's conditions are satisfied, since  $ri$  is naturally isomorphic to  $\text{id}_{\mathcal{C}}$ , we have the following natural isomorphisms for  $c$  and  $c'$  objects in  $\mathcal{C}$ :

$$c \otimes_{\mathcal{C}} c' \cong (ric) \otimes_{\mathcal{C}} (ric') \cong r(ic \otimes_{\mathcal{D}} ic').$$

Then for  $c$  and  $c'$  and  $c''$  objects in  $\mathcal{C}$ , we have

$$\begin{aligned}
 \mathcal{C}(r(ic \otimes_{\mathcal{D}} ic'), c'') &\cong \mathcal{D}(ic \otimes_{\mathcal{D}} ic', ic'') && \text{by adjunction} \\
 &\cong \mathcal{D}(ic, \underline{\mathcal{D}}(ic', ic'')) && \text{by adjunction} \\
 &\cong \mathcal{D}(ic, ir\underline{\mathcal{D}}(ic', ic'')) && \text{by Day's condition (1)} \\
 &\cong \mathcal{C}(c, r\underline{\mathcal{D}}(ic', ic'')) && \text{by full faithfulness of } i
 \end{aligned}$$

which shows that the internal hom  $\underline{\mathcal{C}}(c', c'')$  is naturally isomorphic to  $r\underline{\mathcal{D}}(ic', ic'')$ . In particular this implies that

$$i\underline{\mathcal{C}}(c', c'') \cong ir\underline{\mathcal{D}}(ic', ic'') \cong \underline{\mathcal{D}}(ic', ic'')$$

again by full faithfulness and Day's condition (1).

Thus the motto is “internal homs are computed in the big category  $\mathcal{D}$  but the monoidal product needs to be reflected”.

**A.21. Strong generation of sequences.** We will repeatedly use the same set of strong generators in what follows.

**Definition A.22.** Let  $m$  be an object of  $\mathcal{A}$  and  $x$  an object of  $\mathbb{N}$ . We define a  $\mathcal{A}$ -valued  $\mathbb{N}$ -presheaf  $m_x$  as follows:

$$F_e(m_x) := \begin{cases} m & e \leq x \\ \emptyset & \text{otherwise} \end{cases}$$

with structure maps either the initial map or the identity.

**Lemma A.23.** *The class of presheaves  $\{m_x\}$  as  $m$  and  $x$  vary over the objects of  $\mathcal{A}$  and  $\mathbb{N}$  is strongly generating.*

*Proof.* A map of presheaves from  $m_x$  to an arbitrary presheaf  $X$  is determined uniquely by the level  $x$  map from  $m$  to  $F_x X$ . Every such map in  $\mathcal{A}$  actually determines a map of presheaves, using the initial map for indices greater than  $x$  and using the commutativity of the following diagram for indices  $e$  less than  $x$ :

$$\begin{array}{ccc}
 m & \longrightarrow & F_x X \\
 \downarrow \text{id} & & \downarrow \\
 m & \dashrightarrow & F_e X.
 \end{array}$$

A map of presheaves is an isomorphism just when it is so objectwise, and this argument shows that the given set tests at every object of  $\mathbb{N}$  against every object of  $\mathcal{A}$ .  $\square$

**Lemma A.24.** *Let  $\mathcal{C}$  be a reflective subcategory of  $\mathcal{D}$  with reflector  $r$  and let  $G$  be a strongly generating class for  $\mathcal{D}$ . Then  $rG$  is strongly generating for  $\mathcal{C}$ .*

*Proof.* Let  $f : x \rightarrow y$  in  $\mathcal{C}$  and suppose  $f_* : \mathcal{C}(rg, x) \rightarrow \mathcal{C}(rg, y)$  is an isomorphism for all  $g$  in  $G$ . By adjunction,  $(i(f))_* : \mathcal{D}(g, ix) \rightarrow \mathcal{D}(g, iy)$  is an isomorphism for all  $g$ . Then  $i(f)$  is an isomorphism since  $G$  is generating for  $\mathcal{D}$ . Then  $f$  is an isomorphism since  $i$  is fully faithful.  $\square$

**Corollary A.25.** *The class of presheaves  $\{m_x\}$  as  $m$  and  $x$  vary over the objects of  $\mathcal{A}$  and  $\mathbb{N}$  is strongly generating for the categories of filtered and complete objects.*



*Proof.* The initial map and isomorphisms are monic so  $m_x$  is already filtered, so it is fixed by the reflector to filtered objects. It's easy to see that  $m_x$  is also complete because the limit involved in the completion of  $m_x$  stabilizes. This means that  $m_x$  is also fixed by the reflector to complete objects. The previous lemma completes the proof.  $\square$

**A.26. Monoidal structure on filtered objects.** In order to transfer a monoidal structure from  $\mathcal{A}$ -valued presheaves on  $\mathbb{N}$  to  $\mathbb{N}$ -filtered objects of  $\mathcal{A}$ , we should have a monoidal structure on diagrams.

**Fact A.27.** Day convolution gives the diagram category  $\mathcal{A}^{\mathbb{N}^{\text{op}}}$  a closed symmetric monoidal structure with product

$$F_e(V \otimes W) := \text{colim}_{a+b \geq e} F_a V \otimes F_b W$$

unit

$$F_e(\mathbf{1}_{\mathcal{A}^{\mathbb{N}^{\text{op}}}}) := \begin{cases} \mathbf{1}_{\mathcal{A}} & e = 0 \\ \emptyset & \text{otherwise} \end{cases}$$

and internal hom object

$$F_e(\underline{\mathcal{A}^{\mathbb{N}^{\text{op}}}}(V, W)) := \lim_{e+a \geq b} \underline{\mathcal{A}}(F_a V, F_b W).$$

**Lemma A.28.** *We have the following description of the internal hom object from a generator:*

$$F_e(\underline{\mathcal{A}^{\mathbb{N}^{\text{op}}}}(m_x, V)) \cong \underline{\mathcal{A}}(m, F_{x+e} V),$$

*with morphisms induced by those of  $V$ .*

*Proof.* There are maps from  $\underline{\mathcal{A}}(m, F_{x+e} V)$  to  $\underline{\mathcal{A}}(F_a(m_x), F_b V)$  for  $e+a \geq b$  defined as follows:

- if  $a > x$ , then  $F_a(m_x)$  is initial so  $\underline{\mathcal{A}}(F_a(m_x), F_b V)$  is terminal and the map requires no data.
- if  $a \leq x$ , then  $F_a(m_x) \cong m$ . Since  $e+a \geq b$ , we have  $b \leq x+e$ , so there is a map  $\phi : F_{x+e} V \rightarrow F_b V$ . Then  $\underline{\mathcal{A}}(m, F_{x+e} V) \rightarrow \underline{\mathcal{A}}(m, F_b V)$  is induced by  $\phi$ .

It is immediate that these maps are compatible with the structure morphisms of the subcategory of  $\mathbb{N}^{\text{op}} \times \mathbb{N}$  over which we are taking the limit and realize  $\underline{\mathcal{A}}(m, F_{x+e} V)$  as the limit.  $\square$

**Lemma A.29.** *The inclusion of the reflective subcategory of  $\mathbb{N}$ -filtered objects into  $\mathbb{N}$ -presheaves satisfies Day's simplified condition (2) with respect to the Day convolution symmetric monoidal structure and the generators of Corollary A.25.*

*Proof.* Let  $V$  be an  $\mathbb{N}$ -filtered object and  $m_x$  a  $\mathbb{N}$ -presheaf generator. We must show that

$$\underline{\mathcal{A}^{\mathbb{N}^{\text{op}}}}(m_x, V) \rightarrow \text{ir} \underline{\mathcal{A}^{\mathbb{N}^{\text{op}}}}(m_x, V)$$

is an isomorphism. It is equivalent to show that  $\underline{\mathcal{A}^{\mathbb{N}^{\text{op}}}}(m_x, V)$  is filtered. In turn this is equivalent to showing that the structure map from index  $e$  to index 0 is a monomorphism for all  $e$  in  $\mathbb{N}$ . By the description of Lemma A.28, it is enough to prove that

$$\underline{\mathcal{A}}(m, F_{x+e} V) \rightarrow \underline{\mathcal{A}}(m, F_x V)$$

is a monomorphism for arbitrary  $m$  and  $x$ . But  $F_{x+e}V \rightarrow F_xV$  is a monomorphism by assumption and the functor  $\underline{\mathcal{A}}(m, -)$  is a right adjoint and thus preserves monomorphisms.  $\square$

**Corollary A.30.** *The category of  $\mathbb{N}$ -filtered objects in  $\mathcal{A}$  is closed symmetric monoidal with product*

$$F_e(V \bar{\otimes} W) := \text{im}((\text{colim}_{a+b \geq e} F_a V \otimes F_b W) \rightarrow V \otimes W),$$

unit

$$F_e(\mathbf{1}_{\text{Filt}(\mathcal{A})}) := \begin{cases} \mathbf{1}_{\mathcal{A}} & e = 0 \\ \emptyset & \text{otherwise} \end{cases}$$

and internal hom object

$$F_e(\underline{\text{Filt}}(\mathcal{A})(V, W)) := \lim_{e+a \geq b} \underline{\mathcal{A}}(F_a V, F_b W).$$

**A.31. Monoidal structure on complete filtered objects.** Finally we use Day's theorem again to transfer the monoidal structure on filtered objects to one on complete filtered objects. Again, we make use of the assumptions of section A.4.

**Lemma A.32.** *Let  $W$  be a complete  $\mathbb{N}$ -filtered object and  $m$  an object of  $\mathcal{A}$ . Then the sequence  $\underline{\mathcal{A}}(m, W)$  is a complete filtered object.*

*Proof.* We saw that  $\underline{\mathcal{A}}(m, W)$  was filtered in Lemma A.29. To see that it is complete, we should verify that the canonical map

$$\underline{\mathcal{A}}(m, F_0 W) \xrightarrow{r_0} \lim_e \text{coker} \left( \underline{\mathcal{A}}(m, F_e W) \xrightarrow{(i_e)_*} \underline{\mathcal{A}}(m, F_0 W) \right)$$

is an isomorphism.

Since  $W$  is complete, we have

$$\underline{\mathcal{A}}(m, F_0 W) \cong \underline{\mathcal{A}}(m, \lim_e \text{coker } i_e) \cong \lim_e \underline{\mathcal{A}}(m, \text{coker } i_e)$$

and under this chain of isomorphisms, the natural map

$$\begin{array}{c} \lim_e \text{coker} (\underline{\mathcal{A}}(m, F_e W) \rightarrow \underline{\mathcal{A}}(m, F_0 W)) \\ \downarrow s_0 \\ \lim_e (\underline{\mathcal{A}}(m, \text{coker}(F_e W \rightarrow F_0 W))) \end{array}$$

is isomorphic to a right inverse to  $r_0$ . Since  $\mathcal{A}$  is Abelian, it then suffices to show that  $s_0$  has no kernel. But the kernel of  $s_0$  passes inside the limit and for each index  $e$  is null by left exactness of  $\underline{\mathcal{A}}(m, -)$ .  $\square$

**Lemma A.33.** *Let  $W$  be a complete  $\mathbb{N}$ -filtered object and  $x$  an element of  $\mathbb{N}$ . Then the shifted complex  $W[x]$  with  $F_e(W[x]) := F_{x+e}W$  is a complete filtered object.*

*Proof.* For any  $x$  and  $e$  the map  $\text{coker}(F_{x+e}W \rightarrow F_x W) \rightarrow \text{coker}(F_{x+e}W \rightarrow F_0 W)$  is a monomorphism since  $F_x W \rightarrow F_0 W$  is. Therefore the limit

$$\lim_e \text{coker}(F_{x+e}W \rightarrow F_x W) \xrightarrow{\phi} \lim_e \text{coker}(F_{x+e}W \rightarrow F_0 W) \cong \lim_e \text{coker}(F_e W \rightarrow F_0 W)$$

is a monomorphism. The left hand side is  $F_0\widehat{W}[x]$  and the right hand side is  $F_0\widehat{W}$ . Moreover, the further projection  $\pi_x$  to  $\text{coker}(F_xW \rightarrow F_0W)$  factors through the projection to  $\text{coker}(F_xW \rightarrow F_xW) \cong 0$  so  $\phi$  lifts to  $F_0\widehat{W}[x] \xrightarrow{\tilde{\phi}} F_x\widehat{W} = \ker \pi_x$ :

$$\begin{array}{ccc} \lim_e \text{coker}(F_{x+e}W \rightarrow F_xW) & \xrightarrow{\phi} & \lim_e \text{coker}(F_eW \rightarrow F_0W) \\ \downarrow & & \downarrow \pi_x \\ \text{coker}(F_xW \rightarrow F_xW) & \longrightarrow & \text{coker}(F_xW \rightarrow F_0W). \end{array}$$

The map  $\tilde{\phi}$  is a monomorphism because it lifts the monomorphism  $\phi$ .

On the other hand the composition  $F_xW \cong F_0W[x] \xrightarrow{r_0^{W[x]}} F_0\widehat{W}[x] \xrightarrow{\tilde{\phi}} F_x\widehat{W}$  agrees with the canonical map  $F_xW \rightarrow F_x\widehat{W}$  which is an isomorphism because  $W$  is complete. This shows that  $\tilde{\phi}$  is also an epimorphism. Therefore  $\tilde{\phi}$  is an isomorphism and by two out of three, so is  $r_0^{W[x]}$ .  $\square$

**Corollary A.34.** *The reflective subcategory of complete  $\mathbb{N}$ -filtered objects inside  $\mathbb{N}$ -filtered objects satisfies Day's simplified condition (2) with respect to the symmetric monoidal structure of Corollary A.30 and the generators of Corollary A.25.*

*Proof.* We are checking that for any generator  $m_x$  and any complete filtered object  $W$ , the object

$$\text{Filt}(\mathcal{A})(m_x, W)$$

is complete. We already have a description of this filtered object as a presheaf from Lemma A.28 (and the proof of Lemma A.29): in index  $e$  this presheaf has  $\underline{\mathcal{A}}(m, F_{x+e}W)$ . Then we are done by Lemmas A.32 and A.33.  $\square$

**Corollary A.35.** *The category of complete  $\mathbb{N}$ -filtered objects in  $\mathcal{A}$  is closed symmetric monoidal with product*

$$V \hat{\otimes} W := (V \bar{\otimes} W)^\wedge,$$

and unit and internal hom calculated as in filtered objects.

**A.36. Consequences for  $\mathbb{S}$ -modules.** The tensor product of  $\mathbb{S}$ -modules can be described in terms of colimits and the monoidal product in the ground category. Then because completion is a left adjoint and by Theorem A.20 and Corollary A.34, completion commutes with the tensor product of  $\mathbb{S}$  modules. That is, for  $\mathbb{S}$ -modules  $A$  and  $B$  we have the following natural isomorphism:

$$(A \bar{\otimes} B)^\wedge \cong \widehat{A} \hat{\otimes} \widehat{B}.$$

Similarly, since completion is a left adjoint, it commutes with the colimit defining the composition product of  $\mathbb{S}$ -modules so that for any filtered  $\mathbb{S}$ -modules  $A$  and  $B$ , we have the following sequence of natural isomorphisms:

$$\begin{aligned} (A \circ B)^\wedge &\cong \left( \text{colim}_k A(k) \bar{\otimes}_{\mathbb{S}_k} (B)^{\bar{\otimes} k} \right)^\wedge \\ &\cong \text{colim}_k \widehat{A}(k) \hat{\otimes}_{\mathbb{S}_k} (B^{\bar{\otimes} k})^\wedge \\ &\cong \text{colim}_k \widehat{A}(k) \hat{\otimes}_{\mathbb{S}_k} (\widehat{B})^{\hat{\otimes} k} \\ &\cong \widehat{A} \hat{\otimes} \widehat{B}. \end{aligned}$$

We may collect this into the following corollary.

**Corollary A.37.** *Completion is strong monoidal with respect to the induced composition products  $\circ$  and  $\hat{\circ}$  on  $\mathbb{N}$ -filtered  $\mathbb{S}$ -modules and complete  $\mathbb{N}$ -filtered  $\mathbb{S}$ -modules.*

**A.38. Filtered gr-dg objects.** Recall that  $\text{Filt}^{\text{gr}}(\mathcal{A})$  is the category of filtered gr-dg objects, that is, filtered predifferential graded objects such that the induced predifferential structure on the associated graded object is actually a differential. Likewise  $\widehat{\text{pg}}^{\text{gr}}\mathcal{A}$  consists of complete predifferential graded objects such that the induced predifferential structure on the associated graded object is actually a differential.

**Lemma A.39.** *The category of filtered gr-dg objects  $\text{Filt}^{\text{gr}}(\mathcal{A})$  is a reflective subcategory of the one of filtered pg objects.*

*Proof.* Let  $(V, F, d)$  be a filtered predifferential graded object.

The reflector takes this object to a filtered gr-dg object  $(V, \tilde{F}, d)$  constructed as follows. Define  $\tilde{F}^p(V)$  to be the following subobject of  $F_0V$ :

$$\tilde{F}^p(V) = \sum_{i=0}^p d^{2i} F_{p-i} V.$$

The connecting map  $\tilde{F}^p V \rightarrow V$  is monic by construction, which forces all the connecting maps to be monic. By construction  $d$  respects this filtration. This is clearly functorial.

For adjunction, consider a map  $f$  from a filtered pg object  $X$  into a filtered gr-dg object  $Y$ . Then  $d^{2i} F_{p-i} X$  must be taken by  $f$  to  $d^{2i} F_{p-i} Y \subset F_p Y$  so that  $f$  respects the more restrictive filtration  $\tilde{F}$ . On the other hand given a map  $f$  from  $\tilde{X}$  to  $Y$ , the same formula for  $f$  certainly respects the less restrictive filtration on  $X$ . Thus the identity on the underlying map  $F_0 X \rightarrow F_0 Y$  induces the adjunction.  $\square$

*Remark A.40.* In fact, filtered gr-dg objects is also a coreflective subcategory. The coreflector takes the object  $(V, F, d)$  to a filtered gr-dg object  $(\check{V}, \check{F}, \check{d})$  constructed as follows. We define  $F_p \check{V}$  to be the subset of elements  $v$  of  $F_p V$  such that there exist indices  $p_i$  for all natural numbers  $i$  such that

- (1) The sequence of  $p_i$  begins with  $p$  and is strictly increasing:

$$p = p_0 < p_1 < \dots$$

- (2) The element  $d^{2i} v$  is equivalent to 0 (mod  $F_{>p_i} V$ ).

This subset is a subobject (choosing the maximum of the  $p_i$  sequences for a sum) and closed by construction under  $d$ , which induces  $\check{d}$ .

An equivalent construction proceeds as follows. Consider the functor  $T$  so that  $F_p(TV)$  is the pullback

$$\begin{array}{ccc} F_p(TV) & \dashrightarrow & \text{colim}_{q>p} F_q V \\ \downarrow \text{dashed} & & \downarrow \\ F_p V & \xrightarrow{d^2} & F_p V \end{array}$$

Then  $F_p \check{V} = \lim_r F_p(T^r V)$ .

**Lemma A.41.** *Completion preserves the gr-dg property.*

*Proof.* We use the characterization of Lemma A.6 (which uses the fact that the ground category satisfies AB4). Let  $Y$  be gr-dg. Consider  $F_j \hat{Y} \xrightarrow{d^2} F_j \hat{Y}$ . This is an endomorphism of  $\lim_a \text{coker}(F_a Y \rightarrow F_j Y)$  which admits an epimorphism from  $F_j Y$ . Then since  $Y$  is gr-dg, the map  $F_j Y \xrightarrow{d^2} F_j Y$  factors through  $F_{j+1} Y$ . This means that we get a similar factorization

$$\lim_a \text{coker}(F_a Y \rightarrow F_j Y) \xrightarrow{d^2} \lim_a \text{coker}(F_a Y \rightarrow F_{j+1} Y).$$

□

**Corollary A.42.** *Complete gr-dg objects  $\widehat{\text{Filt}}^{\text{gr}}(\mathcal{A})$  are a reflective subcategory of both complete pg objects and filtered gr-dg objects.*

*Proof.* By Lemma A.41, in both cases the functor  $X \mapsto \widehat{X}$  can serve as reflector (on filtered gr-dg objects this is the same as plain completion). That is, at least it is a functor to the right codomain category.

Then the proof in both cases is a chain of canonical isomorphisms using the formal dg reflector  $X \mapsto \widetilde{X}$  and completion  $X \mapsto \widehat{X}$ . Each one comes either from an adjunction, or full faithfulness of one of the right adjoints.

We check that, for  $Y$  complete and gr-dg and  $X$  filtered, we have:

$$\begin{aligned} \text{Hom}_{\widehat{\text{Filt}}^{\text{gr}}}(\widehat{X}, Y) &\cong \text{Hom}_{\widehat{\text{Filt}}}(\widehat{X}, Y) \\ &\cong \text{Hom}_{\text{Filt}}(\widetilde{X}, Y) \\ &\cong \text{Hom}_{\text{Filt}^{\text{gr}}}(\widetilde{X}, Y) \\ &\cong \text{Hom}_{\text{Filt}}(X, Y). \end{aligned}$$

Now if  $X$  is itself either complete or gr-dg, then this is finally isomorphic by the full faithfulness of the subcategories to the subcategory morphisms. □

**A.43. Monoidal structure on (complete) filtered gr-dg objects.** We use Day's theorem again to transfer the monoidal structure on (complete) filtered objects to one on (complete) gr-dg filtered objects. Again, we use the assumptions of section A.4 .

**Lemma A.44.** *Let  $W$  be an object in  $\text{Filt}^{\text{gr}}(\mathcal{A})$  and  $m$  an object of  $\text{Filt}(\text{pg}\mathcal{A})$ . Then the sequence  $\underline{A}(\widetilde{m}, W)$  is gr-dg.*

*Proof.* On the hom object, the square of the differential is the commutator with  $d^2$ , interpreted in  $\widetilde{m}$  and in  $W$ . The  $p$ th filtered component of  $\underline{A}(\widetilde{m}, W)$  consists of morphisms which increase filtration degree by  $p$ . Since  $d^2$  increases filtration degree by 1 on both sides, the commutator with  $d^2$  of such a morphism increases filtration degree by  $p + 1$ . □

**Lemma A.45.** *Let  $W$  be a filtered gr-dg object and  $x$  an element of  $\mathbb{N}$ . Then the shifted complex  $W[x]$  with  $F_e W[x] := F_{x+e} W$  is gr-dg.*

*Proof.* The filtered gr-dg condition for  $F_{n+x} W$  yields the condition for  $F_n W$ . □

**Corollary A.46.** *The reflective subcategory of (complete) gr-dg objects inside (complete)  $\mathbb{N}$ -filtered objects satisfies Day's simplified condition (2) with respect to the symmetric monoidal structure of Corollary A.30 (Corollary A.35) and the generators of Corollary A.25 reflected into (complete) gr-dg objects via Lemma A.24.*

*Proof.* By Lemma A.45, and because  $m_x$  is complete and filtered, we need only check that for any generator  $m_x$  and any gr-dg object  $W$ , that the object

$$\underline{\text{Filt}}(\mathcal{A})(m_x, W)$$

is gr-dg. We already have a description of this filtered object as a presheaf from Lemma A.28: in index  $e$  this presheaf has  $\underline{\mathcal{A}}(m, F_{x+e}W)$ . Then we are done by Lemmas A.44 and A.45.  $\square$

**Corollary A.47.** *The category of (complete) gr-dg objects in  $\mathcal{A}$  is closed symmetric monoidal with product*

$$(\widetilde{V \otimes W}), \quad (\text{resp. } (\widehat{V \otimes W}),)$$

and unit and internal hom calculated as in filtered objects.

By abuse of notation, we still denote by  $\otimes$  (resp.  $\hat{\otimes}$ ) the tensor product in the category of (complete) gr-dg objects.

## APPENDIX B. GR-FLAT OBJECTS AND THE ASSOCIATED GRADED

This section builds up the necessary material to prove Proposition 3.4.

**Lemma B.1.** *Let the ground category  $\mathcal{A}$  be a Grothendieck category. Let  $\mathcal{C}$  be the class of monomorphisms with flat cokernel. Then  $\mathcal{C}$  is closed under the pushout product.*

*Proof.* This proof follows [Hov02, Theorem 7.2] word for word, although both the hypotheses and conclusion here are weaker. Let  $A_1 \rightarrow X_1$  and  $A_2 \rightarrow X_2$  be monomorphisms with flat cokernel  $C_1$  and  $C_2$  respectively. Then  $0 \rightarrow A_1 \rightarrow X_1 \rightarrow C_1 \rightarrow 0$  is pure exact so remains exact after tensoring with  $A_2$  or  $X_2$ . Then by the  $3 \times 3$  lemma, the middle row of the following diagram is exact:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_1 \otimes A_2 & \longrightarrow & X_1 \otimes A_2 & \longrightarrow & C_1 \otimes A_2 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A_1 \otimes X_2 & \longrightarrow & (X_1 \otimes A_2) \amalg_{(A_1 \otimes A_2)} (A_1 \otimes X_2) & \longrightarrow & C_1 \otimes A_2 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A_1 \otimes X_2 & \longrightarrow & X_1 \otimes X_2 & \longrightarrow & C_1 \otimes X_2 \longrightarrow 0. \end{array}$$

Since  $C_1$  is flat, the map  $C_1 \otimes A_2 \rightarrow C_1 \otimes X_2$  is a monomorphism with cokernel  $C_1 \otimes C_2$ . Then the snake lemma applied to the bottom two rows shows that

$$(X_1 \otimes A_2) \amalg_{(A_1 \otimes A_2)} (A_1 \otimes X_2) \rightarrow X_1 \otimes X_2$$

has zero kernel and cokernel isomorphic to  $C_1 \otimes C_2$ , which is thus flat.  $\square$

**Lemma B.2.** *Let  $A \leftarrow B \rightarrow C$  and  $A' \leftarrow B \rightarrow C'$  be diagrams in  $\mathcal{C}$ , and suppose given a map of diagrams from the former to the latter which is the identity on  $B$  and a monomorphism with flat cokernel on the other two entries. Then the induced morphism between the pushouts is a monomorphism with flat cokernel.*

*Proof.* First, we argue that the induced morphism is a monomorphism. Let  $A \amalg_B C \rightarrow I$  be a monomorphism from the pushout to an injective object (Grothendieck categories have enough injectives). Then since  $A \rightarrow A'$  and  $C \rightarrow C'$  are monomorphisms, there are extensions  $A' \rightarrow I$  of  $A \rightarrow I$  and  $C' \rightarrow I$  of  $C \rightarrow I$ . Because these morphisms are extensions, they remain compatible with  $B \rightarrow I$ , and so pass

to an extension  $A' \amalg_B C' \rightarrow I$  of  $A \amalg_B C \rightarrow I$ . Then  $A \amalg_B C \rightarrow A' \amalg_B C'$  is the first morphism in a monic composition, hence is monic.

To see the statement about the cokernel, note that cokernel (in the pushout diagram category in  $\mathcal{C}$  and in  $\mathcal{C}$ ) commutes with the colimit functor, which is a left adjoint. Then the cokernel of the morphism of pushouts is the sum of the cokernels, and flat objects are closed under sum.  $\square$

**Lemma B.3.** *The  $\mathbb{N}$ -indexed diagram product of two gr-flat filtered objects is a gr-flat filtered object.*

*Proof.* Let  $X$  and  $Y$  be gr-flat filtered objects. We would like to show that  $F_n(X \otimes Y) \rightarrow F_{n-1}(X \otimes Y)$  is a monomorphism. The colimit defining  $F_n(X \otimes Y)$  is of the following diagram:

$$\begin{array}{ccccc} F_n X \otimes F_0 Y & & F_{n-1} X \otimes F_1 Y & & \cdots \\ & \swarrow & & \swarrow & \\ & F_n X \otimes F_1 Y & & F_{n-1} X \otimes F_2 Y & \\ & \searrow & & \searrow & \end{array}$$

We can rewrite this as the following colimit:

$$\begin{array}{ccccc} (F_n X \otimes F_0 Y) & \amalg_{F_n X \otimes F_1 Y} & (F_{n-1} X \otimes F_1 Y) & (F_{n-1} X \otimes F_1 Y) & \amalg_{F_{n-1} X \otimes F_2 Y} & (F_{n-2} X \otimes F_2 Y) \\ & & \swarrow & & \swarrow & \\ & & F_{n-1} X \otimes F_1 Y & & & \cdots \end{array}$$

The map from  $F_n(X \otimes Y)$  to  $F_{n-1}(X \otimes Y)$  is realized under colimit by a map of diagrams from this latter diagram to the following diagram

$$\begin{array}{ccccc} F_{n-1} X \otimes F_0 Y & & F_{n-2} X \otimes F_1 Y & & \cdots \\ & \swarrow & & \swarrow & \\ & F_{n-1} X \otimes F_1 Y & & & \end{array}$$

where the maps involved are

- (1) monomorphisms with flat cokernel on the upper row by Lemma B.1, and
- (2) identities on the lower row.

The map of colimits can thus be constructed inductively from maps of pushout diagrams where the morphism at the corner is the identity and all other morphisms are monomorphisms with flat cokernel. The inductive step is supplied by Lemma B.2.  $\square$

**Lemma B.4.** *The full subcategory of gr-flat (complete) filtered objects in a Grothendieck category is a monoidal subcategory of (complete) filtered objects and the restriction of  $\text{Gr}$  to this subcategory is strong symmetric monoidal.*

*Proof.* We'll do the complete filtered case, but the filtered case is more or less a sub-case of this. Let  $X$  and  $Y$  be gr-flat complete filtered objects. The monoidal product of  $X$  and  $Y$  in complete filtered objects is

$$(\text{Filt}(X \otimes Y))^\wedge$$

But by Lemma B.3, the regular diagram product  $X \otimes Y$  is already filtered so  $\text{Filt}$  is an isomorphism on it. Then  $\text{Gr}(\widehat{X \otimes Y}) \cong \text{Gr}(X \otimes Y)$  since completion preserves associated graded. Again by Lemma B.3,  $X \otimes Y$  is gr-flat. This suffices to establish that gr-flat constitute a monoidal subcategory.

We have already essentially showed that the restriction of  $\text{Gr}$  to this subcategory is strong. Specifically, we've already argued that the second and third step of the composition

$$\text{Gr}(X) \otimes_{\text{Gr}} \text{Gr}(Y) \rightarrow \text{Gr}(X \otimes Y) \xrightarrow{\text{Gr}(\text{Filt}(\ ))} \text{Gr}(X \widehat{\otimes} Y) \xrightarrow{\text{Gr}(\ )} \text{Gr}(X \widehat{\otimes} Y),$$

where  $\widehat{\otimes}$  stands for  $\mathbb{N}^{\text{op}}$ -indexed diagrams, are isomorphisms. The first step is also an isomorphism because  $\text{Gr}$  is the reflector of a closed normal embedding, so the adjoint inclusion is strong monoidal.  $\square$

*Remark B.5.* The gr-flat condition is more or less essential; if  $F_i X / F_{i+1} X$  is not flat then tensoring with a module with trivial filtration can yield a diagram which is not filtered.

Now we are ready to prove Proposition 3.4.

*Proof of Proposition 3.4.* The dual of [Fre17, Lemma 3.1.1] says that a unit preserving colax symmetric monoidal functor takes (infinitesimal) cooperads to (infinitesimal) cooperads. The original lemma does not require cocompleteness of the ground category or the preservation of colimits in each variable by the monoidal product. Therefore the dual does not require completeness or (unlikely) limit-preservation properties. The definition of an operad in that reference uses only the monoidal product and morphisms in the ambient category, so cooperads in a monoidal subcategory of  $\mathcal{C}$  coincide with cooperads in  $\mathcal{C}$  whose underlying objects are in the monoidal subcategory.<sup>3</sup> (The same is true for infinitesimal cooperads.) Now Lemma B.4 says that  $\text{Gr}$  is a strong monoidal functor from the subcategory of gr-flat complete filtered objects in  $\mathcal{C}$  to graded objects in  $\mathcal{C}$ , and that its image lies in the subcategory of degreewise flat objects.  $\square$

### APPENDIX C. MODEL CATEGORY STRUCTURES

In this Appendix, we endow the category  $\widehat{\text{Filt}}^{\text{gr}}(\mathbf{R}\text{-Mod})$  of complete gr-dg  $\mathbf{R}$ -modules with a cofibrantly generated model structure. We propose to consider the graded quasi-isomorphisms as a class of weak equivalences and a set of generating cofibrations similar to the one described in [CESLW19]. We describe the fibrations and we show that the model category structure on complete gr-dg  $\mathbf{R}$ -modules is combinatorial and is a monoidal model category structure. Then we transfer the cofibrantly generated model category structure to the category of complete curved operads via the free curved operad functor and to the category of complete algebras over a curved operad by means of the corresponding free functor and we study the cofibrant objects. We finally show a base change result for the categories of algebras over some curved operads.

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<sup>3</sup>We cannot use the dual of [YJ15, Theorem 12.11(1)] here without modification because this last statement fails—the monadic construction of operads there uses colimits intimately. This implies that the category of Yau–Johnson cooperads in a monoidal subcategory of  $\mathcal{C}$  may not even be well-defined (if the subcategory is not complete), and that if they are well-defined the isomorphism with cooperads whose underlying objects lie in the monoidal subcategory may fail.



We work over unbounded chain complexes over a ring  $\mathbf{R}$ . In this appendix, we assume that  $\mathbf{R}$  is a field of characteristic 0.

**C.1. Cofibrantly generated model structure.** We recall some definitions and a result from [Hov99, 2.1].

**Definition C.2.** Let  $I$  be a class of maps in a category  $\mathcal{M}$ .

- (1) A map is *I-injective* if it has the right lifting property with respect to every map in  $I$ . The class of *I-injective* maps is denoted  $I\text{-inj}$ .
- (2) A map is *I-projective* if it has the left lifting property with respect to every map in  $I$ . The class of *I-projective* maps is denoted  $I\text{-proj}$ .
- (3) A map is an *I-cofibration* if it has the left lifting property with respect to every *I-injective* map. The class of *I-cofibrations* is the class  $(I\text{-inj})\text{-proj}$  and is denoted  $I\text{-cof}$ .
- (4) A map is an *I-fibration* if it has the right lifting property with respect to every *I-projective* map. The class of *I-fibrations* is the class  $(I\text{-proj})\text{-inj}$  and is denoted  $I\text{-fib}$ .
- (5) We assume that  $\mathcal{M}$  is cocomplete. A map is a *relative I-cell complex* if it is a transfinite composition of pushouts of elements of  $I$ . Such a morphism  $f : A \rightarrow B$  is the composition of a  $\lambda$ -sequence  $X : \lambda \rightarrow \mathcal{M}$ , for an ordinal  $\lambda$  such that, for each  $\beta$  such that  $\beta + 1 < \lambda$ , there is a pushout square

$$\begin{array}{ccc} C_\beta & \longrightarrow & X_\beta \\ \downarrow g_\beta & \lrcorner & \downarrow \\ D_\beta & \longrightarrow & X_{\beta+1} \end{array}$$

such that  $g_\beta \in I$ . We denote the collection of relative *I-cell complexes* by *I-cell*.

**Definition C.3.** Let  $\mathcal{M}$  be a cocomplete category and  $\mathcal{N}$  be a collection of morphisms of  $\mathcal{M}$ . An object  $A \in \mathcal{M}$  is  *$\alpha$ -small relative to  $\mathcal{N}$* , for a cardinal  $\alpha$ , if for all  $\alpha$ -filtered ordinals  $\lambda$  and all  $\lambda$ -sequences

$$X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_\beta \rightarrow \cdots$$

such that  $X_\beta \rightarrow X_{\beta+1}$  is in  $\mathcal{N}$  for  $\beta + 1 < \lambda$ , the map of sets

$$\operatorname{colim}_{\beta < \lambda} \operatorname{Hom}_{\mathcal{M}}(A, X_\beta) \rightarrow \operatorname{Hom}_{\mathcal{M}}(A, \operatorname{colim}_{\beta < \lambda} X_\beta)$$

is an isomorphism. We say that  $A$  is *small relative to  $\mathcal{N}$*  if it is  $\alpha$ -small relative to  $\mathcal{N}$  for some  $\alpha$ . We say that  $A$  is *small* if it is small relative to  $\mathcal{M}$ .

**Definition C.4.** A model category  $\mathcal{C}$  is said to be *cofibrantly generated* if there are sets  $I$  and  $J$  of maps such that:

- (1) the domains of the maps in  $I$  are small relative to  $I\text{-cell}$ ,
- (2) the domains of the maps in  $J$  are small relative to  $J\text{-cell}$ ,
- (3) the class of fibrations is  $J\text{-inj}$ ,
- (4) the class of trivial fibrations is  $I\text{-inj}$ .

The set  $I$  is called the set of *generating cofibrations* and the set  $J$  is called the set of *generating trivial cofibrations*.

We now recall the following theorem from [Hov99, Theorem 2.1.19]. It shows how to construct cofibrantly generated model categories.

**Theorem C.5.** *Suppose  $\mathcal{M}$  is a complete and cocomplete category. Suppose  $\mathcal{W}$  is a subcategory of  $\mathcal{M}$ , and  $I$  and  $J$  are sets of maps of  $\mathcal{M}$ . Then there is a cofibrantly generated model structure on  $\mathcal{M}$  with  $I$  as the set of generating cofibrations,  $J$  as the set of generating trivial cofibrations, and  $\mathcal{W}$  as the subcategory of weak equivalences if and only if the following conditions are satisfied:*

- (1) *the subcategory  $\mathcal{W}$  has the two-out-of-three property and is closed under retracts,*
- (2) *the domains of  $I$  are small relative to  $I$ -cell,*
- (3) *the domains of  $J$  are small relative to  $J$ -cell,*
- (4)  *$I$ -inj =  $\mathcal{W} \cap J$ -inj,*
- (5)  *$J$ -cof  $\subseteq \mathcal{W} \cap I$ -cof.*

**C.6. The category of complete gr-dg  $\mathbf{R}$ -modules.** We apply Theorem C.5 to endow the category of complete gr-dg  $\mathbf{R}$ -modules with a proper model structure. In this section, the category  $\mathcal{A}$  denotes the category of  $\mathbf{R}$ -modules.

We recall that a map  $p : (X, F) \rightarrow (Y, F')$  is strict when it satisfies  $p(F_q X) = p(X) \cap F'_q Y$  for all  $q$ . When  $p$  is a surjection, this means that  $p(F_q X) = F'_q Y$  for all  $q$ . We say that a filtered gr-dg  $\mathbf{R}$ -module  $(M, F)$  is *pure of weight  $q$*  if

$$M = F_0 M = F_q M \supseteq F_{q+1} M = 0.$$

We denote by  $\mathbf{R}^{(q)}$  the (complete) filtered  $\mathbf{R}$ -module given by  $\mathbf{R}$  concentrated in pure weight  $q$ . The notation  $\mathbf{R}_n^{(q)}$  means that we consider it in degree  $n$  within a (complete) filtered complex.

Taking notations close to the ones in [CESLW19], we define, for all  $n \in \mathbb{Z}$  and  $q \in \mathbb{N}$ , the complete gr-dg modules

$$\begin{aligned} \hat{\mathcal{Z}}_{q,n}^{0,\infty} &:= \left( \mathbf{R}_n^{(q)} \xrightarrow{1} \mathbf{R}_{n-1}^{(q)} \xrightarrow{1} \mathbf{R}_{n-2}^{(q+1)} \xrightarrow{1} \mathbf{R}_{n-3}^{(q+1)} \xrightarrow{1} \mathbf{R}_{n-4}^{(q+2)} \rightarrow \dots \right)^\wedge \\ &= \coprod_{k \in \mathbb{N}} \mathbf{R}_{n-k}^{(q + \lfloor \frac{k}{2} \rfloor)}, \end{aligned}$$

where  $\coprod$  is the coproduct in complete gr-dg modules, and

$$\begin{aligned} \hat{\mathcal{Z}}_{q,n}^{1,\infty} &:= \left( \mathbf{R}_n^{(q)} \xrightarrow{1} \mathbf{R}_{n-1}^{(q+1)} \xrightarrow{1} \mathbf{R}_{n-2}^{(q+1)} \xrightarrow{1} \mathbf{R}_{n-3}^{(q+2)} \xrightarrow{1} \mathbf{R}_{n-4}^{(q+2)} \rightarrow \dots \right)^\wedge \\ &= \coprod_{k \in \mathbb{N}} \mathbf{R}_{n-k}^{(q + \lceil \frac{k}{2} \rceil)}. \end{aligned}$$

We also define the complete gr-dg  $\mathbf{R}$ -module

$$\hat{\mathcal{B}}_{q,n}^{1,\infty} := \hat{\mathcal{Z}}_{q,n+1}^{0,\infty} \amalg \hat{\mathcal{Z}}_{q+1,n}^{0,\infty}.$$

We denote by  $\varphi_{q,n}^\infty : \hat{\mathcal{Z}}_{q,n}^{1,\infty} \rightarrow \hat{\mathcal{B}}_{q,n}^{1,\infty}$  the morphism of complete gr-dg modules defined by the following diagram

$$\begin{array}{ccccccc} \mathbf{R}_n^{(q)} & \longrightarrow & \mathbf{R}_{n-1}^{(q+1)} & \longrightarrow & \mathbf{R}_{n-2}^{(q+1)} & \longrightarrow & \dots \\ \downarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} & & \downarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} & & \downarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} & & \\ \mathbf{R}_{n+1}^{(q)} & \longrightarrow & \mathbf{R}_n^{(q)} \amalg \mathbf{R}_n^{(q+1)} & \longrightarrow & \mathbf{R}_{n-1}^{(q+1)} \amalg \mathbf{R}_{n-1}^{(q+1)} & \longrightarrow & \mathbf{R}_{n-2}^{(q+1)} \amalg \mathbf{R}_{n-2}^{(q+2)} \longrightarrow \dots \end{array}$$

In order to apply Theorem C.5, we consider the subcategory of  $\widehat{\text{Filt}}^{\text{gr}}(\mathcal{A})$

$$\mathcal{W} := \{ f : (M, F, d_M) \rightarrow (N, F', d_N) \mid f \text{ is a graded quasi-isomorphism} \}$$

of weak equivalences, and the sets

$$I_0^\infty := \{\varphi_{q,n}^\infty : \hat{Z}_{q,n}^{1,\infty} \rightarrow \hat{B}_{q,n}^{1,\infty}\}_{n \in \mathbb{Z}, q \in \mathbb{N}}$$

$$J_0^\infty := \{0 \rightarrow \hat{Z}_{q,n}^{0,\infty}\}_{n \in \mathbb{Z}, q \in \mathbb{N}}$$

of generating cofibrations and generating acyclic cofibrations.

In the following proposition, we characterize the morphisms having the right lifting property with respect to the morphisms in  $J_0^\infty$ . These morphisms will be the fibrations.

**Proposition C.7.** *A map  $p : (Y, F, d_Y) \rightarrow (X, F', d_X)$  has the right lifting property with respect to all the morphisms in  $J_0^\infty$  if and only if the map  $p : (Y, F, d_Y) \rightarrow (X, F', d_X)$  is a strict surjection.*

*Proof.* For every  $q$ , a diagram of the form

$$\begin{array}{ccc} 0 & \longrightarrow & (Y, F) \\ \downarrow & & \downarrow p \\ \hat{Z}_{q,n}^{0,\infty} & \longrightarrow & (X, F') \end{array}$$

is characterized by an element  $x^q \in F'_q X_n$ . A lift in this diagram is equivalent to an element  $y^q$  in  $F_q Y_n$  such that  $p(y^q) = x^q$ .  $\square$

We provide an equivalent description by means of the functor  $\text{Gr}$ .

**Lemma C.8.** *A map  $p : (Y, F) \rightarrow (X, F')$  satisfies  $p_n : Y_n \rightarrow X_n$  is a strict surjection for all  $n \in \mathbb{Z}$  if and only if the map  $(\text{Gr}_q p)_n : \text{Gr}_q Y_n \rightarrow \text{Gr}_q X_n$  is surjective for all  $q \in \mathbb{N}$  and for all  $n \in \mathbb{Z}$ .*

*Proof.* The direct implication is immediate. The reverse implication is given by an induction (we need  $Y$  to be complete for the filtration  $F$  in order to prove this fact). Indeed, assume that  $\text{Gr}_q p$  is surjective for all  $q$  and let  $x \in X$  and  $q = 0$ . If  $\text{Gr}_q x = 0$ , then we fix  $y_q = 0$ . Otherwise, there exists by assumption  $y_q \in F_q Y$  such that  $\text{Gr}_q p(\text{Gr}_q y_q) = \text{Gr}_q(x)$ . The element  $x - p(y_q)$  is in  $F_{q+1} X$ . This reasoning works for any  $q$  provided that  $x \in F_q X$ . By induction and by the fact that  $Y$  is complete, we get that  $p(\sum_q y_q) = x$ . This proof works for  $F_q p$  instead of  $p$  so we get that  $p$  is a strict surjection.  $\square$

We therefore sometimes write *gr-surjection* for a strict surjection.

In the following lemma, we describe the pushouts in the category  $\widehat{\text{Filt}}^{\text{gr}}(\mathcal{A})$  and some properties related to them.

**Lemma C.9.** *Let  $f : X \rightarrow Y$  and  $g : X \rightarrow Y'$  be morphisms in  $\text{Filt}^{\text{gr}}(\mathcal{A})$ . Then*

(1) *the pushout*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow g & \lrcorner & \downarrow g' \\ Y' & \xrightarrow{f'} & Y' \amalg_X Y \end{array}$$

*in  $\text{Filt}^{\text{gr}}(\mathcal{A})$  is given by*

$$Y' \amalg_X Y := ((Y' \amalg Y)/(g(x) - f(x); x \in X), F, d).$$

The filtration  $F$  is given by

$$F_p(Y' \amalg_X Y) := \text{im} \left( F_p Y' \amalg F_p Y \xrightarrow{f'-g'} (Y' \amalg Y) / (g(x) - f(x); x \in X) \right),$$

and the predifferential  $d$  is induced by the predifferentials on  $Y'$  and  $Y$ ;

- (2) if  $f$  is a monomorphism, so is  $f$ ;
- (3) if  $f$  is a strict morphism, so is  $f'$ .

Using the completion map (reflector), the morphisms  $f$  and  $g$  induces morphisms  $\hat{f} : \hat{X} \rightarrow \hat{Y}$  and  $\hat{g} : \hat{X} \rightarrow \hat{Y}'$  in  $\widehat{\text{Filt}}^{\text{gr}}(\mathcal{A})$ . Then

- (a) the pushout

$$\begin{array}{ccc} \hat{X} & \xrightarrow{\hat{f}} & \hat{Y} \\ \downarrow \hat{g} & \lrcorner & \downarrow \hat{g}' \\ \hat{Y}' & \xrightarrow{\hat{f}'} & \hat{Y}' \amalg_{\hat{X}} \hat{Y} \end{array}$$

in  $\widehat{\text{Filt}}^{\text{gr}}(\mathcal{A})$  is given by the completion of the pushout  $Y' \amalg_X Y$  in  $\text{Filt}^{\text{gr}}(\mathcal{A})$ ;

- (b) if  $f$  is a monomorphism (resp. strict morphism), so is  $f$ ;
- (c) when  $f$  is a strict monomorphism, the pushout  $\hat{Y}' \amalg_{\hat{X}} \hat{Y}$  computed in  $\widehat{\text{Filt}}^{\text{gr}}(\mathcal{A})$  is already complete. That is to say the inclusion map, adjoint to the completion map, preserves this pushout.

*Proof.* (1) The pushout is obtained by the cokernel of the map  $X \xrightarrow{g-f} Y' \amalg Y$ . It is computed as the pushout in the category of diagrams  $\mathcal{A}^{\text{nop}}$  to which we apply the reflector  $\mathcal{A}^{\text{nop}} \rightarrow \text{Filt}(\mathcal{A})$ . We recall that  $\mathcal{A} = \mathbf{R}$ -modules. This extends to gr-dg modules.

- (2) This is direct from the description of  $Y' \amalg_X Y$ .
- (3) The statement follows from the fact that

$$(f')^{-1}(F_p(Y' \amalg_X Y)) = g(f^{-1}(F_p B)) + F_p C.$$

- (a) Colimits in  $\widehat{\text{Filt}}^{\text{gr}}(\mathcal{A})$  are computed in this way.
- (b) This follows from (2) and (3) above.
- (c) When  $\hat{f}$  is a strict monomorphism, so is  $\hat{X} \xrightarrow{\hat{g}-\hat{f}} \hat{Y}' \amalg \hat{Y}$ . Thus the filtration on the pushout

$$\hat{Y}' \amalg_{\hat{X}} \hat{Y} = \widehat{Y' \amalg Y} / (\widehat{g-f})(X)$$

is given by

$$\begin{aligned} F_p(\hat{Y}' \amalg_{\hat{X}} \hat{Y}) &= (F_p \hat{Y}' \amalg F_p \hat{Y}) / F_p((\hat{g} - \hat{f})(X)) \\ &= F_p(\widehat{Y' \amalg Y}) / F_p((\widehat{g-f})(X)). \end{aligned}$$

It is the quotient filtration induced by the filtration on  $\widehat{Y' \amalg Y}$  and the pushout is already complete.

□

**Lemma C.10.** *We have the following pushout diagram*

$$\begin{array}{ccc} \hat{\mathcal{Z}}_{q,n}^{1,\infty} & \longrightarrow & 0 \\ \downarrow \varphi_{q,n}^\infty & \lrcorner & \downarrow \\ \hat{\mathcal{B}}_{q,n}^{1,\infty} & \longrightarrow & \hat{\mathcal{Z}}_{q,n}^{1,\infty}. \end{array}$$

Moreover, if the map  $p$  has the right lifting property with respect to the maps  $\{0 \rightarrow \hat{\mathcal{Z}}_{q,n}^{1,\infty}\}$ , then the map  $\text{Gr } p$  is surjective on cycles. Under the assumption that  $p$  is a strict surjection, the reverse is true.

*Proof.* The first part of the lemma follows from Lemma C.9.

Then, the set of diagrams

$$\begin{array}{ccc} 0 & \longrightarrow & (Y, F) \\ \downarrow & & \downarrow p \\ \hat{\mathcal{Z}}_{q,n}^{1,\infty} & \longrightarrow & (X, F'), \end{array} \quad (10)$$

corresponds (bijectively) to the set

$$A_{q,n} := \{x \in F'_q X_n; d_X(x) \in F'_{q+1} X\}.$$

The set of lifts in such diagrams corresponds (bijectively) to the set

$$A_{q,n} := \{(x, y) \in F'_q X_n \amalg F_q Y_n; d_Y(y) \in F_{q+1} Y\}.$$

It is clear that if  $p$  has the right lifting property with respect to the maps  $\{0 \rightarrow \hat{\mathcal{Z}}_{q,n}^{1,\infty}\}$ , then  $\text{Gr } p$  is surjective on cycles. Let assume that  $\text{Gr } p$  is surjective on cycles and  $x \in F'_q X_n$  such that  $d_X(x) \in F'_{q+1} X$ , that is the data of a diagram (10). Using the fact that  $\text{Gr } p$  is surjective on cycles, we get that there exists  $y \in F_q Y_n$  such that  $d_Y(y) \in F_{q+1} Y$  and  $p(y) = x + x^{q+1}$ , with  $x^{q+1} \in F'_{q+1} X_n$ . Under the assumption that  $p$  is a strict surjection, we get that there exists  $y^{q+1} \in F_{q+1} Y_n$  such that  $p(y^{q+1}) = x^{q+1}$ . Finally,  $y - y^{q+1}$  provides the requested lift.  $\square$

We can now characterize what will be the trivial fibrations. We recall that

$$I_0^\infty = \{\varphi_{q,n}^\infty : \hat{\mathcal{Z}}_{q,n}^{1,\infty} \rightarrow \hat{\mathcal{B}}_{q,n}^{1,\infty}\}_{n \in \mathbb{Z}, q \in \mathbb{N}}.$$

**Proposition C.11.** *A map  $p : (Y, F, d_Y) \rightarrow (X, F', d_X)$  has the right lifting property with respect to all the maps in  $I_0^\infty$  if and only if the map  $p$  it is a strict surjection and a graded quasi-isomorphism. In particular,  $I_0^\infty\text{-inj} = J_0^\infty\text{-inj} \cap \mathcal{W}$ .*

*Proof.* The map  $0 \rightarrow \hat{\mathcal{B}}_{q,n}^{1,\infty} = \hat{\mathcal{Z}}_{q,n+1}^{0,\infty} \amalg \hat{\mathcal{Z}}_{q+1,n}^{0,\infty}$  is the composition of the maps  $0 \rightarrow \hat{\mathcal{Z}}_{q,n}^{1,\infty}$  and  $\hat{\mathcal{Z}}_{q,n}^{1,\infty} \rightarrow \hat{\mathcal{B}}_{q,n}^{1,\infty}$ . By the pushout diagram presented in Lemma C.10, we obtain  $I_0^\infty\text{-inj} = I_0^\infty \cup J_0^\infty\text{-inj}$  (since  $0 \rightarrow \hat{\mathcal{Z}}_{q,n+1}^{0,\infty}$  is a retract of  $0 \rightarrow \hat{\mathcal{B}}_{q,n}^{1,\infty}$ ). It follows that maps in  $I_0^\infty\text{-inj}$  are in particular strict surjections (by Proposition C.7).

We now characterize the diagrams

$$\begin{array}{ccc} \hat{\mathcal{Z}}_{q,n}^{1,\infty} & \longrightarrow & (Y, F) \\ \downarrow \varphi_{q,n}^\infty & & \downarrow p \\ \hat{\mathcal{B}}_{q,n}^{1,\infty} & \longrightarrow & (X, F'), \end{array}$$

in  $\widehat{\text{Filt}}^{\text{gr}}(\mathbf{R}\text{-Mod})$  admitting a lifting. The set of such diagrams corresponds (bijectively) to the set

$$B_{q,n} := \left\{ \begin{array}{l} (t, x, y) \in F'_q X_{n+1} \amalg F'_{q+1} X_n \amalg F_q Y_n; \\ p(y) = d_X(t) + x \text{ and } d_Y(y) \in F_{q+1} Y \end{array} \right\}.$$

The set of such diagrams admitting a lifting is in bijection with

$$B'_{q,n} := \left\{ \begin{array}{l} (t, x, y, z) \in F'_q X_{n+1} \amalg F'_{q+1} X_n \amalg F_q Y_n \amalg F_q Y_{n+1}; \\ p(z) = t, p(y) = d_X(t) + x \text{ and } y - d_Y(z) \in F_{q+1} Y \end{array} \right\},$$

since the different conditions satisfied by the tuples in  $B'_{q,n}$  already implies that  $d_Y(y) \in F_{q+1} Y$ .

First, we suppose that  $p$  has the right lifting property with respect to maps in  $I_0^\infty$ . It remains to prove that  $p$  is a graded quasi-isomorphism. By Lemma C.10, we have that  $\text{Gr } p$  is surjective on cycles, so that  $\mathbf{H}_\bullet(\text{Gr } p)$  is surjective as well. We then prove that  $\mathbf{H}_\bullet(\text{Gr } p)$  is injective. Let  $\bar{y} \in \ker \mathbf{H}_n(\text{Gr } p)$ , that is there exists a lift  $y \in F_q Y_n$  such that  $d_Y(y) \in F_{q+1} Y_{n-1}$  and  $p(y) \in \text{im } d_X|_{F_q X} + F_{q+1} X$ . We fix  $p(y) = d_X(x^q) + x^{q+1}$  for some  $x^q \in F_q X$  and  $x^{q+1} \in F_{q+1} X$ . This is precisely the data of an element in  $B_{q,n}$ . By the lifting property, we obtain  $z \in F_q Y_{n+1}$  such that  $y - d_Y(z) \in F_{q+1} Y$ . It follows that the class of  $y = d_Y(z) + (y - d_Y(z))$  is 0 in  $\mathbf{H}_\bullet \text{Gr } Y$  and  $\mathbf{H}_\bullet(\text{Gr } p)$  is injective.

Conversely, assume that  $p$  is a strict surjection and a graded quasi-isomorphism. It remains to show that  $p$  has the right lifting property with respect to the maps  $\{\hat{Z}_{q,n}^{1,\infty} \rightarrow \hat{B}_{q,n}^{1,\infty}\}_{q,n}$ . Let  $(t, x, y) \in B_{q,n}$ . By means of the fact that  $p$  is a strict surjection, we get that there exists  $z \in F_q Y_{n+1}$  such that  $p(z) = t$ . If  $y - d_Y(z) \in F_{q+1} Y$ , we have found the requested lift. Otherwise  $y - d_Y(z)$  provides a non zero element in  $\text{Gr } q Y$ . The fact that  $p$  is a strict surjection implies that  $\text{Gr } q p$  is surjective. We therefore get a short exact sequence

$$0 \rightarrow K \rightarrow \text{Gr } q Y \rightarrow \text{Gr } q X \rightarrow 0$$

in dg-modules where  $K$  is the kernel of  $\text{Gr } q p$ . By the associated long exact sequence and the fact that  $\text{Gr } q p$  is a quasi-isomorphism, we obtain that  $K$  is acyclic. We have  $\bar{y} - d_{\text{Gr } Y}(\bar{z}) \in Z_{n-1}(K)$  since  $d_Y(y - d_Y(z)) \in F_{q+1} Y$  and  $p(y - d_Y(z)) = x \in F'_{q+1} X$ . So there exist  $u^q \in F_q Y_{n+1}$  such that  $p(u^q) \in F_{q+1} X$  and  $y^{q+1} \in F_{q+1} Y_n$  with the property that  $y - d_Y(z) = d_Y(u^q) + y^{q+1}$ . Using the fact that  $p$  is a strict surjection, we obtain that there exists  $u^{q+1} \in F_{q+1} Y_{n+1}$  such that  $p(u^{q+1}) = p(u^q)$ . Finally, we have

$$p(z + u^q - u^{q+1}) = t \text{ and } y - d_Y(z + u^q - u^{q+1}) = y^{q+1} - u^{q+1} \in F_{q+1} Y.$$

This provides the requested lift.  $\square$

**Definition C.12.** Let  $f : (Y, F, d_Y) \rightarrow (X, F', d_X)$  be a map of gr-dg  $\mathbf{R}$ -modules. We denote by  $C(f)$  the mapping cone of the map  $f$  defined by

$$F_q C(f)_n := F_q Y_{n-1} \amalg F_q X_n,$$

with the predifferential  $D(y, x) = (-d_Y(y), f(y) + d_X(x))$ .

**Lemma C.13.** *The cone  $C(\text{id}_Y)$  of the identity map is gr-acyclic (that is to say its gr-homology is 0).*

*Proof.* We have  $\text{Gr}(D)(\bar{y}, \bar{z}) = (-\bar{d}\bar{y}, \bar{y} + \bar{d}\bar{z})$  so  $\text{Gr}(D)(\bar{y}, \bar{z}) = 0$  if and only if  $\bar{d}\bar{y} = 0$  and  $\bar{y} = -\bar{d}\bar{z}$ . This is equivalent to the fact that  $(\bar{y}, \bar{z}) = \text{Gr}(D)(\bar{z}, 0)$ . Hence  $C(\text{id}_Y)$  is gr-acyclic.  $\square$

**Proposition C.14.** *We have  $J_0^\infty\text{-cof} \subseteq \mathcal{W}$ .*

*Proof.* We follow the proof given in [CESLW19]. Let  $f : A \rightarrow B$  be a  $J_0^\infty$ -cofibration. By Proposition C.7, this means that  $f$  has the left lifting property with respect to the maps  $p$  which are strict surjections. We consider the diagram

$$\begin{array}{ccc} A & \longrightarrow & A \amalg C(\text{id}_B)[-1] \\ \downarrow f & & \downarrow (f, \pi_1) \\ B & \xrightarrow{=} & B, \end{array}$$

where  $\pi_1 : C(\text{id}_B)[-1] = B_\bullet \amalg B_{\bullet+1} \rightarrow B_\bullet$  is the projection on the first factor. The map  $\pi_1$  is a strict surjection. It follows that the diagram admits a lift  $h : B \rightarrow A \amalg C(\text{id}_B)[-1]$ . Applying the functor  $\text{Gr}$  to the diagram and using the fact that the cone is gr-acyclic (by Lemma C.13), the two commutative triangles give that  $f$  is a graded quasi-isomorphism.  $\square$

**Theorem C.15.** *The category  $\widehat{\text{Filt}}^{\text{gr}}(\mathbf{R}\text{-Mod})$  of gr-dg  $\mathbf{R}$ -modules admits a proper cofibrantly generated model category structure, where:*

- (1) *weak equivalences are graded quasi-isomorphisms,*
- (2) *fibrations are strict surjections, and*
- (3)  *$I_0^\infty$  and  $J_0^\infty$  are the sets of generating cofibrations and generating acyclic cofibrations respectively.*

*Proof.* We apply Theorem C.5. The graded quasi-isomorphisms satisfy the two-out-of-three property. It follows that the subcategory  $\mathcal{W}$  satisfies the two-out-of-three property. This category is also closed under retract since graded quasi-isomorphisms are closed under retract. The non trivial domains of the maps in  $I_0^\infty$  and in  $J_0^\infty$  are  $\hat{\mathcal{Z}}_{q,n}^{1,\infty}$ . The complete gr-dg modules  $\hat{\mathcal{Z}}_{q,n}^{1,\infty}$  are  $\aleph_1$ -small since completion commutes with  $\aleph_1$ -filtered colimits. Finally, Propositions C.11 and C.14 ensure that we can apply Theorem C.5 and we obtain the wanted cofibrantly generated model structure. Using Proposition C.7, we can see that every object is fibrant. So by [Hir03, Corollary 13.1.3] the model category structure is right proper. It remains to show that the model structure is left proper, that is to say that weak equivalences are preserved by pushout along cofibrations. It is enough to show that for any diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y' \\ \downarrow g & & \downarrow g' \\ X'' & \longrightarrow & Y'', \end{array}$$

in which both squares are cocartesian,  $f$  belongs to  $I_0^\infty$ , and  $g$  belongs to  $\mathcal{W}$ , the map  $g'$  also belongs to  $\mathcal{W}$ . We first show that it is true in the category of chain complexes of  $\mathbf{R}$ -modules. We show that if  $f$  is injective and  $g$  is a quasi-isomorphism, we obtain that the map  $g'$  is a quasi-isomorphism. This follows first

from the fact that in the category of chain complexes pushout along an injection  $f : X \rightarrow Y$  provides an injection  $X' \rightarrow Y'$ . Then in this situation, the second pushout gives a short exact sequence

$$0 \rightarrow X' \rightarrow Y' \amalg X'' \rightarrow Y'' \rightarrow 0$$

which provides a long exact sequence in homology. Finally, the fact that  $g$  is a quasi-isomorphism implies that so is  $g'$ .

We consider the functor

$$\begin{aligned} \text{Gr} : \widehat{\text{Filt}}^{\text{gr}}(\mathcal{A}) &\rightarrow (\text{dg}\mathcal{A})^{\text{ob}\mathbb{N}}, \\ (V, F, d_V) &\mapsto (V, F, d_V)^{\text{gr}} = (\text{Gr } V, d_{\text{Gr } V}). \end{aligned}$$

Given a map  $f = \varphi_{q,n}^\infty$  in  $I_0^\infty$ , we show that the two cocartesian squares in  $\widehat{\text{Filt}}^{\text{gr}}(\mathcal{A})$  give two cocartesian squares under  $\text{Gr}$ . The functor  $\text{Gr}$  is the composition of the inclusion functor  $i_1 : \widehat{\text{Filt}}^{\text{gr}}(\mathcal{A}) \rightarrow \text{Filt}(\text{pg}\mathcal{A})$ , the inclusion functor  $i_2 : \text{Filt}(\text{pg}\mathcal{A}) \rightarrow (\text{pg}\mathcal{A})^{\text{N}^{\text{op}}}$  and the quotient functor  $q_1 : (\text{pg}\mathcal{A})^{\text{N}^{\text{op}}} \rightarrow (\text{dg}\mathcal{A})^{\text{ob}\mathbb{N}}$ . All the maps in  $I_0^\infty$  are strict monomorphisms. From Lemma C.9, we get that the inclusion  $i_1$  preserves pushouts in which one map is a strict monomorphism (the inclusion  $\widehat{\text{Filt}}^{\text{gr}}(\mathcal{A}) \rightarrow \text{Filt}(\text{pg}\mathcal{A})$  preserves the pushouts). Moreover, when  $f$  is a strict monomorphism, so is the map  $X' \rightarrow Y'$ . Therefore the inclusion  $i_1$  sends the two cocartesian squares to two cocartesian squares. Provided that the maps  $f$  and  $X' \rightarrow Y'$  are strict morphisms, Proposition 1.1.11 in [Del71] shows that the functor  $q_1 \cdot i_2$  also sends the two cocartesian squares to two cocartesian squares (the predifferentials don't affect this property). Finally, the functor  $\text{Gr}$  sends maps in  $I_0^\infty$  to cofibrations in  $(\text{dg}\mathcal{A})^{\text{ob}\mathbb{N}}$  (objectwise injections with free, hence projective, cokernel) and maps in  $\mathcal{W}$  to quasi-isomorphisms in  $(\text{dg}\mathcal{A})^{\text{ob}\mathbb{N}}$  so this shows that  $\text{Gr } g'$  is a quasi-isomorphism, that is  $g'$  belongs to  $\mathcal{W}$ .  $\square$

**C.16. Properties of the model structure on gr-dg  $\mathbf{R}$ -modules.** We prove that the model category structure on gr-dg  $\mathbf{R}$ -modules is combinatorial and is a monoidal model structure.

**Definition C.17.** Let  $\mathcal{M}$  be a category endowed with a model structure. We say that  $\mathcal{M}$  is *combinatorial* if it is

- (1) locally presentable as a category, and
- (2) cofibrantly generated as a model category.

**Lemma C.18.** *The category of complete gr-dg  $\mathbf{R}$ -modules is a (locally) presentable category. As a consequence, the category  $\widehat{\text{Filt}}^{\text{gr}}(\mathcal{A})$  is combinatorial.*

*Proof.* The category of  $\mathcal{A} = \mathbf{R}$ -modules is locally presentable (see for instance Example 5.2.2.a in [Bor94]). Applying [AR94, Corollary 1.54], the category  $\mathcal{A}^{\text{N}^{\text{op}}}$  of  $\mathbb{N}^{\text{op}}$ -indexed diagrams is locally presentable. By Lemma A.3 and Corollary A.10, the category  $\widehat{\text{Filt}}(\mathcal{A})$  of complete filtered objects in  $\mathcal{A}$  is a reflective subcategory of the category of  $\mathbb{N}^{\text{op}}$ -indexed diagrams. The monad  $T$  associated with the adjunction sends an  $\mathbb{N}^{\text{op}}$ -indexed diagrams  $\{X(p)\}_p$  to the completion  $\hat{X}$  (seen in  $\mathcal{A}^{\text{N}^{\text{op}}}$ ) of its associated filtered object  $\tilde{X}$  defined by  $F_p \tilde{X} = \text{im}(X(p) \rightarrow X(0))$ . Let us show that the monad  $T$  commutes with filtered colimits. First, the image of a map can be written as the equalizer of a cokernel pair. It therefore commutes with filtered colimits since colimits commute colimits and in the category  $\mathcal{A}$ , finite limits commute with filtered colimits (Corollary 3.4.3 in [Bor94]). Then, the completion



does not commute with filtered colimits but it commutes with  $\kappa$ -filtered colimits, for  $\kappa$  a regular cardinal such that  $\kappa > \aleph_0$  since the completion is defined in the locally presentable category  $\mathcal{A}$  by a diagram obtained by means of two cokernels, a kernel and a limit indexed by  $\aleph_0$  (see for example Corollary 5.2.8 in [Bor94]). It follows that  $T$  commutes with  $\aleph_1$ -filtered colimits and therefore by Theorem 5.5.9 in [Bor94], the category  $\widehat{\text{Filt}}(\mathcal{A})$  of complete filtered objects in  $\mathcal{A}$  is locally presentable. Again by Corollary 1.54 in [AR94], the category  $\widehat{\text{Filt}}(\mathcal{A})^{\text{ob}\mathbb{Z}}$  of graded complete objects in  $\mathcal{A}$  is locally presentable. Let  $A = \mathbf{R}[[d]]$  be the  $\mathbf{R}$ -algebra of formal power series generated by  $d$  of degree  $-1$  and endowed with the filtration  $F_p A = d^{2p}\mathbf{R}[[d]]$  for all  $p \geq 0$ . It is complete for the filtration and therefore provides an algebra in  $\widehat{\text{Filt}}(\mathcal{A})^{\mathbb{Z}^{\text{disc}}}$ . The category of complete gr-dg  $\mathbf{R}$ -modules is the category of  $A$ -modules in  $\widehat{\text{Filt}}(\mathcal{A})^{\mathbb{Z}^{\text{disc}}}$ . Again by Example 5.2.2.a in [Bor94], we obtain that  $\widehat{\text{Filt}}^{\text{gr}}(\mathcal{A})$  is locally presentable.  $\square$

We recall from [Hov99, Definition 4.2.6] the notion of symmetric monoidal model category and we prove that the category  $\widehat{\text{Filt}}^{\text{gr}}(\mathcal{A})$  is a symmetric monoidal model category.

**Definition C.19.** We say that a symmetric monoidal category  $(\mathcal{M}, \otimes, \mathbb{1})$  endowed with a model structure is a *symmetric monoidal model category* when

- (1) the natural morphism

$$i_1 \square i_2 : (X_1 \otimes A_2) \amalg_{(A_1 \otimes A_2)} (A_1 \otimes X_2) \rightarrow X_1 \otimes X_2$$

induced by cofibrations  $i_1 : A_1 \rightarrow X_1$  and  $i_2 : A_2 \rightarrow X_2$  forms a cofibration, respectively an acyclic cofibration if  $i_1$  or  $i_2$  is also acyclic.

- (2) Let  $Q\mathbb{1} \xrightarrow{\pi} \mathbb{1}$  be the cofibrant replacement for the unit obtained by using the functorial factorization to factor  $0 \rightarrow \mathbb{1}$  into a cofibration followed by a trivial fibration. Then the natural map  $Q\mathbb{1} \otimes X \xrightarrow{\pi \otimes \text{id}} \mathbb{1} \otimes X$  is a weak equivalence for all cofibrant  $X$ . Similarly, the natural map  $X \otimes Q\mathbb{1} \xrightarrow{\text{id} \otimes \pi} X \otimes \mathbb{1}$  is a weak equivalence for all cofibrant  $X$ .

**Proposition C.20.** *The category  $\widehat{\text{Filt}}^{\text{gr}}(\mathcal{A})$  is a symmetric monoidal model category.*

*Proof.* To prove the first condition of Definition C.19, by [Hov99, Corollary 4.2.5], it is enough to prove the claim in the case of generating (acyclic) cofibrations.

- (1) We first consider the cofibrations  $i_1 = \varphi_{q,n}^\infty : \hat{\mathcal{Z}}_{q,n}^{1,\infty} \rightarrow \hat{\mathcal{B}}_{q,n}^{1,\infty}$  and  $i_2 = \varphi_{p,m}^\infty$ . The generators, as an  $\mathbf{R}$ -module, of  $X_1 \otimes X_2$  are

$$\{d^k 1_{n+1}^q \otimes d^l 1_{m+1}^p, d^k 1_n^{q+1} \otimes d^l 1_{m+1}^p, d^k 1_{n+1}^q \otimes d^l 1_m^{p+1}, d^k 1_n^{q+1} \otimes d^l 1_m^{p+1}\}_{k,l \in \mathbb{N}},$$

whereas the ones of  $(X_1 \otimes A_2) \amalg_{(A_1 \otimes A_2)} (A_1 \otimes X_2)$  are

$$Z := \{d^k (d 1_{n+1}^q + 1_n^{q+1}) \otimes d^l 1_{m+1}^p, d^k (d 1_{n+1}^q + 1_n^{q+1}) \otimes d^l 1_m^{p+1}, d^k 1_{n+1}^q \otimes d^l (d 1_{m+1}^p + 1_m^{p+1}), d^k 1_n^{q+1} \otimes d^l (d 1_{m+1}^p + 1_m^{p+1})\}_{k,l \in \mathbb{N}}.$$

We prove by induction on the weight grading that the map

$$i_1 \square i_2 : (X_1 \otimes A_2) \amalg_{(A_1 \otimes A_2)} (A_1 \otimes X_2) \rightarrow X_1 \otimes X_2$$

can be written as the coproduct of some cofibrations  $\varphi_{r_s, o_s}^\infty$  and  $\varphi_{r_s, o_s-1}^\infty$ . We first describe  $\varphi_{r_0, o_0}^\infty$  and  $\varphi_{r_0, o_0-1}^\infty$ . We have

$$(d1_{n+1}^q + 1_n^{q+1}) \otimes 1_{m+1}^p + (-1)^{n+1} 1_{n+1}^q \otimes (d1_{m+1}^p + 1_m^{p+1}) = d(1_{n+1}^q \otimes 1_{m+1}^p) + (1_n^{q+1} \otimes 1_{m+1}^p + (-1)^{n+1} 1_{n+1}^q \otimes 1_m^{p+1}).$$

This gives a cofibration  $\varphi_{r_0, o_0}^\infty$  where  $r_0 = q + p$  and  $o_0 = n + m + 1$ , and

- $(d1_{n+1}^q + 1_n^{q+1}) \otimes 1_{m+1}^p + (-1)^{n+1} 1_{n+1}^q \otimes (d1_{m+1}^p + 1_m^{p+1})$  is the generator of  $\hat{Z}_{r_0, o_0}^{1, \infty}$ ,
- $1_{n+1}^q \otimes 1_{m+1}^p$  is the generator of  $\hat{Z}_{r_0, o_0+1}^{0, \infty}$ , and
- $1_n^{q+1} \otimes 1_{m+1}^p - (-1)^n 1_{n+1}^q \otimes 1_m^{p+1}$  is the generator of  $\hat{Z}_{r_0+1, o_0}^{0, \infty}$ .

Similarly,

- the element in  $\hat{Z}_{r_0, o_0-1}^{1, \infty}$

$$z_{n+m}^{p+q} = d(1_{n+1}^q \otimes (d1_{m+1}^p + 1_m^{p+1})) + (-1)^n d((d1_{n+1}^q + 1_n^{q+1}) \otimes 1_{m+1}^p) - ((d1_{n+1}^q + 1_n^{q+1}) \otimes 1_m^{p+1} + 1_n^{q+1} \otimes (d1_{m+1}^p + 1_m^{p+1})),$$

- the element  $b_{n+m+1}^{p+q} = 1_{n+1}^q \otimes d1_{m+1}^p + (-1)^n d1_{n+1}^q \otimes 1_{m+1}^p$  in  $\hat{Z}_{r_0, o_0}^{0, \infty}$ , and
- the element  $b_{n+m}^{p+q+1} = (-1)^{n+1} (1_{n+1}^q \otimes d1_m^{p+1} - d1_n^{q+1} \otimes 1_{m+1}^p) - 2 \times 1_n^{q+1} \otimes 1_{m+1}^p$  in  $\hat{Z}_{r_0+1, o_0-1}^{0, \infty}$

fit into the equality  $z_{n+m+1}^{p+q} = db_{n+m+2}^{p+q} + b_{n+m+1}^{p+q+1}$  and therefore give a cofibration  $\varphi_{r_0, o_0-1}^\infty$ . Let us show that we have the equality

$$\langle d^k 1_{n+1}^q \otimes d^l 1_{m+1}^p, \rangle_{k, l \in \mathbb{N}} \cap \text{im}(i_1 \square i_2) = \{0\}. \quad (11)$$

The term  $d^k 1_{n+1}^q \otimes d^l 1_{m+1}^p$  appears only in two terms of  $Z$  which are  $d^{k-1}(d1_{n+1}^q + 1_n^{q+1}) \otimes d^l 1_{m+1}^p$  and  $d^k 1_{n+1}^q \otimes d^{l-1}(d1_{m+1}^p + 1_m^{p+1})$ . The terms  $d^{k-1} 1_n^{q+1} \otimes d^l 1_{m+1}^p$  and  $d^k 1_{n+1}^q \otimes d^{l-1} 1_m^{p+1}$  appear only in one other term each. We get the two equalities:

$$d^{k-1}(d1_{n+1}^q + 1_n^{q+1}) \otimes d^l 1_{m+1}^p - d^{k-1} 1_n^{q+1} \otimes d^{l-1}(d1_{m+1}^p + 1_m^{p+1}) = d^k 1_{n+1}^q \otimes d^l 1_{m+1}^p - d^{k-1} 1_n^{q+1} \otimes d^{l-1} 1_m^{p+1}$$

and

$$d^k 1_{n+1}^q \otimes d^{l-1}(d1_{m+1}^p + 1_m^{p+1}) - d^{k-1}(d1_{n+1}^q + 1_n^{q+1}) \otimes d^{l-1} 1_m^{p+1} = d^k 1_{n+1}^q \otimes d^l 1_{m+1}^p - d^{k-1} 1_n^{q+1} \otimes d^{l-1} 1_m^{p+1}.$$

The term  $d^{k-1} 1_n^{q+1} \otimes d^{l-1} 1_m^{p+1}$  doesn't appear in an other term in  $Z$ . This proves Equality (11). This implies that the element  $1_{n+1}^q \otimes 1_{m+1}^p$  and its derivatives don't belong to  $\text{im}(i_1 \square i_2)$ . This also gives that  $b_{n+m+1}^{p+q}$  and its derivatives don't belong to  $\text{im}(i_1 \square i_2) \amalg \text{im} \varphi_{r_0, o_0}^\infty$ . Let  $S_0$  be an  $\mathbf{R}$ -linear complement of the domains of  $\varphi_{r_0, o_0}^\infty$  and  $\varphi_{r_0, o_0-1}^\infty$  in  $(X_1 \otimes A_2) \amalg_{(A_1 \otimes A_2)} (A_1 \otimes X_2)$ . It is a tedious but direct computation to show that modulo  $F_{q+p+2}(X_1 \otimes X_2)$ , the map  $i_1 \square i_2$  is given by  $\text{id}_{S_0} \amalg \varphi_{r_0, o_0}^\infty \amalg \varphi_{r_0, o_0-1}^\infty$  (that is to say the codomain of  $\text{id}_{S_0} \amalg \varphi_{r_0, o_0}^\infty \amalg \varphi_{r_0, o_0-1}^\infty$  is equal to  $X_1 \otimes X_2$  modulo  $F_{q+p+2}(X_1 \otimes X_2)$ ). This is the base case of the induction.

We define similarly the maps  $\varphi_{r_s, o_s}^\infty$  and  $\varphi_{r_s, o_s-1}^\infty$ . We have

$$\begin{aligned} d^{2s}(d1_{n+1}^q + 1_n^{q+1}) \otimes d^{2s}1_{m+1}^p + (-1)^{n+1}d^{2s}1_{n+1}^q \otimes d^{2s}(d1_{m+1}^p + 1_m^{p+1}) \\ = d(d^{2s}1_{n+1}^q \otimes d^{2s}1_{m+1}^p) \\ + d^{2s}1_n^{q+1} \otimes d^{2s}1_{m+1}^p + (-1)^{n+1}d^{2s}1_{n+1}^q \otimes d^{2s}1_m^{p+1}. \end{aligned}$$

This gives a cofibration  $\varphi_{r_s, o_s}^\infty$  where  $r_s = q+p+2s$  and  $o_s = n+m+1-4s$ , and

- $d^{2s}(d1_{n+1}^q + 1_n^{q+1}) \otimes d^{2s}1_{m+1}^p + (-1)^{n+1}d^{2s}1_{n+1}^q \otimes d^{2s}(d1_{m+1}^p + 1_m^{p+1})$  is the generator of  $\hat{Z}_{r_s, o_s}^{1, \infty}$ ,
- $d^{2s}1_{n+1}^q \otimes d^{2s}1_{m+1}^p$  is the generator of  $\hat{Z}_{r_s, o_s+1}^{0, \infty}$ , and
- $d^{2s}1_n^{q+1} \otimes d^{2s}1_{m+1}^p - (-1)^n d^{2s}1_{n+1}^q \otimes d^{2s}1_m^{p+1}$  is the generator of  $\hat{Z}_{r_s+1, o_s}^{0, \infty}$ .

Then,

- the element in  $\hat{Z}_{r_s, o_s-1}^{1, \infty}$

$$\begin{aligned} z_{n+m-4s}^{p+q+2s} = d(d^{2s}1_{n+1}^q \otimes d^{2s}(d1_{m+1}^p + 1_m^{p+1})) + (-1)^n (d^{2s}(d1_{n+1}^q + 1_n^{q+1}) \otimes d^{2s}1_{m+1}^p) \\ - (d^{2s}(d1_{n+1}^q + 1_n^{q+1}) \otimes d^{2s}1_m^{p+1} + d^{2s}1_n^{q+1} \otimes d^{2s}(d1_{m+1}^p + 1_m^{p+1})), \end{aligned}$$

- the element  $b_{n+m+1-4s}^{p+q+2s} = d^{2s}1_{n+1}^q \otimes d^{2s+1}1_{m+1}^p + (-1)^n d^{2s+1}1_{n+1}^q \otimes d^{2s}1_{m+1}^p$  in  $\hat{Z}_{r_s, o_s}^{0, \infty}$ , and
- the element  $b_{n+m-4s}^{p+q+1+2s} =$

$$\begin{aligned} (-1)^{n+1} (d^{2s}1_{n+1}^q \otimes d^{2s+1}1_m^{p+1} - d^{2s+1}1_n^{q+1} \otimes d^{2s}1_{m+1}^p) - 2 \times d^{2s}1_n^{q+1} \otimes d^{2s}1_m^{p+1} \\ \text{in } \hat{Z}_{r_s+1, o_s-1}^{0, \infty} \end{aligned}$$

fit into the equality  $z_{n+m+1-4s}^{p+q+2s} = db_{n+m+2-4s}^{p+q+2s} + b_{n+m+1-4s}^{p+q+1+2s}$  and give a cofibration  $\varphi_{r_s, o_s-1}^\infty$ . We denote by  $\text{Codom}_s$  the codomain of the map  $\coprod_{0 \leq k \leq s} (\varphi_{r_k, o_k}^\infty \amalg \varphi_{r_k, o_k-1}^\infty)$ . We want to prove the following statement:

for  $k \geq 0$ , we have the equality

$$\langle d^k 1_{n+1}^q \otimes d^l 1_{m+1}^p \rangle_{k, l \geq 2s} \cap (\text{im}(i_1 \square i_2) + \text{Codom}_{s-1}) = \{0\}. \quad (E_s)$$

and there exists an  $\mathbf{R}$ -linear subspace  $S_s$  of  $X_1 \otimes X_2$  such that modulo

$$F_{q+p+2(s+1)}(X_1 \otimes X_2), \text{ the map } i_1 \square i_2 \text{ is given by} \\ \text{id}_{S_s} \amalg \coprod_{0 \leq k \leq s} (\varphi_{r_k, o_k}^\infty \amalg \varphi_{r_k, o_k-1}^\infty).$$

Let assume Equality  $(E_s)$  for  $s-1$  and the existence of  $S_{s-1}$  such that modulo  $F_{q+p+2s}(X_1 \otimes X_2)$ , the map  $i_1 \square i_2$  is given by

$$\text{id}_{S_{s-1}} \amalg \coprod_{0 \leq k \leq s-1} (\varphi_{r_k, o_k}^\infty \amalg \varphi_{r_k, o_k-1}^\infty).$$

First of all, we obtain  $\text{im}(i_1 \square i_2) + \text{Codom}_{s-1}$  from  $\text{im}(i_1 \square i_2) + \text{Codom}_{s-2}$  by adding the terms  $d^{2s}1_{n+1}^q \otimes d^{2s}1_{m+1}^p$  and  $b_{n+m+1-4s}^{p+q+2s}$  and their boundaries. It is easy to see that all these terms are sums of terms of the form  $d^k 1_{n+1}^q \otimes d^l 1_{m+1}^p$  with at least one term with  $k \geq 2s$  or  $l \geq 2s$ . By Equalities (11) and  $(E_s)$  for  $s-1$ , this gives Equality  $(E_s)$ . It remains to show that there exists  $S_s$  such that modulo  $F_{q+p+2(s+1)}(X_1 \otimes X_2)$ , the map  $i_1 \square i_2$  is given by  $\text{id}_{S_s} \amalg \coprod_{0 \leq k \leq s} (\varphi_{r_k, o_k}^\infty \amalg \varphi_{r_k, o_k-1}^\infty)$ , that is to say the codomain of this last map is equal to  $X_1 \otimes X_2$  modulo  $F_{q+p+2(s+1)}(X_1 \otimes X_2)$ . By the induction hypothesis, it is enough to show that all the elements in

$F_{q+p+2(s+1)}(X_1 \otimes X_2) / F_{q+p+2(s+1)}(X_1 \otimes X_2)$  are in the codomain modulo  $F_{q+p+2(s+1)}(X_1 \otimes X_2)$ . A basis of the module

$$F_{q+p+2s}(X_1 \otimes X_2) / F_{q+p+2(s+1)}(X_1 \otimes X_2)$$

is given by the classes of the elements in the set

$$\begin{aligned} C_s := & \left\{ d^{2k+t} 1_{n+1}^q \otimes d^{2(2s-k+u)+l} 1_{m+1}^p \right\}_{t, u, l \in \llbracket 0, 1 \rrbracket, k \in \llbracket 0, 2s+u \rrbracket} \cup \\ & \left\{ d^{2k+t} 1_n^{q+1} \otimes d^{2(2s-1-k+u)+l} 1_{m+1}^p \right\}, \\ & \left\{ d^{2k+t} 1_{n+1}^q \otimes d^{2(2s-1-k+u)+l} 1_m^{p+1} \right\}_{t, u, l \in \llbracket 0, 1 \rrbracket, k \in \llbracket 0, 2s-1+u \rrbracket} \\ & \cup \left\{ d^{2k+t} 1_n^{q+1} \otimes d^{2(2(s-1)-k+u)+l} 1_m^{p+1} \right\}_{t, u, l \in \llbracket 0, 1 \rrbracket, k \in \llbracket 0, 2(s-1)+u \rrbracket}. \end{aligned}$$

First of all, using the terms  $d^k 1_{n+1}^q \otimes d^l 1_{m+1}^p$  and the terms in  $Z$  such that  $k, l \geq 2s$ , the same tedious computation as for the base case shows that the terms  $d^k 1_{n+1}^q \otimes d^l 1_{m+1}^p$ ,  $d^k 1_n^{q+1} \otimes d^l 1_{m+1}^p$ ,  $d^k 1_{n+1}^q \otimes d^l 1_m^{p+1}$ ,  $d^k 1_n^{q+1} \otimes d^l 1_m^{p+1}$ , for  $k, l \geq 2s$ , are in the codomain modulo  $F_{q+p+2(s+1)}(X_1 \otimes X_2)$ . Finally, it is enough to say that we get the terms  $d^k 1_{n+1}^q \otimes d^l 1_{m+1}^p$ ,  $d^k 1_n^{q+1} \otimes d^l 1_{m+1}^p$ ,  $d^k 1_{n+1}^q \otimes d^l 1_m^{p+1}$ ,  $d^k 1_n^{q+1} \otimes d^l 1_m^{p+1}$ , for  $k, l$  such that  $k$  or  $l$  is  $< 2s$  using the previous terms and the boundaries of some lift of the class modulo  $F_{q+p+2s}(X_1 \otimes X_2)$  of the terms in  $C_{s-1}$ . The picture of this proof is nested cones. This gives the inductive step and therefore proves that the map  $i_1 \square i_2$  is a cofibration since we are working in a complete setting.

When the two maps are acyclic cofibrations, say  $i_1 : 0 \rightarrow \hat{Z}_{q,n}^{0,\infty}$  and  $i_2 : 0 \rightarrow \hat{Z}_{p,m}^{0,\infty}$ , the map  $i_1 \square i_2$  is given by  $0 \rightarrow \mathcal{Z}_{q,n+1}^{0,\infty} \otimes \hat{Z}_{p,m}^{0,\infty}$  which is a coproduct of acyclic cofibrations. When we assume that  $i_1$  or  $i_2$  is an acyclic cofibration, say  $i_1 = \varphi_{q,n}^{1,\infty}$  and  $i_2 : 0 \rightarrow \hat{Z}_{p,m}^{0,\infty}$ , we can see that the map  $i_1 \square i_2$  is isomorphic to the inclusion  $\hat{Z}_{q,n}^{1,\infty} \otimes \hat{Z}_{p,m}^{0,\infty} \rightarrow \hat{Z}_{q,n}^{1,\infty} \otimes \hat{Z}_{p,m}^{0,\infty} \amalg \mathcal{Z}_{q,n+1}^{0,\infty} \otimes \hat{Z}_{p,m}^{0,\infty}$ , which is an acyclic cofibration.

- (2) By [Hov99, Remark 4.2.3] and the fact that a left Quillen functor preserves weak equivalences between cofibrant objects, we can consider any cofibrant replacement of the unit. The map  $\hat{Z}_{0,0}^{1,\infty} \rightarrow \mathbb{1}$  is one. We have to show that for all cofibrant  $X$ , the natural map  $\hat{Z}_{0,0}^{1,\infty} \otimes X \rightarrow X$  (resp.  $X \otimes \hat{Z}_{0,0}^{1,\infty} \rightarrow X$ ) is a weak equivalence. By [Hov99, Lemma 4.2.7], it is enough to show that the map  $X \rightarrow \text{Hom}(\hat{Z}_{0,0}^{1,\infty}, X)$  (where the internal Hom is computed in graded filtered  $\mathbf{R}$ -modules) is a weak equivalence for all  $X$  (since all objects are fibrant). Since this map is an equality, we are done.  $\square$

**C.21. Model structure on complete curved operads.** In this section, we recall from [Hir03, Theorem 11.3.2] the theorem of transfer of cofibrantly generated model structure, essentially due to Quillen [Qui67, Section II.4] and we apply it to endow the category of complete curved operads with a cofibrantly generated model structures. The model structure is obtained from the adjunction between the free complete curved operad and the forgetful functor from complete curved operads to complete gr-dg  $\mathbb{S}$ -modules.

Applying Theorem C.15 for the ring  $\mathbf{R}[\mathbb{S}_m]$ ,  $m \in \mathbb{N}$ , we obtain a proper cofibrantly generated model category structure on the category  $\mathcal{M}_m$  of complete gr-dg  $\mathbf{R}[\mathbb{S}_m]$ -modules. Considering this collection  $(\mathcal{M}_m, W_m, I_m, J_m)_{m \in \mathbb{N}}$  of proper cofibrantly generated model category structures, we have that the product

$$\mathbb{S}\text{-Mod}(\widehat{\mathcal{A}}^{\text{gr}}) := \left( \prod_{m \in \mathbb{N}} \mathcal{M}_m, \prod_{m \in \mathbb{N}} W_m, \prod_{m \in \mathbb{N}} I_m \right)$$

(for  $\mathcal{A} = \mathbf{R}\text{-Mod}$ ) also is a proper cofibrantly generated model category structure (see for example [Hir03, 11.6]). A morphism  $f : M \rightarrow N$  in  $\mathcal{M}$  is a weak equivalence (resp. fibration) if the underlying collection of morphisms  $\{M(m) \rightarrow N(m)\}_{m \in \mathbb{N}}$  consists of weak equivalences (resp. fibrations). Moreover the set  $I$  (resp.  $J$ ) of generating cofibrations (resp. acyclic cofibrations) can be described explicitly as follows:

$$I = \{\hat{Z}_{q,n}^{1,\infty}(m) \rightarrow \hat{B}_{q,n}^{1,\infty}(m)\} \text{ and } J = \{0 \rightarrow \hat{Z}_{q,n}^{0,\infty}(m)\},$$

where  $\hat{Z}_{q,n}^{k,\infty}(m)$  (resp.  $\hat{B}_{q,n}^{1,\infty}(m)$ ) is the complete gr-dg  $\mathbb{S}$ -module obtained by the complete free gr-dg  $\mathbf{R}[\mathbb{S}_m]$ -module  $\hat{Z}_{q,n}^{k,\infty} \otimes \mathbf{R}[\mathbb{S}_m]$  (resp.  $\hat{B}_{q,n}^{1,\infty} \otimes \mathbf{R}[\mathbb{S}_m]$ ) put in arity  $m$ . We denote by  $\mathcal{W}$  the subcategory of weak equivalences. Notice that the domains of elements of  $I$  or  $J$  are small ( $\aleph_1$ -small) in the category  $\mathbb{S}\text{-Mod}(\widehat{\mathcal{A}}^{\text{gr}})$ .

**Theorem C.22** (Theorem 3.3 in [Cra95], Theorem 11.3.2 in [Hir03]). *Let  $\mathcal{C}$  is a cofibrantly generated model category, with  $I$  as set of generating cofibrations,  $J$  as set of generating trivial cofibrations and  $W$  as class of weak equivalences. Let  $\mathcal{D}$  be a complete and cocomplete category and let  $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$  be a pair of adjoint functors. Suppose further that:*

- (1)  $G$  preserves filtered  $\aleph_1$ -colimits,
- (2)  $G$  maps relative  $FJ$ -cell complexes to weak equivalences.

*Then the category  $\mathcal{D}$  is endowed with a cofibrantly generated model structure in which  $FI$  is the set of generating cofibrations,  $FJ$  is the set of generating trivial cofibrations, and the weak equivalences are the maps that  $G$  takes into a weak equivalence in  $\mathcal{C}$ . We have that fibrations are precisely those maps that  $G$  takes into a fibration in  $\mathcal{C}$ . Moreover  $(F, G)$  is a Quillen pair with respect to these model structures.*

In order to apply this theorem, we prove several results. The first one concerns the adjunction between the free curved operad  $\mathfrak{cT}$  and the corresponding forgetful functor.

**Proposition C.23.** *The adjunction between the free curved operad and the forgetful functor (see Theorem 2.20)*

$$\mathfrak{cT} : \mathbb{S}\text{-Mod}(\widehat{\mathcal{A}}^{\text{gr}}) \rightleftarrows \text{Curved operads} : U$$

*provides a monad  $U \cdot \mathfrak{cT}$  whose category of algebras is naturally isomorphic to the category of curved operads.*

*Proof.* We apply the crude monadicity theorem that we recall from [BW05, Section 3.5]. The functor  $U : \text{Curved operads} \rightarrow \mathbb{S}\text{-Mod}(\widehat{\mathcal{A}}^{\text{gr}})$  is monadic if it satisfies the hypotheses:

- the functor  $U$  has a left adjoint,
- the functor  $U$  reflects isomorphisms,

- the category **Curved operads** has coequalizers of those reflexive pairs  $(f, g)$  for which  $(Uf, Ug)$  is a coequalizer and  $U$  preserves those coequalizers.

The free functor is left adjoint to the forgetful functor  $U$ . The forgetful functor clearly reflects isomorphisms. The coequalizer of a pair  $f, g : \mathcal{O} \rightarrow \mathcal{P}$  in the category of curved operads is given by the quotient map

$$(\mathcal{P}, d_{\mathcal{P}}, \theta_{\mathcal{P}}) \rightarrow (\mathcal{P}/(\text{im}(f-g)), \bar{d}_{\mathcal{P}}, \bar{\theta}_{\mathcal{P}})$$

where  $(\text{im}(f-g))$  is the (operadic) ideal generated by  $\text{im}(f-g)$ , the map  $\bar{d}_{\mathcal{P}}$  induced by  $d_{\mathcal{P}}$  is well-defined since  $(\text{im}(f-g))$  is stable under  $d_{\mathcal{P}}$  and the element  $\bar{\theta}_{\mathcal{P}}$  is the class of  $\theta_{\mathcal{P}}$  in  $\mathcal{P}/(\text{im}(f-g))$ . The element  $\bar{\theta}_{\mathcal{P}}$  is a curvature since  $\theta_{\mathcal{P}}$  is. When  $(f, g)$  is a reflexive pair, the ideal generated by  $\text{im}(f-g)$  is equal to  $\text{im}(f-g)$ . It follows that the third condition is satisfied since the coequalizers in  $\mathbb{S}\text{-Mod}(\widehat{\mathcal{A}}^{\text{gr}})$  are given by  $\mathcal{P}/\text{im}(f-g)$ . Hence the forgetful functor  $U$  is monadic and the result follows from the crude monadicity theorem.  $\square$

**Proposition C.24.** *The category of curved operads has all limits and small colimits.*

*Proof.* By Proposition 4.3.1 in [Bor94] and Proposition C.23, the category of curved operads admits the same type of limits as the category  $\mathbb{S}\text{-Mod}(\widehat{\mathcal{A}}^{\text{gr}})$  which is complete (and they are preserved by  $U$ ).

By Proposition 4.3.2 in [Bor94] and Proposition C.23, if some colimits present in  $\mathbb{S}\text{-Mod}(\widehat{\mathcal{A}}^{\text{gr}})$  are preserved by the monad  $U \cdot \mathcal{T}$ , the category of curved operads admits the same type of colimits and they are preserved by  $U$ . Using the fact that  $\circ$  preserves filtered colimits in each variable, since  $\otimes$  does, and that colimits commutes with colimits, we get by the construction of the underlying  $\mathbb{S}$ -module of the free complete operad given in Section 2.12 that the monad  $U' \cdot \mathcal{T}$ , where  $U'$  is the forgetful functor from complete gr-dg operad to complete gr-dg  $\mathbb{S}$ -modules, commutes with filtered colimits. Similarly, the same is true when we replace  $\mathcal{T}$  by  $\mathcal{T}_+$  and the forgetful functor by the corresponding forgetful functor. Then, the free curved operad  $c\mathcal{T}(M, d_M)$  described in Section 2.17 is given by a quotient of  $\mathcal{T}_+(M, d_M)$  which is a coequalizer (hence a colimit). We finally get that the monad  $U \cdot c\mathcal{T}$  commutes with filtered colimits. We therefore obtain that the category of curved operads admits filtered colimits. It is therefore enough to check that it admits pushouts (by Theorem 1 in Chapter IX of [Mac71], a category with filtered colimits and pushouts has all small colimits). The pushouts can be computed explicitly. The construction is well-known in the category of operads in  $\mathcal{A}$  (see for example the proof of Theorem 1.13 in [GJ94]). Given two maps of filtered operads  $\mathbf{c} \rightarrow \mathbf{a}$  and  $\mathbf{c} \rightarrow \mathbf{b}$  and the pushout in operads  $\mathbf{F}_0\mathbf{a} \vee_{\mathbf{F}_0\mathbf{c}} \mathbf{F}_0\mathbf{b}$ . We endow the pushout with the filtration given by  $\text{im}(F_p\mathcal{T}(\mathbf{a} \amalg \mathbf{b}) \rightarrow \mathbf{F}_0\mathbf{a} \vee_{\mathbf{F}_0\mathbf{c}} \mathbf{F}_0\mathbf{b})$  which is the initial one such that there are filtered maps  $\mathbf{a} \rightarrow \mathbf{a} \vee_{\mathbf{c}} \mathbf{b}$  and  $\mathbf{b} \rightarrow \mathbf{a} \vee_{\mathbf{c}} \mathbf{b}$  and  $\mathbf{a} \vee_{\mathbf{c}} \mathbf{b}$  is a filtered operad. It is easy to check that this defines the pushout in filtered operads. Taking completion gives the pushout in complete operads and this extends in a straightforward manner to the gr-dg setting. In order to obtain the pushout of two maps  $(\mathbf{c}, d_{\mathbf{c}}, \theta_{\mathbf{c}}) \rightarrow (\mathbf{a}, d_{\mathbf{a}}, \theta_{\mathbf{a}})$  and  $(\mathbf{c}, d_{\mathbf{c}}, \theta_{\mathbf{c}}) \rightarrow (\mathbf{b}, d_{\mathbf{b}}, \theta_{\mathbf{b}})$  in the curved setting, the only difference is to consider the quotient  $\mathbf{a} \vee_{\mathbf{c}} \mathbf{b}/(\theta_{\mathbf{a}} - \theta_{\mathbf{b}})$  so that

$$(\mathbf{a}, \mathbf{d}_{\mathbf{a}}, \theta_{\mathbf{a}}) \vee_{(\mathbf{c}, \mathbf{d}_{\mathbf{c}}, \theta_{\mathbf{c}})} (\mathbf{b}, \mathbf{d}_{\mathbf{b}}, \theta_{\mathbf{b}}) := (\mathbf{a} \vee_{\mathbf{c}} \mathbf{b}/(\theta_{\mathbf{a}} - \theta_{\mathbf{b}}), d_{\mathbf{a} \vee_{\mathbf{c}} \mathbf{b}/(\theta_{\mathbf{a}} - \theta_{\mathbf{b}})}, \bar{\theta}_{\mathbf{a}} = \bar{\theta}_{\mathbf{b}}).$$

$\square$

We now apply the transfer Theorem C.22 to the adjunction

$$\mathbf{cT} : \mathbb{S}\text{-Mod}(\widehat{\mathcal{A}}^{\text{gr}}) \rightleftarrows \text{Curved operads} : U.$$

We have seen in the beginning of Section C.21 that the category on the left-hand side is a cofibrantly generated model category. In order to apply Theorem C.22, we have to understand  $\mathbf{cT}J$ -cell complexes, with

$$\mathbf{cT}J = \{I \rightarrow \mathbf{cT}(\hat{\mathcal{Z}}_{q,n}^{0,\infty}(m))\}_{n \in \mathbb{Z}, q \in \mathbb{N}, m \in \mathbb{N}}.$$

Then, we study  $\mathbf{cT}I$ -cell complexes, with

$$\mathbf{cT}I = \{\mathbf{cT}(\hat{\mathcal{Z}}_{q,n}^{1,\infty}(m)) \rightarrow \mathbf{cT}(\hat{\mathcal{B}}_{q,n}^{1,\infty}(m))\}_{n \in \mathbb{Z}, q \in \mathbb{N}, m \in \mathbb{N}}$$

in order to describe cofibrant objects. The classes  $\mathbf{cT}I$ , resp.  $\mathbf{cT}J$ , will be the generating cofibrations, resp. generating acyclic cofibrations.

**Lemma C.25.** *A morphism of curved operads is a relative  $\mathbf{cT}J$ -cell complex if and only if it is a map  $\mathcal{O} \rightarrow \mathcal{O} \vee \mathbf{cT}(Z, d_Z)$ , where  $(Z, d_Z)$  is a complete gr-dg  $\mathbb{S}$ -module equal to a direct sum of complete gr-dg  $\mathbb{S}$ -modules  $\hat{\mathcal{Z}}_{q,n}^{0,\infty}(m)$ . In particular,  $(Z, d_Z)$  is a free  $\mathbb{S}$ -module and it is gr-acyclic (that is to say its gr-homology is 0) and it satisfies  $\ker(d_Z) = \{0\}$ . Explicitly, we can write*

$$\mathbf{cT}(Z, d_Z) \cong (\mathcal{T}(\vartheta I \amalg (Z/\text{im } d_Z^2)), d, \vartheta),$$

with  $d\vartheta = 0$  and  $d^2 = [\vartheta, -]$ .

*Proof.* We have  $\mathbf{cT}(0) = \mathcal{T}(\vartheta I)/(\text{im}(0^2 - [\vartheta, -])) \cong \mathcal{T}(\vartheta I)$ . Pushouts of elements of  $\mathbf{cT}J$  are therefore as follows

$$\begin{array}{ccc} \mathcal{T}(\vartheta I) & \longrightarrow & \mathcal{O} \\ \downarrow \bigvee_{\alpha} \mathbf{cT}(j_{\alpha}) & & \downarrow \\ \bigvee_{\alpha} \mathbf{cT}(\hat{\mathcal{Z}}_{\alpha}^{0,\infty}) & \longrightarrow & \mathcal{O} \vee \left( \bigvee_{\alpha} \mathbf{cT}(\hat{\mathcal{Z}}_{\alpha}^{0,\infty}) \right), \end{array}$$

with each  $\hat{\mathcal{Z}}_{\alpha}^{0,\infty}$  equal to a  $\hat{\mathcal{Z}}_{q,n}^{0,\infty}(m)$ . Since the coproduct of free curved operads is the free curved operad on the sum of their generating modules (see the proof of Proposition C.24), the composite of two such pushouts is equal to  $\mathcal{O} \rightarrow \mathcal{O} \vee \mathbf{cT}(\amalg_{\alpha} \hat{\mathcal{Z}}_{\alpha}^{0,\infty})$ . Hence a transfinite composition of such pushouts has the form  $\mathcal{O} \rightarrow \mathcal{O} \vee \mathbf{cT}(Z)$ , with  $Z$  a gr-acyclic gr-dg  $\mathbb{S}$ -module whose components are free complete gr-dg  $\mathbb{S}$ -modules  $\hat{\mathcal{Z}}_{q,n}^{0,\infty}(m)$ . To prove the isomorphism given an explicit description of  $\mathbf{cT}(Z, d_Z)$ , we define a map  $\vartheta I \amalg Z \rightarrow \mathcal{T}(\vartheta I \amalg Z/\text{im } d_Z^2)$  by sending  $\vartheta$  to  $\vartheta$  and an element  $d^k x$ , for a generator  $x$  of a free gr-dg  $\mathbb{S}$ -module  $\hat{\mathcal{Z}}_{q,n}^{0,\infty}(m)$ , to

$$d^k x \mapsto \underbrace{[\vartheta, \dots, [\vartheta, d^{k-2 \lfloor \frac{k}{2} \rfloor} x] \dots]}_{\lfloor \frac{k}{2} \rfloor \text{ times}}.$$

By the universal property of the free complete gr-dg operad, we obtain a (surjective) map

$$\mathcal{T}(\vartheta I \amalg Z) \rightarrow \mathcal{T}(\vartheta I \amalg Z/\text{im } d_Z^2)$$

and a direct computation shows that the ideal  $(\text{im}(d_Z^2 - [\vartheta, -]))$  is sent to zero. It is easy to describe an inverse to this map in complete  $\mathbb{S}$ -modules.  $\square$

**Theorem C.26.** *The category of complete gr-dg operads is endowed with a cofibrantly generated model category structure where the generating (acyclic) cofibrations are the images under the free functor of the generated (acyclic) cofibrations. A map  $f : \mathcal{O} \rightarrow \mathcal{P}$  is a*

- *weak equivalences if and only if, in any arity, it is a graded quasi-isomorphism of gr-dg  $\mathbb{S}$ -modules,*
- *fibration if and only if, in any arity, it is a gr-surjection,*
- *cofibrations if and only if it has the left lifting property with respect to acyclic fibrations.*

Moreover  $(\mathfrak{c}\mathcal{T}, U)$  is a Quillen pair with respect to the cofibrantly model structures. The generating cofibrations are the maps  $\mathfrak{c}\mathcal{T}I$  and the generating acyclic cofibrations are the maps  $\mathfrak{c}\mathcal{T}J$ .

*Proof.* By Proposition C.24, in order to apply Theorem C.22, it is enough to show that  $U$  preserves filtered  $\aleph_1$ -colimits and that it maps relative  $\mathfrak{c}\mathcal{T}J$ -cell complexes to weak equivalences. We have already seen in the proof of Proposition C.24 that  $U$  preserves filtered colimits; in particular, it preserves filtered  $\aleph_1$ -colimits. By Lemma C.25, a relative  $\mathfrak{c}\mathcal{T}J$ -cell complex can be written as a map  $j : \mathcal{O} \rightarrow \mathcal{O} \vee \left( \bigvee_{\alpha} \mathfrak{c}\mathcal{T}(\hat{\mathcal{Z}}_{\alpha}^{0,\infty}) \right) = \mathcal{O} \vee \mathfrak{c}\mathcal{T} \left( \prod_{\alpha} \hat{\mathcal{Z}}_{\alpha}^{0,\infty} \right)$ . Using the description of the pushout of curved operads in the proof of Proposition C.24, we can compute

$$\begin{aligned} \mathcal{O} \vee \mathfrak{c}\mathcal{T} \left( \prod_{\alpha} \hat{\mathcal{Z}}_{\alpha}^{0,\infty} \right) &\cong \left( \mathcal{O} \vee_{\text{gr-dg op}} \mathcal{T}_+ \left( \prod_{\alpha} \hat{\mathcal{Z}}_{\alpha}^{0,\infty} / \text{im } d_{\alpha}^2 \right) \right) / (\theta_{\mathcal{O}} - \vartheta) \\ &\cong \mathcal{O} \vee_{\text{gr-dg op}} \mathcal{T} \left( \prod_{\alpha} \hat{\mathcal{Z}}_{\alpha}^{0,\infty} / \text{im } d_{\alpha}^2 \right). \end{aligned}$$

The functor

$$\text{Gr} : \widehat{\text{Filt}}^{\text{gr}}(\mathcal{A}) \rightarrow (\text{dg}\mathcal{A})^{\text{ob}\mathbb{N}}, \quad (V, F, d_V) \mapsto (\text{Gr}_{\bullet} V, \text{Gr } d_V)$$

commutes with direct sums (see for example [Fre17, Proposition 7.3.8]) and satisfies, for gr-flat complete modules,  $\text{Gr } M \otimes_{\text{Gr}} \text{Gr } N \cong \text{Gr } (M \hat{\otimes} N)$  (see Lemma B.4). By Maschke's Theorem ( $\mathbf{R}$  is a field of characteristic 0), any  $\mathbf{R}[\mathbb{S}_*]$ -module is flat. It follows that when  $M$  and  $N$  are two gr-dg  $\mathbb{S}$ -modules, we have  $\text{Gr } M \circ \text{Gr } N \cong \text{Gr } (M \circ N)$ . Moreover the functor  $\text{Gr}$  preserves filtered colimits. Indeed, it is the composition

$$\widehat{\text{Filt}}^{\text{gr}}(\mathcal{A}) \xrightarrow{i_1} \text{Filt}(\text{pg}\mathcal{A}) \xrightarrow{i_2} (\text{pg}\mathcal{A})^{\text{N}^{\text{op}}} \xrightarrow{q_1} (\text{dg}\mathcal{A})^{\text{ob}\mathbb{N}},$$

where  $i_1$  and  $i_2$  are inclusions and  $q_1$  sends a directed sequence to the ( $\mathbb{N}$ -indexed) product of the cokernel of the maps appearing in the directed sequence. By Proposition 1.62 in [AR94], the functor  $i_2$  preserves filtered colimits. Since cokernels are coequalizers (therefore colimits) in  $(\text{pg}\mathcal{A})^{\text{N}^{\text{op}}}$  and  $(\text{dg}\mathcal{A})^{\text{ob}\mathbb{N}}$ , and colimits commute with colimits, the map  $q_2$  preserves all colimits. Then, because of the presence of the quotient map  $q_1$ , when  $\text{colim}_i M_i$  is a filtered colimit, we have  $\text{Gr } \text{colim}_i M_i = (q_1 \cdot i_2)(\text{colim}_i i_1(M_i))$  (even if  $i_1$  does not preserve ( $\aleph_0$ -)filtered colimits). The free operad functor is obtained as a filtered colimit of finite coproducts of  $\circ$  monoidal products (see Section 2.12), so  $\text{Gr}$  commutes with the free curved operad functor.

The coproduct of two operads can be computed explicitly and is a quotient of the free operad on the direct sum of the two underlying  $\mathbb{S}$ -modules of the operads (see



the proof of Theorem 1.13 in [GJ94]). The graded functor  $\text{Gr}$  therefore preserves the coproduct of two operads. We get

$$\text{Gr} \left( \mathcal{O} \vee_{\text{gr-dg op}} \mathcal{T} \left( \Pi_{\alpha} \hat{\mathcal{Z}}_{\alpha}^{0,\infty} / \text{im } d_{\alpha}^2 \right) \right) \cong \text{Gr} \left( \mathcal{O} \vee_{\text{dg op}} \mathcal{T} \left( \Pi_{\alpha} \text{Gr} \left( \hat{\mathcal{Z}}_{\alpha}^{0,\infty} / \text{im } d_{\alpha}^2 \right) \right) \right).$$

We get, using the fact that  $\mathbf{R}$  is a field of characteristic 0 and the arguments given in [Hin03, Theorem 3.2], that the map  $\text{Gr}(U(j))$  is a quasi-isomorphism.  $\square$

We conclude this section by computing the cofibrant objects in the model category of complete curved operads. We recall that the notion of quasi-free complete curved operad is given in Definition 2.22.

**Proposition C.27.** *A complete curved operad is cofibrant if and only if it is a retract of a quasi-free complete curved operad  $(\mathcal{T}_+(S), d) / (\text{im}(d^2 - [\vartheta, -]))$ , where  $S$  is a complete  $\mathbb{S}$ -module endowed with an exhaustive filtration*

$$S_0 = \{0\} \subset S_1 \subset S_2 \subset \cdots \subset \text{colim}_i S_i = S$$

of free  $\mathbb{S}$ -modules such that  $S_{i-1} \hookrightarrow S_i$  are split monomorphisms of complete  $\mathbb{S}$ -modules with cokernels isomorphic to a sum of complete  $\mathbb{S}$ -modules

$$S_i / S_{i-1} \cong \coprod_{\alpha} \left( \xi^{\alpha} \cdot \mathbf{R}[\mathbb{S}_{m_{\alpha}}] \amalg \hat{\mathcal{Z}}_{q_{\alpha}+1, n_{\alpha}}^{0,\infty}(m_{\alpha}) \right)$$

where  $\xi^{\alpha}$  is in homological degree  $n_{\alpha}+1$  and filtration degree  $q_{\alpha}$ . The predifferential  $d$  is the one of  $\hat{\mathcal{Z}}_{q_{\alpha}+1, n_{\alpha}}^{0,\infty}(m_{\alpha})$  on  $\hat{\mathcal{Z}}_{q_{\alpha}+1, n_{\alpha}}^{0,\infty}(m_{\alpha})$  and

$$d(\xi^{\alpha}) + \zeta^{\alpha} \in (\mathcal{T}_+(S_{i-1}), d) / (\text{im}(d^2 - [\vartheta, -])),$$

with  $\zeta^{\alpha}$  is a generator of the  $\text{gr-dg}$   $\mathbb{S}$ -module  $\hat{\mathcal{Z}}_{q_{\alpha}+1, n_{\alpha}}^{0,\infty}(m_{\alpha})$ .

*Proof.* By Proposition 2.1.18 in [Hov99], cofibrations are retracts of relative  $\text{cTI}$ -cell complexes. We therefore study the pushouts of elements of  $\text{cTI}$  of the form

$$\begin{array}{ccc} \mathbb{V}_{\alpha} \text{cT} \left( \hat{\mathcal{Z}}_{\alpha}^{1,\infty} \right) & \xrightarrow{f} & (\mathcal{T}_+(S_{i-1}), d_{i-1}) / (\text{im}(d_{i-1}^2 - [\vartheta, -])) \\ \downarrow \mathbb{V}_{\alpha} \text{cT}(i_{\alpha}) & & \downarrow \\ \mathbb{V}_{\alpha} \text{cT}(\hat{\mathcal{B}}_{\alpha}^{1,\infty}) & \longrightarrow & \mathcal{P}, \end{array}$$

with each  $\hat{\mathcal{Z}}_{\alpha}^{1,\infty}$  equal to a  $\hat{\mathcal{Z}}_{q_{\alpha}, n_{\alpha}}^{1,\infty}(m_{\alpha})$  and  $\hat{\mathcal{B}}_{\alpha}^{1,\infty}$  equal to a  $\hat{\mathcal{B}}_{q_{\alpha}, n_{\alpha}}^{1,\infty}(m_{\alpha})$ . We denote by  $z^{\alpha}$  the image under  $f$  of the generating (as a  $\text{gr-dg}$   $\mathbb{S}$ -modules) element of  $\hat{\mathcal{Z}}_{q_{\alpha}, n_{\alpha}}^{1,\infty}(m_{\alpha})$ . If we denote by  $\xi^{\alpha}$  and by  $\zeta^{\alpha}$  the generating (as a  $\text{gr-dg}$   $\mathbb{S}$ -modules) elements of  $\hat{\mathcal{B}}_{q_{\alpha}, n_{\alpha}}^{1,\infty}(m_{\alpha}) = \hat{\mathcal{Z}}_{q_{\alpha}, n_{\alpha}+1}^{0,\infty}(m_{\alpha}) \amalg \hat{\mathcal{Z}}_{q_{\alpha}+1, n_{\alpha}}^{0,\infty}(m_{\alpha})$ , and by  $d^k \xi^{\alpha}$ , resp. by  $d^k \zeta^{\alpha}$ , their successive predifferentials, the pushout  $\mathcal{P}$  is equal to

$$\left( \mathcal{T}_+(S_{i-1}) \vee_{\text{compl. op.}} \mathcal{T} \left( \xi^{\alpha} \cdot \mathbf{R}[\mathbb{S}_{m_{\alpha}}] \amalg \hat{\mathcal{Z}}_{q_{\alpha}+1, n_{\alpha}}^{0,\infty}(m_{\alpha}) \right), d_i \right) / (\text{im}(d_i^2 - [\vartheta, -])),$$

where  $d_i$  is the derivation defined on  $S_{i-1}$  by  $d_{i-1}$ , the differential on  $\hat{\mathcal{Z}}_{q_{\alpha}+1, n_{\alpha}}^{0,\infty}(m_{\alpha})$  is the one of  $\hat{\mathcal{Z}}_{q_{\alpha}+1, n_{\alpha}}^{0,\infty}(m_{\alpha})$  and  $d_i \xi^{\alpha} = z^{\alpha} - \zeta^{\alpha}$ . By induction, we get the result.  $\square$

**Corollary C.28.** *Equivalently, a complete curved operad is cofibrant if and only if it is a retract of a quasi-free complete curved operad  $(\mathcal{T}_+(\tilde{S}), d)$ , where  $\tilde{S}$  is a complete  $\mathbb{S}$ -module endowed with an exhaustive filtration*

$$\tilde{S}_0 = \{0\} \subset \tilde{S}_1 \subset \tilde{S}_2 \subset \cdots \subset \text{colim}_i \tilde{S}_i = \tilde{S}$$

of free  $\mathbb{S}$ -modules such that  $\tilde{S}_{i-1} \hookrightarrow \tilde{S}_i$  are split monomorphisms of complete  $\mathbb{S}$ -modules with cokernels isomorphic to a sum of complete  $\mathbb{S}$ -modules

$$\tilde{S}_i/\tilde{S}_{i-1} \cong \coprod_{\alpha} (\xi^{\alpha} \cdot \mathbf{R}[\mathbb{S}_{m_{\alpha}}] \amalg \zeta^{\alpha} \cdot \mathbf{R}[\mathbb{S}_{m_{\alpha}}])$$

where  $\xi^{\alpha}$  is in homological degree  $n_{\alpha} + 1$  and filtration degree  $q_{\alpha}$  and  $\zeta^{\alpha}$  is in homological degree  $n_{\alpha}$  and filtration degree  $q_{\alpha} + 1$ . The predifferential  $d$  is such that  $d(\xi^{\alpha}) + \zeta^{\alpha} \in (\mathcal{T}_+(\tilde{S}_{i-1}), d_{i-1})$ , and  $d(\zeta^{\alpha})$  is obtained by the fact that  $d^2(\xi^{\alpha}) = [\vartheta, \xi^{\alpha}]$ .

**Proposition C.29.** *Any quasi-free complete curved operad  $(\mathcal{T}_+(X), \partial)$  is a retract of a quasi-free complete curved operad  $(\mathcal{T}_+(\tilde{S}), \partial')$ , where the components of  $\tilde{S}$  are free  $\mathbb{S}$ -modules. Moreover, if  $X$  is endowed with an exhaustive filtration*

$$X_0 = \{0\} \subset X_1 \subset X_2 \subset \cdots \subset \operatorname{colim}_i X_i = X$$

such that  $X_{i-1} \hookrightarrow X_i$  are split monomorphisms of complete  $\mathbb{S}$ -modules with cokernels isomorphic to a sum of complete  $\mathbb{S}$ -modules

$$X_i/X_{i-1} \cong \coprod_{\alpha} (\xi^{\alpha} \cdot \mathbf{R}[\mathbb{S}_{m_{\alpha}}] \amalg \zeta^{\alpha} \cdot \mathbf{R}[\mathbb{S}_{m_{\alpha}}]),$$

where  $\xi^{\alpha}$  is in homological degree  $n_{\alpha} + 1$  and filtration degree  $q_{\alpha}$  and  $\zeta^{\alpha}$  is in homological degree  $n_{\alpha}$  and filtration degree  $q_{\alpha} + 1$ . The predifferential  $\partial$  is such that  $\partial(\xi^{\alpha}) + \zeta^{\alpha} \in (\mathcal{T}_+(X_{i-1}), \partial)$ , and  $\partial(\zeta^{\alpha})$  is obtained by the fact that  $\partial^2(\xi^{\alpha}) = [\vartheta, \xi^{\alpha}]$ , then  $\tilde{S}$  can be chosen with the same property and such that the cokernels of the  $\tilde{S}_{i-1} \hookrightarrow \tilde{S}_i$  are free  $\mathbb{S}$ -modules.

Under these hypotheses, the complete curved operad  $(\mathcal{T}_+(X), \partial)$  is cofibrant, as a retract of a cofibrant complete curved operad.

*Proof.* We copy the proof of Lemma 39 in [MV09], adapting the setting. Let  $\overline{X}(m)$  denote the set of equivalence classes under the action of  $\mathbb{S}_m$ . We choose a set of representatives  $\{\bar{x}_i\}_{i \in \mathcal{I}}$ . Let  $\tilde{S}$  be the free  $\mathbb{S}$ -module generated by the  $\{\bar{x}_i\}_{i \in \mathcal{I}}$ . The generator associated with  $\bar{x}_i$  will be denoted by  $s_i$ . For any  $x \in X(m)$ , we consider the subgroup  $\mathbb{S}_x := \{\sigma \in \mathbb{S}_m \mid x \cdot \sigma = \chi(\sigma)x, \chi(\sigma) \in \mathbf{R}\}$ . In this case,  $\chi$  is a character of  $\mathbb{S}_x$ . We define the following element of  $\tilde{S}$ :

$$N(\bar{x}_i) := \frac{1}{|\mathbb{S}_{\bar{x}_i}|} \sum \chi(\sigma^{-1}) \cdot s_i \sigma,$$

where the sum runs over  $\sigma \in \mathbb{S}_{\bar{x}_i}$  ( $N$  preserves the filtration). The image under the boundary map  $\partial$  of an  $\bar{x}_i$  is a (potentially infinite) sum of trees  $\sum T(\bar{x}_{i_1}, \dots, \bar{x}_{i_k})$ . We define the boundary map  $\partial'$  on  $\mathcal{T}_+(\tilde{S})$  by

$$\partial'(s_i) := \sum \frac{1}{|\mathbb{S}_{\bar{x}_i}|} \sum \chi(\sigma^{-1}) \cdot T(N(\bar{x}_i), \dots, N(\bar{x}_{i_k})) \sigma,$$

where the second sum runs over  $\sigma \in \mathbb{S}_{\bar{x}_i}$  and the sum lies in the complete gr-dg module since  $N$  preserves the filtration. Finally, we define the maps of curved operads  $\mathcal{T}_+(\tilde{S}) \rightarrow \mathcal{T}_+(X)$  by  $s_i \mapsto \bar{x}_i$  and  $\mathcal{T}_+(X) \rightarrow \mathcal{T}_+(\tilde{S})$  by  $\bar{x}_i \mapsto N(\bar{x}_i)$ . They form a deformation retract, which preserves the filtration on  $X$  when it exists and the different properties on the cokernels also hold.  $\square$

**C.30. Model structure on categories of complete curved algebras.** Following Hinich [Hin97], we endow the category  $\mathbf{Alg}(\mathcal{O})$  of algebras over an  $\mathbb{S}$ -split complete curved operad  $(\mathcal{O}, d, \theta)$  with a model category structure. We apply Theorem C.22 to the free-forgetful functor adjunction between the categories of gr-dg  $\mathbf{R}$ -modules and  $\mathbf{Alg}(\mathcal{O})$ .

We first describe the free-forgetful adjunction.

**Proposition C.31.** *The forgetful functor  $\# : \mathbf{Alg}(\mathcal{O}) \rightarrow \text{compl. gr-dg } \mathbf{R}\text{-Mod}$  admits a left adjoint free  $\mathcal{O}$ -algebra functor  $F_{\mathcal{O}} : \text{compl. gr-dg } \mathbf{R}\text{-Mod} \rightarrow \mathbf{Alg}(\mathcal{O})$  given by*

$$(V, d_V) \mapsto F_{\mathcal{O}}(V, d_V) := \left( \mathcal{O}(V) / (\text{im}(d_V^2 - \theta \otimes \text{id}_V)), d_{\overline{\mathcal{O}(V)}} \right).$$

*Proof.* The proof is similar to the proof of Theorem 2.20. Let  $(V, d_V)$  be a complete gr-dg  $\mathbf{R}$ -module and  $(A, d_A)$  be an  $(\mathcal{O}, d, \theta)$ -algebra. We denote by  $U\mathcal{O} = U(\mathcal{O}, d, \theta)$  the complete gr-dg operad underlying  $(\mathcal{O}, d, \theta)$ . The above construction  $F_{\mathcal{O}}(V, d_V)$  is a  $U\mathcal{O}$ -algebra as a quotient by the ideal  $(\text{im}(d_V^2 - \theta \otimes \text{id}_V))$  of the free  $U\mathcal{O}$ -algebra

$$U\mathcal{O}(V, d_V) = (\coprod_{n \geq 0} \mathcal{O}(n) \otimes_{\mathbb{S}_n} V^{\otimes n}, d_{\mathcal{O}(V)})$$

where  $d_{\mathcal{O}(V)}$  is the gr-dg predifferential induced by the predifferentials on  $\mathcal{O}$  and on  $V$ . It is a  $(\mathcal{O}, d, \theta)$ -algebra since the condition that  $\theta$  is sent to  $d_{\overline{\mathcal{O}(V)}}^2$  in  $\text{End}_{F_{\mathcal{O}}(V, d_V)}$  follows from the fact that we have considered the quotient by the ideal  $(\text{im}(d_V^2 - \theta \otimes \text{id}_V))$ . Indeed,

$$\begin{aligned} d_{\mathcal{O}(V)}^2 &= d_{\mathcal{O}}^2 \otimes \text{id}_{V^{\otimes \bullet}} + \sum_j \text{id}_{\mathcal{O}} \otimes \text{id}_V^{\otimes j} \otimes d_V^2 \otimes \text{id}^{\otimes(\bullet-j)} \\ &= [\theta, -] \otimes \text{id}_{V^{\otimes \bullet}} + \sum_j \text{id}_{\mathcal{O}} \otimes \text{id}_V^{\otimes j} \otimes (\theta \otimes \text{id}_V) \otimes \text{id}^{\otimes(\bullet-j)} \\ &= \theta \otimes \text{id}_{\mathcal{O}(V)}. \end{aligned}$$

We have the adjunction

$$\text{Hom}_{\text{gr-dg } \mathbf{R}\text{-Mod}}((V, d_V), (A, d_A)^{\#}) \cong \text{Hom}_{U\mathcal{O}\text{-alg.}}(U\mathcal{O}(V, d_V), U(A, d_A)),$$

where  $U(A, d_A)$  is the  $U\mathcal{O}$ -algebra underlying  $(A, d_A)$ . Since  $(A, d_A)$  is a  $(\mathcal{O}, d, \theta)$ -algebra, a morphism of  $U\mathcal{O}$ -algebras  $U\mathcal{O}(V, d_V) \rightarrow U(A, d_A)$  automatically sends the ideal  $(\text{im}(d_V^2 - \theta \otimes \text{id}_V))$  to 0 and coincides (bijectively) with a morphism of  $(\mathcal{O}, d, \theta)$ -algebras  $F_{\mathcal{O}}(V, d_V) \rightarrow (A, d_A)$ . We get the bijection

$$\text{Hom}_{U\mathcal{O}\text{-alg.}}(U\mathcal{O}(V, d_V), U(A, d_A)) \cong \text{Hom}_{\text{gr-dg } \mathbf{R}\text{-Mod}}(F_{\mathcal{O}}(V, d_V), (A, d_A))$$

which gives the result.  $\square$

In order to apply Theorem C.22, we prove several results. The first one concerns the adjunction between the free  $(\mathcal{O}, d, \theta)$ -algebra functor  $F_{\mathcal{O}}$  and the forgetful functor  $\#$ .

**Proposition C.32.** *The adjunction between the free  $(\mathcal{O}, d, \theta)$ -algebra functor and the forgetful functor (see Proposition C.31)*

$$F_{\mathcal{O}} : \text{compl. gr-dg } \mathbf{R}\text{-Mod} \rightleftarrows \mathbf{Alg}(\mathcal{O}) : \#$$

*provides a monad  $\# \cdot F_{\mathcal{O}}$  whose category of algebras is naturally isomorphic to the category of  $(\mathcal{O}, d, \theta)$ -algebras.*

*Proof.* As in the proof of Proposition C.23, we apply the crude monadicity theorem given in [BW05, Section 3.5]. The result follows from the fact that the functor  $\# : \mathbf{Alg}(\mathcal{O}) \rightarrow \mathbf{compl. gr-dg R-Mod}$  is monadic. The free functor is left adjoint to the forgetful functor  $\#$ . The forgetful functor clearly reflects isomorphisms. It remains to show that the category  $\mathbf{Alg}(\mathcal{O})$  has coequalizers of those reflexive pairs  $(f, g)$  for which  $(\#f, \#g)$  is a coequalizer and  $\#$  preserves those coequalizers. The coequalizer of a pair  $f, g : (A, d_A) \rightarrow (B, d_B)$  in the category of  $(\mathcal{O}, d, \theta)$ -algebras is given by the quotient map  $(B, d_B) \rightarrow (B / (\text{im}(f - g)), \bar{d}_B)$  where  $(\text{im}(f - g))$  is the  $(\mathcal{O})$ -ideal generated by  $\text{im}(f - g)$  and the map  $\bar{d}_B$  induced by  $d_B$  is well-defined since  $(\text{im}(f - g))$  is stable under  $d_B$ . The quotient algebra is still a  $(\mathcal{O}, d, \theta)$ -algebra. When  $(f, g)$  is a reflexive pair, the ideal generated by  $\text{im}(f - g)$  is equal to  $\text{im}(f - g)$ . It follows that the remaining condition is satisfied since the coequalizers in  $\mathbf{compl. gr-dg R-Mod}$  are given by  $B / \text{im}(f - g)$ .  $\square$

**Proposition C.33.** *The category of  $(\mathcal{O}, d, \theta)$ -algebras has all limits and small colimits.*

*Proof.* By Proposition 4.3.1 in [Bor94] and Proposition C.32, the category of  $(\mathcal{O}, d, \theta)$ -algebras admits the same type of limits as the category  $\mathbf{compl. gr-dg R-Mod}$  which is complete (and they are preserved by  $\#$ ).

Proposition 4.3.2 in [Bor94] and Proposition C.32 implies that if some colimits in  $\mathbf{compl. gr-dg R-Mod}$  are preserved by the monad  $\# \cdot F_{\mathcal{O}}$ , the category of  $(\mathcal{O}, d, \theta)$ -algebras admits the same type of colimits and they are preserved by  $\#$ .

Using the fact that  $\otimes$  preserves filtered colimits in each variable and that colimits commutes with colimits, we get that the monad  $\# \cdot F_{\mathcal{O}}$  preserves filtered colimits. We therefore obtain that the category of  $(\mathcal{O}, d, \theta)$ -algebras admits filtered colimits. It is therefore enough to check that it admits pushouts (by Theorem 1 in Chapter IX of [Mac71], a category with filtered colimits and pushouts has all small colimits). The pushouts can be computed explicitly as follows. Let  $I_A$  (resp.  $I_B$ ) be the kernel of the map  $\gamma_A : \mathcal{O}(A) \rightarrow A$  (resp.  $\gamma_B$ ) and  $\gamma_A^0 : \mathcal{O}(0) \rightarrow A$  (resp.  $\gamma_B^0 : \mathcal{O}(0) \rightarrow B$ ) the algebra structure given by 0-ary elements in  $\mathcal{O}$ . The coproduct of two  $(\mathcal{O}, d, \theta)$ -algebras  $(A, d_A)$  and  $(B, d_B)$  is given as usual by the quotient

$$A \vee B := \mathcal{O}(A \amalg B) / (I_A \amalg I_B \amalg \text{im}(\gamma_A^0 - \gamma_B^0)).$$

The filtration (resp. predifferential) is induced by the filtrations (resp. predifferentials) on  $A$  and  $B$ . It gives a  $(\mathcal{O}, d, \theta)$ -algebra since  $A$  and  $B$  are and since  $\mathcal{O}$  is curved. Given two maps of complete  $(\mathcal{O}, d, \theta)$ -algebras  $\phi : (C, d_C) \rightarrow (A, d_A)$  and  $\psi : (C, d_C) \rightarrow (B, d_B)$ , we obtain the corresponding pushout  $A \vee_C B$  as the quotient of the coproduct  $A \vee B$  by the ideal generated by the image of the map  $\phi - \psi$ .  $\square$

We apply the transfer Theorem C.22 to the adjunction

$$F_{\mathcal{O}} : \mathbf{compl. gr-dg R-Mod} \rightleftarrows \mathbf{Alg}(\mathcal{O}) : \#.$$

We have seen in the beginning of Section C.21 that the category  $\mathbf{compl. gr-dg R-Mod}$  is a cofibrantly generated model category with generating cofibrations  $I_0^\infty$  and generating acyclic cofibrations  $J_0^\infty$ . In order to apply Theorem C.22, we have to understand  $F_{\mathcal{O}}J_0^\infty$ -cell complexes, with

$$F_{\mathcal{O}}J_0^\infty = \{\mathcal{O}(0) \rightarrow F_{\mathcal{O}}(\hat{\mathcal{Z}}_{q,n}^{0,\infty})\}_{n \in \mathbb{Z}, q \in \mathbb{N}}.$$

Then, we study  $c\mathcal{T}I$ -cell complexes, with

$$c\mathcal{T}I_0^\infty = \{F_{\mathcal{O}}(\hat{\mathcal{Z}}_{q,n}^{1,\infty}) \rightarrow F_{\mathcal{O}}(\hat{\mathcal{B}}_{q,n}^{1,\infty})\}_{n \in \mathbb{Z}, q \in \mathbb{N}}$$

in order to describe cofibrant objects. The classes  $F_{\mathcal{O}}I_0^\infty$ , resp.  $F_{\mathcal{O}}J_0^\infty$ , will be the generating cofibrations, resp. generating acyclic cofibrations.

**Lemma C.34.** *A morphism of  $(\mathcal{O}, d, \theta)$ -algebras is a relative  $F_{\mathcal{O}}J_0^\infty$ -cell complex if and only if it is a map  $A \rightarrow A \vee F_{\mathcal{O}}(Z, d_Z)$ , where  $(Z, d_Z)$  is a complete gr-dg module equal to a direct sum of complete gr-dg modules  $\hat{\mathcal{Z}}_{q,n}^{0,\infty}$ . In particular,  $(Z, d_Z)$  is a free module and it is gr-acyclic (that is to say its gr-homology is 0) and it satisfies  $\ker(d_Z) = \{0\}$ . Explicitly, we can write*

$$F_{\mathcal{O}}(Z, d_Z) \cong (\mathcal{O}(Z/\mathrm{im} d_Z^2), d_{F_{\mathcal{O}}(Z)}),$$

with  $d_{F_{\mathcal{O}}(Z)}^2 = \theta \otimes \mathrm{id}_{Z/\mathrm{im} d_Z^2}$ .

*Proof.* We have  $F_{\mathcal{O}}(0) = \mathcal{O}(0)$ . Pushouts of elements of  $F_{\mathcal{O}}J$  are therefore as follows

$$\begin{array}{ccc} \mathcal{O}(0) & \longrightarrow & A \\ \downarrow \vee_{\alpha} F_{\mathcal{O}}(j_{\alpha}) & & \downarrow \\ \vee_{\alpha} F_{\mathcal{O}}(\hat{\mathcal{Z}}_{\alpha}^{0,\infty}) & \longrightarrow & A \vee \left( \vee_{\alpha} F_{\mathcal{O}}(\hat{\mathcal{Z}}_{\alpha}^{0,\infty}) \right), \end{array}$$

with each  $\hat{\mathcal{Z}}_{\alpha}^{0,\infty}$  equal to a  $\hat{\mathcal{Z}}_{q,n}^{0,\infty}$ . Since the coproduct of free  $(\mathcal{O}, d, \theta)$ -algebras is the free  $(\mathcal{O}, d, \theta)$ -algebra on the sum of their generating modules (see the proof of Proposition C.33), the composite of two such pushouts is equal to  $A \rightarrow A \vee F_{\mathcal{O}}\left(\coprod_{\alpha} \hat{\mathcal{Z}}_{\alpha}^{0,\infty} \coprod \coprod_{\beta} \hat{\mathcal{Z}}_{\beta}^{0,\infty}\right)$ . Hence a transfinite composition of such pushouts has the form  $A \rightarrow A \vee F_{\mathcal{O}}(Z)$ , with  $Z$  a gr-acyclic gr-dg module whose components are free gr-dg modules  $\hat{\mathcal{Z}}_{q,n}^{0,\infty}$ . The last identification is direct.  $\square$

We call  $\mathbb{S}$ -split operad what is defined to be  $\Sigma$ -split in [Hin97]. This definition extends without modification to complete curved operads.

**Theorem C.35.** *Let  $(\mathcal{O}, d, \theta)$  be a complete curved operad which is  $\mathbb{S}$ -split. The category  $\mathrm{Alg}(\mathcal{O})$  of  $(\mathcal{O}, d, \theta)$ -algebras is a cofibrantly generated model category with generating cofibrations  $F_{\mathcal{O}}(I_0^\infty)$  and generating acyclic cofibrations  $F_{\mathcal{O}}(J_0^\infty)$ . The weak equivalences (resp. fibrations) are the maps that are graded quasi-isomorphisms (resp. strict surjections).*

*Proof.* We apply Theorem C.22 to the free-forgetful adjunction given in Proposition C.31. By Proposition C.33, the category  $\mathrm{Alg}(\mathcal{O})$  is complete and cocomplete. We have seen in the proof of Proposition C.33 that the forgetful functor  $\#$  preserves filtered colimit; in particular it preserves filtered  $\aleph_1$ -colimits. We finally have to show that  $\#$  maps relative  $F_{\mathcal{O}}(J_0^\infty)$ -cell complexes to weak equivalences. By Lemma C.34, a relative  $F_{\mathcal{O}}J_0^\infty$ -cell complex can be written as a map  $j : A \rightarrow A \vee F_{\mathcal{O}}\left(\coprod_{\alpha} \hat{\mathcal{Z}}_{\alpha}^{0,\infty}\right)$ . By means of the description of the pushouts of  $(\mathcal{O}, d, \theta)$ -algebras given in the proof of Proposition C.33, we can compute

$$\begin{aligned} A \vee F_{\mathcal{O}}\left(\coprod_{\alpha} \hat{\mathcal{Z}}_{\alpha}^{0,\infty}\right) &\cong A \vee \mathcal{O}\left(\coprod_{\alpha} \hat{\mathcal{Z}}_{\alpha}^{0,\infty} / \mathrm{im} d_{\alpha}^2\right) \\ &\cong \mathcal{O}\left(A\# \coprod \left(\coprod_{\alpha} \hat{\mathcal{Z}}_{\alpha}^{0,\infty} / \mathrm{im} d_{\alpha}^2\right)\right) / (I_A \coprod (\gamma_A^0 - \mathrm{id}_{\mathcal{O}(0)})). \end{aligned}$$

The functor

$$\mathrm{Gr} : \widehat{\mathrm{Filt}}^{\mathrm{gr}}(\mathcal{A}) \rightarrow (\mathrm{dg}\mathcal{A})^{\mathrm{ob}\mathbb{N}}, (V, F, d_V) \mapsto (\mathrm{Gr}_\bullet V, \mathrm{Gr} d_V)$$

commutes with direct sums (see for example [Fre17, Proposition 7.3.8]) and satisfies, for gr-flat complete modules,  $\mathrm{Gr} M \otimes_{\mathrm{Gr}} \mathrm{Gr} N \cong \mathrm{Gr} (M \hat{\otimes} N)$  (see Lemma B.4). Here  $\mathbf{R}$  is a field of characteristic 0, so any module is flat. It follows that the functor  $\mathrm{Gr}$  commutes with the free  $U\mathcal{O}$ -algebra functor and we get

$$\mathrm{Gr} \left( A \vee F_{\mathcal{O}} \left( \Pi_\alpha \hat{\mathcal{Z}}_\alpha^{0,\infty} \right) \right) \cong \mathrm{Gr} (A) \vee_{\mathrm{dg} \mathcal{O}\text{-alg.}} \mathrm{Gr} \mathcal{O} \left( \Pi_\alpha \mathrm{Gr} \left( \hat{\mathcal{Z}}_\alpha^{0,\infty} / \mathrm{im} d_\alpha^2 \right) \right).$$

By the proof of Theorem 4.1.1 in [Hin97], using the fact that  $\mathcal{O}$  is  $\mathbb{S}$ -split, we obtain that the map  $\mathrm{Gr}(j^\#)$  is a quasi-isomorphism.  $\square$

We conclude this section by computing the cofibrant objects in the model category of complete  $(\mathcal{O}, d, \theta)$ -algebras. We use the term *quasi-free* for a complete  $(\mathcal{O}, d, \theta)$ -algebras whose underlying module is free when we forget the predifferential.

**Proposition C.36.** *A complete  $(\mathcal{O}, d, \theta)$ -algebra is cofibrant if and only if it is a retract of a quasi-free complete  $(\mathcal{O}, d, \theta)$ -algebra  $(\mathcal{O}(S), d) / (d^2 - \theta \otimes \mathrm{id})$ , where  $S$  is a complete module endowed with an exhaustive filtration*

$$S_0 = \{0\} \subset S_1 \subset S_2 \subset \cdots \subset \mathrm{colim}_i S_i = S$$

of free modules such that  $S_{i-1} \hookrightarrow S_i$  are split monomorphisms of complete modules with cokernels isomorphic to a sum of complete modules

$$S_i / S_{i-1} \cong \coprod_\alpha \left( \xi^\alpha \cdot \mathbf{R} \amalg \hat{\mathcal{Z}}_{q_\alpha+1, n_\alpha}^{0,\infty} \right)$$

where  $\xi^\alpha$  is in homological degree  $n_\alpha + 1$  and filtration degree  $q_\alpha$ . The predifferential  $d$  is the one of  $\hat{\mathcal{Z}}_{q_\alpha+1, n_\alpha}^{0,\infty}$  on  $\hat{\mathcal{Z}}_{q_\alpha+1, n_\alpha}^{0,\infty}$  and

$$d(\xi^\alpha) + \zeta^\alpha \in (\mathcal{O}(S_{i-1}), d) / (\mathrm{im} (d^2 - \theta \otimes \mathrm{id})),$$

with  $\zeta^\alpha$  is a generator of the gr-dg module  $\hat{\mathcal{Z}}_{q_\alpha+1, n_\alpha}^{0,\infty}$ .

*Proof.* The proof is similar to the proof of Proposition C.27.  $\square$

**Corollary C.37.** *Equivalently, a complete  $(\mathcal{O}, d, \theta)$ -algebra is cofibrant if and only if it is a retract of a quasi-free complete  $(\mathcal{O}, d, \theta)$ -algebra  $(\mathcal{O}(\tilde{S}), d)$ , where  $\tilde{S}$  is a complete module endowed with an exhaustive filtration*

$$\tilde{S}_0 = \{0\} \subset \tilde{S}_1 \subset \tilde{S}_2 \subset \cdots \subset \mathrm{colim}_i \tilde{S}_i = \tilde{S}$$

of free modules such that  $\tilde{S}_{i-1} \hookrightarrow \tilde{S}_i$  are split monomorphisms of complete modules with cokernels isomorphic to a sum of complete modules

$$\tilde{S}_i / \tilde{S}_{i-1} \cong \coprod_\alpha (\xi^\alpha \cdot \mathbf{R} \amalg \zeta^\alpha \cdot \mathbf{R})$$

where  $\xi^\alpha$  is in homological degree  $n_\alpha + 1$  and filtration degree  $q_\alpha$  and  $\zeta^\alpha$  is in homological degree  $n_\alpha$  and filtration degree  $q_\alpha + 1$ . The predifferential  $d$  is such that  $d(\xi^\alpha) + \zeta^\alpha \in (\mathcal{O}(\tilde{S}_{i-1}), d_{i-1})$ , and  $d(\zeta^\alpha)$  is obtained by the fact that  $d^2(\xi^\alpha) = \theta \otimes \xi^\alpha$ .

**C.38. Homotopy category of algebras over a curved operad.** We now come to the study of the homotopy category. We denote by  $\text{Hoalg}(\mathcal{O})$  the homotopy category of  $\text{Alg}(\mathcal{O})$ . We show how a morphism of complete curved operads can provide a Quillen adjunction or a Quillen equivalence between the model category structures.

**Definition C.39.** Let  $\alpha : (\mathcal{O}, d, \theta) \rightarrow (\mathcal{O}', d', \theta')$  be a morphism of complete curved operads.

- (1) We denote by  $\alpha_* : \text{Alg}(\mathcal{O}') \rightarrow \text{Alg}(\mathcal{O})$  the *direct image functor* given by precomposition

$$(\mathcal{O}, d, \theta) \xrightarrow{\alpha} (\mathcal{O}', d', \theta') \rightarrow \text{End}_A.$$

This functor is exact since only the algebra structure changes.

- (2) We denote by  $\alpha^* : \text{Alg}(\mathcal{O}) \rightarrow \text{Alg}(\mathcal{O}')$  the *inverse image functor*, left adjoint to  $\alpha_*$ , given by the following definition: for  $(A, d_A)$  in  $\text{Alg}(\mathcal{O})$ ,

$$\alpha^*(A, d_A) := \mathcal{O}'(A^\#) / (\alpha(\text{id}_{A^\#})(I_A)).$$

We can check that  $\alpha^*(A, d_A)$  satisfies  $d_{\alpha^*(A, d_A)}^2 = \theta' \otimes \text{id}_{\alpha^*(A, d_A)}$ . By a computation similar to the one in the proof of Proposition C.31, this follows from the fact that  $\text{im}(\text{id}_I \otimes d_A^2 - \theta \otimes \text{id}_A) \subset I_A$  by the fact that  $(A, d_A)$  is a  $(\mathcal{O}, d, \theta)$ -algebra and since  $\alpha(\theta) = \theta'$ .

This pair of adjoint functors form a Quillen pair and therefore provides an adjunction on the level of the homotopy categories.

**Theorem C.40.** *Inverse and direct image functors form a Quillen pair, that is we have the adjunction*

$$\mathbb{L}\alpha^* : \text{Hoalg}(\mathcal{O}) \rightleftarrows \text{Hoalg}(\mathcal{O}') : \mathbb{R}\alpha_* = \alpha_*.$$

*Proof.* From Proposition 8.5.3 in [Hir03], it is enough to prove that the left adjoint functor  $\alpha^*$  preserves cofibrations and acyclic cofibrations. By means of the fact that (acyclic) cofibrations are retract of relative  $I_0^\infty$ -cell (resp.  $J_0^\infty$ -cell) complexes (see Proposition 11.2.1 in [Hir03]), it is enough to prove the result on generating (acyclic) cofibrations. We have  $\alpha^*(\mathcal{O}(0), 0) \cong \mathcal{O}'(0)$  since  $I_{\mathcal{O}(0)} = (\mu \otimes (\nu_1 \otimes \cdots \otimes \nu_n) - \gamma_{\mathcal{O}}(\mu \otimes \nu_1 \otimes \cdots \otimes \nu_n) \otimes 1, \mu \in \mathcal{O}(n), \nu_i \in \mathcal{O}(0))$ . Similarly, we show that  $\alpha^*(F_{\mathcal{O}}(\hat{\mathcal{Z}}_{q,n}^{0,\infty})) \cong F_{\mathcal{O}'}(\hat{\mathcal{Z}}_{q,n}^{0,\infty})$  by means of the fact that the ideal  $\text{im}(\text{id}_I \otimes d_{\hat{\mathcal{Z}}_{q,n}^{0,\infty}}^2 - \theta \otimes \text{id}_{\hat{\mathcal{Z}}_{q,n}^{0,\infty}}) \subset I_{F_{\mathcal{O}}(\hat{\mathcal{Z}}_{q,n}^{0,\infty})}$  is sent to the ideal  $\text{im}(\text{id}_I \otimes d_{\hat{\mathcal{Z}}_{q,n}^{0,\infty}}^2 - \alpha(\theta) \otimes \text{id}_{\hat{\mathcal{Z}}_{q,n}^{0,\infty}})$  with  $\alpha(\theta) = \theta'$ . This shows that  $\alpha^*$  preserves acyclic cofibrations. The same reasoning shows that  $\alpha^*$  sends (generating) cofibrations to (generating) cofibrations.  $\square$

Finally, we compare the categories  $\text{Hoalg}(\mathcal{O})$  and  $\text{Hoalg}(\mathcal{O}')$  when the morphism  $\alpha : (\mathcal{O}, d, \theta) \rightarrow (\mathcal{O}', d', \theta')$  is a weak equivalence, that is a graded quasi-isomorphism, of  $\mathbb{S}$ -split complete curved operads compatible with the splitting (that is  $\alpha$  sends the splitting to the splitting).

**Theorem C.41.** *Let  $\alpha : (\mathcal{O}, d, \theta) \rightarrow (\mathcal{O}', d', \theta')$  be a graded quasi-isomorphism of  $\mathbb{S}$ -split complete curved operads compatible with the splittings. Then the functors  $\alpha^*$  and  $\alpha_*$  form a pair of Quillen equivalences, that is the functors*

$$\mathbb{L}\alpha^* : \text{Hoalg}(\mathcal{O}) \rightleftarrows \text{Hoalg}(\mathcal{O}') : \mathbb{R}\alpha_* = \alpha_*$$

are equivalences of the homotopy categories.

*Proof.* The functors  $\alpha_*$  reflects weak equivalences so by Corollary 1.3.16 in [Hov99], it is enough to show that the unit of the adjunction  $A \rightarrow \alpha_*(\alpha^*(A))$  is a weak equivalence, that is a graded quasi-isomorphism, for every cofibrant  $(\mathcal{O}, d, \theta)$ -algebra  $(A, d_A)$ . Since we are working over a field  $\mathbf{R}$  of characteristic 0, we have already seen that the functor  $\text{Gr}$  commutes with direct sums and preserves the tensor products. It follows that

$$\text{Gr } \alpha^*(A, d_A) \cong (\text{Gr } \mathcal{O}')(\text{Gr } A^\#) / ((\text{Gr } \alpha)(\text{id}_{\text{Gr } A^\#})(I_{\text{Gr } A})),$$

since  $\text{Gr } I_A \cong I_{\text{Gr } A}$ . The result now follows from the proof given in Section 4.7 in [Hin97].  $\square$

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