



HAL
open science

Some insights on rightmost spectral values assignment for time delay systems

Souad Amrane, Islam Boussaada, Fazia Bedouhene, Silviu-Iulian Niculescu

► **To cite this version:**

Souad Amrane, Islam Boussaada, Fazia Bedouhene, Silviu-Iulian Niculescu. Some insights on rightmost spectral values assignment for time delay systems. IFAC 2020 - 21st IFAC World Congress, Jul 2020, Berlin / Virtual, Germany. 10.1016/j.ifacol.2020.12.066 . hal-02889940

HAL Id: hal-02889940

<https://hal.science/hal-02889940v1>

Submitted on 6 Jul 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Some insights on rightmost spectral values assignment for time delay systems

Souad Amrane* Islam Boussaada**** Fazia Bedouhene*
Silviu-Iulian Niculescu**

* *Laboratoire de Mathématiques Pures et Appliquées, Université Mouloud Mammeri, Tizi-Ouzou, Algeria, (e-mail: amranesouad@yahoo.fr, fazia.bedouhene@ummo.dz)*

** *Université Paris-Saclay, CNRS, CentraleSupélec, Laboratoire des Signaux et Systèmes (L2S), Inria Saclay, DISCO Team, 3 rue Joliot-Curie, F-91192 Gif-sur-Yvette cedex, France (e-mail: Islam.Boussaada@l2s.centralesupelec.fr, Silviu.Niculescu@l2s.centralesupelec.fr)*

*** *IPSA, 63 Boulevard de Brandebourg, 94200 Ivry-sur-Seine, France*

Abstract: This paper investigates the stability and stabilization of some generic linear second-order time-invariant retarded system with single delay. It provides an appropriate stability criterion based on the manifold defined by the coexistence of the maximal number of negative spectral values. Next, such ideas are exploited in the context of delayed output feedback by an appropriate “partial” pole placement guaranteeing simultaneously the stability in closed-loop and an appropriate exponential decay rate. To perform such an analysis, the argument principle is explicitly used.

Keywords: Time-delay systems, frequency-domain approaches, controller design, argument principle.

1. INTRODUCTION

Time-delay is encountered in many fields and applications, such as in Physics, Biology and Economy. A strong and ever-growing interest on time-delay systems has been reported over the years, see, for instance, Niculescu [2001], Bellman and Cooke [1963], Hale and Lunel [1993], Stépán [1979] and references therein.

In the stability analysis of time-delay systems, many approaches have been developed in both time- and frequency-domain. In particular, a wide range of frequency-domain techniques have been derived, addressing the distribution of the roots of the corresponding characteristic equation and related properties, see for instance Olgac and Sipahi [2002], Michiels and Niculescu [2007], Walton and Marshall [1987], Sipahi et al. [2011], Boussaada et al. [2015], Cooke and van den Driessche [1986].

This work is motivated by recent studies Boussaada and Niculescu [2016b], Boussaada et al. [2018], Boussaada and Niculescu [2016b,a] where a property called *Multiplicity-Induced-Dominancy* (MID) is emphasized, which consists in characterizing the exponential decay rate of the trivial solution. Recently, in Boussaada and Niculescu [2016b], a result by Polya and Szegô is revisited and exploited allowing to explicitly derive a *bound* for the number of real roots of the quasipolynomial function. Such a bound is nothing else than the *degree of the quasipolynomial* Boussaada et al. [2018], that is the sum of the involved polynomials plus the number of the involved delays. In Boussaada and Niculescu [2018] an analytical proof for the dominance of the spectral

value with maximal multiplicity for second-order systems is explicitly provided. Quite recently, Amrane et al. [2018] showed that the multiplicity of a real root itself is not important as such but its connection with the dominance of this root is a meaningful tool for control synthesis. As a matter of fact, it is proven that, under appropriate conditions, the coexistence of the maximal number of negative distinct roots guarantees their dominance. In this case, an adequate factorization is derived in the scalar and second-order delay differential equations with single delay allowing to write the quasipolynomial in an appropriate integral operator form. Furthermore, if these (real) roots are negative, this guarantees the asymptotic stability of the trivial solution.

In some cases, such a factorization is hard to be established, especially for the quasipolynomials with higher degrees. The stability criteria proposed and developed in Stépán [1979] and generalized later in Hassard [1997] allow to compute analytically the number of characteristic roots in the right-half complex plane (i.e., unstable roots) for time-delay system of retarded type avoiding the direct calculation of the corresponding improper integral. Our study is mainly inspired by Boussaada et al. [2020], where the property of MID is investigated for the generic second-order retarded differential equation that we address in this paper. In other words, the present paper focuses on the application of the Stépán-Hassard approach to show the dominance of spectral values which are not necessarily multiple.

The remaining of the paper is organized as follows. In Section 2, we present the problem formulation and we recall some known results on the spectrum distribution of retarded delay systems useful in the forthcoming sections. Next, conditions on the system parameters guaranteeing the co-existence of the maximal number of real roots is carried out in Section 3. In the particular case of equidistributed real roots the argument principle is applied to prove the dominance of such real spectral values. Some concluding remarks end the paper.

2. PRELIMINARIES

2.1 Problem Statement

Consider the following generic LTI system including one delay:

$$\dot{x}(t) = Ax(t) + Bx(t - \tau), \quad (1)$$

where $x(t) \in \mathbb{R}^n$ and A and B are appropriate $n \times n$ real matrices. The corresponding spectral values are characterized by evaluating for s the following characteristic equation:

$$\Delta(s) = \det(sI - A - Be^{-s\tau}) = 0. \quad (2)$$

The asymptotic stability of the trivial solution is guaranteed if all roots of (2) lie in the left-half plane. Notice also that real spectral values correspond to non oscillatory solutions of (1).

It is also well known that second-order linear systems capture the dynamic behavior of many natural phenomena and have found numerous applications in a variety of fields, such as vibration, structural analysis and human balancing. In this paper we focus on planar systems (1) (i.e. $n = 2$) and restrict our analysis to systems where the corresponding quasipolynomial writes under the form:

$$\Delta(s, \tau) = P(s) + Q(s)e^{-s\tau}$$

with $\deg(P) = 2$ and $\deg(Q) = 1$. As mentioned in the Introduction, we will explicitly *investigate the effect of the coexistence of real roots on the stability of the trivial solution*. In other words, we wish to give an answer to the following question: *Is the coexistence of sufficiently many negative roots guarantees the location of the remaining roots in the left-half plane?*

2.2 Prerequisites

In complex analysis, it is well-known that the *argument principle* is a consequence of Cauchy theorem, and it connects the winding number of a closed rectifiable curve (see, e.g., Conway [1984] for a deeper discussion on the topics) with the number of zeros and poles inside the curve. This result gives insights on the location of zeros and poles of a given meromorphic function.

Theorem 1. (Argument principle): let $V \in \mathbb{C}$ be a bounded domain with smooth boundary Γ positively oriented (counter-clockwise) and let f be a meromorphic function inside the contour Γ , then

$$\frac{1}{2i\pi} \oint_{\Gamma} \frac{f'(z)}{f(z)} dz = Z - P, \quad (3)$$

where P and Z denote the number of poles and zeros of f in V , counted with their multiplicities.

In particular, if $f(z)$ is analytic inside Γ , then the left-hand side of (3) gives the number of zeros of $f(z)$ inside Γ .

We need also to recall the statement of Hassard's Theorem: *Theorem 2.* (Hassard's Theorem). Let A_1, \dots, A_m be real n by n matrices, and let τ_1, \dots, τ_m be nonnegative reals. Let

$$\Delta(s, \tau) = \det(sI - \sum_{j=1}^m e^{-s\tau_j} A_j). \quad (4)$$

Let ρ_1, \dots, ρ_J be the positive zeros of $R(y) = \Re(i^{-n}\Delta(iy))$, counted by multiplicity and ordered so that $\rho_1 \geq \dots \geq \rho_J > 0$. For each $j = 1, \dots, J$ such that $\Delta(i\rho_j) = 0$, assume that the multiplicity of $i\rho_j$ a zero of $\Delta(\lambda)$ is the same as the multiplicity of ρ_j as a zero of $R(y)$. Then, the number of roots of the characteristic equation $\Delta(\lambda) = 0$ which lie in $\Re(s) > 0$, counted by multiplicity, is given by the formula

$$\frac{n - K}{2} + \frac{1}{2}(-1)^J \text{sgn}S^{(\kappa)}(0) + \sum_{j=1}^J (-1)^{j-1} \text{sgn}S(\rho_j) \quad (5)$$

where K is the number of zeros of $\Delta(s, \tau)$ on $\Re(\lambda) = 0$, counted by multiplicity, k is the multiplicity of $s = 0$ as a root of $\Delta(s, \tau) = 0$, and $S(y) = \text{Im}(i^{-n}\Delta(iy))$. Furthermore, the count (5) is odd if $\Delta^{(k)}(0) < 0$ and is even if $\Delta^{(k)}(0) > 0$. If $R(y)$ has no positive zeros, set $r = 0$ and omit the summation term in (5). If $\lambda = 0$ is not a root of the characteristic equations, set $\kappa = 0$ and interpret $S(0)^{(0)}$ as $S(0)$ and $\Delta(0)^{(0)}$ as $\Delta(0)$.

3. MAIN RESULTS

Consider now, the second-order system

$$\ddot{x}(t) + a_1\dot{x}(t) + a_0x(t) = u(t) \quad (6)$$

where u is the unknown control. Assume that the system (6) is unstable in the uncontrolled case ($u(t) = 0$). Our aim is to construct an appropriate delayed-state-feedback controller of the form:

$$u(t) = -\alpha_0x(t - \tau) - \alpha_1\dot{x}(t - \tau), \quad (7)$$

allowing to guarantee the stability of the system (6) in closed-loop. The characteristic quasi-polynomial function corresponding to the closed-loop system is described as follows:

$$\Delta(s, \tau) = P(s) + Q(s)e^{-s\tau} = 0, \quad (8)$$

where $P(s)$ and $Q(s)$ are polynomials in s with degree of $Q(s)$ is less than the degree of $P(s)$. In our case, we will have explicitly

$$P(s) = s^2 + a_1s + a_0s \quad \text{and} \quad Q(s) = \alpha_1s + \alpha_0. \quad (9)$$

3.1 On qualitative properties of s_1 as a root of (8)

The following proposition gives conditions on the parameter coefficients that allows assigning a maximum number of spectral values of the second-order system (6)-(7).

Proposition 3. The following assertions hold:

- The quasipolynomial (8) admits four distinct real spectral values s_1, s_2, s_3 and s_4 with $s_4 < s_3 < s_2 < s_1$ if and only if the parameters a_1, a_2, α_1 and α_0 satisfy

$$\left\{ \begin{array}{l} a_1(\tau) = \frac{1}{Q(\tau)} \begin{vmatrix} 1 & s_1^2 & s_1 e^{-s_1\tau} & e^{-s_1\tau} \\ 1 & s_2^2 & s_2 e^{-s_2\tau} & e^{-s_2\tau} \\ 1 & s_3^2 & s_3 e^{-s_3\tau} & e^{-s_3\tau} \\ 1 & s_4^2 & s_4 e^{-s_4\tau} & e^{-s_4\tau} \end{vmatrix}, \\ a_0(\tau) = \frac{1}{Q(\tau)} \begin{vmatrix} s_1^2 & s_1 & s_1 e^{-s_1\tau} & e^{-s_1\tau} \\ s_2^2 & s_2 & s_2 e^{-s_2\tau} & e^{-s_2\tau} \\ s_3^2 & s_3 & s_3 e^{-s_3\tau} & e^{-s_3\tau} \\ s_4^2 & s_4 & s_4 e^{-s_4\tau} & e^{-s_4\tau} \end{vmatrix}, \\ \alpha_1(\tau) = \frac{1}{Q(\tau)} \begin{vmatrix} 1 & s_1 & s_1 e^{-s_1\tau} & s_1^2 \\ 1 & s_2 & s_2 e^{-s_2\tau} & s_2^2 \\ 1 & s_3 & s_3 e^{-s_3\tau} & s_3^2 \\ 1 & s_4 & s_4 e^{-s_4\tau} & s_4^2 \end{vmatrix}, \\ \alpha_0(\tau) = \frac{1}{Q(\tau)} \begin{vmatrix} 1 & s_1 & s_1^2 & s_1 e^{-s_1\tau} \\ 1 & s_2 & s_2^2 & s_2 e^{-s_2\tau} \\ 1 & s_3 & s_3^2 & s_3 e^{-s_3\tau} \\ 1 & s_4 & s_4^2 & s_4 e^{-s_4\tau} \end{vmatrix}, \end{array} \right. \quad (10)$$

where

$$Q(\tau) = \begin{vmatrix} 1 & s_1 & s_1 e^{-s_1\tau} & e^{-s_1\tau} \\ 1 & s_2 & s_2 e^{-s_2\tau} & e^{-s_2\tau} \\ 1 & s_3 & s_3 e^{-s_3\tau} & e^{-s_3\tau} \\ 1 & s_4 & s_4 e^{-s_4\tau} & e^{-s_4\tau} \end{vmatrix}. \quad (11)$$

- The spectral value s_1 is negative if and only if there exists $\tau_0 > 0$ such that

$$a_1(\tau_0) + s_2 = 0. \quad (12)$$

Sketch of the Proof:

- According to the G. Pólya and G. Szegő Theorem Pólya and Szegő [1972], the number of real roots of (8) is four, hence we investigate the existence of four distinct real spectral values $s_1 > s_2 > s_3 > s_4$. The coefficients of the quasi-polynomial function (8) described by (10) and (11) are obtained by solving the following system

$$s_i^2 + a_1 s_i + a_0 + (\alpha_1 s_i + \alpha_0) \exp(-s_i \tau) = 0, \quad i = 1 \dots 4, \quad (13)$$

that we can represent under the more suitable form:

$$\begin{bmatrix} 1 & s_1 & s_1 e^{-s_1\tau} & e^{-s_1\tau} \\ 1 & s_2 & s_2 e^{-s_2\tau} & e^{-s_2\tau} \\ 1 & s_3 & s_3 e^{-s_3\tau} & e^{-s_3\tau} \\ 1 & s_4 & s_4 e^{-s_4\tau} & e^{-s_4\tau} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \alpha_1 \\ \alpha_0 \end{bmatrix} = \begin{bmatrix} -s_1^2 \\ -s_2^2 \\ -s_3^2 \\ -s_4^2 \end{bmatrix} \quad (14)$$

System (13) admits a unique solution because the Vandermonde-type matrix in $Q(\tau)$ is invertible, for every $\tau > 0$. Indeed, its determinant is positive since $\tau \mapsto Q(\tau)$ is increasing from 0 to ∞ , with $Q(0) = 0$. To get the formulas (10), then just apply the Cramer's rule.

- The negativity of the root s_1 is shown through the variation of the function

$$\tau \mapsto a_1(\tau) + s_2. \quad (15)$$

So, for all values of $\tau \in \mathbb{R}^{+*}$ ($\tau > 0$), this later function is continuous and increasing from $-\infty$ to $-s_1$, from which we deduce that $\tau \mapsto a_1(\tau) + s_2$ takes a positive values if and only if $s_1 < 0$. See Amarne et al. [2018].

□

Due to the computation complexity of these time-delay dependent coefficients, we consider only the case of equidis-

tance roots. This allows also to decrease the number of parameters to handle. So, we suppose that the roots of (8) are such that:

$$d = |s_1 - s_2| = |s_2 - s_3| = |s_3 - s_4|. \quad (16)$$

Substituting the formula (16) in (10) and (11), the above parameter coefficients can be rewritten as follows:

$$\left\{ \begin{array}{l} a_1(\tau) = \frac{1}{e^{d\tau} - 1} (2s_1 - 5d + de^{d\tau} - 2s_1 e^{d\tau}) \\ a_0(\tau) = \frac{1}{(e^{d\tau} - 1)^2} (6d^2 - ds_1 e^{2d\tau} + 6ds_1 e^{d\tau} - 5ds_1 + s_1^2 e^{2d\tau} - 2s_1^2 e^{d\tau} + s_1^2) \\ \alpha_1(\tau) = 2de^{-2d\tau} \frac{e^{\tau s_1}}{e^{-d\tau} - 1} \\ \alpha_0(\tau) = \frac{-2de^{-2d\tau} e^{\tau s_1}}{(e^{-d\tau} - 1)^2} (3d - s_1 + s_1 e^{-d\tau}). \end{array} \right. \quad (17)$$

The following theorem gives conditions on the negativeness of s_1 .

Theorem 4. The spectral values s_1 is negative if and only if one of the following equivalent conditions is satisfied:

- (1) The spectral value s_1 is explicitly given by:

$$s_1 = -4 \frac{d}{e^{d\tau} - 1}.$$

- (2) The delay τ takes the value

$$\tau^* = \frac{1}{d} \ln \frac{1}{s_1} (-4d + s_1).$$

- (3) The distance d between the real roots satisfies

$$d = \left[\left\{ \frac{1}{4} s_1 - \frac{1}{\tau} \text{LambertW} \left(1, \frac{\tau s_1}{4} e^{\frac{\tau s_1}{4}} \right) \mid l \in \mathbb{Z} \right\} \setminus \left\{ \frac{2i\pi l}{\tau} \mid l \in \mathbb{Z} \right\} \right] \cap]0, +\infty[.$$

Sketch of the Proof: We use the characterization of the negativeness of s_1 given by equation (12). Observe that, as $\tau \rightarrow \infty$

$$a_1(\tau) = \frac{1}{e^{d\tau} - 1} (2s_1 - 5d + de^{d\tau} - 2s_1 e^{d\tau}) \rightarrow d - 2s_1$$

with the property that $\tau \mapsto a_1(\tau)$ is bounded by $-s_1$. So if there exists τ^* such that $a_1(\tau^*) + s_1 - d = 0$, we obtain that $s_1 < 0$, and vis versa. Now, to get the value of s_1 , τ^* and d respectively, we just have to solve the equation

$$4d - s_1 + s_1 e^{d\tau} = 0. \quad (18)$$

Easy computation gives the value of s_1 and τ^* . For the value of d , it can be calculated using the Lambert W function since the unknown (d) appears both outside and inside the exponential function. The Lambert W function is defined as the multivalued function that satisfies

$$x = W(x) e^{W(x)},$$

for any complex number x . Equivalently, it may be defined as the inverse of the complex function $f(x) = x e^x$.

In order to solve equation (18) for d we rewrite it as follows:

$$\left(d - \frac{s_1}{4} \right) e^{-\tau d} = -\frac{s_1}{4},$$

this implies that

$$-\tau \left(d - \frac{s_1}{4} \right) e^{-\tau \left(d - \frac{s_1}{4} \right)} = \frac{s_1 \tau}{4} e^{\frac{s_1 \tau}{4}}.$$

From the later equation, we have

$$\text{LambertW} \left(\frac{\tau s_1}{4} e^{\frac{\tau s_1}{4}} \right) = -d\tau + \frac{\tau s_1}{4}. \quad (19)$$

Hence

$$d = \left\{ \left[\frac{1}{4} s_1 - \frac{1}{\tau} \text{LambertW} \left(1, \frac{\tau s_1}{4} e^{\frac{\tau s_1}{4}} \right) \mid l \in \mathbb{Z} \right] \setminus \left[\frac{2i\pi l}{\tau} \mid l \in \mathbb{Z} \right] \right\} \cap]0, +\infty[.$$

□

4. ILLUSTRATIVE EXAMPLE

As an illustration of the presented result, we consider the problem of stabilizing the Mach number of a transonic flow in a wind tunnel. The analysis of transonic flows is a challenging problem in compressible fluid dynamics, since a full model of the flow would involve considering the Navier–Stokes equations in a three-dimensional domain and boundary controls for temperature and pressure regulation. A simplified model was considered in Armstrong and Tripp [1981] in order to analyze the response of the Mach number of the flow to changes in the guide vane angle. Instead of using a PDE model, propagation phenomena are modeled through a time delay, leading to the time-delay system

$$\begin{cases} \kappa m'(t) + m(t) = k\vartheta(t - \tau_0), \\ \vartheta''(t) + 2\zeta\omega\vartheta'(t) + \omega^2\vartheta(t) = \omega^2 u(t), \end{cases} \quad (20)$$

in which m , ϑ , and u represent perturbations of the Mach number of the flow, the guide vane angle, and the input of the guide vane actuator, respectively, with respect to steady-state values. The parameters κ and k depend on the steady-state operating point and are assumed to be constant as long as m , ϑ , and u remain small, and satisfy $\kappa > 0$ and $k < 0$. The parameters $\zeta \in (0, 1)$ and $\omega > 0$ come from the design of the guide vane angle actuator and are thus independent of the operating point. The time delay τ_0 is assumed to depend only on the temperature of the flow. Notice that, in the absence of control ($u(t) = 0$), the open-loop system (20) is exponentially stable.

The design of the stabilizing feedback for (20) improving its stability properties has been considered in Manitius [1984], Boussaada et al. [2018]. The design we propose here is a delayed PD controller which can be written $u(t) = \beta_0 x(t - \tau_1) + \beta_1 \dot{x}(t - \tau_1)$. In closed-loop system, the corresponding characteristic equation reads:

$$\Delta(s, \tau_1) = (s - a) \left((s\beta_1 + \beta_0 t) e^{-s\tau_1} + 2\omega s\zeta + \omega^2 + s^2 \right)$$

Since the parameter a is fixed then one focus on the second factor only,

$$\Delta_4(s, \tau_1) = s^2 + 2\omega s\zeta + \omega^2 + (s\beta_1 + \beta_0) e^{-s\tau_1}. \quad (21)$$

To assign four equidistributed rightmost real spectral values $s_k = -k$ for $k \in [1, \dots, 4]$ one uses (17). For instance, for $\omega = 1.2$ and damping factor $\zeta = 0.472$ it is sufficient to set

$$\beta_0 = -\frac{3087}{6050}, \beta_1 = -\frac{343}{3630}, \tau_1 = \ln \left(\frac{22}{7} \right). \quad (22)$$

Figure 1 exhibits the spectrum distribution corresponding to (21).

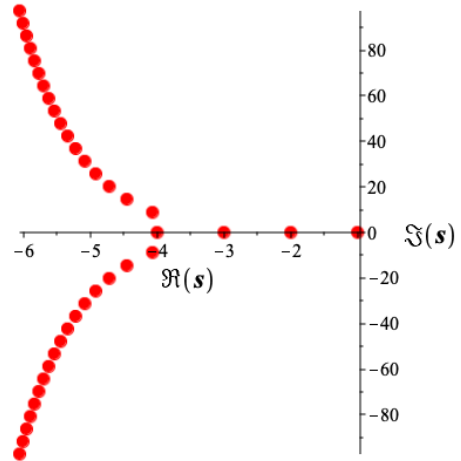


Fig. 1. Zeros distribution of quasipolynomial (21) with $\omega = 1.2$, $\zeta = 0.472$ and the controller gains given by .

5. CONCLUSION

A new dominance result of the spectral values which are not necessarily multiple is established through this paper for a class of second order time delay system with single delay. The stability analysis is based on the argument principle and the Stépàn-Hassard formula which allows to count the number of unstable roots of the corresponding characteristic equation. Further, to show the potential applicability of the result, an illustrative example is provided. The problem of stabilizing the Mach number of a transonic flow in a wind tunnel is discussed and a stabilizing delayed PD-controller is provided. This study is not achieved yet since we considered only the case where the real part of the quasipolynomial function has no real roots. The other case remains under investigation.

REFERENCES

- S. Amarne, F. Bedouhene, I. Boussaada, and S-I. Niculescu. On qualitative properties of low-degree quasipolynomials: Further remarks on the spectral abscissa and rightmost-roots assignment. *Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie*, 61(109):361–381, 2018.
- Ernest S. Armstrong and John S. Tripp. An application of multivariable design techniques to the control of the National Transonic Facility. Technical Paper 1887, NASA, August 1981.
- R. Bellman and K. Cooke. *Differential-difference equations*. Academic Press, New York, 1963.
- I. Boussaada and S. I. Niculescu. Tracking the algebraic multiplicity of crossing imaginary roots for generic quasipolynomials: A Vandermonde-based approach. *IEEE Transactions on Automatic Control*, 61: 1601–1606, 2016a.
- I. Boussaada and S-I. Niculescu. Characterizing the codimension of zero singularities for time-delay systems. *Acta Applicandae Mathematicae*, 145(1):47–88, 2016b.

- I. Boussaada and S-I. Niculescu. On the dominance of multiple spectral values for time-delay systems with applications. *IFAC-PapersOnLine*, 51(14):55–60, 2018.
- I. Boussaada, W. Michiels, and S-I. Niculescu. Further remarks on spectral analysis of delay differential-algebraic systems. *Preprint*, pages 1–12, 2015.
- I. Boussaada, S-I. Niculescu, and K. Trabelsi. Towards a decay rate assignment based design for time-delay systems with multiple spectral values. 2018.
- I. Boussaada, S-I. Niculescu, A. El Ati, R. Pérez-Ramos, and L-K. Trabelsi. Multiplicity-Induced-Dominancy in parametric second-order delay differential equations: Analysis and application in control design. *ESAIM: Control, Optimisation and Calculus of Variations*, 2020. doi: 10.1051/cocv/2019073.
- J.B. Conway. *Functions of one complex variable*. Springer; New York, 2nd edition, 1984.
- K-L. Cooke and Pauline van den Driessche. On zeroes of some transcendental equations. *Funkcial. Ekvac.*, 29(1): 77–90, 1986.
- J. K. Hale and S. M. Verduyn Lunel. *Introduction to functional differential equations*, volume 99 of *Applied Mathematics Sciences*. Springer Verlag, New York, 1993.
- B-D. Hassard. Counting roots of the characteristic equation for linear delay-differential systems. *Journal of Differential Equations*, 136(2):222 – 235, 1997.
- Andrzej Z. Manitius. Feedback controllers for a wind tunnel model involving a delay: analytical design and numerical simulation. *IEEE Trans. Automat. Control*, 29(12):1058–1068, 1984. ISSN 0018-9286. doi: 10.1109/TAC.1984.1103436. URL <https://doi.org/10.1109/TAC.1984.1103436>.
- W. Michiels and S-I. Niculescu. *Stability and stabilization of time-delay systems*, volume 12 of *Advances in Design and Control*. SIAM, 2007.
- S-I. Niculescu. Delay Effects on Stability, A Robust Control Approach. *Springer, Lecture Notes in Control And Information Sciences*, 2001.
- N. Olgac and R. Sipahi. An exact method for the stability analysis of time delayed linear time-invariant (lti) systems. *IEEE Transactions on Automatic Control*, 47(5):793–797, 2002.
- G. Pólya and G. Szegő. *Problems and Theorems in Analysis: Series, Integral Calculus, Theory of Functions; Translation [from the German] by D. Aepli*. Springer, 1972.
- R. Sipahi, S-I. Niculescu, C. T. Abdallah, W. Michiels, and K. Gu. Stability and stabilization of systems with time delay. *IEEE Control Systems*, 31(1):38–65, Feb 2011.
- G. Stépán. *On the stability of linear differential equations with delay*. Qualitative Theory of Differential Equations, Colloquia Mathematica Societas Janos Bolyai, 1979.
- K. Walton and J. E. Marshall. Direct method for tds stability analysis. *IEE Proceedings D - Control Theory and Applications*, 134(2):101–107, 1987.