ERGODIC BEHAVIOR OF CONTROL AND MEAN FIELD GAMES PROBLEMS DEPENDING ON ACCELERATION

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ABSTRACT. The goal of this paper is to study the long time behavior of solutions of the first-order mean field game (MFG) systems with a control on the acceleration. The main issue for this is the lack of small time controllability of the problem, which prevents to define the associated ergodic mean field game problem in the standard way. To overcome this issue, we first study the long-time average of optimal control problems with control on the acceleration: we prove that the time average of the value function converges to an ergodic constant and represent this ergodic constant as a minimum of a Lagrangian over a suitable class of closed probability measure. This characterization leads us to define the ergodic MFG problem as a fixed-point problem on the set of closed probability measures. Then we also show that this MFG ergodic problem has at least one solution, that the associated ergodic constant is unique under the standard monotonicity assumption and that the time-average of the value function of the time-dependent MFG problem with control of acceleration converges to this ergodic constant.

CONTENTS

1. INTRODUCTION

The main goal of this paper is to study the asymptotic behavior of mean field games (MFG) system with acceleration. Let us recall that such systems, first introduced in [33, 1], aim to describe models with infinite number of interacting agents who control their acceleration. The MFG models we study here

Date: July 2, 2020.

read as follows

(1.1)
$$\begin{cases} -\partial_t u^T(t,x,v) + \frac{1}{2}|D_v u^T(t,x,v)|^2 - \langle D_x u^T(t,x,v), v\rangle = F(x,v,m_t^T), & \text{in } [0,T]\times\mathbb{T}^d\times\mathbb{R}^d \\ \partial_t m_t^T - \langle v, D_x m_t^T\rangle - \operatorname{div}\left(m_t^T D_v u^T(t,x,v)\right) = 0, & \text{in } [0,T]\times\mathbb{T}^d\times\mathbb{R}^d \\ u^T(T,x,v) = g(x,v,m_T^T), \quad m_0^T(x,v) = m_0(x,v) \text{ in } \mathbb{T}^d\times\mathbb{R}^d. \end{cases}$$

The above coupled system is a particular case of the more general class of MFG systems, which aim to describe the optimal value u and the distribution m of players, in a Nash equilibrium, for differential games with infinitely many small players. This models were introduced independently by Lasry and Lions [29, 30, 31] and Huang, Malhamé and Caines [27, 26] and since these pioneering works the MFG theory has grown very fast: see for instance the survey papers and the monographs [15, 25, 8, 21] and the references therein. In system (1.1) the pair (u^T, m^T) can be interpreted as follows: u^T is the value function of a typical small player for an optimal control problem of acceleration in which the cost depends on the time-dependent family of probability measures (m_t^T); on the other hand, m_t^T is, for each time t, the distribution of the small players; it evolves in time according to the continuity equation driven by the optimal feedback of the players.

During the last years, the question of the long time behavior of solutions of (standard) MFG systems has attracted a lot of attention. Results describing the long-time average of solutions were obtained in several context: see [17, 18], for second order systems on $\mathbb{T}^d$, and [16, 11, 10], for first order systems on $\mathbb{T}^d$, $\mathbb{R}^d$ and for state constraint case respectively. Recently, the first author and Porretta studied the long time behavior of solutions for the so-called Master equation associated with a second order MFG system, see [20]. In view of the results obtained in these works one would expect the limit of u^T/T to be described by the following ergodic system

(1.2)
$$\begin{cases} \frac{1}{2}|D_v u(x,v)|^2 - \langle D_x u(x,v), v\rangle = F(x,v,m), & (x,v)\in\mathbb{T}^d\times\mathbb{R}^d \\ -\langle v, D_x m\rangle - \operatorname{div}\left(m D_v u(x,v)\right) = 0, & (x,v)\in\mathbb{T}^d\times\mathbb{R}^d \\ \int_{\mathbb{T}^d\times\mathbb{R}^d} m(dx,dv) = 1. \end{cases}$$

The main issue of this paper is that this ergodic system makes no sense. Indeed, as we explain below, even for problems without mean field interaction, we cannot expect to have a solution to the corresponding ergodic Hamilton-Jacobi equation (the first equation in (1.2)). As the drift of the continuity equation (the second equation in (1.2)) is given in terms of solution to the ergodic Hamilton-Jacobi equation, there is no hope to formulate the problem in this way. As far as we know, this is the first time this kind of problem is faced in the literature.

To overcome the issue just described, we first study the ergodic Hamilton-Jacobi equation without mean field interaction. More precisely, in the first part of the paper we investigate the existence of the limit, as T tends to infinity, of $u^T(0,\cdot,\cdot)/T$, where now u^T solves the Hamilton-Jacobi equation (without mean field interaction)

$$\begin{cases} -\partial_t u^T(t,x,v) + \frac{1}{2}|D_v u^T(t,x,v)|^2 - \langle D_x u^T(t,x,v), v\rangle = F(x,v), & \text{in } [0,T]\times\mathbb{T}^d\times\mathbb{R}^d \\ u^T(T,x,v) = 0 \text{ in } \mathbb{T}^d\times\mathbb{R}^d. \end{cases}$$

Here $F : \mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}$ is periodic in space (the first variable) and coercive in velocity (the second one). Following the seminal paper [32], it is known that the existence of the limit of u^T/T is related with the existence of a corrector, namely to a solution of the ergodic Hamilton-Jacobi equation:

$$-\langle D_x u(x,v), v\rangle + \frac{1}{2}|D_v u(x,v)|^2 = F(x,v) + \bar{c}, \quad (x,v)\in \times\mathbb{T}^d\times\mathbb{R}^d,$$

for some constant $\bar{c}$. However, we stress again the fact that due to the lack of coercivity and due to the lack of small time controllability of our model, we do not expect the existence of a continuous viscosity solutions of the ergodic equation. This problem has been overcome in several other frameworks: we can quote for instance [34, 19, 37, 14, 7, 5, 3, 6, 9, 24, 23], for related problems see also [2, 28] and the references therein. Following techniques developed in [6] we prove in the first part of Theorem 2.2 that the limit of u^T/T exists and is equal to a constant. However, this convergence result does not suffice to handle our MFG system of acceleration: indeed, we also need to understand, when the map F also depends on the extra parameter m, how this ergodic constant depends on m. For doing so, we follow ideas from weak-KAM theory (see for instance [22]) and characterize the ergodic constant in terms of closed probability measures: namely, we prove in the second part of Theorem 2.2 that, for any $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$,

$$\lim_{T\to+\infty} \frac{u^T(0,x,v)}{T} = \inf_{\mu\in\mathcal{C}} \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v)\right)\, \mu(dx,dv,dw)$$

where $\mathcal{C}$ is the set of Borel probability measures μ on $\mathbb{T}^d \times \mathbb{R}^d$ with suitable finite moments and which are closed in the sense that, for any test function $\varphi \in C_c^\infty(\mathbb{T}^d \times \mathbb{R}^d)$,

$$\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \Big(\langle D_x\varphi(x,v), v\rangle + \langle D_v\varphi(x,v), w\rangle\Big)\, \eta(dx,dv,dw) = 0,$$

(see also Definition 2.1).

We now come back to our MFG of acceleration (1.1). In view of the characterization of the ergodic constant for the Hamilton-Jacobi equation without mean field interaction, it is natural to describe an equilibrium for the ergodic MFG problem with acceleration as a fixed-point problem on the Wasserstein space: we say that $(\bar\lambda, \bar\mu) \in \mathbb{R} \times \mathcal{C}$ is a solution of the ergodic MFG problem of acceleration if

$$\begin{aligned}\bar\lambda &= \inf_{\mu\in\mathcal{C}} \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,\pi\sharp\bar\mu)\right)\, \mu(dx,dv,dw)\\ &= \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,\pi\sharp\bar\mu)\right)\, \bar\mu(dx,dv,dw),\end{aligned}$$

where $\pi : \mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{T}^d\times\mathbb{R}^d$ is the canonical projection onto the first two variables. We show that such an ergodic MFG problem with acceleration has a solution and that the associated ergodic constant $\bar\lambda$ is unique under the following monotonicity condition (first introduced in [29, 30]): there exists a constant $M_F > 0$ such that for any $m_1, m_2 \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$

$$\begin{aligned}&\int_{\mathbb{T}^d\times\mathbb{R}^d} \big(F(x,v,m_1) - F(x,v,m_2)\big)\, \big(m_1(dx,dv) - m_2(dx,dv)\big)\\ &\quad\geq M_F \int_{\mathbb{T}^d\times\mathbb{R}^d} \big(F(x,v,m_1) - F(x,v,m_2)\big)^2\, dxdv,\end{aligned}$$

see (1) in Theorem 2.5. The main result of the paper is the fact that, if (u^T, m^T) solves the MFG system of acceleration (1.1), then $u^T(0,x,v)/T$ converges, as T tends to infinity, to the unique ergodic constant $\bar\lambda$ of the ergodic MFG problem, see (2) in Theorem 2.5. The main technical step for this is to rewrite the MFG system in terms of time-dependent closed measure (a kind of occupation measure in this set-up), see Theorem 4.3, and to understand the long-time average of these measures.

The rest of this paper is organized as follows. In Section 2, we introduce the notation, some preliminaries and the main results of this paper. In Section 3, we study the long time averaged of the

Hamilton-Jacobi equation without mean field interaction. Section 4 is devoted to the analysis of the ergodic MFG problem and to the asymptotic behavior of the solution of the time dependent MFG system.

Acknowledgement. The first author was partially supported by the ANR (Agence Nationale de la Recherche) project ANR-12-BS01-0008-01, by the CNRS through the PRC grant 1611 and by the Air Force Office for Scientific Research grant FA9550-18-1-0494. The second author was partly supported by Istituto Nazionale di Alta Matematica (GNAMPA 2019 Research Projects).

2. Main results

2.1. **Notations and preliminaries.** We write below a list of symbols used throughout this paper.

- Denote by $\mathbb{N}$ the set of positive integers, by $\mathbb{R}^d$ the d-dimensional real Euclidean space, by $\langle \cdot, \cdot \rangle$ the Euclidean scalar product, by $|\cdot|$ the usual norm in $\mathbb{R}^d$, and by B_R the open ball with center 0 and radius R.
- If Λ is a real $n \times n$ matrix, we define the norm of Λ by
$$\|\Lambda\| = \sup_{|x|=1,\ x\in\mathbb{R}^d} |\Lambda x|.$$

Let $(X, \mathbf{d})$ be a metric space (in the paper, we use $X = \mathbb{T}^d \times \mathbb{R}^d$ or $X = \mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d$).

- For a Lebesgue-measurable subset A of X, we let $\mathcal{L}(A)$ be the Lebesgue measure of A and $\mathbf{1}_A : X \to \{0, 1\}$ be the characteristic function of A, i.e.,
$$\mathbf{1}_A(x) = \begin{cases} 1 & x \in A, \\ 0 & x \notin A. \end{cases}$$
We denote by $L^p(A)$ (for $1 \le p \le \infty$) the space of Lebesgue-measurable functions f with $\|f\|_{p,A} < \infty$, where
$$\|f\|_{\infty,A} := \operatorname*{ess\,sup}_{x\in A} |f(x)|,$$
$$\|f\|_{p,A} := \left(\int_A |f|^p \, dx \right)^{\frac{1}{p}}, \quad 1 \le p < \infty.$$
For brevity, $\|f\|_\infty$ and $\|f\|_p$ stand for $\|f\|_{\infty,X}$ and $\|f\|_{p,X}$ respectively.
- $C_b(X)$ stands for the function space of bounded uniformly continuous functions on X. $C_b^2(X)$ stands for the space of bounded functions on X with bounded uniformly continuous first and second derivatives. $C^k(X)$ ($k \in \mathbb{N}$) stands for the function space of k-times continuously differentiable functions on X, and $C^\infty(X) := \cap_{k=0}^\infty C^k(X)$. $C_c^\infty(X)$ stands for the space of functions in $C^\infty(X)$ with compact support. Let $a < b \in \mathbb{R}$. $AC([a, b]; X)$ denotes the space of absolutely continuous maps $[a, b] \to X$.
- For $f \in C^1(X)$, the gradient of f is denoted by $Df = (D_{x_1} f, ..., D_{x_n} f)$, where $D_{x_i} f = \frac{\partial f}{\partial x_i}$, $i = 1, 2, \cdots, d$. Let k be a nonnegative integer and let $\alpha = (\alpha_1, \cdots, \alpha_d)$ be a multiindex of order k, i.e., $k = |\alpha| = \alpha_1 + \cdots + \alpha_d$, where each component α_i is a nonnegative integer. For $f \in C^k(X)$, define $D^\alpha f := D_{x_1}^{\alpha_1} \cdots D_{x_d}^{\alpha_d} f$.

We recall here the notations and definitions of Wasserstein spaces and Wasserstein distance, for more details we refer to [36, 4]. Here again we denote by $(X, \mathbf{d})$ a metric space (having in mind $X = \mathbb{T}^d \times \mathbb{R}^d$ or $X = \mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d$). Denote by $\mathcal{B}(X)$ the Borel σ-algebra on X and by $\mathcal{P}(X)$ the space of Borel probability measures on X. The support of a measure $\mu \in \mathcal{P}(X)$, denoted by $\mathrm{spt}(\mu)$, is the closed set defined by
$$\mathrm{spt}(\mu) := \Big\{ x \in X : \mu(V_x) > 0 \text{ for each open neighborhood } V_x \text{ of } x \Big\}.$$

We say that a sequence $\{\mu_k\}_{k\in\mathbb{N}} \subset \mathcal{P}(X)$ is weakly-$*$ convergent to $\mu \in \mathcal{P}(X)$, denoted by $\mu_k \xrightarrow{w^*} \mu$, if

$$\lim_{n\to\infty} \int_X f(x)\, \mu_n(dx) = \int_X f(x)\, \mu(dx), \quad \forall f \in C_b(X).$$

For $p \in [1, +\infty)$, the Wasserstein space of order p is defined as

$$\mathcal{P}_p(\mathbb{R}^d) := \left\{ m \in \mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} \mathbf{d}(x_0, x)^p\, m(dx) < +\infty \right\},$$

for some (and thus all) $x_0 \in X$. Given any two measures m and m' in $\mathcal{P}_p(X)$, define

$$\Pi(m, m') := \Big\{ \lambda \in \mathcal{P}(X \times X) : \lambda(A \times X) = m(A),\ \lambda(X \times A) = m'(A),\ \forall A \in \mathcal{B}(X) \Big\}. \tag{2.1}$$

The Wasserstein distance of order p between m and m' is defined by

$$d_p(m, m') = \inf_{\lambda \in \Pi(m,m')} \left(\int_{X\times X} d(x, y)^p\, \lambda(dx, dy) \right)^{1/p}.$$

The distance d_1 is also commonly called the Kantorovich-Rubinstein distance and can be characterized by a useful duality formula (see, for instance, [36]) as follows

$$d_1(m, m') = \sup \left\{ \int_X f(x)\, m(dx) - \int_X f(x)\, m'(dx) \mid f : X \to \mathbb{R} \text{ is 1-Lipschitz} \right\}, \tag{2.2}$$

for all m, $m' \in \mathcal{P}_1(X)$.

2.2. **Calculus of variation with acceleration.** In our first main result we study the large time average of an optimal control problem of acceleration. Let $L : \mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be the Lagrangian function defined as

$$L(x, v, w) = \frac{1}{2}|w|^2 + F(x, v)$$

where $F : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfies the following assumptions:

(**F1**) F is globally continuous with respect to both variables;

(**F2**) there exists $\alpha > 1$ and there exists a constant $c_F \geq 1$ such that for any $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$

$$\frac{1}{c_F}|v|^\alpha - c_F \leq F(x, v) \leq c_F(1 + |v|^\alpha) \tag{2.3}$$

and, without loss of generality, we assume $F(x, v) \geq 0$ for an $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$;

(**F3**) there exists a constant $C_F \geq 0$ such that

$$|D_x F(x, v)| + |D_v F(x, v)| \leq C_F(1 + |v|^\alpha).$$

Let Γ be the set C^1 curves $\gamma : [0, +\infty) \to \mathbb{T}^d$ (endowed with the local uniform convergence of the curve and its derivative) and for $(t, x, v) \in [0, T] \times \mathbb{T}^d \times \mathbb{R}^d$ let $\Gamma_t(x, v)$ be the subset of Γ such that $\gamma(t) = x$ and $\dot\gamma(t) = v$. Define the functional $J^{t,T} : \Gamma \to \mathbb{R}$ as

$$J^{t,T}(\gamma) = \int_t^T \left(\frac{1}{2}|\ddot\gamma(s)|^2 + F(\gamma(s), \dot\gamma(s)) \right) ds, \quad \text{if } \gamma \in H^2(0, T; \mathbb{T}^d), \tag{2.4}$$

and $J^{t,T}(\gamma) = +\infty$ if $\gamma \notin H^2(0, T; \mathbb{T}^d)$, and let $V^T(t, x, v)$ denote the value function associated with the functional $J^{t,T}$, i.e.

$$V^T(t, x, v) = \inf_{\gamma \in \Gamma_t(x,v)} J^{t,T}(\gamma). \tag{2.5}$$

Let H be the Hamiltonian associated with the Lagrangian L, that is for any $(x,v,p_v)\in\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d$,

$$H(x,v,p_v)=\frac{1}{2}|p_v|^2-F(x,v),$$

where $p_v\in\mathbb{R}^d$ denotes the momentum variable associated with $v\in\mathbb{R}^d$. Then, it is not difficult to see that the value function V^T is a continuous viscosity solution of the following Hamilton-Jacobi equation:

$$\begin{cases} -\partial_t V^T(t,x,v)-\langle D_x V^T(t,x,v),v\rangle+\frac{1}{2}|D_v V^T(t,x,v)|^2=F(x,v), & \text{in } [0,T]\times\mathbb{T}^d\times\mathbb{R}^d,\\ V^T(T,x,v)=0 \quad \text{in } \mathbb{T}^d\times\mathbb{R}^d. \end{cases}$$

Our aim is to characterize the behavior of $V^T(0,\cdot,\cdot)$ as $T\to+\infty$. To state the result, we need the notion of closed measure, which requires another notation: we set

$$\mathcal{P}_{\alpha,2}(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)=\left\{\mu\in\mathcal{P}(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d):\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(|w|^2+|v|^\alpha\right)\,\mu(dx,dv,dw)<+\infty\right\}$$

endowed with the weak-* convergence.

Definition 2.1 (Closed measure)
Let $\eta\in\mathcal{P}_{\alpha,2}(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)$. We say that η is a closed measure if for any test function $\varphi\in C_c^\infty(\mathbb{T}^d\times\mathbb{R}^d)$ the following holds

$$\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\Big(\langle D_x\varphi(x,v),v\rangle+\langle D_v\varphi(x,v),w\rangle\Big)\,\eta(dx,dv,dw)=0.$$

We denote by $\mathcal{C}$ the set of closed measures.

Theorem 2.2 (Main result 1). *Assume that F satisfies assumptions* **(F1)** *and* **(F2)**. *Then, the following limits exist:*

$$\lim_{T\to+\infty}\frac{1}{T}V^T(0,x,v)=\lim_{T\to+\infty}\inf_{\gamma\in\Gamma_0(x,v)}\frac{1}{T}J^T(\gamma)$$

and are independent of $(x,v)\in\mathbb{T}^d\times\mathbb{R}^d$. Moreover, if F satisfies also **(F3)** *then*

$$\lim_{T\to\infty}\frac{1}{T}V^T(0,x,v)=\inf_{\mu\in\mathcal{C}}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F(x,v)\right)\,\mu(dx,dv,dw).$$

Remark 2.3.
(1) If we denote by $\bar\lambda$ the above limits, the convergence of $V^T(0,x,v)-\lambda T$ is a completely open problem in this context. This is related to the lack of solution of the ergodic HJ equation.
(2) The (strong) structure condition on L and the fact that the problem is periodic in the x variable can probably be relaxed: this would require however more refined and technical estimates and we have chosen to work in this simpler framework.

2.3. **Mean Field Games of acceleration.** In our second main result, we consider a mean field game problem of acceleration. The Lagrangian function $L:\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d\times\mathcal{P}_1(\mathbb{T}^d\times\mathbb{R}^d)\to\mathbb{R}$ now takes the form

$$L(x,v,w,m)=\frac{1}{2}|w|^2+F(x,v,m)$$

where $F:\mathbb{T}^d\times\mathbb{R}^d\times\mathcal{P}_1(\mathbb{T}^d\times\mathbb{R}^d)\to\mathbb{R}$ satisfies the following assumptions:

- **(F1')** F is globally continuous with respect to all the variables;
- **(F2')** there exists $\alpha>1$ and a constant $c_F\geq 1$ such that for any $(x,v,m)\in\mathbb{R}^d\times\mathbb{R}^d\times\mathcal{P}_1(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)$

$$\frac{1}{c_F}|v|^\alpha-c_F\leq F(x,v,m)\leq c_F(1+|v|^\alpha)$$

and, without loss of generality, we assume $F(x,v,m) \geq 0$ for any $(x,v,m) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d)$;

(F3') there exists a constant $C_F \geq 0$ such that, for any $(x,v,m) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d)$,

$$|D_x F(x,v,m)| + |D_v F(x,v,m)| \leq C_F(1+|v|^\alpha).$$

We consider the time-dependent MFG system

(2.6)
$$\begin{cases} -\partial_t u^T(t,x,v) - \langle D_x u^T(t,x,v), v\rangle + \frac{1}{2}|D_v u^T(t,x,v)|^2 = F(x,v,m_t^T), & \text{in } [0,T]\times\mathbb{T}^d\times\mathbb{R}^d \\ \partial_t m_t^T - \langle v, D_x m_t^T\rangle - \operatorname{div}\left(m_t^T D_v u^T(t,x,v)\right) = 0, & \text{in } [0,T]\times\mathbb{T}^d\times\mathbb{R}^d \\ u^T(T,x,v) = g(x,v,m_T^T), \text{ in } \mathbb{T}^d\times\mathbb{R}^d, \quad m_0^T = m_0 \in \mathcal{P}(\mathbb{T}^d\times\mathbb{R}^d). \end{cases}$$

where the terminal condition of the Hamilton-Jacobi equation satisfies the following:

(G1) $(x,v) \mapsto g(x,v,m)$ belongs to $C_b^1(\mathbb{T}^d \times \mathbb{R}^d)$ for any $m \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ and $m \mapsto g(x,v,m)$ is Lipschitz continuous with respect to the d_1 distance, uniformly in $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$.

We recall that (u^T, m^T) is a solution of (2.6) if u^T is a viscosity solution of the first equation and m^T is a solution in the sense of distributions of the second equation. For more details see [33, 1].

Our aim is to understand the averaged limit of u^T as $T \to +\infty$. For this we define the ergodic MFG problem, inspired by the characterization of the limit in Theorem 2.2. Let us recall that the notion of closed measure was introduced in Definition 2.1 and that $\mathcal{C}$ denotes the set of closed measures.

Definition 2.4 (Solution of the ergodic MFG problem)
We say that $(\bar\lambda, \bar\mu) \in \mathbb{R} \times \mathcal{C}$ is a solution of the ergodic MFG problem if

$$\begin{aligned} \bar\lambda &= \inf_{\mu\in\mathcal{C}} \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,\pi\sharp\bar\mu)\right)\, \mu(dx,dv,dw) \\ &= \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,\pi\sharp\bar\mu)\right)\, \bar\mu(dx,dv,dw). \end{aligned} \tag{2.7}$$

Theorem 2.5 (Main result 2). *Assume that F and G satisfy* **(F1')**, **(F2')** *and* **(G1)**.

(1) There exists at least one solution $(\bar\lambda, \bar\mu) \in \mathbb{R} \times \mathcal{C}$ of the ergodic MFG problem (2.7). Moreover, if F satisfies the following monotonicity assumption: there exists $M_F > 0$ such that for m_1, $m_2 \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$

$$\begin{aligned} &\int_{\mathbb{T}^d\times\mathbb{R}^d} \big(F(x,v,m_1) - F(x,v,m_2)\big)\,(m_1(dx,dv) - m_2(dx,dv)) \\ &\quad \geq M_F \int_{\mathbb{T}^d\times\mathbb{R}^d} \big(F(x,v,m_1) - F(x,v,m_2)\big)^2\, dxdv, \end{aligned} \tag{2.8}$$

then the ergodic constant is unique: If $(\bar\lambda_1, \bar\mu_1)$ and $(\bar\lambda_2, \bar\mu_2)$ are two solutions of the ergodic MFG problem, then $\bar\lambda_1 = \bar\lambda_2$.

(2) Assume in addition that $\alpha = 2$, that **(F3')** *and (2.8) hold and that the initial distribution m_0 is in $\mathcal{P}_2(\mathbb{T}^d \times \mathbb{R}^d)$. Let (u^T, m^T) be a solution of the MFG system (2.6) and let $(\bar\lambda, \bar\mu)$ be a solution of the ergodic MFG problem (2.7). Then $T^{-1}u^T(0,\cdot,\cdot)$ converges locally uniformly to $\bar\lambda$ and we have*

$$\lim_{T\to+\infty} \frac{1}{T} \int_{\mathbb{T}^d\times\mathbb{R}^d} u^T(0,x,v)\, m_0(dx,dv) = \bar\lambda.$$

3. Ergodic behavior of control of acceleration

3.1. **Existence of the limit.** Before proving the main result of this section, Proposition 3.8, we need a few preliminary lemmas.

Lemma 3.1. *Assume that F satisfies* **(F1)** *and* **(F2)**. *Then, for any $(x,v) \in \mathbb{T}^d \times B_R$, with $R \geq 0$, and for any $T > 0$, we have*

$$\frac{1}{T} V^T(0,x,v) \leq c_F(1 + R^\alpha).$$

Remark 3.2. *The result also holds when $F = F(t,x,v)$ depends also on time, provided that F is continuous and satisfies* **(F2)** *with a constant c_F independent of t.*

Proof. Define the curve $\xi(t) = x + tv$, for $t \in [0,T]$. Then, by definition of the value function V^T, we have

$$V^T(0,x,v) \leq J^T(\xi) = \int_0^T F(x+tv, v)\, dt \leq T c_F(1 + R^\alpha).$$

□

Lemma 3.3. *Assume that F satisfies* **(F1)** *and* **(F2)**. *Let $\theta \geq 1$, (x_0, v_0) and (x,v) be in $\mathbb{T}^d \times B_R$ for some $R \geq 1$. Then, there exists a constant $C_2 \geq 0$ (depending only the constants α and c_F in* **(F2)**) *and a curve $\sigma : [0,\theta] \to \mathbb{R}^d$ such that $\sigma(0) = x_0$, $\dot\sigma(0) = v_0$ and $\sigma(\theta) = x$, $\dot\sigma(\theta) = v$ and*

$$J^\theta(\sigma) \leq C_2(R^2\theta^{-1} + R^\alpha \theta). \tag{3.1}$$

Remark 3.4. *The result also holds when $F = F(t,x,v)$ depends also on time, provided that F is continuous and satisfies* **(F2)** *with a constant c_F independent of t.*

Proof. Define the following parametric curve

$$\sigma(t) = x_0 + v_0 t + Bt^2 + Ct^3, \quad t \in [0,\theta].$$

Choosing

$$\begin{cases} B = 3(x - x_0) - \theta v - 2\theta v_0)\theta^{-2} \\ C = (-2(x - x_0) + \theta(v + v_0))\theta^{-3}, \end{cases}$$

we have that $\sigma(0) = x_0$, $\dot\sigma(0) = v_0$ and $\sigma(1) = x$, $\dot\sigma(1) = v$.

By definition of the functional J^θ we get

$$\begin{aligned} J^\theta(\sigma) &= \int_0^\theta \left(\frac{1}{2}|\ddot\sigma(t)|^2 + F(\sigma(t), \dot\sigma(t)) \right) dt \\ &\leq \int_0^\theta \left(\frac{1}{2}|2B + 6Ct|^2 + c_F(1 + |v_0 + 2tB + 3t^2 C|^\alpha) \right) dt \ \leq \ C_2(R^2\theta^{-1} + R^\alpha\theta), \end{aligned}$$

for some constant C_2 depending on the constants α and c_F in **(F2)** only. □

Lemma 3.5. *Let $T \geq 2$ and $(x,v) \in \mathbb{T}^d \times B_{R_0}$ for some $R_0 \geq c_F^{\frac{2}{\alpha}}$. Let $\gamma \in \Gamma(x,v)$ be optimal for $V^T(0,x,v)$. Then for any $\lambda \geq 2$ there exists $\tilde\gamma \in \Gamma(x,v)$ with $\tilde\gamma(T) = x$, $\dot{\tilde\gamma}(T) = v$ and*

$$J^T(\tilde\gamma) \leq J^T(\gamma) + C_3(\lambda^2 R_0^2 + R_0^\alpha \lambda^{-\alpha} T),$$

where the constant C_3 depends on α and c_F only.

Remark 3.6. *The result also holds when $F = F(t,x,v)$ depends also on time, provided that F is continuous and satisfies* **(F2)** *with a constant c_F independent of t. In addition, by the construction in the proof, there exists $\tau > 0$ such that $\tilde\gamma = \gamma$ on $[0,\tau]$ and*

$$\int_\tau^T (\frac{1}{2}|\ddot{\tilde\gamma}(t)|^2 + c_F(1 + |\dot{\tilde\gamma}(t)|^\alpha))dt \leq C_3(\lambda^2 R_0^2 + R_0^\alpha \lambda^{-\alpha} T).$$

Finally, the map which associates $\tilde\gamma$ and τ to γ is measurable.

Proof. Let

$$\tau := \begin{cases} \sup\{t \geq 0,\ |\dot\gamma(t)| \leq \lambda R_0\} & \text{if } |\gamma(T-1)| > \lambda R_0, \\ T-1 & \text{otherwise.} \end{cases}$$

If $\tau \geq T-2$, we set

$$\tilde\gamma(t) = \begin{cases} \gamma(t) & \text{for } t \in [0,\tau], \\ \sigma(t-\tau) & \text{for } t \in [\tau, T], \end{cases}$$

where σ is the map built in Lemma 3.3 with $\theta = T-\tau$, $\sigma(0) = \gamma(\tau)$, $\dot\sigma(0) = \dot\gamma(\tau)$, $\sigma(T-\tau) = x$, $\dot\sigma(T-\tau) = v$. If $\tau < T-2$, then we set

$$\tilde\gamma(t) = \begin{cases} \gamma(t) & \text{for } t \in [0,\tau], \\ \sigma_1(t-\tau) & \text{for } t \in [\tau, \tau+1], \\ \sigma_2(t-\tau-1) & \text{for } t \in [\tau+1, T], \end{cases}$$

where σ_1 and σ_2 are the map built in Lemma 3.3 with $\theta = 1$, $\sigma_1(0) = \gamma(\tau)$, $\dot\sigma_1(0) = \dot\gamma(\tau)$, $\sigma_1(1) = x$, $\dot\sigma(1) = v$ and $\theta = T-\tau-1$ and $\sigma_2(0) = \sigma_2(T-\tau-1) = x$ and $\dot\sigma_2(0) = \dot\sigma_2(T-\tau-1) = v$ respectively. Note that $\tilde\gamma(T) = x$ and $\dot{\tilde\gamma}(T) = v$.

In order to estimate $J^T(\tilde\gamma)$, we first show that τ cannot be too small: Namely we claim that

$$\tau \geq T\left(1 - \frac{c_F(1+R_0^\alpha)}{\frac{1}{c_F}(\lambda R_0)^\alpha - c_F}\right) - 1. \tag{3.2}$$

Indeed, let us first recall that by Lemma 3.1 we have

$$J^T(\gamma) \leq c_F(1+R_0^\alpha)T.$$

On the other hand, by assumption **(F2)** and the fact that $|\dot\gamma(t)| > \lambda R_0$ on $[\tau, T-1]$ and that $F \geq 0$, we also have that

$$\begin{aligned} J^T(\gamma) &= \int_0^T \left(\frac{1}{2}|\ddot\gamma(t)|^2 + F(\gamma(t),\dot\gamma(t))\right)\, dt \\ &\geq \int_\tau^{T-1} \left(\frac{1}{c_F}|\dot\gamma(t)|^\alpha - c_F\right)\, dt \geq (T-\tau-1)\left(\frac{1}{c_F}(\lambda R_0)^\alpha - c_F\right). \end{aligned}$$

So (3.2) holds for $R_0 \geq c_F^{2/\alpha}$.

We estimate $J^T(\tilde\gamma)$ in the case $\tau < T-2$, the other case being similar and easier. Note that $|\dot\gamma(\tau)| \leq \lambda R_0$. By Lemma 3.3 and the fact that $F \geq 0$, we have

$$\begin{aligned} J^T(\tilde\gamma) &= \int_0^\tau (\frac{1}{2}|\ddot\gamma(t)|^2 + F(\gamma(t),\dot\gamma(t)))dt + \int_0^1 (\frac{1}{2}|\ddot\sigma_1(t)|^2 + F(\sigma_1(t),\dot\sigma_1(t)))dt \\ &\quad + \int_0^{T-\tau-1} (\frac{1}{2}|\ddot\sigma_2(t)|^2 + F(\sigma_2(t),\dot\sigma_2(t)))dt \\ &\leq J^T(\gamma) + C_2((\lambda R_0)^2 + (\lambda R_0)^\alpha + R_0^2(T-\tau-1)^{-1} + R_0^\alpha(T-\tau-1)). \end{aligned}$$

In view of (3.2) this implies that

$$J^T(\tilde\gamma) \leq J^T(\gamma) + C_3(\lambda^2 R_0^2 + R_0^\alpha \lambda^{-\alpha} T),$$

for a constant C_3 depending on α and c_F only. □

Next we prove that the $(V^T(0,\cdot,\cdot))$ have locally uniformly bounded oscillations:

Lemma 3.7. *There exists a constant $M_1(R) \geq 0$ such that for any (x,v) and (x_0,v_0) in $\mathbb{T}^d \times \overline{B}_R$ we have that*

$$V^T(0,x,v) - V^T(0,x_0,v_0) \leq M_1(R).$$

Proof. Let γ^* be a minimizer for $V^T(0,x_0,v_0)$ and let $\sigma:[0,1]\to\mathbb{T}^d$ be such that $\sigma(0)=x$, $\dot\sigma(0)=v$ and $\sigma(1)=x_0$, $\dot\sigma(1)=v_0$ as in Lemma 3.3 for $\theta=1$. Define

$$\tilde\gamma(t)=\begin{cases}\sigma(t), & t\in[0,1]\\ \gamma^*(t-1), & t\in[1,T].\end{cases}$$

Then $\tilde\gamma\in\Gamma_0(x,v)$ and, by Lemma 3.3 and the assumption that $F\geq 0$, we have that

$$\begin{aligned}V^T(0,x,v)-V^T(0,x_0,v_0)&\leq\int_0^1\left(\frac{1}{2}|\ddot\sigma(t)|^2+F(\sigma(t),\dot\sigma(t))\right)\,dt\\&+\int_1^T\left(\frac{1}{2}|\ddot\gamma^*(t-1)|^2+F(\gamma^*(t-1),\dot\gamma^*(t-1))\right)\,dt-V^T(0,x_0,v_0)\\&\leq 2C_2R^2+\int_0^{T-1}\left(\frac{1}{2}|\ddot\gamma^*(t)|^2+F(\gamma^*(t),\dot\gamma^*(t))\right)\,dt-V^T(0,x_0,v_0)\\&\leq 2C_2R^2-\int_{T-1}^T\left(\frac{1}{2}|\ddot\gamma^*(t)|^2+F(\gamma^*(t),\dot\gamma^*(t))\right)\,dt\ \leq\ 2C_2R^2,\end{aligned}$$

which is the claim. □

Proposition 3.8 (Existence of the limit). *Assume that F satisfies* **(F1)** *and* **(F2)**. *Then, for any $(x,v)\in\mathbb{T}^d\times\mathbb{R}^d$, the following limits exist:*

$$\lim_{T\to+\infty}\frac{1}{T}V^T(0,x,v)=\lim_{T\to+\infty}\frac{1}{T}\inf_{\gamma\in\Gamma_0(x,v)}J^T(\gamma).$$

In addition the convergence is locally uniform in (x,v) and the limit is independent of (x,v).

Proof. Fix $R_0\geq c_F^{2/\alpha}$ such that $|v|\leq R_0$. Let $\{T_n\}_{n\in\mathbb{N}}$ and let $\{\gamma_n\}_{n\in\mathbb{N}}$ be a sequence of minimizers for $V^{T_n}(0,x,v)$ such that $T_n\to\infty$ as $n\to\infty$ and

$$\liminf_{T\to\infty}\frac{1}{T}V^T(0,x,v)=\lim_{n\to\infty}\frac{1}{T_n}J^{T_n}(\gamma_n).$$

For $\lambda\geq 2$, let us define $\tilde\gamma_n$ is in Lemma 3.5. Then we know that $\tilde\gamma_n(T)=x$, $\dot{\tilde\gamma}_n(T)=v$ and

$$J^{T_n}(\tilde\gamma_n)\leq J^{T_n}(\gamma_n)+C_3(\lambda^2R_0^2+R_0^\alpha\lambda^{-\alpha}T_n). \tag{3.3}$$

Let us define $\hat\gamma_n$ as the periodic extension of the curve $\tilde\gamma_n$, i.e. $\hat\gamma_n$ is T_n-periodic and it is equal to $\tilde\gamma_n$ on $[0,T_n]$. Then, taking $\hat\gamma_n$ as competitors for J^T we obtain that

$$\begin{aligned}\limsup_{T\to\infty}\inf_{\gamma\in\Gamma_0(x,v)}\frac{1}{T}J^T(\gamma)&\leq\limsup_{T\to\infty}\frac{1}{T}J^T(\hat\gamma_n)\\&=\frac{1}{T_n}J^{T_n}(\tilde\gamma_n)\leq\left(\frac{1}{T_n}J^{T_n}(\gamma_n)+C_3(\lambda^2R_0^2T_n^{-1}+R_0^\alpha\lambda^{-\alpha})\right),\end{aligned}$$

where the equality holds true since we are taking the limit of a periodic function and the last inequality holds by (3.3).

We get the conclusion letting $n\to\infty$ and then $\lambda\to\infty$, indeed: as $n\to\infty$ we deduce that

$$\begin{aligned}\limsup_{T\to\infty}\inf_{\gamma\in\Gamma_0(x,v)}\frac{1}{T}J^T(\gamma)&\leq\lim_n\frac{1}{T_n}J^{T_n}(\gamma_n)+C_3R_0^\alpha\lambda^{-\alpha}\\&=\liminf_{T\to+\infty}\inf_{\gamma\in\Gamma_0(x,v)}\frac{1}{T}J^T(\gamma)+C_3R_0^\alpha\lambda^{-\alpha}\end{aligned}$$

and then, taking the limit as $\lambda \to \infty$ we get

$$\limsup_{T\to\infty} \inf_{\gamma\in\Gamma_0(x,v)} \frac{1}{T} J^T(\gamma) \le \liminf_{\gamma\in\Gamma_0(x,v)} \inf_{\gamma\in\Gamma_0(x,v)} \frac{1}{T} J^T(\gamma).$$

As the $(V^T(0,\cdot,\cdot))$ have locally bounded oscillation (Lemma 3.7), the above convergence is locally uniform and the limit does not depend on (x,v). □

3.2. **Characterization of the ergodic limit.** In this part we characterize the limit given in Proposition 3.8 in term of closed measures. The proof of the main result, Proposition 3.17, where this characterization is stated, is technical and requires several steps. Here are the main ideas of the proof. By using standard results on occupational measures, one can obtain in a relatively easy way that

$$\lambda := \lim_{T\to\infty} \inf_{\gamma\in\Gamma_0(x_0,v_0)} \frac{1}{T} J^T(\gamma) \ge \inf_{\mu\in\mathcal{C}} \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v)\right)\, \mu(dx,dv,dw),$$

where $\mathcal{C}$ denotes the set of closed probability measures (see Definition 2.1). The difficult part of the proof is the opposite inequality. The first step for this is a min-max formula (Theorem 3.10) which gives, by using the characterization of closed measures, that

$$\begin{aligned}
&\inf_{\mu\in\mathcal{C}} \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v)\right)\, \mu(dx,dv,dw) \\
&\qquad = \sup_{\varphi\in C_c^\infty(\mathbb{T}^d\times\mathbb{R}^d)} \inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d} \left\{ -\frac{1}{2}|D_v\varphi(x,v)|^2 - \langle D_x\varphi(x,v), v\rangle + F(x,v)\right\}.
\end{aligned}$$

In order to exploit this inequality, one just needs to find a map $\varphi \in C_c^\infty(\mathbb{T}^d\times\mathbb{R}^d)$ for which

$$-\frac{1}{2}|D_v\varphi(x,v)|^2 - \langle D_x\varphi(x,v), v\rangle + F(x,v)$$

is almost equal to λ. This is not easy because the corrector of our ergodic problem does not seem to exist (at least in the usual sense) because of the lack of controllability and, if it existed, it certainly would not be smooth with a compact support. The standard idea in this set-up is to use instead the approximate corrector, i.e., the solution V_δ to

$$\delta V_\delta(x,v) + \frac{1}{2}|D_v V_\delta(x,v)|^2 + \langle D_x V_\delta(x,v), v\rangle = F(x,v) \qquad \text{in } \mathbb{T}^d\times\mathbb{R}^d.$$

However, this approximate corrector has not a compact support either (it is even coercive, see Proposition 3.11) and δV_δ does not converge uniformly to $-\lambda$, but only locally uniformly. We overcome these issues by an extra approximation argument (Lemma 3.13).

Let us first explain why closed measures pop up naturally in our problem. To see this, let $(x_0,v_0) \in \mathbb{T}^d\times\mathbb{R}^d$ be an initial position and let $\gamma^T_{(x_0,v_0)}$ be an optimal trajectory for $V^T(0,x_0,v_0)$. We define the family of Borel probability measures $\{\mu_T\}_{T>0}$ as follows: for any function $\varphi \in C_c^\infty(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)$

$$\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \varphi(x,v,w)\, \mu^T(dx,dv,dw) = \frac{1}{T}\int_0^T \varphi(\gamma^T_{(x_0,v_0)}(t), \dot\gamma^T_{(x_0,v_0)}(t), \ddot\gamma^T_{(x_0,v_0)}(t))\, dt. \tag{3.4}$$

Lemma 3.9. *Assume that F satisfies* **(F1)** *and* **(F2)**. *Let the family of probability measures $\{\mu^T\}_{T>0}$ be defined by (3.4). Then, $\{\mu^T\}_{T>0}$ is tight and there exists a closed measure μ^* such that, up to a subsequence, $\mu^T \rightharpoonup^* \mu^*$ as $T\to+\infty$.*

Proof. We first prove that $\{\mu_T\}_{T>0}$ its a tight family of probability measures. Indeed, by assumption **(F2)** for $(x_0, v_0) \in \mathbb{T}^d \times \mathbb{R}^d$ we know that

$$\begin{aligned}\frac{1}{T}V^T(0,x_0,v_0) =& \frac{1}{T}\int_0^T \left(\frac{1}{2}|\ddot{\gamma}^T_{(x_0,v_0)}(t)|^2 + F(\gamma^T_{(x_0,v_0)}(t), \dot{\gamma}^T_{(x_0,v_0)}(t))\right) dt \\ =& \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v)\right) \mu^T(dx,dv,dw) \\ \geq& \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + \frac{1}{c_F}|v|^\alpha - c_F\right) \mu^T(dx,dv,dw).\end{aligned}$$

On the other hand, by Lemma 3.1 we have that

$$\frac{1}{T}V^T(0,x_0,v_0) \leq C_1$$

where C_1 only depends on the initial point (x_0, v_0). Therefore, we obtain that

$$\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + \frac{1}{c_F}|v|^\alpha\right) \mu^T(dx,dv,dw) \leq C_1$$

which implies that $\{\mu^T\}_{T>0}$ is tight. By Prokhorov theorem there exists a measure $\mu^* \in \mathcal{P}(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)$ such that up to a subsequence $\mu^T \rightharpoonup^* \mu^*$ as $T \to +\infty$.

We now show that the measure μ^* is closed in the sense of Definition 2.1. Let $\varphi \in C^\infty_c(\mathbb{T}^d \times \mathbb{R}^d)$ be a test function and let $R \geq 0$ be such that $\varphi(x,v) = 0$ for any $(x,v) \in \mathbb{T}^d \times B^c_R$. Moreover, define

$$\tau^* = \begin{cases} \sup\{t \in [0,T] : |\dot{\gamma}^T_{(x_0,v_0)}(t)| \leq R\}, & \text{if } |\dot{\gamma}_{(x_0,v_0)}(T)| > R \\ T, & \text{if } |\dot{\gamma}_{(x_0,v_0)}(T)| \leq R \end{cases}$$

and let $\sigma^* : [\tau^*, \tau^*+1] \to \mathbb{T}^d$ be as in Lemma 3.3 such that $\sigma^*(\tau^*) = \gamma^T_{(x_0,v_0)}(\tau^*)$, $\dot{\sigma}^*(\tau^*) = \dot{\gamma}^T_{(x_0,v_0)}(\tau^*)$ and $\sigma^*(\tau^*+1) = x_0$, $\dot{\sigma}^*(\tau^*+1) = v_0$. Moreover, define

$$\tilde{\gamma}(t) = \begin{cases} \gamma^T_{(x_0,v_0)}(t), & t \in [0,\tau^*] \\ \sigma^*(t), & t \in (\tau^*, \tau^*+1]. \end{cases}$$

Then we get

$$\begin{aligned}&\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\langle D_x\varphi(x,v), v\rangle + \langle D_v\varphi(x,v), w\rangle\right) d\mu^T(x,v,w) \\ &= \frac{1}{T}\int_0^T \left(\langle D_x\varphi(\gamma^T_{(x_0,v_0)}(t), \dot{\gamma}^T_{(x_0,v_0)}(t)), \dot{\gamma}^T_{(x_0,v_0)}(t)\rangle + \langle D_v\varphi(\gamma^T_{(x_0,v_0)}(t), \dot{\gamma}^T_{(x_0,v_0)}(t)), \ddot{\gamma}^T_{(x_0,v_0)}(t)\rangle\right) dt \\ &= \underbrace{\frac{1}{T}\int_0^{\tau^*+1} \left(\langle D_x\varphi(\tilde{\gamma}^T(t), \dot{\tilde{\gamma}}^T(t)), \dot{\tilde{\gamma}}^T(t)\rangle + \langle D_v\varphi(\tilde{\gamma}^T(t), \dot{\tilde{\gamma}}^T(t)), \ddot{\tilde{\gamma}}^T(t)\rangle\right) dt}_{\mathbf{A}} \\ &- \underbrace{\frac{1}{T}\int_{\tau^*}^{\tau^*+1} \left(\langle D_x\varphi(\sigma^*(t), \dot{\sigma}^*(t)), \dot{\sigma}^*(t)\rangle + \langle D_v\varphi(\sigma^*(t), \dot{\sigma}^*(t)), \ddot{\sigma}^*(t)\rangle\right) dt}_{\mathbf{B}} \\ &+ \underbrace{\int_{\tau^*}^T \left(\langle D_x\varphi(\gamma^T_{(x_0,v_0)}(t), \dot{\gamma}^T_{(x_0,v_0)}(t)), \dot{\gamma}^T_{(x_0,v_0)}(t)\rangle + \langle D_v\varphi(\gamma^T_{(x_0,v_0)}(t), \dot{\gamma}^T_{(x_0,v_0)}(t)), \ddot{\gamma}^T_{(x_0,v_0)}(t)\rangle\right) dt}_{\mathbf{C}}\end{aligned}$$

One can immediately observe that by construction $\mathbf{C}= 0$ (since φ has a support in $\mathbb{T}^d \times B_R$). By the definition of $\tilde{\gamma}$ one also has that $\mathbf{A} = 0$. The behavior of $\mathbf{B}$ is also immediate because, as φ is bounded,

$$\begin{aligned}&\frac{1}{T}\int_{\tau^*}^{\tau^*+1}\Big(\langle D_x\varphi(\sigma^*(t),\dot\sigma^*(t)),\dot\sigma^*(t)\rangle + \langle D_v\varphi(\sigma^*(t),\dot\sigma^*(t)),\ddot\sigma^*(t)\rangle\Big)\,dt\\ &= \frac{1}{T}(\varphi(\sigma^*(\tau^*+1),\dot\sigma^*(\tau^*+1)) - \varphi(\sigma^*(\tau^*),\sigma^*(\tau^*)) \to 0, \quad \text{as } T\to+\infty.\end{aligned}$$

The proof is thus complete. □

The next step consists in formulating in two different ways the expected limit of Proposition 3.8.

Theorem 3.10 (Minmax formula). *Assume that F satisfies* **(F1)** *and* **(F2)**. *Then, the following equality holds true:*

$$\begin{aligned}&\inf_{\mu\in\mathcal{C}}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2 + F(x,v)\right)\,\mu(dx,dv,dw)\\ &\qquad= \sup_{\varphi\in C_c^\infty(\mathbb{T}^d\times\mathbb{R}^d)}\inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d}\left\{-\frac{1}{2}|D_v\varphi(x,v)|^2 - \langle D_x\varphi(x,v),v\rangle + F(x,v)\right\}.\end{aligned}\tag{3.5}$$

Proof. By definition of a closed measure we can write

$$\begin{aligned}&\inf_{\mu\in\mathcal{C}}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2 + F(x,v)\right)\,\mu(dx,dv,dw)\\ &= \inf_{\mu\in\mathcal{P}_{2,\alpha}(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)}\sup_{\varphi\in C_c^\infty(\mathbb{T}^d\times\mathbb{R}^d)}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\Big(\frac{1}{2}|w|^2 + F(x,v) - \langle D_x\varphi(x,v),v\rangle - \langle D_v\varphi(x,v),w\rangle\Big)\,\mu(dx,dv,dw).\end{aligned}$$

Our aim is to use the min-max Theorem (see Theorem A.1 below). We use for this the notation introduced in Appendix A and set $\mathbb{A} = C_c^\infty(\mathbb{T}^d\times\mathbb{R}^d)$, $\mathbb{B} = \mathcal{P}_{2,\alpha}(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)$ and for any $(\varphi,\mu)\in\mathbb{A}\times\mathbb{B}$

$$\mathcal{L}(\varphi,\mu) := \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\Big(\frac{1}{2}|w|^2 + F(x,v) - \langle D_x\varphi(x,v),v\rangle - \langle D_v\varphi(x,v),w\rangle\Big)\,\mu(dx,dv,dw).$$

Let us choose $\varphi^*(x,v) = 0$ and

$$\begin{aligned}c^* = 1 + \inf_{\mu\in\mathcal{P}_{2,\alpha}(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)}\sup_{\varphi\in C_c^\infty(\mathbb{T}^d\times\mathbb{R}^d)}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}&\Big(\frac{1}{2}|w|^2 + F(x,v)\\ &- \langle D_x\varphi(x,v),v\rangle - \langle D_v\varphi(x,v),w\rangle\Big)\,\mu(dx,dv,dw).\end{aligned}$$

Note that c^* is finite (since it is bounded below by assumption (2.3) and bounded above for $\mu = \delta_{(x_0,0,0)}$ for any $x_0\in\mathbb{T}^d$). In addition, the set $\mathbb{B}^* = \{\mu\in\mathbb{B} : \mathcal{L}(\varphi^*,\mu)\le c^*\}$ is nonempty and tight, and thus compact, in $\mathcal{P}_{2,\alpha}(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)$ for the weak-$*$ convergence. Finally, we have

$$\begin{aligned}c^* \ge 1 + \sup_{\varphi\in C_c^\infty(\mathbb{T}^d\times\mathbb{R}^d)}\inf_{\mu\in\mathcal{P}_{2,\alpha}(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}&\Big(\frac{1}{2}|w|^2 + F(x,v)\\ &- \langle D_x\varphi(x,v),v\rangle - \langle D_v\varphi(x,v),w\rangle\Big)\,\mu(dx,dv,dw).\end{aligned}$$

Therefore, the min-max Theorem A.1 states that

$$\begin{aligned}&\inf_{\mu\in\mathcal{P}_{2,\alpha}(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)}\sup_{\varphi\in C_c^\infty(\mathbb{T}^d\times\mathbb{R}^d)}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\Big(\frac{1}{2}|w|^2 + F(x,v) - \langle D_x\varphi(x,v),v\rangle - \langle D_v\varphi(x,v),w\rangle\Big)\,\mu(dx,dv,dw)\\ &= \sup_{\varphi\in C_c^\infty(\mathbb{T}^d\times\mathbb{R}^d)}\inf_{\mu\in\mathcal{P}_2(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\Big(\frac{1}{2}|w|^2 + F(x,v) - \langle D_x\varphi(x,v),v\rangle - \langle D_v\varphi(x,v),w\rangle\Big)\,\mu(dx,dv,dw)\end{aligned}$$

$$= \sup_{\varphi\in C_c^\infty(\mathbb{T}^d\times\mathbb{R}^d)} \inf_{(x,v,w)\in\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left\{ \frac{1}{2}|w|^2 + F(x,v) - \langle D_x\varphi(x,v), v\rangle - \langle D_v\varphi(x,v), w\rangle \right\}$$

$$= \sup_{\varphi\in C_c^\infty(\mathbb{T}^d\times\mathbb{R}^d)} \inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d} \left\{ -\frac{1}{2}|D_v\varphi(x,v)|^2 - \langle D_x\varphi(x,v), v\rangle + F(x,v) \right\}.$$

This complete the proof. □

Next we introduce and study the discounted problem associated with (2.4). For any $\delta > 0$ and any $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$ we define $J_\delta : \Gamma \to \mathbb{R}\cup\{+\infty\}$ as

$$J_\delta(\gamma) = \int_0^{+\infty} e^{-\delta t}\left(\frac{1}{2}|\ddot\gamma(t)|^2 + F(\gamma(t),\dot\gamma(t))\right) dt$$

if $\dot\gamma$ is absolutely continuous with $\int_0^{+\infty} e^{-\delta t}\left(\frac{1}{2}|\ddot\gamma(t)|^2 + |\dot\gamma(t))|^\alpha\right) dt < +\infty$, and $J_\delta(\gamma) = +\infty$ otherwise. We define the associated value function (the approximate corrector)

$$V_\delta(x,v) = \inf_{\gamma\in\Gamma_0(x,v)} J_\delta(\gamma). \tag{3.6}$$

We recall that V_δ is the unique continuous viscosity solution with a polynomial growth of the following Hamilton-Jacobi equation

$$\delta V_\delta(x,v) + \frac{1}{2}|D_v V_\delta(x,v)|^2 + \langle D_x V_\delta(x,v), v\rangle = F(x,v). \tag{3.7}$$

As the convergence of $V^T(0,\cdot,\cdot)/T$ is locally uniform (by Proposition 3.7), we can apply the Abelian-Tauberian Theorem of [34] and we have that for any $(x,v)\in\mathbb{T}^d\times\mathbb{R}^d$

$$\lim_{\delta\to 0^+} \delta V_\delta(x,v) = \lim_{T\to\infty} \frac{1}{T} V^T(0,x,v) =: \lambda. \tag{3.8}$$

In the proof of the main result of this section (Proposition 3.17) we will have to smoothen the map V^δ. This involves some local regularity properties of V^δ, which is the aim of the next result.

Proposition 3.11. *Assume that F satisfies* **(F1)** – **(F3)**. *Then, we have:*

(i) $\{\delta V_\delta(x,v)\}_{\delta>0}$ *is locally uniformly bounded;*

(ii) $\{V_\delta(x,v)\}_{\delta>0}$ *has locally uniformly bounded oscillation, i.e. there exists a constant $M(R)\geq 0$ such that for any $(x_0,v_0),(x,v)\in\mathbb{T}^d\times\overline{B}_R$*

$$V_\delta(x,v) - V_\delta(x_0,v_0) \leq M(R).$$

(iii) there exists a constant $\tilde C \geq 0$ such that for any $(x,v)\in\mathbb{T}^d\times\mathbb{R}^d$

$$\tilde C^{-1}|v|^\alpha - \tilde C\delta^{-1} \leq V_\delta(x,v) \leq c_F\delta^{-1}(|v|^\alpha + 1); \tag{3.9}$$

(iv) the map $x\mapsto V_\delta(x,v)$ is locally Lipschitz continuous and there exists a constant $C_\delta\geq 0$ such that for a.e. $(x,v)\in\mathbb{T}^d\times\mathbb{R}^d$ the following holds:

$$|D_x V_\delta(x,v)| \leq C_\delta(1+|v|^\alpha). \tag{3.10}$$

Proof. (i) Fix $(x,v)\in\mathbb{T}^d\times\overline{B}_R$ and define a competitor $\gamma:[0,+\infty]\to\mathbb{T}^d$ such that $\gamma(t) = x+tv$. By definition and (2.3) we get

$$\delta V_\delta(x,v) \leq \delta\int_0^\infty e^{-\delta t}F(\gamma(t),\dot\gamma(t))\,ds \leq c_F(1+|v|^\alpha) \;\leq\; c_F(1+R^\alpha).$$

On the other hand, we have by **(F2)** that $F\geq 0$ and thus $V_\delta\geq 0$, which completes the proof of (i).

(ii) Let (x_0, v_0), $(x, v) \in \mathbb{T}^d \times \overline{B}_R$ be fixed points, let γ^* be a minimizer for $V_\delta(x_0, v_0)$ and let σ be defined as in Lemma 3.3 such that $\sigma(0) = x$, $\dot\sigma(0) = v$ and $\sigma(1) = x_0$, $\dot\sigma(1) = v_0$. We define a new curve $\gamma : [0, +\infty) \to \mathbb{T}^d$ as follows

$$\gamma(t) = \begin{cases} \sigma(t), & t \in [0,1] \\ \gamma^*(t-1), & t \in (1, +\infty). \end{cases}$$

Then

$$\begin{aligned} V_\delta(x,v) - V_\delta(x_0,v_0) \le & \int_0^1 e^{-\lambda t}\left(\frac{1}{2}|\ddot\gamma(t)|^2 + F(\gamma(t),\dot\gamma(t))\right)\,dt \\ & + \int_1^{+\infty} e^{-\lambda t}\left(\frac{1}{2}|\ddot\gamma(t)|^2 + F(\gamma(t),\dot\gamma(t))\right)\,dt - V_\lambda(x_0,v_0). \end{aligned} \tag{3.11}$$

By a change of variable, we have that

$$\begin{aligned} &\int_1^{+\infty} e^{-\delta t}\left(\frac{1}{2}|\ddot\gamma^*(t)|^2 + F(\gamma^*(t),\dot\gamma^*(t))\right)\,dt \\ =&e^{-\delta}\int_0^{\infty} e^{-\delta s}\left(\frac{1}{2}|\ddot\gamma^*(s)|^2 + F(\gamma^*(s),\dot\gamma^*(s))\right)\,ds = e^{-\delta}V_\delta(x_0,v_0). \end{aligned}$$

Therefore, we obtain that

$$\begin{aligned} &\left|\int_1^{+\infty} e^{-\delta t}\left(\frac{1}{2}|\ddot\gamma(t)|^2 + F(\gamma(t),\dot\gamma(t))\right)\,dt - V_\delta(x_0,v_0)\right| \le \left|e^{-\delta} - 1\right| V_\delta(x_0,v_0) \\ &\le \delta|V_\delta(x_0,v_0)| \le c_F(1 + R^\alpha), \end{aligned} \tag{3.12}$$

where the last inequality holds true by (i). Moreover, by construction of σ in Lemma 3.3 we have that

$$\int_0^1 e^{-\delta t}\left(\frac{1}{2}|\ddot\sigma(t)|^2 + F(\sigma(t),\dot\sigma(t))\right)\,dt \le J^1(\sigma) \le C_2(R^2 + R^\alpha). \tag{3.13}$$

Combining together inequality (3.12) and (3.13) in (3.11) we get (ii):

$$V_\delta(x,v) - V_\delta(x_0,v_0) \le c_F(1 + R^\alpha) + C_2(R^2 + R^\alpha) =: M(R).$$

(iii) For some constants M_1 and M_2 we have that the map $Z : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ such that $Z(x,v) = M_1^{-1}|v|^\alpha - M_2\delta^{-1}$ is a subsolution of (3.7), indeed

$$\begin{aligned} &\delta Z(x,v) + \frac{1}{2}|D_v Z(x,v)|^2 + \langle D_x Z(x,v), v\rangle - F(x,v) \\ &\le \delta M_1^{-1}|v|^\alpha - M_2 + \frac{1}{2}M_1^{-2}\alpha^2|v|^{2(\alpha-1)} - c_F^{-1}|v|^\alpha + c_F. \end{aligned}$$

As $2(\alpha - 1) \le \alpha$, since $\alpha \in (1, 2]$, we get, for M_1 and M_2 large enough,

$$\delta Z(x,v) + \frac{1}{2}|D_v Z(x,v)|^2 + \langle D_x Z(x,v), v\rangle - F(x,v) \le 0.$$

By comparison we obtain $V_\delta \ge Z$, which proves the first inequality in (3.9).

In the same way, considering the map $Z(x,v) = c_F\delta^{-1}(|v|^\alpha + 1)$, we have

$$\begin{aligned} &\delta Z(x,v) + \frac{1}{2}|D_v Z(x,v)|^2 + \langle D_x Z(x,v), v\rangle - F(x,v) \\ &\ge c_F(|v|^\alpha + 1) + \frac{1}{2}\delta^{-2}(c_F\alpha)^2|v|^{2(\alpha-1)} - c_F|v|^\alpha - c_F \ \ge\ 0, \end{aligned}$$

so that Z is a supersolution. By comparison we conclude that the second inequality in (3.9) holds.

(iv) Let γ^* be optimal for $V_\delta(x,v)$ and let $h \in \mathbb{R}^d$. Then

$$\begin{aligned} V_\delta(x+h,v) &\le \int_0^{+\infty} e^{-\delta t}\left(\frac{1}{2}|\ddot\gamma^*(t)|^2 + F(\gamma^*(t)+h,\dot\gamma^*(t))\right)\,dt \\ &\le V_\delta(x,v) + \int_0^{+\infty} e^{-\delta t}\left(F(\gamma^*(t)+h,\dot\gamma^*(t)) - F(\gamma^*(t),\dot\gamma^*(t))\right)\,dt \\ &\le V_\delta(x,v) + \int_0^{+\infty} e^{-\delta t} c_F(1+|\dot\gamma^*(t)|^\alpha)|h|\,dt, \end{aligned} \tag{3.14}$$

where the last inequality holds true by assumption **(F3)**. Moreover, by (3.9) we deduce that there exists a constant $C_\delta \ge 0$ such that

$$\int_0^{+\infty} e^{-\delta t}(c_F^{-1}|\dot\gamma^*(t)|^\alpha - c_F)\,dt \le V_\delta(x,v) \le C_\delta(1+|v|^\alpha).$$

Therefore, by (3.14) we deduce that

$$V_\delta(x+h,v) - V_\delta(x,v) \le C_\delta(1+|v|^\alpha)|h|,$$

which implies that V_δ is locally Lipschitz continuous in space and proves (iv).

□

We now strengthen a little the convergence in (3.8):

Proposition 3.12. *Assume that F satisfies* **(F1)—(F3)**. *Then*

$$\lambda = \lim_{\delta\to 0^+} \inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d} \delta V_\delta(x,v),$$

with λ defined in (3.8).

Proof. First we note that, by (i) in Proposition 3.11, the convergence in (3.8) is locally uniform. Fix $R \ge 0$ such that

$$c_F^{-1}R^\alpha - c_F > \lambda. \tag{3.15}$$

Then, for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that for any $\delta \in (0,\delta_\varepsilon)$ we have that

$$\inf_{(x,v)\in\mathbb{T}^d\times B_R} \delta V_\delta(x,v) \ge \lambda - \varepsilon\,. \tag{3.16}$$

Fix $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$ and let γ_δ^* be a minimizer for $V_\delta(x,v)$. We define

$$\tau_\delta = \begin{cases} \inf\{t\in[0,+\infty] : |\dot\gamma_\delta^*(t)| \le R\}, & \text{if } \{t\in[0,+\infty] : |\dot\gamma_\delta^*(t)| \le R\} \neq \emptyset \\ +\infty, & \text{if } \{t\in[0,+\infty] : |\dot\gamma_\delta^*(t)| \le R\} = \emptyset. \end{cases}$$

By Dynamic Programming Principle we get

$$V_\delta(x,v) = \int_0^{\tau_\delta} e^{-\delta t}\left(\frac{1}{2}|\ddot\gamma_\delta^*(t)|^2 + F(\gamma_\delta^*(t),\dot\gamma_\delta^*(t))\right)\,dt + e^{-\delta\tau_\delta}V_\delta(\gamma_\delta^*(\tau_\delta),\dot\gamma_\delta^*(\tau_\delta))$$

and by assumption (2.3) and definition of τ_δ we deduce that

$$\delta V_\delta(x,v) \ge (c_F^{-1}R^\alpha - c_F)(1-e^{-\delta\tau_\delta}) + e^{-\delta\tau_\delta}\delta V_\delta(\gamma_\delta^*(\tau_\delta),\dot\gamma_\delta^*(\tau_\delta)). \tag{3.17}$$

If τ_δ is finite, we have that $|\dot\gamma_\delta^*(\tau_\delta)|$ is bounded by R and thus, by (3.15) and (3.16) we deduce that for any $\delta \in (0,\delta_\varepsilon)$

$$\delta V_\delta(x,v) \ge \lambda(1-e^{-\delta\tau_\delta}) + e^{-\delta\tau_\delta}(\lambda-\varepsilon) \ge \lambda - \varepsilon\,.$$

By (3.15) and (3.17) the same inequality also holds if $\tau_\delta = +\infty$. Hence, we obtain that

$$\lim_{\delta\to 0^+} \inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d} \delta V_\delta(x,v) \geq \lambda - \varepsilon\,.$$

By (3.8) we infer that

$$\lambda = \lim_{\delta\to 0^+} \delta V_\delta(0,0) \geq \lim_{\delta\to 0^+} \inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d} \delta V_\delta(x,v) \geq \lambda - \varepsilon,$$

which implies the desired result since ε is arbitrary. □

As V_δ is coercive, we cannot use it directly as a test function to test the fact that a measure is closed. To overcome this issue we approximate V_δ by family of Lipschitz maps (V_δ^R).

Lemma 3.13 (**Approximate problem 1**). *Assume that F satisfies assumption* **(F1)—(F3)**. *Let $R > 0$ and define $F_R(x,v) = \min\{F(x,v), R\}$ for any $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$. Let V_δ^R be the unique continuous and bounded viscosity solution to*

$$\delta V_\delta^R(x,v) + \frac{1}{2}|D_v V_\delta^R(x,v)|^2 + \langle D_x V_\delta^R(x,v), v\rangle = F_R(x,v), \quad (x,v)\in\mathbb{T}^d\times\mathbb{R}^d. \tag{3.18}$$

Then, the following holds:

(i) V_δ^R is globally Lipschitz continuous;

(ii) there are two positive constants $\tilde{c}_{1,\delta}$ and $\tilde{c}_{2,\delta}$ such that

$$\delta V_\delta^R(x,v) \geq \tilde{c}_{1,\delta}\big(1 + \min\{|v|^\alpha, R\}\big) - \tilde{c}_{2,\delta} \tag{3.19}$$

for any $(x,v)\in\mathbb{T}^d\times\mathbb{R}^d$;

(iii) there is a constant $\tilde{C}_\delta \geq 0$ such that

$$|D_x V_\delta^R(x,v)| \leq \tilde{C}_\delta\big(1 + \min\{|v|^\alpha, R\}\big) \tag{3.20}$$

for a.e. $(x,v)\in\mathbb{T}^d\times\mathbb{R}^d$;

(iv) V_δ^R converge, as $R\to+\infty$, uniformly on compact subsets of $\mathbb{T}^d\times\mathbb{R}^d$ to the map V_δ defined in (3.6).

The proofs of (i) and (iv) are direct consequences of optimal control theory while the proofs of (3.19) and (3.20) follow the same argument as for (3.9) and (3.10), respectively and we omit these proofs.

Lemma 3.14. *Assume that F satisfies* **(F1) – (F3)**. *Let F_R and V_δ^R be defined in Lemma 3.13. Then we have that*

$$\inf_{\mu\in\mathcal{C}} \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F_R(x,v)\right)\,\mu(dx,dv,dw) \geq \inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d} \delta V_\delta^R(x,v). \tag{3.21}$$

Proof. Let $\xi^{1,\varepsilon}\in C_c^\infty(\mathbb{R}^d)$ be such that $\mathrm{spt}(\xi^{1,\varepsilon})\subset B_\varepsilon$, $\xi^{1,\varepsilon}(x)\geq 0$ and $\int_{B_\varepsilon}\xi^{1,\varepsilon}(x)\,dx = 1$, and define $V_\delta^{R,\varepsilon}(x,v) = V_\delta^R \star_x \xi^{1,\varepsilon}(x,v)$ where the mollification only holds in x. Then V_δ^ε satisfies the following inequality in the viscosity sense

$$\delta V_\delta^\varepsilon(x,v) + \frac{1}{2}|D_v V_\delta^\varepsilon(x,v)|^2 + \langle D_x V_\delta^\varepsilon(x,v), v\rangle \leq F_R\star\xi^{1,\varepsilon}(x,v) \leq F_R(x,v) + C_F\,\varepsilon(1+\min\{|v|^\alpha, R\})$$

where the last inequality holds true by **(F3)** and the definition of F_R. Now, let $\xi^{2,\varepsilon}\in C_c^\infty(\mathbb{R}^d)$ be such that $\mathrm{spt}(\xi^{2,\varepsilon})\subset B_\varepsilon$, $\xi^{2,\varepsilon}(v)\geq 0$ and $\int_{B_\varepsilon}\xi^{2,\varepsilon}(v)\,dv = 1$ and define $\varphi_R^{\varepsilon,\delta}(x,v) = \xi^{2,\varepsilon}\star_v V_\delta^{R,\varepsilon}(x,v)$ (where the the mollification now only holds in v). Then, by (3.20) we have that

$$|\xi^{2,\varepsilon}\star_v(\langle D_x V_\delta^{R,\varepsilon}(x,\cdot),\cdot\rangle)(v) - \langle D_x\varphi_R^{\varepsilon,\delta}(x,v), v\rangle| \leq \varepsilon\,\|D_x V_\delta^{R,\varepsilon}\|_{L^\infty(B_\varepsilon(x,v))} \leq C_\delta\,\varepsilon(1+\min\{|v|^\alpha, R\}),$$

which implies that

$$\begin{aligned}
&\delta\varphi_R^{\varepsilon,\delta}(x,v)+\frac{1}{2}|D_v\varphi_R^{\varepsilon,\delta}(x,v)|^2+\langle D_x\varphi_R^{\varepsilon,\delta}(x,v),v\rangle\\
&\leq \delta\varphi_R^{\varepsilon,\delta}(x,v)+\frac{1}{2}|D_v\varphi_R^{\varepsilon,\delta}(x,v)|^2+\xi^{2,\varepsilon}\star_v\langle D_xV_\delta^{R,\varepsilon}(x,v),v\rangle+C_\delta\,\varepsilon(1+\min\{|v|^\alpha,R\})\\
&\leq F_R\star\xi^{2,\varepsilon}(x,v)+C_\delta\,\varepsilon(1+\min\{|v|^\alpha,R\})\leq F_R(x,v)+C_{1,\delta}\,\varepsilon(1+\min\{|v|^\alpha,R\})
\end{aligned}$$

where the last inequality holds true by assumption **(F3)**. Thus, so far we have proved that for any $(x,v)\in\mathbb{T}^d\times\mathbb{R}^d$

$$\begin{aligned}
\delta\varphi_R^{\varepsilon,\delta}(x,v)+\frac{1}{2}|D_v\varphi_R^{\varepsilon,\delta}(x,v)|^2+\langle D_x\varphi_R^{\varepsilon,\delta}(x,v),v\rangle\\
\leq\ F_R(x,v)+C_{1,\delta}\,\varepsilon(1+\min\{|v|^\alpha,R\}).
\end{aligned}\tag{3.22}$$

Moreover, in view of (3.19) we deduce that there exists a constant $C_{2,\delta}\geq 0$ such that for any $(x,v)\in\mathbb{T}^d\times\mathbb{R}^d$ we have that

$$\delta\varphi_R^{\varepsilon,\delta}(x,v)\geq C_{2,\delta}^{-1}\min\{|v|^\alpha,R\}-C_{2,\delta}.\tag{3.23}$$

We claim that for $\varepsilon>0$ small enough, the following holds:

$$\begin{aligned}
&\inf_{\mu\in\mathcal{C}}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F_R(x,v)\right)\,d\mu(x,v,w)\\
&\qquad\geq\ \inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d}\left(\delta\varphi_R^{\varepsilon,\delta}(x,v)-C_{1,\delta}\,\varepsilon\left(1+\min\{|v|^\alpha,R\}\right)\right).
\end{aligned}\tag{3.24}$$

By Remark 3.15 below, we can test the fact that a measure is closed by smooth and globally Lipschitz continuous maps. Let $\mathcal{E}(\mathbb{T}^d\times\mathbb{R}^d)$ be such a set. Then

$$\begin{aligned}
&\inf_{\mu\in\mathcal{C}}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F_R(x,v)\right)\,\mu(dx,dv,dw)\\
&=\inf_{\mu\in\mathcal{P}_{\alpha,2}(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)}\sup_{\psi\in\mathcal{E}(\mathbb{T}^d\times\mathbb{R}^d)}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F_R(x,v)-\langle D_x\psi(x,v),v\rangle-\langle D_v\psi(x,v),w\rangle\right)\mu(dx,dv,dw)\\
&\geq\sup_{\psi\in\mathcal{E}(\mathbb{T}^d\times\mathbb{R}^d)}\inf_{\mu\in\mathcal{P}_{\alpha,2}(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F_R(x,v)-\langle D_x\psi(x,v),v\rangle-\langle D_v\psi(x,v),w\rangle\right)\mu(dx,dv,dw)\\
&\geq\inf_{\mu\in\mathcal{P}_{\alpha,2}(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F_R(x,v)-\langle D_x\varphi_R^{\varepsilon,\delta}(x,v),v\rangle-\langle D_v\varphi_R^{\varepsilon,\delta}(x,v),w\rangle\right)\mu(dx,dv,dw)\\
&=\inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d}\left\{-\frac{1}{2}|D_v\varphi_R^{\varepsilon,\delta}(x,v)|^2+F_R(x,v)-\langle D_x\varphi_R^{\varepsilon,\delta}(x,v),v\rangle\right\},
\end{aligned}$$

which proves (3.24) thanks to (3.22). Recalling (3.23), the right hand side of (3.24) is coercive in v uniformly in ε for ε small. As in addition $\varphi_R^{\varepsilon,\delta}$ converges locally uniformly to V_δ^R as $\varepsilon\to 0$, we obtain

$$\lim_{\varepsilon\to 0}\inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d}\left(\delta\varphi_R^{\varepsilon,\delta}(x,v)-C_{2,\delta}\,\varepsilon\left(1+\min\{|v|^\alpha,R\}\right)\right)=\inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d}\delta V_\delta^R(x,v).$$

So we can let $\varepsilon\to 0$ in (3.24) to obtain the result. □

In the proof we used the following:

Remark 3.15. Note that we can allow for a larger class of test functions in Definition 2.1, i.e. $\varphi\in W^{1,\infty}(\mathbb{T}^d\times\mathbb{R}^d)\cap C^\infty(\mathbb{T}^d\times\mathbb{R}^d)$. Indeed, let $\varphi\in W^{1,\infty}(\mathbb{T}^d\times\mathbb{R}^d)\cap C^\infty(\mathbb{T}^d\times\mathbb{R}^d)$ and for $R>1$ let $\xi_R\in C_c^\infty(\mathbb{R}^d)$ be such that $\xi_R(x,v)=1$ for $(x,v)\in\mathbb{T}^d\times B_R$, $\xi_R(x,v)=0$ for $(x,v)\in\mathbb{T}^d\times\mathbb{R}^d\backslash B_{2R}$, $0\leq\xi_R(x,v)\leq 1$ for $\mathbb{T}^d\times B_{2R}\backslash B_R$ and there exists a constant $M\geq 0$ such that $|D\xi_R(x,v)|\leq MR^{-1}$

for any $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$. Set $\varphi_R = \varphi \xi_R$. Then, we have that $\varphi_R \in C_c^\infty(\mathbb{T}^d \times \mathbb{R}^d)$, $D\varphi_R$ is uniformly bounded and converges locally uniformly to $D\varphi$. For $\mu \in \mathcal{C}$ we have:

$$\int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} \Big(\langle D_x \varphi_R(x,v), v \rangle + \langle D_v \varphi_R(x,v), w \rangle \Big) \, \mu(dx,dv,dw) = 0. \tag{3.25}$$

Since $\mu \in \mathcal{P}_{2,\alpha}(\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d)$, we can pass to the limit in (3.25) as $R \to +\infty$ by dominate convergence. This proves that

$$\int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} \Big(\langle D_x \varphi(x,v), v \rangle + \langle D_v \varphi(x,v), w \rangle \Big) \, \mu(dx,dv,dw) = 0$$

for $\varphi \in W^{1,\infty}(\mathbb{T}^d \times \mathbb{R}^d) \cap C^\infty(\mathbb{T}^d \times \mathbb{R}^d)$. □

In the next step, we let $R \to +\infty$ in (3.21):

Lemma 3.16. *Assume that F satisfies* **(F1) – (F3)**. *Let V_δ be defined in* (3.6). *Then*

$$\inf_{\mu \in \mathcal{C}} \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v) \right) \, \mu(dx,dv,dw) \geq \inf_{(x,v) \in \mathbb{T}^d \times \mathbb{R}^d} \delta V_\delta(x,v). \tag{3.26}$$

Proof. We first consider the left-hand side of (3.21), for which we obviously have, by the definition of F_R in Lemma 3.13,

$$\begin{aligned} &\inf_{\mu \in \mathcal{C}} \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F_R(x,v) \right) \, \mu(dx,dv,dw) \\ &\leq \inf_{\mu \in \mathcal{C}} \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v) \right) \, \mu(dx,dv,dw). \end{aligned} \tag{3.27}$$

As for the right hand side of (3.21), we note that, if $(x_R, v_R) \in \mathbb{T}^d \times \mathbb{R}^d$ satisfies

$$V_\delta^R(x_R, v_R) \leq \inf_{(x,v) \in \mathbb{T}^d \times \mathbb{R}^d} V_\delta^R(x,v) + R^{-1},$$

then, as $V_\delta^R \leq V_\delta$ and (3.19) holds, we have

$$\tilde{c}_{1,\delta}\big(1 + \min\{|v_R|^\alpha, R\}\big) - \tilde{c}_{2,\delta} \leq \inf_{(x,v) \in \mathbb{T}^d \times \mathbb{R}^d} V_\delta(x,v) + R^{-1}.$$

This proves that v_R remains bounded in R and we can find a subsequence of (x_R, v_R), denoted in the same way, which converges to some $(\bar{x}, \bar{v}) \in \mathbb{T}^d \times \mathbb{R}^d$ as $R \to +\infty$. Then by local uniform convergence of V_δ^R to V_δ, we obtain that

$$\inf_{(x,v) \in \mathbb{T}^d \times \mathbb{R}^d} V_\delta(x,v) \leq V_\delta(\bar{x}, \bar{v}) = \lim_{R \to +\infty} V_\delta^R(x_R, v_R) = \lim_{R \to +\infty} \inf_{(x,v) \in \mathbb{T}^d \times \mathbb{R}^d} V_\delta^R(x,v). \tag{3.28}$$

Passing to the limit as $R \to +\infty$ in (3.21) proves the Lemma thanks to (3.27) and (3.28). □

We are now ready to prove the main result of this section.

Proposition 3.17 (Characterization with closed measures). *Assume that F satisfies* **(F1) — (F3)**. *For any $(x_0, v_0) \in \mathbb{T}^d \times \mathbb{R}^d$ we have that*

$$\lim_{T \to \infty} \frac{1}{T} V^T(0, x_0, v_0) = \inf_{\mu \in \mathcal{C}} \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v) \right) \, \mu(dx,dv,dw).$$

Proof. Let $\gamma^T_{(x_0,v_0)}$ be a minimum for the problem

$$\inf_{\gamma \in \Gamma_0(x_0,v_0)} J^T(\gamma).$$

Let us define the probability measures μ_T by

$$\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\varphi(x,v,w)\,d\mu^T(x,v,w)=\frac{1}{T}\int_0^T\varphi(\gamma^T_{(x_0,v_0)}(t),\dot\gamma^T_{(x_0,v_0)}(t),\ddot\gamma^T_{(x_0,v_0)}(t))\,dt$$

for any $\varphi\in\mathcal{C}^\infty_c(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)$. By Lemma 3.9, the (μ^T) converge, up to a subsequence (T_n), weak-$*$ to a closed measure μ^*. Therefore

$$\lim_{T\to\infty}\inf_{\gamma\in\Gamma_0(x_0,v_0)}\frac{1}{T}J^T(\gamma)=\lim_{n\to\infty}\frac{1}{T_n}\int_0^{T_n}\left(\frac{1}{2}|\ddot\gamma^{T_n}_{(x_0,v_0)}(t)|^2+F(\gamma^{T_n}_{(x_0,v_0)}(t),\dot\gamma^{T_n}_{(x_0,v_0)}(t))\right)dt$$

$$=\lim_{n\to\infty}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F(x,v)\right)\mu^{T_n}(dx,dv,dw)\geq\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F(x,v)\right)\mu^*(dx,dv,dw).$$

Thus, taking the infimum over the set of closed measures $\mathcal{C}$ we obtain that

$$\lim_{T\to\infty}\inf_{\gamma\in\Gamma_0(x_0,v_0)}\frac{1}{T}J^T(\gamma)\geq\inf_{\mu\in\mathcal{C}}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F(x,v)\right)d\mu(x,v,w).$$

To obtain the opposite inequality, we note that, by (3.26) (which holds for any $\delta>0$) and Proposition 3.12, we have

$$\inf_{\mu\in\mathcal{C}}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F(x,v)\right)\mu(dx,dv,dw)\geq\lim_{\delta\to0^+}\inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d}\delta V_\delta(x,v)=\lambda,$$

where λ defined in (3.8). Then we can conclude thanks to (3.8). □

Proof of Theorem 2.2. The existence of the limit and the fact that it does not depend on (x,v) is the main statement of Proposition 3.8 while the characterization of this limit is given by Proposition 3.17. □

4. Asymptotic behavior of MFG with acceleration

We now turn to MFG problems of acceleration. In order to study the asymptotic behavior of these problems, we first need to describe the expected limit: the ergodic MFG problems of acceleration. The difficulty here is that, as explained in the previous part, we do not expect the existence of a corrector and therefore the ergodic MFG problem cannot be phrased in these terms. We overcome this issue by using the characterization of the ergodic limit given by Theorem 2.2 in terms of closed measures. This suggests the definition of equilibria for ergodic MFG of acceleration (Definition 2.4). We prove the existence and the uniqueness of a solution in Proposition 4.1. In order to pass to the limit in the time-dependent MFG system of acceleration, we first need to rephrase the solution of this system in terms of closed measures (more precisely in terms of the so-called $T-$closed measures, see Definition 4.2). This is the aim of the second part of the section (Theorem 4.3). Thanks to this characterization, we are then able to conclude on the long time average and complete the proof of Definition 2.4.

4.1. Ergodic MFG with acceleration.

Following Definition 2.1 we recall that $\mathcal{C}\subset\mathcal{P}_{\alpha,2}(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)$ denotes the set of closed measures, i.e. $\mu\in\mathcal{C}$ if it satisfies for any test function $\varphi\in\mathcal{C}^\infty_c(\mathbb{T}^d\times\mathbb{R}^d)$ the following condition:

$$\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\Big(\langle D_x\varphi(x,v),v\rangle+\langle D_v\varphi(x,v),w\rangle\Big)\,\mu(dx,dv,dw)=0.$$

The candidate limit problem that we are going to study is the following fixed point problem: we look for a measure $\mu\in\mathcal{C}$ such that

$$\mu\in\operatorname*{argmin}_{\eta\in\mathcal{C}}\left\{\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F(x,v,\pi\sharp\mu)\right)\eta(dx,dv,dw)\right\}\tag{4.1}$$

where $\pi:\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d$, defined as $\pi(x,v,w)=(x,v)$, is the projection function.

Proposition 4.1. *Assume that F satisfies* **(F1')** *and* **(F2')**. *Then, there exists at least one solution $(\bar\lambda, \bar\mu) \in \mathbb{R} \times \mathcal{C}$ of the ergodic MFG problem.*

Moreover, if F satisfies the monotonicity assumption (2.8) and if $(\bar\lambda_1, \bar\mu_1)$ and $(\bar\lambda_2, \bar\mu_2)$ are two solutions of the ergodic MFG problem, then $\bar\lambda_1 = \bar\lambda_2$.

Proof. Let $\mathcal{K}$ be the set of probability measures $\mu \in \mathcal{C}$ such that

$$\int_{\mathbb{T}^d\times\mathbb{R}^d} (\frac{1}{2}|w|^2 + c_F^{-1}|v|^\alpha)\, \mu(dx,dv,dw) \le 2c_F,$$

where α and c_F are given by assumption **(F2')**. We endow $\mathcal{K}$ with the d_1 distance and define, for any $\mu \in \mathcal{K}$, the set $\Psi(\mu)$ as the set of minimizers $\bar\eta \in \mathcal{C}$ of the map defined on $\mathcal{C}$

$$\eta \to \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,\pi\sharp\mu)\right)\, \eta(dx,dv,dw) \tag{4.2}$$

We also denote by $\Lambda(\mu)$ the value of this minimum. First, we show that the set-valued map Ψ is well-defined from $\mathcal{K}$ into $\mathcal{K}$. Indeed, if $\mu \in \mathcal{K}$ and $\bar\eta \in \mathcal{C}$ is any minimum of (4.2), we have by assumption **(F2')** (setting $\tilde\eta = \delta_{(x_0,0,0)} \in \mathcal{C}$ for an arbitrary point $x_0 \in \mathbb{T}^d$):

$$\begin{aligned}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} (\frac{1}{2}|w|^2 + c_F^{-1}|v|^\alpha - c_F)\, \bar\eta(dx,dv,dw) &\le \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} (\frac{1}{2}|w|^2 + F(x,v,\pi\sharp\mu))\, \bar\eta(dx,dv,dw)\\ &\le \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} (\frac{1}{2}|w|^2 + F(x,v,\pi\sharp\mu))\, \tilde\eta(dx,dv,dw) \;\le\; c_F.\end{aligned}$$

So $\bar\eta$ belongs to $\mathcal{K}$. Moreover, we observe that a solution of the ergodic MFG problem exists if the set-valued map Ψ has a fixed-point and we prove that this is the case using the Kakutani fixed-point theorem. Since $\alpha > 1$, by the above considerations, we know that the space $\mathcal{K}$ is compact with respect to the d_1 distance. Thus, for any $\mu \in \mathcal{K}$, the set $\Psi(\mu)$ is convex and compact. It remains to check that Ψ has closed graph. Fix a sequence $\{\mu_j\}_{j\in\mathbb{N}} \subset \mathcal{K}$ and a sequence $\{\eta_j\}_{j\in\mathbb{N}} \subset \mathcal{K}$ such that

$$\mu_j \rightharpoonup^{d_1} \mu, \quad \eta_j \rightharpoonup^{d_1} \bar\eta, \quad \text{and} \quad \eta_j \in \Psi(\mu_j)\ \forall j \in \mathbb{N}.$$

Let us show that $\bar\eta \in \Psi(\mu)$. Note that $\bar\eta \in \mathcal{C}$. It remains to check that $\bar\eta$ minimizes (4.2). By standard lower-semi continuity arguments and continuity of F, we have:

$$\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} (\frac{1}{2}|w|^2 + F(x,v,\pi\sharp\mu))\, \bar\eta(dx,dv,dw) \le \liminf_j \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} (\frac{1}{2}|w|^2 + F(x,v,\pi\sharp\mu_j))\, \eta_j(dx,dv,dw). \tag{4.3}$$

We now check that the right-hand side is not larger that $\Lambda(\mu)$. Indeed, let $\tilde\eta$ belong to $\Psi(\mu)$ and fix $\eta > 0$. As $\tilde\eta$ belongs to $\mathcal{K}$ we can find $R > 0$ such that

$$\int_{(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)\setminus B_R} (\frac{1}{2}|w|^2 + c_F|v|^\alpha + c_F)\, \tilde\eta(dx,dv,dw) \le \varepsilon\,.$$

As $\pi\sharp\mu_j$ converges to $\pi\sharp\mu$ for the d_1 distance, we have by assumption **(F1')** that, for j large enough,

$$\lim_{j\to+\infty} \sup_{(x,v)\in B_R} |F(x,v,\pi\sharp\mu_j) - F(x,v,\pi\sharp\mu)| \le \varepsilon\,.$$

So, by optimality of η_j and the estimates above,

$$\begin{aligned}
&\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}(\frac{1}{2}|w|^2+F(x,v,\pi\sharp\mu_j))\,\eta_j(dx,dv,dw)=\Lambda(\mu_j)\\
&\le\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}(\frac{1}{2}|w|^2+F(x,v,\pi\sharp\mu_j))\,\tilde\eta(dx,dv,dw)\\
&\le\int_{B_R}(\frac{1}{2}|w|^2+F(x,v,\pi\sharp\mu_j))\,\tilde\eta(dx,dv,dw)+\int_{B_R^c}(\frac{1}{2}|w|^2+c_F|v|^\alpha+c_F)\,\tilde\eta(dx,dv,dw)\\
&\le\int_{B_R}(\frac{1}{2}|w|^2+F(x,v,\pi\sharp\mu))\,\tilde\eta(dx,dv,dw)+2\,\varepsilon\le\Lambda(\mu)+2\,\varepsilon\,.
\end{aligned}$$

Coming back to (4.3), this shows that

$$\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}(\frac{1}{2}|w|^2+F(x,v,\pi\sharp\mu))\,\bar\eta(dx,dv,dw)\le\Lambda(\mu),$$

and therefore that $\bar\eta$ belongs to $\Psi(\mu)$. Therefore, applying Kakutani fixed-point theorem we have that there exists a fixed point $\bar\eta$ of Ψ and this is a solution of the ergodic MFG problem.

Now, we prove that under the monotonicity assumption (2.8) the critical value is unique. Let $(\bar\lambda_1,\bar\mu_1)$ and $(\bar\lambda_2,\bar\mu_2)$ be two solutions of the ergodic MFG problem. Then, by definition we have that, for $i=1$ or $i=2$,

$$\begin{aligned}
\bar\lambda_i&=\inf_{\mu\in\mathcal{C}}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F(x,v,\pi\sharp\bar\mu_i)\right)\,\mu(dx,dv,dw)\\
&=\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F(x,v,\pi\sharp\bar\mu_i)\right)\,\bar\mu_i(dx,dv,dw).
\end{aligned}\tag{4.4}$$

Thus, exchanging the role of $\bar\mu_1$ and $\bar\mu_2$ as competitor for $\bar\lambda_1$ and $\bar\lambda_2$, respectively, we get

$$\bar\lambda_1\le\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F(x,v,\pi\sharp\bar\mu_1)\right)\,\bar\mu_2(dx,dv,dw)\tag{4.5}$$

and

$$\bar\lambda_2\le\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F(x,v,\pi\sharp\bar\mu_2)\right)\,\bar\mu_1(dx,dv,dw).\tag{4.6}$$

We first take the difference between (4.5) and (4.4) for $i=2$ and we get

$$\bar\lambda_1-\bar\lambda_2\le\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\big(F(x,v,\pi\sharp\bar\mu_1)-F(x,v,\pi\sharp\bar\mu_2)\big)\,d\bar\mu_2(dx,dv,dw).$$

Taking the difference between (4.5) for $i=1$ and (4.6) we get

$$\bar\lambda_1-\bar\lambda_2\ge\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\big(F(x,v,\pi\sharp\bar\mu_1)-F(x,v,\pi\sharp\bar\mu_2)\big)\,d\bar\mu_1(dx,dv,dw).$$

Thus, taking the difference of the above expressions we deduce that

$$0\ge\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\big(F(x,v,\pi\sharp\bar\mu_1)-F(x,v,\pi\sharp\bar\mu_2)\big)\,(\bar\mu_1(dx,dv,dw)-\bar\mu_2(dx,dv,dw))$$

which implies by monotonicity assumption (2.8) that $F(x,v,\pi\sharp\bar\mu_1)=F(x,v,\pi\sharp\bar\mu_2)$. Coming back to (4.5), it follows that $\bar\lambda_1=\bar\lambda_2$. □

4.2. **Representation of the solution of the time-dependent MFG system.** We now consider the time-dependent MFG system (2.6). In [33, 1], it has been proved that such system has a solution (u^T, m^T) and that the function u^T can be represented as

$$u^T(t,x,v) = \inf_{\gamma\in\Gamma_t(x,v)} \left\{ \int_t^T \left(\frac{1}{2}|\ddot{\gamma}(s)|^2 + F(\gamma(s),\dot{\gamma}(s),m_s^T) \right) ds + g(\gamma(T),\dot{\gamma}(T),m_T^T) \right\}. \tag{4.7}$$

In order to compare the solution of this time-dependent problem with the solution of the ergodic MFG problem, which is written in terms of closed measures, we need to rewrite the time-dependent problem in term of flows of Borel probability measures on $\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d$. The following definition mirrors the definition of closed measure in the ergodic setting:

Definition 4.2 (**T-Closed measures**)
Let T be a finite time horizon and let $m_0 \in \mathcal{P}_1(\mathbb{T}^d \times \mathbb{R}^d)$. If $\eta \in C([0,T];\mathcal{P}_1(\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d))$, we say that η is a T-closed measure associated with m_0 if for any test function $\varphi \in C_c^\infty([0,T]\times\mathbb{T}^d\times\mathbb{R}^d)$ the following holds

$$\begin{aligned} &\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \Big(\partial_t\varphi(t,x,v) + \langle D_x\varphi(t,x,v), v\rangle + \langle D_v\varphi(t,x,v), w\rangle \Big)\, \eta_t(dx,dv,dw)dt \\ &= \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \varphi(T,x,v)\, \eta_T(dx,dv,dw) - \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \varphi(0,x,v)\, m_0(dx,dv). \end{aligned} \tag{4.8}$$

We denote by $\mathcal{C}^T(m_0)$ the set of T-closed measures associated with $m_0 \in \mathcal{P}_1(\mathbb{T}^d \times \mathbb{R}^d)$.

The goal of the subsection is to prove the following equality:

Theorem 4.3. *Assume that F satisfies* **(F1')**, **(F2')** *and g satisfies* **(G1)**. *Let $M \geq 0$ and assume that*

$$\int_{\mathbb{T}^d\times\mathbb{R}^d} |v|^\alpha\, m_0(dx,dv) \leq M. \tag{4.9}$$

Let (u^T, m^T) be a solution to (2.6). Then

$$\begin{aligned} \inf_{\mu\in\mathcal{C}^T(m_0)} \Big\{ &\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,m_t^T) \right) \mu_t(dx,dv,dw)dt \\ &+ \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g(x,v,m_T^T)\, \mu_T(dx,dv,dw) \Big\} \\ &= \int_{\mathbb{T}^d\times\mathbb{R}^d} u^T(0,x,v)\, m_0(dx,dv). \end{aligned} \tag{4.10}$$

In addition, there exists a minimizer $\bar{\mu}^T \in \mathcal{C}^T(m_0)$ of the problem in the left-hand side of (4.10) such that $m_t^T = \pi\sharp\bar{\mu}_t^T$, where $\pi : \mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{T}^d\times\mathbb{R}^d$ is the canonical projection on the two first coordinates, i.e. such that $\pi(x,v,w) = (x,v)$.

The proof of Theorem 4.3 follows standard arguments but is slightly technical because the problem is stated in the whole space in velocity. The main problem is to regularize the map u^T in order to have a smooth function with a compact support which satisfies a suitable (approximate) Hamilton-Jacobi inequality. The first step towards this aim is the following Lemma:

Lemma 4.4 (**Approximate problem 2**). *Let $f : \mathbb{T}^d\times\mathbb{R}^d\times[0,T] \to \mathbb{R}$ be a continuous map with at most a polynomial growth and which is locally Lipschitz continuous in space locally uniformly in time and $g : \mathbb{T}^d\times\mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz continuous map with at most a polynomial growth. Let $R > 0$ and let ξ^R be a smooth cut–off function such that $\xi^R \geq 0$, $\xi^R(x,v) = 1$ if $(x,v) \in \mathbb{T}^d \times \overline{B}_R$, $0 \leq \xi^R(x,v) \leq 1$ if $(x,v) \in \mathbb{T}^d\times\overline{B}_{2R}\backslash\overline{B}_R$ and $\xi^R(x,v) = 0$ if $(x,v) \in \mathbb{T}^d\times\overline{B}_R^c$. Define f_R :*

$\mathbb{T}^d \times \mathbb{R}^d \times [0,T] \to \mathbb{R}$ and $g_R : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ as $f_R = \xi^R f$ and $g_R = \xi^R g$. Let u_R^T be the viscosity solution of the following problem

$$\begin{cases} -\partial_t u_R^T(t,x,v) + \frac{1}{2}|D_v u_R^T(t,x,v)|^2 - \langle D_x u_R^T(t,x,v), v\rangle \\ \qquad\qquad = f_R(t,x,v), & \text{in } [0,T]\times\mathbb{T}^d\times\mathbb{R}^d \\ u^T(T,x,v) = g_R(x,v), & \text{in } \mathbb{T}^d\times\mathbb{R}^d. \end{cases} \tag{4.11}$$

Then, the following hold:

(1) u_R^T has compact support;
(2) u_R^T is Lipschitz continuous in space and velocity variable;
(3) u_R^T converge, as $R \to +\infty$, locally uniformly to the solution u^T of the following problem

$$\begin{cases} -\partial_t u^T(t,x,v) + \frac{1}{2}|D_v u^T(t,x,v)|^2 - \langle D_x u^T(t,x,v), v\rangle = f(t,x,v), & \text{in } [0,T]\times\mathbb{T}^d\times\mathbb{R}^d \\ u^T(T,x,v) = g(x,v), & \text{in } \mathbb{T}^d\times\mathbb{R}^d. \end{cases}$$

The proof of the Lemma follows standard argument in optimal control and we omit it. Next we prove Theorem 4.3 in the simpler case where F and g are replaced by F_R and g_R:

Proposition 4.5. *Assume that F satisfies* **(F1')** *and* **(F2')** *and g satisfies* **(G1)**. *Let (u^T, m^T) be a solution of system (2.6). For $R > 0$, let ξ^R be a smooth cut–off function as in Lemma 4.4 and let us set $F_R = \xi^R F$ and $g_R = \xi^R g$. Let u_R^T be the continuous viscosity solution of the following problem*

$$\begin{cases} -\partial_t u_R^T(t,x,v) + \frac{1}{2}|D_v u_R^T(t,x,v)|^2 - \langle D_x u_R^T(t,x,v), v\rangle \\ \qquad\qquad = F_R(x,v,m_t^T), & \text{in } [0,T]\times\mathbb{T}^d\times\mathbb{R}^d \\ u^R(T,x,v) = g_R(x,v,m_T^T), & \text{in } \mathbb{T}^d\times\mathbb{R}^d. \end{cases} \tag{4.12}$$

Then

$$\begin{aligned} \inf_{\mu\in\mathcal{C}^T(m_0)} \Big\{ \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \Big(\frac{1}{2}|w|^2 + F_R(x,v,m_t^T)\Big)\, \mu_t(dx,dv,dw)dt \\ + \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g_R(x,v,m_T^T)\, \mu_T(dx,dv,dw)\Big\} \\ = \int_{\mathbb{T}^d\times\mathbb{R}^d} u_R^T(0,x,v)\, m_0(dx,dv). \end{aligned}$$

Proof. We first prove that

$$\begin{aligned} \inf_{\mu\in\mathcal{C}^T(m_0)} \Big\{ \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \Big(\frac{1}{2}|w|^2 + F_R(x,v,m_t^T)\Big)\, \mu_t(dx,dv,dw)dt \\ + \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g_R(x,v,m_T^T)\, \mu_T(dx,dv,dw)\Big\} \\ \geq \int_{\mathbb{T}^d\times\mathbb{R}^d} u_R^T(0,x,v)\, m_0(dx,dv). \end{aligned} \tag{4.13}$$

We have that

$$\begin{aligned}
&\inf_{\mu\in\mathcal{C}^T(m_0)}\Big\{\int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\Big(\frac{1}{2}|w|^2+F_R(x,v,m_t^T)\Big)\,\mu_t(dx,dv,dw)dt\\
&\qquad+\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}g_R(x,v,m_T^T)\,\mu_T(dx,dv,dw)\Big\}\\
&=\inf_{\mu\in C([0,T];\mathcal{P}_1(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d))}\sup_{\varphi\in C_c^\infty([0,T]\times\mathbb{T}^d\times\mathbb{R}^d)}\int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\Big(\frac{1}{2}|w|^2+F_R(x,v,m_t^T)+\partial_t\varphi(t,x,v)\\
&\qquad+\langle D_x\varphi(t,x,v),v\rangle+\langle D_v\varphi(t,x,v),w\rangle\Big)\,\mu_t(dx,dv,dw)dt\\
&+\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\Big(g_R(x,v,m_T^T)-\varphi(T,x,v)\Big)\,\mu_T(dx,dv,dw)+\int_{\mathbb{T}^d\times\mathbb{R}^d}\varphi(0,x,v)\,m_0(dx,dv)\\
&\geq\sup_{\varphi\in C_c^\infty([0,T]\times\mathbb{T}^d\times\mathbb{R}^d)}\inf_{\mu\in C([0,T];\mathcal{P}_1(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d))}\int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\Big(\frac{1}{2}|w|^2+F_R(x,v,m_t^T)+\partial_t\varphi(t,x,v)\\
&\qquad+\langle D_x\varphi(t,x,v),v\rangle+\langle D_v\varphi(t,x,v),w\rangle\Big)\,\mu_t(dx,dv,dw)dt\\
&+\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\Big(g_R(x,v,m_T^T)-\varphi(T,x,v)\Big)\,\mu_T(dx,dv,dw)+\int_{\mathbb{T}^d\times\mathbb{R}^d}\varphi(0,x,v)\,m_0(dx,dv).
\end{aligned}$$

In the argument below, the constant c_R depends on R and on the data and may change from line to line. Let $\xi^{1,\varepsilon}=\xi^{1,\varepsilon}(x)$ be a smooth mollifier such that $\mathrm{spt}(\xi^{1,\varepsilon})\subset B_\varepsilon$, $\xi^{1,\varepsilon}(x)\geq 0$ and $\int_{B_\varepsilon}\xi^{1,\varepsilon}(x)\,dx=1$, and define $u_1^{\varepsilon,R}=u_R^T\star_x\xi^{1,\varepsilon}(t,x,v)$ (the convolution being in the x variable only). Let $R^{'}\geq R$ be such that $\mathrm{spt}(u_R^T)$, $\mathrm{spt}(F_R)$ and $\mathrm{spt}(g_R)$ are contained in $B_{R'}$. Then, we have that $u_1^{\varepsilon,R}$ satisfies the following inequality in the viscosity sense

$$\begin{aligned}
&-\partial_t u_1^{\varepsilon,R}(t,x,v)+\frac{1}{2}|D_v u_1^{\varepsilon,R}(t,x,v)|^2-\langle D_x u_1^{\varepsilon,R}(t,x,v),v\rangle\leq F_R\star\xi^{1,\varepsilon}(t,x,v)\\
&\qquad\leq\ F_R(x,v,m_t^T)+C_F\,\varepsilon(1+|v|^\alpha)\mathbf{1}_{(x,v)\in\mathbb{T}^d\times B_{R'}}.
\end{aligned}$$

Now, let $\xi^{2,\varepsilon}=\xi^{2,\varepsilon}(v)$ be a smooth mollifier such that $\mathrm{spt}(\xi^{2,\varepsilon})\subset B_\varepsilon$, $\xi^{2,\varepsilon}(v)\geq 0$ and $\int_{B_\varepsilon}\xi^{2,\varepsilon}(v)\,dv=1$ and define $u_2^{R,\varepsilon}=\xi^{2,\varepsilon}\star_v u_1^{R,\varepsilon}(t,x,v)$ (the convolution being now in the v variable only). Then, by the Lipschitz regularity of u_R^T stated in Lemma 4.4 we have that

$$|\xi^{2,\varepsilon}\star_v\langle D_x u_1^{R,\varepsilon}(t,x,\cdot),\cdot\rangle(v)-\langle D_x u_2^{R,\varepsilon}(t,x,v),v\rangle|\leq\varepsilon\,\|D_x u_1^{R,\varepsilon}\|_\infty\leq c_R\,\varepsilon\,\mathbf{1}_{(x,v)\in\mathbb{T}^d\times B_{R'}}.$$

Hence $u_2^{\varepsilon,R}$ satisfies in the viscosity sense:

$$\begin{aligned}
&-\partial_t u_2^{\varepsilon,R}(t,x,v)+\frac{1}{2}|D_v u_2^{\varepsilon,R}(t,x,v)|^2-\langle D_x u_2^{\varepsilon,R}(t,x,v),v\rangle\\
&\qquad\leq\ F_R(x,v,m_t^T)+c_R\,\varepsilon\,\mathbf{1}_{(x,v)\in\mathbb{T}^d\times B_{R'}}.
\end{aligned}$$

We finally regularize $u_2^{\varepsilon,R}$ in time. Let $\xi^{3,\varepsilon}=\xi^{3,\varepsilon}(t)$ be a smooth mollifier such that $\mathrm{spt}(\xi^{2,\varepsilon})\subset B_\varepsilon$, $\xi^{2,\varepsilon}(t)\geq 0$ and $\int_{B_\varepsilon}\xi^{2,\varepsilon}(t)\,dt=1$ and define $u_3^{R,\varepsilon}=\xi^{3,\varepsilon}\star_t u_2^{R,\varepsilon}(t,x,v)$ (convolution in time). Thus, $u_3^{R,\varepsilon}$, for any $(t,x,v)\in(-\infty,T-\varepsilon]\times\mathbb{T}^d\times\mathbb{R}^d$, satisfies (in the classical sense)

$$\begin{aligned}
&-\partial_t u_3^{R,\varepsilon}(t,x,v)+\frac{1}{2}|D_v u_3^{R,\varepsilon}(t,x,v)|^2-\langle D_x u_3^{R,\varepsilon}(t,x,v),v\rangle\\
&\qquad\leq\xi^{3,\varepsilon}\star_t F_R(x,v,m_\cdot^T)(t)+c_R\,\varepsilon\,\mathbf{1}_{(x,v)\in\mathbb{T}^d\times B_{R'}}.
\end{aligned}$$

By [33, Theorem 5.9] we know that m^T is Lipschitz continuous in time with respect to the d_1 distance. Setting $\hat{u}^R_\varepsilon(t,x,v) = u_3^{R,\varepsilon}(t-\varepsilon,x,v)$, $\hat{u}^R_\varepsilon$ satisfies therefore

$$\begin{aligned} &-\partial_t \hat{u}^R_\varepsilon(t,x,v) + \frac{1}{2}|D_v \hat{u}^R_\varepsilon(t,x,v)|^2 - \langle D_x \hat{u}^R_\varepsilon(t,x,v), v\rangle \\ &\qquad \le F_R(x,v,m_t^T) + c_R\,\varepsilon\,\mathbf{1}_{(x,v)\in\mathbb{T}^d\times B_{R'}}. \end{aligned} \tag{4.14}$$

We note that $\hat{u}^R_\varepsilon$ is smooth and has a compact support and converges uniformly to u^R as $\varepsilon \to 0$. Using $\hat{u}^R_\varepsilon$ as test function we get

$$\begin{aligned} &\sup_{\varphi\in C_c^\infty([0,T]\times\mathbb{T}^d\times\mathbb{R}^d)} \inf_{\mu\in C([0,T];\mathcal{P}_1(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d))} \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \Big(\frac{1}{2}|w|^2 + F_R(x,v,m_t^T) + \partial_t\varphi(t,x,v) \\ &\qquad + \langle D_x\varphi(t,x,v), v\rangle + \langle D_v\varphi(t,x,v), w\rangle\Big)\,\mu_t(dx,dv,dw)dt \\ &+ \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \Big(g_R(x,v,m_T^T) - \varphi(T,x,v)\Big)\,\mu_T(dx,dv,dw) + \int_{\mathbb{T}^d\times\mathbb{R}^d} \varphi(0,x,v)\,m_0(dx,dv) \\ &\ge \inf_{\mu\in C([0,T];\mathcal{P}_1(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d))} \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \Big(\frac{1}{2}|w|^2 + F_R(x,v,m_t^T) + \partial_t\hat{u}^R_\varepsilon(t,x,v) \\ &\qquad + \langle D_x\hat{u}^R_\varepsilon(t,x,v), v\rangle + \langle D_v\hat{u}^R_\varepsilon(t,x,v), w\rangle\Big)\,\mu_t(dx,dv,dw)dt \\ &+ \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \Big(g_R(x,v,m_T^T) - \hat{u}^R_\varepsilon(T,x,v)\Big)\,\mu_T(dx,dv,dw) + \int_{\mathbb{T}^d\times\mathbb{R}^d} \hat{u}^R_\varepsilon(0,x,v)\,m_0(dx,dv) \\ &= \inf_{(t,x,v)\in[0,T]\times\mathbb{T}^d\times\mathbb{R}^d} \Big\{\frac{1}{2}|D_v\hat{u}^R_\varepsilon(t,x,v)|^2 + F_R(x,v,m_t^T) - \partial_t\hat{u}^R_\varepsilon(t,x,v) + \langle D_x\hat{u}^R_\varepsilon(t,x,v), v\rangle \\ &\qquad + g_R(x,v,m_T^T) - \hat{u}^R_\varepsilon(T,x,v)\Big\} + \int_{\mathbb{T}^d\times\mathbb{R}^d} \hat{u}^R_\varepsilon(0,x,v)\,m_0(dx,dv). \end{aligned}$$

By (4.14) we obtain that

$$\begin{aligned} &\inf_{(t,x,v)\in[0,T]\times\mathbb{T}^d\times\mathbb{R}^d} \Big\{\Big(\frac{1}{2}|D_v\hat{u}^R_\varepsilon(t,x,v)|^2 + F_R(x,v,m_t^T) + \partial_t\hat{u}^R_\varepsilon(t,x,v) + \langle D_x\hat{u}^R_\varepsilon(t,x,v), v\rangle\Big) \\ &\qquad + g_R(x,v,m_T^T) - \hat{u}^R_\varepsilon(T,x,v)\Big\} + \int_{\mathbb{T}^d\times\mathbb{R}^d} \hat{u}^R_\varepsilon(0,x,v)\,m_0(dx,dv) \\ &\ge -c_R\,\varepsilon + \inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d} \Big\{g_R(x,v,m_T^T) - \hat{u}^R_\varepsilon(T,x,v)\Big\} + \int_{\mathbb{T}^d\times\mathbb{R}^d} \hat{u}^R_\varepsilon(0,x,v)\,m_0(dx,dv). \end{aligned}$$

As $\varepsilon \to 0^+$ we obtain (4.13).

On the other hand, since u_R^T is a continuous viscosity solution we know that it can be represented as follows:

$$u_R^T(0,x,v) = \inf_{\gamma\in\Gamma_0(x,v)} \left\{\int_0^T \left(\frac{1}{2}|\ddot{\gamma}(t)|^2 + F_R(\gamma(t),\dot{\gamma}(t),m_t^T)\right)\,dt + g_R(\gamma(T),\dot{\gamma}(T),m_T^T)\right\}. \tag{4.15}$$

We define the measure $\nu \in C([0,T];\mathcal{P}_1(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d))$ as

$$\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \varphi(x,v,w)\,\nu_t(dx,dv,dw) = \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \varphi(\gamma_{(x,v)}(t),\dot{\gamma}_{(x,v)}(t),\ddot{\gamma}_{(x,v)}(t))\,m_0(dx,dv),$$

for any $\varphi \in C_c^\infty(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)$ and any $t\in[0,T]$, where $\gamma_{(x,v)}$ is a measurable selection of minimizers of problem (4.15), see Lemma 4.6. By the regularity of the minimizers it is not difficult to prove that

$\nu \in \mathcal{C}^T(m_0)$. Moreover, integrating the equality

$$u_R^T(0,x,v) = \int_0^T \left(\frac{1}{2}|\ddot{\gamma}_{(x,v)}(t)|^2 + F_R(\gamma_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t), m_t^T)\right) dt + g_R(\gamma_{(x,v)}(T), \dot{\gamma}_{(x,v)}(T), m_T^T)$$

against the measure m_0 we deduce that

$$\begin{aligned}
&\int_{\mathbb{T}^d\times\mathbb{R}^d} u_R^T(0,x,v)\, m_0(dx,dv)\\
&= \int_{\mathbb{T}^d\times\mathbb{R}^d}\int_0^T \left(\frac{1}{2}|\ddot{\gamma}_{(x,v)}(t)|^2 + F_R(\gamma_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t), m_t^T)\right) dt\, m_0(dx,dv)\\
&\qquad + \int_{\mathbb{T}^d\times\mathbb{R}^d} g_R(\gamma_{(x,v)}(T), \dot{\gamma}_{(x,v)}(T), m_T^T)\, m_0(dx,dv)\\
&= \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F_R(x,v,m_t^T)\right) \nu_t(dx,dv,dw)dt + \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g_R(x,v,m_T^T)\, \nu_T(dx,dv,dw)\\
&\geq \inf_{\mu\in\mathcal{C}^T(m_0)} \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F_R(x,v,m_t^T)\right) \mu_t(dx,dv,dw)dt\\
&\qquad + \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g_R(x,v,m_T^T)\, \mu_T(dx,dv,dw).
\end{aligned}$$

This completes the proof. □

Proof of Theorem 4.3. Using the notation of Proposition 4.5 we know that for any $R \geq 0$

$$\begin{aligned}
&\inf_{\mu\in\mathcal{C}^T(m_0)} \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F_R(x,v,m_t^T)\right) \mu_t(dx,dv,dw)dt\\
&\qquad + \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g_R(x,v,m_T^T)\, \mu_T(dx,dv,dw)\\
&\qquad = \int_{\mathbb{T}^d\times\mathbb{R}^d} u_R^T(0,x,v)\, m_0(dx,dv).
\end{aligned}$$

Then, on the one hand it is easy to see, by standard optimal control arguments, that for any $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$ we have that $|u_R^T(0,x,v)| \leq \tilde{C}_1(1+|v|^\alpha)$. By Dominated Convergence Theorem we get

$$\lim_{R\to+\infty} \int_{\mathbb{T}^d\times\mathbb{R}^d} u_R^T(0,x,v)\, m_0(dx,dv) = \int_{\mathbb{T}^d\times\mathbb{R}^d} u^T(0,x,v)\, m_0(dx,dv).$$

On the other hand, without loss of generality we can define a cut-off function ξ_R as in Proposition 4.5 such that F_R and g_R are non-decreasing in R. Thus

$$\begin{aligned}
&\limsup_{R\to+\infty} \inf_{\mu\in\mathcal{C}^T(m_0)} \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F_R(x,v,m_t^T)\right) \mu_t(dx,dv,dw)dt\\
&\qquad + \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g_R(x,v,m_T^T)\, \mu_T(dx,dv,dw)\\
&\qquad \leq \inf_{\mu\in\mathcal{C}^T(m_0)} \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,m_t^T)\right) \mu_t(dx,dv,dw)dt\\
&\qquad + \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g(x,v,m_T^T)\, \mu_T(dx,dv,dw).
\end{aligned}$$

To prove the reverse inequality, let $\{R_j\}_{j\in\mathbb{N}}$ and $\{\mu_t^j\}_{j\in\mathbb{N}} \subset \mathcal{C}^T(m_0)$ be such that

$$\begin{aligned}
\liminf_{R\to+\infty} \inf_{\mu\in\mathcal{C}^T(m_0)} & \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F_R(x,v,m_t^T)\right)\, \mu_t(dx,dv,dw)dt \\
&+ \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g_R(x,v,m_T^T)\, \mu_T(dx,dv,dw) \\
&= \lim_{j\to+\infty} \inf_{\mu\in\mathcal{C}^T(m_0)} \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F_{R_j}(x,v,m_t^T)\right)\, \mu_t(dx,dv,dw)dt \\
&+ \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g_{R_j}(x,v,m_T^T)\, \mu_T(dx,dv,dw) \\
&= \lim_{j\to+\infty} \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F_{R_j}(x,v,m_t^T)\right)\, \mu_t^j(dx,dv,dw)dt \\
&+ \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g_{R_j}(x,v,m_T^T)\, \mu_T^j(dx,dv,dw).
\end{aligned}$$

We claim that $\{\mu_t^j\}_{j\in\mathbb{N}}$ is tight. Indeed, the lower bound on F and g, there exists a constant $C \geq 0$ such that

$$\sup_j \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} |w|^2\, \mu_t^j(dx,dv,dw)dt \leq C \tag{4.16}$$

and thus it is enough to prove that the moment with respect to v is also bounded. In order to prove this bound, let $\psi \in C_c^\infty(\mathbb{R}^d)$ with $\psi(0) = 0$ and such that $|D\psi(p)| \leq 1$. For $\varphi(t,x,v) = (T-t)\psi(v)$, we have, by the definition of a $T-$closed measure in (4.8),

$$\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \big(-\psi(v) + (T-t)\langle D\psi(v), w\rangle\big)\, \mu_t^j(dx,dv,dw)dt = -T\int_{\mathbb{T}^d\times\mathbb{R}^d} \psi(v)\, m_0(dx,dv) \tag{4.17}$$

and by (4.16) and Cauchy-Schwarz inequality we get

$$\Big|\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} (T-t)\langle D\psi(v), w\rangle\, \mu_t^j(dx,dv,dw)dt\Big| \leq TC^{1/2}.$$

Thus, by (4.17) we obtain that

$$\Big|\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \psi(v)\, \mu_t^j(dx,dv,dw)dt\Big| \leq C,$$

for some new constant C. If we choose ψ_n such that $\psi_n(v)$ increases in n and converges to $|v|$, we get therefore

$$\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} |v|\, \mu_t^j(dx,dv,dw)dt \leq C.$$

This implies that $\{\mu_t^j\}_{j\in\mathbb{N}}$ is tight and, up to a subsequence still denoted by μ_t^j, converges to some $\bar\mu \in \mathcal{C}^T(m_0)$. Then, we have that

$$
\begin{aligned}
&\inf_{\mu\in\mathcal{C}^T(m_0)} \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,m_t^T)\right)\, \mu_t(dx,dv,dw)dt \\
&+ \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g(x,v,m_T^T)\, \mu_T(dx,dv,dw) \\
&\le \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,m_t^T)\right)\, \bar\mu_t(dx,dv,dw)dt \\
&+ \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g(x,v,m_T^T)\, \bar\mu_T(dx,dv,dw) \\
&\le \liminf_{j\to+\infty} \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F_{R_j}(x,v,m_t^T)\right)\, \mu_t^j(dx,dv,dw)dt \\
&+ \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g_{R_j}(x,v,m_T^T)\, \mu_T^j(dx,dv,dw) \\
&= \liminf_{j\to+\infty} \inf_{\mu\in\mathcal{C}^T(m_0)} \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F_{R_j}(x,v,m_t^T)\right)\, \mu_t(dx,dv,dw)dt \\
&+ \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g_{R_j}(x,v,m_T^T)\, \mu_T(dx,dv,dw).
\end{aligned}
$$

This completes the proof of equality (4.10).

It remain to check the existence of a minimizer $\bar\mu^T \in \mathcal{C}^T(m_0)$ of the problem in the left-hand side such that $m_t^T = \pi\sharp\bar\mu_t^T$. For this, let $\gamma_{(x,v)}$ denote the measurable selection of minimizers of $u^T(0,x,v)$ in (4.7) as in Lemma 4.6 below and define the measure

$$
\bar\mu_t^T = \left((x,v)\to(\gamma_{(x,v)}(t),\dot\gamma_{(x,v)}(t), D_v u^T(t,\gamma_{(x,v)}(t),\dot\gamma_{(x,v)}(t)))\right)\sharp m_0
$$

for any $t\in[0,T]$. Note that by [1, Lemma 3.5] $\bar\mu_t^T$ is well-defined since $u(t,x,\cdot)$ is differentiable along the optimal trajectory $\gamma_{(x,v)}$ with

$$
\ddot\gamma_{(x,v)}(t) = D_v u^T(t,\gamma_{(x,v)}(t),\dot\gamma_{(x,v)}(t)), \quad t\in[0,T]
$$

In particular, it is easy to see that $\bar\mu^T \in \mathcal{C}^T(m_0)$ and moreover, by [1, Proposition 4.2] we have that $m_t^T = \pi\sharp\bar\mu_t^T$ since $m_t^T = ((x,v)\to(\gamma_{(x,v)}(t),\dot\gamma_{(x,v)}(t)))\sharp m_0$. By the representation formula of the value function we have that

$$
\begin{aligned}
u^T(0,x,v) &= \int_0^T \left(\frac{1}{2}|\ddot\gamma_{(x,v)}(t)|^2 + F(\gamma_{(x,v)}(t),\dot\gamma_{(x,v)}(t),m_t^T)\right)dt + g(\gamma_{(x,v)}(T),\dot\gamma_{(x,v)}(T),m_T^T) \\
&= \int_0^T \left(\frac{1}{2}|D_v u^T(t,\gamma_{(x,v)}(t),\dot\gamma_{(x,v)}(t))|^2 + F(\gamma_{(x,v)}(t),\dot\gamma_{(x,v)}(t),m_t^T)\right)dt \\
&\quad + g(\gamma_{(x,v)}(T),\dot\gamma_{(x,v)}(T),m_T^T).
\end{aligned}
$$

Integrating both side against the measure m_0 and using the definition of $\bar\mu^T$, we obtain that $\bar\mu^T$ satisfies the equality in (4.10) and therefore is optimal. □

Lemma 4.6. *Assume that F satisfies* **(F1')** *and* **(F2')** *and g satisfies* **(G1)**. *For $(x,v)\in\mathbb{T}^d\times\mathbb{R}^d$ let $\Gamma^*(x,v)\subset\Gamma_0(x,v)$ be the set of minimizers of problem (4.7) for $t=0$. Then, the set-valued map*

$$
\Gamma^* : (\mathbb{T}^d\times\mathbb{R}^d, |\cdot|) \rightrightarrows (\Gamma, \|\cdot\|_\infty), \quad (x,v)\mapsto\Gamma^*(x,v)
$$

has a measurable selection $\gamma_{(x,v)}$, i.e. $(x,v) \to \gamma_{(x,v)}$ is measurable and, for any $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$, $\gamma_{(x,v)} \in \Gamma^*(x,v)$.

Proof. By using classical results from optimal control theory it is not difficult to see that Γ^* has a closed graph, see for instance [33, Lemma 4.1]. Therefore, by [12, Proposition 9.5] the set-valued map $(x,v) \rightrightarrows \Gamma^*(x,v)$ is measurable with closed values. This implies by [13, Theorem A 5.2] the existence of a measurable selection $\gamma_{(x,v)} \in \Gamma^*(x,v)$. □

4.3. **Convergence of the solution of the time dependent MFG system.** We now investigate the limit as the horizon $T \to +\infty$ of the time-dependent MFG problem. The main result of this subsection is the following proposition:

Proposition 4.7 (Convergence of MFG solution). *Assume that F satisfies* **(F1')**, **(F2')**, **(F3')** *with $\alpha = 2$ and the monotonicity condition (2.8), that g satisfies* **(G1)** *and that the initial distribution m_0 in (2.6) belongs to $\mathcal{P}_2(\mathbb{T}^d \times \mathbb{R}^d)$. Let (u^T, m^T) be a solution of the MFG system (2.6) and let $(\bar\lambda, \bar\mu)$ be the solution of the ergodic MFG problem (4.1). Then*

$$\lim_{T\to+\infty} \frac{1}{T} \int_{\mathbb{T}^d\times\mathbb{R}^d} u^T(0,x,v)\, m_0(dx,dv) = \bar\lambda.$$

Throughout the section, we assume that the assumption of Proposition 4.7 are in force. The proof of the proposition—given at the end of the subsection—is made at the level of the closed and $T-$closed measures. For this we first need to discuss how to manipulate them. The first lemma is a straightforward application of the definition of $T-$closed measures:

Lemma 4.8 (Concatenation of T-closed measure). *Let $T, T' > 0$, $m_0 \in \mathcal{P}_2(\mathbb{T}^d \times \mathbb{R}^d)$, $\mu_1 \in \mathcal{C}^T(m_0)$ and $\mu_2 \in \mathcal{C}^{T'}(m_1)$ with $m_1 = \pi\sharp\mu_1(T)$. Then, the measure*

$$\mu_t := \begin{cases} \mu_1(t), & t \in [0,T] \\ \mu_2(t-T), & t \in (T, T+T'] \end{cases}$$

belongs to $\mathcal{C}^{T+T'}(m_0)$.

Next we explain how to link two measures by a $T-$closed measure:

Lemma 4.9 (Linking two measures by a T-closed measure). *Let m_0^1 and m_0^2 belong to $\mathcal{P}_2(\mathbb{T}^d \times \mathbb{R}^d)$. Then, there exists $\mu^{m_0^1\to m_0^2} \in \mathcal{C}^{T=1}(m_0^1)$ such that $m_0^2 = \pi\sharp\mu_1^{m_0^1\to m_0^2}$ and*

$$\int_0^1 \int_{\mathbb{T}^d\times\mathbb{R}^{2d}} \Big(\frac{1}{2}|w|^2 + c_F(1+|v|^2)\Big)\, \mu_t^{m_0^1\to m_0^2}(dx,dv,dw)dt \le C_2(1 + M_2(m_0^1) + M_2(m_0^2)), \tag{4.18}$$

where $M_2(m) = \int_{\mathbb{T}^d\times\mathbb{R}^d} |v|^2 dm(x,v)$ (for $m \in \mathcal{P}_2(\mathbb{T}^d \times \mathbb{R}^d)$) and where C_2 depends only on α and c_F.

Proof. Let $(x_0,v_0) \in \mathrm{spt}(m_0^1)$ and let $(x,v) \in \mathrm{spt}(m_0^2)$. Then, following the proof of Lemma 3.3, there exists a curve $\sigma_{(x_0,v_0)}^{(x,v)} : [0,1] \to \mathbb{T}^d$ such that $\sigma_{(x_0,v_0)}^{(x,v)}(0) = x_0$, $\dot\sigma_{(x_0,v_0)}^{(x,v)}(0) = v_0$ and $\sigma_{(x_0,v_0)}^{(x,v)}(1) = y$, $\dot\sigma_{(x_0,v_0)}^{(x,v)}(1) = w$ with

$$\int_0^1 \Big(\frac{1}{2}|\ddot\sigma_{(x_0,v_0)}^{(x,v)}(t)|^2 + c_F(1+|\dot\sigma_{(x_0,v_0)}^{(x,v)}(t)|^2)\Big)dt \le C_2(1+|v|^2+|v_0|^2). \tag{4.19}$$

Moreover, by construction, σ depends continuously on (x_0,v_0,x,v). Let $\lambda \in \Pi(m_0^1, m_0^2)$ be a transport plan between m_0^1 and m_0^2 (see (2.1)). We define the measure $\mu^{m_0^1\to m_0^2} \in \mathcal{C}^1(m_0^1)$ by

$$\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \varphi(x,v,w)\mu_t^{m_0^1\to m_0^2}(dx,dv,dw) = \int_{(T^d\times\mathbb{R}^d)^2} \varphi(\sigma_{(x_0,v_0)}^{(x,v)}(t), \dot\sigma_{(x_0,v_0)}^{(x,v)}(t), \ddot\sigma_{(x_0,v_0)}^{(x,v)}(t))\, \lambda(dx_0,dv_0,dx,dv)$$

for any $\varphi \in C_c^\infty(\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d)$. Then, on easily checks that $m_0^2 = \pi\sharp\mu_1^{m_0^1 \to m_0^2}$ and that, by (4.19):

$$\int_0^1 \int_{\mathbb{T}^d\times\mathbb{R}^{2d}} (\frac{1}{2}|w|^2 + c_F(1+|v|^2))\, \mu_t^{m_0^1\to m_0^2}(dx,dv,dw)dt$$
$$\leq C_2 \int_0^1 \int_{\mathbb{T}^d\times\mathbb{R}^{2d}} (1+|v|^2+|v_0|^2)\, \mu_t^{m_0^1\to m_0^2}(dx,dv,dw)dt = C_2(1+M_2(m_0^1)+M_2(m_0^2)).$$

□

Proposition 4.10 (Energy estimate). *Under the notation and assumption of Proposition 4.7, there exists a constant $C \geq 0$ (independent of T) such that*

$$\int_0^T \sup_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d} \frac{|F(x,v,m_t^T) - F(x,v,\bar m)|^{2d+2}}{(1+|v|^2)^{2d}} dt \leq CT^{\frac{1}{2}}, \tag{4.20}$$

where $\bar m = \pi\sharp\bar\mu$, with $\pi(x,v,w) = (x,v)$.

Proof. The proof consists in building from $\bar\mu$ and μ^T competitors in problems (4.1) and (4.10) respectively. Let us recall that μ^T and $\bar\mu$ are minimizers for these respective problems.

We start with problem (4.10). Fix $T \geq 2$. We define the measure $\tilde\mu^T$ by

$$\tilde\mu_t^T = \begin{cases} \mu_t^{m_0\to\bar m}, & t\in[0,1] \\ \bar\mu, & t\in(1,T], \end{cases} \tag{4.21}$$

where $\mu^{m_0\to\bar m}$ is the measure defined by Lemma 4.9. We know by Lemma 4.8 that $\tilde\mu^T$ belongs to $\mathcal{C}^T(m_0)$. So we can use $\tilde\mu^T$ as a competitor in problem (4.10) to get

$$\begin{aligned} &\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,m_t^T)\right) \mu_t^T(dx,dv,dw)dt \\ &\qquad + \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g(x,v,m_T^T)\, \mu_T^T(dx,dv,dw) \\ &\leq \int_0^1 \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,m_t^T)\right) \mu_t^{m_0\to\bar m}(dx,dv,dw)dt \\ &\qquad + \int_1^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,m_t^T)\right) \bar\mu(dx,dv,dw)dt \\ &\qquad + \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g(x,v,m_T^T)\, \bar\mu(dx,dv,dw). \end{aligned} \tag{4.22}$$

Next we build from μ^T a competitor for the minimization problem (4.1) for which $\bar\mu$ is a minimizer. In view of [1, Proposition 4.2] (see also [33, Theorem 6.3]) there exists a Borel measurable maps $(x,v) \to \gamma_{(x,v)}$ such that, for each $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$, $\gamma_{(x,v)}$ is a minimizer for $u^T(0,x,v)$ in (4.10) and satisfies

$$\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^{2d}} \varphi(x,v,w)\, \mu_t^T(dx,dv,dw)dt = \int_{\mathbb{T}^d\times\mathbb{R}^d} \int_0^T \varphi(\gamma_{(x,v)}(t), \dot\gamma_{(x,v)}(t), \ddot\gamma_{(x,v)}(t))dt m_0(dx,dv) \tag{4.23}$$

for any test function $\varphi \in C_b^0(\mathbb{T}^d \times \mathbb{R}^{2d})$. By Lemma 3.5 and Remark 3.6, for any $\lambda \geq 2$, there exist Borel measurable maps $(x,v) \to \tilde\gamma_{(x,v)}$ and $(x,v) \to \tau_{(x,v)}$ such that

$$\tilde\gamma_{(x,v)}(0) = \tilde\gamma_{(x,v)}(T) = x,\ \dot{\tilde\gamma}_{(x,v)}(0) = \dot{\tilde\gamma}_{(x,v)}(T) = v \text{ and } \tilde\gamma_{(x,v)} = \gamma_{(x,v)} \text{ on } [0,\tau_{(x,v)}] \tag{4.24}$$

and

$$\int_{\tau_{(x,v)}}^{T} (\frac{1}{2}|\ddot{\tilde{\gamma}}_{(x,v)}(t)|^2 + c_F(1+|\dot{\tilde{\gamma}}_{(x,v)}(t)|^2)dt \leq C_3(1+|v|)^2(\lambda^2+\lambda^{-2}T). \tag{4.25}$$

Let us define $\hat{\mu}^T$ by

$$\int_{\mathbb{T}^d\times\mathbb{R}^{2d}} \varphi(x,v,w)\,\hat{\mu}^T(dx,dv,dw) = T^{-1}\int_{\mathbb{T}^d\times\mathbb{R}^d}\int_0^T \varphi(t,\tilde{\gamma}_{(x,v)}(t),\dot{\tilde{\gamma}}_{(x,v)}(t),\ddot{\tilde{\gamma}}_{(x,v)}(t))dt m_0(dx,dv) \tag{4.26}$$

for any test function $\varphi \in C_b^0(\mathbb{T}^d\times\mathbb{R}^{2d})$. Note that, by (4.24), $\hat{\mu}^T$ belongs to $\mathcal{C}$. So using the closed measure $\hat{\mu}^T$ as a competitor in problem (4.1) we deduce that

$$\begin{aligned}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F(x,v,\bar{m})\right)\,\bar{\mu}(dx,dv,dw)\\ \leq \int_{\mathbb{T}^d\times\mathbb{R}^{2d}}\left(\frac{1}{2}|w|^2+F(x,v,\bar{m})\right)\,\hat{\mu}^T(dx,dv,dw).\end{aligned} \tag{4.27}$$

Note that by the definition of $\hat{\mu}^T$ in (4.26) and by (4.24) and (4.25), we have

$$\begin{aligned}&T\int_{\mathbb{T}^d\times\mathbb{R}^{2d}}\left(\frac{1}{2}|w|^2+F(x,v,\bar{m})\right)\,\hat{\mu}^T(dx,dv,dw)\\ &=\int_{\mathbb{T}^d\times\mathbb{R}^d}\int_0^T(\frac{1}{2}|\ddot{\tilde{\gamma}}_{(x,v)}(t)|^2+F(\tilde{\gamma}_{(x,v)}(t),\dot{\tilde{\gamma}}_{(x,v)}(t),\bar{m}))dt\,m_0(dx,dv)\\ &\leq\int_{\mathbb{T}^d\times\mathbb{R}^d}\Big(\int_0^{\tau_{(x,v)}}(\frac{1}{2}|\ddot{\gamma}_{(x,v)}(t)|^2+F(\gamma_{(x,v)}(t),\dot{\gamma}_{(x,v)}(t),\bar{m}))dt\\ &\qquad+\int_{\tau_{(x,v)}}^T(\frac{1}{2}|\ddot{\tilde{\gamma}}_{(x,v)}(t)|^2+c_F(1+|\dot{\tilde{\gamma}}_{(x,v)}(t)|^2))dt\Big)\,m_0(dx,dv)\\ &\leq\int_{\mathbb{T}^d\times\mathbb{R}^d}\Big(\int_0^T(\frac{1}{2}|\ddot{\gamma}_{(x,v)}(t)|^2+F(\gamma_{(x,v)}(t),\dot{\gamma}_{(x,v)}(t),\bar{m}))dt\\ &\qquad+C_3(1+|v|)^2(\lambda^2+\lambda^{-2}T)\Big)\,m_0(dx,dv).\end{aligned}$$

Plugging this inequality into (4.27) and using the representation of μ^T in (4.23) then gives

$$\begin{aligned}&\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F(x,v,\bar{m})\right)\,\bar{\mu}(dx,dv,dw)\\ &\quad\leq\int_{\mathbb{T}^d\times\mathbb{R}^{2d}}\left(\frac{1}{2}|w|^2+F(x,v,\bar{m})\right)\,\hat{\mu}^T(dx,dv,dw)\\ &\quad\leq T^{-1}\int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F(x,v,\bar{m})\right)\,\mu_t^T(dx,dv,dw)dt\\ &\qquad+2C_3(1+M_2(m_0))(\lambda^2T^{-1}+\lambda^{-2}),\end{aligned} \tag{4.28}$$

where $M_2(m_0) = \int_{\mathbb{T}^d\times\mathbb{R}^d} |v|^2 dm_0(x,v)$. Putting together (4.22) and (4.28) (multiplied by T) then implies that

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^{2d}} (\frac{1}{2}|w|^2 + F(x,v,m_t^T)) d\mu_t^T(x,v,w) + \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^{2d}} \left(\frac{1}{2}|w|^2 + F(x,v,\bar m)\right)\, \bar\mu(dx,dv,dw)dt \\
&\le \int_0^1 \int_{\mathbb{T}^d\times\mathbb{R}^{2d}} \left(\frac{1}{2}|w|^2 + F(x,v,m_t^T)\right)\, \mu_t^{m_0\to\bar m}(dx,dv,dw)dt \\
&\quad + \int_1^T \int_{\mathbb{T}^d\times\mathbb{R}^{2d}} \left(\frac{1}{2}|w|^2 + F(x,v,m_t^T)\right)\, \bar\mu(dx,dv,dw)dt \\
&\qquad + \int_{\mathbb{T}^d\times\mathbb{R}^{2d}} g(x,v,m_T^T)\, \bar\mu(dx,dv,dw) - \int_{\mathbb{T}^d\times\mathbb{R}^{2d}} g(x,v,m_T^T)\, \mu_T^T(dx,dv,dw) \\
&\quad + \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^{2d}} \left(\frac{1}{2}|w|^2 + F(x,v,\bar m)\right)\, \mu_t^T(dx,dv,dw)dt \\
&\quad + 2C_3(1+M_2(m_0))(\lambda^2 + \lambda^{-\alpha}T).
\end{aligned}
$$

Using (4.18) to bound the first term in the right-hand side (note that $\bar m$ belongs to $\mathcal{P}_{\alpha,2}(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)$ with $\alpha = 2$, so that $\bar m \in \mathcal{P}_2(\mathbb{T}^d\times\mathbb{R}^d)$) we obtain therefore

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^{2d}} (F(x,v,m_t^T) - F(x,v,\bar m))\, (\mu_t^T(dx,dv,dw) - \bar\mu(dx,dv,dw))dt \\
&\qquad \le\ C_2(1 + M_2(m_0) + M_2(\bar m)) + 2\|g\|_\infty + 2C_3(1+M_2(m_0))(\lambda^2 + \lambda^{-2}T).
\end{aligned}
$$

We now use the strong monotonicity condition (2.8) and choose $\lambda = T^{1/4}$ to get

$$
\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^{2d}} (F(x,v,m_t^T) - F(x,v,\bar m))^2 dxdvdt \le\ CT^{\frac{1}{2}}
$$

for a constant C independent of T. Recalling that F satisfies **(F3')**, we obtain (4.20) by the interpolation inequality Lemma B.1 in the Appendix. □

Proof of Proposition 4.7. Throughout the proof, C denotes a constant independent of T and which may change from line to line. Let $\mu^T \in \mathcal{C}^T(m_0)$ be associated with a solution (u^T, m^T) of the MFG system (2.6) as in Theorem 4.3. By Theorem 4.3 we have that

$$
\begin{aligned}
&\frac{1}{T}\int_{\mathbb{T}^d\times\mathbb{R}^d} u^T(0,x,v)\, m_0(dx,dv) \\
&\qquad = \frac{1}{T}\Big\{\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,m_t^T)\right)\, \mu_t^T(dx,dv,dw)dt \\
&\qquad\qquad + \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g(x,v,m_T^T)\, \mu_T^T(dx,dv,dw)\Big\} \qquad (4.29)\\
&\qquad = \inf_{\mu\in\mathcal{C}^T(m_0)} \frac{1}{T}\Big\{\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,m_t^T)\right)\, \mu_t(dx,dv,dw)dt \\
&\qquad\qquad + \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g(x,v,m_T^T)\, \mu_T(dx,dv,dw)\Big\}.
\end{aligned}
$$

We first claim that

$$\begin{aligned}\limsup_{T\to+\infty}\inf_{\mu\in\mathcal{C}^T(m_0)}\frac{1}{T}\Big\{&\int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F(x,v,m_t^T)\right)\,\mu_t(dx,dv,dw)dt\\&+\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}g(x,v,m_T^T)\,\mu_T(dx,dv,dw)\Big\}\\&\le\inf_{\tilde\mu\in\mathcal{C}}\Big\{\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F(x,v,\bar m)\right)\,\tilde\mu(dx,dv,dw)\Big\}.\end{aligned}\tag{4.30}$$

In order to prove the claim, we first note that, by Young's inequality and Proposition 4.10, we have, for any $\mu\in\mathcal{C}^T(m_0)$,

$$\begin{aligned}&\Big|\int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\big(F(x,v,m_t^T)-F(x,v,\bar m)\big)\,\mu_t(dx,dv,dw)dt\Big|\\&\le\int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\sup_{(x',v')\in\mathbb{T}^d\times\mathbb{R}^d}\frac{|F(x',v',m_t^T)-F(x',v',\bar m)|}{(1+|v'|^2)^{\frac{d}{d+1}}}\,(1+|v|^2)^{\frac{d}{d+1}}\,\mu_t(dx,dv,dw)dt\\&\le\frac{T^{\frac14}}{2d+2}\int_0^T\sup_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d}\frac{|F(x,v,m_t^T)-F(x,v,\bar m)|^{2d+2}}{(1+|v|^2)^{2d}}dt\\&\qquad+\frac{(2d+1)T^{-\frac{1}{4(2d+1)}}}{2d+2}\int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}(1+|v|^2)^{\frac{2d}{(2d+1)}}\,\mu_t(dx,dv,dw)dt\\&\le CT^{\frac34}+T^{-\frac{1}{4(2d+1)}}\int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}(1+|v|^2)\,\mu_t(dx,dv,dw)dt.\end{aligned}\tag{4.31}$$

As g is bounded, we have therefore, for any $\mu\in\mathcal{C}^T(m_0)$,

$$\begin{aligned}&\frac{1}{T}\Big\{\int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F(x,v,m_t^T)\right)\,\mu_t(dx,dv,dw)dt\\&\qquad+\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}g(x,v,m_T^T)\,\mu_T(dx,dv,dw)\Big\}\\&\le\frac{1}{T}\Big\{\int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F(x,v,\bar m)\right)\,\mu_t(dx,dv,dw)dt\\&\qquad+T^{-\frac{1}{4(2d+1)}}\int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}(1+|v|^2)\,\mu_t(dx,dv,dw)dt\Big\}+CT^{-\frac14}+T^{-1}\|g\|_\infty.\end{aligned}\tag{4.32}$$

Given $\tilde\mu\in\mathcal{C}$, we know from Lemma 4.9 that there exists $\mu^{m_0\to\pi\sharp\tilde\mu}$ such that

$$\int_0^1\int_{\mathbb{T}^d\times\mathbb{R}^{2d}}(\frac{1}{2}|w|^2+c_F(1+|v|^2))\mu_t^{m_0\to\pi\sharp\tilde\mu}(dx,dv,dw)dt\le C_2(1+M_2(m_0)+M_2(\pi\sharp\tilde\mu)).\tag{4.33}$$

Let us then define $\tilde\mu^T$ by

$$\tilde\mu_t^T=\begin{cases}\mu_t^{m_0\to\pi\sharp\tilde\mu}, & t\in[0,1]\\ \tilde\mu, & t\in(1,T],\end{cases}$$

By Lemma 4.9, $\tilde{\mu}^T$ belongs to $\mathcal{C}^T(m_0)$ and we have, in view of (4.33),

$$
\begin{aligned}
&T^{-1}\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,\bar{m})\right)\, \tilde{\mu}_t^T(dx,dv,dw)dt \\
&\leq C_2 T^{-1}(1 + M_2(m_0) + M_2(\pi\sharp\tilde{\mu})) + T^{-1}(T-1)\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,\bar{m})\right)\, \tilde{\mu}(dx,dv,dw)
\end{aligned}
$$

while

$$
\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} (1+|v|^2)\, \tilde{\mu}_t^T(dx,dv,dw)dt \leq C_2(1 + M_2(m_0) + M_2(\pi\sharp\tilde{\mu})) + (T-1)M_2(\pi\sharp\tilde{\mu}).
$$

Therefore, coming back to (4.32) and using the $\tilde{\mu}^T$ built as above from the $\tilde{\mu} \in \mathcal{C}$ as competitors, we have

$$
\begin{aligned}
&\inf_{\mu\in\mathcal{C}^T(m_0)} \frac{1}{T}\Big\{\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,m_t^T)\right)\, \mu_t(dx,dv,dw)dt \\
&\qquad + \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g(x,v,m_T^T)\, \mu_T(dx,dv,dw)\Big\} \\
&\leq \inf_{\tilde{\mu}\in\mathcal{C}}\Big\{\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,\bar{m})\right)\, \tilde{\mu}(dx,dv,dw) \\
&\qquad + CT^{-\frac{1}{4(2d+1)}}(1 + M_2(m_0) + M_2(\pi\sharp\tilde{\mu}))\Big\} + CT^{-\frac{1}{4}} + T^{-1}\|g\|_\infty.
\end{aligned}
\tag{4.34}
$$

Since, by assumption **(F2')**,

$$
\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} F(x,v,\bar{m})\, \tilde{\mu}(dx,dv,dw) \geq c_F^{-1} M_2(\pi\sharp\tilde{\mu}) - c_F,
$$

one easily checks that the limit of the right-hand side of (4.34) as $T \to +\infty$ is

$$
\inf_{\tilde{\mu}\in\mathcal{C}}\Big\{\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,\bar{m})\right)\, \tilde{\mu}(dx,dv,dw)\Big\}.
$$

This proves our claim (4.30).

Next we claim that there exists a closed measure $\hat{\mu} \in \mathcal{C}$ such that

$$
\begin{aligned}
&\liminf_{T\to+\infty} \frac{1}{T}\Big\{\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,m_t^T)\right)\, \mu_t^T(dx,dv,dw)dt \\
&\qquad + \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g(x,v,m_T^T)\, \mu_T^T(dx,dv,dw)\Big\} \\
&\geq \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,\bar{m})\right)\, \hat{\mu}(dx,dv,dw).
\end{aligned}
\tag{4.35}
$$

For the proof of (4.35), we work with a subsequence of $T \to +\infty$ (still denoted by T) along which the lower limit in the left-hand side is achieved. Coming back to (4.31), we have

$$\begin{aligned}
&\frac{1}{T}\Big\{\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,m_t^T)\right)\, \mu_t^T(dx,dv,dw)dt \\
&\qquad + \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g(x,v,m_T^T)\, \mu_T^T(dx,dv,dw)\Big\} \\
&\geq \frac{1}{T}\Big\{\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,\bar m)\right)\, \mu_t^T(dx,dv,dw)dt \\
&\qquad - T^{-\frac{1}{4(2d+1)}} \int_0^T (1+M_2(\pi\sharp\mu_t^T))dt\Big\} - CT^{-\frac{1}{4}} - \|g\|_\infty T^{-1}.
\end{aligned}$$

By the coercivity of F in assumption **(F2')**, we can absorb the second term in the right-hand side into the first one and obtain:

$$\begin{aligned}
&\frac{1}{T}\Big\{\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,m_t^T)\right)\, \mu_t^T(dx,dv,dw)dt \\
&\qquad + \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g(x,v,m_T^T)\, \mu_T^T(dx,dv,dw)\Big\} \\
&\geq \frac{1}{T}(1-C^{-1}T^{-\frac{1}{4(2d+1)}}) \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,\bar m)\right)\, \mu_t^T(dx,dv,dw)dt \\
&\qquad - CT^{-\frac{1}{4(2d+1)}} - CT^{-\frac{1}{4}} - \|g\|_\infty T^{-1}.
\end{aligned} \tag{4.36}$$

As in the proof of Proposition 4.10 (see (4.28)), for any $\lambda \geq 1$, we can find a closed measure $\hat\mu^T \in \mathcal{C}$ such that

$$\begin{aligned}
&\int_{\mathbb{T}^d\times\mathbb{R}^{2d}} \left(\frac{1}{2}|w|^2 + F(x,v,\bar m)\right)\, \hat\mu^T(dx,dv,dw) \\
&\qquad \leq T^{-1}\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,\bar m)\right)\, \mu_t^T(dx,dv,dw)dt \\
&\qquad + 2C_3(1+M_2(m_0))(\lambda^2 T^{-1} + \lambda^{-2}).
\end{aligned}$$

Plugging this inequality into (4.36) we find therefore

$$\begin{aligned}
&\frac{1}{T}\Big\{\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,m_t^T)\right)\, \mu_t^T(dx,dv,dw)dt \\
&\qquad + \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g(x,v,m_T^T)\, \mu_T^T(dx,dv,dw)\Big\} \\
&\geq (1-C^{-1}T^{-\frac{1}{4(2d+1)}}) \int_{\mathbb{T}^d\times\mathbb{R}^{2d}} \left(\frac{1}{2}|w|^2 + F(x,v,\bar m)\right)\, \hat\mu^T(dx,dv,dw) \\
&\qquad - 2C_3(1+M_2(m_0))(\lambda^2 T^{-1} + \lambda^{-2}) - CT^{-\frac{1}{4(2d+1)}}.
\end{aligned}$$

By assumption **(F2')**, the functional in the right-hand side of the inequality is coercive for T large enough. So $\hat\mu^T$ weakly-$*$ converges (up to a subsequence) to a closed measure $\hat\mu$. Taking the lower-limit in the last inequality then implies (4.35).

Putting together (4.30) and (4.35), we find that $\hat\mu$ is a minimizer in the right-hand side of (4.30) and that the semi-limits and the inequalities in (4.30) and (4.35) are in fact limits and equalities. So coming

back to (4.29) we find that

$$\lim_{T\to+\infty}\frac{1}{T}\int_{\mathbb{T}^d\times\mathbb{R}^d}u^T(0,x,v)\,m_0(dx,dv)=\inf_{\tilde\mu\in\mathcal{C}}\Big\{\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\Big(\frac{1}{2}|w|^2+F(x,v,\bar m)\Big)\;\tilde\mu(dx,dv,dw)\Big\}.$$

The right-hand side of this equality is nothing than but $\bar\lambda$ since $(\bar\lambda,\bar\mu)$ is a solution to the the ergodic MFG problem with $\bar m=\pi\sharp\bar\mu$: this completes the proof of the proposition. □

To complete the proof of Theorem 2.5, we need estimates on the oscillation of u^T. This comes next:

Lemma 4.11. *For any $R\geq 1$ and $(x,v),(x',v')\in\mathbb{T}^d\times B_R$, we have*

$$|u^T(0,x,v)-u^T(0,x',v')|\leq CR^2T^{\frac{4d+3}{4(d+1)}},$$

where C is independent of T and R.

Proof. Let $\gamma\in\Gamma(x,v)$ be optimal for $u^T(0,x,v)$ in (4.7). We define $\tilde\gamma\in\Gamma(x',v')$ by

$$\tilde\gamma(t)=\begin{cases}\sigma(t) & \text{if } t\in[0,1]\\ \gamma(t-1) & \text{if } t\in[1,T].\end{cases}$$

where σ is as in Lemma 3.3 with $\sigma(0)=x'$, $\dot\sigma(0)=v'$, $\sigma(1)=x$, $\dot\sigma(1)=v$ and

$$\int_0^1\Big(\frac{1}{2}|\ddot\sigma(t)|^2+F(\sigma(t),\dot\sigma(t),m_t^T)\Big)dt\leq 2C_2R^2.$$

Note that, as the problem for u^T depends on time through (m_t^T), the cost associated with $\tilde\gamma$ could be quite far from the cost associated with γ. To overcome this issue, we use in a crucial way Proposition 4.10. Indeed, applying (4.20) in Proposition 4.10, we have

$$\begin{aligned}&\int_0^T\big|F(\gamma(t),\dot\gamma(t),m_t^T)-F(\gamma(t),\dot\gamma(t),\bar m))\big|\;dt\\ &\quad\leq\int_0^T(1+|\dot\gamma(t)|^2)^{\frac{d}{d+1}}\sup_{(y,z)\in\mathbb{T}^d\times\mathbb{R}^d}\frac{|F(y,z,m_t^T)-F(y,z,\bar m)|}{(1+|v|^2)^{\frac{d}{d+1}}}dt\\ &\quad\leq\Big(\int_0^T(1+|\dot\gamma(t)|^2)^{\frac{2d}{2d+1}}dt\Big)^{\frac{2d+1}{2d+2}}\Big(\int_0^T\sup_{(y,z)\in\mathbb{T}^d\times\mathbb{R}^d}\frac{|F(y,z,m_t^T)-F(y,z,\bar m)|^{2d+2}}{(1+|v|^2)^{2d}}dt\Big)^{\frac{1}{2d+2}}\\ &\quad\leq CT^{\frac{1}{4(d+1)}}\Big(\int_0^T(1+|\dot\gamma(t)|^2)dt\Big)^{\frac{2d+1}{2d+2}}.\end{aligned}$$

We have by assumption **(F2')** and Lemma 3.1 that

$$\int_0^T(c_F^{-1}|\dot\gamma(t)|^2-c_F)dt\leq u^T(0,x,v)\leq c_FT(1+|v|^2).\tag{4.37}$$

Therefore

$$\int_0^T\big|F(\gamma(t),\dot\gamma(t),m_t^T)-F(\gamma(t),\dot\gamma(t),\bar m))\big|\leq CT^{\frac{4d+3}{4(d+1)}}(1+R^2)^{\frac{2d+1}{2d+2}}.\tag{4.38}$$

For the very same reason we also have

$$\int_1^T\big|F(\gamma(t-1),\dot\gamma(t-1),m_t^T)-F(\gamma(t-1),\dot\gamma(t-1),\bar m))\big|\leq CT^{\frac{4d+3}{4(d+1)}}(1+R^2)^{\frac{2d+1}{2d+2}},\tag{4.39}$$

because we only used the optimality of γ only in the estimate (4.37). So, by (4.38) and (4.39) we obtain

$$\begin{aligned}
u^T(0,x',v') &\le \int_0^T \Big(\frac{1}{2}|\ddot{\tilde\gamma}(t)|^2 + F(\tilde\gamma(t),\dot{\tilde\gamma}(t),m_t^T)\Big)dt\\
&= \int_0^1 \Big(\frac{1}{2}|\ddot\sigma(t)|^2 + F(\sigma(t),\dot\sigma(t),m_t^T)\Big)dt + \int_0^{T-1}\Big(\frac{1}{2}|\ddot\gamma(t-1)|^2 + F(\gamma(t-1),\dot\gamma(t-1),m_t^T)\Big)dt\\
&\le 2C_2R^2 + \int_0^T \Big(\frac{1}{2}|\ddot\gamma(t)|^2 + F(\gamma(t),\dot\gamma(t),\bar m)\Big)dt + CT^{\frac{4d+3}{4(d+1)}}(1+R^2)^{\frac{2d+1}{2d+2}}\\
&\le 2C_2R^2 + \int_0^T \Big(\frac{1}{2}|\ddot\gamma(t)|^2 + F(\gamma(t),\dot\gamma(t),m_t^T)\Big)dt + 2CT^{\frac{4d+3}{4(d+1)}}(1+R^2)^{\frac{2d+1}{2d+2}}\\
&\le u^T(0,x,v) + 2C_2R^2 + 2CT^{\frac{4d+3}{4(d+1)}}(1+R^2)^{\frac{2d+1}{2d+2}},
\end{aligned}$$

from which the result derives easily. □

Proof of Theorem 2.5. Proposition 4.1 states the existence of a solution for the ergodic MFG system and its uniqueness under assumption (2.8). From Proposition 4.7 we know that

$$\lim_{T\to+\infty}\frac{1}{T}\int_{\mathbb{T}^d\times\mathbb{R}^d} u^T(0,x,v)m_0(dx,dv) = \bar\lambda.$$

It remains to prove the local uniform convergence of u^T to $\bar\lambda$. Fix $R>0$ and $\varepsilon>0$. We have by Lemma 3.1 that

$$0 \le u^T(0,x,v) \le c_F T(1+|v|^2). \tag{4.40}$$

As $m_0\in\mathcal{P}_2(\mathbb{T}^d\times\mathbb{R}^d)$, there exists $R'\ge R$ such that

$$\int_{\mathbb{T}^d\times(\mathbb{R}^d\setminus B_{R'})}(1+|v|^2)m_0(dx,dv)\le\varepsilon\,. \tag{4.41}$$

Then, for any $(x_0,v_0)\in\mathbb{T}^d\times B_R$, we have, by Lemma 4.11, (4.40) and (4.41),

$$\begin{aligned}
|\frac{1}{T}u^T(0,x_0,v_0)-\bar\lambda| &\le \left|\frac{1}{T}\int_{\mathbb{T}^d\times\mathbb{R}^d} u^T(0,x,v)m_0(dx,dv)-\bar\lambda\right|\\
&+\frac{1}{T}\int_{\mathbb{T}^d\times B_{R'}}\left|u^T(0,x,v)-u^T(0,x_0,v_0)\right|m_0(dx,dv)\\
&+\frac{1}{T}\int_{\mathbb{T}^d\times(\mathbb{R}^d\setminus B_{R'})}(|u^T(0,x,v)|+|u^T(0,x_0,v_0)|)m_0(dx,dv)\\
&\le \left|\frac{1}{T}\int_{\mathbb{T}^d\times\mathbb{R}^d} u^T(0,x,v)m_0(dx,dv)-\bar\lambda\right| + CT^{-1}(R')^2T^{\frac{4d+3}{4(d+1)}} + c_F\,\varepsilon(2+R^2),
\end{aligned}$$

from which the local uniform convergence of $u^T(0,\cdot,\cdot)/T$ to $\bar\lambda$ can be obtained easily. □

Appendix A. Von Neumann minmax theorem

Let $\mathbb{A}$, $\mathbb{B}$ be convex sets of some vector spaces and let us suppose that $\mathbb{B}$ is endowed with some Hausdorff topology. Let $\mathcal{L}:\mathbb{A}\times\mathbb{B}\to\mathbb{R}$ be a saddle function satisfying

(1) $a\mapsto\mathcal{L}(a,b)$ is concave in $\mathbb{A}$ for every $b\in\mathbb{B}$,
(2) $b\mapsto\mathcal{L}(a,b)$ is convex in $\mathbb{B}$ for every $a\in\mathbb{A}$.

It is always true that

$$\inf_{b\in\mathbb{B}}\sup_{a\in\mathbb{A}}\mathcal{L}(a,b)\ge\sup_{a\in\mathbb{A}}\inf_{b\in\mathbb{B}}\mathcal{L}(a,b).$$

Theorem A.1 ([35]). *Assume that there exists $a^* \in \mathbb{A}$ and $c^* > \sup_{a\in\mathbb{A}} \inf_{b\in\mathbb{B}} \mathcal{L}(a,b)$ such that*

$$\mathbb{B}^* := \{b \in \mathbb{B} : \mathcal{L}(a^*, b) \leq c^*\}$$

is not empty and compact in $\mathbb{B}$, and that $b \mapsto \mathcal{L}(a,b)$ is lower semicontinuous in $\mathbb{B}^*$ for every $a \in \mathbb{A}$.

Then

$$\min_{b\in\mathbb{B}} \sup_{a\in\mathbb{A}} \mathcal{L}(a,b) = \sup_{a\in\mathbb{A}} \inf_{b\in\mathbb{B}} \mathcal{L}(a,b).$$

APPENDIX B. AN INTERPOLATION INEQUALITY

Lemma B.1. *Assume that $f : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is locally Lipschitz continuous with*

$$|f(x,v)| + |D_x f(x,v)| + |D_v f(x,v)| \leq c_0(1+|v|^\alpha) \qquad \text{for a.e. } (x,v) \in \mathbb{T}^d \times \mathbb{R}^d \tag{B.1}$$

for some constants $c_0 > 0$ and $\alpha \in (1,2]$. There exists a constants $C_d > 0$ (depending on dimension only) such that

$$\sup_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d} \frac{|f(x,v)|^{2d+2}}{(1+|v|^\alpha)^{2d}} \leq C_d c_0^{2d} \int_{\mathbb{T}^d\times\mathbb{R}^d} |f(x,v)|^2 dxdv.$$

Proof. Let $(x_0,v_0) \in \mathbb{T}^d \times \mathbb{R}^d$ be such that $f(x_0,v_0) \neq 0$ and let $R = \frac{|f(x_0,v_0)|}{2c_0(3+2|v_0|^\alpha)}$. Note that, by our assumption on $|f|$ in (B.1), R is less than 1. Then, for any $(x,v) \in B_R(x_0,v_0)$, we have by assumption (B.1) that

$$|D_x f(x,v)| + |D_v f(x,v)| \leq c_0(1+(1+|v_0|)^\alpha) \leq c_0(1+2^{\alpha-1}+2^{\alpha-1}|v_0|^\alpha) \leq c_0(3+2|v_0|^\alpha),$$

(where we used the fact that $R \leq 1$ and that $(a+b)^\alpha \leq 2^{\alpha-1}(a^\alpha+b^\alpha)$ in the first inequality and the fact that $\alpha \leq 2$ in the second one). Therefore

$$|f(x,v)| \geq |f(x_0,v_0)| - c_0(3+2|v_0|^\alpha)R = \frac{|f(x_0,v_0)|}{2}.$$

Taking the square and integrating over $B_R(x_0,v_0)$ gives

$$\int_{\mathbb{T}^d\times\mathbb{R}^d} |f(x,v)|^2 dxdv \geq |B_1| R^{2d} \frac{|f(x_0,v_0)|^2}{4} = |B_1| \frac{|f(x_0,v_0)|^{2d+2}}{2^{2d+2} c_0^{2d} (3+2|v_0|^\alpha)^{2d}},$$

which implies the result. □

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