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ON THE DIRAC BAG MODEL IN STRONG MAGNETIC FIELDS

J.-M. BARBAROUX, L. LE TREUST, N. RAYMOND, AND E. STOCKMEYER

ABSTRACT. In this work we study two-dimensional Dirac operators on bounded domains coupled to a magnetic field perpendicular to the plane. We focus on the MIT bag boundary condition and provide accurate asymptotic estimates for the low-lying (positive and negative) energies in the limit of a strong magnetic field.

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1. INTRODUCTION

Consider an open, smooth and simply connected domain $\Omega \subset \mathbb{R}^2$ and a magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$, smooth and pointing in direction $\hat{\mathbf{z}}$ orthogonal to the plane. In this work we consider a Dirac operator restricted to Ω and coupled to the magnetic field \mathbf{B} through a magnetic vector potential $\mathbf{A} = (A_1, A_2)^T$ satisfying $\nabla \times \mathbf{A} = \mathbf{B}$. The magnetic Dirac operator acts on a dense subspace of $L^2(\Omega, \mathbb{C}^2)$ as,

$$\sigma \cdot (-i\nabla - b\mathbf{A}) = \begin{pmatrix} 0 & -i(\partial_1 - i\partial_2) - bA_1 + ibA_2 \\ -i(\partial_1 + i\partial_2) - bA_1 - ibA_2 & 0 \end{pmatrix}, \quad (1.1)$$

where $b > 0$ is a positive coupling constant. We write $\sigma \cdot \mathbf{x} = \sigma_1 x_1 + \sigma_2 x_2$ for $\mathbf{x} = (x_1, x_2)$ with the usual Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If we assume that the spinors satisfy a boundary relation of the type $\varphi = \mathcal{B}\varphi$ on $\partial\Omega$ with a unitary and self-adjoint boundary matrix $\mathcal{B}: \partial\Omega \rightarrow \mathbb{C}^{2 \times 2}$, then simple integration by parts shows that the local current density $\langle \varphi, \sigma \cdot \mathbf{n} \varphi \rangle_{\mathbb{C}^2}$ vanishes at each point of the boundary if and only if

$$\mathcal{B} \sigma \cdot \mathbf{n} + \sigma \cdot \mathbf{n} \mathcal{B} = 0 \quad \text{on} \quad \partial\Omega, \quad (1.2)$$

where \mathbf{n} is the normal vector pointing outward to the boundary and $\langle \cdot, \cdot \rangle_{\mathbb{C}^2}$ is the standard scalar product on \mathbb{C}^2 (antilinear w.r.t. the left argument). In particular, for these cases, the Dirac operator is formally symmetric and satisfies the bag condition, i.e., that no current flows through $\partial\Omega$ [10]. In the physics literature these type of models have been earlier considered to describe neutrino billiards [10] and (in the three dimensional setting) quark confinement [13]. More recently, they have regained attention with the advent of graphene and other Dirac materials, see e.g., [1, 12, 34, 16].

Using the properties of the Pauli matrices and those of \mathcal{B} it is easy to see that the most general form of \mathcal{B} acts as a multiplication on $L^2(\partial\Omega)$ with

$$\mathcal{B} \equiv \mathcal{B}_\eta = (\sigma \cdot \mathbf{t}) \cos \eta + \sigma_3 \sin \eta, \quad (1.3)$$

for certain sufficiently smooth $\eta: \partial\Omega \rightarrow \mathbb{R}$ and \mathbf{t} being the unit tangent vector pointing clockwise (we have that $\mathbf{n} \times \mathbf{t} = \hat{\mathbf{z}}$). The most frequently used boundary conditions in the physics literature are the cases when $\cos \eta = 0$ and $\sin \eta = 0$ known as zig-zag and MIT bag or infinite-mass boundary conditions, respectively. For recent mathematical literature on the subject in the two and three dimensional settings see for instance [8, 4, 24, 27, 7] about self-adjointness, [35, 3, 5] for the derivations as an infinite mass limit, and [9, 25, 2] for eigenvalue estimates.

In this work we consider Dirac operators D_η acting as in (1.1) on spinors φ satisfying $\varphi = \mathcal{B}_\eta \varphi$, with $\eta \in [0, 2\pi)$. We give the precise definition of the self-adjoint realization below. Assuming that the magnetic field satisfies $\inf_{x \in \bar{\Omega}} B(x) = b_0 > 0$ (besides certain geometrical conditions, see Assumption 1.7), we provide asymptotic estimates for the corresponding low-lying eigenvalues in the strong coupling constant limit $b \rightarrow \infty$.

The behavior of the corresponding operators in the physically most relevant cases mentioned above are quite different from each other. Indeed, on the one hand, the spectrum of a zig-zag operator is symmetric with respect to zero and zero is an eigenvalue

of infinite multiplicity. On the other hand, the spectrum of the MIT-bag is far from being symmetric for large magnetic fields and zero is never in its spectrum.

Our main results can be roughly summarized as follows: For $k \in \{1, 2, 3, \dots\}$ we denote by $\eta_k^+ \geq 0$ and $\eta_k^- < 0$ the non-negative and negative eigenvalues of D_0 , the MIT bag operator with $\eta = 0$. They are ordered such that $|\eta_k^\pm| \leq |\eta_{k+1}^\pm|$. Then, there is a constant $C_k^+ > 0$ such that, as $b \rightarrow \infty$,

$$\eta_k^+ = C_k^+ b^k e^{-2\alpha b} (1 + o(1)). \quad (1.4)$$

We provide explicit expressions for the constants C_k^+ and $\alpha > 0$ in terms of the geometry and the magnetic field B (see Theorem 1.10). In particular, the positive eigenvalues of D_0 accumulate exponentially fast to zero in the strong magnetic field limit. This behavior is in contrast to the one of the negative eigenvalues. Indeed, for the first negative eigenvalue we show that there is a constant $C^- > 0$ such that

$$\eta_1^- = -C^- b^{\frac{1}{2}} (1 + o(1)). \quad (1.5)$$

The constant C^- obeys an effective minimization problem (see Theorem 1.14). We can even describe the repartition of the first negative eigenvalues by means of an effective operator obtained by microlocal technics (see Theorem 1.19).

The proof of (1.4) and (1.5) is based upon the asymptotic analysis of a min-max principle for the corresponding operator D_0 . We show a new min-max characterization, well adapted to our setting, whose proof is inspired by [15] and [19]. A result in the same spirit has been recently used in [2]. It is easy to see that the min-max characterization applies well to any boundary conditions with $\cos \eta \neq 0$, as described in Appendix A one obtains the same type of asymptotic formulas (1.4) and (1.5) with different constants.

As for the zig-zag case, when $\cos \eta = 0$, we obtain analogous results for the energies through a simple application of the asymptotic analysis performed in [6] and the relation between zig-zag and Pauli-Dirichlet operators. This is explained in Section 1.4 and the results can be summarized as follows: For $k \in \{1, 2, 3, \dots\}$ we denote by μ_k^- and μ_k^+ the k -th positive eigenvalue of $D_{\pi/2}$ and $D_{3\pi/2}$, respectively. Then, we find constants $0 < c_k \leq C_k < \infty$ that, as $b \rightarrow \infty$,

$$c_k b^{(k+1)/2} e^{-\alpha b} (1 + o(1)) \leq \mu_k^- \leq C_k b^{(k+1)/2} e^{-\alpha b} (1 + o(1)),$$

and

$$\mu_k^+ \geq \sqrt{2bb_0},$$

where $\alpha > 0$ is the same constant appearing in (1.4).

Our results compare qualitatively well with the findings in the physics literature [33, 20, 37] for constant magnetic fields on the disk. Let us mention nevertheless that the exponential decays of η_k^+ and μ_k^- of the type $e^{-\alpha_{\text{phy}} b^2}$ ($\alpha_{\text{phy}} > 0$) conjectured by the physicists are too strong. We also show that the eigenvalues $(\eta_k^-)_{k \geq 1}$ do not merge to form the Landau levels since the eigenfunctions associated with the lowest eigenvalues concentrate on the edge (see Theorem 1.14 and Proposition 6.2).

1.1. Basic definitions and assumptions. We study the equivalent semiclassical problem given by the the action of

$$\mathcal{D}_{h,\mathbf{A}} = \sigma \cdot (\mathbf{p} - \mathbf{A}) = \begin{pmatrix} 0 & d_{h,\mathbf{A}} \\ d_{h,\mathbf{A}}^\times & 0 \end{pmatrix}, \quad (1.6)$$

where $\mathbf{p} = -ih\nabla$ for $h > 0$,

$$d_{h,\mathbf{A}} = -2ih\partial_z - A_1 + iA_2, \quad d_{h,\mathbf{A}}^\times = -2ih\partial_{\bar{z}} - A_1 - iA_2,$$

with $\partial_{\bar{z}} = \frac{\partial_1 + i\partial_2}{2}$ and $\partial_z = \frac{\partial_1 - i\partial_2}{2}$. We focus on the boundary conditions described above for $\eta = 0$, that is

$$\mathcal{B} = \sigma \cdot \mathbf{t} = -i\sigma_3(\sigma \cdot \mathbf{n}),$$

where \mathbf{n} is the outward pointing normal to the boundary $\partial\Omega$. The associated magnetic Dirac operator with MIT bag (or infinite mass) boundary condition is $(\mathcal{D}_{h,\mathbf{A}}, \text{Dom}(\mathcal{D}_{h,\mathbf{A}}))$ with

$$\text{Dom}(\mathcal{D}_{h,\mathbf{A}}) = \{\varphi \in H^1(\Omega, \mathbb{C}^2), \quad \mathcal{B}\varphi = \varphi \text{ on } \partial\Omega\}.$$

Remark 1.1. Note that

$$\sigma \cdot \mathbf{n} = \begin{pmatrix} 0 & \bar{\mathbf{n}} \\ \mathbf{n} & 0 \end{pmatrix},$$

so that the boundary condition reads

$$u_2 = i\mathbf{n}u_1,$$

where $\varphi = (u_1, u_2)^T$, and $\mathbf{n} = (n_1, n_2)^T$ denotes the normal vector in \mathbb{R}^2 and also $\mathbf{n} = n_1 + in_2 \in \mathbb{C}$.

Notation 1. We denote by $\langle \cdot, \cdot \rangle$ the standard L^2 -scalar product (antilinear w.r.t. the left argument) on Ω and by $\|\cdot\|$ the associated norm. In the same way, we denote by $\langle \cdot, \cdot \rangle_{\partial\Omega}$ the L^2 -scalar product on $L^2(\partial\Omega)$.

The main purpose of the paper is to study the asymptotic behavior of the eigenvalues near 0 in the semiclassical limit $h \rightarrow 0$.

Remark 1.2. The eigenvalues in the strong magnetic field limit given by the operator D_0 described in the introduction can be found by a simple scaling argument. We have

$$\text{sp } D_0 = b \text{ sp } \mathcal{D}_{1/b, \mathbf{A}}.$$

Then, (1.4) and (1.5) are direct consequences of Theorems 1.10 and 1.14, respectively (see below).

Assumption 1.3.

- (i) Ω is bounded, simply connected, $\partial\Omega$ is \mathcal{C}^2 -regular,
- (ii) $B \in W^{1,\infty}(\bar{\Omega})$.

Under Assumption 1.3, the operator $\mathcal{D}_{h,0}$, without magnetic field, is self-adjoint on $L^2(\Omega)^2$ (see for instance [8]). We work in the so-called *Coulomb gauge* that is given through the unique solution of the Poisson equation

$$\Delta\phi = B, \quad \phi|_{\partial\Omega} = 0, \quad (1.7)$$

by choosing $\mathbf{A} = (-\partial_2\phi, \partial_1\phi)^T = \nabla\phi^\perp$. Notice that by standard regularity theory the components of \mathbf{A} are bounded. Hence $\mathcal{D}_{h,\mathbf{A}}$ is self-adjoint and it has compact resolvent

since $\text{Dom}(\mathcal{D}_{h,\mathbf{A}}) \subset H^1$. In particular, the spectrum $\mathcal{D}_{h,\mathbf{A}}$ of is discrete. We denote by $(\lambda_k^+(h))_{k \geq 1}$ and $(-\lambda_k^-(h))_{k \geq 1}$ the positive and negative eigenvalues of $\mathcal{D}_{h,\mathbf{A}}$ counted with multiplicities. In fact, $\mathcal{D}_{h,\mathbf{A}}$ has no zero modes. This can be seen using the following lemma, which is a consequence of [21] and [6].

Lemma 1.4. *For all $h > 0$, there exists $C(h) > 0$ such that, for all $u \in H_0^1(\Omega)$, we have*

$$\|d_{h,\mathbf{A}}^\times u\|^2 \geq C(h)\|u\|^2.$$

Proposition 1.5. *The operator $\mathcal{D}_{h,\mathbf{A}}$ has no zero modes.*

Proof. Consider $\varphi = (u, v)^T \in \text{Dom}(\mathcal{D}_{h,\mathbf{A}})$ such that $\mathcal{D}_{h,\mathbf{A}}\varphi = 0$. We have $d_{h,\mathbf{A}}v = d_{h,\mathbf{A}}^\times u = 0$. Thus, integrating by parts, and using the boundary condition, we get

$$0 = \langle d_{h,\mathbf{A}}v, u \rangle = \langle v, d_{h,\mathbf{A}}^\times u \rangle + h \langle -i\bar{\mathbf{n}}v, u \rangle_{\partial\Omega} = h\|u\|_{\partial\Omega}^2.$$

Therefore $u \in H_0^1(\Omega)$, and Lemma 1.4 implies that $u = 0$. \square

Since $\mathcal{D}_{h,\mathbf{A}}$ has no zero mode, its spectrum is

$$\text{sp}(\mathcal{D}_{h,\mathbf{A}}) = \{\dots, -\lambda_2^-(h), -\lambda_1^-(h)\} \cup \{\lambda_1^+(h), \lambda_2^+(h), \dots\}. \quad (1.8)$$

Assumption 1.6. *B is positive. We define $b_0 = \inf_{\Omega} B > 0$ and $b'_0 = \min_{\partial\Omega} B$.*

Under this assumption, ϕ is subharmonic so that

$$\max_{x \in \bar{\Omega}} \phi = \max_{x \in \partial\Omega} \phi = 0,$$

and the minimum of ϕ will be negative and attained in Ω .

Assumption 1.7.

- (i) *The minimum ϕ_{\min} of ϕ is attained at a unique point x_{\min} .*
- (ii) *The Hessian matrix $\text{Hess}_{\min} \phi$ of ϕ at x_{\min} is positive definite i.e. x_{\min} is non-degenerate minimum. We also denote by z_{\min} , the minimum x_{\min} seen as a complex number.*

1.2. Main results. The magnetic Hardy space is

$$\mathcal{H}_{h,\mathbf{A}}^2(\Omega) = \{u \in L^2(\Omega) : d_{h,\mathbf{A}}^\times u = 0, u|_{\partial\Omega} \in L^2(\partial\Omega)\}.$$

We let

$$\mathfrak{H}_{h,\mathbf{A}} = H^1(\Omega) + \mathcal{H}_{h,\mathbf{A}}^2(\Omega),$$

and endow it with the Hermitian scalar product given by

$$\forall (u_1, u_2) \in \mathfrak{H}_{h,\mathbf{A}} \times \mathfrak{H}_{h,\mathbf{A}}, \quad \langle u_1, u_2 \rangle_{\mathfrak{H}_{h,\mathbf{A}}} = \langle u_1, u_2 \rangle + \langle d_{h,\mathbf{A}}^\times u_1, d_{h,\mathbf{A}}^\times u_2 \rangle + \langle u_1, u_2 \rangle_{\partial\Omega}.$$

1.2.1. About the positive eigenvalues. The following result gives us a non-linear min-max characterization for the positive eigenvalues of $\mathcal{D}_{h,\mathbf{A}}$.

Theorem 1.8. *Under Assumption 1.3. We have, for all $h > 0$ and $k \geq 1$,*

$$\lambda_k^+(h) = \min_{\substack{W \subset \mathfrak{H}_{h,\mathbf{A}} \\ \dim W = k}} \max_{u \in W \setminus \{0\}} \frac{h\|u\|_{\partial\Omega}^2 + \sqrt{h^2\|u\|_{\partial\Omega}^4 + 4\|u\|^2\|d_{h,\mathbf{A}}^\times u\|^2}}{2\|u\|^2}.$$

Remark 1.9. Due to the symmetry of the problem we can also use this min-max characterization for the negative eigenvalues of $\mathcal{D}_{h,\mathbf{A}}$ after changing the sign of the magnetic field. Indeed, consider the charge conjugation operator

$$C: \varphi \in \mathbb{C}^2 \mapsto \sigma_1 \bar{\varphi} \in \mathbb{C}^2,$$

where $\bar{\varphi}$ is the vector of \mathbb{C}^2 made of the complex conjugate of the coefficients of φ . We have $C\text{Dom}(\mathcal{D}_{h,\mathbf{A}}) = \text{Dom}(\mathcal{D}_{h,\mathbf{A}})$, and $C\mathcal{D}_{h,\mathbf{A}}C = -\mathcal{D}_{h,-\mathbf{A}}$. In particular, we get that

$$\text{sp}(\mathcal{D}_{h,\mathbf{A}}) = -\text{sp}(\mathcal{D}_{h,-\mathbf{A}}).$$

In order to state our next result on the asymptotic estimates of the $\lambda_k^+(h)$ we introduce some notation to explicitly define the constant C_k^+ from (1.4).

Notation 2. Let us denote by $\mathcal{O}(\Omega)$ and $\mathcal{O}(\mathbb{C})$ the sets of holomorphic functions on Ω and \mathbb{C} . We consider the following (anisotropic) Segal-Bargmann space

$$\mathcal{B}^2(\mathbb{C}) = \{u \in \mathcal{O}(\mathbb{C}) : N_{\mathcal{B}}(u) < +\infty\},$$

where

$$N_{\mathcal{B}}(u) = \left(\int_{\mathbb{R}^2} |u(y_1 + iy_2)|^2 e^{-\text{Hess}_{x_{\min}} \phi(y,y)} dy \right)^{1/2}.$$

We also introduce the Hardy space

$$\mathcal{H}^2(\Omega) = \{u \in \mathcal{O}(\Omega) : \|u\|_{\partial\Omega} < +\infty\},$$

where

$$\|u\|_{\partial\Omega} = \left(\int_{\partial\Omega} |u(y_1 + iy_2)|^2 dy \right)^{1/2}.$$

We also define for $P \in \mathcal{H}^2(\Omega)$, $A \subset \mathcal{H}^2(\Omega)$,

$$\text{dist}_{\mathcal{H}}(P, A) = \inf \{ \|P - Q\|_{\partial\Omega}, \text{ for all } Q \in A \},$$

and for $P \in \mathcal{B}^2(\mathbb{C})$, $A \subset \mathcal{B}^2(\mathbb{C})$,

$$\text{dist}_{\mathcal{B}}(P, A) = \inf \{ N_{\mathcal{B}}(P - Q), \text{ for all } Q \in A \}.$$

The following constant is important in our asymptotic analysis

$$C_k(B, \Omega) = \left(\frac{\text{dist}_{\mathcal{H}}((z - z_{\min})^{k-1}, \mathcal{H}_k^2(\Omega))}{\text{dist}_{\mathcal{B}}(z^{k-1}, \mathcal{P}_{k-2})} \right)^2, \quad (1.9)$$

where $\mathcal{P}_{k-2} = \text{span}(1, \dots, z^{k-2}) \subset \mathcal{B}^2(\mathbb{C})$, $\mathcal{P}_{-1} = \{0\}$ and

$$\mathcal{H}_k^2(\Omega) = \{u \in \mathcal{H}^2(\Omega), u^{(n)}(z_{\min}) = 0, \text{ for } n \in \{0, \dots, k-1\}\}. \quad (1.10)$$

Theorem 1.10. *Under Assumptions 1.3, 1.6 and 1.7, we have for all $k \geq 1$,*

$$\lambda_k^+(h) = C_k(B, \Omega) h^{1-k} e^{2\phi_{\min}/h} (1 + o_{h \rightarrow 0}(1)).$$

Remark 1.11. Let us assume that Ω is the disk of radius R centered at 0, and that B is radial. In this case $z_{\min} = 0$ and $\text{Hess}_{x_{\min}} \phi = B(0)\text{Id}/2$. Moreover, using Fourier series, we see that $(z^n)_{n \geq 0}$ is an orthogonal basis for $N_{\mathcal{B}}$ and $\|\cdot\|_{\partial\Omega}$. In particular, $\mathcal{H}_k^2(\Omega)$ is $\|\cdot\|_{\partial\Omega}$ -orthogonal to z^{k-1} so that

$$\text{dist}_{\mathcal{H}}(z^{k-1}, \mathcal{H}_k^2(\Omega))^2 = \|z^{k-1}\|_{\partial\Omega}^2 = R^{2k-2} |\partial\Omega| = 2\pi R^{2k-1}.$$

In addition, \mathcal{P}_{k-2} is $N_{\mathcal{B}}$ -orthogonal to z^{k-1} so that

$$\text{dist}_{\mathcal{B}} \left(z^{k-1}, \mathcal{P}_{k-2} \right)^2 = N_{\mathcal{B}}(z^{k-1})^2 = 2\pi \frac{2^{k-1}(k-1)!}{B(0)^k},$$

Thus, we get that

$$C_k(B, \Omega) = \frac{B(0)^k}{(k-1)!} \left(\frac{R^2}{2} \right)^{k-1} R.$$

Remark 1.12. Theorem 1.10 can evoke some kind of tunneling estimate. In the semi-classical study of electric Schrödinger operators with symmetric wells, it is well-known that the lowest eigenvalues differ from each other modulo terms in the form $e^{-S/h}$. The quantity S reflects the interaction between the wells and is related to lengths of geodesics connecting the wells. Here, the eigenvalues are themselves exponentially small and the S is replaced by ϕ_{\min} . In our analysis we will even see that the corresponding eigenfunctions are essentially localized near x_{\min} . Nevertheless, this x_{\min} is determined by the *global* magnetic geometry. That is why, we could interpret Theorem 1.10 as measure of a tunneling effect between every points of $\bar{\Omega}$.

1.2.2. *About the negative eigenvalues.* We now turn to the negative eigenvalues of $\mathcal{D}_{h,\mathbf{A}}$. Consider, for all $\alpha \geq 0$,

$$\nu(\alpha) = \inf_{\substack{u \in \mathfrak{H}_{-\mathbf{A}_0}^2(\mathbb{R}_+^2) \\ u \neq 0}} \frac{\int_{\mathbb{R}_+^2} |(-i\partial_{x_1} - x_2 + i(-i\partial_{x_2}))u|^2 dx_1 dx_2 + \alpha \int_{\mathbb{R}} |u(x_1, 0)|^2 dx_1}{\|u\|^2}, \quad (1.11)$$

with $\mathbf{A}_0 = (-x_2, 0)$. Notice that the quadratic form minimized in (1.11) corresponds to the magnetic Schrödinger operator on a half-plane with a constant magnetic field (equaling 1) and equipped by a Robin-like boundary condition. More details are given in Section 4.

Remark 1.13. We can prove (see Proposition 4.15) that the equation $\nu(\alpha) = \alpha^2$ has a unique positive solution, denoted by a_0 . In fact, we will see that a_0 equals

$$\inf_{\substack{u \in \mathfrak{H}_{-\mathbf{A}_0}^2(\mathbb{R}_+^2) \\ u \neq 0}} \frac{\int_{\mathbb{R}} |u(s, 0)|^2 ds + \sqrt{\left(\int_{\mathbb{R}} |u(s, 0)|^2 ds \right)^2 + 4\|u\|^2 \int_{\mathbb{R}_+^2} |(-i\partial_s - \tau + i(-i\partial_\tau))u|^2 ds d\tau}}{2\|u\|^2}.$$

Moreover, $a_0 \in (0, \sqrt{2})$. Numerical calculations suggest that a_0 is approximately equal to 1.31236.

Theorem 1.14. *Under Assumptions 1.3 and 1.6, we have*

$$\lambda_1^-(h) = h^{\frac{1}{2}} \min(\sqrt{2b_0}, a_0 \sqrt{b'_0}) + o_{h \rightarrow 0}(h^{\frac{1}{2}}),$$

where $\lambda_1^-(h)$ is defined in Section 1.1, $b_0 = \min_{\bar{\Omega}} B(x)$ and $b'_0 = \min_{\partial\Omega} B(x)$. In particular, when $B \equiv b_0$ is constant, we have

$$\lambda_1^-(h) = a_0 \sqrt{b_0 h} + o_{h \rightarrow 0}(h^{\frac{1}{2}}).$$

Remark 1.15. The asymptotic analysis leading to Theorems 1.10 and 1.14 strongly differ from each other. Indeed, the eigenfunctions are localized near x_{\min} for the positive energies, whereas, when B is constant, they are localized near the boundary for the

negative ones. Moreover, in this last case, for non-constant magnetic fields (see the discussion in Appendix B), the eigenfunctions might be localized inside if b_0/b'_0 is small enough. Consequently, the underlying semiclassical problems do not share the same structure.

1.2.3. *Negative eigenvalues for a constant magnetic field.* Let us now focus on the case with constant magnetic field, and improve Theorem 1.14. In order to establish our improvement, and to make the analysis more elegant, we make the following assumption (see Appendix D for more detail). This assumption will allow to define “holomorphic tubular coordinates”, which are particularly adapted to our operator.

Assumption 1.16. *The boundary $\partial\Omega$ is an analytic curve.*

Various properties of the eigenvalues of the operator $\mathcal{M}_{\mathbb{R}_+, \alpha, \xi}^-$ associated with the following (analytic) family of quadratic forms

$$q_{\mathbb{R}_+, \alpha, \xi}^-(u) = \int_{\mathbb{R}_+} (|u'|^2 + |(\xi - \tau)u|^2) d\tau + (\alpha - \xi)|u(0)|^2 + \|u\|^2, \quad \alpha > 0, \xi \in \mathbb{R},$$

play a fundamental role in the analysis of the negative eigenvalues. The Robin boundary condition reads

$$\partial_t u(0) = (\alpha - \xi)u(0).$$

We denote by $(\nu_{\mathbb{R}_+, j}^-(\alpha, \xi))_{j \geq 1}$ the non-decreasing sequence of its eigenvalues. For shortness, we let $\nu^-(\alpha, \xi) = \nu_{\mathbb{R}_+, 1}^-$, and we denote by $u_{\alpha, \xi}$ the corresponding normalized positive eigenfunction. We can prove (see Section 4) that $\nu^-(\alpha, \cdot)$ has a unique minimum at some ξ_α , which is non-degenerate.

This operator appears after using the partial Fourier transform in relation with (1.11).

Let us consider the following differential operator

$$\mathfrak{Q}_h^{\text{eff}} = (D_s + \mathfrak{t}_h)^2 - \frac{\kappa^2}{12}, \quad (1.12)$$

where

$$\mathfrak{t}_h = \frac{|\Omega|}{h|\partial\Omega|} - \frac{a_0}{h^{\frac{1}{2}}} + \frac{\pi}{|\partial\Omega|},$$

a_0 is defined in Remark 1.13 and

$$\mathfrak{c}_0 := \frac{a_0 u_{a_0, a_0}^2(0)}{2a_0 - u_{a_0, a_0}^2(0)} > 0.$$

Remark 1.17. We will see that the denominator of \mathfrak{c}_0 is indeed positive. Moreover, this constant is directly related to the second derivative of the first negative dispersion curve ϑ_1^- at a_0 of the Dirac operator on the half-plane with constant magnetic field (equal to 1), see Section 1.3 and Theorem 4.3.

We denote by $\lambda_n(\mathfrak{Q}_h^{\text{eff}})$ the n -th eigenvalue of $\mathfrak{Q}_h^{\text{eff}}$.

Remark 1.18. By gauge invariance, the spectrum of $\mathfrak{Q}_h^{\text{eff}}$ does not change whenever \mathfrak{t}_h is replaced by $\mathfrak{t}_h + \frac{2k\pi}{|\partial\Omega|}$. Therefore, we can easily check that, for all $n \geq 1$, there exists $C > 0$ such that, for all $h \in (0, h_0)$,

$$|\lambda_n(\mathfrak{Q}_h^{\text{eff}})| \leq C.$$

Here comes our last main result.

Theorem 1.19. *We have*

$$\lambda_n^-(h) = h^{\frac{1}{2}}a_0 + h^{\frac{3}{2}}\mathbf{c}_0\lambda_n(\mathfrak{Q}_h^{\text{eff}}) + o(h^{\frac{3}{2}}).$$

In the disk case, we can compute the eigenvalues $\lambda_n(\mathfrak{Q}_h^{\text{eff}})$ recursively.

Proposition 1.20. *Let $(m_n(h))_{n \geq 1}$ be a sequence of \mathbb{Z} which satisfies*

$$m_n(h) \in \arg \min\{|m + \mathbf{t}_h|, m \in \mathbb{Z} \setminus \{m_1(h), \dots, m_{n-1}(h)\}\} \setminus \{m_1(h), \dots, m_{n-1}(h)\}.$$

If Ω is a disk of radius $R > 0$, we have for all $n \geq 1$,

$$\lambda_n(\mathfrak{Q}_h^{\text{eff}}) = |m_n(h) + \mathbf{t}_h|^2 - \frac{1}{12R^2}.$$

Remark 1.21. Since \mathbf{t}_h depends continuously on h and $\mathbf{t}_h \rightarrow +\infty$ as $h \rightarrow 0$, there are infinitely many $h > 0$ for which there exists $k \in \mathbb{Z}$ such that

$$\mathbf{t}_h = \frac{1}{2} + k. \tag{1.13}$$

In these cases, the spectrum of $\mathfrak{Q}_h^{\text{eff}}$ for the disk of radius $R > 0$ is

$$\text{sp}(\mathfrak{Q}_h^{\text{eff}}) = \left\{ \left| \frac{1}{2} + m \right|^2 - \frac{1}{12R^2}, m \in \mathbb{N} \right\},$$

each eigenvalue has multiplicity 2 and the sequence $(m_n(h))_{n \geq 1}$ is not uniquely defined. If (1.13) is not satisfied then, all the eigenvalues are simple.

Actually, the microlocal strategy used to obtain Theorem 1.19 also allows to get results for variable magnetic fields. Such results are described with some details in Appendix B. Somehow, the case with variable magnetic field is simpler since the variations of the field have a stronger effect than the geometry.

Theorem 1.19 should also be compared to [17, Theorem 1.1] which deals with the Neumann Laplacian with constant magnetic field. In their paper, Fournais and Helffer show the crucial influence of the curvature on the spectral asymptotics and on the spectral gap. This gap is directly related to the localization of the eigenfunctions near the points of maximal curvature. We stress that it is not the case with Theorem 1.19 since the effective operator does not induce a particular localization on the boundary. This reflects that our problem is more degenerate from the semiclassical point of view. In order to deal with such a degeneracy, we use a microlocal dimensional reduction to the boundary (also known as the Grushin method). As far as we know, such a method, combined with a non-linear characterization of the eigenvalues, does not seem to have been used before to study semiclassical Dirac operators. The version of this method that we use in this paper is inspired by [26] and closely related to the Ph. D. thesis by Keraval [23]. It was also recently used to establish a formula describing a pure magnetic tunnel effect in [11].

1.3. Dirac operators with constant magnetic field on \mathbb{R}^2 and \mathbb{R}_+^2 . When considering Theorem 1.14, we can wonder what the interpretation of the positive constant a_0 is. In fact, an important part of the semiclassical analysis of spectral problems relies on the study of some operators obtained (formally) after a semiclassical zoom around each point of $\overline{\Omega}$. In the present article, these are magnetic Dirac operators with constant magnetic field. Thus, let us consider homogenous Dirac operators on \mathbb{R}^2 and $\mathbb{R}_+^2 = \mathbb{R} \times \mathbb{R}_+$ with the same formalism by choosing the gauge associated with the vector potential $\mathbf{A}_0 = (-x_2, 0)^T$.

Definition 1.22. The operators $\mathcal{D}_{\mathbb{R}^2}$ and $\mathcal{D}_{\mathbb{R}_+^2}$ act as $\sigma \cdot (-i\nabla - \mathbf{A}_0)$ on

$$\text{Dom}(\mathcal{D}_{\mathbb{R}^2}) = \{\varphi \in H^1(\mathbb{R}^2, \mathbb{C}^2), \quad x_2\varphi \in L^2(\mathbb{R}^2, \mathbb{C}^2)\}$$

and

$$\text{Dom}(\mathcal{D}_{\mathbb{R}_+^2}) = \{\varphi \in H^1(\mathbb{R}_+^2, \mathbb{C}^2), \quad x_2\varphi \in L^2(\mathbb{R}_+^2, \mathbb{C}^2), \quad \sigma_1\varphi = \varphi \text{ on } \partial\mathbb{R}_+^2\}.$$

The spectral properties of $\mathcal{D}_{\mathbb{R}^2}$ can be found for instance in [36, Theorem 7.2]. A novelty in this paper is the study of $\mathcal{D}_{\mathbb{R}_+^2}$.

Theorem 1.23. *The operators $\mathcal{D}_{\mathbb{R}^2}$ and $\mathcal{D}_{\mathbb{R}_+^2}$ are self-adjoint and satisfy*

$$\text{sp}(\mathcal{D}_{\mathbb{R}^2}) = \{\pm\sqrt{2k}, k \in \mathbb{N}\},$$

and

$$\text{sp}(\mathcal{D}_{\mathbb{R}_+^2}) = (-\infty, -a_0] \cup [0, +\infty),$$

where $a_0 \in (0, \sqrt{2})$ is defined in Remark 1.13. The spectrum of $\mathcal{D}_{\mathbb{R}^2}$ is made of infinitely degenerate eigenvalues. The spectrum of $\mathcal{D}_{\mathbb{R}_+^2}$ is purely absolutely continuous.

We will present with more details the study of the negative part of the spectrum of $\mathcal{D}_{\mathbb{R}_+^2}$ since many of the associated results will be used in the proof of the asymptotics of the negative eigenvalues.

1.4. The zig-zag case. In this paper, we consider the Dirac operator with MIT bag boundary condition (and its variants in Appendix A). The so-called zig-zag boundary condition also appears commonly in the description of the electrical properties of pieces of graphene. It is worth noticing that the spectral properties of the related operators exhibit completely different asymptotic behaviors compared with the ones studied here. More precisely, the operators $(\mathcal{Z}_{h,\mathbf{A}}^\pm, \text{Dom}(\mathcal{Z}_{h,\mathbf{A}}^\pm))$ acting as $\sigma \cdot (\mathbf{p} - \mathbf{A})$ on

$$\text{Dom}(\mathcal{Z}_{h,\mathbf{A}}^-) = H_0^1(\Omega, \mathbb{C}) \times \{u \in L^2(\Omega, \mathbb{C}), \partial_z u \in L^2(\Omega, \mathbb{C})\},$$

$$\text{Dom}(\mathcal{Z}_{h,\mathbf{A}}^+) = \{u \in L^2(\Omega, \mathbb{C}), \partial_{\bar{z}} u \in L^2(\Omega, \mathbb{C})\} \times H_0^1(\Omega, \mathbb{C}),$$

are self-adjoint. This is easily seen since by construction the operators $\mathcal{Z}_{h,\mathbf{A}}^\pm$ have the supersymmetric structure

$$\mathcal{Z}_{h,\mathbf{A}}^\pm = \begin{pmatrix} 0 & D_\pm \\ D_\pm^* & 0 \end{pmatrix},$$

where D_+ and D_-^* have Dirichlet boundary conditions. Moreover, 0 is an eigenvalue of infinite multiplicity for both of them and their kernels can be determined explicitly (see [36, Chapter 5], [32] and [6, Proposition 4.4]).

Next notice that since $\sigma_3 \mathcal{Z}_{h,\mathbf{A}}^\pm = -\mathcal{Z}_{h,\mathbf{A}}^\pm \sigma_3$ holds, the spectra of both operators is symmetric with respect to zero. Moreover, by simply squaring the operators one sees that, due to the isospectrality of $D_\pm D_\pm^*$ and $D_\pm^* D_\pm$ away from zero,

$$\left\{ \lambda^2, \lambda \in \text{sp}(\mathcal{Z}_{h,\mathbf{A}}^+) \setminus \{0\} \right\} = \text{sp}\{D_+^* D_+\}, \text{ and } \left\{ \lambda^2, \lambda \in \text{sp}(\mathcal{Z}_{h,\mathbf{A}}^-) \setminus \{0\} \right\} = \text{sp}\{D_- D_-^*\}.$$

Thus, their discrete spectrum satisfy

$$\text{sp}_d(\mathcal{Z}_{h,\mathbf{A}}^\pm) = \text{sp}(\mathcal{Z}_{h,\mathbf{A}}^\pm) \setminus \{0\} = \left\{ \sqrt{\alpha_k^\pm(h)}, k \in \mathbb{N}^* \right\} \cup \left\{ -\sqrt{\alpha_k^\pm(h)}, k \in \mathbb{N}^* \right\},$$

where $(\alpha_k^+(h))_{k \geq 1}$ and $(\alpha_k^-(h))_{k \geq 1}$ are the ordered sequences of the eigenvalues (counted with multiplicity) of the operators $D_+^* D_+$ and $D_- D_-^*$ that act as

$$|\mathbf{p} - \mathbf{A}|^2 + hB, \quad \text{and} \quad |\mathbf{p} - \mathbf{A}|^2 - hB,$$

on $H_0^1(\Omega, \mathbb{C}) \cap H^2(\Omega, \mathbb{C})$. Therefore, we deduce from [6, Theorem 1.3.], that there exists $(\tilde{C}_k(B, \Omega))_{k \geq 1}$ and $\theta_0 \in (0, 1]$ such that for all $k \geq 1$

$$\begin{aligned} \left(\theta_0 \tilde{C}_k(B, \Omega) h^{1-k} e^{2\phi_{\min}/h} \right)^{1/2} (1 + o_{h \rightarrow 0}(1)) &\leq \sqrt{\alpha_k^-(h)} \\ &\leq \left(\tilde{C}_k(B, \Omega) h^{1-k} e^{2\phi_{\min}/h} \right)^{1/2} (1 + o_{h \rightarrow 0}(1)), \end{aligned}$$

as $h \rightarrow 0$. Finally, it is well known that

$$\sqrt{\alpha_k^+(h)} \geq \sqrt{2b_0 h}.$$

1.5. Structure of the article. The article is organized as follows.

Section 2 is devoted to establish a non-linear min-max characterization of the positive eigenvalues, see Proposition 2.8. A crucial step in our way of proving this proposition is Proposition 2.12 which establishes an isomorphism between a positive eigenspace and a kernel of Schrödinger operator.

In Section 3, we prove Theorem 1.10 by using the non-linear min-max characterization. First, we establish an upper bound, see Lemma 3.1 and Proposition 3.5. Then, we prove that the minimizers of our non-linear min-max are approximated by functions such that $d_{h,\mathbf{A}}^\times u = 0$ (see Section 3.2). This allows to establish the lower bound, see Corollary 3.16 and Proposition 3.10.

In Section 4, we prove Theorem 1.23 about the spectrum of homogeneous magnetic Dirac operators on \mathbb{R}^2 and \mathbb{R}_+^2 . Various properties of the corresponding dispersion curves are also established. The characterization of $a_0 \in (0, \sqrt{2})$ as the unique solution of $\nu(\alpha) = \alpha^2$ is explained in Section 4.7. Numerical illustrations are also provided, see Section 4.6.

Section 5 is devoted to the proof of Theorem 1.14. One of the main ingredients is Proposition 5.1 which establishes a one-term asymptotics of the ground-state energy of a Pauli-Robin operator. This proposition is proved by means of a semiclassical partition of the unity. Near the boundary, due to the lack of ellipticity of the Cauchy-Riemann operators, we are led to introduce *conformal* tubular coordinates thanks to the Riemann mapping theorem. This is the price to pay to be able to approximate the magnetic field by the constant magnetic field, and to control the remainders.

In Section 6, we consider the case with constant magnetic field and we start the proof of Theorem 1.19. The first step is to show that the first eigenfunctions of our Pauli-Robin operator are localized near the boundary at the scale $h^{\frac{1}{2}}$, see Proposition 6.2. This allows to reduce the analysis to a tubular neighborhood of the boundary, see Corollary 6.5. In this neighborhood, we use the holomorphic tubular coordinates given in Appendix D (where it is explained how to construct such coordinates by imposing a *parametrization by arc length* of $\partial\Omega$), and put the operator under a normal form by means of changes of gauge and of functions. The operator is rescaled with respect to the normal variable and we get the operator $\mathcal{N}_{a,h}$, see (6.7).

In Section 7, up to inserting cutoff functions, this operator is seen as a pseudo-differential operator with respect to the curvilinear coordinate, see (7.1) and Section 7.1.3. Corollary 7.3 tells us that it is enough to study our pseudo-differential with cutoff functions $\check{\mathcal{N}}_{a,h}$. Then, a parametrix is constructed by means of the Grushin formalism, see Proposition 7.5. This parametrix is used to reveal an effective operator, see (7.4). The connection with the spectrum of $\check{\mathcal{N}}_{a,h}$ is made in Proposition 7.9. The spectral analysis of the effective operator is done in Section 7.3, see especially Proposition 7.11. Finally, the relation between the spectrum of the Pauli-Robin operator and the one of the Dirac operator is explained in Section 7.4.

In Appendix C, for the convenience of the reader, we recall why the magnetic Hardy space is a Hilbert space.

In Appendix A, we discuss some straightforward extensions of our results related to variable boundary conditions.

In Appendix B, we explain how to describe all the negative eigenvalues when the magnetic field is variable, under generic assumptions. The main steps are only sketched since the analysis does not crucially involve subprincipal terms as for the constant magnetic field case.

2. A NON-LINEAR MIN-MAX CHARACTERIZATION

The aim of this section is to establish Theorem 1.8. To do so, we first establish in Section 2.1 some fundamental properties of the natural minimization space $\mathfrak{H}_{h,\mathbf{A}}$. Then, we prove that the λ -eigenspace of $\mathcal{D}_{h,\mathbf{A}}$ are isomorphic with the 0-eigenspace of an auxiliary operator \mathcal{L}_λ depending quadratically on λ , see Proposition 2.12. Section 2.3 is devoted to describe the spectrum of \mathcal{L}_λ , and in particular when $0 \in \text{sp}(\mathcal{L}_\lambda)$.

Throughout this section, $h > 0$ is fixed.

2.1. Magnetic Hardy spaces.

Definition 2.1. The magnetic Hardy space is

$$\mathcal{H}_{h,\mathbf{A}}^2(\Omega) = \{u \in L^2(\Omega) : d_{h,\mathbf{A}}^\times u = 0, u|_{\partial\Omega} \in L^2(\partial\Omega)\}.$$

We let

$$\mathfrak{H}_{h,\mathbf{A}} = H^1(\Omega) + \mathcal{H}_{h,\mathbf{A}}^2(\Omega),$$

and endow it with the Hermitian scalar product given by

$$\forall (u_1, u_2) \in \mathfrak{H}_{h,\mathbf{A}} \times \mathfrak{H}_{h,\mathbf{A}}, \quad \langle u_1, u_2 \rangle_{\mathfrak{H}_{h,\mathbf{A}}} = \langle u_1, u_2 \rangle + \langle d_{h,\mathbf{A}}^\times u_1, d_{h,\mathbf{A}}^\times u_2 \rangle + \langle u_1, u_2 \rangle_{\partial\Omega}.$$

Let us recall the following proposition.

Proposition 2.2 ([6, Proposition 2.1.]). *The free Dirac operator and the magnetic Dirac operator are related by the formula*

$$e^{\sigma_3\phi/h}\sigma \cdot \mathbf{p} e^{\sigma_3\phi/h} = \sigma \cdot (\mathbf{p} - \mathbf{A}), \quad (2.1)$$

as operators acting on $H^1(\Omega, \mathbb{C}^2)$ functions.

Remark 2.3. By using the change of function $u = e^{-\phi/h}w$ suggested by Proposition 2.2, we have

$$\mathcal{H}_{h,\mathbf{A}}^2(\Omega) = e^{-\phi/h}\mathcal{H}_0^2(\Omega), \quad \mathfrak{H}_{h,\mathbf{A}} = e^{-\phi/h}\mathfrak{H}_0,$$

where

$$\mathfrak{H}_0 = H^1(\Omega) + \mathcal{H}_0^2(\Omega),$$

and

$$\mathcal{H}_0^2(\Omega) = \{w \in L^2(\Omega) : \partial_{\bar{z}}w = 0, w|_{\partial\Omega} \in L^2(\partial\Omega)\}.$$

Note that, for all $(u_1, u_2) \in \mathfrak{H}_{h,\mathbf{A}} \times \mathfrak{H}_{h,\mathbf{A}}$,

$$\langle u_1, u_2 \rangle_{\mathfrak{H}_{h,\mathbf{A}}} = \langle w_1, w_2 \rangle_{L^2(e^{-2\phi/h})} + \langle -2ih\partial_{\bar{z}}w_1, -2ih\partial_{\bar{z}}w_2 \rangle_{L^2(e^{-2\phi/h})} + \langle w_1, w_2 \rangle_{\partial\Omega},$$

where $w_j = e^{\phi/h}u_j$ for $j = 1, 2$. Then, by using the Riemann biholomorphism $F : \mathbb{D} \rightarrow \Omega$, the classical Hardy space $\mathcal{H}_0^2(\Omega) = \mathcal{H}^2(\Omega)$ becomes the canonical Hardy space

$$\mathcal{H}^2(\mathbb{D}) = \left\{ f \in \mathcal{O}(\mathbb{D}) : \left(\frac{f^{(n)}(0)}{n!} \right)_{n \geq 0} \in \ell^2(\mathbb{N}) \right\}.$$

Note that, for $f \in \mathcal{H}^2(\mathbb{D})$,

$$\|f\|^2 = 2\pi \sum_{n \geq 1} (2n+2)^{-1} |u_n|^2, \quad \|f\|_{\partial\Omega}^2 = 2\pi \sum_{n \geq 0} |u_n|^2, \quad u_n = \frac{f^{(n)}(0)}{n!}. \quad (2.2)$$

The following lemma is a classical result. For the reader's convenience, we recall the proof in Appendix C.

Lemma 2.4. *The space $(\mathcal{H}_{h,\mathbf{A}}^2(\Omega), \langle \cdot, \cdot \rangle_{\partial\Omega})$ is a Hilbert space. Moreover, $\mathcal{H}_{h,\mathbf{A}}^2(\Omega)$ is compactly embedded in $L^2(\Omega)$.*

The next lemma is related to elliptic estimates for magnetic Cauchy-Riemann operators.

Lemma 2.5 ([6, Theorem 4.6.]). *There exists $c > 0$ such that, for all $h > 0$, and for all $u \in \{v \in L^2(\Omega), d_{h,\mathbf{A}}^\times v \in L^2(\Omega)\}$,*

$$\sqrt{2hb_0} \|\Pi_{h,\mathbf{A}}^\perp u\| \leq \|d_{h,\mathbf{A}}^\times u\|, \quad ch^2 (\|\Pi_{h,\mathbf{A}}^\perp u\|_{\partial\Omega} + \|\nabla \Pi_{h,\mathbf{A}}^\perp u\|) \leq \|d_{h,\mathbf{A}}^\times u\|,$$

where $\Pi_{h,\mathbf{A}}$ is the (orthogonal) spectral projection on the kernel of the adjoint of the operator $d_{h,\mathbf{A}}$ with Dirichlet boundary conditions, i.e. $(d_{h,\mathbf{A}}, H_0^1(\Omega))^*$, and

$$\text{Id} = \Pi_{h,\mathbf{A}} + \Pi_{h,\mathbf{A}}^\perp.$$

Let us now prove some properties of the spaces $\mathfrak{H}_{h,\mathbf{A}}$.

Proposition 2.6. *The following holds.*

- (i) $(\mathfrak{H}_{h,\mathbf{A}}, \langle \cdot, \cdot \rangle_{\mathfrak{H}_{h,\mathbf{A}}})$ is a Hilbert space.
- (ii) $H^1(\Omega)$ is dense in $\mathfrak{H}_{h,\mathbf{A}}$.

(iii) *The embedding $\mathfrak{H}_{h,\mathbf{A}} \hookrightarrow L^2(\Omega)$ is compact.*

Proof. Let us prove (i). We consider a Cauchy sequence (u_n) for $\|\cdot\|_{\mathfrak{H}_{h,\mathbf{A}}}$. It is obviously a Cauchy sequence for $\|\cdot\|$ and $\|\cdot\|_{\partial\Omega}$. We write $u_n = \Pi_{h,\mathbf{A}}u_n + \Pi_{h,\mathbf{A}}^\perp u_n$. From Lemma 2.5, we see that $(\Pi_{h,\mathbf{A}}^\perp u_n)$ is a Cauchy sequence in $H^1(\Omega)$, and thus converges to some $u^\perp \in H^1(\Omega)$. Moreover, by using again Lemma 2.5, $(\Pi_{h,\mathbf{A}}u_n)$ is a Cauchy sequence in $\mathcal{H}_{h,\mathbf{A}}^2(\Omega)$. From Lemma 2.4, $(\Pi_{h,\mathbf{A}}u_n)$ converges to some $u \in \mathcal{H}_{h,\mathbf{A}}^2(\Omega)$. It follows that (u_n) converges to $u + u^\perp$ in $\mathfrak{H}_{h,\mathbf{A}}$.

Item (ii) is a consequence of [6, Lemma C.1].

By using again the orthogonal decomposition induced by $\Pi_{h,\mathbf{A}}$, and the compactness of $H^1(\Omega) \hookrightarrow L^2(\Omega)$, and of $\mathcal{H}_{h,\mathbf{A}}^2(\Omega) \hookrightarrow L^2(\Omega)$ (see Lemma 2.4), we get (iii). \square

2.2. Statement of the min-max characterization. The proof of Theorem 1.8 is a consequence of Propositions 2.7 and 2.8, see below.

Notation 3. For all $k \geq 1$ and all $h > 0$, we define

$$\mu_k(h) = \inf_{\substack{W \subset H^1(\Omega) \\ \dim W = k}} \sup_{u \in W \setminus \{0\}} \rho_+(u),$$

where

$$\rho_+(u) = \frac{h\|u\|_{\partial\Omega}^2 + \sqrt{h^2\|u\|_{\partial\Omega}^4 + 4\|u\|^2\|d_{h,\mathbf{A}}^\times u\|^2}}{2\|u\|^2}. \quad (2.3)$$

Proposition 2.7. *We have, for all $k \geq 1$,*

$$\mu_k(h) = \inf_{\substack{W \subset \mathfrak{H}_{h,\mathbf{A}} \\ \dim W = k}} \sup_{u \in W \setminus \{0\}} \rho_+(u) = \min_{\substack{W \subset \mathfrak{H}_{h,\mathbf{A}} \\ \dim W = k}} \sup_{u \in W \setminus \{0\}} \rho_+(u) > 0.$$

Proof. We use Proposition 2.6 (ii) & (iii), and observe that $\rho_+(u) > 0$ for all $u \in \mathfrak{H}_{h,\mathbf{A}} \setminus \{0\}$. \square

Proposition 2.8. *For all $k \geq 1$, and $h > 0$, we have*

$$\lambda_k^+(h) = \mu_k(h).$$

The following sections are devoted to the proof of Proposition 2.8.

In the following, we drop the h -dependence in the notation.

2.3. A characterization of the μ_k .

Notation 4. Let $\lambda \geq 0$. Consider the quadratic form defined by

$$\forall u \in \mathfrak{H}_{h,\mathbf{A}}, \quad Q_\lambda(u) = \|d_{h,\mathbf{A}}^\times u\|^2 + h\lambda\|u\|_{\partial\Omega}^2 - \lambda^2\|u\|^2,$$

and, for all $k \geq 1$,

$$\ell_k(\lambda) = \inf_{\substack{W \subset H^1(\Omega) \\ \dim W = k}} \sup_{u \in W \setminus \{0\}} \frac{Q_\lambda(u)}{\|u\|^2}.$$

Note that, for all $u \in \mathfrak{H}_{h,\mathbf{A}} \setminus \{0\}$,

$$Q_\lambda(u) = -\|u\|^2(\lambda - \rho_-(u))(\lambda - \rho_+(u)), \quad (2.4)$$

where $\rho_+(u)$ is defined in (2.3) and $\rho_-(u)$ is the other zero of the polynomial above.

From Proposition 2.6, we deduce the following.

Lemma 2.9. *For $\lambda > 0$, the (bounded below) quadratic form Q_λ is closed. The associated (unbounded) self-adjoint operator \mathcal{L}_λ has compact resolvent, and its eigenvalues are characterized by the usual min-max formulas*

$$\ell_k(\lambda) = \inf_{\substack{W \subset H^1(\Omega) \\ \dim W = k}} \sup_{u \in W \setminus \{0\}} \frac{Q_\lambda(u)}{\|u\|^2} = \min_{\substack{W \subset \mathfrak{H}_{h,\mathbf{A}} \\ \dim W = k}} \max_{u \in W \setminus \{0\}} \frac{Q_\lambda(u)}{\|u\|^2}.$$

We prove some properties of ℓ_k seen as a function of λ .

Lemma 2.10. *For all $k \geq 1$, the function $\ell_k : (0, +\infty) \rightarrow \mathbb{R}$ satisfies the following:*

- (i) ℓ_1 is concave,
- (ii) for all $\mu \in (0, \mu_1)$, and all $k \geq 1$, $\ell_k(\mu) > 0$,
- (iii) $\lim_{\lambda \rightarrow +\infty} \ell_k(\lambda) = -\infty$,
- (iv) ℓ_k is continuous,
- (v) the equation $\ell_k(\lambda) = 0$ has exactly one positive solution, denoted by E_k .
- (vi) for all $\lambda > 0$,

$$|\ell_k(\lambda)| \geq \lambda |E_k - \lambda|.$$

Proof. Item (i) follows by observing that the infimum of a family of concave functions is itself concave.

It is enough to check Item (ii) for $k = 1$. Consider $\mu > 0$. Thanks to Proposition 2.7, there exists a normalized function $u \in \mathfrak{H}_{h,\mathbf{A}}$ such that $\ell_1(\mu) = Q_\mu(u)$. If $\ell_1(\mu) \leq 0$, then, by (2.4), we have that $\mu \geq \rho_+(u) \geq \mu_1$.

By taking any finite dimensional space $W \subset H_0^1(\Omega)$, we readily see that

$$\ell_k(\lambda) \leq \sup_{u \in W, \|u\|=1} \|d_{h,\mathbf{A}}^\times u\|^2 - \lambda^2.$$

We get Item (iii).

Since ℓ_1 is concave, it is also continuous. Then, the family $(\mathcal{L}_\lambda)_{\lambda > 0}$ is analytic of type (B) in the sense of Kato (i.e., $\text{Dom}(Q_\lambda)$ is independent of $\lambda > 0$). This implies that the ℓ_k are continuous functions. Actually, this can directly be seen from the following equality

$$\lambda_1^{-1} Q_{\lambda_1}(u) - \lambda_2^{-1} Q_{\lambda_2}(u) = (\lambda_2 - \lambda_1) \left(\|d_{h,\mathbf{A}}^\times u\|^2 (\lambda_1 \lambda_2)^{-1} + \|u\|^2 \right), \quad (2.5)$$

for all $\lambda_1, \lambda_2 > 0$ and $u \in \mathfrak{H}_{h,\mathbf{A}}$.

Let us now deal with Item (v). Firstly, let $0 < \lambda_1 < \lambda_2$ and $W \subset \mathfrak{H}_{h,\mathbf{A}}$ with $\dim W = k$. By (2.5), for all $u \in W \setminus \{0\}$, we have

$$\lambda_1^{-1} Q_{\lambda_1}(u) \geq (\lambda_2 - \lambda_1) \|u\|^2 + \lambda_2^{-1} Q_{\lambda_2}(u).$$

and taking the supremum over the vectors $u \in W \setminus \{0\}$,

$$\lambda_1^{-1} \sup_{u \in W \setminus \{0\}} \frac{Q_{\lambda_1}(u)}{\|u\|^2} \geq (\lambda_2 - \lambda_1) + \lambda_2^{-1} \sup_{u \in W \setminus \{0\}} \frac{Q_{\lambda_2}(u)}{\|u\|^2}.$$

Hence, taking the infimum over the subsets $W \subset \mathfrak{H}_{h,\mathbf{A}}$ of dimension k , we get

$$\lambda_1^{-1} \ell_k(\lambda_1) \geq (\lambda_2 - \lambda_1) + \lambda_2^{-1} \ell_k(\lambda_2). \quad (2.6)$$

By Items (ii), (iii) and (iv), there is at least one $\lambda > 0$ such that $\ell_k(\lambda) = 0$. Assume by contradiction that ℓ_k has two zeros $0 < \lambda_1 < \lambda_2$. By (2.6), we get the contradiction $0 \geq \lambda_2 - \lambda_1 > 0$. Therefore, ℓ_k has only one positive zero.

To deal with Item (vi), we take first $\lambda_1 = E_k < \lambda_2$ so that

$$-\ell_k(\lambda_2) \geq \lambda_2(\lambda_2 - E_k),$$

and $|\ell_k(\lambda_2)| \geq \lambda_2|\lambda_2 - E_k|$. Then, consider $0 < \lambda_1 < \lambda_2 = E_k$,

$$\ell_k(\lambda_1) \geq \lambda_1(E_k - \lambda_1),$$

and $|\ell_k(\lambda_1)| \geq \lambda_1|\lambda_1 - E_k|$. These two inequalities give Item (vi). □

Proposition 2.11. *For all $k \geq 1$, μ_k is the only positive zero of ℓ_k , i.e.,*

$$E_k = \mu_k.$$

Proof. In virtue of Proposition 2.7, we notice that $\mu_k > 0$. Then, we proceed in two steps.

Firstly, consider a subspace $W_k \subset \mathfrak{H}_{h,\mathbf{A}}$ of dimension k such that

$$\max_{u \in W_k \setminus \{0\}} \rho_+(u) = \mu_k.$$

For all $u \in W_k \setminus \{0\}$, $\rho_+(u) \leq \mu_k$. By the definition of $\ell_k(\mu_k)$ and (2.4), we have

$$\ell_k(\mu_k) \leq \max_{u \in W_k \setminus \{0\}} Q_{\mu_k}(u) \leq 0.$$

Secondly, for all subspace $W \subset \mathfrak{H}_{h,\mathbf{A}}$ of dimension k , we have

$$\mu_k \leq \max_{u \in W \setminus \{0\}} \rho_+(u).$$

There exists $u_k \in W \setminus \{0\}$ such that $\mu_k \leq \rho_+(u_k)$. Then, we have

$$\max_{u \in W \setminus \{0\}} Q_{\mu_k}(u) \geq Q_{\mu_k}(u_k) \geq 0,$$

and taking the infimum over W , we find $\ell_k(\mu_k) \geq 0$.

We deduce that $\ell_k(\mu_k) = 0$ and conclude by using Lemma 2.10 (v). □

2.4. Proof of Proposition 2.8.

2.4.1. *An isomorphism.* The following proposition is crucial.

Proposition 2.12. *Let $\lambda > 0$. Then, the map*

$$\mathcal{J}_\lambda : \begin{cases} \ker \mathcal{L}_\lambda & \longrightarrow \ker(\mathcal{D}_{h,\mathbf{A}} - \lambda) \\ u & \longmapsto \begin{pmatrix} u \\ \lambda^{-1} d_{h,\mathbf{A}}^\times u \end{pmatrix} \end{cases}$$

is well-defined and it is an isomorphism.

Proof. First, we show that the range of \mathcal{J}_λ is indeed contained $\ker(\mathcal{D}_{h,\mathbf{A}} - \lambda)$. Let $u \in \ker(\mathcal{L}_\lambda)$. Notice that $u \in \ker(\mathcal{L}_\lambda)$ is equivalent to

$$\forall w \in \mathfrak{H}_{h,\mathbf{A}}, \quad Q_\lambda(u, w) = \langle d_{h,\mathbf{A}}^\times u, d_{h,\mathbf{A}}^\times w \rangle + h\lambda \langle u, w \rangle_{\partial\Omega} - \lambda^2 \langle u, w \rangle = 0. \quad (2.7)$$

We set

$$\varphi = \begin{pmatrix} u \\ v \end{pmatrix}, \quad v = \frac{d_{h,\mathbf{A}}^\times u}{\lambda}.$$

For all $\psi = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \text{Dom}(\mathcal{D}_{h,\mathbf{A}})$, we have

$$\begin{aligned} \langle \varphi, \mathcal{D}_{h,\mathbf{A}}\psi \rangle &= \langle u, d_{h,\mathbf{A}}w_2 \rangle + \langle v, d_{h,\mathbf{A}}^\times w_1 \rangle \\ &= \langle d_{h,\mathbf{A}}^\times u, w_2 \rangle + \langle u, -ih\bar{n}w_2 \rangle_{\partial\Omega} + \left\langle \frac{d_{h,\mathbf{A}}^\times u}{\lambda}, d_{h,\mathbf{A}}^\times w_1 \right\rangle \\ &= \langle \lambda v, w_2 \rangle + h\langle u, w_1 \rangle_{\partial\Omega} + \left\langle \frac{d_{h,\mathbf{A}}^\times u}{\lambda}, d_{h,\mathbf{A}}^\times w_1 \right\rangle \\ &= \langle \lambda v, w_2 \rangle + \lambda\langle u, w_1 \rangle = \lambda\langle \varphi, \psi \rangle, \end{aligned}$$

where

- the second equality comes from an integration by parts using Proposition 2.6 (ii),
- the third uses the boundary condition $w_2 = inw_1$,
- the fourth uses (2.7).

This shows, by the definition of the adjoint, that $\varphi \in \text{Dom}(\mathcal{D}_{h,\mathbf{A}}^*) = \text{Dom} \mathcal{D}_{h,\mathbf{A}}$ and in particular that $\mathcal{D}_{h,\mathbf{A}}\varphi = \lambda\varphi$. Therefore, the map is well-defined, and we observe that it is injective.

Let us show that \mathcal{I}_λ is surjective. Consider $\begin{pmatrix} u \\ v \end{pmatrix} \in \ker(\mathcal{D}_{h,\mathbf{A}} - \lambda)$. The eigenvalue equation reads

$$d_{h,\mathbf{A}}^\times u = \lambda v, \quad d_{h,\mathbf{A}}v = \lambda u, \quad \text{and } v = inu \text{ on } \partial\Omega.$$

Let $w \in \mathfrak{H}_{h,\mathbf{A}}$. Using the eigenvalue equation, and again an integration by parts, we get

$$\begin{aligned} Q_\lambda(u, w) &= \langle d_{h,\mathbf{A}}^\times u, d_{h,\mathbf{A}}^\times w \rangle + h\lambda\langle u, w \rangle_{\partial\Omega} - \lambda^2\langle u, w \rangle \\ &= \lambda\langle v, d_{h,\mathbf{A}}^\times w \rangle + h\lambda\langle -i\bar{n}v, w \rangle_{\partial\Omega} - \lambda^2\langle u, w \rangle \\ &= \lambda\langle d_{h,\mathbf{A}}v, w \rangle - \lambda^2\langle u, w \rangle = \lambda^2\langle u, w \rangle - \lambda^2\langle u, w \rangle = 0. \end{aligned}$$

Hence, $u \in \text{Dom}(\mathcal{L}_\lambda^*) = \text{Dom}(\mathcal{L}_\lambda)$ and $u \in \ker \mathcal{L}_\lambda$. \square

Corollary 2.13. *We set $\Lambda = \{\lambda_j^+, j \geq 1\}$ and $M = \{\mu_k, k \geq 1\}$. We have $\Lambda = M$. In particular, $\mu_1 = \lambda_1^+$.*

Proof. Let $\lambda \in \Lambda$. Proposition 2.12 implies that $0 \in \text{sp}(\mathcal{L}_\lambda)$. Then, there exists $j \geq 1$ such that $\ell_j(\lambda) = 0$ and thus (Proposition 2.11) $\lambda = E_j = \mu_j \in M$.

Let $\mu \in M$. Then, there exists $j \geq 1$ such that $\mu = E_j$, and hence $\ell_j(\mu) = 0$. In particular, $0 \in \text{sp}(\mathcal{L}_\mu)$ and thus $\mu \in \text{sp}(\mathcal{D}_{h,\mathbf{A}})$ by the isomorphism. \square

Notation 5. Let us denote by $(a_k)_{k \geq 1}$ the unique increasing sequence such that $\Lambda = M = \{a_k, k \geq 1\}$. $(\mu_k)_{k \geq 1}$ is just a priori a non decreasing sequence. In addition, for all $k \geq 1$, we set $m_k = \dim \ker(\mathcal{D}_{h,\mathbf{A}} - a_k)$.

2.4.2. *Induction argument.* Now, we can prove Proposition 2.8 by induction.

For $n \geq 0$, the induction statement is

$$\mathcal{P}(n) \quad : \quad \forall j \in \{1, \dots, m_1 + \dots + m_n + 1\}, \quad \mu_j = \lambda_j^+.$$

Thanks to Corollary 2.13, $\mathcal{P}(0)$ is satisfied.

Let $n \geq 0$. Assume that, for all $0 \leq k \leq n$, $\mathcal{P}(k)$ holds.

Notice that

$$\mu_{m_1+\dots+m_n+1} = \lambda_{m_1+\dots+m_n+1}^+ = a_{n+1}. \quad (2.8)$$

By definition, we have $a_{n+1} \in \text{sp}(\mathcal{D}_{h,\mathbf{A}})$. Moreover, by using the isomorphism,

$$m_{n+1} = \dim \ker(\mathcal{L}_{a_{n+1}}).$$

By the min-max theorem, there exists $j_0 \geq 0$ such that

$$\ell_{j_0+1}(a_{n+1}) = \dots = \ell_{j_0+m_{n+1}}(a_{n+1}) = 0.$$

By Lemma 2.10 (v), we have

$$a_{n+1} = E_{j_0+1} = \dots = E_{j_0+m_{n+1}},$$

so that, using again Proposition 2.11,

$$a_{n+1} = \mu_{j_0+1} = \dots = \mu_{j_0+m_{n+1}}.$$

Let us now show that $j_0 = m_1 + \dots + m_n$. By the induction hypothesis, we have

$$\mu_{m_1+\dots+m_n} = a_n < a_{n+1}.$$

Thus, $j_0 \geq m_1 + \dots + m_n$.

Let us suppose, by contradiction, that $j_0 \geq m_1 + \dots + m_n + 1$. With (2.8), we get

$$\mu_{m_1+\dots+m_n+1} = \mu_{j_0+1} = \dots = \mu_{j_0+m_{n+1}} = a_{n+1}.$$

In particular, we have the $m_{n+1} + 1$ relations:

$$\ell_{m_1+\dots+m_n+1}(a_{n+1}) = \ell_{j_0+1}(a_{n+1}) = \dots = \ell_{j_0+m_{n+1}}(a_{n+1}) = 0.$$

By the min-max theorem, this shows that

$$\dim \ker \mathcal{L}_{a_{n+1}} \geq m_{n+1} + 1 > m_{n+1} = \dim \ker(\mathcal{D}_{h,\mathbf{A}} - a_{n+1}).$$

This contradicts the isomorphism property. Therefore, $j_0 = m_1 + \dots + m_n$. This argument also shows that the multiplicity of $\mu_{m_1+\dots+m_n+1}$ equals m_{n+1} . With the induction hypothesis, we get

$$\forall j \in \{1, \dots, m_1 + \dots + m_{n+1}\}, \quad \mu_j = \lambda_j^+.$$

By definition, we have

$$\lambda_{m_1+\dots+m_n+1}^+ = \min(\Lambda \setminus \{a_1, \dots, a_{n+1}\}) = \min(M \setminus \{a_1, \dots, a_{n+1}\}).$$

We observe that $\mu_{m_1+\dots+m_n+1} > a_{n+1}$ since the multiplicity of $\mu_{m_1+\dots+m_n+1}$ equals m_{n+1} . This proves that

$$\lambda_{m_1+\dots+m_n+1}^+ = \min(M \setminus \{a_1, \dots, a_{n+1}\}) = \mu_{m_1+\dots+m_n+1}.$$

This concludes the induction argument.

3. SEMICLASSICAL ANALYSIS OF THE POSITIVE EIGENVALUES

In this section we prove Theorem 1.10 by applying Theorem 1.8, and considering the asymptotic analysis of a simpler problem. If one wants to estimate $\lambda_k^+(h)$, it is natural to use the functions of the Hardy space $\mathcal{H}_{h,\mathbf{A}}^2(\Omega)$ introduced in Definition 2.1 as test functions. This cancels the $d_{h,\mathbf{A}}^\times$ -term in ρ_+ and leads to define

$$\nu_k(h) = \inf_{\substack{W \subset \mathcal{H}_{h,\mathbf{A}}^2(\Omega) \\ \dim W = k}} \sup_{u \in W \setminus \{0\}} \frac{h \|u\|_{\partial\Omega}^2}{\|u\|^2}.$$

Theorem 1.10 is a consequence of the following three results.

Lemma 3.1. *For all $k \in \mathbb{N} \setminus \{0\}$ and all $h > 0$, we have*

$$\lambda_k^+(h) \leq \nu_k(h).$$

Proof. It follows from the definition of $\nu_k(h)$. \square

Actually, we can prove that $\nu_k(h)$ is also a good asymptotic lower bound for $\lambda_k^+(h)$, see Section 3.2 where the following is proved.

Proposition 3.2. *For all $k \geq 1$, we have*

$$\nu_k(h) \leq \lambda_k^+(h)(1 + \mathcal{O}(h^\infty)).$$

In the next section, we study the asymptotic behavior of $\nu_k(h)$, which is summarized in the following proposition.

Proposition 3.3. *For all $k \geq 1$,*

$$\nu_k(h) = C_k(B, \Omega) h^{1-k} e^{2\phi_{\min}/h} (1 + o(1)),$$

where $C_k(B, \Omega)$ is defined in (1.9).

Remark 3.4. Proposition 3.3 shows that each $\nu_k(h)$ goes to zero exponentially when h goes to zero. The analysis in Section 3.2 strongly relies on this fact.

3.1. About the proof of Proposition 3.3. Using the change of function $u = e^{-\phi/h}v$ suggested by Proposition 2.2 and detailed in Remark 2.3, we get

$$\nu_k(h) = \inf_{\substack{W \subset \mathcal{H}^2(\Omega) \\ \dim W = k}} \sup_{v \in W \setminus \{0\}} \frac{h \|v\|_{\partial\Omega}^2}{\|e^{-\phi/h}v\|^2}.$$

In what follows we give upper and lower bounds for $\nu_k(h)$. The technics borrow ideas from our previous work [6].

3.1.1. *Upper bound.* Let us consider $k \geq 1$ fixed.

Notation 6. Let us denote by $(P_n)_{n \in \mathbb{N}}$ the $N_{\mathcal{B}}$ -orthogonal family such that $P_n(Z) = Z^n + \sum_{j=0}^{n-1} b_{n,j} Z^j$ obtained after a Gram-Schmidt process on $(1, Z, \dots, Z^n, \dots)$. Since P_n is $N_{\mathcal{B}}$ -orthogonal to \mathcal{P}_{n-1} , we have

$$\begin{aligned} \text{dist}_{\mathcal{B}}(Z^n, \mathcal{P}_{n-1}) &= \text{dist}_{\mathcal{B}}(P_n, \mathcal{P}_{n-1}) = \inf\{N_{\mathcal{B}}(P_n - Q), Q \in \mathcal{P}_{n-1}\} \\ &= \inf\{\sqrt{N_{\mathcal{B}}(P_n)^2 + N_{\mathcal{B}}(Q)^2}, Q \in \mathcal{P}_{n-1}\} = N_{\mathcal{B}}(P_n), \text{ for } n \in \mathbb{N}. \end{aligned} \quad (3.1)$$

By the Cauchy-Schwarz inequality and the Cauchy formula, the subspace $\mathcal{H}_k^2(\Omega)$ is closed in $\mathcal{H}^2(\Omega)$. Therefore, there exists a unique $Q_n \in \mathcal{H}_k^2(\Omega)$ such that

$$\text{dist}_{\mathcal{H}}((z - z_{\min})^n, \mathcal{H}_k^2(\Omega)) = \|(z - z_{\min})^n - Q_n(z)\|_{\partial\Omega},$$

for $n \in \{0, \dots, k-1\}$. We recall that N_B , \mathcal{P}_{n-1} , and $\mathcal{H}_k^2(\Omega)$ are defined in Notation 2.

The following proposition gives the wanted upper bound.

Proposition 3.5.

$$\nu_k(h) \leq \left(\frac{\text{dist}_{\mathcal{H}}((z - z_{\min})^{k-1}, \mathcal{H}_k^2(\Omega))}{\text{dist}_B(z^{k-1}, \mathcal{P}_{k-2})} \right)^2 h^{1-k} e^{2\phi_{\min}/h} (1 + o(1)).$$

Proposition 3.5 is a consequence of the following lemmas and relies on the introduction of an appropriate k -dimensional vector space $V_{h,k}$ of test functions. Let us define $V_{h,k}$ by

$$V_{h,k} = \text{span}(w_{0,h}, \dots, w_{k-1,h}) \subset \mathcal{H}^2(\Omega), \quad (3.2)$$

$$w_{n,h}(z) = h^{-\frac{1}{2}} P_n \left(\frac{z - z_{\min}}{h^{1/2}} \right) - h^{-\frac{1+n}{2}} Q_n(z), \text{ for } n \in \{0, \dots, k-1\}.$$

Lemma 3.6 ([6, Lemma 3.5]). *Let $h \in (0, 1]$, $v_h = \sum_{j=0}^{k-1} c_j w_{j,h} \in V_{h,k}$ with $c_0, \dots, c_{k-1} \in \mathbb{C}$, and $(w_{j,h})_{j \in \{0, \dots, k-1\}}$ defined in (3.2). We have*

$$\int_{\Omega} |v_h|^2 e^{-2(\phi(x) - \phi_{\min})/h} dx = (1 + o(1)) \sum_{j=0}^{k-1} |c_j|^2 N_B(P_j)^2, \quad (3.3)$$

where $o(1)$ does not depend on $c = (c_0, \dots, c_{k-1})$.

For the numerator, we have the following result.

Lemma 3.7. *Let $h \in (0, 1]$, $v_h = \sum_{j=0}^{k-1} c_j w_{j,h} \in V_{h,k}$ with $c_0, \dots, c_{k-1} \in \mathbb{C}$. We have*

$$\|v_h\|_{\partial\Omega}^2 \leq |c_{k-1}|^2 h^{-k} \left\| (z - z_{\min})^{k-1} - Q_{k-1} \right\|_{\partial\Omega}^2 + o(h^{-k}) \|c\|_{\ell^2}^2.$$

Here, $o(1)$ does not depend on c_0, \dots, c_{k-1} .

Proof. Let us estimate $\|v_h\|_{\partial\Omega}$. From the triangle inequality, we get

$$\|v_h\|_{\partial\Omega} \leq |c_{k-1}| \|w_{k-1,h}\|_{\partial\Omega} + \sum_{j=0}^{k-2} |c_j| \|w_{j,h}\|_{\partial\Omega}.$$

Then, from degree considerations and the triangle inequality, we get, for $1 \leq j \leq k-2$,

$$\|w_{j,h}\|_{\partial\Omega} = \mathcal{O}\left(h^{\frac{1-k}{2}}\right),$$

and

$$\|w_{k-1,h}\|_{\partial\Omega} = (1 + o(1)) h^{-\frac{k}{2}} \left\| (z - z_{\min})^{k-1} - Q_{k-1} \right\|_{\partial\Omega}.$$

The conclusion follows. \square

Proposition 3.5 follows from these last two lemmas and a straightforward study of a finite dimensional min-max problem on vectors $(c_0, \dots, c_{k-1}) \in \mathbb{C}^k$.

3.1.2. *Lower bound.* Let $k \geq 1$. Let us consider an orthonormal family $(v_{j,h})_{1 \leq j \leq k}$ (for the scalar product of $L^2(e^{-2\phi/h} dx)$) associated with the eigenvalues $(\nu_j(h))_{1 \leq j \leq k}$. We define

$$\mathcal{E}_k(h) = \text{span}_{1 \leq j \leq k} v_{j,h}.$$

The next two lemmas gives a priori bounds on the functions in $\mathcal{E}_k(h)$.

Lemma 3.8. *There exist $C, h_0 > 0$ such that for all $v_h \in \mathcal{E}_k(h)$ and $h \in (0, h_0)$, we have,*

$$\|v_h\|^2 \leq Ch^{-k} e^{2\phi_{\min}/h} \int_{\Omega} e^{-2\phi/h} |v_h|^2 dx.$$

Proof. From the continuous embedding $\mathcal{H}^2(\Omega) \hookrightarrow L^2(\Omega)$, and Proposition 3.5, there exist $c, C, h_0 > 0$ such that, for all $h \in (0, h_0)$ and all $v \in \mathcal{E}_k(h)$,

$$ch\|v\|^2 \leq h\|v\|_{\partial\Omega}^2 \leq \nu_k(h) \int_{\Omega} e^{-2\phi/h} |v_h|^2 dx \leq Ch^{1-k} e^{2\phi_{\min}/h} \int_{\Omega} e^{-2\phi/h} |v_h|^2 dx.$$

□

Lemma 3.9. *Let $\alpha \in (1/3, 1/2)$. We have*

$$\lim_{h \rightarrow 0} \sup_{v_h \in \mathcal{E}_k(h) \setminus \{0\}} \left| \frac{\int_{D(x_{\min}, h^\alpha)} e^{-2\phi/h} |v_h(x)|^2 dx}{\int_{\Omega} e^{-2\phi/h} |v_h(x)|^2 dx} - 1 \right| = 0,$$

Proof. Assume that $\alpha \in (\frac{1}{3}, \frac{1}{2})$. We have for all $x \in D(x_{\min}, h^\alpha)$,

$$\phi(x) = \phi_{\min} + \frac{1}{2} \text{Hess}_{x_{\min}} \phi(x - x_{\min}, x - x_{\min}) + \mathcal{O}(h^{3\alpha}).$$

By the maximum principle,

$$\min_{x \in D(x_{\min}, h^\alpha)^c} \phi(x) = \min_{x \in \partial D(x_{\min}, h^\alpha)^c} \phi(x) \geq \phi_{\min} + h^{2\alpha} \min \text{sp}(\text{Hess}_{x_{\min}}) + \mathcal{O}(h^{3\alpha}).$$

The result follows from Lemma 3.8. □

We are now in a good position to study the lower bound.

Proposition 3.10. *We have*

$$\nu_k(h) \geq \left(\frac{\text{dist}_{\mathcal{H}}((z - z_{\min})^{k-1}, \mathcal{H}_k^2(\Omega))}{\text{dist}_{\mathcal{B}}(z^{k-1}, \mathcal{P}_{k-2})} \right)^2 h^{1-k} e^{2\phi_{\min}/h} (1 + o(1)).$$

Proof. Let $\alpha \in (1/3, 1/2)$. With Lemma 3.9,

$$h e^{2\phi_{\min}/h} \|v_h\|_{\partial\Omega}^2 (1 + o(1)) \leq \nu_k(h) \left\| e^{-\frac{1}{2h} \text{Hess}_{x_{\min}} \phi(x - x_{\min}, x - x_{\min})} v_h \right\|_{L^2(D(x_{\min}, h^\alpha))}^2. \quad (3.4)$$

In the following, we split the proof into two parts. Firstly, we replace v_h by its Taylor expansion at the order $k - 1$ at x_{\min} in the R. H. S. of (3.4). Secondly, we do the same for the L.H.S. of the same equation.

- i. In view of the Cauchy formula, and the Cauchy-Schwarz inequality, there exist $C > 0, h_0 > 0$ such that, for all $h \in (0, h_0)$, for all $v \in \mathcal{H}^2(\Omega)$, all $z_0 \in D(x_{\min}, h^\alpha)$, and all $n \in \{0, \dots, k\}$,

$$|v^{(n)}(z_0)| \leq C \|v\|_{\partial\Omega}. \quad (3.5)$$

Let us define, for all $v \in \mathcal{H}^2(\Omega)$,

$$N_h(v) = \left\| e^{-\frac{1}{2h} \text{Hess}_{x_{\min}} \phi(x-x_{\min}, x-x_{\min})} v \right\|_{L^2(D(x_{\min}, h^\alpha))}.$$

By the Taylor formula, we can write

$$v_h = \text{Tayl}_{k-1} v_h + R_{k-1}(v_h),$$

where

$$\text{Tayl}_{k-1} v_h = \sum_{n=0}^{k-1} \frac{v_h^{(n)}(z_{\min})}{n!} (z - z_{\min})^n,$$

and, for all $z_0 \in D(z_{\min}, h^\alpha)$,

$$|R_{k-1}(v_h)(z_0)| \leq C |z - z_{\min}|^k \sup_{D(z_{\min}, h^\alpha)} |v_h^{(k)}|.$$

With (3.5) and a rescaling, the Taylor remainder satisfies

$$N_h(R_{k-1}(v_h)) \leq Ch^{\frac{k}{2}} h^{\frac{1}{2}} \|v_h\|_{\partial\Omega}.$$

Thus, by the triangle inequality,

$$N_h(v_h) \leq N_h(\text{Tayl}_{k-1} v_h) + Ch^{\frac{k}{2}} h^{\frac{1}{2}} \|v_h\|_{\partial\Omega}.$$

Thus, with (3.4), we get

$$(1 + o(1)) e^{\phi_{\min}/h} \sqrt{h} \|v_h\|_{\partial\Omega} \leq \sqrt{\nu_k(h)} N_h(\text{Tayl}_{k-1} v_h) + C \sqrt{\nu_k(h)} h^{\frac{1+k}{2}} \|v_h\|_{\partial\Omega},$$

so that, thanks to Proposition 3.5,

$$(1 + o(1)) e^{\phi_{\min}/h} \sqrt{h} \|v_h\|_{\partial\Omega} \leq \sqrt{\nu_k(h)} N_h(\text{Tayl}_{k-1} v_h) \leq \sqrt{\nu_k(h)} \hat{N}_h(\text{Tayl}_{k-1} v_h), \quad (3.6)$$

with

$$\hat{N}_h(w) = \left\| e^{-\frac{1}{2h} \text{Hess}_{x_{\min}} \phi(x-x_{\min}, x-x_{\min})} w \right\|_{L^2(\mathbb{R}^2)}.$$

This inequality shows in particular that Tayl_{k-1} is injective on $\mathcal{E}_k(h)$ and

$$\dim \text{Tayl}_{k-1} \mathcal{E}_k(h) = k. \quad (3.7)$$

- ii. Let us recall that

$$\mathcal{H}_k^2(\Omega) = \{u \in \mathcal{H}^2(\Omega) : \forall n \in \{0, \dots, k-1\}, u^{(n)}(x_{\min}) = 0\}.$$

Since $(v_h - \text{Tayl}_{k-1} v_h) \in \mathcal{H}_k^2(\Omega)$, we have, by the triangle inequality,

$$\begin{aligned} \|v_h\|_{\partial\Omega} &\geq \left\| \frac{v_h^{(k-1)}(z_{\min})}{(k-1)!} (z - z_{\min})^{k-1} + (v_h - \text{Tayl}_{k-1} v_h) \right\|_{\partial\Omega} - \|\text{Tayl}_{k-2} v_h\|_{\partial\Omega} \\ &\geq \frac{|v_h^{(k-1)}(z_{\min})|}{(k-1)!} \text{dist}_{\mathcal{H}}((z - z_{\min})^{k-1}, \mathcal{H}_k^2(\Omega)) - \|\text{Tayl}_{k-2} v_h\|_{\partial\Omega}, \end{aligned}$$

where

$$\begin{aligned} & \text{dist}((z - z_{\min})^{k-1}, \mathcal{H}_k^2(\Omega)) \\ &= \inf \left\{ \left\| (z - z_{\min})^{k-1} - Q(z) \right\|_{\partial\Omega}, \text{ for all } Q \in \mathcal{H}_k^2(\Omega) \right\}. \end{aligned}$$

Using again the triangle inequality,

$$\|\text{Tayl}_{k-2} v_h\|_{\partial\Omega} \leq C \sum_{n=0}^{k-2} |v_h^{(n)}(z_{\min})|.$$

Moreover,

$$\begin{aligned} \sum_{n=0}^{k-2} |v_h^{(n)}(z_{\min})| &\leq h^{-\frac{k-2}{2}} \sum_{n=0}^{k-2} h^{\frac{n}{2}} |v_h^{(n)}(z_{\min})| \leq h^{-\frac{k-2}{2}} \sum_{n=0}^{k-1} h^{\frac{n}{2}} |v_h^{(n)}(z_{\min})| \\ &\leq Ch^{-\frac{k-2}{2}} h^{-\frac{1}{2}} \hat{N}_h(\text{Tayl}_{k-1} v_h), \end{aligned}$$

where we used the rescaling property

$$\hat{N}_h \left(\sum_{n=0}^{k-1} c_n (z - z_{\min})^n \right) = h^{\frac{1}{2}} \hat{N}_1 \left(\sum_{n=0}^{k-1} c_n h^{\frac{n}{2}} (z - z_{\min})^n \right), \quad (3.8)$$

and the equivalence of the norms in finite dimension:

$$\exists C > 0, \forall d \in \mathbb{C}^k, \quad C^{-1} \sum_{n=0}^{k-1} |d_n| \leq \hat{N}_1 \left(\sum_{n=0}^{k-1} d_n (z - z_{\min})^n \right) \leq C \sum_{n=0}^{k-1} |d_n|.$$

We find

$$\|v_h\|_{\partial\Omega} \geq \frac{|v_h^{(k-1)}(z_{\min})|}{(k-1)!} \text{dist}((z - z_{\min})^{k-1}, \mathcal{H}_k^2(\Omega)) - Ch^{-\frac{k-2}{2}} h^{-\frac{1}{2}} \hat{N}_h(\text{Tayl}_{k-1} v_h),$$

and thus, by (3.6),

$$\begin{aligned} (1 + o(1)) e^{\phi_{\min}/h} \sqrt{h} \frac{|v_h^{(k-1)}(z_{\min})|}{(k-1)!} \text{dist}((z - z_{\min})^{k-1}, \mathcal{H}_k^2(\Omega)) \\ \leq \left(\sqrt{\nu_k(h)} + Ch^{\frac{2-k}{2}} e^{\phi_{\min}/h} \right) \hat{N}_h(\text{Tayl}_{k-1} v_h). \end{aligned} \quad (3.9)$$

Let us now end the proof of the lower bound by using (3.9) and (3.7).

Since we have (3.7), we deduce that

$$\begin{aligned} (1 + o(1)) e^{\phi_{\min}/h} \sqrt{h} \text{dist}_{\mathcal{H}}((z - z_{\min})^{k-1}, \mathcal{H}_k^2(\Omega)) \sup_{c \in \mathbb{C}^k} \frac{|c_{k-1}|}{\hat{N}_h(\sum_{n=0}^{k-1} c_n (z - z_{\min})^n)} \\ \leq \sqrt{\nu_k(h)} + Ch^{\frac{2-k}{2}} e^{\phi_{\min}/h}. \end{aligned} \quad (3.10)$$

By (3.8), we infer

$$h^{\frac{1}{2}} \sup_{c \in \mathbb{C}^k} \frac{|c_{k-1}|}{\hat{N}_h(\sum_{n=0}^{k-1} c_n (z - z_{\min})^n)} = \sup_{c \in \mathbb{C}^k} \frac{h^{\frac{1-k}{2}} |c_{k-1}|}{\hat{N}_1(\sum_{n=0}^{k-1} c_n (z - z_{\min})^n)}.$$

Since \hat{N}_1 is related to the Segal-Bargmann norm $N_{\mathcal{B}}$ via a translation, and recalling Notation 6, we get

$$\sup_{c \in \mathbb{C}^k} \frac{|c_{k-1}|}{\hat{N}_1(\sum_{n=0}^{k-1} c_n (z - z_{\min})^n)} = \sup_{c \in \mathbb{C}^k} \frac{|c_{k-1}|}{N_{\mathcal{B}}(\sum_{n=0}^{k-1} c_n z^n)} = \frac{1}{N_{\mathcal{B}}(P_{k-1})}.$$

Thus,

$$(1 + o(1)) h^{\frac{1-k}{2}} e^{\phi_{\min}/h} \frac{\text{dist}_{\mathcal{H}}((z - z_{\min})^{k-1}, \mathcal{H}_k^2(\Omega))}{N_{\mathcal{B}}(P_{k-1})} \leq \sqrt{\nu_k(h)}. \quad (3.11)$$

The conclusion follows. \square

3.2. Approximation results. Let us roughly explain the strategy to establish Proposition 3.2. Recall Theorem 1.8 which gives a nonlinear min-max formulation for $\lambda_k^+(h)$. Let us remark that the functions in the range of the orthogonal projector $\Pi_{h,\mathbf{A}}$ defined in Lemma 2.5,

$$\text{ran } \Pi_{h,\mathbf{A}} = \{u \in L^2(\Omega), d_{h,\mathbf{A}}^\times u = 0\} = e^{-\phi/h} \mathcal{O}(\Omega) \cap L^2(\Omega),$$

do not have in general an $L^2(\partial\Omega)$ -trace. Nevertheless, since

$$\text{ran } \Pi_{h,\mathbf{A}}^\perp \subset H^1(\Omega),$$

the projection $\Pi_{h,\mathbf{A}} u = u - \Pi_{h,\mathbf{A}}^\perp u$ has an $L^2(\partial\Omega)$ -trace whenever $u \in \mathfrak{H}_{h,\mathbf{A}}$ (u itself has an $L^2(\partial\Omega)$ -trace) :

$$\text{rg } \Pi_{h,\mathbf{A}} \upharpoonright_{\mathfrak{H}_{h,\mathbf{A}}} = \mathcal{H}_{h,\mathbf{A}}^2(\Omega).$$

Consider a minimizing subspace $W_h \subset \mathfrak{H}_{h,\mathbf{A}} = \mathcal{H}_{h,\mathbf{A}}^2(\Omega) + H^1(\Omega)$ (of dimension k). Then, we can prove that W_h is quasi invariant under the orthogonal projector $\Pi_{h,\mathbf{A}}$, see Lemma 3.14. So, we would like to write $\rho_+(u) \simeq \rho_+(\Pi_{h,\mathbf{A}} u)$ for all $u \in W_h$. In the following, we will use approximate subspaces to highlight the stability of the projection procedure. For that purpose, we will use a number $M_k(h) \geq \lambda_k^+(h)$ such that

$$M_k(h) = \lambda_k^+(h)(1 + o(1)).$$

Remark 3.11. By Remark 3.4, $M_k(h)$ goes itself exponentially fast to zero.

Notation 7. For notational simplicity, we write $M \equiv M_k(h)$.

There exists $W_h \subset \mathfrak{H}_{h,\mathbf{A}}$ with $\dim W_h = k$ such that

$$\lambda_k^+(h) \leq \sup_{W_h \setminus \{0\}} \rho_+(u) \leq M. \quad (3.12)$$

The following lemma is straightforward.

Lemma 3.12. *For all $u \in \mathfrak{H}_{h,\mathbf{A}}$, we have*

$$2h \|u\|_{\partial\Omega}^2 \leq \mathcal{Q}_h(u), \quad 2 \|u\| \|d_{h,\mathbf{A}}^\times u\| \leq \mathcal{Q}_h(u),$$

where

$$\mathcal{Q}_h(u) = h \|u\|_{\partial\Omega}^2 + \sqrt{h^2 \|u\|_{\partial\Omega}^4 + 4 \|u\|^2 \|d_{h,\mathbf{A}}^\times u\|^2}.$$

Thanks to Lemma 3.12 and (3.12), we get the following.

Lemma 3.13. *For all $u \in W_h$,*

$$h\|u\|_{\partial\Omega}^2 \leq M\|u\|^2, \quad (3.13)$$

and thus

$$\|d_{h,\mathbf{A}}^\times u\|^2 \leq M^2\|u\|^2. \quad (3.14)$$

Lemma 3.14. *For all $u \in W_h$, we have*

$$\|\Pi_{h,\mathbf{A}}^\perp u\| \leq \frac{M}{\sqrt{2hb_0}}\|u\|, \quad (3.15)$$

$$\|\Pi_{h,\mathbf{A}}^\perp u\|_{\partial\Omega} \leq \frac{M}{ch^2}\|u\|. \quad (3.16)$$

Moreover, for h small enough, $\Pi_{h,\mathbf{A}}\mathbb{1}_{W_h}$ is injective.

Proof. Combining (3.14) and Lemma 2.5, we readily get (3.15) and (3.16). The injectivity follows from (3.15) and Remark 3.11. \square

Proposition 3.15. *For all $u \in W_h$, we have*

$$\lambda_k^+(h)(1 + \mathcal{O}(h^\infty))\|\Pi_{h,\mathbf{A}}u\|^2 \geq h\|\Pi_{h,\mathbf{A}}u\|_{\partial\Omega}^2.$$

Proof. Let us consider (3.13). We have

$$M^{\frac{1}{2}}\|u\| \geq \sqrt{h}\|u\|_{\partial\Omega} = \sqrt{h}\|\Pi_{h,\mathbf{A}}u + \Pi_{h,\mathbf{A}}^\perp u\|_{\partial\Omega} \geq \sqrt{h}(\|\Pi_{h,\mathbf{A}}u\|_{\partial\Omega} - \|\Pi_{h,\mathbf{A}}^\perp u\|_{\partial\Omega}).$$

By (3.16), we get

$$M^{\frac{1}{2}} \left(1 + h^{-\frac{3}{2}} \frac{1}{c} M^{\frac{1}{2}}\right) \|u\| \geq \sqrt{h}\|\Pi_{h,\mathbf{A}}u\|_{\partial\Omega}.$$

From (3.15), and the triangle inequality, we have

$$\left(1 - \frac{M}{\sqrt{2hb_0}}\right) \|u\| \leq \|\Pi_{h,\mathbf{A}}u\|.$$

By Remark 3.11, we see that, for h small enough, $1 - \frac{M}{\sqrt{2hb_0}} > 0$. Hence,

$$M^{\frac{1}{2}} \left(1 + h^{-\frac{3}{2}} \frac{1}{c} M^{\frac{1}{2}}\right) \left(1 - \frac{M}{\sqrt{2hb_0}}\right)^{-1} \|\Pi_{h,\mathbf{A}}u\| \geq \sqrt{h}\|\Pi_{h,\mathbf{A}}u\|_{\partial\Omega}.$$

Squaring this, and using Remark 3.11, we obtain the desired estimate. \square

Corollary 3.16. *For all $k \geq 1$, we have*

$$\nu_k(h) \leq \lambda_k^+(h)(1 + \mathcal{O}(h^\infty)).$$

Proof. Since $\Pi_{h,\mathbf{A}}\mathbb{1}_{W_h}$ is injective, we have $\dim \Pi_{h,\mathbf{A}}(W_h) = k$. Moreover, $\Pi_{h,\mathbf{A}}(W_h) \subset \mathcal{H}_{h,\mathbf{A}}^2(\Omega)$. The conclusion follows from Proposition 3.15 and the definition of $\nu_k(h)$. \square

4. HOMOGENEOUS DIRAC OPERATORS

This section is devoted to the proof of Theorem 1.23. After using the fibration induced by the partial Fourier transform with respect to the tangential variable, it is a consequence of Theorem 4.3. More formulas related to the fibered operators are established in Section 4.8 in our way of proving Theorem 1.19.

4.1. The fibered operators. Using the Fourier transform in the x_1 -direction, we are lead to introduce the following family of fiber operators that are one-dimensional Dirac operators.

Definition 4.1. Let $\xi \in \mathbb{R}$ be the Fourier variable in the x_1 -direction. The operators $\mathcal{D}_{\xi, \mathbb{R}}$ and $\mathcal{D}_{\xi, \mathbb{R}_+}$ act as

$$-i\sigma_2\partial_t + \sigma_1(\xi + t),$$

on

$$\text{Dom}(\mathcal{D}_{\xi, \mathbb{R}}) = B^1(\mathbb{R}, \mathbb{C}^2) := \{\varphi \in H^1(\mathbb{R}, \mathbb{C}^2), \quad t\varphi \in L^2(\mathbb{R}, \mathbb{C}^2)\}$$

and

$$\text{Dom}(\mathcal{D}_{\xi, \mathbb{R}_+}) = \{\varphi \in B^1(\mathbb{R}_+, \mathbb{C}^2), \quad \sigma_1\varphi(0) = \varphi(0)\},$$

where

$$B^1(\mathbb{R}_+, \mathbb{C}^2) := \{\varphi \in H^1(\mathbb{R}_+, \mathbb{C}^2), \quad t\varphi \in L^2(\mathbb{R}_+, \mathbb{C}^2)\}.$$

Let us state the main properties of these operators.

Proposition 4.2. *Let $\xi \in \mathbb{R}$. The operators $\mathcal{D}_{\xi, \mathbb{R}}$ and $\mathcal{D}_{\xi, \mathbb{R}_+}$ are self-adjoint with compact resolvent.*

Notation 8. $(\vartheta_{k, \mathbb{R}}^+(\xi))_{k \geq 1}$ and $(\vartheta_{k, \mathbb{R}_+}^+(\xi))_{k \geq 1}$ are the increasing sequences of the non-negative eigenvalues of $\mathcal{D}_{\xi, \mathbb{R}}$ and $\mathcal{D}_{\xi, \mathbb{R}_+}$. $(-\vartheta_{k, \mathbb{R}}^-(\xi))_{k \geq 1}$ and $(-\vartheta_{k, \mathbb{R}_+}^-(\xi))_{k \geq 1}$ are the decreasing sequences of the negative eigenvalues of $\mathcal{D}_{\xi, \mathbb{R}}$ and $\mathcal{D}_{\xi, \mathbb{R}_+}$. The eigenvalues are counted according to their multiplicity.

Let us present the properties of the dispersion curves.

Theorem 4.3. *Let $k \geq 1$. We have*

- (i) $\vartheta_{k, \mathbb{R}}^+(\xi) = \sqrt{2(k-1)}$ and $\vartheta_{k, \mathbb{R}}^-(\xi) = \sqrt{2k}$, for all $\xi \in \mathbb{R}$,
- (ii) $\xi \mapsto \vartheta_{k, \mathbb{R}_+}^+(\xi)$ is a regular increasing function with no critical point such that

$$\lim_{\xi \rightarrow -\infty} \vartheta_{k, \mathbb{R}_+}^+(\xi) = \sqrt{2(k-1)} \quad \text{and} \quad \lim_{\xi \rightarrow +\infty} \vartheta_{k, \mathbb{R}_+}^+(\xi) = +\infty,$$

- (iii) $\xi \mapsto \vartheta_{k, \mathbb{R}_+}^-(\xi)$ is a regular function with a single critical point ξ_k , that decreases on $(-\infty, \xi_k)$ and increases on $(\xi_k, +\infty)$ and such that

$$\lim_{\xi \rightarrow -\infty} \vartheta_{k, \mathbb{R}_+}^-(\xi) = +\infty \quad \text{and} \quad \lim_{\xi \rightarrow +\infty} \vartheta_{k, \mathbb{R}_+}^-(\xi) = \sqrt{2k}.$$

Moreover, $\xi_1 = a_0$ where $0 < a_0 := \min \vartheta_{1, \mathbb{R}_+}^- = \vartheta_{1, \mathbb{R}_+}^-(\xi_1) < \sqrt{2}$, and

$$\partial_\xi^2 \vartheta_1^-(\xi_1) = \frac{2a_0 u_{a_0, a_0}^2(0)}{2a_0 - u_{a_0, a_0}^2(0)} > 0. \quad (4.1)$$

Theorem 1.23 follows from Proposition 4.2, Theorem 4.3, Theorem XIII.86 and Theorem XIII.85 of [31].

4.2. Proof of Proposition 4.2. Let $\xi \in \mathbb{R}$. Let us prove the proposition for $\mathcal{D}_{\xi, \mathbb{R}_+}$. The proof for $\mathcal{D}_{\xi, \mathbb{R}}$ follows the same line. Since $\sigma_1 \xi$ is a bounded and symmetric matrix, $\mathcal{D}_{\xi, \mathbb{R}_+}$ is self-adjoint if and only if $\mathcal{D}_{0, \mathbb{R}_+}$ is self-adjoint. Let \mathcal{D} be the operator acting as $-i\sigma_2 \partial_t + \sigma_1 t$ on

$$\text{Dom}(\mathcal{D}) = \{\varphi \in H_0^1(\mathbb{R}_+, \mathbb{C}^2), \quad t\varphi \in L^2(\mathbb{R}_+, \mathbb{C}^2)\}.$$

By the anti-commutation relations of the Pauli matrices, the operator \mathcal{D} is symmetric and for all $\varphi \in \text{Dom}(\mathcal{D})$,

$$\begin{aligned} \|\mathcal{D}\varphi\|_{\mathbb{R}_+}^2 &= \|\partial_t \varphi\|_{\mathbb{R}_+}^2 + \|t\varphi\|_{\mathbb{R}_+}^2 + 2 \text{Re}\langle -i\sigma_2 \partial_t \varphi, \sigma_1 t \varphi \rangle \\ &= \|\partial_t \varphi\|_{\mathbb{R}_+}^2 + \|t\varphi\|_{\mathbb{R}_+}^2 + \int_0^{+\infty} t \partial_t \langle \varphi, \sigma_3 \varphi \rangle_{\mathbb{C}^2} dt \\ &= \|\partial_t \varphi\|_{\mathbb{R}_+}^2 + \|t\varphi\|_{\mathbb{R}_+}^2 - \langle \varphi, \sigma_3 \varphi \rangle. \end{aligned}$$

Hence, \mathcal{D} is closed. The adjoint \mathcal{D}^* acts as $-i\sigma_2 \partial_t + \sigma_1 t$ on

$$\text{Dom}(\mathcal{D}^*) = \{\varphi \in L^2(\mathbb{R}_+, \mathbb{C}^2), \quad (-i\sigma_2 \partial_t + \sigma_1 t)\varphi \in L^2(\mathbb{R}_+, \mathbb{C}^2)\}.$$

Studying the deficiency indices, we consider a solution $\varphi \in L^2(\mathbb{R}_+, \mathbb{C}^2)$ of

$$\mathcal{D}^* \varphi = i\varphi.$$

We get

$$(\mathcal{D}^*)^2 \varphi = (-\partial_t^2 + t^2 - \sigma_3) \varphi = -\varphi.$$

By [14, Eq. 12.2.2, 12.8.2, 12.8.3], we deduce that the only L^2 -solutions of $\mathcal{D}^* \varphi = i\varphi$ are of the form

$$t \mapsto a \begin{pmatrix} U(0, \sqrt{2}t) \\ \frac{i}{\sqrt{2}} U(1, \sqrt{2}t) \end{pmatrix} := a\varphi_+,$$

where $a \in \mathbb{C}$ and $U(0, \cdot)$, $U(1, \cdot)$ are parabolic cylinder functions. Similarly, the only L^2 -solutions of $\mathcal{D}^* \varphi = -i\varphi$ are of the form

$$t \mapsto b \begin{pmatrix} U(0, \sqrt{2}t) \\ \frac{-i}{\sqrt{2}} U(1, \sqrt{2}t) \end{pmatrix} := b\varphi_-,$$

where $b \in \mathbb{C}$. By [30, Theorem X.2], there is a one-to-one correspondance between the set of self-adjoint extensions of \mathcal{D} and the circle $\{e^{i\theta}, \theta \in \mathbb{R}\}$. The corresponding operators \mathcal{D}^θ act as $-i\sigma_2 \partial_t + \sigma_1 t$ on

$$\text{Dom}(\mathcal{D}^\theta) = \left\{ \varphi + a(\varphi_+ + e^{i\theta} \varphi_-), \quad \varphi \in \text{Dom}(\mathcal{D}), a \in \mathbb{C} \right\}.$$

We have $\mathcal{D}_{0, \mathbb{R}_+} = \mathcal{D}^{\theta_0}$ where $\theta_0 \in (-\pi, \pi)$ is the unique $\theta \in (-\pi, \pi]$ such that

$$\sigma_1(\varphi_+ + e^{i\theta} \varphi_-)(0) = \varphi_+(0) + e^{i\theta} \varphi_-(0).$$

This means that

$$U(1, 0) \frac{i}{\sqrt{2}} (1 - e^{i\theta}) = U(0, 0)(1 + e^{i\theta}),$$

so that $\theta \neq \pi$ (since $U(1, 0) \neq 0$) and

$$\frac{\sqrt{2}U(0, 0)}{U(1, 0)} = i \frac{1 - e^{i\theta}}{1 + e^{i\theta}} = i \frac{-2i \sin\left(\frac{\theta}{2}\right)}{2 \cos\left(\frac{\theta}{2}\right)} = \tan\left(\frac{\theta}{2}\right).$$

This ends the proof of Proposition 4.2.

4.3. Min-max characterization of the eigenvalues of the fibered operators.

Notation 9. Let $\alpha > 0, \xi \in \mathbb{R}$. We introduce

$$q_{\mathbb{R}, \xi}^{\pm}(u) = \int_{\mathbb{R}} |(\xi \pm t + \partial_t)u|^2 dt = \int_{\mathbb{R}} (|\partial_t u|^2 + |(\xi \pm t)u|^2 \mp |u|^2) dt,$$

for all $u \in B^1(\mathbb{R}, \mathbb{C})$ and

$$\begin{aligned} q_{\mathbb{R}_+, \alpha, \xi}^{\pm}(v) &= \int_{\mathbb{R}_+} |(\xi \pm t + \partial_t)v|^2 dt + \alpha|v(0)|^2 \\ &= \int_{\mathbb{R}_+} (|\partial_t v|^2 + |(\xi \pm t)v|^2 \mp |v|^2) dt + (\alpha - \xi)|v(0)|^2, \end{aligned}$$

for all $v \in B^1(\mathbb{R}_+, \mathbb{C})$.

These quadratic forms are closed and non-negative. The associated operators

$$\mathcal{M}_{\mathbb{R}, \xi}^{\pm} = -\partial_t^2 + (t \pm \xi)^2 \mp 1$$

and

$$\mathcal{M}_{\mathbb{R}_+, \alpha, \xi}^{\pm} = -\partial_t^2 + (t \pm \xi)^2 \mp 1$$

with the boundary condition $\varphi'(0) = (\alpha - \xi)\varphi(0)$ are self-adjoint with compact resolvent.

Note also that the family $(\mathcal{M}_{\mathbb{R}_+, \alpha, \xi}^{\pm})_{\alpha > 0, \xi \in \mathbb{R}}$ is of type (B) in the sense of Kato:

- (i) $\text{Dom}(q_{\mathbb{R}_+, \alpha, \xi}^{\pm}) = B^1(\mathbb{R}_+)$ does not depend on ξ or α ,
- (ii) for all $u \in B^1(\mathbb{R}_+)$, $(\alpha, \xi) \mapsto q_{\mathbb{R}_+, \alpha, \xi}^{\pm}(u)$ is analytic.

Remark 4.4. For $\alpha > 0$, the operator $\mathcal{M}_{\mathbb{R}_+, \alpha, \alpha}^{\pm} \pm 1$ coincides with the famous de Gennes operator (see [18]).

Notation 10. Let $\alpha > 0, \xi \in \mathbb{R}, k \in \mathbb{N} \setminus \{0\}$. We introduce

$$\nu_{\mathbb{R}, k}^{\pm}(\xi) = \inf_{\substack{V \subset B^1(\mathbb{R}) \\ \dim V = k}} \sup_{u \in V \setminus \{0\}} \frac{q_{\mathbb{R}, \xi}^{\pm}(u)}{\|u\|_{\mathbb{R}}^2},$$

and

$$\nu_{\mathbb{R}_+, k}^{\pm}(\alpha, \xi) = \inf_{\substack{V \subset B^1(\mathbb{R}_+) \\ \dim V = k}} \sup_{u \in V \setminus \{0\}} \frac{q_{\mathbb{R}_+, \alpha, \xi}^{\pm}(u)}{\|u\|_{\mathbb{R}_+}^2}.$$

The arguments of Section 2 can be easily adapted to this setting and imply that

$$\vartheta_{k, \mathbb{R}}^{\pm}(\xi) = \sqrt{\nu_{\mathbb{R}, k}^{\pm}(\xi)},$$

and $\vartheta_{k, \mathbb{R}_+}^\pm(\xi)$ is the only solution $\alpha > 0$ of

$$\nu_{\mathbb{R}_+, k}^\pm(\alpha, \xi) = \alpha^2,$$

where $\vartheta_{k, \mathbb{R}}^\pm(\xi)$ and $\vartheta_{k, \mathbb{R}_+}^\pm(\xi)$ are defined in Notation 8.

By translation invariance, we have

$$\nu_{\mathbb{R}, k}^\pm(\xi) = \nu_{\mathbb{R}, k}^\pm(0) \text{ for } k \geq 1 \text{ and } \xi \in \mathbb{R}.$$

These are related with the eigenvalues of the harmonic oscillator and Point (i) of Theorem 4.3 follows.

4.4. About the dispersion curves $\nu_{\mathbb{R}_+, k}^\pm$. In this section, we prove the following proposition concerning the dispersion curves $\nu_{\mathbb{R}_+, k}^\pm$.

Proposition 4.5. *Let $\alpha > 0$ and $k \geq 1$.*

- (i) *The function $\xi \mapsto \nu_{\mathbb{R}_+, k}^+(\alpha, \xi)$ is smooth, has no critical point, is increasing, and tends to $+\infty$ as $\xi \rightarrow +\infty$ and to $2(k-1)$ as $\xi \rightarrow -\infty$.*
- (ii) *The function $\xi \mapsto \nu_{\mathbb{R}_+, k}^-(\alpha, \xi)$ is smooth, has a unique critical point, which is a minimum ξ_α , decreases on $(-\infty, \xi_\alpha)$ and increases on $(\xi_\alpha, +\infty)$, tends to $+\infty$ as $\xi \rightarrow -\infty$ and to $2k$ as $\xi \rightarrow +\infty$. Moreover, $\nu_{\mathbb{R}_+, 1}^-(\alpha, \xi_\alpha) < 2$.*

Remark 4.6. Actually, to prove $\nu_{\mathbb{R}_+, 1}^-(\alpha, \xi_\alpha) < 2$, one could avoid the asymptotic analysis by using the knowledge of the de Gennes function. Consider $\xi = \alpha > 0$. Then,

$$\nu_{\mathbb{R}_+, 1}^-(\alpha, \alpha) = \mu(\alpha) + 1,$$

where μ is the celebrated de Gennes function. We know that, on \mathbb{R}_+ , $\mu < 1$. Thus, for all $\alpha > 0$,

$$\nu_{\mathbb{R}_+, 1}^-(\alpha, \xi_\alpha) \leq \mu(\alpha) + 1 < 2 = (\sqrt{2})^2.$$

Let $\alpha > 0$ and $k \geq 1$ be fixed. In this part, we remove the subscript \mathbb{R}_+ for the sake of notation simplicity. By the analytic perturbation theory, we know that $\nu_k^\pm(\cdot, \cdot)$ are analytic functions of α and ξ . Let $\xi \mapsto \nu^\pm(\alpha, \xi)$ be one the analytic branch of eigenvalues of $\mathcal{M}_{\alpha, \xi}^\pm$ and $u_{\alpha, \xi}^\pm$ is a corresponding normalized eigenfunction.

The following elementary lemma will be used many times in this section.

Lemma 4.7. *We have*

$$\langle -\psi'', \varphi \rangle = \langle \psi, -\varphi'' \rangle + \psi'(0)\varphi(0) - \psi(0)\varphi'(0).$$

Proof. We have

$$\langle -\psi'', \varphi \rangle = \langle \psi', \varphi' \rangle + \psi'(0)\varphi(0) = -\langle \psi, \varphi'' \rangle + \psi'(0)\varphi(0) - \psi(0)\varphi'(0).$$

□

In the following lemmas, we compute derivatives of ν^\pm with respect to α and ξ .

Lemma 4.8. *We have*

$$\partial_\xi \nu^\pm(\alpha, \xi) = \int_0^{+\infty} 2(\xi \pm t) u_{\alpha, \xi}^2(t) dt - u_{\alpha, \xi}(0)^2,$$

and

$$(\mathcal{M}_{\alpha, \xi}^\pm - \nu^\pm) v_{\alpha, \xi} = (\partial_\xi \nu^\pm) u_{\alpha, \xi} - 2(\xi \pm t) u_{\alpha, \xi}, \quad v_{\alpha, \xi} = \partial_\xi u_{\alpha, \xi},$$

with

$$(\partial_t + \xi - \alpha)v_{\alpha,\xi}(0) = -u_{\alpha,\xi}(0).$$

Proof. We have

$$(\mathcal{M}_{\alpha,\xi}^\pm - \nu^\pm)u_{\alpha,\xi} = 0.$$

Then,

$$(\mathcal{M}_{\alpha,\xi}^\pm - \nu^\pm)\partial_\xi u_{\alpha,\xi} + \partial_\xi \mathcal{M}_{\alpha,\xi}^\pm u_{\alpha,\xi} = \partial_\xi \nu^\pm(\xi)u_{\alpha,\xi},$$

so that

$$\langle (\mathcal{M}_{\alpha,\xi}^\pm - \nu^\pm)\partial_\xi u_{\alpha,\xi}, u_{\alpha,\xi} \rangle + 2 \int_0^{+\infty} (\xi \pm t)u_{\alpha,\xi}^2(t)dt = \partial_\xi \nu^\pm(\xi).$$

By integrations by parts, we have

$$\langle (\mathcal{M}_{\alpha,\xi}^\pm - \nu^\pm)\partial_\xi u_{\alpha,\xi}, u_{\alpha,\xi} \rangle = \partial_t \partial_\xi u_{\alpha,\xi}(0)u_{\alpha,\xi}(0) - \partial_\xi u_{\alpha,\xi}(0)\partial_t u_{\alpha,\xi}(0).$$

Note that

$$\partial_\xi \partial_t u_{\alpha,\xi}(0) = -u_{\alpha,\xi}(0) + (\alpha - \xi)\partial_\xi u_{\alpha,\xi}(0).$$

Thus,

$$\langle (\mathcal{M}_{\alpha,\xi}^\pm - \nu^\pm)\partial_\xi u_{\alpha,\xi}, u_{\alpha,\xi} \rangle = -u_{\alpha,\xi}(0)^2.$$

□

In the following lemma, we focus on the second derivative in ξ .

Lemma 4.9. *We have*

$$(\mathcal{M}_{\alpha,\xi}^\pm - \nu^\pm)w_{\alpha,\xi} = 2(\partial_\xi \nu^\pm)v_{\alpha,\xi} + \partial_\xi^2 \nu^\pm u_{\alpha,\xi} - 4(\xi \pm t)v_{\alpha,\xi} - 2u_{\alpha,\xi}, \quad w_{\alpha,\xi} = \partial_\xi^2 u_{\alpha,\xi},$$

with

$$(\partial_t + \xi - \alpha)w_{\alpha,\xi}(0) = -2v_{\alpha,\xi}(0).$$

Moreover,

$$-\frac{\partial_\xi^2 \nu}{2} + 1 + 2\langle (\xi \pm t)v_{\alpha,\xi}, u_{\alpha,\xi} \rangle = u_{\alpha,\xi}(0)v_{\alpha,\xi}(0).$$

Proof. We have

$$\langle (\mathcal{M}_{\alpha,\xi}^\pm - \nu^\pm)w_{\alpha,\xi}, u_{\alpha,\xi} \rangle = \partial_t w_{\alpha,\xi}(0)u_{\alpha,\xi}(0) - w_{\alpha,\xi}(0)\partial_t u_{\alpha,\xi}(0) = -2u_{\alpha,\xi}(0)v_{\alpha,\xi}(0).$$

In addition,

$$\langle (\mathcal{M}_{\alpha,\xi}^\pm - \nu^\pm)w_{\alpha,\xi}, u_{\alpha,\xi} \rangle = \partial_\xi^2 \nu^\pm - 2 - 4\langle (\xi \pm t)v_{\alpha,\xi}, u_{\alpha,\xi} \rangle.$$

□

In the next lemma, we explicitly use the α -dependence of the eigenfunctions.

Lemma 4.10. *We have*

$$\partial_\alpha \nu^\pm(\alpha, \xi) = u_{\alpha,\xi}^2(0),$$

and

$$(\mathcal{M}_{\alpha,\xi}^\pm - \nu^\pm(\alpha, \xi))\partial_\alpha u_{\alpha,\xi} = \partial_\alpha \nu^\pm(\alpha, \xi)u_{\alpha,\xi},$$

with

$$\partial_\alpha \partial_t u_{\alpha,\xi}(0) = u_{\alpha,\xi}(0) + (\alpha - \xi)\partial_\alpha u_{\alpha,\xi}(0).$$

Proof. We get

$$\partial_\alpha \nu^\pm(\alpha, \xi) = \langle (\mathcal{M}_{\alpha, \xi}^\pm - \nu^\pm(\alpha, \xi)) \partial_\alpha u_{\alpha, \xi}, u_{\alpha, \xi} \rangle = \partial_t \partial_\alpha u_{\alpha, \xi}(0) u_{\alpha, \xi}(0) - \partial_\alpha u_{\alpha, \xi}(0) \partial_t u_{\alpha, \xi}(0),$$

and the conclusion follows. \square

A consequence (which will be used later) of the previous lemma is the following.

Lemma 4.11. *We have*

$$\partial_\alpha^2 \nu^\pm(\alpha, \xi) = -2u_{\alpha, \xi}(0) v_{\alpha, \xi}(0) + 4\langle (\xi \pm t) u_{\alpha, \xi}, \partial_\alpha u_{\alpha, \xi} \rangle, \quad v_{\alpha, \xi} := \partial_\xi u_{\alpha, \xi}.$$

Proof. Thanks to Lemma 4.8, and using that $\langle u_{\alpha, \xi}, \partial_\alpha u_{\alpha, \xi} \rangle = 0$, we have

$$\langle (\mathcal{M}_{\alpha, \xi}^\pm - \nu^\pm) v_{\alpha, \xi}, \partial_\alpha u_{\alpha, \xi} \rangle = -2\langle (\xi \pm t) u_{\alpha, \xi}, \partial_\alpha u_{\alpha, \xi} \rangle.$$

Integrating by parts, we get using Lemmas 4.8 and 4.10,

$$\begin{aligned} -2\langle (\xi \pm t) u_{\alpha, \xi}, \partial_\alpha u_{\alpha, \xi} \rangle &= \partial_t v_{\alpha, \xi}(0) \partial_\alpha u_{\alpha, \xi}(0) - v_{\alpha, \xi}(0) \partial_t \partial_\alpha u_{\alpha, \xi}(0) \\ &= ((\alpha - \xi) v_{\alpha, \xi}(0) - u_{\alpha, \xi}(0)) \partial_\alpha u_{\alpha, \xi}(0) - v_{\alpha, \xi}(0) ((\alpha - \xi) \partial_\alpha u_{\alpha, \xi}(0) + u_{\alpha, \xi}(0)) \\ &= -u_{\alpha, \xi}(0) \partial_\alpha u_{\alpha, \xi}(0) - u_{\alpha, \xi}(0) v_{\alpha, \xi}(0). \end{aligned}$$

Lemma 4.10 gives

$$2u_{\alpha, \xi}(0) \partial_\alpha u_{\alpha, \xi}(0) = \partial_\alpha^2 \nu^\pm(\alpha, \xi),$$

and the conclusion follows. \square

Let us study the critical points of $\xi \mapsto \nu^\pm(\alpha, \xi)$.

Proposition 4.12. *We have*

$$\partial_\xi \nu^\pm(\alpha, \xi) = \pm (\nu^\pm(\alpha, \xi) + \alpha^2 - 2\alpha\xi) u_{\alpha, \xi}^2(0). \quad (4.2)$$

In particular, if ξ_α is a critical point of $\nu^\pm(\alpha, \cdot)$, we have

$$\nu^\pm(\alpha, \xi_\alpha) = -\alpha^2 + 2\alpha\xi_\alpha. \quad (4.3)$$

Moreover,

$$\partial_\xi^2 \nu^\pm(\alpha, \xi_\alpha) = \mp 2\alpha u_{\alpha, \xi_\alpha}^2(0). \quad (4.4)$$

All the critical points of ν^- are local non-degenerate minima, and all the critical points of ν^+ are local non-degenerate maxima. In particular, there is at most one critical point. If such a point exists for $\nu^+(\alpha, \cdot)$, then $\nu^+(\alpha, \cdot)$ is bounded from above.

Proof. With Lemma 4.8, we get

$$\begin{aligned} \partial_\xi \nu^\pm(\alpha, \xi) &= \pm \int_0^{+\infty} \partial_t [(\xi \pm t)^2] u_{\alpha, \xi}^2(t) dt - u_{\alpha, \xi}(0)^2 \\ &= \mp 2 \int_0^{+\infty} (\xi - t)^2 u_{\alpha, \xi}(t) u'_{\alpha, \xi}(t) dt \mp \xi^2 u_{\alpha, \xi}(0)^2 - u_{\alpha, \xi}(0)^2 \\ &= \mp 2 \int_0^{+\infty} (u''_{\alpha, \xi}(t) + (\nu^\pm(\alpha, \xi) \pm 1) u_{\alpha, \xi}(t) u'_{\alpha, \xi}(t)) dt \mp \xi^2 u_{\alpha, \xi}(0)^2 - u_{\alpha, \xi}(0)^2 \\ &= \mp \int_0^{+\infty} \partial_t \left((u'_{\alpha, \xi})^2 + (\nu^\pm(\alpha, \xi) \pm 1) u_{\alpha, \xi}^2 \right) dt \mp \xi^2 u_{\alpha, \xi}(0)^2 - u_{\alpha, \xi}(0)^2 \\ &= (\pm (\nu^\pm(\alpha, \xi) \pm 1) \pm (\alpha - \xi)^2 \mp \xi^2 - 1) u_{\alpha, \xi}(0)^2 \\ &= (\pm \nu^\pm(\alpha, \xi) \pm \alpha^2 \mp 2\alpha\xi) u_{\alpha, \xi}(0)^2. \end{aligned}$$

We get (4.3). Taking the derivative of (4.3), we deduce (4.4). The last sentence follows from (4.3) and (4.4). \square

Next, we show that $\nu^-(\alpha, \cdot)$ has always a critical point.

Corollary 4.13. *For all $j \geq 1$, the function $\nu_j^-(\alpha, \cdot)$ has a unique critical point $\xi_{j,\alpha}^-$, and it is a non-degenerate minimum. The function $\xi \mapsto \nu_j^-(\alpha, \xi)$ is decreasing on $(-\infty, \xi_{j,\alpha}^-)$ and is increasing on $(\xi_{j,\alpha}^-, +\infty)$.*

Proof. If $\nu_j^-(\alpha, \cdot)$ has no critical points, then it is decreasing (it is decreasing on $(-\infty, 0)$ by Proposition 4.12). From Proposition 4.12, we deduce that, for all $\xi \geq 0$,

$$-\nu_j^-(\alpha, \xi) - \alpha^2 + 2\alpha\xi \leq 0,$$

and that $\lim_{\xi \rightarrow +\infty} \nu_j^-(\alpha, \xi) = +\infty$. This is in contradiction with the function being decreasing. This shows that $\nu_j^-(\alpha, \cdot)$ has a unique critical point. It is a *local* non-degenerate minimum. Since there is only one critical point, this shows that it is a global minimum. \square

Let us study the asymptotic behavior of ν_k^\pm .

Lemma 4.14. *Let $k \geq 1$. We have that*

$$\lim_{\xi \rightarrow +\infty} \nu_k^+(\alpha, \xi) = \lim_{\xi \rightarrow -\infty} \nu_k^-(\alpha, \xi) = +\infty,$$

and

$$\lim_{\xi \rightarrow -\infty} \nu_k^+(\alpha, \xi) = 2(k-1), \quad \lim_{\xi \rightarrow +\infty} \nu_k^-(\alpha, \xi) = 2k.$$

Proof. Let us first remark that for $\xi < 0$ and $u \in B^1(\mathbb{R}_+)$, we have

$$q_{\alpha,\xi}^-(u) \geq (\xi^2 - 1)\|u\|_{\mathbb{R}_+}^2,$$

and

$$\lim_{\xi \rightarrow -\infty} \nu_k^-(\alpha, \xi) = +\infty.$$

For $\xi > 0$ and $u \in B^1(\mathbb{R}_+) \setminus \{0\}$, we denote by $\tau = \xi t$ and $v(\tau) = u(t)$ so that

$$\begin{aligned} \frac{q_{\alpha,\xi}^+(u)}{\|u\|_{\mathbb{R}_+}^2} &= \frac{\int_{\mathbb{R}_+} (|\partial_t u|^2 + |(\xi + t)u|^2 - |u|^2) dt + (\alpha - \xi)|u(0)|^2}{\|u\|_{\mathbb{R}_+}^2} \\ &= \frac{\int_{\mathbb{R}_+} \left(\xi^2 |\partial_\tau v|^2 + |(\xi + \frac{\tau}{\xi})v|^2 - |v|^2 \right) d\tau + \xi(\alpha - \xi)|v(0)|^2}{\|v\|_{\mathbb{R}_+}^2} \\ &= \xi^2 \frac{\int_{\mathbb{R}_+} \left(|\partial_\tau v|^2 + |(1 + \frac{\tau}{\xi^2})v|^2 - \xi^{-2}|v|^2 \right) d\tau + (\frac{\alpha}{\xi} - 1)|v(0)|^2}{\|v\|_{\mathbb{R}_+}^2} \\ &\geq \xi^2 \left(1 - \xi^{-2} + \frac{\int_{\mathbb{R}_+} |\partial_\tau v|^2 d\tau + (\frac{\alpha}{\xi} - 1)|v(0)|^2}{\|v\|_{\mathbb{R}_+}^2} \right). \end{aligned}$$

Let us consider the Robin Laplacian associated with the quadratic form

$$v \mapsto \int_{\mathbb{R}_+} |\partial_\tau v|^2 d\tau + \left(\frac{\alpha}{\xi} - 1 \right) |v(0)|^2.$$

Its essential spectrum is $[0, +\infty)$ and the only point in the negative spectrum is the eigenvalue $-(1 - \alpha/\xi)^2$ whose eigenspace is spanned by $\tau \mapsto e^{-(1-\alpha/\xi)\tau}$. We get

$$\frac{q_{\alpha,\xi}^+(u)}{\|u\|_{\mathbb{R}_+}^2} \geq \xi^2 (1 - \xi^{-2} - (1 - \alpha/\xi)^2) = 2\alpha\xi - (1 + \alpha^2),$$

and thus

$$\lim_{\xi \rightarrow +\infty} \nu_k^+(\alpha, \xi) = +\infty.$$

Let $\xi < 0$. Let us consider $u \in B^1(\mathbb{R}_+)$, we define $u(t) = v(\xi + t)$

$$\begin{aligned} \frac{q_{\alpha,\xi}^+(u)}{\|u\|_{\mathbb{R}_+}^2} &= \frac{\int_{\mathbb{R}_+} (|\partial_t u|^2 + |(\xi + t)u|^2 - |u|^2) dt + (\alpha - \xi)|u(0)|^2}{\|u\|_{\mathbb{R}_+}^2} \\ &= \frac{\int_{\xi}^{+\infty} (|\partial_\tau v|^2 + |\tau v|^2 - |v|^2) dt + (\alpha - \xi)|v(\xi)|^2}{\|v\|_{(\xi, +\infty)}^2}. \end{aligned}$$

Using truncated Hermite's functions as test functions, we get that

$$\limsup_{\xi \rightarrow -\infty} \nu_k^+(\alpha, \xi) \leq 2(k - 1). \quad (4.5)$$

Since

$$\frac{q_{\alpha,\xi}^+(u)}{\|u\|_{\mathbb{R}_+}^2} \geq \frac{\int_{\xi}^{+\infty} (|\partial_\tau v|^2 + |\tau v|^2 - |v|^2) dt}{\|v\|_{(\xi, +\infty)}^2},$$

the eigenvalue $\nu_k^+(\alpha, \xi)$ is larger than the k -th eigenvalue of the operator $-\partial_\tau^2 + \tau^2 - 1$ with Neumann boundary condition on $(\xi, +\infty)$. In other words,

$$\nu_k^+(\alpha, \xi) \geq \mu_k(\xi) - 1,$$

where μ_k is the k -th dispersion curve of the de Gennes operator. It is well-known that ¹

$$\lim_{\xi \rightarrow -\infty} \mu_k(\xi) = 2k - 1.$$

Thus,

$$\liminf_{\xi \rightarrow -\infty} \nu_k^+(\alpha, \xi) \geq 2(k - 1).$$

As in (4.5), using Hermite's functions, we get that

$$\limsup_{\xi \rightarrow +\infty} \nu_k^-(\alpha, \xi) \leq 2k. \quad (4.6)$$

Let $\xi > \alpha/2$. Let us consider $u, v \in B^1(\mathbb{R}_+)$ such that $u(t) = v(\xi t)$. We have

$$\begin{aligned} q_{\alpha,\xi}^-(u) &= \int_{\mathbb{R}_+} (|\partial_t u|^2 + |(t - \xi)u|^2 + |u|^2) dt + (\alpha - \xi)|u(0)|^2 \\ &= \int_{\alpha/4}^{+\infty} (|\partial_t u|^2 + |(t - \xi)u|^2 + |u|^2) dt \\ &\quad + \xi^{-1} \int_0^{\xi\alpha/4} (\xi^2 |\partial_\tau v|^2 + |(\tau/\xi - \xi)v|^2 + |v|^2) d\tau + (\alpha - \xi)|v(0)|^2, \end{aligned}$$

¹see, for instance, [18, Prop. 3.2.2 & 3.2.4]

and

$$\begin{aligned} & \xi^{-1} \int_0^{\alpha\xi/4} (\xi^2 |\partial_\tau v|^2 + |(\tau/\xi - \xi)v|^2 + |v|^2) d\tau + (\alpha - \xi)|v(0)|^2 \\ & \geq \xi \left(\int_0^{\alpha\xi/4} |\partial_\tau v|^2 d\tau + (\alpha/\xi - 1)|v(0)|^2 \right) + \xi^{-1}(1 + (\xi - \alpha/4)^2) \int_0^{\alpha\xi/4} |v|^2 d\tau. \end{aligned}$$

By studying the spectrum of the Robin-Neumann Laplacian whose quadratic form is

$$v \mapsto \int_0^{\alpha\xi/4} |\partial_\tau v|^2 d\tau + (\alpha/\xi - 1)|v(0)|^2,$$

we get,

$$\xi \left(\int_0^{\alpha\xi/4} |\partial_\tau v|^2 d\tau + (\alpha/\xi - 1)|v(0)|^2 \right) \geq \xi(-1 + \alpha/\xi + o(\xi^{-1})) \int_0^{\alpha\xi/4} |v|^2 d\tau.$$

We deduce that

$$\begin{aligned} & \xi^{-1} \int_0^{\xi\alpha/4} (\xi^2 |\partial_\tau v|^2 + |(\tau/\xi - \xi)v|^2 + |v|^2) d\tau + (\alpha - \xi)|v(0)|^2 \\ & \geq (\alpha/2 + o(1)) \int_0^{\alpha\xi/4} |v|^2 d\tau, \end{aligned}$$

so that

$$q_{\alpha,\xi}^-(u) \geq \int_{\alpha/4}^{+\infty} (|\partial_t u|^2 + |(t - \xi)u|^2 + |u|^2) dt + \xi(\alpha/2 + o(1)) \int_0^{\alpha/4} |u|^2 dt. \quad (4.7)$$

Let $(u_{1,\xi}, \dots, u_{k,\xi})$ be an orthonormal family of eigenfunctions of $\mathcal{M}_{\alpha,\xi}^-$ associated with the k first eigenvalues.

We have, for all $u \in \text{span}_{1 \leq j \leq k} u_{j,\xi}$,

$$\begin{aligned} \nu_k^-(\alpha, \xi) \|u\|^2 & \geq q_{\alpha,\xi}^-(u) \geq \int_{\frac{\alpha}{4}}^{+\infty} (|\partial_t u|^2 + |(t - \xi)u|^2 + |u|^2) dt + \xi \left(\frac{\alpha}{2} + o(1) \right) \int_0^{\frac{\alpha}{4}} |u|^2 dt \\ & \geq \xi \left(\frac{\alpha}{2} + o(1) \right) \int_0^{\alpha/4} |u|^2 dt. \end{aligned}$$

By (4.7) and the upper bound (4.6), we get that for ξ large enough, the family of the restrictions of $u_{1,\xi}, \dots, u_{k,\xi}$ to the interval $(\alpha/4, +\infty)$ is of dimension k and

$$(1 + o(1)) \nu_k^-(\alpha, \xi) \geq \inf_{\substack{V \subset B^1(\alpha/4, +\infty) \\ \dim V = k}} \sup_{\substack{u \in V \\ \|u\|_{(\alpha/4, +\infty)} = 1}} \int_{\alpha/4}^{+\infty} (|\partial_t u|^2 + |(t - \xi)u|^2 + |u|^2) dt.$$

In the same way as for the de Gennes operator ($\alpha = 0$), we get

$$\lim_{\xi \rightarrow +\infty} \nu_k^-(\alpha, \xi) = 2k,$$

and the conclusion follows. \square

End of the proof of Proposition 4.5. By Proposition 4.12 and Lemma 4.14, we get that $\xi \mapsto \nu_k^+(\alpha, \xi)$ has no critical point. Using again Lemma 4.14, we deduce that $\xi \mapsto \nu_k^+(\alpha, \xi)$ is increasing.

The behavior of $\xi \mapsto \nu_k^-(\alpha, \xi)$ is described in Corollary 4.13. The monotonicity of $\xi \mapsto \nu_1^-(\alpha, \xi)$ and its limit in $+\infty$ ensure that $\nu_1^-(\alpha, \xi_\alpha) < 2$.

4.5. Proof of points (ii) and (iii) of Theorem 4.3. In this part, we remove the subscript \mathbb{R}_+ for the sake of notation simplicity. Let $k \geq 1$ and $\xi \in \mathbb{R}$. By the min-max characterization of $\vartheta_k^\pm(\xi)$, we have that

$$\begin{aligned} 0 &< \nu_k^\pm(\alpha, \xi) - \alpha^2, \text{ for all } 0 < \alpha < \vartheta_k^\pm(\xi), \\ 0 &> \nu_k^\pm(\alpha, \xi) - \alpha^2, \text{ for all } \alpha > \vartheta_k^\pm(\xi), \\ 0 &= \nu_k^\pm(\vartheta_k^\pm(\xi), \xi) - \vartheta_k^\pm(\xi)^2. \end{aligned}$$

4.5.1. *Limits of ϑ_k^\pm .* Let $\alpha > 0$ be fixed. By Proposition 4.5, there is $M \in \mathbb{R}$ such that for all $\xi > M$

$$\nu_k^+(\alpha, \xi) - \alpha^2 > 0.$$

Hence, $\alpha < \vartheta_k^+(\xi)$. This shows that

$$\lim_{\xi \rightarrow +\infty} \vartheta_k^+(\xi) = +\infty.$$

The same kind of arguments ensure that

$$\begin{aligned} \lim_{\xi \rightarrow -\infty} \vartheta_k^+(\xi) &= \sqrt{2(k-1)} \\ \lim_{\xi \rightarrow -\infty} \vartheta_k^-(\xi) &= +\infty \text{ and } \lim_{\xi \rightarrow +\infty} \vartheta_k^-(\xi) = \sqrt{2k}. \end{aligned}$$

4.5.2. *Regularity of ϑ_k^\pm .* We have, for all $\xi \in \mathbb{R}$,

$$\nu_k^\pm(\vartheta_k^\pm(\xi), \xi) - \vartheta_k^\pm(\xi)^2 = 0.$$

Let us explain why ϑ_k^\pm is smooth. Consider the function

$$F(\alpha, \xi) = \nu_k^\pm(\alpha, \xi) - \alpha^2.$$

We have

$$\partial_\alpha F(\alpha, \xi) = \partial_\alpha \nu_k^\pm(\alpha, \xi) - 2\alpha.$$

By Lemma 4.10, we get

$$\partial_\alpha F(\alpha, \xi) = [u_{k,\alpha,\xi}^\pm(0)]^2 - 2\alpha.$$

Let us analyze the sign of $\partial_\alpha F(\vartheta_k^\pm(\xi), \xi)$. Notice that $P : \alpha \mapsto q_{\alpha,\xi}^\pm(u_{k,\vartheta_k^\pm(\xi),\xi}^\pm) - \alpha^2$ is a polynomial of degree 2, which is zero at $\alpha = \vartheta_k^\pm(\xi)$. Moreover, by the min-max principle, we have

$$P(\alpha) \geq \nu_k^\pm(\alpha, \xi) - \alpha^2,$$

so that, for all $\alpha \in (0, \vartheta_k^\pm(\xi))$, $P(\alpha) > 0$. It follows that $P'(\vartheta_k^\pm(\xi)) < 0$. Since

$$P'(\vartheta_k^\pm(\xi)) = [u_{k,\alpha,\xi}^\pm(0)]^2 - 2\alpha = \partial_\alpha F(\vartheta_k^\pm(\xi), \xi),$$

we get that $\partial_\alpha F(\vartheta_k^\pm(\xi), \xi) < 0$. With the Implicit Function Theorem, we deduce that ϑ_k^\pm is a smooth function (since F is smooth). Moreover, the derivative of the implicit function is given by the usual formula

$$\partial_\xi \vartheta_k^\pm(\xi) = -\frac{\partial_\xi F(\vartheta_k^\pm(\xi), \xi)}{\partial_\alpha F(\vartheta_k^\pm(\xi), \xi)} = \frac{\partial_\xi \nu_k^\pm(\vartheta_k^\pm(\xi), \xi)}{2\vartheta_k^\pm(\xi) - \partial_\alpha \nu_k^\pm(\vartheta_k^\pm(\xi), \xi)}, \quad (4.8)$$

where we see that the denominator of the last expression is a positive function.

4.5.3. Critical points of ϑ_k^\pm . We deduce that $\partial_\xi \vartheta_k^\pm(\xi)$ has the sign of $\partial_\xi \nu_k^\pm(\vartheta_k^\pm(\xi), \xi)$. Thus, by Proposition 4.5, $\xi \mapsto \vartheta_k^\pm(\xi)$ increases and has no critical points.

Moreover, ξ_0 is a critical point of $\xi \mapsto \vartheta_k^-(\xi)$ if and only if ξ_0 is a (actually *the*) critical point of $\xi \mapsto \nu_k^-(\vartheta_k^-(\xi_0), \xi)$. Let $\xi_0 \in \mathbb{R}$ be such a critical point. By (4.8) and Proposition 4.12, we have

$$\partial_\xi^2 \vartheta_k^-(\xi_0) = \frac{[\partial_\xi^2 \nu_k^-](\vartheta_k^-(\xi_0), \xi_0)}{2\vartheta_k^-(\xi_0) - \partial_\alpha \nu_k^-(\vartheta_k^-(\xi_0), \xi_0)} > 0.$$

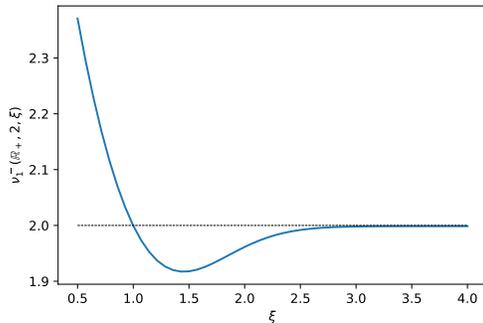
Hence, there is at most one critical point which is a non-degenerate minimum. The function ϑ_k^- has exactly one minimum ξ_k , increases on $(\xi_k, +\infty)$ and decreases on $(-\infty, \xi_k)$. For $\alpha = \vartheta_1^-(\xi_1)$, we also get that ξ_1 is the minimum of the function $\xi \mapsto \nu_1^-(\alpha, \xi)$. By Proposition 4.12,

$$0 = \nu_1^-(\alpha, \xi_1) - \alpha^2 = -2\alpha^2 + 2\alpha\xi_1,$$

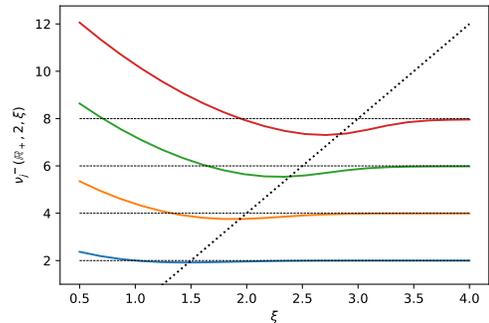
so that $\xi_1 = \alpha = \vartheta_1^-(\xi_1)$. Using again Proposition 4.12 and Lemma 4.10, we get

$$\partial_\xi^2 \vartheta_1^-(\xi_1) = \frac{2\alpha u_{\alpha, \alpha}(0)^2}{2\alpha - u_{\alpha, \alpha}(0)^2} > 0.$$

4.6. Numerical illustrations. By using naive finite difference method and dichotomy method, it is possible to compute the eigenvalues $\nu_{\mathbb{R}_+, k}^\pm(\alpha, \cdot)$ and $\vartheta_{k, \mathbb{R}_+}^\pm$ by using a short Python script. Subfigures (a) and (b) of Figure 1 below present $\nu_{\mathbb{R}_+, k}^-(\alpha, \cdot)$ in colored lines. The horizontal dashed lines represent the Landau levels and the dotted affine line of (b) is the graph of the function $\xi \mapsto -\alpha^2 + 2\alpha\xi$ (see Proposition 4.12).



(a) The function $\nu_{\mathbb{R}_+, 1}^-(2, \cdot)$



(b) Functions $\nu_{\mathbb{R}_+, k}^-(2, \cdot)$ and the function $\xi \mapsto -\alpha^2 + 2\alpha\xi$

Figure 1. Dispersion curves $\nu_{\mathbb{R}_+, k}^-$

Figure 2 displays the increasing behavior of the $\nu_{\mathbb{R}_+,j}^+(\alpha, \cdot)$ and the associated Landau levels $2(j-1)$ for $j \geq 1$.

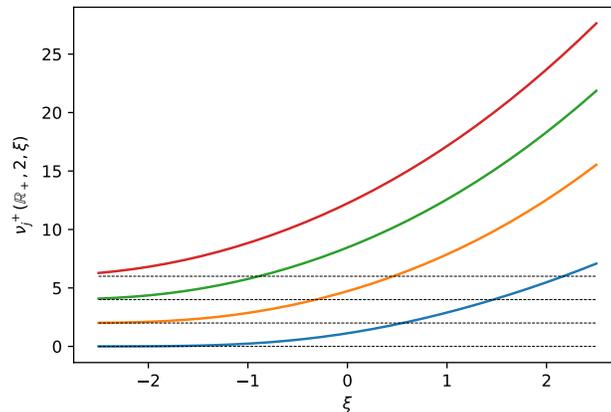


Figure 2. The functions $\nu_{\mathbb{R}_+,j}^+(2, \cdot)$

Figure 3 shows the dispersion curves $\pm\vartheta_{\mathbb{R}_+,j}^\pm$ representing the spectrum of the fibered Dirac operators $\mathcal{D}_{\xi, \mathbb{R}_+}$ with the associated Landau levels $\pm\sqrt{2k}$ as dashed lines.

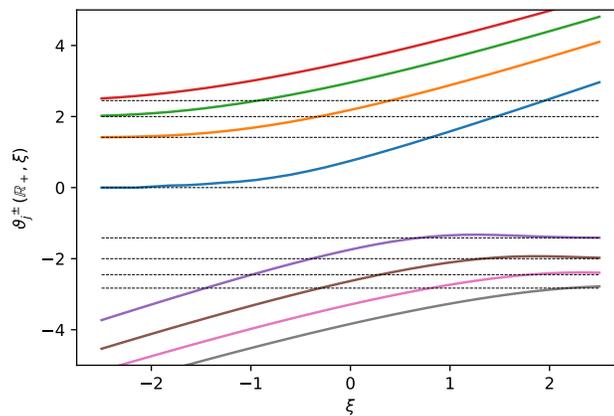


Figure 3. Functions $\pm\vartheta_{\mathbb{R}_+,j}^\pm$

All these simulations agree with all our theoretical results.

4.7. **On the function ν .** We recall that ν is defined in (1.11).

Proposition 4.15. *The function ν is non-negative on $[0, +\infty)$, increasing, concave and it satisfies*

$$\nu(0) = 0, \quad \nu(+\infty) = 2, \quad \liminf_{\alpha \rightarrow 0^+} \frac{\nu(\alpha)}{\alpha} > 0.$$

In particular, the constant a_0 defined in Theorem 4.3 is the unique positive solution of the equation $\nu(\alpha) = \alpha^2$. Moreover, $\xi_{a_0} = a_0$, where, for all $\alpha > 0$, ξ_α is the unique ξ such that $\nu(\alpha) = \nu_1^-(\alpha, \xi)$ and

$$a_0 = \inf_{\substack{u \in \mathcal{S}_{-A_0}^2(\mathbb{R}_+^2) \\ u \neq 0}} \frac{\|u\|_{\partial\mathbb{R}_+^2}^2 + \sqrt{\|u\|_{\partial\mathbb{R}_+^2}^4 + 4\|u\|^2 \int_{\mathbb{R}_+^2} |(-i\partial_s - \tau + i(-i\partial_\tau))u|^2 ds d\tau}}{2\|u\|^2}.$$

Proof. The function ν is concave as an infimum of linear functions. The equality $\nu(0) = 0$ follows by considering the zero modes², and $\nu(+\infty) = 2$ comes from the fact that, when $\alpha \rightarrow +\infty$, $\nu(\alpha)$ converges to the groundstate energy on the half-space with Dirichlet boundary condition. Then, the concavity implies that

$$\liminf_{\alpha \rightarrow 0^+} \frac{\nu(\alpha)}{\alpha} > 0. \quad (4.9)$$

Let us explain why ν is a smooth function on $(0, +\infty)$. Let us recall that by Proposition 4.12,

$$\nu(\alpha) = \min_{\xi \in \mathbb{R}} \nu_1^-(\alpha, \xi) = \nu_1^-(\alpha, \xi_\alpha) < 2, \quad (4.10)$$

and that, for all $\alpha > 0$, $\xi_\alpha > 0$ is the unique solution of

$$\partial_\xi \nu_1^-(\alpha, \xi) = 0. \quad (4.11)$$

For all $\alpha > 0$, we have $\partial_\xi^2 \nu_1^-(\alpha, \xi_\alpha) > 0$, and thus the analytic implicit function theorem applied to (4.11) implies that $\alpha \mapsto \xi_\alpha$ is analytic. Since ν_1^- is analytic, we deduce that $\alpha \mapsto \nu(\alpha)$ is analytic. We notice that

$$\nu'(\alpha) = \partial_\alpha \nu_1^-(\alpha, \xi_\alpha) + \partial_\xi \nu_1^-(\alpha, \xi_\alpha) \frac{d\xi_\alpha}{d\alpha} = \partial_\alpha \nu_1^-(\alpha, \xi_\alpha).$$

Thanks to Lemma 4.10, we get

$$\nu'(\alpha) = u_{\alpha, \xi_\alpha}^2(0) > 0. \quad (4.12)$$

Let us now consider the function

$$f(\alpha) = \nu(\alpha) - \alpha^2.$$

From (4.9), we see that f is positive on some interval $(0, a)$ with $a > 0$. Then, by $\nu(+\infty) = 2$, we see that f is negative on some interval $(b, +\infty)$. By the Intermediate Value Theorem, we deduce that f has at least one zero in $(0, +\infty)$. Let us prove that there is only one zero. Consider $\alpha > 0$ such that $f(\alpha) = 0$. We have $f'(\alpha) = \nu'(\alpha) - 2\alpha$. Due to (4.3), we have $\xi_\alpha = \alpha$, and with Lemma 4.8, we get

$$2\alpha - u_{\alpha, \xi_\alpha}^2(0) = \int_0^{+\infty} t u_{\alpha, \xi_\alpha}^2(t) dt > 0.$$

²We can also check that ν is right continuous at 0.

This, with (4.12), implies that $f'(\alpha) < 0$. We deduce that f has at most one positive zero (and thus exactly one, denoted by a_1). By (4.10), we get that

$$\nu(a_1) = \nu_1^-(a_1, a_1) = a_1^2,$$

so that $a_1 = \vartheta_1^-(a_1)$. By (4.8) and Theorem 4.3, a_1 is the only critical point of ϑ_1^- so that $a_0 = a_1$.

Let us denote by a_2 the constant introduced in Remark 1.13 :

$$a_2 := \inf_{\substack{u \in \mathfrak{H}_{-\mathbf{A}_0}^2(\mathbb{R}_+^2) \\ u \neq 0}} \rho(u),$$

with

$$\rho(u) := \frac{\int_{\mathbb{R}} |u(s, 0)|^2 ds + \sqrt{\left(\int_{\mathbb{R}} |u(s, 0)|^2 ds\right)^2 + 4\|u\|^2 \int_{\mathbb{R}_+^2} |(-i\partial_s - \tau + i(-i\partial_\tau))u|^2 ds d\tau}}{2\|u\|^2}.$$

Let $u \in \mathfrak{H}_{-\mathbf{A}_0}^2(\mathbb{R}_+^2) \setminus \{0\}$ such that $\|u\| = 1$. $\rho(u)$ is the only positive root of the second order polynomial

$$Q_u : \alpha \mapsto \int_{\mathbb{R}_+^2} |(-i\partial_{x_1} - x_2 + i(-i\partial_{x_2}))u|^2 dx_1 dx_2 + \alpha \int_{\mathbb{R}} |u(x_1, 0)|^2 dx_1 - \alpha^2.$$

By (1.11), we get that $0 = Q_u(\alpha) \geq \nu(\alpha) - \alpha^2$ with $\alpha = \rho(u)$. Hence, $a_0 \leq \alpha$ and taking the infimum over u ensures that $a_0 \leq a_2$. Let $\varepsilon > 0$. We have $\nu(\alpha) - \alpha^2 < 0$ with $\alpha = a_0 + \varepsilon$. Therefore, there exists a normalized function $u \in \mathfrak{H}_{-\mathbf{A}_0}^2(\mathbb{R}_+^2)$ such that

$$Q_u(\alpha) < 0.$$

By definition of $\rho(u)$, we get that $\rho(u) < \alpha$. By definition of a_2 , we get $a_2 < a_0 + \varepsilon$ and the result follows. \square

4.8. More formulas for ν_1^- . In this usection, we study other properties of the eigenfunctions associated with the negative eigenvalues. Let $\alpha > 0$, $\xi \in \mathbb{R}$, $\nu = \nu_1^-$ and $u_{\alpha, \xi}$ is a regular branch of real normalized eigenfunctions associated with the eigenvalue $\nu(\alpha, \xi)$:

$$\begin{aligned} \mathcal{M}_{\alpha, \xi}^- u_{\alpha, \xi} &= \nu(\alpha, \xi) u_{\alpha, \xi}, \text{ on } \mathbb{R}_+, \\ (\partial_\tau + \xi - \alpha) u_{\alpha, \xi}(0) &= 0. \end{aligned}$$

4.8.1. *About the momenta of $u_{\alpha, \xi}$ and \mathcal{C}_ξ .* We consider the operator

$$\mathcal{C}_\xi = 2 \left(\tau \mathcal{M}_{\alpha, \xi}^- + \xi - \partial_\tau + \tau^2(\xi - \tau) \right), \quad (4.13)$$

which will appear in the computation of the asymptotics of the negative eigenvalues.

Lemma 4.16. *The operator \mathcal{C}_ξ is symmetric on $H^2(\mathbb{R}_+)$.*

Proof. For shortness, we let $n_0 = \mathcal{M}_{\alpha, \xi}^-$. We write

$$\begin{aligned} \langle (\tau n_0 - \partial_\tau)u, v \rangle &= \langle n_0 u, \tau v \rangle - \langle u', v \rangle \\ &= \langle u, n_0(\tau v) \rangle + u'(0)(\tau v)(0) - u(0)(\tau v)'(0) - \langle u', v \rangle, \end{aligned}$$

so that

$$\langle (\tau n_0 - \partial_\tau)u, v \rangle = \langle n_0 u, \tau v \rangle - u(0)v(0) + \langle u, v' \rangle + u(0)v(0),$$

and

$$\langle (\tau n_0 - \partial_\tau)u, v \rangle = \langle u, (n_0 \tau + \partial_\tau)v \rangle = \langle u, (\tau n_0 - \partial_\tau)v \rangle.$$

□

We let

$$M_j = \int_0^{+\infty} (\xi - \tau)^j u_{\alpha, \xi}^2 d\tau,$$

and

$$P_j(\tau) = (\xi - \tau)^j.$$

In order to compute the momenta M_j , the following lemma will be convenient.

Lemma 4.17. *Let p be any polynomial in the τ variable. We have*

$$\begin{aligned} & \left(\mathcal{M}_{\alpha, \xi}^- - \nu(\alpha, \xi) \right) (2pu'_{\alpha, \xi} - p'u_{\alpha, \xi}) \\ &= \left(p^{(3)} - 4((\xi - \tau)^2 + 1 - \nu(\alpha, \xi))p' + 4(\xi - \tau)p \right) u_{\alpha, \xi}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} & \left\langle \left(\mathcal{M}_{\alpha, \xi}^- - \nu(\alpha, \xi) \right) (2pu'_{\alpha, \xi} - p'u_{\alpha, \xi}), u_{\alpha, \xi} \right\rangle \\ &= u_{\alpha, \xi}^2(0) \left(-p''(0) + 2p'(0)(\alpha - \xi) + 2p(0)(\xi^2 + 1 - \nu(\alpha, \xi) - (\alpha - \xi)^2) \right). \end{aligned}$$

Proof. The first part follows by a straightforward computation. For the second identity, we use Lemma 4.7 and the equation and the boundary condition satisfied by $u_{\alpha, \xi}$,

$$\begin{aligned} & \left\langle \left(\mathcal{M}_{\alpha, \xi}^- - \nu(\alpha, \xi) \right) (2pu'_{\alpha, \xi} - p'u_{\alpha, \xi}), u_{\alpha, \xi} \right\rangle \\ &= - (2pu'_{\alpha, \xi} - p'u_{\alpha, \xi})u'_{\alpha, \xi}(0) + (2pu'_{\alpha, \xi} - p'u_{\alpha, \xi})'(0)u_{\alpha, \xi}(0) \\ &= - 2p(0)(\xi - \alpha)^2 u_{\alpha, \xi}^2(0) + p'(0)(\alpha - \xi)u_{\alpha, \xi}^2(0) \\ & \quad + (2p(0)u''_{\alpha, \xi}(0) + 2p'(0)(\alpha - \xi)u_{\alpha, \xi}(0) - p''(0)u_{\alpha, \xi}(0) - p'(0)(\alpha - \xi)u_{\alpha, \xi}(0))u_{\alpha, \xi}(0) \\ &= u_{\alpha, \xi}^2(0) \left(-2p(0)(\xi - \alpha)^2 + 2p'(0)(\alpha - \xi) + 2p(0)(\xi^2 + 1 - \nu(\alpha, \xi) - p''(0)) \right). \end{aligned}$$

□

In the following lemma, we compute the first momenta M_j .

Lemma 4.18. *We have*

$$\begin{aligned} M_1 &= \frac{u_{\alpha, \xi}^2(0)}{2} (\xi^2 + 1 - \nu(\alpha, \xi) - (\alpha - \xi)^2), \\ M_2 &= \frac{\nu(\alpha, \xi) - 1}{2} + \frac{u_{\alpha, \xi}^2(0)}{4} \left(-(\alpha - \xi) + \xi(\xi^2 + 1 - \nu(\alpha, \xi) - (\alpha - \xi)^2) \right), \\ M_3 &= \frac{u_{\alpha, \xi}^2(0)}{12} \left((4\nu(\alpha, \xi) - 4 + 2\xi^2)(\xi^2 + 1 - \nu(\alpha, \xi) - (\alpha - \xi)^2) - 2 - 4\xi(\alpha - \xi) \right), \end{aligned}$$

and

$$M_4 = \frac{3}{8} + \frac{3}{8}(\nu - 1)^2 + \frac{u_{\alpha,\xi}^2(0)}{16} \left(-6\xi + (3(1 - \nu) - 6\xi^2)(\alpha - \xi) \right) \\ + \frac{u_{\alpha,\xi}^2(0)}{16} (2\xi^3 + 3(\nu - 1)\xi)(\xi^2 + 1 - \nu - (\alpha - \xi)^2)$$

Assume that $\alpha = \xi = a_0$. Then, we have

$$M_1 = \frac{u_{\alpha,\xi}^2(0)}{2}, \\ M_2 = \frac{\xi^2 - 1}{2} + \frac{\xi u_{\alpha,\xi}^2(0)}{4}, \\ M_3 = \frac{\xi^2 - 1}{2} u_{\alpha,\xi}^2(0), \\ M_4 = \frac{3}{8} + \frac{3}{8}(\xi^2 - 1)^2 + \frac{u_{\alpha,\xi}^2(0)}{16}(5\xi^3 - 9\xi).$$

Remark 4.19. The assumption on α and ξ comes from that fact that we will need the momenta with $\xi = \xi_\alpha$ when α satisfies $\nu(\alpha) = \alpha^2$. In this case, $\xi = \alpha = a_0$. See (4.3).

Proof. In Lemma 4.17, we take $p = 1$ and get

$$4M_1 = 2u_{\alpha,\xi}^2(0)(\xi^2 + 1 - \nu(\alpha, \xi) - (\alpha - \xi)^2).$$

We recover the results of Lemma 4.8 and Proposition 4.2. Taking $p = (\xi - \tau)$, we get

$$8M_2 + 4M_0(1 - \nu(\alpha, \xi)) = u_{\alpha,\xi}^2(0) \left(-2(\alpha - \xi) + 2\xi(\xi^2 + 1 - \nu(\alpha, \xi) - (\alpha - \xi)^2) \right),$$

and the result follows. Taking $p = (\xi - \tau)^2$, we get

$$8(1 - \nu(\alpha, \xi))M_1 + 12M_3 \\ = u_{\alpha,\xi}^2(0) \left(-2 - 4\xi(\alpha - \xi) + 2\xi^2(\xi^2 + 1 - \nu(\alpha, \xi) - (\alpha - \xi)^2) \right),$$

and

$$M_3 = \frac{u_{\alpha,\xi}^2(0)}{12} \left((4\nu(\alpha, \xi) - 4 + 2\xi^2)(\xi^2 + 1 - \nu(\alpha, \xi) - (\alpha - \xi)^2) - 2 - 4\xi(\alpha - \xi) \right).$$

Finally, taking $p = (\xi - \tau)^3$, we get

$$-6 + 16M_4 + 12(1 - \nu(\alpha, \xi))M_2 \\ = u_{\alpha,\xi}^2(0) \left(-6\xi + 2(-3\xi^2)(\alpha - \xi) + 2\xi^3(\xi^2 + 1 - \nu(\alpha, \xi) - (\alpha - \xi)^2) \right),$$

and

$$M_4 = \frac{6}{16} \\ + \frac{12}{16}(\nu - 1) \left(\frac{\nu - 1}{2} + \frac{u_{\alpha,\xi}^2(0)}{4} \left(-(\alpha - \xi) + \xi(\xi^2 + 1 - \nu - (\alpha - \xi)^2) \right) \right) \\ + \frac{u_{\alpha,\xi}^2(0)}{16} \left(-6\xi - 6\xi^2(\alpha - \xi) + 2\xi^3(\xi^2 + 1 - \nu(\alpha, \xi) - (\alpha - \xi)^2) \right).$$

The remaining identities follows from the fact that $\nu(a_0, a_0) = a_0^2$ (see Theorem 4.3 and Notation 10). \square

The following quantities appear in the construction of the effective operator on the boundary that we obtain expanding the operator (written in appropriate coordinates) in powers of h .

Lemma 4.20. *We have*

$$\langle \mathcal{C}_\xi u_\xi, u_\xi \rangle = 2B(\alpha, \xi) u_{\alpha, \xi}^2(0),$$

where

$$\begin{aligned} B(\alpha, \xi) = & -\frac{\nu(\alpha, \xi)}{2}(\xi^2 + 1 - \nu(\alpha, \xi) - (\alpha - \xi)^2) + \frac{1}{2} \\ & + \frac{1}{12}((4\nu(\alpha, \xi) - 4 + 2\xi^2)(\xi^2 + 1 - \nu(\alpha, \xi) - (\alpha - \xi)^2) - 2 - 4\xi(\alpha - \xi)) \\ & - \frac{\xi}{2}(-(\alpha - \xi) + \xi(\xi^2 + 1 - \nu(\alpha, \xi) - (\alpha - \xi)^2)) \\ & + \frac{\xi^2}{2}(\xi^2 + 1 - \nu(\alpha, \xi) - (\alpha - \xi)^2). \end{aligned}$$

If $\alpha = \xi = a_0$, then

$$\langle \mathcal{C}_\xi u_{\alpha, \xi}, u_{\alpha, \xi} \rangle = 0, \quad \partial_\xi \langle \mathcal{C}_\xi u_{\alpha, \xi}, u_{\alpha, \xi} \rangle = -\frac{\partial_\xi^2 \nu(\alpha, \xi)}{2}.$$

Proof. We let $n_0 = \mathcal{M}_{\alpha, \xi}^-$. Let us write

$$\begin{aligned} & \langle (\tau n_0 - \xi - \partial_\tau + \tau^2(\xi - \tau)) u_{\alpha, \xi}, u_{\alpha, \xi} \rangle \\ & = \nu(\alpha, \xi) \int_0^{+\infty} \tau u_{\alpha, \xi}^2 d\tau - \xi + \frac{u_{\alpha, \xi}^2(0)}{2} + \int_0^{+\infty} (\xi - \tau - \xi)^2 (\xi - \tau) u_{\alpha, \xi}^2 d\tau, \end{aligned}$$

so that

$$\begin{aligned} & \langle (\tau n_0 - \xi - \partial_\tau + \tau^2(\xi - \tau)) u_{\alpha, \xi}, u_{\alpha, \xi} \rangle \\ & = \xi(\nu(\alpha, \xi) - 1) - \nu(\alpha, \xi)M_1 + \frac{u_{\alpha, \xi}^2(0)}{2} + M_3 - 2\xi M_2 + \xi^2 M_1. \end{aligned}$$

Therefore,

$$\langle \mathcal{C}_\xi u_{\alpha, \xi}, u_{\alpha, \xi} \rangle = 2 \left(A(\alpha, \xi) + B(\alpha, \xi) u_{\alpha, \xi}^2(0) \right),$$

where

$$A(\alpha, \xi) = \xi(\nu(\alpha, \xi) - 1) - \xi(\nu(\alpha, \xi) - 1) = 0.$$

When $\alpha = \xi$ and $\nu(\alpha, \xi) = \xi^2$, we easily check that

$$B(\alpha, \xi) = 0, \quad \langle \mathcal{C}_\xi u_{\alpha, \xi}, u_{\alpha, \xi} \rangle = 0.$$

Let us compute the derivative $\partial_\xi B(\alpha, \xi)$ when $\alpha = \xi$ and $\nu(\alpha, \xi) = \xi^2$. We have

$$\begin{aligned} \partial_\xi B(\alpha, \xi) = & -\xi\nu(\alpha, \xi) + \frac{1}{12}(4\xi + 2\xi(6\xi^2 - 4) + 4\xi) - \frac{\xi}{2} - \frac{\xi}{2}(1 + 1 + 2\xi^2) + \xi + \xi^3 \\ = & -\frac{\xi}{2}. \end{aligned}$$

Thus,

$$\partial_\xi \langle \mathcal{C}_\xi u_{\alpha, \xi}, u_{\alpha, \xi} \rangle = -\xi u_{\alpha, \xi}^2(0) = -\frac{\partial_\xi^2 \nu(\alpha, \xi)}{2}.$$

□

Remark 4.21. Note that, when $\alpha = \xi = \xi_\alpha$, we have

$$u'_{\xi_\alpha}(0) = 0,$$

and this means that $\nu(\alpha, \xi_\alpha)$ is an eigenvalue of the de Gennes operator with parameter $\xi_\alpha > 0$. This eigenvalue lies on the parabola $\xi \mapsto \xi^2$.

The functions $g_{\alpha,\xi}$ and $k_{\alpha,\xi}$ studied in the next two sections appear in the construction of the symbol of the effective operator on the boundary.

4.8.2. *About the function $g_{\alpha,\xi}$.*

Lemma 4.22. *For all $\alpha > 0$, $\xi \in \mathbb{R}$, there exists a unique $g_{\alpha,\xi}$ such that*

$$(\mathcal{M}_{\alpha,\xi}^- - \nu(\alpha, \xi))g_{\alpha,\xi} = u_{\alpha,\xi}^2(0)u_{\alpha,\xi}, \quad (\xi - \alpha + \partial_\tau)g_{\alpha,\xi}(0) = u_{\alpha,\xi}(0), \quad \langle g_{\alpha,\xi}, u_{\alpha,\xi} \rangle = 0.$$

Moreover,

$$u_{\alpha,\xi}(0)v_{\alpha,\xi}(0) - 2\langle (\xi - \tau)g_{\alpha,\xi}, u_{\alpha,\xi} \rangle = -u_{\alpha,\xi}(0)g_{\alpha,\xi}(0),$$

and

$$(\mathcal{M}_{\alpha,\xi}^- - \nu(\alpha, \xi))\partial_\xi g_{\alpha,\xi} = \partial_\xi \nu(\alpha, \xi)g_{\alpha,\xi} + 2u_{\alpha,\xi}(0)v_{\alpha,\xi}(0)u_{\alpha,\xi} + u_{\alpha,\xi}^2(0)v_{\alpha,\xi} - 2(\xi - \tau)g_{\alpha,\xi}.$$

Proof. For shortness, we write u , v and g instead of $u_{\alpha,\xi}$, $v_{\alpha,\xi}$ and $g_{\alpha,\xi}$. For any g satisfying the boundary condition, we have

$$\langle (\mathcal{M}_{\alpha,\xi}^- - \nu(\alpha, \xi))g, u \rangle = g'(0)u(0) - g(0)u'(0) = (g'(0) - g(0)(\alpha - \xi))u(0) = u^2(0).$$

We have

$$(\mathcal{M}_{\alpha,\xi}^- - \nu(\alpha, \xi))\partial_\xi g = \partial_\xi \nu(\alpha, \xi)g + 2u(0)v(0)u + u^2(0)v - 2(\xi - \tau)g.$$

Then, using the fact that $0 = \partial_\xi \|u\|^2 = 2\langle u, v \rangle$, and Lemma 4.7,

$$\begin{aligned} \langle (\mathcal{M}_{\alpha,\xi}^- - \nu(\alpha, \xi))\partial_\xi g, u \rangle &= 2u(0)v(0) - 2\langle (\xi - \tau)g, u \rangle \\ &= \partial_\tau \partial_\xi g(0)u(0) - \partial_\xi g(0)u'(0) \\ &= (\partial_\tau \partial_\xi g(0) + (\xi - \alpha)\partial_\xi g(0))u(0). \end{aligned}$$

Note that

$$(\xi - \alpha + \partial_\tau)\partial_\xi g(0) = v(0) - g(0).$$

Thus

$$2u(0)v(0) - 2\langle (\xi - \tau)g, u \rangle = u(0)v(0) - u(0)g(0).$$

□

Lemma 4.23. *We have*

$$(\mathcal{M}_{\alpha,\xi}^- - \nu(\alpha, \xi))(v_{\alpha,\xi} + g_{\alpha,\xi}) = \partial_\xi \nu(\alpha, \xi)u_{\alpha,\xi} - 2(\xi - t)u_{\alpha,\xi} + u_{\alpha,\xi}^2(0)u_{\alpha,\xi},$$

with the boundary condition

$$(\partial_t + \xi - \alpha)(v_{\alpha,\xi} + g_{\alpha,\xi})(0) = 0,$$

and $\langle v_{\alpha,\xi} + g_{\alpha,\xi}, u_{\alpha,\xi} \rangle = 0$.

Proof. It follows from Lemmata 4.8 and 4.22. □

Lemma 4.24. *We have*

$$u_{\alpha,\xi}(0)g_{\alpha,\xi}(0) = \frac{1}{2}\partial_\alpha^2\nu(\alpha,\xi).$$

Proof. For shortness, we write u , v and g instead of $u_{\alpha,\xi}$, $v_{\alpha,\xi}$ and $g_{\alpha,\xi}$. Thanks to Lemma 4.10, we have

$$(\mathcal{M}_{\alpha,\xi}^- - \nu(\alpha,\xi))g = \partial_\alpha\nu(\alpha,\xi)u.$$

Taking the derivative with respect to α and then the inner product with u , we get

$$\langle (\mathcal{M}_{\alpha,\xi}^- - \nu(\alpha,\xi))\partial_\alpha g, u \rangle = \partial_\alpha\nu(\alpha,\xi)\langle g, u \rangle + \partial_\alpha^2\nu(\alpha,\xi) + \partial_\alpha\nu(\alpha,\xi)\langle \partial_\alpha u, u \rangle = \partial_\alpha^2\nu(\alpha,\xi),$$

and an integration by parts provides us with

$$\partial_t\partial_\alpha g(0)u(0) - \partial_\alpha g(0)\partial_t u(0) = \partial_\alpha^2\nu(\alpha,\xi).$$

We have

$$(\partial_t + \xi - \alpha)\partial_\alpha g(0) = \partial_\alpha u(0) + g(0),$$

so that

$$\partial_\alpha u(0)u(0) + u(0)g(0) = \partial_\alpha^2\nu(\alpha,\xi).$$

$$\partial_\alpha^2\nu(\alpha,\xi) = 2u(0)\partial_\alpha u(0),$$

so that

$$u(0)g(0) = \frac{\partial_\alpha^2\nu(\alpha,\xi)}{2}.$$

□

4.8.3. *About the function $k_{\alpha,\xi}$.* Let $k_{\alpha,\xi}$ be the unique solution orthogonal to $u_{\alpha,\xi}$ of

$$(\mathcal{M}_{\alpha,\xi}^- - \nu(\alpha,\xi))k_{\alpha,\xi} = -\Pi^\perp(\mathcal{C}_\xi u_{\alpha,\xi}). \quad (4.14)$$

with

$$(\xi - \alpha + \partial_\tau)k_{\alpha,\xi}(0) = 0,$$

where Π^\perp is the orthogonal projection on the orthogonal of $u_{\alpha,\xi}$,

Lemma 4.25. *We have*

$$\langle \mathcal{C}_\xi u_{\alpha,\xi}, v_{\alpha,\xi} \rangle = -k_{\alpha,\xi}(0)u_{\alpha,\xi}(0) + 2\langle (\xi - \tau)k_{\alpha,\xi}, u_{\alpha,\xi} \rangle.$$

Proof. We recall (4.14), and Lemma 4.8. Then, since $\langle u_{\alpha,\xi}, v_{\alpha,\xi} \rangle = 0$ and $\langle u_{\alpha,\xi}, k_{\alpha,\xi} \rangle = 0$, we get, by an integration by parts,

$$\begin{aligned} -\langle \mathcal{C}_\xi u_{\alpha,\xi}, v_{\alpha,\xi} \rangle &= -\langle \Pi^\perp \mathcal{C}_\xi u_{\alpha,\xi}, v_{\alpha,\xi} \rangle = \langle (\mathcal{M}_{\alpha,\xi}^- - \nu(\alpha,\xi))k_{\alpha,\xi}, v_{\alpha,\xi} \rangle \\ &= k'_{\alpha,\xi}(0)v_{\alpha,\xi}(0) - k_{\alpha,\xi}(0)v'_{\alpha,\xi}(0) + \langle k_{\alpha,\xi}, (\mathcal{M}_{\alpha,\xi}^- - \nu(\alpha,\xi))v_{\alpha,\xi} \rangle \\ &= u_{\alpha,\xi}(0)k_{\alpha,\xi}(0) - 2\langle (\xi - \tau)u_{\alpha,\xi}, k_{\alpha,\xi} \rangle. \end{aligned}$$

□

In fact the function $k_{\alpha,\xi}$ can be computed explicitly when $\alpha = \xi$ and $\nu(\alpha,\xi) = \alpha^2$.

Lemma 4.26. *Assume that $\alpha = \xi = a_0$. Consider the function*

$$k_0 = \left(-\frac{\xi}{2} + \frac{2}{3}P_1 \right) u_{\alpha,\xi} + \left(\frac{2}{3}(1 - \xi^2) + \xi P_1 - \frac{1}{3}P_2 \right) u'_{\alpha,\xi}.$$

It solves

$$(\mathcal{M}_{\alpha,\xi} - \nu(\alpha, \xi))k_0 = -\mathcal{C}_\xi u_{\alpha,\xi},$$

and satisfies the Neumann condition.

Proof. We take $\alpha = \xi = a_0$. For shortness, we remove the reference to (α, ξ) . We want to solve the equation

$$(\mathcal{M} - \nu)k = -\mathcal{C}_\xi u.$$

We have

$$(\mathcal{M} - \nu)(pu + qu') = -p''u - q''u' - 2p'u' - 2q'u'' + q(\mathcal{M} - \nu)u',$$

and

$$(\mathcal{M} - \nu)u' - 2(\xi - \tau)u = 0, \quad -u'' = (\nu - 1)u - (\xi - \tau)^2u.$$

Thus,

$$(\mathcal{M} - \nu)(pu + qu') = (-p'' + 2q'((\nu - 1) - P_2) + 2qP_1)u + (-2p' - q'')u'.$$

Looking at the expression of \mathcal{C}_ξ in (4.13), we want that

$$-2p' - q'' = 2,$$

so that we take

$$p = -\tau - \frac{q'}{2}.$$

Let us now determine the function q . We want that

$$-\frac{q^{(3)}}{2} + 2q'((\nu - 1) - P_2) + 2qP_1 = -2\nu\tau + 2\xi - 2\tau^2P_1.$$

We get

$$-\frac{q^{(3)}}{2} + 2q'((\nu - 1) - P_2) + 2qP_1 = 2\nu P_1 - 2\xi\nu + 2\xi - 2(\xi - \tau - \xi)^2P_1,$$

which can be written as

$$-\frac{q^{(3)}}{2} + 2q'((\nu - 1) - P_2) + 2qP_1 = -2\xi(\nu - 1) + (2\nu - 2\xi^2)P_1 + 4\xi P_2 - 2P_3.$$

We look for q in the form

$$q = \alpha_3 P_3 + \alpha_2 P_2 + \alpha_1 P_1 + \alpha_0.$$

We have $q^{(3)} = -6\alpha_3$ and

$$q' = -3\alpha_3 P_2 - 2\alpha_2 P_1 - \alpha_1.$$

We get

$$\begin{aligned} 3\alpha_3 - 2\alpha_1(\nu - 1) &= -2\xi(\nu - 1), \\ -4\alpha_2(\nu - 1) + 2\alpha_0 &= 2(\nu - \xi^2) = 0, \\ 2\alpha_1 - 6\alpha_3(\nu - 1) + 2\alpha_1 &= 4\xi, \\ 4\alpha_2 + 2\alpha_2 &= -2, \\ 6\alpha_3 + 2\alpha_3 &= 0. \end{aligned}$$

This gives

$$\alpha_3 = 0, \quad \alpha_2 = -\frac{1}{3}, \quad \alpha_1 = \xi, \quad \alpha_0 = \frac{2}{3}(1 - \xi^2).$$

Thus,

$$q = \frac{2}{3}(1 - \xi^2) + \xi P_1 - \frac{1}{3}P_2,$$

and

$$p = -\tau - \frac{q'}{2} = P_1 - \xi - \frac{1}{2}(-\xi + \frac{2}{3}P_1) = -\frac{\xi}{2} + \frac{2}{3}P_1.$$

Let us check that k_0 satisfies the Neumann condition. Notice that $u'(0) = 0$ and $u''(0) = u(0)$. We have

$$\partial_\tau k_0(0) = \left(-\frac{2}{3} + \frac{2}{3}(1 - \xi^2) + \xi^2 - \frac{\xi^2}{3} \right) u(0) = 0.$$

□

The following two lemmata will be used in the proof of Theorem 1.19, especially to find the constant $-\frac{1}{12}$ in (1.12).

Lemma 4.27. *Assume that $\alpha = \xi = a_0$. Then,*

$$\langle \mathcal{C}_\xi k_{\alpha, \xi}, u_{\alpha, \xi} \rangle = -\frac{3}{4} + \frac{11}{4}\xi^2 - \frac{19}{8}\xi^4 + \left(-\frac{37}{48}\xi + \frac{19}{16}\xi^3 \right) u_{\alpha, \xi}^2(0).$$

Proof. We drop the index (α, ξ) . We have

$$\langle \mathcal{C}k, u \rangle = \langle \mathcal{C}k_0, u \rangle = \langle k_0, \mathcal{C}u \rangle = 2\langle k, (\xi(\xi^2 - 1) - 2\xi P_2 + P_3 - \partial_\tau) u \rangle,$$

where we used Lemmata 4.16 and 4.20, and the explicit expression of \mathcal{C}_ξ in (4.13). Therefore, with Lemma 4.26, we have to estimate

$$I = \left\langle \left(-\frac{\xi}{2} + \frac{2}{3}P_1 \right) u + \left(\frac{2}{3}(1 - \xi^2) + \xi P_1 - \frac{1}{3}P_2 \right) u', (\xi(\xi^2 - 1) - 2\xi P_2 + P_3)u - u' \right\rangle.$$

It can be written as

$$I = I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned} I_1 &= \left\langle \left(-\frac{\xi}{2} + \frac{2}{3}P_1 \right) (\xi(\xi^2 - 1) - 2\xi P_2 + P_3)u, u \right\rangle, \\ I_2 &= - \left\langle \left(\frac{2}{3}(1 - \xi^2) + \xi P_1 - \frac{1}{3}P_2 \right) u', u' \right\rangle, \\ I_3 &= - \left\langle \left(-\frac{\xi}{2} + \frac{2}{3}P_1 \right) u, u' \right\rangle, \\ I_4 &= \left\langle \left(\frac{2}{3}(1 - \xi^2) + \xi P_1 - \frac{1}{3}P_2 \right) u', (\xi(\xi^2 - 1) - 2\xi P_2 + P_3)u \right\rangle. \end{aligned}$$

We have

$$\left(-\frac{\xi}{2} + \frac{2}{3}P_1 \right) (\xi(\xi^2 - 1) - 2\xi P_2 + P_3) = -\frac{\xi}{2}(\xi^3 - \xi) + \frac{2}{3}(\xi^3 - \xi)P_1 + \xi^2 P_2 - \frac{11}{6}\xi P_3 + \frac{2}{3}P_4,$$

so that

$$I_1 = -\frac{\xi}{2}(\xi^3 - \xi) + \frac{2}{3}(\xi^3 - \xi)M_1 + \xi^2 M_2 - \frac{11}{6}\xi M_3 + \frac{2}{3}M_4. \quad (4.15)$$

Then, by integrating by parts and using $u'(0) = 0$,

$$I_2 = \left\langle \left(\left(\frac{2}{3}(1 - \xi^2) + \xi P_1 - \frac{1}{3}P_2 \right) u' \right)', u \right\rangle$$

We get

$$I_2 = \left\langle \left(-\xi + \frac{2}{3}P_1 \right) u', u \right\rangle + \left\langle \left(\frac{2}{3}(1 - \xi^2) + \xi P_1 - \frac{1}{3}P_2 \right) u'', u \right\rangle.$$

By integration by parts,

$$\left\langle \left(-\xi + \frac{2}{3}P_1 \right) u', u \right\rangle = \frac{1}{3} + \frac{\xi}{6}u^2(0).$$

We recall that $u'' = (P_2 + 1 - \xi^2)u$ so that

$$\left\langle \left(\frac{2}{3}(1 - \xi^2) + \xi P_1 - \frac{1}{3}P_2 \right) u'', u \right\rangle = \left\langle \left(\frac{2}{3}(1 - \xi^2) + \xi P_1 - \frac{1}{3}P_2 \right) (P_2 + 1 - \xi^2)u, u \right\rangle,$$

and then

$$\left\langle \left(\frac{2}{3}(1 - \xi^2) + \xi P_1 - \frac{1}{3}P_2 \right) u'', u \right\rangle = \frac{2}{3}(1 - \xi^2)^2 + \xi(1 - \xi^2)M_1 + \frac{1}{3}(1 - \xi^2)M_2 + \xi M_3 - \frac{1}{3}M_4.$$

We deduce that

$$I_2 = \frac{1}{3} + \frac{\xi}{6}u^2(0) + \frac{2}{3}(1 - \xi^2)^2 + \xi(1 - \xi^2)M_1 + \frac{1}{3}(1 - \xi^2)M_2 + \xi M_3 - \frac{1}{3}M_4. \quad (4.16)$$

Integrating by parts, we get

$$I_3 = -\frac{1}{3} + \frac{\xi}{12}u^2(0). \quad (4.17)$$

Finally, we again integrate by parts to find

$$I_4 = \frac{\xi}{3}u^2(0) - \frac{1}{2} \int_0^{+\infty} \left[\left(\frac{2}{3}(1 - \xi^2) + \xi P_1 - \frac{1}{3}P_2 \right) (\xi(\xi^2 - 1) - 2\xi P_2 + P_3) \right]' u^2 d\tau,$$

so that

$$I_4 = \frac{\xi}{3}u^2(0) - \frac{1}{2} \left(-\xi^2(\xi^2 - 1) + 2\xi(1 - \xi^2)M_1 + (8\xi^2 - 2)M_2 - \frac{20\xi}{3}M_3 + \frac{5}{3}M_4 \right). \quad (4.18)$$

It remains to notice that $\langle \mathcal{C}k, u \rangle = 2I$, to use (4.15), (4.16), (4.17), (4.18), and to remember Lemma 4.18. \square

Lemma 4.28. *Assume that $\alpha = \xi = a_0$. Let*

$$\mathcal{C}_{\xi,2} = -4\tau(\partial_\tau + \xi - \tau) + 2\tau^2 n_0 + \frac{8}{3}(\xi - \tau)\tau^3 - 4\tau^2 + \tau^4, \quad n_0 = \mathcal{M}_{\alpha,\xi}.$$

We have

$$\langle \mathcal{C}_{\xi,2} u_{\alpha,\xi}, u_{\alpha,\xi} \rangle = \frac{3}{4} - \frac{11}{4}\xi^2 + \frac{19}{8}\xi^4 + \left(\frac{15}{16}\xi - \frac{19}{16}\xi^3 \right) u_{\alpha,\xi}^2(0).$$

Proof. We have

$$\begin{aligned} \tau^2 &= (\tau - \xi + \xi)^2 = P_2 - 2\xi P_1 + \xi^2, \\ \tau^3 &= (\tau - \xi + \xi)^3 = -P_3 + 3\xi P_2 - 3\xi^2 P_1 + \xi^3, \\ \tau^4 &= (\xi - \tau - \xi)^4 = P_4 - 4\xi P_3 + 6\xi^2 P_2 - 4\xi^3 P_1 + \xi^4. \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{C}_{\xi,2} &= -4\tau\partial_\tau + 4P_2 - 4\xi P_1 + 2(P_2 - 2\xi P_1 + \xi^2)n_0 + \frac{8}{3}(-P_4 + 3\xi P_3 - 3\xi^2 P_2 + \xi^3 P_1) \\ &\quad - 4(P_2 - 2\xi P_1 + \xi^2) + P_4 - 4\xi P_3 + 6\xi^2 P_2 - 4\xi^3 P_1 + \xi^4. \end{aligned}$$

Since we consider $\langle \mathcal{C}_{\xi,2} u_{\alpha,\xi}, u_{\alpha,\xi} \rangle$, we can replace n_0 by ν . Rearranging the terms, we get

$$\langle \mathcal{C}_{\xi,2} u_{\alpha,\xi}, u_{\alpha,\xi} \rangle = 2 - \frac{5}{3}M_4 + 4\xi M_3 + \left(4\xi - \frac{16}{3}\xi^3 \right) M_1 - 4\xi^2 + 3\xi^4.$$

It remains to use Lemma 4.18. \square

Lemmata 4.27 and 4.28 can be combined with Proposition 4.12 to get the following.

Lemma 4.29. *Assume that $\alpha = \xi = a_0$. Then,*

$$\langle \mathcal{C}_\xi u_{\alpha,\xi}, k_{\alpha,\xi} \rangle + \langle \mathcal{C}_{\xi,2} u_{\alpha,\xi}, u_{\alpha,\xi} \rangle = \frac{\xi u_{\alpha,\xi}^2(0)}{6} = \frac{\partial_\xi^2 \nu(\alpha, \xi)}{12}.$$

5. SEMICLASSICAL ANALYSIS OF THE FIRST NEGATIVE EIGENVALUE

5.1. About the proof of Theorem 1.14. Thanks to the charge conjugation (see Remark 1.9), the negative eigenvalues $\lambda_k^-(h)$ can be characterized as follows. For $\lambda \geq 0$, consider the quadratic form

$$\tilde{Q}_\lambda(u) = q_{\lambda,h}(u) - \lambda^2 \|u\|^2, \quad q_{\lambda,h}(u) = \|d_{h,-\mathbf{A}}^\times u\|^2 + \lambda h \|u\|_{\partial\Omega}^2.$$

Let us denote by $(\tilde{\ell}_k(\lambda))_{k \geq 1}$ the eigenvalues of the corresponding operator. As in Section 2, for all $k \geq 1$, the equation $\tilde{\ell}_k(\lambda) = 0$ has a unique positive solution; this solution is $\lambda_k^-(h)$. On the other hand, we have

$$\tilde{\ell}_k(\lambda) = \gamma_k(\lambda, h) - \lambda^2,$$

where the $(\gamma_k(\lambda, h))_{k \geq 1}$ are the eigenvalues of the operator associated with $q_{\lambda,h}$. Note that, by Lemma 2.10(vi), for all $\lambda > 0$,

$$|\gamma_k(\lambda, h) - \lambda^2| = |\tilde{\ell}_k(\lambda)| \geq \lambda |\lambda - \lambda_k^-(h)|, \quad (5.1)$$

and $\lambda_k^-(h)$ is the unique solution of

$$\gamma_k(\lambda, h) = \lambda^2.$$

We write $\lambda_k^-(h) = e_k(h)h^{\frac{1}{2}}$, and the equation becomes

$$\gamma_1(e_k(h)h^{\frac{1}{2}}, h) = e_k(h)^2 h. \quad (5.2)$$

Note that, by setting $\lambda = ah^{\frac{1}{2}}$ with $a > 0$, we have the reformulation of (5.1):

$$|h^{-1}\gamma_k(ah^{\frac{1}{2}}, h) - a^2| \geq a|a - e_k(h)|. \quad (5.3)$$

The main goal of the next section is to establish the following estimate.

Proposition 5.1. *We have, for all $a > 0$,*

$$\gamma_1(ah^{\frac{1}{2}}, h) = h\Lambda(a) + o(h), \quad \Lambda(a) = \min\left(2b_0, b'_0\nu(a(b'_0)^{-1/2})\right).$$

Proposition 5.1 implies Theorem 1.14. Indeed, observe that, substituting this asymptotic expansion into (5.3), we get

$$|\Lambda(a) - a^2 + o(1)| \geq a|a - e_1(h)|.$$

Notice that, if $a > 0$ is such that $\Lambda(a) = a^2$, then

$$e_1(h) = a + o(1).$$

Actually, there is a unique positive a such that

$$\min\left(2b_0, b'_0\nu(a(b'_0)^{-1/2})\right) = a^2,$$

which is given by

$$a = \min(\sqrt{2b_0}, a_0\sqrt{b'_0}),$$

where a_0 is the unique positive solution of $\nu(\alpha) = \alpha^2$, see Proposition 4.15. We deduce that

$$\lim_{h \rightarrow 0} e_1(h) = \min(\sqrt{2b_0}, a_0\sqrt{b'_0}),$$

or equivalently

$$\lambda_1^-(h) = h^{\frac{1}{2}} \min(\sqrt{2b_0}, a_0\sqrt{b'_0}) + o(h^{\frac{1}{2}}).$$

5.2. Ground energy of a Pauli-Robin type operator. Let $a > 0$. We consider the quadratic form

$$\mathcal{Q}_{a,h}(u) = q_{ah^{1/2},h}(u) = \|d_{h,-\mathbf{A}}^\times u\|^2 + ah^{\frac{3}{2}}\|u\|_{\partial\Omega}^2,$$

and we have

$$\gamma_1(ah^{\frac{1}{2}}, h) = \inf_{\substack{u \in \mathfrak{H}_{-\mathbf{A}}^2(\Omega) \\ u \neq 0}} \frac{\mathcal{Q}_{a,h}(u)}{\|u\|^2}.$$

5.2.1. Localization formula. Let $\rho \in (0, \frac{1}{2})$. Let us consider a semiclassical partition of the unity $(\chi_j)_{j \in \mathbb{Z}^2}$ with $\text{supp } \chi_j \subset D(x_j, h^\rho)$, and such that

$$\sum_{j \in \mathbb{Z}^2} \chi_j^2 = 1, \quad \sum_{j \in \mathbb{Z}^2} |\nabla \chi_j|^2 \leq Ch^{-2\rho}, \quad (C > 1).$$

Lemma 5.2. *We have*

$$\mathcal{Q}_{a,h}(u) = \sum_{j \in \mathbb{Z}^2} \mathcal{Q}_{a,h}(\chi_j u) - h^2 \sum_{j \in \mathbb{Z}^2} \|(\nabla \chi_j)u\|^2.$$

In particular,

$$\mathcal{Q}_{a,h}(u) \geq \sum_{j \in \mathbb{Z}^2} \mathcal{Q}_{a,h}(\chi_j u) - Ch^{2-2\rho}\|u\|^2.$$

Proof. Let us write

$$\begin{aligned} \|d_{h,-\mathbf{A}}^\times u\|^2 &= \sum_{j \in \mathbb{Z}^2} \langle d_{h,-\mathbf{A}}^\times u, d_{h,-\mathbf{A}}^\times (\chi_j^2 u) \rangle \\ &= \sum_{j \in \mathbb{Z}^2} \left(\langle d_{h,-\mathbf{A}}^\times u, [d_{h,-\mathbf{A}}^\times, \chi_j] \chi_j u \rangle + \langle \chi_j d_{h,-\mathbf{A}}^\times u, d_{h,-\mathbf{A}}^\times (\chi_j u) \rangle \right) \\ &= \sum_{j \in \mathbb{Z}^2} \left(\langle \chi_j d_{h,-\mathbf{A}}^\times u, [d_{h,-\mathbf{A}}^\times, \chi_j] u \rangle + \langle \chi_j d_{h,-\mathbf{A}}^\times u, d_{h,-\mathbf{A}}^\times (\chi_j u) \rangle \right) \\ &= \sum_{j \in \mathbb{Z}^2} \left(-\|[d_{h,-\mathbf{A}}^\times, \chi_j]u\|^2 + \langle d_{h,-\mathbf{A}}^\times (\chi_j u), [d_{h,-\mathbf{A}}^\times, \chi_j]u \rangle + \langle \chi_j d_{h,-\mathbf{A}}^\times u, d_{h,-\mathbf{A}}^\times (\chi_j u) \rangle \right) \\ &= \sum_{j \in \mathbb{Z}^2} \left(\|d_{h,-\mathbf{A}}^\times (\chi_j u)\|^2 - \|[d_{h,-\mathbf{A}}^\times, \chi_j]u\|^2 + 2i \text{Im} \langle d_{h,-\mathbf{A}}^\times (\chi_j u), [d_{h,-\mathbf{A}}^\times, \chi_j]u \rangle \right), \end{aligned}$$

where we used that the commutator $[d_{h,-\mathbf{A}}^\times, \chi_j] = -2ih\partial_{\bar{z}}\chi_j$ is a function. Taking the real part, we get

$$\|d_{h,-\mathbf{A}}^\times u\|^2 = \sum_{j \in \mathbb{Z}^2} \left(\|d_{h,-\mathbf{A}}^\times (\chi_j u)\|^2 - \|h(\nabla \chi_j)u\|^2 \right).$$

□

5.2.2. *Lower bound.* Let j be such that $\text{supp}(\chi_j) \subset \Omega$. Then, we have

$$\mathcal{Q}_{a,h}(\chi_j u) = \|d_{h,-\mathbf{A}}^\times(\chi_j u)\|^2 \geq 2hb_0 \|\chi_j u\|^2, \quad (5.4)$$

since the Dirichlet realization of

$$d_{h,-\mathbf{A}}^\times d_{h,-\mathbf{A}}^\times = (-ih\nabla + \mathbf{A})^2 + hB$$

is bounded from below by $2hb_0$.

Therefore, let us focus on the j such that $\text{supp}(\chi_j) \cap \partial\Omega \neq \emptyset$. We may assume that $x_j \in \partial\Omega$.

Let us bound the local energy $\mathcal{Q}_{a,h}(\chi_j u)$ from below.

Proposition 5.3. *We have*

$$\mathcal{Q}_{a,h}(\chi_j u) \geq \left[b'_0 \nu (a(b'_0)^{-1/2}) h - Ch^{\frac{1}{2}+2\rho} \right] \|\chi_j u\|^2. \quad (5.5)$$

Proof. Before starting the proof, let us say a few words about the strategy. The general idea is to approximate the magnetic field, on the support of χ_j , by a constant magnetic field, and to flatten the boundary by means of tubular coordinates. Due to the lack of ellipticity of the Cauchy-Riemann operators, we cannot choose the canonical tubular coordinates (given by the curvilinear abscissa and the distance to the boundary). However, with the exponential coordinates (5.6), we are able to avoid this problem for the disc, and then, by means of the Riemann mapping, for Ω . This amounts to constructing ‘‘conformal’’ tubular coordinates for Ω .

It is convenient to use the change of function

$$u = e^{\phi/h} v.$$

For notational simplicity, we let $u_j = \chi_j u$ and $v_j = \chi_j v$. We have

$$\mathcal{Q}_{a,h}(u_j) = h^2 \int_{\Omega} e^{2\phi/h} |2\partial_{\bar{z}} v_j|^2 dx + ah^{\frac{3}{2}} \|v_j\|_{\partial\Omega}^2.$$

Let us use the Riemann biholomorphism $F : \mathbb{D} \rightarrow \Omega$. We let $w_j = v_j \circ F$. We get

$$\mathcal{Q}_{a,h}(u_j) = 4h^2 \int_{\mathbb{D}} e^{2\phi \circ F(y)/h} |\partial_{\bar{y}} w_j|^2 dy + ah^{\frac{3}{2}} \int_{\partial\mathbb{D}} |w_j|^2 |F'| d\sigma, \quad \partial_{\bar{y}} = \frac{1}{2}(\partial_{y_1} + i\partial_{y_2}).$$

Note that w_j is supported in a neighborhood of order h^ρ of $\partial\mathbb{D}$. Let us now use a change of coordinates near the boundary. Let $\delta > 0$. Consider the ‘‘exponential polar coordinates’’, $y = P(s, \tau)$, given by

$$y_1 = e^{-\tau} \cos s, \quad y_2 = e^{-\tau} \sin s \quad (s, \tau) \in \mathcal{T}_\delta := [0, 2\pi) \times (0, \delta). \quad (5.6)$$

P is a smooth diffeomorphism in a neighborhood of the boundary. We have

$$-e^\tau \partial_s = \sin s \partial_{y_1} - \cos s \partial_{y_2}, \quad -e^\tau \partial_\tau = \cos s \partial_{y_1} + \sin s \partial_{y_2},$$

and we get

$$\partial_{y_1} + i\partial_{y_2} = ie^{\tau+is}(\partial_s + i\partial_\tau).$$

The coordinates of the center x_j of the support of χ_j are denoted by $(s_j, 0)$.

In terms of these new coordinates, we have

$$\begin{aligned} \mathcal{Q}_{a,h}(u_j) &= h^2 \int_{\mathcal{T}_\delta} e^{2\phi \circ F(P(s,\tau))/h} |(\partial_s + i\partial_\tau)(w_j \circ P)|^2 ds d\tau \\ &\quad + ah^{\frac{3}{2}} \int_0^{2\pi} |w_j \circ P(s, 0)|^2 |F'(e^{is})| ds. \end{aligned}$$

We let $\check{\phi} = \phi \circ F \circ P$. Since ϕ is zero at the boundary, we have that $e^{2\check{\phi}(s,0)/h} = 1$. Then, by using that $|F'(e^{is})| \geq (1 - Ch^\rho)|F'(e^{is_j})|$, and by commuting the exponential with the Cauchy-Riemann derivative, we get

$$\begin{aligned} (1 - Ch^\rho)^{-1} \mathcal{Q}_{a,h}(u_j) &\geq \int_{\mathcal{T}_\delta} |(h\partial_s - \partial_s \check{\phi} + ih\partial_\tau - i\partial_\tau \check{\phi})e^{\check{\phi}(s,\tau)/h}(w_j \circ P)|^2 ds d\tau \\ &\quad + ah^{\frac{3}{2}} \int_0^{2\pi} |e^{\check{\phi}(s,0)/h} w_j \circ P(s, 0)|^2 |F'(e^{is_j})| ds. \end{aligned}$$

Then,

$$\begin{aligned} (1 - Ch^\rho)^{-1} \mathcal{Q}_{a,h}(u_j) &\geq \int_{\mathcal{T}_\delta} |(-ih\partial_s + \check{A}_1 + i(-ih\partial_\tau + \check{A}_2))W_j|^2 ds d\tau \\ &\quad + |F'(e^{is_j})| ah^{\frac{3}{2}} \int_0^{2\pi} |W_j(s, 0)|^2 ds, \end{aligned}$$

where $W_j = e^{\check{\phi}(s,\tau)/h}(w_j \circ P)$ and $\check{A} = \nabla \check{\phi}^\perp = (-\partial_\tau \check{\phi}, \partial_s \check{\phi})$. Now, we have a magnetic Cauchy-Riemann problem on a flat space, with a uniform Robin condition.

A computation that uses the identity $(\partial_s + i\partial_\tau)(e^{-\tau+is}) = 0$ and

$$\begin{aligned} \nabla \times \check{A} &= (\partial_s^2 + \partial_\tau^2)(\phi \circ F \circ P) = e^{-2\tau} \Delta_y(\phi \circ F)(P(s, \tau)) \\ &= e^{-2\tau} |F'(P(s, \tau))|^2 B(F(P(s, \tau))) \\ &= \beta_j + \mathcal{O}(|s - s_j| + |\tau - \tau_j|), \end{aligned}$$

gives the new constant magnetic field $\beta_j = |F'(y_j)|^2 B(x_j)$.

Using the Young inequality, we get

$$\begin{aligned} &\int_{\mathcal{T}_\delta} |(-ih\partial_s + \check{A}_1 + i(-ih\partial_\tau + \check{A}_2))W_j|^2 ds d\tau \\ &\geq (1 - \varepsilon) \int_{\mathcal{T}_\delta} |(-ih\partial_s + \check{A}_{1,j} + i(-ih\partial_\tau + \check{A}_{2,j}))W_j|^2 ds d\tau - \varepsilon^{-1} \int_{\mathcal{T}_\delta} |\check{A} - \check{A}_j|^2 |W_j|^2 ds d\tau, \end{aligned}$$

where $\check{A}_j = (\check{A}_{1,j}, \check{A}_{2,j})$ is the Taylor approximation of \check{A} at the order one at (s_j, τ_j) :

$$|\check{A} - \check{A}_j| \leq Ch^{2\rho},$$

on the support of W_j . We get that

$$(1 - Ch^\rho)^{-1} \mathcal{Q}_{a,h}(u_j) \geq (1 - \varepsilon) \mathcal{Q}_j(W_j) - Ch^{4\rho} \varepsilon^{-1} \int_{\mathcal{T}_\delta} |W_j|^2 ds d\tau, \quad (5.7)$$

with

$$\begin{aligned} \mathcal{Q}_j(W) &= \int_{\mathbb{R}_+^2} |(-ih\partial_s + \check{A}_{1,j} + i(-ih\partial_\tau + \check{A}_{2,j}))W|^2 ds d\tau \\ &\quad + |F'(e^{is_j})| ah^{\frac{3}{2}} \int_{\mathbb{R}} |W(s, 0)|^2 ds. \end{aligned}$$

Let us remark that, by construction,

$$\nabla \times \check{A}_j = \beta_j,$$

so that after a change of gauge, we can assume that $\check{A}_j = (-\beta_j\tau, 0)$.

Thus, we get a new quadratic form on $L^2(\mathbb{R}_+^2)$ which is associated with a new operator \mathcal{L}_j . We are interested in the bottom of its spectrum:

$$\inf \text{sp}(\mathcal{L}_j) = \inf_{\substack{W \in \mathfrak{H}_{-\check{A}_j}^2(\mathbb{R}_+^2) \\ W \neq 0}} \frac{\mathcal{Q}_j(W)}{\|W\|^2}.$$

Let us consider the rescaling

$$(s, \tau) = h^{\frac{1}{2}} \beta_j^{-\frac{1}{2}} (\tilde{s}, \tilde{\tau}).$$

We get

$$\inf \text{sp}(\mathcal{L}_j) = h\beta_j\mu_j, \quad \mu_j = \inf_{\substack{W \in \mathfrak{H}_{-\check{A}_j}^2(\mathbb{R}_+^2) \\ W \neq 0}} \frac{\tilde{\mathcal{Q}}_j(W)}{\|W\|^2},$$

where

$$\tilde{\mathcal{Q}}_j(W) = \int_{\mathbb{R}_+^2} |(-i\partial_s - \tau + i(-i\partial_\tau))W|^2 ds d\tau + aB(x_j)^{-\frac{1}{2}} \int_{\mathbb{R}} |W(s, 0)|^2 ds.$$

Then,

$$(1 - Ch^\rho)^{-1} \mathcal{Q}_{a,h}(u_j) \geq [(1 - \varepsilon)h\beta_j\mu_j - Ch^{4\rho}\varepsilon^{-1}] \|W_j\|^2.$$

We choose ε such that

$$\varepsilon h = \varepsilon^{-1} h^{4\rho},$$

so that

$$\varepsilon = h^{-\frac{1}{2}+2\rho},$$

and

$$(1 - Ch^\rho)^{-1} \mathcal{Q}_{a,h}(u_j) \geq [h\beta_j\mu_j - Ch^{\frac{1}{2}+2\rho}] \|W_j\|^2.$$

In particular, we get

$$\mathcal{Q}_{a,h}(u_j) \geq [h\beta_j\mu_j - Ch^{\frac{1}{2}+2\rho} - Ch^{1+\rho}] \|W_j\|^2.$$

Then,

$$\begin{aligned}
\mathcal{Q}_{a,h}(u_j) &\geq \left[h\beta_j\mu_j - Ch^{\frac{1}{2}+2\rho} \right] \int_{\mathcal{T}_\delta} e^{2\check{\phi}(s,\tau)/h} |(v_j \circ F \circ P)|^2 ds d\tau \\
&\geq \left[h\beta_j\mu_j - Ch^{\frac{1}{2}+2\rho} - Ch^{1+\rho} \right] \int_{\mathbb{D}} e^{2\phi(F(y))/h} |(v_j \circ F(y))|^2 dy \\
&= \left[h\beta_j\mu_j - Ch^{\frac{1}{2}+2\rho} - Ch^{1+\rho} \right] \int_{\Omega} e^{2\phi/h} |v_j(x)|^2 |(F^{-1})'(x)|^2 dx \\
&\geq |(F^{-1})'(x_j)|^2 \left[h\beta_j\mu_j - Ch^{\frac{1}{2}+2\rho} - Ch^{1+\rho} \right] \int_{\Omega} e^{2\phi/h} |v_j(x)|^2 dx \\
&\geq \left[hB(x_j)\mu_j - Ch^{\frac{1}{2}+2\rho} - Ch^{1+\rho} \right] \int_{\Omega} e^{2\phi/h} |v_j(x)|^2 dx \\
&\geq \left[hB(x_j)\mu_j - Ch^{\frac{1}{2}+2\rho} \right] \int_{\Omega} e^{2\phi/h} |v_j(x)|^2 dx \\
&= \left[hB(x_j)\mu_j - Ch^{\frac{1}{2}+2\rho} \right] \int_{\Omega} |\chi_j u(x)|^2 dx.
\end{aligned}$$

Then, letting $\mathbf{A}_0 = (-\tau, 0)$, we have

$$\begin{aligned}
B(x_j)\mu_j &= \inf_{\substack{u \in \mathfrak{H}_{-\mathbf{A}_0}^2(\mathbb{R}_+^2) \\ u \neq 0}} \frac{B(x_j) \int_{\mathbb{R}_+^2} |(-i\partial_s - \tau + i(-i\partial_\tau))u|^2 ds d\tau + aB(x_j)^{\frac{1}{2}} \int_{\mathbb{R}} |u(s, 0)|^2 ds}{\|u\|^2} \\
&\geq \inf_{\substack{u \in \mathfrak{H}_{-\mathbf{A}_0}^2(\mathbb{R}_+^2) \\ u \neq 0}} \frac{b'_0 \int_{\mathbb{R}_+^2} |(-i\partial_s - \tau + i(-i\partial_\tau))u|^2 ds d\tau + a(b'_0)^{\frac{1}{2}} \int_{\mathbb{R}} |u(s, 0)|^2 ds}{\|u\|^2} \\
&= b'_0 \nu (a(b'_0)^{-1/2}).
\end{aligned}$$

The result follows. \square

Remark 5.4. It is clear from the proof that we also have a reverse inequality of (5.7):

$$(1 - Ch^\rho)^{-1} \mathcal{Q}_{a,h}(u_j) \leq (1 + \varepsilon) \mathcal{Q}_j(W_j) + Ch^{4\rho} \varepsilon^{-1} \int_{\mathcal{T}_\delta} |W_j|^2 ds d\tau. \quad (5.8)$$

Gathering the estimates (5.4) and (5.5), and using Lemma 5.2, we find that

$$\mathcal{Q}_{a,h}(u) \geq \left[\Lambda(a)h - Ch^{\frac{1}{2}+2\rho} - Ch^{2-2\rho} \right] \|u\|^2.$$

We choose ρ such that

$$\frac{1}{2} + 2\rho = 2 - 2\rho.$$

Thus, $\rho = \frac{3}{8}$ and

$$\mathcal{Q}_{a,h}(u) \geq \left[\Lambda(a)h - Ch^{\frac{5}{4}} \right] \|u\|^2.$$

The min-max principle implies the lower bound in Proposition 5.1.

5.2.3. *Upper bound.* The upper bound in Proposition 5.1 follows by inserting appropriate localized test functions in $\mathcal{Q}_{a,h}$. Let us provide the main lines of the strategy for this classical analysis.

We recall that

$$\gamma_1(ah^{\frac{1}{2}}, h) = \inf_{\substack{u \in \mathfrak{H}_{-\mathbf{A}}^2(\Omega) \\ u \neq 0}} \frac{\mathcal{Q}_{a,h}(u)}{\|u\|^2}.$$

In particular, we have

$$\gamma_1(ah^{\frac{1}{2}}, h) \leq \inf_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\mathcal{Q}_{a,h}(u)}{\|u\|^2} = \inf_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\|(-ih\nabla + \mathbf{A})u\|^2 + \int_{\Omega} hB|u|^2 dx}{\|u\|^2}.$$

The last quantity is the groundstate energy of $(-ih\nabla + \mathbf{A})^2 + hB$. Pick up a point $x_0 \in \Omega$. We can always find a normalized test function φ_h in $\mathcal{C}_0^\infty(\Omega)$, localized at the scale $h^{\frac{1}{2}}$ near x_0 , and such that

$$\|(-ih\nabla + \mathbf{A})\varphi_h\|^2 + \int_{\Omega} hB|\varphi_h|^2 dx \leq 2B(x_0)h + o(h).$$

Now, if B attains its minimum inside at x_0 , then we deduce that

$$\gamma_1(ah^{\frac{1}{2}}, h) \leq 2b_0h + o(h). \quad (5.9)$$

If not, for any $\varepsilon > 0$, we may find $x_0 \in \Omega$ such that $|B(x_0) - b_0| \leq \varepsilon$, and (5.9) is true as well.

On the other hand, let us consider $x_0 \in \partial\Omega$ where the minimum of $B|_{\partial\Omega}$ is attained. Take a fixed cutoff function χ centered at x_0 , and a minimizing sequence $(W_n) \subset \mathcal{S}(\overline{\mathbb{R}_+^2})$ associated with μ . Then, we consider the function $\psi_h(s, \tau) = \chi(s, \tau)W_n((b'_0)^{\frac{1}{2}}h^{-\frac{1}{2}}(s, \tau))$ and its avatar φ_h in the original coordinates (after the maps P and F). Using Remark 5.4 (where u_j is replaced by φ_h), we get

$$\gamma_1(ah^{\frac{1}{2}}, h) \leq hb'_0\nu(a(b'_0)^{-1/2}) + o(h).$$

This, together with (5.9), gives the desired upper bound.

6. A FIRST NORMAL FORM

The aim of this section is to start the proof of Theorem 1.19 by reducing the analysis to a tubular neighborhood of the boundary.

6.1. **Description of the operator.** We consider the closed quadratic form

$$\mathcal{Q}_{a,h}(u) = \|d_{h,-\mathbf{A}}^\times u\|^2 + ah^{\frac{3}{2}}\|u\|_{\partial\Omega}^2, \quad d_{h,-\mathbf{A}}^\times = -2ih\partial_z + A_1 + iA_2, \quad (6.1)$$

for $u \in \text{Dom}(\mathcal{Q}_{a,h}) = \mathfrak{H}_{h,-\mathbf{A}}$ with

$$\begin{aligned} \mathfrak{H}_{h,-\mathbf{A}} &= H^1(\Omega) + \mathcal{H}_{h,-\mathbf{A}}^2(\Omega), \\ \mathcal{H}_{h,-\mathbf{A}}^2(\Omega) &= \{u \in L^2(\Omega) : d_{h,-\mathbf{A}}^\times u = 0, u|_{\partial\Omega} \in L^2(\partial\Omega)\}. \end{aligned}$$

We also let

$$d_{h,-\mathbf{A}} = -2ih\partial_z + A_1 - iA_2.$$

Let us describe the associated self-adjoint operator $\mathcal{L}_{a,h}$. For that purpose, we will need the following lemma.

Lemma 6.1. *For all $u, v \in H^1(\Omega)$,*

$$\langle u, d_{h,-\mathbf{A}}^\times v \rangle = \langle d_{h,-\mathbf{A}} u, v \rangle + ih \int_{\partial\Omega} \bar{n} u \bar{v} d\sigma.$$

Proof. We recall that

$$\langle u, \partial_j v \rangle = -\langle \partial_j u, v \rangle + \int_{\partial\Omega} n_j u \bar{v} d\sigma.$$

Thus,

$$\langle u, -2i\partial_{\bar{z}} v \rangle = \langle -2i\partial_z u, v \rangle + i \int_{\partial\Omega} \bar{n} u \bar{v} d\sigma.$$

□

By Proposition 2.6 and Lemma 6.1, we write for all $u \in \text{Dom}(\mathcal{L}_{a,h})$, and all $v \in \mathfrak{H}_{h,-\mathbf{A}}$,

$$\begin{aligned} & \langle d_{h,-\mathbf{A}}^\times u, d_{h,-\mathbf{A}}^\times v \rangle + ah^{\frac{3}{2}} \int_{\partial\Omega} u \bar{v} d\sigma \\ &= \left\langle \left(d_{h,-\mathbf{A}} d_{h,-\mathbf{A}}^\times \right) u, v \right\rangle + ih \int_{\partial\Omega} d_{h,-\mathbf{A}}^\times u \bar{v} d\sigma + ah^{\frac{3}{2}} \int_{\partial\Omega} u \bar{v} d\sigma, \end{aligned}$$

so that the operator $\mathcal{L}_{a,h}$ acts on $L^2(\Omega)$ as

$$\left(d_{h,-\mathbf{A}} d_{h,-\mathbf{A}}^\times \right) = (-ih\nabla + \mathbf{A})^2 + hB, \quad (6.2)$$

and the boundary condition is

$$-i\bar{n} d_{h,-\mathbf{A}}^\times u = ah^{\frac{1}{2}} u, \quad \text{on } \partial\Omega. \quad (6.3)$$

6.2. Localization near the boundary. We can prove that the eigenfunctions associated with the low-lying eigenvalues have an exponential localization near the boundary, at the scale $h^{\frac{1}{2}}$.

Proposition 6.2. *Let $\varepsilon_0 \in (0, 2b_0)$ and $\gamma \in (0, \sqrt{\varepsilon_0})$. There exist $C, h_0 > 0$ such that for all $h \in (0, h_0)$, all $a > 0$ and all eigenvalue $\lambda \leq (2b_0 - \varepsilon_0)h$ of $\mathcal{L}_{a,h}$ and any eigenfunction ψ_h of $\mathcal{L}_{a,h}$ associated with λ , we have*

$$\left\| \psi_h \exp\left(\frac{\gamma \text{dist}(\cdot, \partial\Omega)}{h^{1/2}}\right) \right\|_{L^2(\Omega)}^2 + h^{-1} \left| \mathcal{Q}_{a,h} \left(\psi_h \exp\left(\frac{\gamma \text{dist}(\cdot, \partial\Omega)}{h^{1/2}}\right) \right) \right| \leq C \|\psi_h\|_{L^2(\Omega)}^2.$$

Before giving the proof, let us recall the following lemma.

Lemma 6.3. *Let $h > 0$, χ be a real Lipschitzian function on Ω and $\psi \in \mathfrak{H}_{h,-\mathbf{A}}$, we have*

$$\text{Re} \langle d_{h,-\mathbf{A}}^\times \psi, d_{h,-\mathbf{A}}^\times (\chi^2 \psi) \rangle_\Omega = \|d_{h,-\mathbf{A}}^\times (\chi \psi)\|^2 - h^2 \|\psi \nabla \chi\|^2.$$

Let us now give the proof of Proposition 6.2.

Proof. Let us define the following Lipschitzian functions

$$\Omega \ni \mathbf{x} \mapsto \Phi(\mathbf{x}) = \gamma \text{dist}(\mathbf{x}, \partial\Omega) \in \mathbb{R},$$

and

$$\Omega \ni \mathbf{x} \mapsto \chi_h(\mathbf{x}) = e^{\Phi(\mathbf{x})h^{-1/2}} \in \mathbb{R}.$$

Since $H^1(\Omega)$ is dense in $\text{Dom}(\mathcal{Q}_{a,h})$ and χ_h is Lipschitzian, we get that $\chi_h^2 \psi_h$ belongs to $\text{Dom}(\mathcal{Q}_{a,h})$. We have that

$$\begin{aligned} \mathcal{Q}_{a,h}(\psi_h, \chi_h^2 \psi_h) &= \text{Re} \langle \mathcal{L}_{a,h} \psi_h, \chi_h^2 \psi_h \rangle_{\Omega} \\ &= \text{Re} \left\{ \langle d_{h,-\mathbf{A}}^{\times} \psi_h, d_{h,-\mathbf{A}}^{\times} (\chi_h^2 \psi_h) \rangle_{\Omega} + ah^{3/2} \|\chi_h \psi_h\|_{\partial\Omega}^2 \right\}. \end{aligned}$$

By Lemma 6.3, we get that

$$\mathcal{Q}_{a,h}(\psi_h, \chi_h^2 \psi_h) = \mathcal{Q}_{a,h}(\chi_h \psi_h) - h^2 \|\psi_h \nabla \chi_h\|_{L^2(\Omega)}^2.$$

Recall that ψ_h is an eigenfunction of $\mathcal{L}_{a,h}$ associated with the eigenvalue λ , so that

$$\mathcal{Q}_{a,h}(\chi_h \psi_h) - h^2 \|\psi_h \nabla \chi_h\|_{L^2(\Omega)}^2 = \lambda \|\chi_h \psi_h\|_{L^2(\Omega)}^2. \quad (6.4)$$

Let $R \geq 1$ and $c > 1$. We introduce a quadratic partition of unity of Ω ,

$$\chi_{1,h,R}^2 + \chi_{2,h,R}^2 = 1,$$

in order to study the asymptotic behavior of ψ_h in the interior and near the boundary $\partial\Omega$ separately. We assume that $\chi_{1,h,R}$ satisfies

$$\chi_{1,h,R}(\mathbf{x}) = \begin{cases} 1 & \text{if } \text{dist}(\mathbf{x}, \partial\Omega) \geq h^{1/2}R \\ 0 & \text{if } \text{dist}(\mathbf{x}, \partial\Omega) \leq h^{1/2}R/2, \end{cases}$$

and that,

$$\max(|\nabla \chi_{1,h,R}(\mathbf{x})|, |\nabla \chi_{2,h,R}(\mathbf{x})|) \leq 2ch^{-1/2}/R,$$

for all $\mathbf{x} \in \Omega$. Using again Lemma 6.3, we get

$$\mathcal{Q}_{a,h}(\chi_h \psi_h) = \sum_{k=1,2} \mathcal{Q}_{a,h}(\chi_{k,h,R} \chi_h \psi_h) - h^2 \|\chi_h \psi_h \nabla \chi_{k,h,R}\|_{L^2(\Omega)}^2.$$

We have $\mathcal{Q}_{a,h}(\chi_{1,h,R} \chi_h \psi_h) \geq 2b_0 h \|\chi_{1,h,R} \chi_h \psi_h\|^2$ by support considerations. Let us also remark that

$$h^2 \|\chi_h \psi_h \nabla \chi_{k,h,R}\|_{L^2(\Omega)}^2 \leq 4hc^2/R^2 \|\chi_h \psi_h\|_{L^2(\Omega)}^2$$

and

$$h^2 \|\psi_h \nabla \chi_h\|_{L^2(\Omega)}^2 \leq h\gamma^2 \|\chi_h \psi_h\|_{L^2(\Omega)}^2.$$

We deduce from (6.4) that

$$\lambda \|\chi_h \psi_h\|_{L^2(\Omega)}^2 \geq \mathcal{Q}_{c,h}(\chi_{1,h,R} \chi_h \psi_h) + \mathcal{Q}_{c,h}(\chi_{2,h,R} \chi_h \psi_h) - h \|\psi_h \chi_h\|_{L^2(\Omega)}^2 (\gamma^2 + 8c^2 R^{-2}),$$

Since $\mathcal{Q}_{a,h}(\chi_{2,h,R} \chi_h \psi_h) \geq 0$, we get

$$\lambda \|\chi_h \psi_h\|_{L^2(\Omega)}^2 \geq 2b_0 h \|\chi_{1,h,R} \chi_h \psi_h\|^2 - h \|\psi_h \chi_h\|_{L^2(\Omega)}^2 (\gamma^2 + 8c^2 R^{-2}),$$

Then,

$$\begin{aligned} &\left(2b_0 h - \lambda - h \left(\gamma^2 + \frac{8c^2}{R^2} \right) \right) \|\chi_{1,h,R} \chi_h \psi_h\|^2 \\ &\leq \lambda \|\chi_{2,h,R} \chi_h \psi_h\|^2 + h \left(\gamma^2 + \frac{8c^2}{R^2} \right) \|\chi_{2,h,R} \chi_h \psi_h\|^2. \end{aligned}$$

Using the assumption on λ and the support of $\chi_{2,h,R}$, we get

$$\begin{aligned} h \left(\varepsilon_0 - \left(\gamma^2 + \frac{8c^2}{R^2} \right) \right) \|\chi_{1,h,R} \chi_h \psi_h\|^2 \\ \leq \lambda \|\chi_{2,h,R} \chi_h \psi_h\|^2 + h \left(\gamma^2 + \frac{8c^2}{R^2} \right) \|\chi_{2,h,R} \chi_h \psi_h\|^2 \leq C(R)h \|\psi_h\|^2. \end{aligned}$$

With $\gamma \in (0, \sqrt{\varepsilon_0})$, and choosing R such that

$$\varepsilon_0 - \left(\gamma^2 + \frac{8c^2}{R^2} \right) > 0,$$

we infer the existence of $\tilde{c} > 0$ such that

$$\tilde{c} \|\chi_{1,h,R} \chi_h \psi_h\|^2 \leq C(R) \|\psi_h\|^2.$$

We deduce that

$$\|\chi_h \psi_h\|_{L^2(\Omega)} \leq C \|\psi_h\|_{L^2(\Omega)},$$

and the conclusion follows by coming back to (6.4). \square

Remark 6.4. When $B = b_0 = 1$ is constant, by Proposition 5.1, for all $a > 0$,

$$\gamma_1(ah^{\frac{1}{2}}, h) = h\Lambda(a) + o(h) = h \min(2, \nu(a)) + o(h).$$

By Proposition 4.15, $\nu(a) < 2$ for all $a > 0$ so that $\Lambda(a) = \nu(a)$.

Thus, for any $0 < \varepsilon < 2 - \nu(a)$, there exist $h_0 > 0$ such that, for all $h \in (0, h_0)$,

$$\gamma_1(ah^{\frac{1}{2}}, h) \leq (2b_0 - \varepsilon)h.$$

With the same proof as the one of Theorem 1.14, we also have for all $n \geq 1$,

$$\gamma_n(ah^{\frac{1}{2}}, h) = h\nu(a) + o(h),$$

and thus up to choosing a smaller $h_0 > 0$ we get for all $0 < h < h_0$,

$$\gamma_n(ah^{\frac{1}{2}}, h) \leq (2b_0 - \varepsilon)h,$$

and the hypothesis of Proposition 6.2 are satisfied for the n -first eigenvalues of $\mathcal{L}_{a,h}$.

From now on, we assume that $B = b_0 = 1$. Let us fix $0 < \varepsilon < 2 - \nu(a)$. Consider

$$N_h(a) = \{n \geq 1 : \gamma_n(ah^{\frac{1}{2}}, h) \leq (2b_0 - \varepsilon)h\}.$$

With Remark 6.4, we see that, for h small enough, $N_h(a)$ contains any given $n \geq 1$.

Let us consider the operator $\widetilde{\mathcal{L}}_{a,h}$ acting on the square integrable functions of the small neighborhood of the boundary $\Omega_\delta = \varphi(S_\delta)$, with $S_\delta = \mathbb{R}/(|\partial\Omega|\mathbb{Z}) \times (0, \delta)$, and where $x = \varphi(s, t)$ corresponds to the holomorphic tubular coordinates given in Section D (where we explain how to construct these coordinates in such a way that $|\partial_s \varphi(s, 0)| = 1$). The operator $\widetilde{\mathcal{L}}_{a,h}$ acts on $L^2(\Omega_\delta)$ as

$$(-ih\nabla + \mathbf{A})^2 + hB, \tag{6.5}$$

and the boundary conditions are

$$\begin{aligned} -i\bar{n}d_{h,-\mathbf{A}}^\times u &= ah^{\frac{1}{2}}u, & \text{on } \partial\Omega, \\ u &= 0, & \text{on } \partial\Omega_\delta \setminus \partial\Omega. \end{aligned} \quad (6.6)$$

We denote by $(\tilde{\gamma}_n(ah^{\frac{1}{2}}, h))_{n \geq 1}$ the increasing sequence of the eigenvalues of the operator $\widetilde{\mathcal{L}}_{a,h}$ counted with multiplicity.

We take $\delta = h^{\frac{1}{2}-\eta}$ with $\eta \in (0, \frac{1}{2})$.

As a consequence of Proposition 6.2 and Remark 6.4, we have the following.

Corollary 6.5. *For all $n \geq 1$, we have*

$$\gamma_n(ah^{\frac{1}{2}}, h) \leq \tilde{\gamma}_n(ah^{\frac{1}{2}}, h),$$

and

$$\tilde{\gamma}_n(ah^{\frac{1}{2}}, h) \leq \gamma_n(ah^{\frac{1}{2}}, h) + \mathcal{O}(h^\infty),$$

uniformly in a , for a in any interval $(0, M)$ with $M > 0$.

6.3. An operator near the boundary. In this section, we write the operator $\widetilde{\mathcal{L}}_{a,h}$ in holomorphic tubular coordinates.

6.3.1. Tubular coordinates. On the neighborhood Ω_δ , we use the holomorphic boundary coordinates

$$x = \varphi(s, t), \quad (s, t) \in S_\delta = \mathbb{R}/(|\partial\Omega|\mathbb{Z}) \times (0, \delta).$$

Lemma 6.6. *We have*

$$-2ih\partial_{\bar{z}} + A_1 + iA_2 = \varphi'|\varphi'|^{-2} \left(hD_s + ihD_t + \tilde{A}_1 + i\tilde{A}_2 \right), \quad \tilde{A}(s, t) = (d\varphi)^T \circ A \circ \varphi(s, t),$$

where $D = -i\partial$. Moreover, we have

$$\partial_s \tilde{A}_2(s, t) - \partial_t \tilde{A}_1(s, t) = |\varphi'(s + it)|^2.$$

Proof. We have

$$\begin{pmatrix} \partial_s \\ \partial_t \end{pmatrix} = \begin{pmatrix} \partial_s \varphi_1 & \partial_s \varphi_2 \\ \partial_t \varphi_1 & \partial_t \varphi_2 \end{pmatrix} \begin{pmatrix} \partial_1 \\ \partial_2 \end{pmatrix},$$

and also

$$\begin{pmatrix} \partial_1 \\ \partial_2 \end{pmatrix} = |\varphi'|^{-2} \begin{pmatrix} \partial_t \varphi_2 & -\partial_s \varphi_2 \\ -\partial_t \varphi_1 & \partial_s \varphi_1 \end{pmatrix} \begin{pmatrix} \partial_s \\ \partial_t \end{pmatrix},$$

where φ' stands for $\varphi'(s + it)$. We deduce that

$$2|\varphi'|^2 \partial_{\bar{z}} = \partial_t \varphi_2 \partial_s - \partial_s \varphi_2 \partial_t + i(-\partial_t \varphi_1 \partial_s + \partial_s \varphi_1 \partial_t) = (\partial_t \varphi_2 - i\partial_t \varphi_1) \partial_s + (-\partial_s \varphi_2 + i\partial_s \varphi_1) \partial_t.$$

Thus,

$$2|\varphi'|^2 \partial_{\bar{z}} = (\partial_s \varphi_1 - i\partial_t \varphi_1) \partial_s + (-\partial_t \varphi_1 + i\partial_s \varphi_1) \partial_t = (\partial_s \varphi_1 + i\partial_s \varphi_2) \partial_s + i(\partial_s \varphi_1 + i\partial_s \varphi_2) \partial_t,$$

so that

$$2|\varphi'|^2 \partial_{\bar{z}} = \partial_s \varphi (\partial_s + i\partial_t).$$

Note also that $\partial_s \varphi(s, t) = \varphi'(s + it)$ and thus

$$2\partial_{\bar{z}} = \varphi'|\varphi'|^{-2} (\partial_s + i\partial_t).$$

Recalling that

$$(d\varphi)^T = \begin{pmatrix} \partial_s \varphi_1 & \partial_s \varphi_2 \\ \partial_t \varphi_1 & \partial_t \varphi_2 \end{pmatrix},$$

we have

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = |\varphi'|^{-2} \begin{pmatrix} \partial_t \varphi_2 & -\partial_s \varphi_2 \\ -\partial_t \varphi_1 & \partial_s \varphi_1 \end{pmatrix} \begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{pmatrix}.$$

Hence

$$\begin{aligned} A_1 + iA_2 &= |\varphi'|^{-2} \left(\partial_t \varphi_2 \tilde{A}_1 - \partial_s \varphi_2 \tilde{A}_2 + i(-\partial_t \varphi_1 \tilde{A}_1 + \partial_s \varphi_1 \tilde{A}_2) \right) \\ &= |\varphi'|^{-2} \left((\partial_t \varphi_2 - i\partial_t \varphi_1) \tilde{A}_1 + (-\partial_s \varphi_2 + i\partial_s \varphi_1) \tilde{A}_2 \right) \\ &= |\varphi'|^{-2} \partial_s \varphi \left(\tilde{A}_1 + i\tilde{A}_2 \right) = |\varphi'|^{-2} \varphi' \left(\tilde{A}_1 + i\tilde{A}_2 \right). \end{aligned}$$

The conclusion follows. \square

We recall that that quadratic form associated with $\widetilde{\mathcal{L}}_{a,h}$ is given by

$$\mathcal{Q}_{a,h}(u) = \int_{\Omega_\delta} |(-2ih\partial_{\bar{z}} + A_1 + iA_2)u|^2 dx + ah^{\frac{3}{2}} \int_{\partial\Omega} |u|^2 d\sigma.$$

With Lemma 6.6, we deduce that

$$\mathcal{Q}_{a,h}(u) = \widetilde{\mathcal{Q}}_{a,h}(\tilde{u}) = \int_{S_\delta} |(hD_s + ihD_t + \tilde{A}_1 + i\tilde{A}_2)\tilde{u}|^2 dsdt + ah^{\frac{3}{2}} \int_0^{|\partial\Omega|} |\tilde{u}(s,0)|^2 ds,$$

where $\tilde{u}(s,t) = u \circ \varphi(s,t)$, and where we used that $|\partial_s \varphi(s,0)| = 1$ to deal with the boundary term (see Appendix D). The ambient Hilbert space is now $L^2(|\varphi'|^2 dsdt)$. To go to the flat L^2 -space, we let $\hat{u} = \varphi' \tilde{u}$. Since φ' is holomorphic, it commutes with the $\partial_s + i\partial_t$, and we deduce that

$$\mathcal{Q}_{a,h}(u) = \widehat{\mathcal{Q}}_{a,h}(\hat{u}) = \int_{S_\delta} |\varphi'|^{-2} (hD_s + ihD_t + \tilde{A}_1 + i\tilde{A}_2)\hat{u}|^2 dsdt + ah^{\frac{3}{2}} \int_0^{|\partial\Omega|} |\hat{u}(s,0)|^2 ds.$$

The operators $\widetilde{\mathcal{L}}_{a,h}$ and $\widehat{\mathcal{L}}_{a,h}$ are unitarily equivalent.

6.3.2. *Change of gauge.* Let us now use an appropriate change of gauge to cancel \tilde{A}_2 . Consider

$$\psi_1(s,t) = \int_0^t \tilde{A}_2(s,\tau) d\tau, \quad \widehat{\mathbf{A}} = \tilde{\mathbf{A}} - \nabla \psi_1.$$

Notice that

$$\hat{A}_1(s,t) = \tilde{A}_1(s,t) - \int_0^t \partial_s \tilde{A}_2(s,u) du, \quad \hat{A}_2(s,t) = 0.$$

Clearly, with Lemma 6.6,

$$-\partial_t \hat{A}_1 = \partial_s \tilde{A}_2 - \partial_t \tilde{A}_1 = |\varphi'|^2.$$

so that, we have

$$\hat{A}_1(s,t) = \hat{A}_1(s,0) - \int_0^t |\varphi'(s+ir)|^2 dr.$$

Now, consider

$$\psi_2(s) = \int_0^s \left(\hat{A}_1(u, 0) - \frac{1}{|\partial\Omega|} \int_0^{|\partial\Omega|} \hat{A}_1(v, 0) dv \right) du.$$

The function ψ_2 is $|\partial\Omega|$ -periodic. We let

$$\check{\mathbf{A}} = \hat{\mathbf{A}} - \nabla\psi_2 = \tilde{\mathbf{A}} - \nabla\psi,$$

where

$$\psi = \psi_1 + \psi_2.$$

We find that

$$\check{A}_1(s, t) = \frac{1}{|\partial\Omega|} \int_0^{|\partial\Omega|} \hat{A}_1(v, 0) dv - \int_0^t |\varphi'(s + ir)|^2 dr, \quad \check{A}_2(s, t) = 0.$$

By the Green-Riemann formula,

$$\int_0^{|\partial\Omega|} \hat{A}_1(v, 0) dv = \int_0^{|\partial\Omega|} \tilde{A}_1(v, 0) dv = \int_{\partial\Omega} \mathbf{A} \cdot \boldsymbol{\gamma}' ds = \int_{\Omega} \operatorname{curl} \mathbf{A} dx = |\Omega|.$$

Thus,

$$\check{A}_1(s, t) = \gamma_0 - \int_0^t |\varphi'(s + ir)|^2 dr, \quad \check{A}_2(s, t) = 0, \quad \gamma_0 = \frac{|\Omega|}{|\partial\Omega|}.$$

Letting $\hat{u} = e^{-i\psi/h} \check{u}$, we get

$$\hat{\mathcal{Q}}_{a,h}(\hat{u}) = \int_{S_\delta} |\varphi'|^{-2} (hD_s + ihD_t + \check{A}_1) \check{u}^2 ds dt + ah^{\frac{3}{2}} \int_0^{|\partial\Omega|} |\check{u}(s, 0)|^2 ds.$$

The associated operator is

$$\mathfrak{M}_{a,h} = (-ih\partial_s - h\partial_t + \check{A}_1(s, t)) |\varphi'|^{-2} (-ih\partial_s + h\partial_t + \check{A}_1(s, t))$$

and the boundary conditions are

$$(-ih\partial_s + \gamma_0 + h\partial_t) u(s, 0) = ah^{1/2} u(s, 0), \quad u(s, \delta) = 0.$$

Remark 6.7. By unitary equivalence, the eigenvalues of $\mathfrak{M}_{a,h}$ are $(\tilde{\gamma}_n(ah^{\frac{1}{2}}, h))_{n \geq 1}$.

6.3.3. *Rescaling.* We let $t = h\tau$ with $\hbar = h^{\frac{1}{2}}$. We also divide the operator by h and we get the operator, called $\mathcal{M}_{a,\hbar}$, acting on the Hilbert space $L^2(S_{\delta\hbar^{-1}}, ds d\tau)$ as

$$\mathcal{M}_{a,\hbar} = (-i\hbar\partial_s - \partial_\tau + \hbar^{-1} \check{A}_1(s, \hbar\tau)) |\varphi'(s + i\hbar\tau)|^{-2} (-i\hbar\partial_s + \partial_\tau + \hbar^{-1} \check{A}_1(s, \hbar\tau)),$$

and the boundary conditions are

$$\left(-i\hbar\partial_s + \frac{\gamma_0}{\hbar} + \partial_\tau \right) u(s, 0) = au(s, 0), \quad u(s, \delta\hbar^{-1}) = 0.$$

6.3.4. *Another change of gauge.* For all $m \in \mathbb{Z}$, the function $s \mapsto e^{2im\pi \frac{s}{|\partial\Omega|}}$ is $|\partial\Omega|$ -periodic. Thus, the operators

$$\mathcal{M}_{a,\hbar,m} := e^{2i\pi m \frac{s}{|\partial\Omega|}} \mathcal{M}_{a,\hbar} e^{-2i\pi m \frac{s}{|\partial\Omega|}},$$

are unitarily equivalent to $\mathcal{M}_{a,\hbar}$ and act as

$$\mathcal{M}_{a,\hbar,m} = (-i\hbar\partial_s - \partial_\tau + A_{1,\hbar,m}(s, \tau)) |\varphi'(s + i\hbar\tau)|^{-2} (-i\hbar\partial_s + \partial_\tau + A_{1,\hbar,m}(s, \tau)),$$

where

$$A_{1,\hbar,m}(s, \tau) = -\frac{2\pi m\hbar}{|\partial\Omega|} + \hbar^{-1} \check{A}_1(s, \hbar\tau).$$

The boundary conditions are

$$\left(-i\hbar\partial_s + \left(\frac{\gamma_0}{\hbar} - \frac{2\pi m\hbar}{|\partial\Omega|} \right) + \partial_\tau \right) u(s, 0) = au(s, 0), \quad u(s, \delta\hbar^{-1}) = 0.$$

Let us make a particular choice of m . This choice is made so that $\frac{\gamma_0}{\hbar} - \frac{2m\pi\hbar}{|\partial\Omega|}$ is the closest possible to 0. Consider

$$d_\hbar := \min_{m \in \mathbb{Z}} \left| \frac{|\partial\Omega|}{2\pi\hbar} \frac{\gamma_0}{\hbar} - m \right| \in \left[0, \frac{1}{2} \right].$$

Let us denote by m_\hbar the smallest minimizer (there are at most two minimizers), and let

$$r_\hbar := \frac{|\partial\Omega|}{2\pi\hbar} \frac{\gamma_0}{\hbar} - m_\hbar.$$

We have $d_\hbar = |r_\hbar|$ and we can write

$$\frac{\gamma_0}{\hbar} - \frac{2m_\hbar\pi\hbar}{|\partial\Omega|} = \hbar\theta, \quad \theta = \frac{2\pi}{|\partial\Omega|} r_\hbar.$$

The constant θ is uniformly bounded:

$$|\theta| \leq \frac{\pi}{|\partial\Omega|}.$$

With this choice of m_\hbar , we get the new (self-adjoint) operator

$$\mathcal{N}_{a,\hbar} = (-i\hbar\partial_s - \partial_\tau + A_{1,\hbar,m_\hbar}) |\varphi'(s + i\hbar\tau)|^{-2} (-i\hbar\partial_s + \partial_\tau + A_{1,\hbar,m_\hbar}), \quad (6.7)$$

where

$$A_{1,\hbar,m_\hbar}(s, \tau) = \hbar\theta - \hbar^{-1} \int_0^{\hbar\tau} |\varphi'(s, u)|^2 du,$$

and

$$\theta = \frac{2\pi r_\hbar}{|\partial\Omega|} = \frac{|\Omega|}{\hbar^2 |\partial\Omega|} - \frac{2m_\hbar\pi}{|\partial\Omega|}. \quad (6.8)$$

The boundary conditions are

$$(-i\hbar\partial_s + \hbar\theta + \partial_\tau) u(s, 0) = au(s, 0), \quad u(s, \delta\hbar^{-1}) = 0.$$

Remark 6.8. The eigenvalues of $\mathcal{N}_{a,\hbar}$ are the $(h^{-1} \tilde{\gamma}_n(a\hbar^{\frac{1}{2}}, h))_{n \geq 1}$.

7. MICROLOCAL DIMENSIONAL REDUCTION

Let us now focus on the spectral properties of $\mathcal{N}_{a,\hbar}$ defined in (6.7), and its boundary conditions are with the boundary conditions

$$(-i\hbar\partial_s + \partial_\tau)u(s, 0) = \alpha u(s, 0), \quad u(s, \delta\hbar^{-1}) = 0, \quad \alpha := a - \theta\hbar.$$

We underline that θ depends on \hbar and α , but that it is uniformly bounded. In what follows, θ will be consider a parameter.

7.1. Inserting cutoff functions and pseudo-differential interpretation.

7.1.1. *Cutoff with respect to the normal variable.* We can prove that the first eigenfunctions of $\mathcal{N}_{a,\hbar}$ satisfy Agmon estimates with respect to τ (in Proposition 6.2, $\text{dist}(x, \partial\Omega)$ can essentially be replaced by $t(x)$ near the boundary since we have (D.1)).

This leads to consider the operator, acting on $L^2(\mathbb{R}/(|\partial\Omega|\mathbb{Z}) \times (0, +\infty))$,

$$\widetilde{\mathcal{N}}_{a,\hbar} = (-i\hbar\partial_s - \partial_\tau + A_{1,\hbar,m_\hbar}^X)|\varphi'(s + i\hbar\chi_h(\tau)\tau)|^{-2}(-i\hbar\partial_s + \partial_\tau + A_{1,\hbar,m_\hbar}^X),$$

with

$$A_{1,\hbar,m_\hbar}^X(s, \tau) = \hbar\theta - \tau - \chi_h(\tau)\hbar^{-1} \left(\int_0^{\hbar\tau} |\varphi'(s, u)|^2 du - \hbar\tau \right)$$

and $\chi_h(\tau) = \chi(\tau\hbar^\eta)$ where χ is a smooth cutoff function equaling 1 near 0, and $\eta > 0$ being as small as necessary.

We can then check that the low-lying eigenvalues of $\mathcal{N}_{a,\hbar}$ coincide with those of $\widetilde{\mathcal{N}}_{a,\hbar}$ modulo $\mathcal{O}(\hbar^\infty)$. Let us drop the tildas to lighten the presentation.

7.1.2. *Pseudo-differential interpretation.* This operator $\widetilde{\mathcal{N}}_{a,\hbar}$ can be seen as a pseudo-differential operator with operator symbol (for more detail, the reader can consult the Ph.D. thesis by Keraval [23], or the paper by Martinez [26]). To describe its symbol, let us consider some Taylor expansions (see Lemma D.4):

$$|\varphi'(s + i\hbar\chi_h(\tau)\tau)|^{-2} = 1 + 2\hbar\kappa\chi_h\tau + 2\hbar\kappa^2\chi_h^2\tau^2 + o(\hbar^2),$$

$$A_{1,\hbar,m_\hbar}^X(s, \tau) = \hbar\theta - \tau + \hbar\kappa\chi_h\tau^2 - \frac{2}{3}\hbar^2\kappa^2\chi_h\tau^3 + o(\hbar^2).$$

Then, we have

$$\mathcal{N}_{a,\hbar} = \text{Op}_\hbar^W(n_\hbar), \quad n_\hbar = n_0 + \hbar n_1 + \hbar^2 n_2 + \dots \quad (7.1)$$

where the first symbols are given by

$$\begin{aligned} n_0 &= (\xi - \partial_\tau - \tau)(\xi + \partial_\tau - \tau) = -\partial_\tau^2 + (\xi - \tau)^2 + 1, \\ n_1 &= \kappa \left(2(\xi - \tau - \partial_\tau)\chi_h\tau(\xi - \tau + \partial_\tau) + \chi_h\tau^2(\xi - \tau + \partial_\tau) + (\xi - \tau - \partial_\tau)(\chi_h\tau^2) \right) \\ &\quad + 2\theta(\xi - \tau), \\ n_2 &= \kappa^2 \left[2(-\partial_\tau + \xi - \tau)\chi_h^2\tau^2(\partial_\tau + \xi - \tau) \right. \\ &\quad \left. + \frac{4}{3} \left((-\partial_\tau + \xi - \tau)(\chi_h\tau^3) + (\chi_h\tau^3)(\partial_\tau + \xi - \tau) \right) + \chi_h^2\tau^4 \right] \\ &\quad + \chi_h\kappa\theta \left(2\tau^2 - 2 + 4\tau(\xi - \tau) \right) + \theta^2 - 2\chi_h'\kappa\theta\tau, \end{aligned} \quad (7.2)$$

where n_0 is equipped with the boundary condition

$$(\partial_\tau + \xi - \alpha)\psi = 0.$$

Actually, we have

$$n_1 = \kappa\chi_h\mathcal{C}_\xi + 2\theta(\xi - \tau) - \kappa\chi'_h(2\tau(\xi + \partial_\tau - \tau) + \tau^2), \quad \mathcal{C}_\xi = 2(\tau n_0 - \xi - \partial_\tau + \tau^2(\xi - \tau)),$$

We also notice that

$$\begin{aligned} n_2 = \kappa^2 & \left[-4\tau\chi_h^2(\partial_\tau + \xi - \tau) + 2\tau^2\chi_h^2n_0 + \frac{8}{3}(\xi - \tau)\chi_h\tau^3 - 4\chi_h\tau^2 + \chi_h^2\tau^4 \right] \\ & + \chi_h\kappa\theta(2\tau^2 - 2 + 4\tau(\xi - \tau)) + \theta^2 - 2\chi'_h\kappa\theta\tau - 4\chi_h\chi'_h\tau^2(\partial_\tau + \xi - \tau) - \frac{4}{3}\chi'_h\tau^3. \end{aligned}$$

Remark 7.1. In our periodic framework, the usual Weyl quantization formula on \mathbb{R} may be expressed by means of Fourier series (on the torus $\mathbb{R}/(2L\mathbb{Z})$):

$$\text{Op}_h^W(p)\psi(x) = \sum_{(k,j) \in \mathbb{Z}^2} e^{ix(j+k)\omega} \widehat{p}\left(j, \omega\frac{j\hbar}{2} + \omega\hbar k\right) \widehat{\psi}(k), \quad \omega = \frac{2\pi}{2L},$$

where the Fourier coefficient is defined by

$$\widehat{\psi}(k) = \frac{1}{2L} \int_0^{2L} \psi(x) e^{-ik\omega x} dx.$$

Let us briefly recall where this formula comes from. We have

$$\begin{aligned} \text{Op}_h^W(p)\psi(x) &= \frac{1}{2\pi\hbar} \int_{\mathbb{R}^2} e^{i(x-y)\eta/\hbar} p\left(\frac{x+y}{2}, \eta\right) \psi(y) dy d\eta \\ &= \frac{1}{2\pi\hbar} \sum_{(j,k) \in \mathbb{Z}^2} \widehat{\psi}(k) \int_{\mathbb{R}^2} e^{i(x-y)\eta/\hbar} \widehat{p}(j, \eta) e^{ij\omega\frac{x+y}{2}} e^{ik\omega y} dy d\eta \\ &= \frac{1}{2\pi\hbar} \sum_{(j,k) \in \mathbb{Z}^2} e^{ij\omega\frac{x}{2}} \widehat{\psi}(k) \int_{\mathbb{R}} dy e^{i\omega y(k+\frac{j}{2})} \int_{\mathbb{R}} d\eta e^{-i(y-x)\eta/\hbar} \widehat{p}(j, \eta) \\ &= \frac{1}{2\pi\hbar} \sum_{(j,k) \in \mathbb{Z}^2} e^{ij\omega\frac{x}{2}} \widehat{\psi}(k) \int_{\mathbb{R}} dy e^{i\omega y(k+\frac{j}{2})} \mathcal{F}\widehat{p}\left(j, \frac{y-x}{\hbar}\right) \\ &= \sum_{(j,k) \in \mathbb{Z}^2} e^{ij\omega\frac{x}{2}} e^{i\omega x(k+\frac{j}{2})} \widehat{\psi}(k) \frac{1}{2\pi} \int_{\mathbb{R}} dz e^{i\hbar\omega z(k+\frac{j}{2})} \mathcal{F}\widehat{p}(\eta, z) \\ &= \sum_{(j,k) \in \mathbb{Z}^2} e^{ij\omega\frac{x}{2}} e^{i\omega x(k+\frac{j}{2})} \widehat{\psi}(k) \widehat{p}\left(j, \hbar\omega\left(k + \frac{j}{2}\right)\right). \end{aligned}$$

When p only depends on ξ , $p(x, \xi) = \chi(\xi)$, this formula becomes

$$\text{Op}_h^W(p)\psi(x) = \sum_{k \in \mathbb{Z}} e^{ixk\omega} \chi(\omega\hbar k) \widehat{\psi}(k).$$

7.1.3. *Microlocal cutoff.* Then, we insert cutoff functions with respect to ξ , and we consider

$$\check{\mathcal{N}}_{a,h} = \text{Op}_h^W(\check{n}_h), \quad \check{n}_h(s, \xi) = n_h(s, \chi_0(\xi)\xi),$$

where χ_0 is a smooth cutoff function supported near ξ_a . In this way, \check{n}_h belongs to a suitable symbol class (essentially, this means that everything is going as if $\check{n}_h \in S(1)$, $S(1)$ being the class of bounded symbols), see [11, Section 3] and [23] where similar classes are used.

Proposition 7.2. *The low-lying eigenvalues of $\mathcal{N}_{a,h}$ and those of $\check{\mathcal{N}}_{a,h}$ coincide module $\mathcal{O}(\hbar^\infty)$.*

Proof. The key is to prove that the eigenfunctions of $\widetilde{\mathcal{N}}_{a,h}$, which is a perturbation of $\text{Op}_h^W n_0$, are microlocalized near ξ_a . This fact comes from the behavior of the principal (operator) symbol n_0 (its first eigenvalue, as a function of ξ has a unique minimum, which is not attained at infinity). A similar analysis can be found in [11, Section 5]. \square

Corollary 7.3. *For all $n \geq 1$, we have*

$$h^{-1}\lambda_n(a, h) = \lambda_n(\check{\mathcal{N}}_{a,h}) + \mathcal{O}(\hbar^\infty),$$

uniformly with respect to a in a bounded interval.

7.2. **Construction of a parametrix.** Let us now work on the operator with the cutoff functions.

Lemma 7.4. *Consider the operator*

$$\mathcal{P}_0 = \begin{pmatrix} \check{n}_0 - z & \cdot \check{u}_\xi \\ \langle \cdot, \check{u}_{\alpha,\xi} \rangle & 0 \end{pmatrix}, \quad \check{u}_{\alpha,\xi} = u_{\alpha,\xi\chi_0(\xi)}.$$

For z sufficiently close to $\nu(a) = \nu(a, \xi_a)$, \mathcal{P}_0 is bijective and

$$\mathcal{Q}_0 := \mathcal{P}_0^{-1} = \begin{pmatrix} p_0^{-1} & \cdot \check{u}_{\alpha,\xi} \\ \langle \cdot, \check{u}_{\alpha,\xi} \rangle & z - \nu(\alpha, \xi\chi_0(\xi)) \end{pmatrix}, \quad p_0^{-1} = (n_0 - z)^{-1}\Pi^\perp.$$

The aim of this section is to prove the following proposition.

Proposition 7.5. *We let*

$$\mathcal{P}_h = \text{Op}_h^W \begin{pmatrix} \check{n}_h - z & \cdot \check{u}_{\alpha,\xi} \\ \langle \cdot, \check{u}_{\alpha,\xi} \rangle & 0 \end{pmatrix},$$

and, for $j \geq 2$,

$$\mathcal{P}_j = \begin{pmatrix} \check{n}_j & 0 \\ 0 & 0 \end{pmatrix}.$$

Consider the operator symbols defined by

$$\mathcal{Q}_1 = -\mathcal{Q}_0\mathcal{P}_1\mathcal{Q}_0,$$

and

$$\mathcal{Q}_2 = -\mathcal{Q}_0\mathcal{P}_2\mathcal{Q}_0 - \mathcal{Q}_1\mathcal{P}_1\mathcal{Q}_0 - \frac{1}{2i}(\partial_\xi\mathcal{Q}_0 \cdot \partial_s\mathcal{P}_1 - \partial_s\mathcal{Q}_1 \cdot \partial_\xi\mathcal{P}_0)\mathcal{Q}_0.$$

Then, we have, for some $N_3 \in \mathbb{N}$,

$$\text{Op}_h^W(\mathcal{Q}_0 + \hbar\mathcal{Q}_1 + \hbar^2\mathcal{Q}_2)\mathcal{P}_h = \text{Id} + \hbar^3\mathcal{O}(\langle \tau \rangle^{N_3}).$$

More generally, we can find $(\mathcal{Q}_j)_{1 \leq j \leq J}$, and $N_J \in \mathbb{N}$ such that

$$\text{Op}_h^W(\mathcal{Q}_0 + \hbar \mathcal{Q}_1 + \hbar^2 \mathcal{Q}_2 + \dots + \hbar^J \mathcal{Q}_J) \mathcal{P}_h = \text{Id} + \hbar^{J+1} \mathcal{O}(\langle \tau \rangle^{N_J}). \quad (7.3)$$

Moreover, the bottom right coefficient of

$$\mathcal{Q}_0 + \hbar \mathcal{Q}_1 + \hbar^2 \mathcal{Q}_2$$

is

$$q_h^\pm = z - \nu(\alpha, \xi) - \hbar \langle \check{n}_1 \check{u}_{\alpha, \xi}, \check{u}_{\alpha, \xi} \rangle + \hbar^2 \langle (\check{n}_1 p_0^{-1} \check{n}_1 - \check{n}_2(s, \xi \chi_0(\xi))) \check{u}_{\alpha, \xi}, \check{u}_{\alpha, \xi} \rangle.$$

Proof. We have

$$\mathcal{Q}_0 \mathcal{P}_0 = \text{Id}.$$

Let us consider the operator \mathcal{Q}_1 defined by the relation:

$$\mathcal{Q}_0 \mathcal{P}_1 + \mathcal{Q}_1 \mathcal{P}_0 + \frac{1}{2i} \{\mathcal{Q}_0, \mathcal{P}_0\} = 0,$$

Note that $\{\mathcal{Q}_0, \mathcal{P}_0\} = 0$ since \mathcal{P}_0 does not depend on s . Then, consider also \mathcal{Q}_2 defined by

$$\mathcal{Q}_0 \mathcal{P}_2 + \mathcal{Q}_1 \mathcal{P}_1 + \mathcal{Q}_2 \mathcal{P}_0 + \frac{1}{2i} (\{\mathcal{Q}_0, \mathcal{P}_1\} + \{\mathcal{Q}_1, \mathcal{P}_0\}) = 0.$$

With these choices, we have, thanks to the Weyl calculus,

$$\text{Op}_h^W(\mathcal{Q}_0 + \hbar \mathcal{Q}_1 + \hbar \mathcal{Q}_2) \mathcal{P}_h = \text{Id} + \mathcal{O}(\hbar^3).$$

Let us compute the bottom right coefficient of

$$\mathcal{Q}_0 + \hbar \mathcal{Q}_1 + \hbar \mathcal{Q}_2.$$

We have

$$\mathcal{Q}_1 = -\mathcal{Q}_0 \mathcal{P}_1 \mathcal{Q}_0 = - \begin{pmatrix} p_0^{-1} \check{n}_1 p_0^{-1} & p_0^{-1} \check{n}_1 q_0^+ \\ q_0^- \check{n}_1 p_0^{-1} & q_0^- \check{n}_1 q_0^+ \end{pmatrix}.$$

Then,

$$\mathcal{Q}_2 = -\mathcal{Q}_0 \mathcal{P}_2 \mathcal{Q}_0 - \mathcal{Q}_1 \mathcal{P}_1 \mathcal{Q}_0 - \frac{1}{2i} (\partial_\xi \mathcal{Q}_0 \cdot \partial_s \mathcal{P}_1 - \partial_s \mathcal{Q}_1 \cdot \partial_\xi \mathcal{P}_0) \mathcal{Q}_0.$$

Note that

$$\partial_\xi \mathcal{Q}_0 \cdot \partial_s \mathcal{P}_1 \cdot \mathcal{Q}_0 = \kappa' \begin{pmatrix} \partial_\xi p_0^{-1} & \cdot \partial_\xi \check{u}_{\alpha, \xi} \\ \langle \cdot, \partial_\xi \check{u}_{\alpha, \xi} \rangle & -\partial_\xi \nu(\alpha, \xi) \end{pmatrix} \begin{pmatrix} \mathcal{C}_\xi & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p_0^{-1} & \cdot \check{u}_{\alpha, \xi} \\ \langle \cdot, \check{u}_{\alpha, \xi} \rangle & z - \nu(\alpha, \xi \chi_0(\xi)) \end{pmatrix}.$$

Then, we have

$$q_2^\pm = -q_0^- \check{n}_2 q_0^+ + q_0^- \check{n}_1 p_0^{-1} \check{n}_1 q_0^+ - \frac{\kappa'}{2i} (\langle \mathcal{C}_\xi \check{u}_{\alpha, \xi}, \partial_\xi \check{u}_{\alpha, \xi} \rangle - \partial_s \mathcal{Q}_1 \cdot \partial_\xi \mathcal{P}_0 \cdot \mathcal{Q}_0).$$

Note that

$$-\partial_s \mathcal{Q}_1 \cdot \partial_\xi \mathcal{P}_0 \cdot \mathcal{Q}_0 = \partial_s \mathcal{Q}_1 \cdot \mathcal{P}_0 \cdot \partial_\xi \mathcal{Q}_0,$$

and then

$$\partial_s \mathcal{Q}_1 \cdot \mathcal{P}_0 \cdot \partial_\xi \mathcal{Q}_0 = -\kappa' \begin{pmatrix} p_0^{-1} \mathcal{C} p_0^{-1} & p_0^{-1} \mathcal{C} q_0^+ \\ q_0^- \mathcal{C} p_0^{-1} & q_0^- \mathcal{C} q_0^+ \end{pmatrix} \begin{pmatrix} \check{n}_0 - z & \cdot \check{u}_{\alpha, \xi} \\ \langle \cdot, \check{u}_{\alpha, \xi} \rangle & 0 \end{pmatrix} \begin{pmatrix} \partial_\xi p_0^{-1} & \cdot \partial_\xi \check{u}_{\alpha, \xi} \\ \langle \cdot, \partial_\xi \check{u}_{\alpha, \xi} \rangle & -\partial_\xi \nu(\alpha, \xi) \end{pmatrix}.$$

We deduce that

$$\begin{aligned} q_2^\pm &= -q_0^- \check{n}_2 q_0^+ + q_0^- \check{n}_1 p_0^{-1} \check{n}_1 q_0^+ - \frac{\kappa'}{2i} (\langle \mathcal{C}_\xi \check{u}_{\alpha, \xi}, \partial_\xi \check{u}_{\alpha, \xi} \rangle - \langle \mathcal{C}_\xi \partial_\xi \check{u}_{\alpha, \xi}, \check{u}_{\alpha, \xi} \rangle) \\ &= -q_0^- \check{n}_2 q_0^+ + q_0^- \check{n}_1 p_0^{-1} \check{n}_1 q_0^+, \end{aligned}$$

where we used Lemma 4.16. The existence of the \mathcal{Q}_j in (7.3) can be obtained by induction. \square

From now on, we fix $J \geq 2$.

Proposition 7.6. *We have*

$$\text{dist}(0, \text{sp}(Q_{z,J}^\pm)) \|\psi\| \leq C \left(1 + \text{dist}(0, \text{sp}(Q_{z,J}^\pm)) \right) (\|(\mathcal{N}_{a,h} - z)\psi\| + \hbar^{J+1} \|\langle \tau \rangle^{N_J} \psi\|).$$

Proof. We let

$$\mathcal{Q}_h = \text{Op}_h^W(\mathcal{Q}_0 + \hbar \mathcal{Q}_1 + \hbar^2 \mathcal{Q}_2 + \dots + \hbar^J \mathcal{Q}_J) = \begin{pmatrix} Q & Q^+ \\ Q^- & Q^\pm \end{pmatrix},$$

and recall that

$$\mathcal{P}_h = \begin{pmatrix} \mathcal{N}_{a,h} - z & \Pi^* \\ \Pi & 0 \end{pmatrix}, \quad \Pi^* = \text{Op}_h^W(\cdot u_\xi), \quad \Pi = \text{Op}_h^W\langle \cdot, u_\xi \rangle.$$

We have

$$\mathcal{Q}_h \mathcal{P}_h = \text{Id} + \hbar^{J+1} \mathcal{O}(\langle \tau \rangle^{N_J}).$$

This implies that

$$Q^-(\mathcal{N}_{a,h} - z) + Q^\pm \Pi = \hbar^{J+1} \mathcal{O}(\langle \tau \rangle^{N_J}),$$

and

$$Q(\mathcal{N}_{a,h} - z) + Q^+ \Pi = \text{Id} + \hbar^{J+1} \mathcal{O}(\langle \tau \rangle^{N_J}).$$

We get that

$$\|\psi\| \leq C \|\Pi \psi\| + C \|(\mathcal{N}_{a,h} - z)\psi\| + C \hbar^{J+1} \|\langle \tau \rangle^{N_J} \psi\|,$$

and

$$\|Q^\pm \Pi \psi\| \leq C \|(\mathcal{N}_{a,h} - z)\psi\| + C \hbar^{J+1} \|\langle \tau \rangle^{N_J} \psi\|.$$

Then, since Q^\pm is self-adjoint and by using the spectral theorem,

$$\|\psi\| \leq C \left(1 + \frac{1}{\text{dist}(0, \text{sp}(Q^\pm))} \right) (\|(\mathcal{N}_{a,h} - z)\psi\| + \hbar^3 \|\langle \tau \rangle^{N_J} \psi\|),$$

so that the conclusion follows. \square

Proposition 7.7. *We have*

$$\text{dist}(\text{sp}(\mathcal{N}_{a,h}), z) \|\psi\| \leq C \|Q_{z,J}^\pm \psi\| + C \hbar^{J+1} \|\psi\|.$$

Proof. We have

$$\mathcal{P}_h \mathcal{Q}_h = \text{Id} + \mathcal{O}(\hbar^{J+1}).$$

It follows that

$$(\mathcal{N}_h - z)Q^+ + \Pi^* Q_{z,J}^\pm = \mathcal{O}(\hbar^{J+1}),$$

and

$$\Pi Q^+ = \text{Id} + \mathcal{O}(\hbar^{J+1}).$$

We get

$$\|(\mathcal{N}_{a,\hbar} - z)(Q^+\psi)\| \leq C\|Q_{z,J}^\pm\psi\| + C\hbar^{J+1}\|\psi\|,$$

and

$$\text{dist}(\text{sp}(\mathcal{N}_{a,\hbar}), z)\|Q^+\psi\| \leq C\|Q_{z,J}^\pm\psi\| + C\hbar^{J+1}\|\psi\|.$$

Then,

$$\text{dist}(\text{sp}(\mathcal{N}_\hbar), z)\|\psi\| \leq C\|Q_{z,J}^\pm\psi\| + C\hbar^{J+1}\|\psi\|.$$

□

Let us consider the case when $J = 2$, and consider the pseudo-differential operator whose Weyl symbol is

$$p_h^{\text{eff}}(s, \xi) = \nu(\alpha, \check{\xi}) + \hbar\langle \check{n}_1 u_{\alpha, \check{\xi}}, u_{\alpha, \check{\xi}} \rangle - \hbar^2 \langle (\check{n}_1(\mathcal{N}_{\alpha, \check{\xi}} - \nu(\alpha, \xi_\alpha))^{-1} \Pi^\perp \check{n}_1 - \check{n}_2) u_{\alpha, \check{\xi}}, u_{\alpha, \check{\xi}} \rangle, \quad (7.4)$$

where $\check{\xi} = \xi\chi_0(\xi)$. Due to the fact that $\nu(\alpha, \xi)$ has a unique and non-degenerate minimum, we can check that the low-lying eigenfunctions are microlocalized near ξ_α .

Lemma 7.8. *For all normalized eigenfunction ψ of $\text{Op}_\hbar^W p_h^{\text{eff}}$ associated with an eigenvalue $\lambda \leq \nu(\alpha, \xi_\alpha) + C\hbar$, we have*

$$\text{Op}_\hbar^W(\chi_h)\psi = \mathcal{O}(\hbar^\infty),$$

with $\chi_h(\xi) = \chi(\hbar^{-\frac{1}{2}+\eta}(\xi - \xi_\alpha))$, where χ is a smooth function equal to 1 away from a fixed neighborhood of 0, and equal to 0 near 0.

Moreover, we have

$$\lambda_n^{\text{eff}}(\alpha, \hbar) = \nu(\alpha, \xi_\alpha) + \mathcal{O}(\hbar).$$

In particular, Lemma 7.8 tells us that the first eigenvalues lie in $D(\nu(\alpha, \xi_\alpha), C\hbar)$.

From Propositions 7.6 and 7.7, we deduce the following.

Proposition 7.9. *For $z \in D(\nu(\alpha, \xi_\alpha), C\hbar)$,*

$$\text{dist}(z, \text{sp}(\text{Op}_\hbar^W(p_h^{\text{eff}})))\|\psi\| \leq C \left(1 + \text{dist}(z, \text{sp}(\text{Op}_\hbar^W(p_h^{\text{eff}}))) \right) (\|(\mathcal{N}_{a,\hbar} - z)\psi\| + \hbar^3 \|\langle \tau \rangle^{N_3} \psi\|),$$

and

$$\text{dist}(z, \text{sp}(\mathcal{N}_{a,\hbar}))\|\psi\| \leq C\|(\text{Op}_\hbar^W(p_h^{\text{eff}}) - z)\psi\| + C\hbar^3\|\psi\|.$$

In addition, we have

$$|\lambda_n(\mathcal{N}_{a,\hbar}) - \lambda_n^{\text{eff}}(\alpha, \hbar)| \leq C\hbar^3, \quad \alpha = a - \theta\hbar,$$

uniformly with respect to $a \in (a_0 - \eta, a_0 + \eta)$.

7.3. On the effective operator. Let us now consider an a in the form

$$a = a_0 + \hbar a_1 + \hbar^2 a_2,$$

or equivalently

$$\alpha = \alpha_0 + \hbar\alpha_1 + \hbar^2\alpha_2, \quad \alpha_0 = a_0, \quad \alpha_1 = a_1 - \theta, \quad \alpha_2 = a_2.$$

For such a choice, let us perform the spectral analysis of $\text{Op}_\hbar^W(p_h^{\text{eff}})$. Note that, thanks to Lemma 7.8, we can remove the frequency cutoff $\chi_0(\xi)$, up to a remainder of order $\mathcal{O}(\hbar^\infty)$. We can also replace χ_h by 1 thanks to the exponential decay of $u_{\alpha, \xi}$ modulo $\mathcal{O}(\hbar^\infty)$.

Then, we expand the symbol

$$\begin{aligned}
p_h^{\text{eff}}(s, \xi) &= \nu(a_0, \xi) + \hbar p_1^{\text{eff}}(s, \xi) + \hbar^2 p_2^{\text{eff}}(s, \xi) + \mathcal{O}(\hbar^3), \\
p_1^{\text{eff}}(s, \xi) &= \partial_\alpha \nu(a_0, \xi)(a_1 - \theta) + \langle n_1 u_{a_0, \xi}, u_{a_0, \xi} \rangle, \\
p_2^{\text{eff}}(s, \xi) &= \partial_\alpha \nu(a_0, \xi) a_2 + \frac{\partial_\alpha^2 \nu(a_0, \xi)}{2} (a_1 - \theta)^2 + (a_1 - \theta) \partial_\alpha \langle n_1 u_{a_0, \xi}, u_{a_0, \xi} \rangle(a_0) \\
&\quad - \langle (n_1(\mathcal{M}_{a_0, \xi} - \nu(a_0, \xi_{a_0})))^{-1} \Pi^\perp n_1 - n_2 \rangle u_{a_0, \xi}, u_{a_0, \xi} \rangle.
\end{aligned} \tag{7.5}$$

Lemma 7.8 invites us to write a Taylor expansion of p_h^{eff} near ξ_{a_0} . We can write

$$\begin{aligned}
p_h^{\text{eff}}(s, \xi) &= \nu(a_0, \xi_{a_0}) + \hbar^2 \partial_\alpha \nu(a_0, \xi_{a_0}) a_2 + \frac{\partial_\xi^2 \nu(a_0, \xi_{a_0})}{2} (\xi - \xi_{a_0})^2 \\
&\quad + \hbar (p_1^{\text{eff}}(s, \xi_{a_0}) + \partial_\xi p_1^{\text{eff}}(s, \xi_{a_0})(\xi - \xi_{a_0})) + \hbar^2 \tilde{p}_2^{\text{eff}}(s, \xi_{a_0}) + r_h,
\end{aligned}$$

where $r_h \in S(1)$ (*i.e.*, r_h and its derivatives at any order are bounded) satisfies

$$|r_h| \leq C |\xi - \xi_{a_0}|^3 + C \hbar |\xi - \xi_{a_0}|^2, \tag{7.6}$$

and

$$\begin{aligned}
\tilde{p}_2^{\text{eff}}(s, \xi_{a_0}) &= \frac{\partial_\alpha^2 \nu(a_0, \xi_{a_0})}{2} (a_1 - \theta)^2 + (a_1 - \theta) \partial_\alpha \langle n_1 u_{a_0, \xi_{a_0}}, u_{a_0, \xi_{a_0}} \rangle(a_0) \\
&\quad - \langle (n_1(\mathcal{M}_{a_0, \xi_{a_0}} - \nu(a_0, \xi_{a_0})))^{-1} \Pi^\perp n_1 - n_2 \rangle u_{a_0, \xi_{a_0}}, u_{a_0, \xi_{a_0}} \rangle
\end{aligned} \tag{7.7}$$

Let us consider $p_1^{\text{eff}}(s, \xi_{a_0})$. By using (7.2) (with χ_h replaced by 1), Proposition 4.12, Lemmata 4.8 and 4.10, we get

$$\begin{aligned}
p_1^{\text{eff}}(s, \xi_{a_0}) &= \partial_\alpha \nu(a_0, \xi_{a_0})(a_1 - \theta) + \langle n_1 u_{a_0, \xi_{a_0}}, u_{a_0, \xi_{a_0}} \rangle \\
&= \partial_\alpha \nu(a_0, \xi_{a_0})(a_1 - \theta) + 2\theta \langle (\xi - \tau) u_{a_0, \xi_{a_0}}, u_{a_0, \xi_{a_0}} \rangle \\
&= a_1 u_{a_0, \xi_{a_0}}^2(0).
\end{aligned}$$

Therefore,

$$p_h^{\text{eff}}(s, \xi) = \nu(a_0, \xi_{a_0}) + a_1 u_{a_0, \xi_{a_0}}^2(0) \hbar + u_{a_0, \xi_{a_0}}^2(0) a_2 \hbar^2 + \widehat{p}_h^{\text{eff}}(s, \xi) + r_h, \tag{7.8}$$

where

$$\widehat{p}_h^{\text{eff}}(s, \xi) = \frac{\partial_\xi^2 \nu(a_0, \xi_{a_0})}{2} (\xi - \xi_{a_0})^2 + \hbar \partial_\xi p_1^{\text{eff}}(s, \xi_{a_0})(\xi - \xi_{a_0}) + \hbar^2 \tilde{p}_2^{\text{eff}}(s, \xi_{a_0}). \tag{7.9}$$

In particular, we have found our ultimate effective (differential) operator

$$\begin{aligned}
\widehat{\mathcal{P}}_h^{\text{eff}} &= \text{Op}_h^{\text{W}}(\widehat{p}_h^{\text{eff}}) = \frac{\partial_\xi^2 \nu(a_0, \xi_{a_0})}{2} (\hbar D_s - \xi_{a_0})^2 \\
&\quad + \frac{\hbar}{2} \left(\partial_\xi p_1^{\text{eff}}(s, \xi_{a_0})(\hbar D_s - \xi_{a_0}) + (\hbar D_s - \xi_{a_0}) \partial_\xi p_1^{\text{eff}}(s, \xi_{a_0}) \right) + \hbar^2 \tilde{p}_2^{\text{eff}}(s, \xi_{a_0}).
\end{aligned}$$

We can rewrite it in the following form

$$\begin{aligned} \widehat{\mathcal{P}}_h^{\text{eff}} = \frac{\partial_\xi^2 \nu(a_0, \xi_{a_0})}{2} \left(\hbar D_s - \xi_{a_0} + \hbar \frac{\partial_\xi p_1^{\text{eff}}(s, \xi_{a_0})}{\partial_\xi^2 \nu(a_0, \xi_{a_0})} \right)^2 \\ + \hbar^2 \left(\tilde{p}_2^{\text{eff}}(s, \xi_{a_0}) - \frac{(\partial_\xi p_1^{\text{eff}}(s, \xi_{a_0}))^2}{2\partial_\xi^2 \nu(a_0, \xi_{a_0})} \right). \end{aligned} \quad (7.10)$$

Lemma 7.10. *Let $(\widehat{\lambda}_n^{\text{eff}}(a, \hbar))_{n \geq 1}$ be the non-decreasing sequence of the eigenvalues of $\widehat{\mathcal{P}}_h^{\text{eff}}$. We have, for all $n \geq 1$,*

$$\widehat{\lambda}_n^{\text{eff}}(a, \hbar) = \mathcal{O}(\hbar^2).$$

Proposition 7.11. *There exist $C, \hbar_0 > 0$, $\eta \in (0, \frac{1}{2})$ such that the following holds. Consider the operators $\widehat{\mathcal{P}}_h^{\text{eff}, \pm}$ defined by*

$$\widehat{\mathcal{P}}_h^{\text{eff}, \pm} = \text{Op}_h^{\text{W}} \widehat{p}_h^{\text{eff}, \pm},$$

where

$$\widehat{p}_h^{\text{eff}, \pm} = (1 \pm C\hbar^{\frac{1}{2}-\eta}) \frac{\partial_\xi^2 \nu(a_0, \xi_{a_0})}{2} (\xi - \xi_{a_0})^2 + \hbar \partial_\xi p_1^{\text{eff}}(s, \xi_{a_0}) (\xi - \xi_{a_0}) + \hbar^2 \tilde{p}_2^{\text{eff}}(s, \xi_{a_0}) \pm C\hbar^{2+\eta}.$$

For all $\hbar \in (0, \hbar_0)$, we have

$$\nu(a_0, \xi_{a_0}) + a_1 u_{a_0, \xi_{a_0}}^2(0)\hbar + a_2 u_{a_0, \xi_{a_0}}^2(0)\hbar^2 + \lambda_n^{\text{eff}, -}(a, \hbar) \leq \lambda_n^{\text{eff}}(a, \hbar),$$

and

$$\lambda_n^{\text{eff}}(a, \hbar) \leq \nu(a_0, \xi_{a_0}) + a_1 u_{a_0, \xi_{a_0}}^2(0)\hbar + a_2 u_{a_0, \xi_{a_0}}^2(0)\hbar^2 + \lambda_n^{\text{eff}, +}(a, \hbar).$$

Proof. Let us recall that p_h^{eff} is given in (7.8) and that the remainder r_h is defined in (7.6). We have

$$p_h^{\text{eff}} - \left(\nu(a_0, \xi_{a_0}) + a_1 u_{a_0, \xi_{a_0}}^2(0)\hbar + a_2 u_{a_0, \xi_{a_0}}^2(0)\hbar^2 \right) = \widehat{p}_h^{\text{eff}}(s, \xi) + r_h.$$

We have

$$\langle \text{Op}_h^{\text{W}}(\widehat{p}_h^{\text{eff}}(s, \xi) + r_h)\psi, \psi \rangle = \langle \text{Op}_h^{\text{W}} \widehat{p}_h^{\text{eff}}(s, \xi)\psi, \psi \rangle + \langle \text{Op}_h^{\text{W}} r_h \psi, \psi \rangle.$$

Let us discuss the upper bound, the lower bound following from similar arguments. We consider

$$\mathcal{E}_N(\hbar) = \text{span}_{1 \leq j \leq N} \psi_{j, \hbar},$$

where $(\psi_{j, \hbar})$ is an orthonormal family of eigenfunctions of $\text{Op}_h^{\text{W}} \widehat{p}_h^{\text{eff}}$ associated with the eigenvalues $(\widehat{\lambda}_n^{\text{eff}}(a, \hbar))$. We can prove that, for all $\psi \in \mathcal{E}_N(\hbar)$, we have

$$\text{Op}_h^{\text{W}}[\chi(\hbar^{-\frac{1}{2}+\eta}(\xi - \xi_{a_0}))]\psi = \psi + \mathcal{O}(\hbar^\infty)\|\psi\|, \quad (7.11)$$

where χ is a smooth cutoff function equal to 1 near 0 with compact support, and $\eta \in (0, \frac{1}{2})$. Then, by using (7.6), (7.11), and classical pseudo-differential estimates,

$$|\langle \text{Op}_h^{\text{W}} r_h \psi, \psi \rangle| \leq C\hbar^{\frac{1}{2}-\eta} \|(\hbar D_s - \xi_{a_0})\psi\|^2 + C\hbar \|(\hbar D_s - \xi_{a_0})\psi\|^2 + C\hbar^{2+\eta} \|\psi\|^2,$$

for some $\eta \in (0, \frac{1}{2})$. Indeed, we can write

$$r_{\hbar}(s, \xi) = (\xi - \xi_{a_0})^2 \tilde{r}_{\hbar}(s, \xi) + \check{r}_{\hbar},$$

where \tilde{r}_{\hbar} and \check{r}_{\hbar} belong to $S(1)$, with $|\tilde{r}_{\hbar}| \leq C|\xi - \xi_{a_0}|$, and \check{r}_{\hbar} equals 0 near ξ_{a_0} . Due to (7.11) and support considerations, we get

$$\text{Op}_{\hbar}^{\text{W}}(\check{r}_{\hbar})\psi = \mathcal{O}(\hbar^{\infty})\|\psi\|.$$

Then, a computation gives

$$\text{Op}_{\hbar}^{\text{W}}(\xi - \xi_{a_0})\text{Op}_{\hbar}^{\text{W}}(\tilde{r}_{\hbar})\text{Op}_{\hbar}^{\text{W}}(\xi - \xi_{a_0}) = \text{Op}_{\hbar}^{\text{W}}\left((\xi - \xi_{a_0})^2 \tilde{r}_{\hbar} + \frac{\hbar^2}{4} \partial_s^2 \tilde{r}_{\hbar}\right),$$

from which we deduce that

$$\langle \text{Op}_{\hbar}^{\text{W}}(\tilde{r}_{\hbar}(\xi - \xi_{a_0})^2)\psi, \psi \rangle \leq \langle \text{Op}_{\hbar}^{\text{W}}(\tilde{r}_{\hbar})(\hbar D_s - \xi_{a_0})\psi, (\hbar D_s - \xi_{a_0})\psi \rangle + C\hbar^2 \|\text{Op}_{\hbar}^{\text{W}}(\partial_s^2 \tilde{r}_{\hbar})\psi\| \|\psi\|.$$

The last term can be controlled by the Gårding inequality (in the class $S^{\delta}(1)$, with $\delta = \frac{1}{2} - \eta$) and (7.11):

$$\|\text{Op}_{\hbar}^{\text{W}}(\partial_s^2 \tilde{r}_{\hbar})\psi\| \leq C\hbar^{\frac{1}{2} - \eta} \|\psi\|.$$

We deduce that

$$\begin{aligned} & \langle \text{Op}_{\hbar}^{\text{W}}(\widehat{p}_{\hbar}^{\text{eff}}(s, \xi) + r_{\hbar})\psi, \psi \rangle \\ & \leq \langle \text{Op}_{\hbar}^{\text{W}}(\widehat{p}_{\hbar}^{\text{eff}}(s, \xi))\psi, \psi \rangle + C\hbar^{\frac{1}{2} - \eta} \|(\hbar D_s - \xi_{a_0})\psi\|^2 + C\hbar \|(\hbar D_s - \xi_{a_0})\psi\|^2 + C\hbar^{2+\eta} \|\psi\|^2, \end{aligned}$$

which gives the desired upper bound after recalling (7.9) and using the min-max principle.

The lower bound follows in the same way (by using the eigenfunctions of $\text{Op}_{\hbar}^{\text{W}} p_{\hbar}^{\text{eff}}$). \square

Proposition 7.12. *For all $n \geq 1$, we have*

$$\lambda_n^{\text{eff}, \pm}(a, \hbar) = \widehat{\lambda}_n^{\text{eff}}(a, \hbar) + o(\hbar^2).$$

Proof. This comes from the fact that, if ψ is an eigenfunction of $\text{Op}_{\hbar}^{\text{W}}(\widehat{p}_{\hbar}^{\text{eff}, \pm})$, resp. $\text{Op}_{\hbar}^{\text{W}}(\widehat{p}_{\hbar}^{\text{eff}})$, associated with $\lambda_n^{\text{eff}, \pm}(a, \hbar) = \mathcal{O}(\hbar^2)$, resp. $\widehat{\lambda}_n^{\text{eff}}(a, \hbar) = \mathcal{O}(\hbar^2)$, we have $\|(\hbar D_s - \xi_{a_0})\psi\|^2 \leq C\hbar^2 \|\psi\|^2$. \square

With Corollary 7.3 and Propositions 7.9 and 7.11, we deduce the following important corollary.

Corollary 7.13. *For all $n \geq 1$, we have*

$$\hbar^{-1} \lambda_n(a, \hbar) = \nu(a_0, \xi_{a_0}) + a_1 u_{a_0, \xi_{a_0}}^2(0) \hbar + a_2 u_{a_0, \xi_{a_0}}^2(0) \hbar^2 + \widehat{\lambda}_n^{\text{eff}}(a, \hbar) + o(\hbar^2).$$

7.4. Proof of Theorem 1.19.

7.4.1. *A choice of a .* From Corollary 7.13 and (5.3), we deduce that

$$a|a - e_n(\hbar)| \leq |\nu(a_0, \xi_{a_0}) + a_1 u_{a_0, \xi_{a_0}}^2(0)\hbar + a_2 u_{a_0, \xi_{a_0}}^2(0)\hbar^2 + \widehat{\lambda}_n^{\text{eff}}(a, \hbar) - a^2| + o(\hbar^2).$$

It remains to make a clever choice of a_1 and a_2 . We have

$$a = a_0 + \hbar a_1 + \hbar^2 a_2.$$

We want that

$$\nu(a_0, \xi_{a_0}) + a_1 u_{a_0, \xi_{a_0}}^2(0)\hbar + a_2 u_{a_0, \xi_{a_0}}^2(0)\hbar^2 + \widehat{\lambda}_n^{\text{eff}}(a, \hbar) - (a_0 + \hbar a_1 + \hbar^2 a_2)^2$$

to be $o(\hbar^2)$. By the choice of a_0 , we have

$$\nu(a_0, \xi_{a_0}) = \nu(a_0) = a_0^2.$$

Then, we would like to have

$$a_1 u_{a_0, \xi_{a_0}}^2(0) = 2a_0 a_1.$$

By using Lemma 4.8, we have

$$u_{a_0, \xi_{a_0}}^2(0) = 2 \int_{\mathbb{R}_0} (\xi_{a_0} - t) u_{a_0, \xi_{a_0}}^2(t) dt,$$

and by Proposition 4.12 (with our choice of a_0), we have $\xi_{a_0} = a_0$ so that

$$u_{a_0, \xi_{a_0}}^2(0) - 2a_0 = -2 \int_{\mathbb{R}_0} t u_{a_0, \xi_{a_0}}^2(t) dt < 0.$$

Thus, we must choose $a_1 = 0$.

Now, we must choose a_2 so that

$$2a_0 a_2 \hbar^2 = a_2 u_{a_0, \xi_{a_0}}^2(0)\hbar^2 + \widehat{\lambda}_n^{\text{eff}}(a, \hbar) + o(\hbar^2).$$

Note that $\widehat{\lambda}_n^{\text{eff}}(a, \hbar)$ only depends on a_0 . This leads to choose

$$a_2 = \hbar^{-2} (2a_0 - u_{a_0, \xi_{a_0}}^2(0))^{-1} \widehat{\lambda}_n^{\text{eff}}(a_0, \hbar).$$

With this choice, we get

$$e_n(\hbar) = a_0 + \hbar^2 a_2 + o(\hbar^2).$$

It remains to describe $\widehat{\mathcal{P}}_h^{\text{eff}}$ in order to get Theorem 1.19.

7.4.2. *End of the proof.* Let us now analyze the dependence on θ of $\widehat{\mathcal{P}}_h^{\text{eff}}$ defined in (7.10). Let us look at the first term. We have

$$\begin{aligned} \partial_\xi p_1^{\text{eff}}(s, \xi_{a_0}) &= -\theta \partial_\alpha \partial_\xi \nu(a_0, \xi_{a_0}) + \partial_\xi \langle n_1 u_{a_0, \xi}, u_{a_0, \xi} \rangle \\ &= \theta \left(-\partial_\alpha \partial_\xi \nu(a_0, \xi_{a_0}) + 2\partial_\xi \int_0^{+\infty} (\xi - \tau) u_{a_0, \xi}^2(\tau) d\tau \right) + \kappa \partial_\xi \langle \mathcal{C}_\xi u_{a_0, \xi}, u_{a_0, \xi} \rangle \\ &= \theta \partial_\xi^2 \nu(a_0, \xi_{a_0}) + \kappa \partial_\xi \langle \mathcal{C}_\xi u_{a_0, \xi}, u_{a_0, \xi} \rangle. \end{aligned}$$

where we used $n_1 = \kappa \mathcal{C}_\xi + 2\theta(\xi - \tau)$ and Lemmata 4.8 and 4.10.

This shows that

$$\begin{aligned} \left(\hbar D_s - \xi_{a_0} + \hbar \frac{\partial_\xi p_1^{\text{eff}}(s, \xi_{a_0})}{\partial_\xi^2 \nu(a_0, \xi_{a_0})} \right)^2 &= \left(\hbar D_s - \xi_{a_0} + \hbar \theta + \hbar \kappa \frac{\partial_\xi \langle \mathcal{C}_\xi u_{a_0, \xi}, u_{a_0, \xi} \rangle}{\partial_\xi^2 \nu(a_0, \xi_{a_0})} \right)^2 \\ &= \left(\hbar D_s - \xi_{a_0} + \hbar \theta - \hbar \frac{\kappa}{2} \right)^2 = \left(\hbar D_s - a_0 + \hbar \theta - \hbar \frac{\kappa}{2} \right)^2, \end{aligned} \quad (7.12)$$

where we used Lemma 4.20 and the fact that $\xi_{a_0} = a_0$.

Remark 7.14. We recall that the expression of θ is given in (6.8). By the Gauss-Bonnet formula,

$$\underline{\kappa} = \frac{1}{|\partial\Omega|} \int_0^{|\partial\Omega|} \kappa(s) ds = \frac{2\pi}{|\partial\Omega|}.$$

Thus, in (7.12), due to the gauge invariance, $\frac{\kappa}{2}$ can be replaced by $\frac{\underline{\kappa}}{2} = \frac{\pi}{|\partial\Omega|}$. This, with Remark 1.18, is consistent with the expression of \mathfrak{t}_h in (1.12).

Let us now look at the second term in (7.10), *i.e.*,

$$\tilde{p}_2^{\text{eff}}(s, \xi_{a_0}) - \frac{(\partial_\xi p_1^{\text{eff}}(s, \xi_{a_0}))^2}{2\partial_\xi^2 \nu(a_0, \xi_{a_0})}, \quad (7.13)$$

where we recall (7.5) and (7.7).

Let us explain why this term does not depend on θ . It depends a priori on θ quadratically. Let us gather the terms depending on θ linearly (they have all κ as common factor):

$$\begin{aligned} L &= -\partial_\alpha \langle \mathcal{C}_{\xi_{a_0}} u_{\alpha, \xi_{a_0}}, u_{\alpha, \xi_{a_0}} \rangle - 4 \langle (\xi - \tau)(\mathcal{M} - \nu(a_0))^{-1} \Pi^\perp \mathcal{C}_{\xi_{a_0}} u_{a_0, \xi_{a_0}}, u_{a_0, \xi_{a_0}} \rangle \\ &\quad - \partial_\xi \langle \mathcal{C}_\xi u_{a_0, \xi}, u_{a_0, \xi} \rangle + \langle (2\tau^2 - 2 + 4\tau(\xi - \tau)) u_{a_0, \xi_{a_0}}, u_{a_0, \xi_{a_0}} \rangle. \end{aligned} \quad (7.14)$$

Note that

$$\partial_\xi \mathcal{C}_\xi = 2\tau^2 - 2 + 4\tau(\xi - \tau),$$

so that (7.14) becomes

$$L = -2 \langle \mathcal{C}_{\xi_{a_0}} u_{\alpha, \xi_{a_0}}, \partial_\alpha u_{\alpha, \xi_{a_0}} \rangle - 4 \langle (\xi - \tau)(\mathcal{M} - \nu(a_0))^{-1} \Pi^\perp \mathcal{C}_{\xi_{a_0}} u_{a_0, \xi_{a_0}}, u_{a_0, \xi_{a_0}} \rangle - 2 \langle \mathcal{C}_\xi u_{a_0, \xi_{a_0}}, u_{a_0, \xi_{a_0}} \rangle.$$

We recall that $\langle \mathcal{C}_{\xi_{a_0}} u_{a_0, \xi_{a_0}}, u_{a_0, \xi_{a_0}} \rangle = 0$ so that $\Pi^\perp \mathcal{C}_{\xi_{a_0}} u_{a_0, \xi_{a_0}} = \mathcal{C}_{\xi_{a_0}} u_{a_0, \xi_{a_0}}$, and we use Lemma 4.25 to get

$$\begin{aligned} L &= -2 \langle \mathcal{C}_{\xi_{a_0}} u_{\alpha, \xi_{a_0}}, \partial_\alpha u_{\alpha, \xi_{a_0}} \rangle + 4 \langle (\xi - \tau) k_{a_0, \xi_{a_0}}, u_{a_0, \xi_{a_0}} \rangle - 2 \langle \mathcal{C}_\xi u_{a_0, \xi_{a_0}}, u_{a_0, \xi_{a_0}} \rangle \\ &= -2 \langle \mathcal{C}_{\xi_{a_0}} u_{\alpha, \xi_{a_0}}, \partial_\alpha u_{\alpha, \xi_{a_0}} \rangle + 2 k_{a_0, \xi_{a_0}}(0) u_{a_0, \xi_{a_0}}(0) \\ &= 2 \left[\langle (\mathcal{M}_{a_0, \xi_{a_0}} - \nu(a_0)) k_{\alpha, \xi_{a_0}}, \partial_\alpha u_{\alpha, \xi_{a_0}} \rangle + k_{a_0, \xi_{a_0}}(0) u_{a_0, \xi_{a_0}}(0) \right] \\ &= 2 \left[k'_{a_0, \xi_{a_0}}(0) \partial_\alpha u_{\alpha, \xi_{a_0}}(0) - k_{a_0, \xi_{a_0}}(0) \partial_\alpha \partial_\tau u_{\alpha, \xi_{a_0}}(0) + k_{a_0, \xi_{a_0}}(0) u_{a_0, \xi_{a_0}}(0) \right], \end{aligned}$$

where we used Lemma 4.7, and the fact that $(\mathcal{M}_{a_0, \xi_{a_0}} - \nu(a_0)) \partial_\alpha u_{\alpha, \xi_{a_0}} = 0$. Let us now recall that $\xi_{a_0} = a_0$, by Proposition 4.12. In addition, since $\partial_\tau u_{\alpha, \xi}(0) = (\alpha - \xi) u_{\alpha, \xi}(0)$, we get

$$\partial_\alpha \partial_\tau u_{\alpha, \xi_{a_0}}(0) = u_{a_0, \xi_{a_0}}(0).$$

This implies that

$$L = 0.$$

Let us now gather the terms depending on θ quadratically:

$$\begin{aligned} C &= \frac{\partial_\alpha^2 \nu(a_0, \xi_{a_0})}{2} - 2\partial_\alpha \langle (\xi - \tau)u_{\alpha, \xi_{a_0}}, u_{\alpha, \xi_{a_0}} \rangle \\ &\quad + 2\langle (\xi - \tau)(\mathcal{M}_{a_0, \xi_{a_0}} - \nu(a_0))^{-1} \Pi^\perp(-2(\xi - \tau)u_{\alpha, \xi_{a_0}}), u_{\alpha, \xi_{a_0}} \rangle + 1 - \frac{\partial_\xi^2 \nu(a_0, \xi_{a_0})}{2}. \end{aligned}$$

By Lemma 4.23, we get

$$\begin{aligned} &(\mathcal{M}_{a_0, \xi_{a_0}} - \nu(a_0))^{-1} \Pi^\perp(-2(\xi - \tau)u_{\alpha, \xi_{a_0}}) \\ &= (\mathcal{M}_{a_0, \xi_{a_0}} - \nu(a_0))^{-1} \Pi^\perp(-2(\xi - \tau)u_{\alpha, \xi_{a_0}} + u_{\xi_{a_0}}^2(0)u_{a_0, \xi_{a_0}}) \\ &= v_{a_0, \xi_{a_0}} + g_{a_0, \xi_{a_0}}. \end{aligned}$$

Thus, with Lemmata 4.9 and 4.22,

$$\begin{aligned} C &= \frac{\partial_\alpha^2 \nu(a_0, \xi_{a_0})}{2} - 2\partial_\alpha \langle (\xi - \tau)u_{\alpha, \xi_{a_0}}, u_{\alpha, \xi_{a_0}} \rangle \\ &\quad + 2\langle (\xi - \tau)(v_{a_0, \xi_{a_0}} + g_{a_0, \xi_{a_0}}), u_{a_0, \xi_{a_0}} \rangle + 1 - \frac{\partial_\xi^2 \nu(a_0, \xi_{a_0})}{2} \\ &= \frac{\partial_\alpha^2 \nu(a_0, \xi_{a_0})}{2} - 2\partial_\alpha \langle (\xi - \tau)u_{\alpha, \xi_{a_0}}, u_{\alpha, \xi_{a_0}} \rangle + 2\langle (\xi - \tau)g_{a_0, \xi_{a_0}}, u_{a_0, \xi_{a_0}} \rangle \\ &\quad + u_{a_0, \xi_{a_0}}(0)v_{a_0, \xi_{a_0}}(0) \\ &= \frac{\partial_\alpha^2 \nu(a_0, \xi_{a_0})}{2} - 2\partial_\alpha \langle (\xi - \tau)u_{\alpha, \xi_{a_0}}, u_{\alpha, \xi_{a_0}} \rangle + 2u_{a_0, \xi_{a_0}}(0)v_{a_0, \xi_{a_0}}(0) + u_{a_0, \xi_{a_0}}(0)g_{a_0, \xi_{a_0}}(0) \\ &= \frac{\partial_\alpha^2 \nu(a_0, \xi_{a_0})}{2} - 4\langle (\xi - \tau)u_{\alpha, \xi_{a_0}}, \partial_\alpha u_{\alpha, \xi_{a_0}} \rangle + 2u_{a_0, \xi_{a_0}}(0)v_{a_0, \xi_{a_0}}(0) + u_{a_0, \xi_{a_0}}(0)g_{a_0, \xi_{a_0}}(0) \end{aligned}$$

Using Lemmata 4.11 and 4.24, we get

$$C = 0.$$

Therefore, in (7.13), we can replace θ by 0. Using again Lemma 4.20, we see that it remains to consider $\tilde{p}_2^{\text{eff}}(s, \xi_{a_0})$ defined in (7.7) and given by

$$\tilde{p}_2^{\text{eff}}(s, \xi_{a_0}) = -\langle (n_1(\mathcal{M}_{a_0, \xi_{a_0}} - \nu(a_0, \xi_{a_0}))^{-1} \Pi^\perp n_1 - n_2)u_{a_0, \xi_{a_0}}, u_{a_0, \xi_{a_0}} \rangle, \quad (7.15)$$

where n_1 and n_2 can be replaced by³

$$n_1 = \kappa \mathcal{C}_\xi, \quad n_2 = \kappa^2 \mathcal{C}_{\xi, 2}.$$

Hence,

$$\tilde{p}_2^{\text{eff}}(s, \xi_{a_0}) = \kappa^2 \left(\langle \mathcal{C}_{\xi_{a_0}} u_{a_0, \xi_{a_0}}, k_{a_0, \xi_{a_0}} \rangle + \langle \mathcal{C}_{\xi, 2} u_{a_0, \xi_{a_0}}, u_{a_0, \xi_{a_0}} \rangle \right).$$

Lemmata 4.29 and 4.20 show that

$$\tilde{p}_2^{\text{eff}}(s, \xi_{a_0}) - \frac{(\partial_\xi p_1^{\text{eff}}(s, \xi_{a_0}))^2}{2\partial_\xi^2 \nu(a_0, \xi_{a_0})} = \frac{\kappa^2}{12} \partial_\xi^2 \nu(a_0, \xi_{a_0}) - \kappa^2 \frac{\partial_\xi^2 \nu(a_0, \xi_{a_0})}{8} = -\frac{\kappa^2}{12} \frac{\partial_\xi^2 \nu(a_0, \xi_{a_0})}{2}.$$

³See Lemma 4.28 and (7.2).

Recalling (7.10), (7.12), and the discussion in Section 7.4.1, this ends the proof of Theorem 1.19.

APPENDIX A. THE RESULTS UNDER VARIOUS LOCAL BOUNDARY CONDITIONS

For $\eta \in \mathbb{R}$, and \mathbf{n} is a unit vector, we define the boundary matrix

$$\mathcal{B}_{\eta, \mathbf{n}} = -i\sigma_3(\sigma \cdot \mathbf{n}) \cos(\eta) + \sigma_3 \sin(\eta).$$

$\mathcal{B}_{\eta, \mathbf{n}}$ is an unitary and Hermitian matrix so that its spectrum is $\{\pm 1\}$. For any regular function $\eta: \partial\Omega \rightarrow \mathbb{R}$, we introduce the local boundary condition

$$\mathcal{B}_{\eta(s), \mathbf{n}(s)}\varphi(s) = \varphi(s), \quad s \in \partial\Omega,$$

where $\mathbf{n}: \partial\Omega \rightarrow S^1$ is the outward pointing normal and $\varphi: \partial\Omega \rightarrow \mathbb{C}^2$. The associated magnetic Dirac operator $(\mathcal{D}_{h, \mathbf{A}, \eta}, \text{Dom}(h, \mathcal{D}_{h, \mathbf{A}, \eta}))$ acts as $\mathcal{D}_{h, \mathbf{A}}$ on

$$\text{Dom}(\mathcal{D}_{h, \mathbf{A}, \eta}) = \{\varphi \in H^1(\Omega)^2, \quad \mathcal{B}_{\eta, \mathbf{n}}\varphi = \varphi \text{ on } \partial\Omega\}.$$

The case $\eta \equiv 0$ correspond to the MIT bag boundary condition. Note that

$$\mathcal{B}_{\eta, \mathbf{n}} = \begin{pmatrix} \sin(\eta) & -i\bar{\mathbf{n}} \cos(\eta) \\ i\mathbf{n} \cos(\eta) & -\sin(\eta) \end{pmatrix},$$

so that the boundary condition reads

$$u_2 = i\mathbf{n} \frac{\cos(\eta)}{1 + \sin(\eta)} u_1,$$

where $\varphi = (u_1, u_2)^T$.

Assumption A.1. $\eta \in \mathcal{C}^1(\partial\Omega)$ and $\cos(\eta(s)) > 0$ for all $s \in \partial\Omega$.

In [8], the authors proved that under Assumptions 1.3 and A.1, $\mathcal{D}_{h, \mathbf{A}, \eta}$ is self-adjoint. We define

$$\gamma: s \in \partial\Omega \mapsto \frac{\cos(\eta(s))}{1 + \sin(\eta(s))} \in \mathbb{R}_+.$$

Since $\partial\Omega$ is compact, we get that

$$0 < \inf_{\partial\Omega} \gamma \leq \gamma(s) \leq \sup_{\partial\Omega} \gamma < +\infty. \quad (\text{A.1})$$

Notation 11. Let

$$\|u\|_{\partial\Omega, \gamma}^2 = \int_{\partial\Omega} |u^2| \gamma \, ds,$$

where $u \in L^2(\partial\Omega)$. By (A.1), this norm is equivalent with the one introduced in Notation 2.

It is straightforward to see that the proofs of the min-max characterization and of Theorem 1.10 are exactly the same up to the replacement of the norm on the boundary. In particular, the constants in the asymptotic analysis are defined with respect to the corresponding weighted Hardy norm on the boundary.

Theorem 1.14 has also its counterpart in this context. Here, the proof has to be slightly adapted by Taylor approximating γ around each point of the boundary. We choose to present our proof for the MIT bag condition only in order not to burden the reader with complicated notations that do not give more insight on the problem. More precisely, we get :

Theorem A.2. *Under Assumptions 1.3, 1.6, and A.1:*

(i) *Under the further assumption 1.7 we have, for all $k \geq 1$,*

$$\lambda_k^+(h) = \left(\frac{\text{dist}_{\mathcal{H}}((z - z_{\min})^{k-1}, \mathcal{H}_k^2(\Omega))}{\text{dist}_{\mathcal{B}}(z^{k-1}, \mathcal{P}_{k-2})} \right)^2 h^{1-k} e^{2\phi_{\min}/h} (1 + o_{h \rightarrow 0}(1)),$$

(ii)

$$\lambda_1^-(h) = h^{\frac{1}{2}} \min \left(\sqrt{2b_0}, c_{\gamma(x)} \sqrt{B(x)}; x \in \partial\Omega \right) + o_{h \rightarrow 0}(h^{\frac{1}{2}}),$$

where for any $x \in \partial\Omega$, $c_{\gamma(x)} > 0$ is the unique positive solution of the equation $\nu_{\gamma(x)}(c) = c^2$ with

$$\nu_{\gamma(x)}(c) = \inf_{\substack{u \in H^1(\mathbb{R}_+^2) \\ u \neq 0}} \frac{\int_{\mathbb{R}_+^2} |(-i\partial_s - \tau + i(-i\partial_\tau))u|^2 ds d\tau + c\gamma(x) \int_{\mathbb{R}} |u(s, 0)|^2 ds}{\|u\|^2}.$$

Remark A.3. Using Remark 1.9, we also cover the case $\cos(\eta(s)) < 0$ for all $s \in \partial\Omega$.

APPENDIX B. NEGATIVE EIGENVALUES AND VARIABLE MAGNETIC FIELDS

Let us assume that B is smooth and positive. As we saw in Sections 6 and 7, the asymptotic analysis of the negative eigenvalues is related to the one of a Schrödinger operator with a Robin-like boundary condition (see the quadratic form (6.1)). We proved (see Proposition 5.1) the following one-term asymptotic expansion

$$\lambda_1(a, h) = h \min \left(2b_0, b'_0 \nu(a(b'_0)^{-1/2}) \right) + o(h),$$

where $b_0 = \min_{x \in \bar{\Omega}} B(x)$ and $b'_0 = \min_{x \in \partial\Omega} B(x)$.

It implies that

$$\lambda_1^-(h) = h^{\frac{1}{2}} \min(\sqrt{2b_0}, a_0 \sqrt{b'_0}) + o(h^{\frac{1}{2}}).$$

B.1. Case of boundary localization. As for the case with constant magnetic field, when $a_0 \sqrt{b'_0} < \sqrt{2b_0}$, we can prove that the first eigenfunctions of $\mathcal{L}_{a,h}$ are exponentially localized near the boundary when a is close enough to $a_0 \sqrt{b'_0}$. In this case, we have

$$\min \left(2b_0, b'_0 \nu(a(b'_0)^{-1/2}) \right) = b'_0 \nu(a(b'_0)^{-1/2}).$$

Let us now explain how our strategy can be adapted to describe the asymptotic behavior of $\lambda_n(a, h)$. The key point is still the study of $\mathcal{N}_{a,h}$, see (6.7) and to use its interpretation as a pseudo-differential operator, see (7.1). The main difference with the case of constant magnetic field appears in the principal operator symbol. We can check that n_0 is replaced by

$$-\partial_\tau^2 + (\xi - b(s)\tau)^2 + b(s),$$

where $s \mapsto b(s)$ is the restriction of the magnetic to the boundary, and where n_0 is equipped with the Robin condition

$$(\partial_\tau + \xi - \alpha)\psi = 0,$$

where we recall that $\alpha = a - \hbar\theta$. We also recall that θ is defined in (6.8) (see also Section 6.3.4 where γ_0 has to be replaced by $\frac{\int_{\Omega} B(x) dx}{|\partial\Omega|}$). Implementing the Grushin method, we see that the principal symbol of the effective operator is the first eigenvalue of n_0 , denoted

by $\mu(s, \xi)$. It can be explicitly described thanks to ν . Consider the rescaling $\tau = b(s)^{-\frac{1}{2}}\tilde{\tau}$, we get

$$\mu(s, \xi) = b(s)\nu\left(\frac{\alpha}{b(s)^{\frac{1}{2}}}, \frac{\xi}{b(s)^{\frac{1}{2}}}\right).$$

In order to study the function μ (and its critical points), we will need some lemmas.

The first lemma comes from the concavity of $\alpha \mapsto \nu(\alpha)$.

Lemma B.1. *For all $\alpha > 0$,*

$$\nu'(\alpha) \leq \frac{\nu(\alpha)}{\alpha}.$$

Lemma B.2. *The function $f : (0, +\infty) \ni b \mapsto b\nu\left(\frac{\alpha}{\sqrt{b}}\right)$ is increasing and its derivative is positive.*

Proof. We have seen in Section 4.7 that ν is analytic and that $\nu' > 0$. We have

$$f'(b) = \nu\left(\frac{\alpha}{\sqrt{b}}\right) - \frac{\alpha}{2\sqrt{b}}\nu'\left(\frac{\alpha}{\sqrt{b}}\right) \geq \frac{\alpha}{2\sqrt{b}}\nu'\left(\frac{\alpha}{\sqrt{b}}\right) > 0,$$

where we used Lemma B.1. □

Proposition B.3. *For all $s \in \partial\Omega$ and $\xi \in \mathbb{R}$, we have*

$$b'_0\nu\left(\frac{\alpha}{\sqrt{b'_0}}\right) \leq b(s)\nu\left(\frac{\alpha}{\sqrt{b}}\right) \leq \mu(s, \xi).$$

In particular, $b'_0\nu\left(\frac{\alpha}{\sqrt{b'_0}}\right)$ is the minimal value of μ .

Moreover, if b has a unique minimum at s_0 , then μ has also a unique minimum at $(s, \xi) = \left(s_0, \sqrt{b'_0}\xi\frac{\alpha}{\sqrt{b'_0}}\right)$.

Proof. The inequality follows from the fact that

$$\nu(\alpha) = \min_{\xi \in \mathbb{R}} \nu(\alpha, \xi),$$

and from Lemma B.2. Taking s_0 a minimum of b and $\xi = \sqrt{b'_0}\xi\frac{\alpha}{\sqrt{b'_0}}$ (see Proposition 4.5 for the definition of $\alpha \mapsto \xi_\alpha$), we get the minimal value. By using the strict monotonicity in Lemma B.2, we get the conclusion about the unique minimum. □

Let us now make the following generic assumption.

Assumption B.4. b has a unique minimum at s_0 , which is non-degenerate.

Proposition B.5. *The function μ has a unique minimum. This minimum is non-degenerate.*

Proof. From Proposition B.3, the minimum is uniquely attained at the point $(s_0, \xi_0) = \left(s_0, \sqrt{b'_0}\xi\frac{\alpha}{\sqrt{b'_0}}\right)$. We have just to check the non-degeneracy.

We have

$$\partial_\xi \mu(s, \xi) = \sqrt{b(s)} \partial_\xi \nu \left(\frac{\alpha}{b(s)^{\frac{1}{2}}}, \frac{\xi}{b(s)^{\frac{1}{2}}} \right),$$

and

$$\begin{aligned} & \partial_s \mu(s, \xi) \\ &= b'(s) \nu \left(\frac{\alpha}{b(s)^{\frac{1}{2}}}, \frac{\xi}{b(s)^{\frac{1}{2}}} \right) - \frac{b'(s)}{2\sqrt{b(s)}} \left(\alpha \partial_\alpha \nu \left(\frac{\alpha}{b(s)^{\frac{1}{2}}}, \frac{\xi}{b(s)^{\frac{1}{2}}} \right) + \partial_\xi \nu \left(\frac{\alpha}{b(s)^{\frac{1}{2}}}, \frac{\xi}{b(s)^{\frac{1}{2}}} \right) \right). \end{aligned}$$

At a critical point (s, ξ) , we have

$$b'(s) \left(\nu \left(\frac{\alpha}{b(s)^{\frac{1}{2}}}, \frac{\xi}{b(s)^{\frac{1}{2}}} \right) - \frac{1}{2\sqrt{b(s)}} \alpha \partial_\alpha \nu \left(\frac{\alpha}{b(s)^{\frac{1}{2}}}, \frac{\xi}{b(s)^{\frac{1}{2}}} \right) \right) = 0.$$

At this point, we have⁴

$$\nu \left(\frac{\alpha}{b(s)^{\frac{1}{2}}}, \frac{\xi}{b(s)^{\frac{1}{2}}} \right) = \nu \left(\frac{\alpha}{b(s)^{\frac{1}{2}}} \right) \quad \partial_\alpha \nu \left(\frac{\alpha}{b(s)^{\frac{1}{2}}}, \frac{\xi}{b(s)^{\frac{1}{2}}} \right) = \nu' \left(\frac{\alpha}{b(s)^{\frac{1}{2}}} \right).$$

Lemma B.2 shows that $b'(s) = 0$ which is consistent with the fact that b is assumed to be minimal at $s = s_0$.

Let us compute the derivatives of order two at $(s, \xi) = (s_0, \xi_0)$.

We have

$$\partial_\xi^2 \mu(s_0, \xi_0) = \partial_\xi^2 \nu \left(\frac{\alpha}{b(s_0)^{\frac{1}{2}}}, \frac{\xi_0}{b(s_0)^{\frac{1}{2}}} \right) > 0, \quad (\text{B.1})$$

$$\partial_s \partial_\xi \mu(s_0, \xi_0) = 0,$$

$$\partial_s^2 \mu(s_0, \xi_0) = b''(s_0) \left(\nu \left(\frac{\alpha}{\sqrt{b(s_0)}} \right) - \frac{\alpha}{2\sqrt{b(s_0)}} \nu' \left(\frac{\alpha}{\sqrt{b(s_0)}} \right) \right) > 0. \quad (\text{B.2})$$

This shows that the minimum is non-degenerate. \square

We can prove that the eigenfunctions of $\mathcal{N}_{a,h}$ are microlocalized near (s_0, ξ_0) . The localization in space near the minimum of b allows to take $\theta = 0$ by using an appropriate local (near s_0) change of gauge. Thus $\alpha = a$.

By using the Grushin reduction (and the harmonic approximation of μ), we get the following asymptotic expansion.

Proposition B.6. *We have, for some $d_0 \in \mathbb{R}$ (a priori depending on a),*

$$\lambda_n(a, h) = b'_0 h \nu \left(\frac{a}{\sqrt{b'_0}} \right) + h^{\frac{3}{2}} \left(\left(n - \frac{1}{2} \right) \sqrt{\partial_s^2 \mu(s_0, \xi_0) \partial_\xi^2 \mu(s_0, \xi_0) + d_0} \right) + \mathcal{O}(h^2),$$

where the second order derivatives are given in (B.2) and (B.1).

⁴Recall that, for all $\alpha > 0$, $\nu'(\alpha) = \partial_\alpha \nu(\alpha, \xi_\alpha)$.

From this, we can deduce the following asymptotic expansion of the negative eigenvalues.

$$\lambda_n^-(h) = a_0 h^{\frac{1}{2}} \sqrt{b'_0} + \frac{h}{2a_0} \left((n - \frac{1}{2}) \sqrt{\partial_s^2 \mu(s_0, \xi_0) \partial_{\xi}^2 \mu(s_0, \xi_0)} + \tilde{d}_0 \right) + o(h),$$

with $\tilde{d}_0 \in \mathbb{R}$ and where α has to be replaced by a_0 in the expression of the second order derivatives.

B.2. Case of interior localization. When $a_0 \sqrt{b'_0} > \sqrt{2b_0}$, and when a is close enough to $\sqrt{2b_0}$, we can prove that the first eigenfunctions of $\mathcal{L}_{a,h}$ are exponentially localized near the set $\{B = b_0\}$. In this case, the boundary can essentially be forgotten and the model operator is the electro-magnetic Schrödinger operator

$$(-ih\nabla - \mathbf{A})^2 + hB(x).$$

Here \mathbf{A} and B are extended to \mathbb{R}^2 in such a way that the minimal level set of B is not changed. Modulo $\mathcal{O}(h^\infty)$, this operator governs the spectral asymptotics of $\mathcal{L}_{a,h}$. The spectral analysis of such an operator can be done by means of various methods, one of them being the Birkhoff normal form, see [29] and [22, Section 4] where the *electro-magnetic* case is tackled. In the generic case when B has a unique and non-degenerate minimum at $x_0 \in \Omega$, we have

$$\lambda_n(a, h) = 2b_0 h + h^2 (c_0(2n - 1) + c_1) + \mathcal{O}(h^3),$$

where $c_1 \in \mathbb{R}$ and

$$c_0 = \frac{\sqrt{\det(\text{Hess}_{x_0} B)}}{B(x_0)}.$$

From this, we deduce that

$$\lambda_n^-(h) = h^{\frac{1}{2}} \sqrt{2b_0} + h^{\frac{3}{2}} \left(\frac{c_0}{2\sqrt{2b_0}} (2n - 1) + \frac{c_1}{2\sqrt{2b_0}} \right) + \mathcal{O}(h^{\frac{5}{2}}).$$

APPENDIX C. PROOF OF LEMMA 2.4

We use Remark 2.3 to consider the case when $\Omega = \mathbb{D}$. We let

$$\ell_w^2(\mathbb{N}) = \left\{ u \in \ell^2(\mathbb{N}) : \sum_{n \geq 0} (n+1)^{-1} |u_n|^2 < +\infty \right\}.$$

Thanks to the isomorphism expressed in (2.2), $(\mathcal{H}^2(\mathbb{D}), \langle \cdot, \cdot \rangle_{\partial \mathbb{D}})$ is a Hilbert space.

Consider

$$K = \left\{ u \in \ell^2(\mathbb{N}) : \sum_{n \geq 0} |u_n|^2 \leq 1 \right\}.$$

It is sufficient to show that K is precompact in $\ell_w^2(\mathbb{N})$. Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that, for all $u \in K$,

$$\sum_{n \geq N+1} \frac{1}{n+1} |u_n|^2 \leq \frac{\varepsilon^2}{4}.$$

Moreover, the unit ball of \mathbb{C}^{N+1} for the standard ℓ^2 -norm is precompact, and we can write

$$\exists (a_0, \dots, a_M) \in \mathbb{C}^{N+1}, \quad B_{N+1}(0, 1) \subset \bigcup_{j=0}^M B_{N+1,w} \left(a_j, \frac{\varepsilon}{2} \right),$$

where $B_{N+1,w}$ are the balls for the ℓ_w^2 -norm. We have

$$K \subset \bigcup_{j=0}^M B_w \left(\underline{a}_j, \varepsilon \right),$$

where \underline{a}_j denotes the extension by zero of the finite sequence a_j . Indeed, there exists $N \in \mathbb{N}$ such that, for all $u \in K$,

$$\left\| u - \sum_{j=0}^N u_j e_j \right\|_{\ell_w^2} \leq \frac{\varepsilon}{2}.$$

Then, $\sum_{j=0}^N u_j e_j \in B_{N+1}(0, 1)$, and the conclusion follows from the triangle inequality. Here, $(e_j)_{j \geq 0}$ is the canonical basis of $l^2(\mathbb{N})$.

APPENDIX D. HOLOMORPHIC TUBULAR COORDINATES

The aim of this section is to define an appropriate system of coordinates $x = \varphi(s, t)$ near $\partial\Omega$. We want φ to be holomorphic, and that $s \mapsto \varphi(s, 0) = \gamma(s)$ is a counterclockwise parametrization of the boundary by arc length, *i.e.*, $|\gamma'(s)| = 1$.

D.1. Definition of the coordinates.

Assumption D.1. *The boundary $\partial\Omega$ is an analytic curve, *i.e.* there exist $\rho > 1$ and an analytic and injective function*

$$g: \{\rho^{-1} < |z| < \rho\} \rightarrow \mathbb{R}^2,$$

such that $g|_{\{|z|=1\}}$ is a regular parametrization of $\partial\Omega$.

Proposition D.2. *Under Assumption D.1, there exist $\delta_0 > 0$ and a function*

$$\begin{aligned} \varphi: \mathbb{R}/(|\partial\Omega|\mathbb{Z}) \times (-\delta_0, \delta_0) &\longrightarrow \mathbb{R}^2, \\ w = (s, t) &\longmapsto \varphi(s, t) = x, \end{aligned}$$

such that

- (i) φ is holomorphic *i.e.* $\partial_{\bar{w}}\varphi = 1/2(\partial_s + i\partial_t)\varphi = 0$,
- (ii) $\varphi|_{\{t=0\}} =: \gamma$ is a positively oriented parametrization by arc length of $\partial\Omega$,
- (iii) φ is injective and $2 \geq |\partial_w\varphi(s, t)| \geq 1/2$ for all $(s, t) \in \mathbb{R}/(|\partial\Omega|\mathbb{Z}) \times (-\delta_0, \delta_0)$,
- (iv) $\varphi|_{\{t>0\}} \subset \Omega$ and φ induces a parametrization of a neighborhood \mathcal{V} of the boundary.
- (v) For all $x \in \mathcal{V}$,

$$\frac{1}{2}t(x) \leq \text{dist}(x, \partial\Omega) \leq 2t(x). \quad (\text{D.1})$$

D.2. Proof of Proposition D.2. We will define φ as the flow of a gradient. The following lemma will be crucial.

Lemma D.3. *There exists an open set $U \subset \mathbb{R}^2$ and a regular function $\zeta: U \rightarrow \mathbb{R}$ such that $\partial\Omega \subset U$ and*

$$\begin{cases} \Delta\zeta &= 0 \text{ on } U, \\ \zeta &= 0 \text{ on } \partial\Omega, \\ \partial_{\mathbf{n}}\zeta &= -1 \text{ on } \partial\Omega. \end{cases} \quad (\text{D.2})$$

Proof. By [28, Proposition 3.1] and Assumption D.1, there exist $r_0 > 1$ and a biholomorphism $F: \mathbb{D}(0, r_0) \rightarrow F(\mathbb{D}(0, r_0))$ such that $F(\mathbb{D}) = \Omega$. Denoting $\tilde{\zeta} = \zeta \circ F$, the problem of the existence of (U, ζ) such that (D.2) holds is equivalent to finding an open set $\tilde{U} \supset \partial\mathbb{D}$ and a function $\zeta: \tilde{U} \rightarrow \mathbb{R}$ such that

$$\begin{cases} \Delta\tilde{\zeta} &= 0 \text{ on } \tilde{U}, \\ \tilde{\zeta} &= 0 \text{ on } \partial\mathbb{D}, \\ \partial_{\mathbf{n}}\tilde{\zeta} &= -|F'| \text{ on } \partial\mathbb{D}. \end{cases} \quad (\text{D.3})$$

Since, the function F' does not vanish on $\mathbb{D}(0, r_0)$, there exists a holomorphic function G on $\mathbb{D}(0, r_0)$ such that $G^2 = F'$ and $G\overline{G} = |F'|$. The function $z \mapsto G(z)\overline{G(\bar{z}^{-1})}$ is holomorphic on $\{r_0^{-1} < |z| < r_0\}$ and coincides with $|F'|$ on $\partial\mathbb{D}$. We can write for $r_0^{-1} < |z| < r_0$,

$$G(z)\overline{G(\bar{z}^{-1})} = \sum_{j \in \mathbb{Z}} a_j z^j.$$

Since $|F'| \in \mathbb{R}$, we have that $a_j = \overline{a_{-j}}$ for all j and the radius of convergence $r_1 = \limsup_{j \rightarrow +\infty} |a_j|^{1/j}$ of $\sum_{j \geq 0} a_j z^j$ satisfies $r_1 > r_0$. For $z = re^{is}$ such that $r_0^{-1} < r < r_0$, we define

$$\begin{aligned} \tilde{\zeta}(z) &= -a_0 \log(r) - \sum_{j \in \mathbb{Z} \setminus \{0\}} a_j \left(\frac{r^j}{2j} + \frac{r^{-j}}{-2j} \right) e^{ijs} \\ &= -a_0 \log(|z|) - \sum_{j \in \mathbb{Z} \setminus \{0\}} a_j \left(\frac{z^j}{2j} + \frac{\overline{z^{-j}}}{-2j} \right) \\ &= -a_0 \log(|z|) - 2\text{Re} \sum_{j \geq 1} a_j \left(\frac{z^j}{2j} + \frac{\overline{z^{-j}}}{-2j} \right), \end{aligned}$$

and the conclusion follows. \square

We can now prove Proposition D.2.

Let $\gamma: \mathbb{R}/(|\partial\Omega|\mathbb{Z}) \rightarrow \partial\Omega$ be a positively oriented parametrization by arc length of $\partial\Omega$. We define the function φ as the solution of the following Cauchy problem

$$\begin{cases} \partial_t \varphi(s, t) &= \frac{\nabla \zeta}{|\nabla \zeta|^2} \circ \varphi(s, t), \\ \varphi(s, 0) &= \gamma(s). \end{cases} \quad (\text{D.4})$$

By Cauchy-Lipschitz Theorem, there exists $\delta_0 > 0$ such that the function $\varphi: \mathbb{R}/(|\partial\Omega|\mathbb{Z}) \times (-\delta_0, \delta_0) \rightarrow U$ is well-defined, regular and injective.

By (D.4), we have for all (s, t) that $\partial_t \zeta \circ \varphi(s, t) = 1$. By (D.2), we deduce that $\zeta \circ \varphi(s, t) = t$ and $0 = \partial_s(\zeta \circ \varphi)(s, t) = \nabla \zeta(\varphi(s, t)) \cdot \partial_s \varphi(s, t)$ so that

$$\partial_t \varphi(s, t) \cdot \partial_s \varphi(s, t) = 0.$$

Therefore, there exists a regular function α such that

$$\partial_s \varphi(s, t) = \alpha(s, t) J \partial_t \varphi(s, t),$$

with $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. By (D.2), we have

$$\partial_s \varphi(s, 0) = \gamma'(s), \quad \partial_t \varphi(s, 0) = -\mathbf{n}(s),$$

so that $\alpha(s, 0) = -1$. We also have $\alpha(s, t) = \partial_s \varphi(s, t) \cdot J \nabla \zeta(\varphi(s, t))$ and

$$\partial_t \alpha(s, t) = \partial_{st}^2 \varphi(s, t) \cdot J \nabla \zeta(\varphi(s, t)) + \partial_s \varphi(s, t) \cdot J \text{Hess } \zeta(\varphi(s, t)) \partial_t \varphi(s, t).$$

Notice that

$$\partial_{st}^2 \varphi = \left(\frac{\text{Hess } \zeta}{|\nabla \zeta|^2} - 2 \frac{\nabla \zeta (\nabla \zeta)^T \text{Hess } \zeta}{|\nabla \zeta|^4} \right) \partial_s \varphi,$$

and using the fact that $(\nabla \zeta)^T J \nabla \zeta = 0$,

$$\begin{aligned} \partial_{st}^2 \varphi \cdot J \nabla \zeta &= \partial_s \varphi \cdot \left(\frac{\text{Hess } \zeta}{|\nabla \zeta|^2} - 2 \frac{\text{Hess } \zeta \nabla \zeta (\nabla \zeta)^T}{|\nabla \zeta|^4} \right) J \nabla \zeta \\ &= \partial_s \varphi \cdot \left(\frac{\text{Hess } \zeta}{|\nabla \zeta|^2} \right) J \nabla \zeta. \end{aligned}$$

By (D.2), we conclude that

$$\partial_t \alpha = \frac{\partial_s \varphi}{|\nabla \zeta|^2} \cdot (\text{Hess } \zeta J + J \text{Hess } \zeta) \nabla \zeta = \frac{\partial_s \varphi}{|\nabla \zeta|^2} \cdot (J \Delta \zeta) \nabla \zeta = 0,$$

and $\partial_s \varphi = -J \partial_t \varphi$. This ends the proof of the proposition.

D.3. Taylor expansions with respect to t . We let $\gamma(s) = \varphi(s, 0)$ and we have $|\gamma'(s)| = 1$. The curvature κ is defined through

$$\gamma''(s) = -\kappa n(s). \tag{D.5}$$

Lemma D.4. *We have*

$$\varphi'(s + it) = \left(1 - \kappa t + \frac{t^2}{2} \kappa^2 \right) \gamma' + \frac{t^2}{2} \kappa' n + o(t^2),$$

and

$$|\varphi'(s + it)|^2 = 1 - 2\kappa t + 2\kappa^2 t^2 + o(t^2), \quad |\varphi'(s + it)|^{-2} = 1 + 2\kappa t + 2\kappa^2 t^2 + o(t^2).$$

Proof. We have

$$\varphi'(s + it) = \partial_s \varphi(s, t) = \partial_s \varphi(s, 0) + t \partial_t \partial_s \varphi(s, 0) + \frac{t^2}{2} \partial_t^2 \partial_s \varphi(s, 0) + o(t^2).$$

Since

$$\partial_t \varphi = i \partial_s \varphi,$$

we get

$$\varphi'(s + it) = \gamma'(s) + it\gamma''(s) - \frac{t^2}{2}\gamma^{(3)}(s) + o(t^2).$$

By using (D.5), we have

$$\gamma^{(3)}(s) = -\kappa^2\gamma' - \kappa'n,$$

and thus

$$\varphi'(s + it) = \gamma' - i\kappa tn + \frac{t^2}{2}(\kappa^2\gamma' + \kappa'n) + o(t^2),$$

and the conclusion follows. \square

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