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Abstract. In this work we study two-dimensional Dirac operators on bounded domains coupled to a magnetic field perpendicular to the plane. We focus on the MIT bag boundary condition and provide accurate asymptotic estimates for the low-lying (positive and negative) energies in the limit of a strong magnetic field.

Contents

1. Introduction

Consider an open, smooth and simply connected domain $\Omega \subset \mathbb{R}^2$ and a magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$, smooth and pointing in direction $\hat{\mathbf{z}}$ orthogonal to the plane. In this work we consider a Dirac operator restricted to Ω and coupled to the magnetic field $\mathbf{B}$ through a magnetic vector potential $\mathbf{A} = (A_1, A_2)^T$ satisfying $\nabla \times \mathbf{A} = \mathbf{B}$. The magnetic Dirac operator acts on a dense subspace of $L^2(\Omega, \mathbb{C}^2)$ as,

$$\sigma \cdot (-i\nabla - b\mathbf{A}) = \begin{pmatrix} 0 & -i(\partial_1 - i\partial_2) - bA_1 + ibA_2 \\ -i(\partial_1 + i\partial_2) - bA_1 - ibA_2 & 0 \end{pmatrix}, \quad (1.1)$$

where $b > 0$ is a positive coupling constant. We write $\sigma \cdot \mathbf{x} = \sigma_1 x_1 + \sigma_2 x_2$ for $\mathbf{x} = (x_1, x_2)$ with the usual Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If we assume that the spinors satisfy a boundary relation of the type $\varphi = \mathcal{B}\varphi$ on $\partial\Omega$ with a unitary and self-adjoint boundary matrix $\mathcal{B}\colon \partial\Omega \to \mathbb{C}^{2\times 2}$, then simple integration by parts shows that the local current density $\sigma \cdot \mathbf{n}$ vanishes at each point of the boundary if and only if

$$\mathcal{B}\,\sigma \cdot \mathbf{n} + \sigma \cdot \mathbf{n}\,\mathcal{B} = 0 \quad \text{on} \quad \partial\Omega\,, \quad (1.2)$$

where $\mathbf{n}$ is the normal vector pointing outward to the boundary. In particular, for these cases, the Dirac operator is formally symmetric and satisfies the bag condition, i.e., that no current flows through $\partial\Omega$ [10]. In the physics literature these type of models have being earlier considered to describe neutrino billards [10] and (in the three dimensional setting) quark confinement [12]. More recently, they have regained attention with the advent of graphene and other Dirac materials, see e.g., [1, 11, 24, 14].

Using the properties of the Pauli matrices and those of $\mathcal{B}$ it is easy to see that the most general form of $\mathcal{B}$ acts as a multiplication on $L^2(\partial\Omega)$ with

$$\mathcal{B} \equiv \mathcal{B}_\eta = (\sigma \cdot \mathbf{t}) \cos\eta + \sigma_3 \sin\eta\,, \quad (1.3)$$

for certain sufficiently smooth $\eta : \partial\Omega \to \mathbb{R}$ and $\mathbf{t}$ being the unit tangent vector pointing clockwise (we have that $\mathbf{n} \times \mathbf{t} = \hat{\mathbf{z}}$). The most frequently used boundary conditions in the physics literature are the cases when $\cos\eta = 0$ and $\sin\eta = 0$ known as zig-zag and MIT bag or infinite-mass boundary conditions, respectively. For recent mathematical literature on the subject in the two and three dimensional settings see for instance [8, 4, 19, 21, 7] about self-adjointness, [25, 3, 5] for the derivations as an infinite mass limit, and [9, 20, 2] for eigenvalue estimates.

In this work we consider Dirac operators D_η acting as in (1.1) on spinors φ satisfying $\varphi = B_\eta\varphi$, with $\eta \in [0, 2\pi)$. We give the precise definition of the self-adjoint realization below. Assuming that the magnetic field satisfies $\inf_{x\in\overline{\Omega}} B(x) = B_0 > 0$ (besides certain geometrical conditions, see Assumption 1.6), we provide asymptotic estimates for the corresponding low-lying eigenvalues in the strong coupling constant limit $b \to \infty$.

The behavior of the corresponding operators in the physically most relevant cases mentioned above are quite different from each other. Indeed, on the one hand, the spectrum of a zig-zag operator is symmetric with respect to zero and zero is an eigenvalue

of infinite multiplicity. On the other hand, the spectrum of the MIT-bag is far from being symmetric for large magnetic fields and zero is never in its spectrum.

Our main results can be roughly summarized as follows: For $k \in \{1, 2, 3, \dots\}$ we denote by $\eta_k^+ \geqslant 0$ and $\eta_k^- < 0$ the non-negative and negative eigenvalues of, D_0, the MIT bag operator with $\eta = 0$. They are ordered such that $|\eta_k^\pm| \leqslant |\eta_{k+1}^\pm|$. Then, there is a constant $C_k^+ > 0$ such that, as $b \to \infty$,

$$\eta_k^+ = C_k^+ b^k e^{-2\alpha b}(1 + o(1))\,. \tag{1.4}$$

We provide explicit expressions for the constants C_k^+ and $\alpha > 0$ in terms of the geometry and the magnetic field B (see Theorem 1.9). In particular, the positive eigenvalues of $\mathscr{D}_{h,\mathbf{A}}$ accumulate exponentially fast to zero in the strong magnetic field limit. This behavior is in contrast to the one of the negative eigenvalues. Indeed, for the first negative eigenvalue we show that there is a constant $C^- > 0$ such that

$$\eta_1^- = -C^- b^{\frac{1}{2}} + o(b^{-\frac{1}{2}})\,. \tag{1.5}$$

The constant C^- obeys an effective minimization problem and we know that it is smaller or equal than $\sqrt{2}$ and for constant magnetic field $C^- < \sqrt{2}$ (see Theorem 1.11).

The proof of (1.4) and (1.5) is based upon the asymptotic analysis of a mini-max principle for the corresponding operator D_0. We show a new min-max characterization, well adapted to our setting, whose proof is inspired by [13] and [16]. A result in the same spirit has being recently used in [2]. It is easy to see that the mini-max characterization applies well to any boundary conditions with $\cos\eta \neq 0$, as described in Appendix A one obtains the same type of asymptotic formulas (1.4) and (1.5) with different constants.

As for the zig-zag case, when $\cos\eta = 0$, we obtain analogous results for the energies through a simple application of the asymptotic analysis performed in [6] and the relation between zig-zag and Pauli-Dirichlet operators. This is explained in Section 1.3 and the results can be summarized as follows: For $k \in \{1, 2, 3, \dots\}$ we denote by μ_k^- and μ_k^+ the k-th positive eigenvalue of $D_{\pi/2}$ and $D_{3\pi/2}$, respectively. Then, we find constants $0 < c_k \leqslant C_k < \infty$ that, as $b \to \infty$,

$$c_k b^{(k+1)/2} e^{-\alpha b}(1 + o(1)) \leqslant \mu_k^- \leqslant C_k b^{(k+1)/2} e^{-\alpha b}(1 + o(1))\,,$$

and

$$\mu_k^+ \geqslant \sqrt{2bB_0}\,,$$

where $\alpha > 0$ is the same constant appearing in (1.4).

Let us finally mention that our results compare well with the findings in the physics literature [23, 17, 27] for constant magnetic fields.

1.1. **Basic definitions and assumptions.**

We study the equivalent semiclassical problem given by the the action of

$$\mathscr{D}_{h,\mathbf{A}} = \sigma \cdot (\mathbf{p} - \mathbf{A}) = \begin{pmatrix} 0 & d_{h,\mathbf{A}} \\ d_{h,\mathbf{A}}^\times & 0 \end{pmatrix}\,, \tag{1.6}$$

where $\mathbf{p} = -ih\nabla$ for $h > 0$,

$$d_{h,\mathbf{A}} = -2ih\partial_z - A_1 + iA_2\,, \quad d_{h,\mathbf{A}}^\times = -2ih\partial_{\bar{z}} - A_1 - iA_2\,,$$

with $\partial_{\overline{z}} = \frac{\partial_1 + i\partial_2}{2}$ and $\partial_z = \frac{\partial_1 - i\partial_2}{2}$. We focus on the boundary conditions described above for $\eta = 0$, that is

$$\mathcal{B} = \sigma \cdot \mathbf{t} = -i\sigma_3(\sigma \cdot \mathbf{n}) \,,$$

where $\mathbf{n}$ is the outward pointing normal to the boundary $\partial\Omega$. The associated magnetic Dirac operator with MIT bag (or infinite mass) boundary condition is $(\mathscr{D}_{h,\mathbf{A}}, \mathsf{Dom}(\mathscr{D}_{h,\mathbf{A}}))$ with

$$\mathsf{Dom}(\mathscr{D}_{h,\mathbf{A}}) = \left\{ \varphi \in H^1(\Omega, \mathbb{C}^2) \,, \quad \mathcal{B}\varphi = \varphi \text{ on } \partial\Omega \right\} .$$

Remark 1.1. Note that

$$\sigma \cdot \mathbf{n} = \begin{pmatrix} 0 & \overline{\mathbf{n}} \\ \mathbf{n} & 0 \end{pmatrix} ,$$

so that the boundary condition reads

$$u_2 = i\mathbf{n}u_1 \,,$$

where $\varphi = (u_1, u_2)^T$, and $\mathbf{n} = (n_1, n_2)^T$ denotes the normal vector in $\mathbb{R}^2$ and also $\mathbf{n} = n_1 + in_2 \in \mathbb{C}$.

The main purpose of the paper is to study the asymptotic behavior of the eigenvalues near 0 in the semiclassical limit $h \to 0$.

Assumption 1.2.

(i) *Ω is bounded, simply connected, $\partial\Omega$ is $\mathcal{C}^2$-regular,*
(ii) $B \in W^{1,\infty}(\overline{\Omega})$.

Under Assumption 1.2, the operator $\mathscr{D}_{h,0}$, without magnetic field, is self-adjoint on $L^2(\Omega)^2$ (see for instance [8]). We work in the so-called *Coulomb gauge* that is given through the unique solution of the Poisson equation

$$\Delta\phi = B \,, \qquad \phi_{|\partial\Omega} = 0 \,, \tag{1.7}$$

by choosing $\mathbf{A} = (-\partial_2\phi, \partial_1\phi)^T = \nabla\phi^\perp$. Notice that by standard regularity theory the components of $\mathbf{A}$ are bounded. Hence $\mathscr{D}_{h,\mathbf{A}}$ is self-adjoint and it has compact resolvent since $\mathsf{Dom}(\mathscr{D}_{h,\mathbf{A}}) \subset H^1$. In particular, the spectrum $\mathscr{D}_{h,\mathbf{A}}$ of is discrete. We denote by $(\lambda_k^+(h))_{k\geqslant 1}$ and $(-\lambda_k^-(h))_{k\geqslant 1}$ the positive and negative eigenvalues of $\mathscr{D}_{h,\mathbf{A}}$ counted with multiplicities. In fact, $\mathscr{D}_{h,\mathbf{A}}$ has no zero modes. This can be seen using the following lemma, which is a consequence of [18] and [6].

Notation 1. We denote by $\langle \cdot, \cdot \rangle$ the standard scalar product on Ω (antilinear w.r.t. the left argument), and by $\| \cdot \|$ the associated norm. In the same way, we denote by $\langle \cdot, \cdot \rangle_{\partial\Omega}$ the L^2-scalar product on $L^2(\partial\Omega)$.

Lemma 1.3. *For all $h > 0$, there exists $C(h) > 0$ such that, for all $u \in H_0^1(\Omega)$, we have*

$$\|d_{h,\mathbf{A}}^\times u\|^2 \geqslant C(h)\|u\|^2 \,.$$

Proposition 1.4. *The operator $\mathscr{D}_{h,\mathbf{A}}$ has no zero modes.*

Proof. Consider $\varphi = (u, v)^T \in \mathrm{Dom}(\mathscr{D}_{h,\mathbf{A}})$ such that $\mathscr{D}_{h,\mathbf{A}}\varphi = 0$. We have $d_{h,\mathbf{A}}v = d_{h,\mathbf{A}}^\times u = 0$. Thus, integrating by parts, and using the boundary condition, we get

$$0 = \langle d_{h,\mathbf{A}}v, u\rangle = \langle v, d_{h,\mathbf{A}}^\times u\rangle + h\langle -i\overline{\mathbf{n}}\, v, u\rangle_{\partial\Omega} = h\|u\|_{\partial\Omega}^2 \,.$$

Therefore $u \in H_0^1(\Omega)$, and Lemma 1.3 implies that $u = 0$. □

Since $\mathscr{D}_{h,\mathbf{A}}$ has no zero mode, its spectrum is

$$\mathrm{sp}(\mathscr{D}_{h,\mathbf{A}}) = \{\dots,\ -\lambda_2^-(h)\,,\ -\lambda_1^-(h)\} \cup \{\lambda_1^+(h)\,,\ \lambda_2^+(h)\,,\dots\}\,. \tag{1.8}$$

Assumption 1.5. *B is positive. We define $b_0 = \inf_\Omega B > 0$ and $b_0' = \min_{\partial\Omega} B$.*

Under this assumption, ϕ is subharmonic so that

$$\max_{x\in\overline{\Omega}} \phi = \max_{x\in\partial\Omega} \phi = 0\,,$$

and the minimum of ϕ will be negative and attained in Ω.

Assumption 1.6.

(i) *The minimum $\phi_{\min}$ of ϕ is attained at a unique point $x_{\min}$.*
(ii) *The Hessian matrix $\mathsf{Hess}_{\min}\phi$ of ϕ at $x_{\min}$ is positive definite i.e. $x_{\min}$ is non-degenerate minimum. We also denote by $z_{\min}$, the minimum $x_{\min}$ seen as a complex number.*

1.2. **Main results.** The magnetic Hardy space is

$$\mathscr{H}^2_{h,\mathbf{A}}(\Omega) = \{u \in L^2(\Omega) : d^\times_{h,\mathbf{A}} u = 0\,, u_{|\partial\Omega} \in L^2(\partial\Omega)\}\,.$$

We let

$$\mathfrak{H}_{h,\mathbf{A}} = H^1(\Omega) + \mathscr{H}^2_{h,\mathbf{A}}(\Omega)\,,$$

and endow it with the Hermitian scalar product given by

$$\forall (u_1,u_2) \in \mathfrak{H}_{h,\mathbf{A}} \times \mathfrak{H}_{h,\mathbf{A}}\,, \quad \langle u_1,u_2\rangle_{\mathfrak{H}_{h,\mathbf{A}}} = \langle u_1,u_2\rangle + \langle d^\times_{h,\mathbf{A}} u_1, d^\times_{h,\mathbf{A}} u_2\rangle + \langle u_1,u_2\rangle_{\partial\Omega}\,.$$

The following result gives us a non-linear min-max characterization for the positive eigenvalues of $\mathscr{D}_{h,\mathbf{A}}$.

Theorem 1.7. *Under Assumption 1.2. We have, for all $h>0$ and $k \geqslant 1$,*

$$\lambda_k^+(h) = \min_{\substack{W\subset\mathfrak{H}_{h,\mathbf{A}}\\ \dim W = k}} \max_{u\in W\setminus\{0\}} \frac{h\|u\|^2_{\partial\Omega} + \sqrt{h^2\|u\|^4_{\partial\Omega} + 4\|u\|^2\|d^\times_{h,\mathbf{A}}u\|^2}}{2\|u\|^2}\,.$$

Remark 1.8. Due to the symmetry of the problem we use also this min-max characterization for the negative eigenvalues of $\mathscr{D}_{h,\mathbf{A}}$ after changing the sign of the magnetic field. Indeed, consider the charge conjugation operator

$$C\colon \varphi \in \mathbb{C}^2 \mapsto \sigma_1\overline{\varphi} \in \mathbb{C}^2\,,$$

where $\overline{\varphi}$ is the vector of $\mathbb{C}^2$ made of the complex conjugate of the coefficients of φ. We have $C\mathsf{Dom}(\mathscr{D}_{h,\mathbf{A}}) = \mathsf{Dom}(\mathscr{D}_{h,\mathbf{A}})$, and $C\mathscr{D}_{h,\mathbf{A}}C = -\mathscr{D}_{h,-\mathbf{A}}$. In particular, we get that

$$\mathrm{sp}(\mathscr{D}_{h,\mathbf{A}}) = -\mathrm{sp}(\mathscr{D}_{h,-\mathbf{A}})\,.$$

In order to state our next result on the asymptotic estimates of the $\lambda_k^+(h)$ we introduce some notation to explicitly define the constant C_k^+ from (1.4).

Notation 2. Let us denote by $\mathscr{O}(\Omega)$ and $\mathscr{O}(\mathbb{C})$ the sets of holomorphic functions on Ω and $\mathbb{C}$. We consider the following (anisotropic) Segal-Bargmann space

$$\mathscr{B}^2(\mathbb{C}) = \{u \in \mathscr{O}(\mathbb{C}) : N_{\mathcal{B}}(u) < +\infty\}\,,$$

where

$$N_{\mathcal{B}}(u) = \left(\int_{\mathbb{R}^2} \left| u\left(y_1 + iy_2\right) \right|^2 e^{-\mathsf{Hess}_{x_{\min}} \phi(y,y)} \mathrm{d}y \right)^{1/2} .$$

We also introduce the Hardy space

$$\mathscr{H}^2(\Omega) = \left\{ u \in \mathscr{O}(\Omega) : \|u\|_{\partial\Omega} < +\infty \right\},$$

where

$$\|u\|_{\partial\Omega} = \left(\int_{\partial\Omega} \left| u\left(y_1 + iy_2\right) \right|^2 \mathrm{d}y \right)^{1/2} .$$

We also define for $P \in \mathscr{H}^2(\Omega)$, $A \subset \mathscr{H}^2(\Omega)$,

$$\mathsf{dist}_{\mathcal{H}}(P, A) = \inf \left\{ N_{\mathcal{H}}(P - Q)\,, \text{ for all } Q \in A \right\} ,$$

and for $P \in \mathscr{B}^2(\mathbb{C})$, $A \subset \mathscr{B}^2(\mathbb{C})$,

$$\mathsf{dist}_{\mathcal{B}}(P, A) = \inf \left\{ N_{\mathcal{B}}(P - Q)\,, \text{ for all } Q \in A \right\} .$$

The following constant is important in our asymptotic analysis

$$C_k(B, \Omega) = \left(\frac{\mathrm{dist}_{\mathcal{H}}\left((z - z_{\min})^{k-1}, \mathscr{H}^2_k(\Omega) \right)}{\mathrm{dist}_{\mathcal{B}}\left(z^{k-1}, \mathcal{P}_{k-2} \right)} \right)^2 , \tag{1.9}$$

where $\mathcal{P}_{k-2} = \mathrm{span}\left(1, \ldots, z^{k-2}\right) \subset \mathscr{B}^2(\mathbb{C})$, $\mathcal{P}_{-1} = \{0\}$ and

$$\mathscr{H}^2_k(\Omega) = \left\{ u \in \mathscr{H}^2(\Omega),\ u^{(n)}(z_{\min}) = 0,\ \text{for } n \in \{0, \ldots, k-1\} \right\}. \tag{1.10}$$

Theorem 1.9. *Under Assumptions 1.2, 1.5 and 1.6, we have for all $k \geqslant 1$,*

$$\lambda_k^+(h) = C_k(B, \Omega) h^{1-k} e^{2\phi_{\min}/h} (1 + o_{h\to 0}(1)) .$$

Remark 1.10. Let us assume that Ω is the disk of radius R centered at 0, and that B is radial. In this case $z_{\min} = 0$ and $\mathsf{Hess}_{x_{\min}} \phi = B(0)\mathrm{Id}/2$. Moreover, using Fourier series, we see that $(z^n)_{n\geqslant 0}$ is an orthogonal basis for $N_{\mathcal{B}}$ and $N_{\mathcal{H}}$. In particular, $\mathscr{H}^2_k(\Omega)$ is $N_{\mathcal{H}}$-orthogonal to z^{k-1} so that

$$\mathsf{dist}_{\mathcal{H}}\left(z^{k-1}, \mathscr{H}^2_k(\Omega) \right)^2 = \|z^{k-1}\|^2_{\partial\Omega} = R^{2k-2} |\partial\Omega| = 2\pi R^{2k-1} .$$

In addition, $\mathcal{P}_{k-2}$ is $N_{\mathcal{B}}$-orthogonal to z^{k-1} so that

$$\mathsf{dist}_{\mathcal{B}}\left(z^{k-1}, \mathcal{P}_{k-2} \right)^2 = N_{\mathcal{B}}(z^{k-1})^2 = 2\pi \frac{2^{k-1}(k-1)!}{B(0)^k} ,$$

Thus, we get that

$$C_k(B, \Omega) = \frac{B(0)^k}{(k-1)!} \left(\frac{R^2}{2} \right)^{k-1} .$$

We now turn to the negative eigenvalues of $\mathscr{D}_{h,\mathbf{A}}$. Consider, for all $c \geqslant 0$,

$$\nu(c) = \inf_{\substack{u \in \mathfrak{H}^2_{-\mathbf{A}_0}(\mathbb{R}^2_+) \\ u \neq 0}} \frac{\int_{\mathbb{R}^2_+} |(-i\partial_s - \tau + i(-i\partial_\tau))u|^2 \mathrm{d}s\mathrm{d}\tau + c \int_{\mathbb{R}} |u(s,0)|^2 \mathrm{d}s}{\|u\|^2} , \tag{1.11}$$

with $\mathbf{A}_0 = (-\tau, 0)$. Notice that the quadratic form minimized in (1.11) corresponds to the magnetic Schrödinger operator on a half-plane with a constant magnetic field

(equaling 1) and equipped by a Robin-like boundary condition. More details are given in Appendix C.

We can prove (see Proposition C.7) that the equation $\nu(c) = c^2$ has a unique positive solution, denoted by c_0. Moreover, $c_0 \in (0, \sqrt{2})$.

Theorem 1.11. *Under Assumptions 1.2 and 1.5, we have*

$$\lambda_1^-(h) = h^{\frac{1}{2}} \min(\sqrt{2b_0}, c_0\sqrt{b_0'}) + o_{h\to 0}(h^{\frac{1}{2}}) \,,$$

where $\lambda_1^-(h)$ is defined in Section 1.1, $b_0 = \min_{\overline{\Omega}} B(x)$ and $b_0' = \min_{\partial\Omega} B(x)$. In particular, when $B \equiv b_0$ is constant, we have

$$\lambda_1^-(h) = c_0\sqrt{b_0 h} + o_{h\to 0}(h^{\frac{1}{2}}) \,.$$

Remark 1.12. In fact, c_0 equals

$$\inf_{\substack{u\in\mathfrak{H}^2_{-,\mathbf{A}_0}(\mathbb{R}^2_+)\\ u\neq 0}} \frac{\int_{\mathbb{R}} |u(s,0)|^2 \mathrm{d}s + \sqrt{\left(\int_{\mathbb{R}} |u(s,0)|^2 \mathrm{d}s\right)^2 + 4\|u\|^2 \int_{\mathbb{R}^2_+} |(-i\partial_s - \tau + i(-i\partial_\tau))u|^2 \mathrm{d}s\mathrm{d}\tau}}{2\|u\|^2} \,.$$

Remark 1.13. The asymptotic analysis leading to Theorems 1.9 and 1.11 strongly differ from each other. Indeed, the eigenfunctions are localized near $x_{\min}$ for the positive energies, whereas, when B is constant, they are localized near the boundary for the negative ones. Moreover, in this last case, for non-constant magnetic fields, the eigenfunctions might be localized inside if b_0/b_0' is small enough. Consequently, the underlying semiclassical problems do not share the same structure. Actually, in a forthcoming paper, we reduce the spectral analysis of the negative energies to the one of a pseudo-differential operator on the boundary.

Remark 1.14. The eigenvalues in the strong magnetic field limit given by the operator D_0 described in the introduction can be found by a simple scaling argument. Fix the magnetic field B to be lower bounded by $b > 0$. Then we have

$$\operatorname{sp} D_0 = b \operatorname{sp} \mathscr{D}_{1/b,\tilde{\mathbf{A}}} \,,$$

where the components of $\tilde{\mathbf{A}}$ satisfy $\partial_1\tilde{A}_2 - \partial_2\tilde{A}_1 = B/b$. Then equations (1.4) and (1.5) are direct consequences of theorems 1.9 and 1.11, respectively.

1.3. **The zig-zag case.** In this paper, we consider the Dirac operator with MIT bag boundary condition (and its variants in Appendix A). The so-called zig-zag boundary condition also appears commonly in the description of the electrical properties of pieces of graphene. It is worth noticing that the spectral properties of the related operators exhibit completely different asymptotic behaviors compared with the ones studied here. More precisely, the operators $(\mathscr{Z}^{\pm}_{h,\mathbf{A}}, \mathsf{Dom}(\mathscr{Z}^{\pm}_{h,\mathbf{A}}))$ acting as $\sigma\cdot(\mathbf{p}-\mathbf{A})$ on

$$\mathsf{Dom}(\mathscr{Z}^{-}_{h,\mathbf{A}}) = H^1_0(\Omega,\mathbb{C}) \times \{u \in L^2(\Omega,\mathbb{C}) \,, \partial_z u \in L^2(\Omega,\mathbb{C})\} \,,$$
$$\mathsf{Dom}(\mathscr{Z}^{+}_{h,\mathbf{A}}) = \{u \in L^2(\Omega,\mathbb{C}) \,, \partial_{\bar{z}} u \in L^2(\Omega,\mathbb{C})\} \times H^1_0(\Omega,\mathbb{C}) \,,$$

are self-adjoint. This easily seen since by construction the operators $\mathscr{Z}^{\pm}_{h,\mathbf{A}}$ have the supersymmetric structure

$$\mathscr{Z}^{\pm}_{h,\mathbf{A}} = \begin{pmatrix} 0 & D_\pm \\ D_\pm^* & 0 \end{pmatrix} \,,$$

where D_+ and D_-^* have Dirichlet boundary conditions. Moreover, 0 is an eigenvalue of infinite multiplicity for both of them and their kernels can be determined explicitly (see [26, Chapter 5],[22] and [6, Proposition 4.4]).

Next notice that since $\sigma_3 \mathscr{Z}^{\pm}_{h,\mathbf{A}} = -\mathscr{Z}^{\pm}_{h,\mathbf{A}}\sigma_3$ holds, the spectra of both operators is symmetric with respect to zero. Moreover, by simply squaring the operators one sees that, due to the isospectrality of $D_\pm D_\pm^*$ and $D_\pm^* D_\pm$ away from zero,

$$\left\{\lambda^2, \lambda \in \mathrm{sp}\,(\mathscr{Z}^{+}_{h,\mathbf{A}}) \setminus \{0\}\right\} = \mathrm{sp}\{D_+^* D_+\}, \quad \text{and} \quad \left\{\lambda^2, \lambda \in \mathrm{sp}\,(\mathscr{Z}^{-}_{h,\mathbf{A}}) \setminus \{0\}\right\} = \mathrm{sp}\{D_- D_-^*\}\,.$$

Thus, their discrete spectrum satisfy

$$\mathrm{sp}_d\,(\mathscr{Z}^{\pm}_{h,\mathbf{A}}) = \mathrm{sp}\,(\mathscr{Z}^{\pm}_{h,\mathbf{A}}) \setminus \{0\} = \left\{\sqrt{\alpha_k^{\pm}(h)}\,, k \in \mathbb{N}^*\right\} \cup \left\{-\sqrt{\alpha_k^{\pm}(h)}\,, k \in \mathbb{N}^*\right\}\,,$$

where $(\alpha_k^+(h))_{k\geqslant 1}$ and $(\alpha_k^-(h))_{k\geqslant 1}$ are the ordered sequences of the eigenvalues (counted with multiplicity) of the operators $D_+^* D_+$ and $D_- D_-^*$ that act as

$$|\,\mathbf{p} - \mathbf{A}|^2 + hB\,, \quad \text{and} \quad |\,\mathbf{p} - \mathbf{A}|^2 - hB\,,$$

on $H^1_0(\Omega, \mathbb{C}) \cap H^2(\Omega, \mathbb{C})$. Therefore, we deduce from [6, Theorem 1.3.], that there exists $\theta_0 \in (0, 1]$ such that for all $k \geqslant 1$

$$\begin{aligned}\left(\theta_0 2C_k(B,\Omega) h^{1-k} e^{2\phi_{\min}/h}\right)^{1/2} (1 + o_{h\to 0}(1)) &\leqslant \sqrt{\alpha_k^-(h)} \\ &\leqslant \left(2C_k(B,\Omega) h^{1-k} e^{2\phi_{\min}/h}\right)^{1/2} (1 + o_{h\to 0}(1))\,,\end{aligned}$$

as $h \to 0$. Finally, it is well known that

$$\sqrt{\alpha_k^+(h)} \geqslant \sqrt{2B_0 h}\,.$$

2. A non-linear min-max characterization

The aim of this section is to establish Theorem 1.7. To do so, we first establish in Section 2.1 some fundamental properties of the natural minimization space $\mathfrak{H}_{h,\mathbf{A}}$. Then, we prove that the λ-eigenspace of $\mathscr{D}_{h,\mathbf{A}}$ are isomorphic with the 0-eigenspace of an auxiliary operator $\mathscr{L}_\lambda$ depending quadratically on λ, see Proposition 2.12. Section 2.3 is devoted to describe the spectrum of $\mathscr{L}_\lambda$, and in particular when $0 \in \mathrm{sp}(\mathscr{L}_\lambda)$.

Throughout this section, $h > 0$ is fixed.

2.1. Magnetic Hardy spaces.

Definition 2.1. The magnetic Hardy space is

$$\mathscr{H}^2_{h,\mathbf{A}}(\Omega) = \{u \in L^2(\Omega) : d^{\times}_{h,\mathbf{A}} u = 0\,, u_{|\partial\Omega} \in L^2(\partial\Omega)\}\,.$$

We let

$$\mathfrak{H}_{h,\mathbf{A}} = H^1(\Omega) + \mathscr{H}^2_{h,\mathbf{A}}(\Omega)\,,$$

and endow it with the Hermitian scalar product given by

$$\forall (u_1, u_2) \in \mathfrak{H}_{h,\mathbf{A}} \times \mathfrak{H}_{h,\mathbf{A}}\,, \quad \langle u_1, u_2\rangle_{\mathfrak{H}_{h,\mathbf{A}}} = \langle u_1, u_2\rangle + \langle d^{\times}_{h,\mathbf{A}} u_1, d^{\times}_{h,\mathbf{A}} u_2\rangle + \langle u_1, u_2\rangle_{\partial\Omega}\,.$$

Proposition 2.2. *The free Dirac operator and the magnetic Dirac operator are related by the formula*

$$e^{\sigma_3\phi/h}\sigma\cdot\mathbf{p}\,e^{\sigma_3\phi/h}=\sigma\cdot(\mathbf{p}-\mathbf{A})\,, \tag{2.1}$$

as operators acting on $H^1(\Omega,\mathbb{C}^2)$ functions.

Remark 2.3. By using the change of function $u=e^{-\phi/h}w$ suggested by Proposition 2.2, we have

$$\mathscr{H}^2_{h,\mathbf{A}}(\Omega)=e^{-\phi/h}\mathscr{H}^2_0(\Omega)\,,\quad \mathfrak{H}_{h,\mathbf{A}}=e^{-\phi/h}\mathfrak{H}_{\mathbf{0}}\,,$$

where

$$\mathfrak{H}_{\mathbf{0}}=H^1(\Omega)+\mathscr{H}^2_0(\Omega)\,,$$

and

$$\mathscr{H}^2_0(\Omega)=\{w\in L^2(\Omega):\partial_{\overline{z}}w=0\,,w_{|\partial\Omega}\in L^2(\partial\Omega)\}\,.$$

Note that, for all $(u_1,u_2)\in\mathfrak{H}_{h,\mathbf{A}}\times\mathfrak{H}_{h,\mathbf{A}}$,

$$\langle u_1,u_2\rangle_{\mathfrak{H}_{h,\mathbf{A}}}=\langle w_1,w_2\rangle_{L^2(e^{-2\phi/h})}+\langle -2ih\partial_{\overline{z}}w_1,-2ih\partial_{\overline{z}}w_2\rangle_{L^2(e^{-2\phi/h})}+\langle w_1,w_2\rangle_{\partial\Omega}\,,$$

where $w_j=e^{\phi/h}u_j$ for $j=1,2$. Then, by using the Riemann biholomorphism $F:\mathbb{D}\to\Omega$, the classical Hardy space $\mathscr{H}^2_0(\Omega)=\mathscr{H}^2(\Omega)$ becomes the canonical Hardy space

$$\mathscr{H}^2(\mathbb{D})=\left\{f\in\mathscr{O}(\mathbb{D}):\left(\frac{f^{(n)}(0)}{n!}\right)_{n\geqslant 0}\in\ell^2(\mathbb{N})\right\}\,.$$

Note that, for $f\in\mathscr{H}^2(\mathbb{D})$,

$$\|f\|^2=2\pi\sum_{n\geqslant 1}(2n+2)^{-1}|u_n|^2\,,\quad \|f\|^2_{\partial\Omega}=2\pi\sum_{n\geqslant 0}|u_n|^2\,,\quad u_n=\frac{f^{(n)}(0)}{n!}\,. \tag{2.2}$$

The following lemma is a classical result. For the reader's convenience, we recall the proof in Appendix B.

Lemma 2.4. *The space $(\mathscr{H}^2_{h,\mathbf{A}}(\Omega),\langle\cdot,\cdot\rangle_{\partial\Omega})$ is a Hilbert space. Moreover, $\mathscr{H}^2_{h,\mathbf{A}}(\Omega)$ is compactly embedded in $L^2(\Omega)$.*

Lemma 2.5. *There exists $c>0$ such that, for all $h>0$, and for all $u\in H^1(\Omega)$,*

$$\sqrt{2hB_0}\|\Pi^{\perp}_{h,\mathbf{A}}u\|\leqslant\|d^{\times}_{h,\mathbf{A}}u\|\,,\qquad ch^2(\|\Pi^{\perp}_{h,\mathbf{A}}u\|_{\partial\Omega}+\|\nabla\Pi^{\perp}_{h,\mathbf{A}}u\|)\leqslant\|d^{\times}_{h,\mathbf{A}}u\|\,,$$

where $\Pi_{h,\mathbf{A}}$ is the (orthogonal) spectral projection on the kernel of the adjoint of the $d_{h,\mathbf{A}}$ with Dirichlet boundary conditions, i.e. *$(d_{h,\mathbf{A}},H^1_0(\Omega))^\star$, and*

$$\mathrm{Id}=\Pi_{h,\mathbf{A}}+\Pi^{\perp}_{h,\mathbf{A}}\,.$$

Proposition 2.6. *The following holds.*

(i) *$(\mathfrak{H}_{h,\mathbf{A}},\langle\cdot,\cdot\rangle_{\mathfrak{H}_{h,\mathbf{A}}})$ is a Hilbert space.*
(ii) *$H^1(\Omega)$ is dense in $\mathfrak{H}_{h,\mathbf{A}}$.*
(iii) *The embedding $\mathfrak{H}_{h,\mathbf{A}}\hookrightarrow L^2(\Omega)$ is compact.*

Proof. Let us prove (i). We consider a Cauchy sequence (u_n) for $\|\cdot\|_{\mathfrak{H}_{h,\mathbf{A}}}$. It is obviously a Cauchy sequence for $\|\cdot\|$ and $\|\cdot\|_{\partial\Omega}$. We write $u_n = \Pi_{h,\mathbf{A}} u_n + \Pi_{h,\mathbf{A}}^{\perp} u_n$. From Lemma 2.5, we see that $(\Pi_{h,\mathbf{A}}^{\perp} u_n)$ is a Cauchy sequence in $H^1(\Omega)$, and thus converges to some $u^{\perp} \in H^1(\Omega)$. Moreover, by using again Lemma 2.5, $(\Pi_{h,\mathbf{A}} u_n)$ is a Cauchy sequence in $\mathscr{H}^2_{h,\mathbf{A}}(\Omega)$. From Lemma 2.4, $(\Pi_{h,\mathbf{A}} u_n)$ converges to some $u \in \mathscr{H}^2_{h,\mathbf{A}}(\Omega)$. It follows that (u_n) converges to $u + u^{\perp}$ in $\mathfrak{H}_{h,\mathbf{A}}$.

Item (ii) is a consequence of [6, Lemma C.1].

By using again the orthogonal decomposition induced by $\Pi_{h,\mathbf{A}}$, and the compactness of $H^1(\Omega) \hookrightarrow L^2(\Omega)$, and of $\mathscr{H}^2_{h,\mathbf{A}}(\Omega) \hookrightarrow L^2(\Omega)$ (see Lemma 2.4), we get (iii). □

2.2. **Statement of the min-max characterization.** The proof of Theorem 1.7 is a consequence of Propositions 2.7 and 2.8, see below.

Notation 3. For all $k \geqslant 1$ and all $h > 0$, we define

$$\mu_k(h) = \inf_{\substack{W \subset H^1(\Omega) \\ \dim W = k}} \sup_{u \in W \setminus \{0\}} \rho_+(u)\,,$$

where

$$\rho_+(u) = \frac{h\|u\|^2_{\partial\Omega} + \sqrt{h^2\|u\|^4_{\partial\Omega} + 4\|u\|^2 \|d^{\times}_{h,\mathbf{A}} u\|^2}}{2\|u\|^2}\,. \tag{2.3}$$

Proposition 2.7. *We have, for all $k \geqslant 1$,*

$$\mu_k(h) = \inf_{\substack{W \subset \mathfrak{H}_{h,\mathbf{A}} \\ \dim W = k}} \sup_{u \in W \setminus \{0\}} \rho_+(u) = \min_{\substack{W \subset \mathfrak{H}_{h,\mathbf{A}} \\ \dim W = k}} \sup_{u \in W \setminus \{0\}} \rho_+(u) > 0\,.$$

Proof. We use Proposition 2.6 (ii) & (iii), and observe that $\rho_+(u) > 0$ for all $u \in \mathfrak{H}_{h,\mathbf{A}}$. □

Proposition 2.8. *For all $k \geqslant 1$, and $h > 0$, we have*

$$\lambda_k^+(h) = \mu_k(h)\,.$$

The following sections are devoted to the proof of Proposition 2.8.

In the following, we drop the h-dependence in the notation.

2.3. **A characterization of the μ_k.**

Notation 4. Let $\lambda \geqslant 0$. Consider the quadratic form defined by

$$\forall u \in \mathfrak{H}_{h,\mathbf{A}}\,, \quad Q_\lambda(u) = \|d^{\times}_{h,\mathbf{A}} u\|^2 + h\lambda\|u\|^2_{\partial\Omega} - \lambda^2\|u\|^2\,,$$

and, for all $k \geqslant 1$,

$$\ell_k(\lambda) = \inf_{\substack{W \subset H^1(\Omega) \\ \dim W = k}} \sup_{u \in W \setminus \{0\}} \frac{Q_\lambda(u)}{\|u\|^2}\,.$$

Note that, for all $u \in \mathfrak{H}_{h,\mathbf{A}} \setminus \{0\}$,

$$Q_\lambda(u) = -\|u\|^2(\lambda - \rho_-(u))(\lambda - \rho_+(u))\,, \tag{2.4}$$

where $\rho_+(u)$ is defined in (2.3) and $\rho_-(u)$ is the other zero of the polynomial above.

From Proposition 2.6, we deduce the following.

Lemma 2.9. *For $\lambda > 0$, the (bounded below) quadratic form Q_λ is closed. The associated (unbounded) self-adjoint operator $\mathscr{L}_\lambda$ has compact resolvent, and its eigenvalues are characterized by the usual min-max formulas*

$$\ell_k(\lambda) = \inf_{\substack{W \subset H^1(\Omega) \\ \dim W = k}} \sup_{u \in W \setminus \{0\}} \frac{Q_\lambda(u)}{\|u\|^2} = \min_{\substack{W \subset \mathfrak{H}_{h,\mathbf{A}} \\ \dim W = k}} \max_{u \in W \setminus \{0\}} \frac{Q_\lambda(u)}{\|u\|^2} \,.$$

Lemma 2.10. *For all $k \geqslant 1$, the function $\ell_k : (0, +\infty) \to \mathbb{R}$ satisfies the following:*

(i) *ℓ_1 is concave,*
(ii) *for all $\mu \in (0, \mu_1)$, and all $k \geqslant 1$, $\ell_k(\mu) > 0$,*
(iii) *$\lim_{\lambda \to +\infty} \ell_k(\lambda) = -\infty$,*
(iv) *ℓ_k is continuous,*
(v) *the equation $\ell_k(\lambda) = 0$ has exactly one positive solution, denoted by E_k.*

Proof. Item (i) follows by observing that the infimum of a family of concave functions is itself concave.

It is enough to check Item (ii) for $k = 1$. Consider $\mu > 0$. Thanks to Proposition 2.7, there exists $u \in \mathfrak{H}_{h,\mathbf{A}}$ such that $\ell_1(\mu) = Q_\mu(u)$. If $\ell_1(\mu) \leqslant 0$, then, by (2.4), we have that $\mu \geqslant \rho_+(u) \geqslant \mu_1$.

By taking any finite dimensional space $W \subset H^1_0(\Omega)$, we readily see that

$$\ell_k(\lambda) \leqslant \sup_{u \in W,\, \|u\|=1} \|d^\times_{h,\mathbf{A}} u\| - \lambda^2 \,.$$

We get Item (iii).

Since ℓ_1 is concave, it is also continuous. Then, the family $(\mathscr{L}_\lambda)_{\lambda>0}$ is analytic of type (B) in the sense of Kato (i.e., $\mathrm{Dom}(Q_\lambda)$ is independent of $\lambda > 0$). This implies that the ℓ_k are continuous functions. Actually, this can directly be seen from the following equality

$$\lambda_1^{-1} Q_{\lambda_1}(u) - \lambda_2^{-1} Q_{\lambda_2}(u) = (\lambda_2 - \lambda_1) \left(\|d^\times_{h,\mathbf{A}} u\|^2 (\lambda_1 \lambda_2)^{-1} + \|u\|^2 \right) , \tag{2.5}$$

for all $\lambda_1, \lambda_2 > 0$ and $u \in \mathfrak{H}_{h,\mathbf{A}}$.

Let us now deal with Item (v). Consider $\lambda_1 > 0$ such that $\ell_k(\lambda_1) = 0$.

Firstly, there exists $W \subset \mathfrak{H}_{h,\mathbf{A}}$ with $\dim W = k$ such that $\max_W Q_{\lambda_1} = 0$. In particular, for all $u \in W$, $Q_{\lambda_1}(u) \leqslant 0$. Using (2.5), we find, for all $u \in W$,

$$\lambda_2^{-1} Q_{\lambda_2}(u) \leqslant -(\lambda_2 - \lambda_1) \left(\|d^\times_{h,\mathbf{A}} u\|^2 (\lambda_1 \lambda_2)^{-1} + \|u\|^2 \right) .$$

This implies that, for all $\lambda_2 > \lambda_1$,

$$\lambda_2^{-1} \sup_{u \in W,\, \|u\|=1} Q_{\lambda_2}(u) \leqslant -(\lambda_2 - \lambda_1) \inf_{u \in W,\, \|u\|=1} \left(\|d^\times_{h,\mathbf{A}} u\|^2 (\lambda_1 \lambda_2)^{-1} + \|u\|^2 \right) \leqslant -(\lambda_2 - \lambda_1) \,.$$

Hence,

$$\forall \lambda_2 > \lambda_1 \,, \quad \ell_k(\lambda_2) \leqslant -\lambda_2(\lambda_2 - \lambda_1) < 0 \tag{2.6}$$

Secondly, for all $W \subset \mathfrak{H}_{h,\mathbf{A}}$ with $\dim W = k$, we have $\max_{u \in W, \|u\|=1} Q_{\lambda_1}(u) \geqslant 0$. We have, for all $\lambda_2 < \lambda_1$, and all $u \in W$ with $\|u\| = 1$,

$$\begin{aligned} \lambda_1^{-1} Q_{\lambda_1}(u) &= (\lambda_2 - \lambda_1) \left(\|d^\times_{h,\mathbf{A}} u\|^2 (\lambda_1 \lambda_2)^{-1} + \|u\|^2 \right) + \lambda_2^{-1} Q_{\lambda_2}(u) \\ &\leqslant (\lambda_2 - \lambda_1) + \lambda_2^{-1} \max_{u \in W, \|u\|=1} Q_{\lambda_2}(u) \,, \end{aligned}$$

so that, taking the supremum for $u \in W$,

$$0 \leqslant (\lambda_2 - \lambda_1) + \lambda_2^{-1} \max_{u \in W, \|u\|=1} Q_{\lambda_2}(u)\,,$$

and thus

$$\forall \lambda_2 \in (0, \lambda_1)\,, \quad 0 < \lambda_2(\lambda_1 - \lambda_2) \leqslant \ell_k(\lambda_2)\,. \tag{2.7}$$

From (2.6) and (2.7), we see that the zeros of ℓ_k are isolated. From Item (ii), these zeros do not accumulate at 0. Thus, we can consider E_k to be the smallest positive zero of ℓ_k. Consider $\tilde{E}_k > E_k$, the possible next positive zero of ℓ_k. From (2.6) and by continuity, we see that $\ell_k < 0$ on $(E_k, \tilde{E}_k)$, but, from (2.7), $\ell_k > 0$ on $(E_k, \tilde{E}_k)$. Therefore, ℓ_k has only one positive zero. □

Proposition 2.11. *For all $k \geqslant 1$, μ_k is the only positive zero of ℓ_k, i.e.,*

$$E_k = \mu_k\,.$$

Proof. In virtue of Proposition 2.7, we notice that $\mu_k > 0$. Then, we proceed in two steps.

Firstly, consider a subspace $W_k \subset \mathfrak{H}_{h,\mathbf{A}}$ of dimension k such that

$$\max_{u \in W_k \setminus \{0\}} \rho_+(u) = \mu_k\,.$$

For all $u \in W_k \setminus \{0\}$, $\rho_+(u) \leqslant \mu_k$. By the definition of $\ell_k(\mu_k)$ and (2.4), we have

$$\ell_k(\mu_k) \leqslant \max_{u \in W_k \setminus \{0\}} Q_{\mu_k}(u) \leqslant 0\,.$$

Secondly, for all subspace $W \subset \mathfrak{H}_{h,\mathbf{A}}$ of dimension k, we have

$$\mu_k \leqslant \max_{u \in W \setminus \{0\}} \rho_+(u)\,.$$

There exists $u_k \in W \setminus \{0\}$ such that $\mu_k \leqslant \rho_+(u_k)$. Then, we have

$$\max_{u \in W \setminus \{0\}} Q_{\mu_k}(u) \geqslant Q_{\mu_k}(u_k) \geqslant 0\,,$$

and taking the infimum, we find $\ell_k(\mu_k) \geqslant 0$.

We deduce that $\ell_k(\mu_k) = 0$ and conclude by using Lemma 2.10 (v). □

2.4. **Proof of Proposition 2.8.**

2.4.1. *An isomorphism.* The following proposition is crucial.

Proposition 2.12. *Let $\lambda > 0$. Then, the map*

$$\mathscr{J}_\lambda : \left\{ \begin{array}{ccc} \ker \mathscr{L}_\lambda & \longrightarrow & \ker(\mathscr{D}_{h,\mathbf{A}} - \lambda) \\ u & \longmapsto & \begin{pmatrix} u \\ \lambda^{-1} d^\times_{h,\mathbf{A}} u \end{pmatrix} \end{array} \right.$$

is well-defined and it is an isomorphism.

Proof. First, we show that the range of $\mathscr{J}_\lambda$ is indeed contained $\ker(\mathscr{D}_{h,\mathbf{A}} - \lambda)$. Let $u \in \ker(\mathscr{L}_\lambda)$. Notice that $u \in \ker(\mathscr{L}_\lambda)$ is equivalent to

$$\forall w \in \mathfrak{H}_{h,\mathbf{A}}\,, \quad Q_\lambda(u, w) = \langle d^\times_{h,\mathbf{A}} u, d^\times_{h,\mathbf{A}} w \rangle + h\lambda \langle u, w \rangle_{\partial\Omega} - \lambda^2 \langle u, w \rangle = 0\,. \tag{2.8}$$

We set

$$\varphi = \begin{pmatrix} u \\ v \end{pmatrix}, \quad v = \frac{d^{\times}_{h,\mathbf{A}} u}{\lambda}.$$

For all $\psi = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathrm{Dom}(\mathscr{D}_{h,\mathbf{A}})$, we have

$$\begin{aligned}
\langle \varphi, \mathscr{D}_{h,\mathbf{A}} \psi \rangle &= \langle u, d_{h,\mathbf{A}} w_2 \rangle + \langle v, d^{\times}_{h,\mathbf{A}} w_1 \rangle \\
&= \langle d^{\times}_{h,\mathbf{A}} u, w_2 \rangle + \langle u, -ih\overline{n}\, w_2 \rangle_{\partial\Omega} + \left\langle \frac{d^{\times}_{h,\mathbf{A}} u}{\lambda}, d^{\times}_{h,\mathbf{A}} w_1 \right\rangle \\
&= \langle \lambda v, w_2 \rangle + h \langle u, w_1 \rangle_{\partial\Omega} + \left\langle \frac{d^{\times}_{h,\mathbf{A}} u}{\lambda}, d^{\times}_{h,\mathbf{A}} w_1 \right\rangle \\
&= \langle \lambda v, w_2 \rangle + \lambda \langle u, w_1 \rangle = \lambda \langle \varphi, \psi \rangle ,
\end{aligned}$$

where

– the second equality comes from an integration by parts using Proposition 2.6 (ii),
– the third uses the boundary condition $w_2 = inw_1$,
– the fourth uses (2.8).

This shows, by the definition of the adjoint, that $\varphi \in \mathrm{Dom}(\mathscr{D}_{h,\mathbf{A}}{}^*) = \mathrm{Dom}\, \mathscr{D}_{h,\mathbf{A}}$ and in particular that $\mathscr{D}_{h,\mathbf{A}} \varphi = \lambda \varphi$. Therefore, the map is well-defined, and we observe that it is injective.

Let us show that $\mathscr{J}_\lambda$ is surjective. Consider $\begin{pmatrix} u \\ v \end{pmatrix} \in \ker(\mathscr{D}_{h,\mathbf{A}} - \lambda)$. The eigenvalue equation reads

$$d^{\times}_{h,\mathbf{A}} u = \lambda v, \quad d_{h,\mathbf{A}} v = \lambda u, \quad \text{and } v = inu \text{ on } \partial\Omega .$$

Let $w \in \mathfrak{H}_{h,\mathbf{A}}$. Using the eigenvalue equation, and again an integration by parts, we get

$$\begin{aligned}
Q_\lambda(u,w) &= \langle d^{\times}_{h,\mathbf{A}} u, d^{\times}_{h,\mathbf{A}} w \rangle + h\lambda \langle u, w \rangle_{\partial\Omega} - \lambda^2 \langle u, w \rangle \\
&= \lambda \langle v, d^{\times}_{h,\mathbf{A}} w \rangle + h\lambda \langle -i\overline{n} v, w \rangle_{\partial\Omega} - \lambda^2 \langle u, w \rangle \\
&= \lambda \langle d_{h,\mathbf{A}} v, w \rangle - \lambda^2 \langle u, w \rangle = \lambda^2 \langle u, w \rangle - \lambda^2 \langle u, w \rangle = 0 .
\end{aligned}$$

Hence, $u \in \mathrm{Dom}(\mathscr{D}_{h,\mathbf{A}})$ and $u \in \ker \mathscr{L}_\lambda$. □

Corollary 2.13. *We set $\Lambda = \{\lambda^+_j, j \geqslant 1\}$ and $M = \{\mu_k, k \geqslant 1\}$. We have $\Lambda = M$. In particular, $\mu_1 = \lambda^+_1$.*

Proof. Let $\lambda \in \Lambda$. Proposition 2.12 implies that $0 \in \mathrm{sp}(\mathscr{L}_\lambda)$. Then, there exists $j \geqslant 1$ such that $\ell_j(\lambda) = 0$ and thus (Proposition 2.11) $\lambda = E_j = \mu_j \in M$.

Let $\mu \in M$. Then, there exists $j \geqslant 1$ such that $\mu = E_j$, and hence $\ell_j(\mu) = 0$. In particular, $0 \in \mathrm{sp}(\mathscr{L}_\mu)$ and thus $\mu \in \mathrm{sp}(\mathscr{D}_{h,\mathbf{A}})$ by the isomorphism. □

Notation 5. Let us denote by $(a_k)_{k\geqslant 1}$ the unique increasing sequence such that $\Lambda = M = \{a_k, k \geqslant 1\}$. In addition, for all $k \geqslant 1$, we set $m_k = \dim \ker(\mathscr{D}_{h,\mathbf{A}} - a_k)$.

2.4.2. *Induction argument.* Now, we can prove Proposition 2.8 by induction.

For $n \geqslant 0$, the induction statement is

$$\mathscr{P}(n) \quad : \quad \forall j \in \{1, \dots, m_1 + \dots + m_n + 1\}\,, \qquad \mu_j = \lambda_j^+\,.$$

Thanks to Corollary 2.13, $\mathscr{P}(0)$ is satisfied.

Let $n \geqslant 0$. Assume that, for all $0 \leqslant k \leqslant n$, $\mathscr{P}(k)$ holds.

Notice that

$$\mu_{m_1+\dots+m_n+1} = \lambda^+_{m_1+\dots+m_n+1} = a_{n+1}\,. \tag{2.9}$$

By definition, we have $a_{n+1} \in \mathrm{sp}(\mathscr{D}_{h,\mathbf{A}})$. Moreover, by using the isomorphism,

$$m_{n+1} = \dim\ker(\mathscr{L}_{a_{n+1}})\,.$$

By the min-max theorem, there exists $j_0 \geqslant 0$ such that

$$\ell_{j_0+1}(a_{n+1}) = \dots = \ell_{j_0+m_{n+1}}(a_{n+1}) = 0\,.$$

By Lemma 2.10 (v), we have

$$a_{n+1} = E_{j_0+1} = \dots = E_{j_0+m_{n+1}}\,,$$

so that, using again Proposition 2.11,

$$a_{n+1} = \mu_{j_0+1} = \dots = \mu_{j_0+m_{n+1}}\,.$$

Let us now show that $j_0 = m_1 + \dots + m_n$. By the induction hypothesis, we have

$$\mu_{m_1+\dots+m_n} = a_n < a_{n+1}\,.$$

Thus, $j_0 \geqslant m_1 + \dots + m_n$.

Let us suppose, by contradiction, that $j_0 \geqslant m_1 + \dots + m_n + 1$. With (2.9), we get

$$\mu_{m_1+\dots+m_n+1} = \mu_{j_0+1} = \dots = \mu_{j_0+m_{n+1}} = a_{n+1}\,.$$

In particular, we have the $m_{n+1} + 1$ relations:

$$\ell_{m_1+\dots+m_n+1}(a_{n+1}) = \ell_{j_0+1}(a_{n+1}) = \dots = \ell_{j_0+m_{n+1}}(a_{n+1}) = 0\,.$$

By the min-max theorem, this shows that

$$\dim\ker\mathscr{L}_{a_{n+1}} \geqslant m_{n+1} + 1 > m_{n+1} = \dim\ker(\mathscr{D}_{h,\mathbf{A}} - a_{n+1})\,.$$

This contraditcs the isomorphism property. Therefore, $j_0 = m_1 + \dots + m_n$. This argument also shows that the multiplicity of $\mu_{m_1+\dots+m_{n+1}}$ equals m_{n+1}. With the induction hypothesis, we get

$$\forall j \in \{1, \dots, m_1 + \dots + m_{n+1}\}\,, \qquad \mu_j = \lambda_j^+\,.$$

By definition, we have

$$\lambda^+_{m_1+\dots+m_{n+1}+1} = \min\big(\Lambda \setminus \{a_1, \dots, a_{n+1}\}\big) = \min\big(M \setminus \{a_1, \dots, a_{n+1}\}\big)\,.$$

We observe that $\mu_{m_1+\dots+m_{n+1}+1} > a_{n+1}$ since the muliplicity of $\mu_{m_1+\dots+m_{n+1}}$ equals m_{n+1}.

This proves that

$$\lambda^+_{m_1+\dots+m_{n+1}+1} = \min\big(M \setminus \{a_1, \dots, a_{n+1}\}\big) = \mu_{m_1+\dots+m_{n+1}+1}\,.$$

This concludes the induction argument.

3. Semiclassical analysis of the positive eigenvalues

In this section we prove Theorem 1.9 by applying Proposition 2.8, and considering the asymptotic analysis of a simpler problem. If one wants to estimate $\lambda_k^+(h)$, it is natural to use the functions of the Hardy space $\mathscr{H}^2_{h,\mathbf{A}}(\Omega)$ as test functions. This cancels the $d^\times_{h,\mathbf{A}}$-term in ρ_+ and leads to define

$$\nu_k(h) = \inf_{\substack{W\subset\mathscr{H}^2_{h,\mathbf{A}}(\Omega)\\ \dim W=k}} \sup_{u\in W\setminus\{0\}} \frac{h\|u\|^2_{\partial\Omega}}{\|u\|^2}\,.$$

Theorem 1.9 is a consequence of the following three results.

Lemma 3.1. *For all $k\in\mathbb{N}\setminus\{0\}$ and all $h>0$, we have*

$$\lambda_k^+(h)\leqslant\nu_k(h)\,.$$

Proof. It follows from the definition of $\nu_k(h)$. □

Actually, we can prove that $\nu_k(h)$ is also a good asymptotic lower bound for $\lambda_k^+(h)$, see Section 3.2 where the following is proved.

Proposition 3.2. *For all $k\geqslant 1$, we have*

$$\nu_k(h)\leqslant\lambda_k^+(h)(1+\mathscr{O}(h^\infty))\,.$$

In the next section, we study the asymptotic behavior of $\nu_k(h)$, which is summarized in the following proposition.

Proposition 3.3. *For all $k\geqslant 1$,*

$$\nu_k(h) = C_k(B,\Omega)h^{1-k}e^{2\phi_{\min}/h}(1+o(1))\,,$$

where $C_k(B,\Omega)$ is defined in (1.9).

Remark 3.4. Proposition 3.3 shows that each $\nu_k(h)$ goes to zero exponentially when h goes to zero. The analysis in Section 3.2 strongly relies on this fact.

3.1. About the proof of Proposition 3.3.

Using the change of function $u=e^{-\phi/h}v$, we get

$$\nu_k(h) = \inf_{\substack{W\subset\mathscr{H}^2(\Omega)\\ \dim W=k}} \sup_{v\in W\setminus\{0\}} \frac{h\|v\|^2_{\partial\Omega}}{\|e^{-\phi/h}v\|^2}\,.$$

In what follows we give upper and lower bounds for $\nu_k(h)$. The technics borrow ideas from our previous work [6].

3.1.1. *Upper bound.*

Let us consider $k\geqslant 1$ fixed.

Notation 6. Let us denote by $(P_n)_{n\in\mathbb{N}}$ the $N_{\mathcal{B}}$-orthogonal family such that $P_n(Z)=Z^n+\sum_{j=0}^{n-1}b_{n,j}Z^j$ obtained after a Gram-Schmidt process on $(1,Z,\dots,Z^n,\dots)$. Since P_n is $N_{\mathcal{B}}$-orthogonal to $\mathcal{P}_{n-1}$, we have

$$\begin{aligned}\mathsf{dist}_{\mathcal{B}}\left(Z^n,\mathcal{P}_{n-1}\right) &= \mathsf{dist}_{\mathcal{B}}\left(P_n,\mathcal{P}_{n-1}\right) = \inf\{N_{\mathcal{B}}(P_n-Q)\,,Q\in\mathcal{P}_{n-1}\}\\ &= \inf\{\sqrt{N_{\mathcal{B}}(P_n)^2+N_{\mathcal{B}}(Q)^2}\,,Q\in\mathcal{P}_{n-1}\} = N_{\mathcal{B}}(P_n)\,,\ \text{for } n\in\mathbb{N}\,.\end{aligned} \tag{3.1}$$

Let $Q_n \in \mathscr{H}^2_k(\Omega)$ be the unique function such that

$$\mathsf{dist}_{\mathcal{H}}\left((z-z_{\min})^n, \mathscr{H}^2_k(\Omega)\right) = N_{\mathcal{H}}((z-z_{\min})^n - Q_n(z))\,,$$

for $n \in \{0,\dots,k-1\}$. We recall that $N_{\mathcal{B}}$, $N_{\mathcal{H}}$, $\mathcal{P}_{n-1}$, and $\mathscr{H}^2_k(\Omega)$ are defined in Notation 2.

Proposition 3.5.

$$\nu_k(h) \leqslant \left(\frac{\mathsf{dist}_{\mathcal{H}}\left((z-z_{\min})^{k-1}, \mathscr{H}^2_k(\Omega)\right)}{\mathsf{dist}_{\mathcal{B}}\left(z^{k-1}, \mathcal{P}_{k-2}\right)}\right)^2 h^{1-k} e^{2\phi_{\min}/h}(1+o(1))\,.$$

Proposition 3.5 is a consequence of the lemmas.

Let us define the k-dimensional vector space $V_{h,k}$ by

$$V_{h,k} = \mathrm{span}(w_{0,h},\dots,w_{k-1,h}) \subset \mathscr{H}^2(\Omega)\,, \tag{3.2}$$

$$w_{n,h}(z) = h^{-\frac{1}{2}} P_n\left(\frac{z-z_{\min}}{h^{1/2}}\right) - h^{-\frac{1+n}{2}} Q_n(z), \text{ for } n \in \{0,\dots,k-1\}\,.$$

Lemma 3.6. *Let $h \in (0,1]$, $v_h = \sum_{j=0}^{k-1} c_j w_{j,h} \in V_{h,k}$ with $c_0,\dots c_{k-1} \in \mathbb{C}$, and $(w_{j,h})_{j\in\{0,\dots,k-1\}}$ defined in (3.2). We have*

$$\int_\Omega |v_h|^2 e^{-2(\phi(x)-\phi_{\min})/h}\mathrm{d}x = (1+o(1))\sum_{j=0}^{k-1}|c_j|^2 N_{\mathcal{B}}(P_j)^2\,, \tag{3.3}$$

where $o(1)$ does not depend on $c = (c_0,\dots,c_{k-1})$.

Lemma 3.7. *Let $h \in (0,1]$, $v_h = \sum_{j=0}^{k-1} c_j w_{j,h} \in V_{h,k}$ with $c_0,\dots c_{k-1} \in \mathbb{C}$. We have*

$$N_{\mathcal{H}}(v_h)^2 \leqslant |c_{k-1}|^2 h^{-k} N_{\mathcal{H}}\left((z-z_{\min})^{k-1} - Q_{k-1}\right)^2 + o(h^{-k})\|c\|^2_{\ell^2}\,.$$

Here, $o(1)$ does not depend on $c_0,\dots c_{k-1}$.

Proof. Let us now estimate $N_{\mathcal{H}}(v_h)$. From the triangle inequality, we get

$$N_{\mathcal{H}}(v_h) \leqslant |c_{k-1}| N_{\mathcal{H}}(w_{k-1,h}) + \sum_{j=0}^{k-2}|c_j| N_{\mathcal{H}}(w_{j,h})\,.$$

Then, from degree considerations and the triangle inequality, we get, for $1 \leqslant j \leqslant k-2$,

$$N_{\mathcal{H}}(w_{j,h}) = \mathscr{O}\left(h^{\frac{1-k}{2}}\right)\,,$$

and

$$N_{\mathcal{H}}(w_{k-1,h}) = (1+o(1))h^{-\frac{k}{2}} N_{\mathcal{H}}\left((z-z_{\min})^{k-1} - Q_{k-1}\right)\,.$$

The conclusion follows. □

3.1.2. *Lower bound.* Let $k \geqslant 1$. Let us consider an orthonormal family $(v_{j,h})_{1\leqslant j\leqslant k}$ (for the scalar product of $L^2(e^{-2\phi/h}\mathrm{d}x)$) associated with the eigenvalues $(\nu_j(h))_{1\leqslant j\leqslant k}$. We define

$$\mathscr{E}_k(h) = \operatorname*{span}_{1\leqslant j\leqslant k} v_{j,h}\,.$$

Lemma 3.8. *There exist $C, h_0 > 0$ such that for all $v_h \in \mathscr{E}_k(h)$ and $h \in (0, h_0)$, we have,*

$$\|v_h\|^2 \leqslant Ch^{-k}e^{2\phi_{\min}/h}\int_\Omega e^{-2\phi/h}|v_h|^2\mathrm{d}x\,.$$

Proof. From the continuous embedding $\mathscr{H}^2(\Omega) \hookrightarrow L^2(\Omega)$, and Proposition 3.5, there exist $c, C, h_0 > 0$ such that, for all $h \in (0, h_0)$ and all $v \in \mathscr{E}_k(h)$,

$$ch\|v\|^2 \leqslant h\|v\|^2_{\partial\Omega} \leqslant \nu_k(h)\int_\Omega e^{-2\phi/h}|v_h|^2\mathrm{d}x \leqslant Ch^{1-k}e^{2\phi_{\min}/h}\int_\Omega e^{-2\phi/h}|v_h|^2\mathrm{d}x\,.$$

□

Lemma 3.9. *Let $\alpha \in (0, 1/2)$. We have*

$$\lim_{h\to 0}\sup_{v_h\in\mathscr{E}_k(h)\setminus\{0\}}\left|\frac{\int_{D(x_{\min},\ h^\alpha)} e^{-2\phi/h}|v_h(x)|^2\mathrm{d}x}{\int_\Omega e^{-2\phi/h}|v_h(x)|^2\mathrm{d}x} - 1\right| = 0\,,$$

We can now start the proof of the lower bound.

Proof. Assume that $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$. We have, for all $x \in D(x_{\min}, h^\alpha)$,

$$\phi(x) = \phi_{\min} + \frac{1}{2}\mathsf{Hess}_{x_{\min}}\phi(x - x_{\min}, x - x_{\min}) + \mathscr{O}(h^{3\alpha})\,.$$

Then, with Lemma 3.9,

$$he^{2\phi_{\min}/h}\|v_h\|^2_{\partial\Omega}\,(1 + o(1)) \leqslant \nu_k(h)\left\|e^{-\frac{1}{2h}\mathsf{Hess}_{x_{\min}}\phi(x-x_{\min},x-x_{\min})}v_h\right\|^2_{L^2(D(x_{\min},h^\alpha))}\,. \tag{3.4}$$

In the following, we split the proof into two parts. Firstly, we replace v_h by its Taylor expansion at the order $k-1$ at $x_{\min}$ in the R. H. S. of (3.4). Secondly, we do the same for the L.H.S. of the same equation.

i. In view of the Cauchy formula, and the Cauchy-Schwarz inequality, there exist $C > 0, h_0 > 0$ such that, for all $h \in (0, h_0)$, for all $v \in \mathscr{H}^2(\Omega)$, all $z_0 \in D(x_{\min}, h^\alpha)$, and all $n \in \{0, \dots k\}$,

$$|v^{(n)}(z_0)| \leqslant C\|v\|_{\partial\Omega}\,. \tag{3.5}$$

Let us define, for all $v \in \mathscr{H}^2(\Omega)$,

$$N_h(v) = \left\|e^{-\frac{1}{2h}\mathsf{Hess}_{x_{\min}}\phi(x-x_{\min},x-x_{\min})}v\right\|_{L^2(D(x_{\min},h^\alpha))}\,.$$

By the Taylor formula, we can write

$$v_h = \mathrm{Tayl}_{k-1}v_h + R_{k-1}(v_h)\,,$$

where

$$\mathrm{Tayl}_{k-1}v_h = \sum_{n=0}^{k-1}\frac{v_h^{(n)}(z_{\min})}{n!}(z - z_{\min})^n\,,$$

and, for all $z_0 \in D(z_{\min}, h^\alpha)$,

$$|R_{k-1}(v_h)(z_0)| \leqslant C|z - z_{\min}|^k \sup_{D(z_{\min},h^\alpha)} |v_h^{(k)}| \,.$$

With (3.5) and a rescaling, the Taylor remainder satisfies

$$N_h(R_{k-1}(v_h)) \leqslant Ch^{\frac{k}{2}} h^{\frac{1}{2}} \|v_h\|_{\partial\Omega} \,.$$

Thus, by the triangle inequality,

$$N_h(v_h) \leqslant N_h(\mathrm{Tayl}_{k-1} v_h) + Ch^{\frac{k}{2}} h^{\frac{1}{2}} \|v_h\|_{\partial\Omega} \,.$$

Thus, with (3.4), we get

$$(1+o(1))e^{\phi_{\min}/h}\sqrt{h}\|v_h\|_{\partial\Omega} \leqslant \sqrt{\nu_k(h)} N_h(\mathrm{Tayl}_{k-1} v_h) + C\sqrt{\nu_k(h)} h^{\frac{1+k}{2}} \|v_h\|_{\partial\Omega} \,,$$

so that, thanks to Proposition 3.5,

$$(1+o(1))e^{\phi_{\min}/h}\sqrt{h}\|v_h\|_{\partial\Omega} \leqslant \sqrt{\nu_k(h)} N_h(\mathrm{Tayl}_{k-1} v_h) \leqslant \sqrt{\nu_k(h)} \hat{N}_h(\mathrm{Tayl}_{k-1} v_h) \,, \quad (3.6)$$

with

$$\hat{N}_h(w) = \left\| e^{-\frac{1}{2h}\mathsf{Hess}_{x_{\min}}\phi(x-x_{\min},x-x_{\min})} w \right\|_{L^2(\mathbb{R}^2)} \,.$$

This inequality shows in particular that Tayl_{k-1} is injective on $\mathscr{E}_k(h)$ and

$$\dim \mathrm{Tayl}_{k-1} \mathscr{E}_k(h) = k \,. \quad (3.7)$$

ii. Let us recall that

$$\mathscr{H}^2_k(\Omega) = \{u \in \mathscr{H}^2(\Omega) : \forall n \in \{0, \dots, k-1\}, u^{(n)}(x_{\min}) = 0\} \,.$$

Since $(v_h - \mathrm{Tayl}_{k-1} v_h) \in \mathscr{H}^2_k(\Omega)$, we have, by the triangle inequality,

$$\begin{aligned} \|v_h\|_{\partial\Omega} &\geqslant \left\| \frac{v_h^{(k-1)}(z_{\min})}{(k-1)!}(z - z_{\min})^{k-1} + (v_h - \mathrm{Tayl}_{k-1} v_h) \right\|_{\partial\Omega} - \left\|\mathrm{Tayl}_{k-2} v_h\right\|_{\partial\Omega} \\ &\geqslant \frac{|v_h^{(k-1)}(z_{\min})|}{(k-1)!} \mathrm{dist}_{\mathcal{H}}((z - z_{\min})^{k-1}, \mathscr{H}^2_k(\Omega)) - \left\|\mathrm{Tayl}_{k-2} v_h\right\|_{\partial\Omega} \,, \end{aligned}$$

where

$$\begin{aligned} &\mathrm{dist}((z - z_{\min})^{k-1}, \mathscr{H}^2_k(\Omega)) \\ &\qquad = \inf \left\{ \left\| (z - z_{\min})^{k-1} - Q(z) \right\|_{\partial\Omega} \,, \text{ for all } Q \in \mathscr{H}^2_k(\Omega) \right\} \,. \end{aligned}$$

Using again the triangle inequality,

$$\|\mathrm{Tayl}_{k-2} v_h\|_{\partial\Omega} \leqslant C \sum_{n=0}^{k-2} |v_h^{(n)}(z_{\min})| \,.$$

Moreover,

$$\begin{aligned} \sum_{n=0}^{k-2} |v_h^{(n)}(z_{\min})| \leqslant h^{-\frac{k-2}{2}} \sum_{n=0}^{k-2} h^{\frac{n}{2}} |v_h^{(n)}(z_{\min})| &\leqslant h^{-\frac{k-2}{2}} \sum_{n=0}^{k-1} h^{\frac{n}{2}} |v_h^{(n)}(z_{\min})| \\ &\leqslant Ch^{-\frac{k-2}{2}} h^{-\frac{1}{2}} \hat{N}_h(\mathrm{Tayl}_{k-1} v_h) \,, \end{aligned}$$

where we used the rescaling property

$$\hat{N}_h\left(\sum_{n=0}^{k-1} c_n (z-z_{\min})^n\right) = h^{\frac{1}{2}} \hat{N}_1\left(\sum_{n=0}^{k-1} c_n h^{\frac{n}{2}} (z-z_{\min})^n\right), \tag{3.8}$$

and the equivalence of the norms in finite dimension:

$$\exists C>0\,, \forall d\in\mathbb{C}^k\,, \quad C^{-1}\sum_{n=0}^{k-1}|d_n| \leqslant \hat{N}_1\left(\sum_{n=0}^{k-1} d_n (z-z_{\min})^n\right) \leqslant C\sum_{n=0}^{k-1}|d_n|\,.$$

We find

$$\|v_h\|_{\partial\Omega} \geqslant \frac{|v_h^{(k-1)}(z_{\min})|}{(k-1)!}\mathrm{dist}((z-z_{\min})^{k-1}, \mathscr{H}_k^2(\Omega)) - Ch^{-\frac{k-2}{2}} h^{-\frac{1}{2}} \hat{N}_h(\mathrm{Tayl}_{k-1} v_h)\,,$$

and thus, by (3.6),

$$\begin{aligned}(1+o(1))e^{\phi_{\min}/h}\sqrt{h}\frac{|v_h^{(k-1)}(z_{\min})|}{(k-1)!}\mathrm{dist}((z-z_{\min})^{k-1}, \mathscr{H}_k^2(\Omega))& \\ \leqslant \left(\sqrt{\nu_k(h)} + Ch^{\frac{2-k}{2}} e^{\phi_{\min}/h}\right)\hat{N}_h(\mathrm{Tayl}_{k-1} v_h)\,.&\end{aligned} \tag{3.9}$$

Let us now end the proof of the lower bound by using (3.9) and (3.7).

Since we have (3.7), we deduce that

$$\begin{aligned}(1+o(1))e^{\phi_{\min}/h}\sqrt{h}\mathrm{dist}_{\mathcal{H}}((z-z_{\min})^{k-1}, \mathscr{H}_k^2(\Omega))\sup_{c\in\mathbb{C}^k}\frac{|c_{k-1}|}{\hat{N}_h(\sum_{n=0}^{k-1} c_n (z-z_{\min})^n)}& \\ \leqslant \sqrt{\nu_k(h)} + Ch^{\frac{2-k}{2}} e^{\phi_{\min}/h}\,.&\end{aligned} \tag{3.10}$$

By (3.8), we infer

$$h^{\frac{1}{2}}\sup_{c\in\mathbb{C}^k}\frac{|c_{k-1}|}{\hat{N}_h(\sum_{n=0}^{k-1} c_n (z-z_{\min})^n)} = \sup_{c\in\mathbb{C}^k}\frac{h^{\frac{1-k}{2}}|c_{k-1}|}{\hat{N}_1(\sum_{n=0}^{k-1} c_n (z-z_{\min})^n)}\,.$$

Since $\hat{N}_1$ is related to the Segal-Bargmann norm $N_{\mathcal{B}}$ via a translation, and recalling Notation 6, we get

$$\sup_{c\in\mathbb{C}^k}\frac{|c_{k-1}|}{\hat{N}_1(\sum_{n=0}^{k-1} c_n (z-z_{\min})^n)} = \sup_{c\in\mathbb{C}^k}\frac{|c_{k-1}|}{N_{\mathcal{B}}(\sum_{n=0}^{k-1} c_n z^n)} = \frac{1}{N_{\mathcal{B}}(P_{k-1})}\,.$$

Thus,

$$(1+o(1))h^{\frac{1-k}{2}} e^{\phi_{\min}/h}\frac{\mathrm{dist}_{\mathcal{H}}((z-z_{\min})^{k-1}, \mathscr{H}_k^2(\Omega))}{N_{\mathcal{B}}(P_{k-1})} \leqslant \sqrt{\nu_k(h)}\,. \tag{3.11}$$

The conclusion follows. □

3.2. **Approximation results.** Let us roughly explain the strategy to establish Proposition 3.2. Recall Proposition 2.7 which states that $\lambda_k^+(h) = \mu_k(h)$. Consider a minimizing subspace $W \subset \mathfrak{H}_{h,\mathbf{A}} = \mathscr{H}^2_{h,\mathbf{A}}(\Omega) + H^1(\Omega)$ (of dimension k). Then, we can prove that W is quasi invariant under $\Pi_{h,\mathbf{A}}$, see Lemma 3.13. So, we would like to write $\rho_+(u) \simeq \rho_+(\Pi_{h,\mathbf{A}}u)$ for all $u \in W$. Unfortunately, the elements of $\Pi_{h,\mathbf{A}}W$ do not necessarily belong to $L^2(\partial\Omega)$. Then, we cannot relate $\rho_+(\Pi_{h,\mathbf{A}}u)$ to the simpler optimization problem defining $\nu_k(h)$. The elements in $\mathfrak{H}_{h,\mathbf{A}}$ do not necessarily belong to $H^1(\Omega)$ (see for instance [22]). However, according to Proposition 2.6, $H^1(\Omega)$ is dense in $\mathfrak{H}_{h,\mathbf{A}}$ and then we can check (see the proof of Corollary 3.15) that $\Pi_{h,\mathbf{A}}H^1(\Omega) \subset H^1(\Omega)$, and the desired trace property is satisfied.

The price to pay is to use an approximate subspace $W_h \subset H^1(\Omega)$. For that purpose, we will use a number $M_k(h) > \mu_k(h)$ such that

$$M_k(h) = \mu_k(h)(1 + \mathscr{O}(h^\infty))\,.$$

Remark 3.10. By Remark 3.4, we may choose $M_k(h) = \mu_k(h)(1+\mu_k(h))$. Notice also that $M_k(h)$ goes itself exponentially fast to zero.

Notation 7. For notational simplicity, we write $M \equiv M_k(h)$.

There exists $W_h \subset H^1(\Omega)$ with $\dim W_h = k$ such that

$$\mu_k(h) \leqslant \sup_{W_h\setminus\{0\}} \rho_+(u) \leqslant M\,. \tag{3.12}$$

The following lemma is straightforward.

Lemma 3.11. *For all $u \in H^1(\Omega)$, we have*

$$2h\|u\|^2_{\partial\Omega} \leqslant \mathscr{Q}_h(u)\,, \quad 2\|u\|\|d^\times_{h,\mathbf{A}}u\| \leqslant \mathscr{Q}_h(u)\,,$$

where

$$\mathscr{Q}_h(u) = h\|u\|^2_{\partial\Omega} + \sqrt{h^2\|u\|^4_{\partial\Omega} + 4\|u\|^2\|d^\times_{h,\mathbf{A}}u\|^2}\,.$$

Thanks to Lemma 3.11 and (3.12), we get the following.

Lemma 3.12. *For all $u \in W_h$,*

$$h\|u\|^2_{\partial\Omega} \leqslant M\|u\|^2\,, \tag{3.13}$$

and thus

$$\|d^\times_{h,\mathbf{A}}u\|^2 \leqslant M^2\|u\|^2\,. \tag{3.14}$$

Lemma 3.13. *For all $u \in W_h$, we have*

$$\|\Pi^\perp_{h,\mathbf{A}}u\| \leqslant \frac{M}{\sqrt{2hB_0}}\|u\|\,, \tag{3.15}$$

$$\|\Pi^\perp_{h,\mathbf{A}}u\|_{\partial\Omega} \leqslant \frac{M}{ch^2}\|u\|\,. \tag{3.16}$$

Moreover, for h small enough, $\Pi_{h,\mathbf{A}}\restriction_{W_h}$ is injective.

Proof. Combining (3.14) and Lemma 2.5, we readily get (3.15) and (3.16). The injectivity follows from (3.15) and Remark 3.10. □

Proposition 3.14. *For all $u \in W_h$, we have*

$$\mu_k(h)(1+\mathscr{O}(h^\infty))\|\Pi_{h,\mathbf{A}}u\|^2 \geqslant h\|\Pi_{h,\mathbf{A}}u\|^2_{\partial\Omega}\,.$$

Proof. Let us consider (3.13). We have

$$M^{\frac12}\|u\| \geqslant \sqrt{h}\|u\|_{\partial\Omega} = \sqrt{h}\|\Pi_{h,\mathbf{A}}u + \Pi^\perp_{h,\mathbf{A}}u\|_{\partial\Omega} \geqslant \sqrt{h}(\|\Pi_{h,\mathbf{A}}u\|_{\partial\Omega} - \|\Pi^\perp_{h,\mathbf{A}}u\|_{\partial\Omega})\,.$$

By (3.16), we get

$$M^{\frac12}\left(1+h^{-\frac32}\frac{1}{c}M^{\frac12}\right)\|u\| \geqslant \sqrt{h}\|\Pi_{h,\mathbf{A}}u\|_{\partial\Omega}\,.$$

From (3.15), and the triangle inequality, we have

$$\left(1-\frac{M}{\sqrt{2hB_0}}\right)\|u\| \leqslant \|\Pi_{h,\mathbf{A}}u\|\,.$$

By Remark 3.10, we see that, for h small enough, $1-\frac{M}{\sqrt{2hB_0}} > 0$. Hence,

$$M^{\frac12}\left(1+h^{-\frac32}\frac{1}{c}M^{\frac12}\right)\left(1-\frac{M}{\sqrt{2hB_0}}\right)^{-1}\|\Pi_{h,\mathbf{A}}u\| \geqslant \sqrt{h}\|\Pi_{h,\mathbf{A}}u\|_{\partial\Omega}\,.$$

Squaring this, and using Remark 3.10, we obtain the desired estimate. □

Corollary 3.15. *For all $k \geqslant 1$, we have*

$$\nu_k(h) \leqslant \mu_k(h)(1+\mathscr{O}(h^\infty))\,.$$

Proof. Since $\Pi_{h,\mathbf{A}}\restriction_{W_h}$ is injective, we have $\dim\Pi_{h,\mathbf{A}}(W_h) = k$. Moreover, $\Pi_{h,\mathbf{A}}(W_h) \subset H^1(\Omega)$. Indeed, for $u \in H^1(\Omega)$, we write

$$\Pi_{h,\mathbf{A}}u = u - \Pi^\perp_{h,\mathbf{A}}u \in H^1(\Omega)\,,$$

by Lemma 2.5. This shows that $\Pi_{h,\mathbf{A}}(W_h) \subset \mathscr{H}^2_{h,\mathbf{A}}(\Omega)$.

The conclusion follows from Proposition 3.14 and the definition of $\nu_k(h)$. □

4. Semiclassical analysis of the first negative eigenvalue

4.1. **About the proof of Theorem 1.11.** Thanks to the charge conjugation (see Remark 1.8), the negative eigenvalues $\lambda^-_k(h)$ can be characterized as follows. For $\lambda \geqslant 0$, consider the quadratic form

$$\widetilde{Q}_\lambda(u) = q_{\lambda,h}(u) - \lambda^2\|u\|^2\,, \quad q_{\lambda,h}(u) = \|d^\times_{h,-\mathbf{A}}u\|^2 + \lambda h\|u\|^2_{\partial\Omega}\,.$$

Let us denote by $(\tilde\ell_k(\lambda))_{k\geqslant1}$ the eigenvalues of the corresponding operator. As in Section 2, for all $k \geqslant 1$, the equation $\tilde\ell_k(\lambda) = 0$ has a unique positive solution; this solution is $\lambda^-_k(h)$. On the other hand, we have

$$\tilde\ell_k(\lambda) = \gamma_k(\lambda,h) - \lambda^2\,,$$

where the $(\gamma_k(\lambda,h))_{k\geqslant1}$ are the eigenvalues of the operator associated with $q_{\lambda,h}$. Note that, by (2.6) and (2.7), for all $\lambda > 0$,

$$|\gamma_k(\lambda,h) - \lambda^2| = |\tilde\ell_k(\lambda)| \geqslant \lambda|\lambda - \lambda^-_k(h)|\,. \tag{4.1}$$

Therefore, $\lambda^-_k(h)$ is the unique solution of

$$\gamma_k(\lambda,h) = \lambda^2\,.$$

Let us now consider the case $k = 1$. We write $\lambda_1^-(h) = e_1(h)h^{\frac{1}{2}}$, and the equation becomes

$$\gamma_1(e_1(h)h^{\frac{1}{2}}, h) = e_1(h)^2 h\,. \tag{4.2}$$

Note that, by setting $\lambda = ah^{\frac{1}{2}}$ with $a > 0$, we have the reformulation of (4.1):

$$|h^{-1}\gamma_1(ah^{\frac{1}{2}}, h) - a^2| \geqslant a|a - e_1(h)|\,. \tag{4.3}$$

The main goal of the next section is to establish the following estimate.

Proposition 4.1. *We have, for all $a > 0$,*

$$\gamma_1(ah^{\frac{1}{2}}, h) = h\Lambda(a) + o(h)\,, \quad \Lambda(a) = \min\left(2b_0, b_0'\nu(a(b_0')^{-1/2})\right)\,.$$

Proposition 4.1 implies Theorem 1.11. Observe that, substituing this asymptotic expansion into (4.3), we get

$$|\Lambda(a) - a^2 + o(1)| \geqslant a|a - e_1(h)|\,.$$

Notice that, if $a > 0$ is such that $\Lambda(a) = a^2$, then $e_1(h)$ is approximated by a.

Actually, there is a unique positive a such that

$$\min\left(2b_0, b_0'\nu(a(b_0')^{-1/2})\right) = a^2\,,$$

which is given by

$$a = \min(\sqrt{2b_0}, c_0\sqrt{b_0'})\,,$$

where c_0 is the unique positive solution of $\nu(c) = c^2$, see Proposition C.7. We deduce that

$$\lim_{h\to 0} e_1(h) = \min(\sqrt{2b_0}, c_0\sqrt{b_0'})\,,$$

or equivalently

$$\lambda_1^-(h) = h^{\frac{1}{2}}\min(\sqrt{2b_0}, c_0\sqrt{b_0'}) + o(h^{\frac{1}{2}})\,.$$

4.2. **Ground energy of Pauli-Robin type operator.** Let $a > 0$. We consider the quadratic form

$$\mathscr{Q}_{a,h}(u) = q_{ah^{1/2},h}(u) = \|d^\times_{h,-\mathbf{A}}u\|^2 + ah^{\frac{3}{2}}\|u\|^2_{\partial\Omega}\,,$$

and we have

$$\gamma_1(ah^{\frac{1}{2}}, h) = \inf_{\substack{u\in\mathfrak{H}^2_{-\mathbf{A}}(\Omega)\\ u\neq 0}} \frac{\mathscr{Q}_{a,h}(u)}{\|u\|^2}\,.$$

4.2.1. *Localization formula.* Let $\rho \in \left(0, \frac{1}{2}\right)$. Let us consider a semiclassical partition of the unity $(\chi_j)_{j\in\mathbb{Z}^2}$ with $\operatorname{supp}\chi_j \subset D(x_j, h^\rho)$, and such that

$$\sum_{j\in\mathbb{Z}^2}\chi_j^2 = 1\,, \quad \sum_{j\in\mathbb{Z}^2}|\nabla\chi_j|^2 \leqslant Ch^{-2\rho}\,.$$

Lemma 4.2. *We have*

$$\mathscr{Q}_{a,h}(u) = \sum_{j\in\mathbb{Z}^2}\mathscr{Q}_{a,h}(\chi_j u) - h^2\sum_{j\in\mathbb{Z}^2}\|(\nabla\chi_j)u\|^2\,.$$

In particular,

$$\mathscr{Q}_{a,h}(u) \geqslant \sum_{j\in\mathbb{Z}^2}\mathscr{Q}_{a,h}(\chi_j u) - Ch^{2-2\rho}\|u\|^2\,.$$

Proof. Let us write

$$\begin{aligned}
\|d^{\times}_{h,-\mathbf{A}}u\|^2 &= \sum_{j\in\mathbb{Z}^2}\langle d^{\times}_{h,-\mathbf{A}}u, d^{\times}_{h,-\mathbf{A}}(\chi_j^2 u)\rangle \\
&= \sum_{j\in\mathbb{Z}^2}\Big(\langle d^{\times}_{h,-\mathbf{A}}u, [d^{\times}_{h,-\mathbf{A}},\chi_j]\chi_j u\rangle + \langle \chi_j d^{\times}_{h,-\mathbf{A}}u, d^{\times}_{h,-\mathbf{A}}(\chi_j u)\rangle\Big) \\
&= \sum_{j\in\mathbb{Z}^2}\Big(\langle \chi_j d^{\times}_{h,-\mathbf{A}}u, [d^{\times}_{h,-\mathbf{A}},\chi_j]u\rangle + \langle \chi_j d^{\times}_{h,-\mathbf{A}}u, d^{\times}_{h,-\mathbf{A}}(\chi_j u)\rangle\Big) \\
&= \sum_{j\in\mathbb{Z}^2}\Big(-\|[d^{\times}_{h,-\mathbf{A}},\chi_j]u\|^2 + \langle d^{\times}_{h,-\mathbf{A}}(\chi_j u), [d^{\times}_{h,-\mathbf{A}},\chi_j]u\rangle + \langle \chi_j d^{\times}_{h,-\mathbf{A}}u, d^{\times}_{h,-\mathbf{A}}(\chi_j u)\rangle\Big) \\
&= \sum_{j\in\mathbb{Z}^2}\Big(\|d^{\times}_{h,-\mathbf{A}}(\chi_j u)\|^2 - \|[d^{\times}_{h,-\mathbf{A}},\chi_j]u\|^2 + 2i\mathrm{Im}\,\langle d^{\times}_{h,-\mathbf{A}}(\chi_j u), [d^{\times}_{h,-\mathbf{A}},\chi_j]u\rangle\Big)\,,
\end{aligned}$$

where we used that the commutator $[d^{\times}_{h,-\mathbf{A}},\chi_j] = -2ih\partial_{\bar z}\chi_j$ is a function. Taking the real part, we get

$$\|d^{\times}_{h,-\mathbf{A}}u\|^2 = \sum_{j\in\mathbb{Z}^2}\Big(\|d^{\times}_{h,-\mathbf{A}}(\chi_j u)\|^2 - \|h(\nabla\chi_j)u\|^2\Big)\,.$$

□

4.2.2. *Lower bound.* Let j be such that $\mathrm{supp}(\chi_j)\subset\Omega$. Then, we have

$$\mathscr{Q}_{a,h}(\chi_j u) = \|d^{\times}_{h,-\mathbf{A}}(\chi_j u)\|^2 \geqslant 2hb_0\|\chi_j u\|^2\,, \tag{4.4}$$

since the Dirichlet realization of

$$d_{h,-\mathbf{A}}d^{\times}_{h,-\mathbf{A}} = (-ih\nabla+\mathbf{A})^2 + hB$$

is bounded from below by $2hb_0$.

Therefore, let us focus on the j such that $\mathrm{supp}(\chi_j)\cap\partial\Omega\neq\emptyset$. We may assume that $x_j\in\partial\Omega$.

Let us bound the local energy $\mathscr{Q}_{a,h}(\chi_j u)$ from below.

Proposition 4.3. *We have*

$$\mathscr{Q}_{a,h}(\chi_j u) \geqslant \Big[b_0'\nu(a(b_0')^{-1/2})h - Ch^{\frac{1}{2}+2\rho}\Big]\|\chi_j u\|^2\,. \tag{4.5}$$

Proof. Before starting the proof, let us say a few words about the strategy. The general idea is to approximate the magnetic field, on the support of χ_j, by a constant magnetic field, and to flatten the boundary by means of tubular coordinates. Due to the lack of ellipticity of the Cauchy-Riemann operators, we cannot choose the canonical tubular coordinates (given by the curvilinear abscissa and the distance to the boundary). However, with the exponential coordinates (4.6), we are able to avoid this problem for the disc, and then, by means of the Riemann mapping, for Ω. This amounts to constructing "conformal" tubular coordinates for Ω.

It is convenient to use the change of function

$$u = e^{\phi/h}v\,.$$

For notational simplicity, we let $u_j = \chi_j u$ and $v_j = \chi_j v$. We have

$$\mathscr{Q}_{a,h}(u_j) = h^2 \int_\Omega e^{2\phi/h} |2\partial_{\bar{z}} v_j|^2 \mathrm{d}x + a h^{\frac{3}{2}} \|v_j\|^2_{\partial\Omega}\,.$$

Let us use the Riemann biholomorphism $F : \mathbb{D} \to \Omega$. We let $w_j = v_j \circ F$. We get

$$\mathscr{Q}_{a,h}(u_j) = 4h^2 \int_{\mathbb{D}} e^{2\phi\circ F(y)/h} |\partial_{\overline{y}} w_j|^2 \mathrm{d}y + a h^{\frac{3}{2}} \int_{\partial\mathbb{D}} |w_j|^2 |F'| \mathrm{d}\sigma\,, \quad \partial_{\overline{y}} = \frac{1}{2}(\partial_{y_1} + i\partial_{y_2})\,.$$

Note that w_j is supported in a neighborhood of order h^ρ of $\partial\Omega$. Let us now use a change of coordinates near the boundary. Let $\delta > 0$. Consider the "exponential polar coordinates", $y = P(s,\tau)$, given by

$$y_1 = e^{-\tau}\cos s\,, \quad y_2 = e^{-\tau}\sin s \quad (s,\tau) \in \mathcal{T}_\delta := [0, 2\pi) \times (0,\delta)\,. \tag{4.6}$$

P is a smooth diffeomorphism in a neighborhood of the boundary. We have

$$-e^\tau \partial_s = \sin s \partial_{y_1} - \cos s \partial_{y_2}\,, \quad -e^\tau \partial_\tau = \cos s \partial_{y_1} + \sin s \partial_{y_2}\,,$$

and we get

$$\partial_{y_1} + i\partial_{y_2} = i e^{\tau + is}(\partial_s + i\partial_\tau)\,.$$

The coordinates of the center x_j of the support of χ_j are denoted by $(s_j, 0)$.

In terms of these new coordinates, we have

$$\mathscr{Q}_{a,h}(u_j) = h^2 \int_{\mathcal{T}_\delta} e^{2\phi\circ F(P(s,\tau))/h} |e^\tau(\partial_s + i\partial_\tau)(w_j \circ P)|^2 e^{-2\tau} \mathrm{d}s\mathrm{d}\tau \\ + a h^{\frac{3}{2}} \int_0^{2\pi} |w_j \circ P(s,0)|^2 |F'(e^{is})| \mathrm{d}s\,.$$

Approximating $e^{2\tau}$ by 1 on the support of $w_j \circ P$, we get

$$(1 - Ch^\rho)^{-1} \mathscr{Q}_{a,h}(u_j) \geqslant h^2 \int_{\mathcal{T}_\delta} e^{2\phi\circ F(P(s,\tau))/h} |(\partial_s + i\partial_\tau)(w_j \circ P)|^2 \mathrm{d}s\mathrm{d}\tau \\ + a h^{\frac{3}{2}} \int_0^{2\pi} |w_j \circ P(s,0)|^2 |F'(e^{is})| \mathrm{d}s\,.$$

We let $\check{\phi} = \phi \circ F \circ P$. Since ϕ is zero at the boundary, we have that $e^{2\check{\phi}(s,0)/h} = 1$. Then, by using that $|F'(e^{is})| \geqslant (1 - Ch^\rho)|F'(e^{is_j})|$, and by commuting the exponential with the Cauchy-Riemann derivative, we get

$$(1 - Ch^\rho)^{-1} \mathscr{Q}_{a,h}(u_j) \geqslant \int_{\mathcal{T}_\delta} |(h\partial_s - \partial_s\check{\phi} + ih\partial_\tau - i\partial_\tau\check{\phi}) e^{\check{\phi}(s,\tau))/h} (w_j \circ P)|^2 \mathrm{d}s\mathrm{d}\tau \\ + a h^{\frac{3}{2}} \int_0^{2\pi} |e^{\check{\phi}(s,0))/h} w_j \circ P(s,0)|^2 |F'(e^{is_j})| \mathrm{d}s\,,$$

with a possible different constant $C > 0$.

Then,

$$(1-Ch^\rho)^{-1}\mathscr{Q}_{a,h}(u_j) \geqslant \int_{\mathcal{T}_\delta} |(-ih\partial_s + \check{A}_1 + i(-ih\partial_\tau + \check{A}_2))W_j|^2 \mathrm{d}s\mathrm{d}\tau \\ + |F'(e^{is_j})|ah^{\frac{3}{2}} \int_0^{2\pi} |W_j(s,0)|^2 \mathrm{d}s\,,$$

where $W_j = e^{\check{\phi}(s,\tau))/h}(w_j \circ P)$ and $\check{A} = \nabla\check{\phi}^\perp = (-\partial_\tau\check{\phi}, \partial_s\check{\phi})$. Now, we have a magnetic Cauchy-Riemann problem on a flat space, with a uniform Robin condition.

A computation that uses the identity $(\partial_s + i\partial_\tau)(e^{-\tau+is}) = 0$,

$$\begin{aligned}\nabla \times \check{A} = (\partial_s^2 + \partial_\tau^2)(\phi \circ F \circ P) &= e^{-2\tau}\Delta_y(\phi \circ F)(P(s,\tau)) \\ &= e^{-2\tau}|F'(P(s,\tau))|^2 B(F(P(s,\tau))) \\ &= \beta_j + \mathscr{O}(|s-s_j| + |\tau - \tau_j|)\,,\end{aligned}$$

gives the new constant magnetic field $\beta_j = |F'(y_j)|^2 B(x_j)$.

Using the Young inequality, we get

$$\int_{\mathcal{T}_\delta} |(-ih\partial_s + \check{A}_1 + i(-ih\partial_\tau + \check{A}_2))W_j|^2 \mathrm{d}s\mathrm{d}\tau \\ \geqslant (1-\varepsilon)\int_{\mathcal{T}_\delta} |(-ih\partial_s + \check{A}_{1,j} + i(-ih\partial_\tau + \check{A}_{2,j}))W_j|^2 \mathrm{d}s\mathrm{d}\tau - \varepsilon^{-1}\int_{\mathcal{T}_\delta} |\check{A} - \check{A}_j|^2 |W_j|^2 \mathrm{d}s\mathrm{d}\tau\,,$$

where $\check{A}_j = (\check{A}_{1,j}, \check{A}_{2,j})$ is the Taylor approximation of $\check{A}$ at the order one at (s_j, τ_j):

$$|\check{A} - \check{A}_j| \leqslant Ch^{2\rho}\,,$$

on the support of W_j. We get that

$$(1-Ch^\rho)^{-1}\mathscr{Q}_{a,h}(u_j) \geqslant (1-\varepsilon)\mathcal{Q}_j(W_j) - Ch^{4\rho}\varepsilon^{-1}\int_{\mathcal{T}_\delta} |W_j|^2 \mathrm{d}s\mathrm{d}\tau\,, \tag{4.7}$$

with

$$\mathcal{Q}_j(W) = \int_{\mathbb{R}^2_+} |(-ih\partial_s + \check{A}_{1,j} + i(-ih\partial_\tau + \check{A}_{2,j}))W|^2 \mathrm{d}s\mathrm{d}\tau \\ + |F'(e^{is_j})|ah^{\frac{3}{2}} \int_{\mathbb{R}} |W(s,0)|^2 \mathrm{d}s\,.$$

Let us remark that, by construction,

$$\nabla \times \check{A}_j = \beta_j\,,$$

so that after a change of gauge, we can assume that $\check{A}_j = (-\beta_j\tau, 0)$.

Thus, we get a new quadratic form on $L^2(\mathbb{R}^2_+)$ which is associated with a new operator $\mathcal{L}_j$. We are interested in the bottom of its spectrum:

$$\inf \mathrm{sp}(\mathcal{L}_j) = \inf_{\substack{W \in \mathfrak{H}^2_{-\check{A}_j}(\mathbb{R}^2_+) \\ W \neq 0}} \frac{\mathcal{Q}_j(W)}{\|W\|^2}\,.$$

Let us consider the rescaling

$$(s,\tau) = h^{\frac{1}{2}}\beta_j^{-\frac{1}{2}}(\tilde{s},\tilde{\tau})\,.$$

We get

$$\inf \mathrm{sp}(\mathcal{L}_j) = h\beta_j\mu_j\,, \quad \mu_j = \inf_{\substack{W\in\mathfrak{H}^2_{-\check{A}_j}(\mathbb{R}^2_+)\\ W\neq 0}} \frac{\widetilde{\mathcal{Q}}_j(W)}{\|W\|^2}\,,$$

where

$$\widetilde{\mathcal{Q}}_j(W) = \int_{\mathbb{R}^2_+} |(-i\partial_s - \tau + i(-i\partial_\tau))W|^2 \mathrm{d}s\mathrm{d}\tau + aB(x_j)^{-\frac{1}{2}}\int_{\mathbb{R}} |W(s,0)|^2\mathrm{d}s\,.$$

Then,

$$(1-Ch^\rho)^{-1}\mathscr{Q}_{a,h}(u_j) \geqslant \left[(1-\varepsilon)h\beta_j\mu_j - Ch^{4\rho}\varepsilon^{-1}\right]\|W_j\|^2\,.$$

We choose ε such that

$$\varepsilon h = \varepsilon^{-1}h^{4\rho}\,,$$

so that

$$\varepsilon = h^{-\frac{1}{2}+2\rho}\,,$$

and

$$(1-Ch^\rho)^{-1}\mathscr{Q}_{a,h}(u_j) \geqslant \left[h\beta_j\mu_j - Ch^{\frac{1}{2}+2\rho}\right]\|W_j\|^2\,.$$

In particular, we get

$$\mathscr{Q}_{a,h}(u_j) \geqslant \left[h\beta_j\mu_j - Ch^{\frac{1}{2}+2\rho} - Ch^{1+\rho}\right]\|W_j\|^2\,.$$

Then,

$$\begin{aligned}
\mathscr{Q}_{a,h}(u_j) &\geqslant \left[h\beta_j\mu_j - Ch^{\frac{1}{2}+2\rho}\right]\int_{\mathcal{T}_\delta} e^{2\check{\phi}(s,\tau))/h}|(v_j\circ F\circ P)|^2\mathrm{d}s\mathrm{d}\tau\\
&\geqslant \left[h\beta_j\mu_j - Ch^{\frac{1}{2}+2\rho} - Ch^{1+\rho}\right]\int_{\mathbb{D}} e^{2\phi(F(y)))/h}|(v_j\circ F(y))|^2\mathrm{d}y\\
&= \left[h\beta_j\mu_j - Ch^{\frac{1}{2}+2\rho} - Ch^{1+\rho}\right]\int_\Omega e^{2\phi/h}|v_j(x)|^2|(F^{-1})'(x)|^2\mathrm{d}x\\
&\geqslant |(F^{-1})'(x_j)|^2\left[h\beta_j\mu_j - Ch^{\frac{1}{2}+2\rho} - Ch^{1+\rho}\right]\int_\Omega e^{2\phi/h}|v_j(x)|^2\mathrm{d}x\\
&\geqslant \left[hB(x_j)\mu_j - Ch^{\frac{1}{2}+2\rho} - Ch^{1+\rho}\right]\int_\Omega e^{2\phi/h}|v_j(x)|^2\mathrm{d}x\\
&\geqslant \left[hB(x_j)\mu_j - Ch^{\frac{1}{2}+2\rho}\right]\int_\Omega e^{2\phi/h}|v_j(x)|^2\mathrm{d}x\\
&= \left[hB(x_j)\mu_j - Ch^{\frac{1}{2}+2\rho}\right]\int_\Omega |\chi_j u(x)|^2\mathrm{d}x\,.
\end{aligned}$$

Then, letting $\mathbf{A}_0 = (-\tau, 0)$, we have

$$\begin{aligned} B(x_j)\mu_j &= \inf_{\substack{u\in\mathfrak{H}^2_{-\mathbf{A}_0}(\mathbb{R}^2_+)\\ u\neq 0}} \frac{B(x_j)\int_{\mathbb{R}^2_+}|(-i\partial_s-\tau+i(-i\partial_\tau))u|^2\mathrm{d}s\mathrm{d}\tau + aB(x_j)^{\frac12}\int_{\mathbb{R}}|u(s,0)|^2\mathrm{d}s}{\|u\|^2} \\ &\geqslant \inf_{\substack{u\in\mathfrak{H}^2_{-\mathbf{A}_0}(\mathbb{R}^2_+)\\ u\neq 0}} \frac{b_0'\int_{\mathbb{R}^2_+}|(-i\partial_s-\tau+i(-i\partial_\tau))u|^2\mathrm{d}s\mathrm{d}\tau + a(b_0')^{\frac12}\int_{\mathbb{R}}|u(s,0)|^2\mathrm{d}s}{\|u\|^2} \\ &= b_0'\nu(a(b_0')^{-1/2})\,. \end{aligned}$$

The result follows. □

Remark 4.4. It is clear from the proof that we also have a reverse inequality of (4.7):

$$(1-Ch^\rho)^{-1}\mathscr{Q}_{a,h}(u_j) \leqslant (1+\varepsilon)\mathcal{Q}_j(W_j) + Ch^{4\rho}\varepsilon^{-1}\int_{\mathcal{T}_\delta}|W_j|^2\mathrm{d}s\mathrm{d}\tau\,. \tag{4.8}$$

Gathering the estimates (4.4) and (4.5), and using Lemma 4.2, we find that

$$\mathscr{Q}_{a,h}(u) \geqslant \left[\Lambda(a)h - Ch^{\frac12+2\rho} - Ch^{2-2\rho}\right]\|u\|^2\,.$$

We choose ρ such that

$$\frac12 + 2\rho = 2 - 2\rho\,.$$

Thus, $\rho = \frac38$ and

$$\mathscr{Q}_{a,h}(u) \geqslant \left[\Lambda(a)h - Ch^{\frac54}\right]\|u\|^2\,.$$

The min-max principle implies the lower bound in Proposition 4.1.

4.2.3. *Upper bound.* The upper bound in Proposition 4.1 follows by inserting appropriate localized test functions in $\mathscr{Q}_{a,h}$. Let us provide the main lines of the strategy for this classical analysis.

We recall that

$$\gamma_1(ah^{\frac12},h) = \inf_{\substack{u\in\mathfrak{H}^2_{-\mathbf{A}}(\Omega)\\ u\neq 0}} \frac{\mathscr{Q}_{a,h}(u)}{\|u\|^2}\,.$$

In particular, we have

$$\gamma_1(ah^{\frac12},h) \leqslant \inf_{\substack{u\in H^1_0(\Omega)\\ u\neq 0}} \frac{\mathscr{Q}_{a,h}(u)}{\|u\|^2} = \inf_{\substack{u\in H^1_0(\Omega)\\ u\neq 0}} \frac{\|(-ih\nabla+\mathbf{A})u\|^2 + \int_\Omega hB|u|^2\mathrm{d}x}{\|u\|^2}\,.$$

The last quantity is the groundstate energy of $(-ih\nabla+\mathbf{A})^2 + hB$. Pick up a point $x_0\in\Omega$. We can always find a normalized test function φ_h in $\mathscr{C}^\infty_0(\Omega)$, localized at the scale $h^{\frac12}$ near x_0, and such that

$$\|(-ih\nabla+\mathbf{A})\varphi_h\|^2 + \int_\Omega hB|\varphi_h|^2\mathrm{d}x \leqslant 2B(x_0)h + o(h)\,.$$

Now, if B attains its minimum inside at x_0, then we deduce that

$$\gamma_1(ah^{\frac12},h) \leqslant 2b_0h + o(h)\,. \tag{4.9}$$

If not, for any $\varepsilon > 0$, we may find $x_0 \in \Omega$ such that $|B(x_0) - b_0| \leqslant \varepsilon$, and (4.9) is true as well.

On the other hand, let us consider $x_0 \in \partial\Omega$ where the minimum of $B_{|\partial\Omega}$ is attained. Take a fixed cutoff function χ centered at x_0, and a minimizing sequence $(W_n) \subset \mathscr{S}(\overline{\mathbb{R}^2_+})$ associated with μ. Then, we consider the function $\psi_h(s,\tau) = \chi(s,\tau)W_n((b_0')^{\frac{1}{2}}h^{-\frac{1}{2}}(s,\tau))$ and its avatar φ_h in the original coordinates (afer the maps P and F). Using Remark 4.4 (where u_j is replaced by φ_h), we get

$$\gamma_1(ah^{\frac{1}{2}}, h) \leqslant hb_0'\nu(a(b_0')^{-1/2}) + o(h)\,.$$

This, together with (4.9), gives the desired upper bound.

Appendix A. The results under various local boundary conditions

For $\eta \in \mathbb{R}$, and $\mathbf{n}$ is a unit vector, we define the boundary matrix

$$\mathcal{B}_{\eta,\mathbf{n}} = -i\sigma_3(\sigma\cdot\mathbf{n})\cos(\eta) + \sigma_3\sin(\eta)\,.$$

$\mathcal{B}_{\eta,\mathbf{n}}$ is an unitary and Hermitian matrix so that its spectrum is $\{\pm 1\}$. For any regular function $\eta\colon \partial\Omega \to \mathbb{R}$, we introduce the local boundary condition

$$\mathcal{B}_{\eta(s),\mathbf{n}(s)}\varphi(s) = \varphi(s)\,, \quad s \in \partial\Omega\,,$$

where $\mathbf{n}\colon \partial\Omega \to S^1$ is the outward pointing normal and $\varphi\colon \partial\Omega \to \mathbb{C}^2$. The associated magnetic Dirac operator $(\mathscr{D}_{h,\mathbf{A},\eta}, \mathsf{Dom}(h,\mathscr{D}_{h,\mathbf{A},\eta}))$ acts as $\mathscr{D}_{h,\mathbf{A}}$ on

$$\mathsf{Dom}(\mathscr{D}_{h,\mathbf{A},\eta}) = \left\{\varphi \in H^1(\Omega)^2\,, \quad \mathcal{B}_{\eta,\mathbf{n}}\varphi = \varphi \text{ on } \partial\Omega\right\}\,.$$

The case $\eta \equiv 0$ correspond to the MIT bag boundary condition. Note that

$$\mathcal{B}_{\eta,\mathbf{n}} = \begin{pmatrix} \sin(\eta) & -i\overline{\mathbf{n}}\cos(\eta) \\ i\mathbf{n}\cos(\eta) & -\sin(\eta)\end{pmatrix}\,,$$

so that the boundary condition reads

$$u_2 = i\mathbf{n}\frac{\cos(\eta)}{1+\sin(\eta)}u_1\,,$$

where $\varphi = (u_1, u_2)^T$.

Assumption A.1. *$\eta \in \mathcal{C}^1(\partial\Omega)$ and $\cos(\eta(s)) > 0$ for all $s \in \partial\Omega$.*

In [8], the authors proved that under Assumptions 1.2 and A.1, $\mathscr{D}_{h,\mathbf{A},\eta}$ is self-adjoint. We define

$$\gamma\colon s \in \partial\Omega \mapsto \frac{\cos(\eta(s))}{1+\sin(\eta(s))} \in \mathbb{R}_+\,.$$

Since $\partial\Omega$ is compact, we get that

$$0 < \inf_{\partial\Omega}\gamma \leqslant \gamma(s) \leqslant \sup_{\partial\Omega}\gamma < +\infty\,. \tag{A.1}$$

Notation 8. Let

$$\|u\|^2_{\partial\Omega,\gamma} = \int_{\partial\Omega}|u^2|\,\gamma\,\mathrm{d}s\,,$$

where $u \in L^2(\partial\Omega)$. By (A.1), this norm is equivalent with the one introduced in Notation 2.

It is straightforward to see that the proofs of the min-max characterization and of Theorem 1.9 are exactly the same up to the replacement of the norm on the boundary. In particular, the constants in the asymptotic analysis are defined with respect to the corresponding weighted Hardy norm on the boundary.

Theorem 1.11 has also its counterpart in this context. Here, the proof has to be slightly adapted by Taylor approximating γ around each point of the boundary. We choose to present our proof for the MIT bag condition only in order not to burden the reader with complicated notations that do not give more insight on the problem. More precisely, we get :

Theorem A.2. *Under Assumptions 1.2, 1.5, and A.1:*

(i) *Under the further assumption 1.6 we have, for all $k \geqslant 1$,*

$$\lambda_k^+(h) = \left(\frac{\mathrm{dist}_{\mathcal{H}} \left((z - z_{\min})^{k-1}, \mathscr{H}_k^2(\Omega) \right)}{\mathrm{dist}_{\mathcal{B}} \left(z^{k-1}, \mathcal{P}_{k-2} \right)} \right)^2 h^{1-k} e^{2\phi_{\min}/h} (1 + o_{h\to 0}(1)) \,,$$

(ii)

$$\lambda_1^-(h) = h^{\frac{1}{2}} \min \left(\sqrt{2b_0}, c_{\gamma(x)} \sqrt{B(x)} \,; x \in \partial\Omega \right) + o_{h\to 0}(h^{\frac{1}{2}}) \,,$$

where for any $x \in \partial\Omega$, $c_{\gamma(x)} > 0$ is the unique positive solution of the equation $\nu_{\gamma(x)}(c) = c^2$ with

$$\nu_{\gamma(x)}(c) = \inf_{\substack{u \in H^1(\mathbb{R}^2_+) \\ u \neq 0}} \frac{\int_{\mathbb{R}^2_+} |(-i\partial_s - \tau + i(-i\partial_\tau))u|^2 \mathrm{d}s\mathrm{d}\tau + c\gamma(x) \int_{\mathbb{R}} |u(s,0)|^2 \mathrm{d}s}{\|u\|^2} \,.$$

Remark A.3. Using Remark 1.8, we also cover the case $\cos(\eta(s)) < 0$ for all $s \in \partial\Omega$.

Appendix B. Proof of Lemma 2.4

We use Remark 2.3 to consider the case when $\Omega = \mathbb{D}$. We let

$$\ell^2_w(\mathbb{N}) = \left\{ u \in \ell^2(\mathbb{N}) : \sum_{n\geqslant 0} (n+1)^{-1} |u_n|^2 < +\infty \right\} \,.$$

Thanks to the isomorphism expressed in (2.2), $(\mathscr{H}^2(\mathbb{D}), \langle \cdot, \cdot \rangle_{\partial\mathbb{D}})$ is a Hilbert space.

Consider

$$K = \left\{ u \in \ell^2(\mathbb{N}) : \sum_{n\geqslant 0} |u_n|^2 \leqslant 1 \right\} \,.$$

It is sufficient to show that K is precompact in $\ell^2_w(\mathbb{N})$. Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that, for all $u \in K$,

$$\sum_{n\geqslant N+1} \frac{1}{n+1} |u_n|^2 \leqslant \frac{\varepsilon^2}{4} \,.$$

Moreover, the unit ball of $\mathbb{C}^{N+1}$ for the standard ℓ^2-norm is precompact, and we can write

$$\exists (a_0, \dots, a_M) \in \mathbb{C}^{N+1} \,, \quad B_{N+1}(0,1) \subset \bigcup_{j=0}^{M} B_{N+1,w} \left(a_j, \frac{\varepsilon}{2} \right) \,,$$

where $B_{N+1,w}$ are the balls for the ℓ^2_w-norm. We have

$$K \subset \bigcup_{j=0}^{M} B_w\left(\underline{a_j}, \varepsilon\right) ,$$

where $\underline{a_j}$ denotes the extension by zero of the finite sequence a_j. Indeed, there exists $N \in \mathbb{N}$ such that, for all $u \in K$,

$$\left\| u - \sum_{j=0}^{N} u_j e_j \right\|_{\ell^2_w} \leqslant \frac{\varepsilon}{2} .$$

Then, $\sum_{j=0}^{N} u_j e_j \in B_{N+1}(0,1)$, and the conclusion follows from the triangle inequality. Here, $(e_j)_{j\geqslant 0}$ is the canonical basis of $l^2(\mathbb{N})$.

Appendix C. About the functions $\nu_j(c,\cdot)$

For all $c \geqslant 0$, we have

$$\nu(c) = \inf_{u \in \mathfrak{H}^2_{-\mathbf{A}_0}(\mathbb{R}^2_+)} \frac{\int_{\mathbb{R}^2_+} |(-i\partial_s - \tau + i(-i\partial_\tau))u|^2 \mathrm{d}s\mathrm{d}\tau + c\int_{\mathbb{R}} |u(s,0)|^2 \mathrm{d}s}{\|u\|^2} .$$

The aim of this section is to prove that the equation $\nu(c) = c^2$ has a unique positive solution.

Due to the min-max theorem, $\nu(c)$ is the bottom of the spectrum of the self-adjoint operator $(-i\partial_s - \tau - i(-i\partial_\tau))(-i\partial_s - \tau + i(-i\partial_\tau))$ acting on $L^2(\mathbb{R}^2_+)$ with boundary condition

$$(-i\partial_s + i(-i\partial_\tau))\psi(s,0) = c\psi(s,0) .$$

The Fourier transform in s gives a direct integral and

$$\nu(c) = \inf_{\xi\in\mathbb{R}} \nu_1(c,\xi) , \tag{C.1}$$

where $\nu_1(c,\xi)$ is the bottom of the spectrum of the corresponding operator:

$$\nu_1(c,\xi) = \inf_{u\in B^1(\mathbb{R}_+)} \frac{q_{c,\xi}(u)}{\|u\|^2} , \quad B^1(\mathbb{R}_+) = \left\{u \in H^1(\mathbb{R}_+) , \tau u \in L^2(\mathbb{R}_+)\right\} ,$$

where

$$q_{c,\xi}(u) = \int_{\mathbb{R}_+} |(\xi - \tau + \partial_\tau)u|^2 \mathrm{d}\tau + c|u(0)|^2 .$$

Note that, for all $u \in L^2(\mathbb{R}_+)$, $(\partial_\tau - \tau)u \in L^2(\mathbb{R}_+)$ is equivalent to $u \in B^1(\mathbb{R}_+)$. In fact, an integration by parts gives

$$q_{c,\xi}(u) = \int_{\mathbb{R}_+} \left(|u'|^2 + |(\xi-\tau)u|^2\right) \mathrm{d}\tau + (c-\xi)|u(0)|^2 + \|u\|^2 .$$

Let us consider the associated self-adjoint operator, acting on $L^2(\mathbb{R}_+)$,

$$\mathscr{N}_{c,\xi} = -\partial_\tau^2 + (\tau-\xi)^2 + 1$$

with the boundary condition $\varphi'(0) = (c-\xi)\varphi(0)$. This operator has compact resolvent and thus, we may consider the non-decreasing sequence of its eigenvalues $(\nu_j(c,\xi))_{j\geqslant 1}$. Note also that the family $(\mathscr{N}_{c,\xi})_{\xi\in\mathbb{R}}$ is of type (B) in the sense of Kato:

(i) $\mathrm{Dom}(q_{c,\xi}) = B^1(\mathbb{R}_+)$ does not depend on ξ (or c),
(ii) for all $u \in B^1(\mathbb{R}_+)$, $\xi \mapsto q_{c,\xi}(u)$ is analytic.

Remark C.1. When $\xi = c$, this operator coincides with the famous de Gennes operator (see [15]).

Due to the aforementionned Fourier decomposition, we may focus our analysis on the properties of the ν_j. In Section C.1, we prove that each ν_j, as a function of ξ, has a unique minimum, which is non-degenerate. In Section C.2, we perform the large frequency analysis of the ν_j ($\xi \to +\infty$): they go to Landau levels from below, as displayed in the numerical illustrations of Section C.3. Section C.4 is devoted to the proof that $\nu(c) = c^2$ has a unique positive solution.

C.1. **Critical points of $\nu_j(c,\cdot)$.** By the analytic pertubation theory, we know that $\nu_j(c,\cdot)$ are analytic functions. Let $\nu(c,\cdot)$ be one these eigenvalues of $\mathscr{N}_{c,\xi}$, and $u_{c,\xi} \equiv u_\xi$ is a corresponding normalized eigenfunction.

Lemma C.2. *We have*

$$\partial_\xi \nu(c,\xi) = \int_0^{+\infty} 2(\xi - t) u_\xi^2(t) \mathrm{d}t - u_\xi^2(0)\,.$$

Proof. We have

$$(\mathscr{N}_{c,\xi} - \nu) u_\xi = 0\,.$$

Then, in the sense of quadratic forms,

$$(\mathscr{N}_{c,\xi} - \nu)\partial_\xi u_\xi + \partial_\xi \mathscr{N}_{c,\xi} u_\xi = \partial_\xi \nu(\xi) u_\xi\,,$$

so that

$$\langle (\mathscr{N}_{c,\xi} - \nu)\partial_\xi u_\xi, u_\xi\rangle + 2\int_0^{+\infty} (\xi - t) u_\xi^2(t)\mathrm{d}t = \partial_\xi \nu(\xi)\,.$$

By integrations by parts, we have

$$\langle (\mathscr{N}_{c,\xi} - \nu)\partial_\xi u_\xi, u_\xi\rangle = \partial_t \partial_\xi u_\xi(0) u_\xi(0) - \partial_\xi u_\xi(0)\partial_t u_\xi(0)\,.$$

Note that

$$\partial_\xi \partial_t u_\xi(0) = -u_\xi(0) + (c - \xi)\partial_\xi u_\xi(0)\,.$$

Thus,

$$\langle (\mathscr{N}_{c,\xi} - \nu)\partial_\xi u_\xi, u_\xi\rangle = -u_\xi^2(0)\,.$$

□

In the next lemma, we explicitely use the c-dependence of the eigenfunction.

Lemma C.3. *We have*

$$\partial_c \nu(c,\xi) = u_{c,\xi}^2(0)\,.$$

Proof. We have

$$(\mathscr{N}_{c,\xi} - \nu(c,\xi))\partial_c u_{c,\xi} = \partial_c \nu(c,\xi) u_{c,\xi}\,.$$

We get

$$\partial_c \nu(c,\xi) = \langle (\mathscr{N}_{c,\xi} - \nu(c,\xi))\partial_c u_{c,\xi}, u_{c,\xi}\rangle = \partial_t \partial_c u_{c,\xi}(0) u_{c,\xi}(0) - \partial_c u_{c,\xi}(0)\partial_t u_{c,\xi}(0)\,.$$

Note that

$$\partial_c \partial_t u_{c,\xi}(0) = u_{c,\xi}(0) + (c - \xi)\partial_c u_{c,\xi}(0)\,,$$

and the conclusion follows. □

Proposition C.4. *We have*

$$\partial_\xi \nu(c,\xi) = \left(-\nu(c,\xi) - c^2 + 2c\xi\right) u_\xi^2(0)\,. \tag{C.2}$$

In particular, if ξ_c is a critical point of $\nu(c,\cdot)$, we have

$$\nu(c,\xi_c) = -c^2 + 2c\xi_c\,. \tag{C.3}$$

Moreover, all the critical points are local non-degenerate minima and

$$\partial_\xi^2 \nu(c,\xi_c) = 2cu_{\xi_c}^2(0) > 0\,. \tag{C.4}$$

In particular, there is at most one critical point.

Proof. By using the previous lemma, we get

$$\begin{aligned}
\partial_\xi \nu(c,\xi) &= -\int_0^{+\infty} \partial_t[(\xi-t)^2]u_\xi^2(t)\mathrm{d}t - u_\xi(0)^2 \\
&= 2\int_0^{+\infty} (\xi-t)^2 u_\xi(t)u_\xi'(t)\mathrm{d}t + \xi^2 u_\xi(0)^2 - u_\xi(0)^2 \\
&= 2\int_0^{+\infty} (u_\xi''(t) + (\nu(c,\xi)-1)u_\xi)u_\xi'(t)\mathrm{d}t + \xi^2 u_\xi(0)^2 - u_\xi(0)^2 \\
&= \int_0^{+\infty} \partial_t\left((u_\xi')^2 + (\nu(c,\xi)-1)u_\xi^2\right)\mathrm{d}t + \xi^2 u_\xi(0)^2 - u_\xi(0)^2 \\
&= \left(-(\nu(c,\xi)-1) - (c-\xi)^2 + \xi^2 - 1\right) u_\xi(0)^2 \\
&= \left(-\nu(c,\xi) - c^2 + 2c\xi\right) u_\xi(0)^2\,.
\end{aligned}$$

We get (C.3). Taking the derivative of (C.3), we deduce (C.4). The last sentence follows from (C.3) and (C.4). □

The previous statement tells us that $\nu(c,\cdot)$ has at most one critical point, which is a non-degenerate minimum. Next, we show that there is always a critical point.

Corollary C.5. *For all $j \geqslant 1$, the function $\nu_j(c,\cdot)$ has a unique critical point $\xi_{j,c}$, and it is a non-degenerate minimum. The function ν_j is decreasing on $(-\infty,\xi_{j,c})$ and is increasing on $(\xi_{j,c},+\infty)$.*

Proof. If $\nu_j(c,\cdot)$ has no critical points, then it is non-increasing (it is non-increasing on $(-\infty,0)$ by Proposition C.4). From Proposition C.4, we deduce that, for all $\xi \geqslant 0$,

$$-\nu_j(c,\xi) - c^2 + 2c\xi \leqslant 0\,,$$

and that $\lim_{\xi\to+\infty} \nu_j(x,\xi) = +\infty$. This is in contradiction with the function being decreasing. This show that $\nu_j(c,\cdot)$ has a unique critical point. It is a *local* non-degenerate minimum. Since there is only one critical point, this shows that it is a global minimum. □

C.2. **Asymptotic analysis of $\nu_j(c,\xi)$.** This section is devoted to the asymptotic analysis of $\nu_j(c,\xi)$ when $\xi \to +\infty$.

When $\xi > 0$, we use the rescaling $\tau = \xi t$, and $\mathscr{N}_{c,\xi}$ is unitarily equivalent to

$$\xi^2(\mathscr{P}_{c,h} + h)\,, \quad h = \xi^{-2}$$

where $\mathscr{P}_{c,h} = -h^2\partial_t^2 + (t-1)^2$ is equipped with the boundary condition

$$h^2\varphi'(0) = (ch^{\frac{3}{2}} - h)\varphi(0)\,.$$

The corresponding quadratic form is

$$\mathscr{Q}_{c,h}(\varphi) = \int_0^{+\infty} (h^2|\varphi'|^2 + |(t-1)\varphi|^2)\mathrm{d}t + (ch^{\frac{3}{2}} - h)|\varphi(0)|^2\,.$$

Let us denote by $(\pi_j(c,h))_{j\geqslant 1}$ the non-decreasing sequence of the eigenvalues of $\mathscr{P}_{c,h}$.

Proposition C.6. *For all $j \geqslant 1$,*

$$\pi_j(c,h) = (2j-1)h + \mathscr{O}(h^\infty)\,.$$

In particular,

$$\nu_j(c,\xi) = 2j + \mathscr{O}(\xi^{-\infty})\,.$$

Moreover, for all ξ large enough, we have

$$\nu_j(c,\xi) < 2j\,. \tag{C.5}$$

Proof. By using the Hermite functions and the spectral theorem, we have, for all $n \geqslant 1$,

$$\mathrm{dist}((2n-1)h, \mathrm{sp}(\mathscr{P}_{h,c})) = \mathscr{O}(h^\infty)\,.$$

Moreover, for h small enough, we have

$$\mathscr{Q}_{c,h}(\varphi) \leqslant \int_0^{+\infty} (h^2|\varphi'|^2 + |(t-1)\varphi|^2)\mathrm{d}t\,.$$

Thus, by the min-max principle,

$$\pi_j(c,h) \leqslant \mu_j^{\mathrm{Neu}}(h)\,,$$

where $\mu_j^{\mathrm{Neu}}(h)$ is the j-th eigenvalue of the Neumann realization of $-h^2\partial_t^2 + (t-1)^2$. From our knowledge of the de Gennes operator[1], we have, for h small enough,

$$\mu_j^{\mathrm{Neu}}(h) = (2j-1)h + \mathscr{O}(h^\infty)\,, \quad \mu_j^{\mathrm{Neu}}(h) < (2j-1)h\,. \tag{C.6}$$

Therefore, for all $j \geqslant 1$, and for all h small enough, the j-th eigenvalue of $\mathscr{P}_{h,c}$ satisfies

$$\pi_j(c,h) < (2j-1)h\,.$$

Consider $\varepsilon > 0$ to be determined later. We have

$$\mathscr{Q}_{c,h}(\varphi) = \int_0^{\varepsilon h^{\frac{1}{2}}} (h^2|\varphi'|^2 + |(t-1)\varphi|^2)\mathrm{d}t + (ch^{\frac{3}{2}} - h)|\varphi(0)|^2 + \int_{\varepsilon h^{\frac{1}{2}}}^{+\infty} (h^2|\varphi'|^2 + |(t-1)\varphi|^2)\mathrm{d}t\,.$$

We get

$$\mathscr{Q}_{c,h}(\varphi) \geqslant \mathscr{Q}^1_{c,h}(\varphi) + (1 - 2\varepsilon h^{\frac{1}{2}})\|\varphi\|^2_{L^2(0,\varepsilon h^{\frac{1}{2}})} + \int_{\varepsilon h^{\frac{1}{2}}}^{+\infty} (h^2|\varphi'|^2 + |(t-1)\varphi|^2)\mathrm{d}t\,,$$

with

$$\mathscr{Q}^1_{c,h}(\varphi) = \int_0^{\varepsilon h^{\frac{1}{2}}} h^2|\varphi'|^2\mathrm{d}t + (ch^{\frac{3}{2}} - h)|\varphi(0)|^2\,.$$

[1] see, for instance, [15, Prop. 3.2.2 & 3.2.4] where the result is expressed in terms of ξ

The corresponding operator $\mathscr{P}^1_{h,c}$ is $-h^2\partial_t^2$ with the Robin condition at zero $h^2\varphi'(0) = (ch^{\frac{3}{2}} - h)\varphi(0)$ and Neumann condition at $\varepsilon h^{\frac{1}{2}}$, $\varphi'(\varepsilon h^{\frac{1}{2}}) = 0$. Consider a negative eigenvalue $\lambda = -\omega^2$, with $\omega > 0$. We have

$$-h^2\varphi'' = \lambda\varphi\,, \qquad h^2\varphi'(0) = (ch^{\frac{3}{2}} - h)\varphi(0)\,, \quad \varphi'(\varepsilon h^{\frac{1}{2}}) = 0\,.$$

We have

$$\varphi(t) = Be^{-\omega t/h} + Ae^{\omega t/h}\,.$$

The boundary conditions become

$$\omega(A - B) = (A + B)(ch^{\frac{1}{2}} - 1)\,, \quad B = Ae^{2\omega\varepsilon/h^{1/2}}\,.$$

Thus,

$$\omega = \frac{1 + e^{2\omega\varepsilon/h^{1/2}}}{1 - e^{2\omega\varepsilon/h^{1/2}}}(ch^{\frac{1}{2}} - 1)\,.$$

This equation has a unique positive solution. We can check that this solution satisfies $\omega = 1 - ch^{\frac{1}{2}} + \mathscr{O}(h^\infty)$. Therefore, the first negative (and only negative) eigenvalue λ of $\mathscr{P}^1_{h,c}$ satisfies

$$\lambda = -1 + 2ch^{\frac{1}{2}} - c^2h + \mathscr{O}(h^\infty)\,.$$

Take $\varepsilon = \frac{c}{4}$. For h small enough, we have

$$\mathscr{Q}_{c,h}(\varphi) \geqslant ch^{\frac{1}{2}}\|\varphi\|^2_{L^2(0,\varepsilon h^{\frac{1}{2}})} + \int_{\varepsilon h^{\frac{1}{2}}}^{+\infty} (h^2|\varphi'|^2 + |(t-1)\varphi|^2)\mathrm{d}t\,.$$

Consider

$$\mathscr{E}_N(h) = \mathrm{span}_{1\leqslant j\leqslant N}\,\psi_j\,,$$

where $(\psi_j)_{1\leqslant j\leqslant N}$ is an orthonormal family of eigenfunctions associated with $(\pi_j(c,h))_{1\leqslant j\leqslant N}$. For all $\varphi \in \mathscr{E}_N(h)$, we have

$$\pi_N(c,h)\|\varphi\|^2 \geqslant ch^{\frac{1}{2}}\|\varphi\|^2_{L^2(0,\varepsilon h^{\frac{1}{2}})} + \int_{\varepsilon h^{\frac{1}{2}}}^{+\infty} (h^2|\varphi'|^2 + |(t-1)\varphi|^2)\mathrm{d}t \geqslant ch^{\frac{1}{2}}\|\varphi\|^2_{L^2(0,\varepsilon h^{\frac{1}{2}})}\,.$$

In particular, since $\pi_N(c,h) \leqslant (2N-1)h$, $\mathscr{E}_N(h) \ni \varphi \mapsto \varphi_{|(0,\varepsilon h^{\frac{1}{2}})}$ is injective for h small enough.

Moreover,

$$(\pi_N(c,h) - ch^{\frac{1}{2}})\|\varphi\|^2_{L^2(0,\varepsilon h^{\frac{1}{2}})} + \pi_N(c,h)\|\varphi\|^2_{L^2(\varepsilon h^{\frac{1}{2}},+\infty)} \geqslant \int_{\varepsilon h^{\frac{1}{2}}}^{+\infty} (h^2|\varphi'|^2 + |(t-1)\varphi|^2)\mathrm{d}t\,.$$

For h small enough, we have $\pi_N(c,h) - ch^{\frac{1}{2}} \leqslant 0$. Then,

$$\pi_N(c,h)\|\varphi\|^2_{L^2(\varepsilon h^{\frac{1}{2}},+\infty)} \geqslant \int_{\varepsilon h^{\frac{1}{2}}}^{+\infty} (h^2|\varphi'|^2 + |(t-1)\varphi|^2)\mathrm{d}t\,.$$

By the min-max theorem, and using the aforementioned injectivity, we have

$$\mu_N(\varepsilon,h) \leqslant \pi_N(c,h)\,,$$

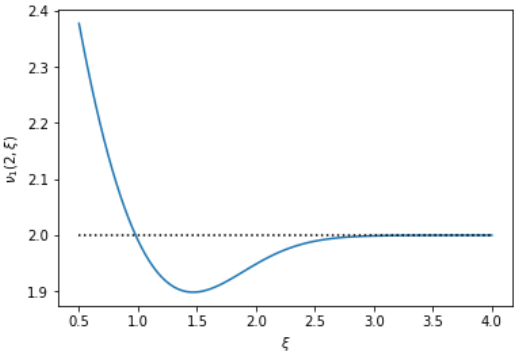
where the $(\mu_j(\varepsilon,h))_{1\leqslant j\leqslant N}$ are the eigenvalues of the Neumann realization on $L^2(\varepsilon h^{\frac{1}{2}},+\infty)$ of $-h^2\partial_t^2 + (t-1)^2$. As for (C.6) (*i.e.*, when $\varepsilon = 0$), we check that

$$\mu_N(\varepsilon,h) = (2N-1)h + \mathscr{O}(h^\infty)\,.$$

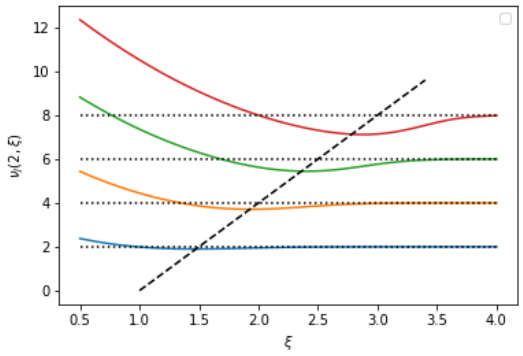
The conclusion follows.

The inequality (C.5) is a consequence of Corollary C.5. □

C.3. **Numerical illustrations.** By using a naive finite difference method, it is a possible to compute the eigenvalues $\nu_j(c,\cdot)$ by using a short Python script, see the figures below. These simulations are consistent with all our theoretical results.



(A) The function $\nu_1(2,\cdot)$



(B) Functions $\nu_j(2,\cdot)$ and the function $\xi \mapsto -c^2 + c\xi$

C.4. **On the function ν.** We recall that ν is defined in (1.11).

Proposition C.7. *The function ν is non-negative on $[0,+\infty)$, concave and it satisfies*

$$\nu(0) = 0\,, \quad \nu(+\infty) = 2\,, \quad \liminf_{c\to 0^+} \frac{\nu(c)}{c} > 0\,.$$

In particular, the equation $\nu(c) = c^2$ has a unique positive solution c_0 and $c_0 \in (0,\sqrt{2})$.

Remark C.8. Numerical calculations suggest that c_0 is approximately equal to 1.31236.

Proof. The function ν is concave as an infimum of linear functions. The equality $\nu(0) = 0$ follows by considering the zero modes[2], and $\nu(+\infty) = 2$ comes from the fact that, when $c \to +\infty$, $\nu(c)$ converges to the groundstate energy on the half-space with Dirichlet boundary condition. Then, the concavity implies that

$$\liminf_{c\to 0^+} \frac{\nu(c)}{c} > 0\,. \tag{C.7}$$

Let us explain why ν is a smooth function on $(0,+\infty)$. Let us recall that

$$\nu(c) = \min_{\xi\in\mathbb{R}} \nu_1(c,\xi) = \nu_1(c,\xi_c)\,,$$

and that, for all $c > 0$, $\xi_c > 0$ is the unique solution of

$$\partial_\xi \nu_1(c,\xi) = 0\,. \tag{C.8}$$

For all $c > 0$, we have $\partial_\xi^2 \nu_1(c,\xi_c) > 0$, and thus the analytic implicit function theorem applied to (C.8) implies that $c \mapsto \xi_c$ is analytic. Since ν_1 is analytic, we deduce that $c \mapsto \nu(c)$ is analytic. We notice that

$$\nu'(c) = \partial_c \nu_1(c,\xi_c) + \partial_\xi \nu_1(c,\xi_c)\frac{\mathrm{d}\xi_c}{\mathrm{d}c} = \partial_c \nu_1(c,\xi_c)\,.$$

[2]We can also check that ν is right continuous at 0.

Thanks to Lemma C.3, we get

$$\nu'(c) = u^2_{c,\xi_c}(0)\,. \tag{C.9}$$

Let us now consider the function

$$f(c) = \nu(c) - c^2\,.$$

From (C.7), we see that f is positive on some interval $(0, a)$ with $a > 0$. Then, by $\nu(+\infty) = 2$, we see that f is negative on some interval $(b, +\infty)$. By the Intermediate Value Theorem, we deduce that f has at least one zero in $(0, +\infty)$. Let us prove that there is only one zero. Consider $c > 0$ such that $f(c) = 0$. We have $f'(c) = \nu'(c) - 2c$. Due to (C.3), we have $\xi_c = c$, and with Lemma C.2, we get

$$2c - u^2_{c,\xi_c}(0) = \int_0^{+\infty} t u^2_{c,\xi_c}(t)\mathrm{d}t > 0\,.$$

This, with (C.9), implies that $f'(c) < 0$. We deduce that f has at most one positive zero (and thus exactly one, denoted by c_0).

Let us now prove that $c_0 \in (0, \sqrt{2})$. Let us recall (C.1). From (C.5) and Corollary C.5, we have $\nu(c_0) = \nu_1(c_0, \xi_{c_0}) < 2$. Thus, $c_0^2 = \nu(c_0) < 2$. This shows that $c_0 < \sqrt{2}$. □

Remark C.9. Actually, one could have avoided our asymptotic analysis to prove that $c_0 < \sqrt{2}$ by using the knowledge of the de Gennes function. Consider $\xi = c > 0$. Then,

$$\nu_1(c, c) = \mu(c) + 1\,,$$

where μ is the celebrated de Gennes function. We know that, on $\mathbb{R}_+$, $\mu < 1$. Thus, for all $c > 0$,

$$\nu(c) < \mu(c) + 1 < 2 = (\sqrt{2})^2\,.$$

Nevertheless, the dispersion curves play a very important role in the description of the subprincipal terms of Theorem 1.11.

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