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The gardener’s hyperbolas

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Abstract

We propose two new simple constructions of hyperbolas with a string passing through two fixed points (the foci): because of the similarities to the ellipse case, we call such constructions “gardener’s.” The first construction defines the curve by posing some constraints on the position of the tracing point, the second by the solution of certain tangent conditions. We also study a vertical transposition of the previous horizontal constructions.

1 String constructions of hyperbolas

The so-called gardener’s method to draw an ellipse is one of the most known and elegant examples of geometric constructions beyond ruler and compass. The required tools are very simple: one just needs a string with its ends fixed in two points of the plane or, as a little variation, a loop of string put around two pins. Bifocal conics are defined as the loci satisfying the constant sum (ellipse) or difference (hyperbola) of the lengths of the rays, where the rays are the vectors between a point of the curve and the two foci. Back to our gardener, the taut thread naturally embodies the property of keeping constant the sum of the two rays lengths (such a sum is the length of the ellipse major axis). Out of this construction, the string construction of conics (not only ellipses) have been introduced mainly for optics: such methods to trace hyperbolas with strings are essentially two.

As introduced by the Persian Ibn Sahl at the end of the 1st millennium [7], a first way (a printed version taken from [9] is visible in the left of Fig. 1) to construct a hyperbola of foci C and F consists in introducing a rotating rod FN and a string connecting N and C . If we consider the point P (not named in the image, but corresponding to the hand) such that the string NPC is taut, then $|NP| + |PC|$ is its length and thus is constant, but also $|NP| + |PF|$ is constant (it’s the length of the string), hence also the difference $|PC| - |PF|$ has to be constant, making P stay on a hyperbola.¹

An alternative way to trace a hyperbola, introduced by Kepler at the beginning of the 17th century [3], comes even more directly from the definition.

¹See also <https://imaginary.org/film/mathlapse-constructions-by-pin-and-string-conics>.

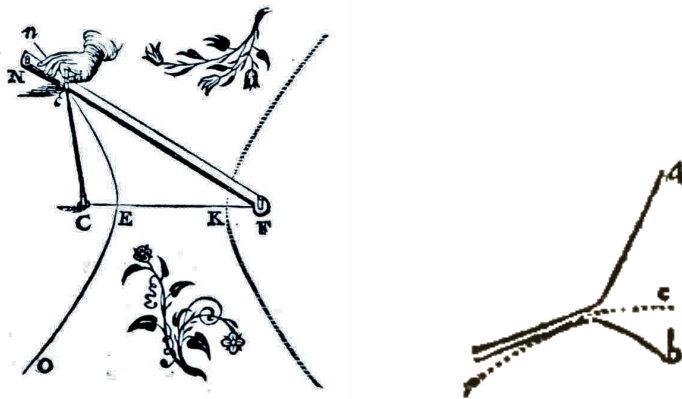


Figure 1: How to trace a hyperbola with strings. Left: by string and ruler [9, p. 67]. Right: by two strings to be equally extended [3, Ch. IV, Sec. 4].

Given the foci A, B (right of Fig. 1: even though in the image the letters of points are lower-case, we use them upper-case) and a point C through which the hyperbola has to pass, we can attach in A and B two strings making them taut in C . If we extend the two strings of the same length L (the two strings must not slide on one another), the point C' obtained when the strings are taut lies on the hyperbola of foci A, B passing through C . Indeed, the difference between $|AC'| = |AC| + L$ and $|BC'| = |BC| + L$ is the same as the difference between $|AC|$ and $|BC|$. From a practical perspective, while the first method can be actually performed (however, it requires accuracy and coordination, as you can try at your own or see in on-line videos), the latter is poorly achievable with precision in a continuous way, at least if one does not introduce something like toothed belts, hook-and-loop strips or a zipper instead of the two non-sliding strings.

2 First new method

Our purpose is to introduce two new simple methods (designed by the first author) to draw hyperbolas given the foci and a starting point. The allowed tools in the gardener's ellipse are simply a string, two pegs to be put on the foci and something to trace the curve: we want to keep the same "gardening tools" to trace hyperbolas. Specifically, as a tracing tool, consider a heavy load which, when dragged, leaves a trail on the ground: in the garden that could be a full flowerpot, on the desk any small but heavy object (eventually wet or inked to leave a trace on a paper sheet). Mathematically, we call such a tool "dragged point:" note that the heaviness is not important to satisfy the gravity law when falling, but just to be subject to the resistance of the ground/plane when moving. Furthermore, we also consider the motion of the weight ideally not subject to any inertia (practically, that means that the weight has to be

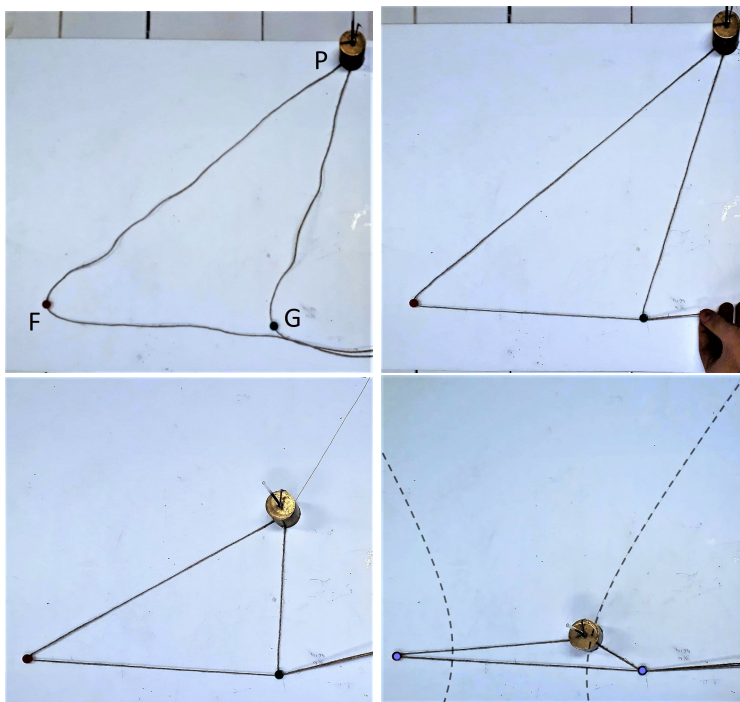


Figure 2: Pictures of a new string construction of hyperbolas. In this specific realization, the heavy load has a hole in the centre in which we posed a pen to leave a trace: the string is attached twice to such a pen (the threads coming both from F and G). In the last picture, we superimposed the hyperbola of foci F, G passing through P to compare it with the traced curve.

pulled slowly enough).

Let's begin with the first method (to easily get some pictures we implemented the desk version): as visible in the first picture of Fig. 2, on a plane hammer two nails on F and G and attach the two ends of a string to the dragged point. Then make the string pass on the left of the nails before reaching the right side of the plane (trivially you can invert all left and right sides). Given an initial position P of the dragged point, make the string taut by pulling the final loop of the string to the right so that both strings touch the right nail G , and then continue pulling the string (by the final loop or in general by keeping the two strings together).

Proposition 1. *The curve traced by the first method is a part of the hyperbola of foci F, G passing through the initial position P .*

Proof. While pulling, named P' the new position of the weight, we have that $|FP'| - |FP| = |GP'| - |GP|$ (we pull the same length on both string components), thus $|FP'| - |GP'|$ is kept constant. Therefore, the thesis follows. \square

For this method traces only the branch of hyperbola comprised between the initial position and the axis of symmetry through the foci, to trace symmetrically the two branches of hyperbola it is necessary to construct four starting point (by symmetry, top and bottom for any branch).

3 Second new method

After the first method, a curious gardener should be interested in exploring possible variations. We can note that, in the proposed construction, both ends of the string have to be attached to the tracing point in order to directly embody the definition of a hyperbola (constant difference of rays lengths). What happens if, given a string passing through the foci, we simply make it pull the heavy tracing point? From a material perspective, assuming that the weight can freely slide along the string, it is no longer necessary to pull both string extremities: we can nail one of them in a focus and pull only one extremity over the other focus, as visible in Fig. 3. As we are going to observe, this time the condition is no longer directly posed on the length of the rays, but on the direction that the weight has to follow. Maintaining the already introduced notation, let F, G be the fixed points through which the string passes. Furthermore, let W be the position of the dragged point (W is not a fixed point, but moves according to the string constraint).

We can establish the nature of the curve traced by W thanks to some physical remarks². Thanks to the friction of the weight on the plane, if we pull the free string extremity slowly enough we can guarantee negligible inertia. On such a condition, in any moment W has to move in the direction of the resultant forces according to Newton's second law of mechanics. The force applied to the pulled string extremity is transmitted to W with direction \overrightarrow{WG} , we call it \mathbf{p} , thus we have $\mathbf{p} = \frac{\overrightarrow{WG}}{\|\overrightarrow{WG}\|}p$. The string, being fixed in the motionless F , generates on W also a force \mathbf{r} resisting to the pulling along \overrightarrow{WF} ; we have $\mathbf{r} = \frac{\overrightarrow{WF}}{\|\overrightarrow{WF}\|}r$. Assuming the string to be frictionless on W , $r = p$; the resulting force in W , $\mathbf{p} + \mathbf{r}$, has the direction of the bisector of the angle \widehat{FWG} . Therefore, the direction of the weight has to be the bisector of the rays: it is well known that the only curve satisfying such direction condition is the hyperbola of foci F and G . This is not the only method to trace hyperbolas thanks to this condition. In [6], authors present a mathematical machine tracing conics by the property that the tangent has to bisect the rays (internal for hyperbolas and external for ellipses).

However, we pretend that there exists a “purely geometrical” proof of this behavior. We express this idea enunciating a principle which does not involve any

²We thank Massimo Salvi, Alfrédéric Josse and Jean-Philippe Jay for their ideas and inputs in this matter.

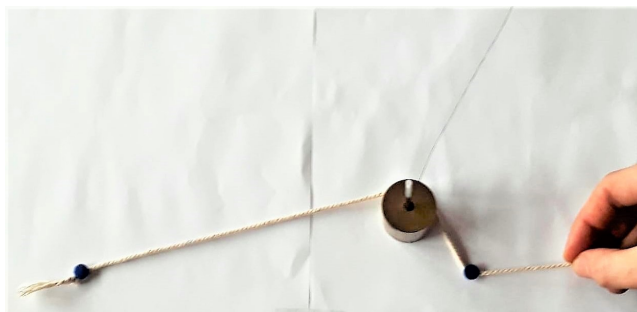


Figure 3: Another construction of a hyperbola (even though the string should pass through the centre of the weight, not along its perimeter). This time the curve is traced for its tangent condition (the string imposes the weight to move along the bisector of the rays).

of the mechanical concepts used above but asserts how the point W moves under the traction exerted on it. This principle is meant to become a postulate, basis of a geometry of the dragged motion: the dragged point moves minimizing in a certain way its displacement. We now precise our statement.

Our dragged point W has to behave as a non-inertial body: it moves only when subject to a force, and when the force stops, no inertial velocity is kept. To rephrase such a concept avoiding velocities and forces, we can use a parameter whose variation determines the displacement of W : by introducing the length k of the string between the points F and G , we can formally consider the geometric condition of pulling the string up to a certain level. Indeed, for any k , the dragged point is in the ellipse $E(k)$ of foci F, G and major axis of length k (because $|FW| + |WG| = k$). Subject to the constrain k (i.e. we pull the string), the dragged point, being non-inertial, has to avoid any unnecessary displacement. More precisely, considering the resistance of the dragged point, we propose the following:

Postulate (Dragged-point principle). *The dragged point W has to locally minimize its displacement while staying on $E(k)$ (at the changes of k).*

In another context, such a postulate reminds Fermat’s principle for optics: a light ray moves from a point to another on trajectories that locally minimize the traversing time (Fermat’s formulation is not physically exact, one should consider “stationary” and not only “minimal” trajectories). However, assuming the postulate, we can geometrically analyse the trajectory of the weight.

Proposition 2. *The curve traced by the second method is a part of the hyperbola of foci F, G passing through the initial position P .*

Proof. In a generic point P on $E(k_0)$ with $k_0 > |FG|$, for the introduced postulate, the direction of the dragged point has to minimize its displacement while staying on $E(k)$ (with $k = k_0 - \delta k$ decreasing, so $\delta k > 0$). To find

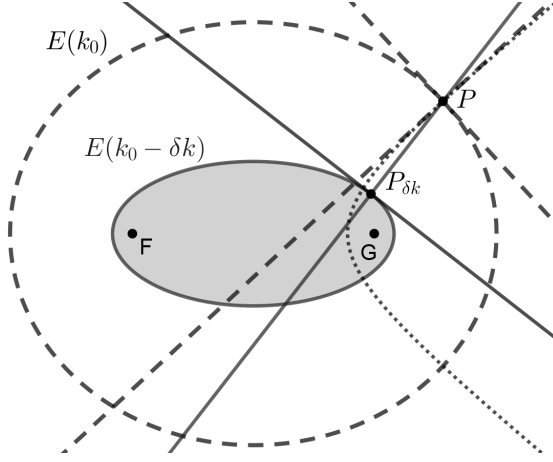


Figure 4: Named $P_{\delta k}$ the closest point to P on $E(k)$, the line $PP_{\delta k}$ is normal to $E(k)$. The limit direction of $PP_{\delta k}$ (dashed case) when $E(k)$ tends to be the ellipse of P has been normal to $E(k_0)$. The trajectory of P follows the dotted branch of hyperbola: note that the trajectory does not have to pass through the various $P_{\delta k}$.

such a direction, for any k consider $P_{\delta k}$ the closest point of $E(k)$ to P , i.e. $|PP_{\delta k}| = \min\{|PQ| : Q \in E(k)\}$. Such a minimum exists because $E(k)$ is a closed set in \mathbb{R}^2 , and is unique because $E(k)$ is convex.

Fixed any Cartesian reference frame, we can consider the direction function $\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}/[0, \pi[$ s.t. $\alpha(r \cos \theta, r \sin \theta) = [\theta]$ (the latter being an equivalence class on $\mathbb{R}/[0, \pi[$). Adopting such a notation, the direction of P is

$$\alpha_P = \lim_{\delta k \rightarrow 0^+} \alpha \left(\overrightarrow{PP_{\delta k}} \right).$$

Now we want to prove that, called in general $\overrightarrow{n_{\delta k}}$ the unit-vector normal to $E(k)$ in $P_{\delta k}$ oriented toward the interior of the ellipse, and in particular \overrightarrow{n} the normal to $E(k_0)$ in P , $\alpha_P = \alpha(\overrightarrow{n})$.

As visible in Fig. 4, to minimize the distance to P , a point $P_{\delta k}$ on $E(k)$ has to satisfy the property that the tangent to the border of $E(k)$ in $P_{\delta k}$ has to be perpendicular to the line $PP_{\delta k}$ (this condition is verified also by the maximum). That means that, for every $\delta k > 0$, it holds $\alpha \left(\overrightarrow{PP_{\delta k}} \right) = \alpha(\overrightarrow{n_{\delta k}})$. Considering the limit $\delta k \rightarrow 0^+$, we have that $P_{\delta k} \rightarrow P$ and $\overrightarrow{n_{\delta k}} \rightarrow \overrightarrow{n}$, thus $\alpha_P = \alpha(\overrightarrow{n})$, as sought.

Keeping in mind that confocal ellipses and hyperbolas cut each other orthogonally, the only curve satisfying the tangent condition imposed by the pulling of the string is the hyperbola of foci F, G (P is a general point of the plane out along the segment between F and G). \square

4 Other methods by weights

Beyond the classic constructions of Section 1 that involve only strings and eventually rods, there are several interesting mechanical methods to construct hyperbolas. Some of such methods use our tools, i.e. weights, but to what extent do similar implementations realize the same curves?

For instance, in [8, pp. 161–163] we find the following problem (see right of Fig. 5).

On a vertical plane, let P be a weight free to slide along the string fixed in F and passing through the pulley G : what is the the locus of P by the pulling of the string beyond G ?

Such a problem can be interpreted as the vertical transposition of our second method (Section 3). In the previous sections we considered constructions on a horizontal plane; however, with some very little modifications, we can consider them on a vertical plane. In these cases, assuming the strings to have negligible weight, it is important to consider the direction of the gravity force for the heavy point. A simple example of curve obtained on a vertical plane by a heavy point is the one described by an acrobat walking along a rope: considering the (not taut) rope fixed in its extremities and assuming the weight of the tightrope walker to make the string taut, his/her feet move along an ellipse.

Back to our problem, the point P has to be in an equilibrium position for any length of the string between F and G , thus the direction of the gravity force in P has to bisect \widehat{FPG} . As also visible online³, introducing O the middle point of FG , the resulting curve is a branch of the rectangular hyperbola passing through F and G and with asymptotes passing through O with directions parallel and perpendicular to the gravity force. Note that, if F and G have the same height, the rectangular hyperbola degenerates in a line.

We can easily construct a vertical version also for our first method (Section 2): instead of two nails on a horizontal plane, let F and G be two pulleys (cf. left of Fig. 5). Attaching the weight P to two pieces of string passing respectively through F and G , we can pull together the two threads as done in Fig. 2. The string, the weight and the pulleys stay in the same vertical plane during the pulling. In this case the curve described by P is a branch of hyperbola with F and G as foci: the weight force in P imposes the point to stay in the bottom position and so makes the strings taut (this time without any resistance of the plane). Therefore, the hyperbola can be described by a “real” heavy point without any reference to the traction of the point on the plane⁴.

To sum up, the first method and its vertical variation allow to construct the same curve whereas the second methods, horizontal and vertical, do not (in our

³Cf. the curve of the bucket, <https://www.mathcurve.com/courbes2d.gb/hyperbole/hyperboleequilatere.shtml>.

⁴For such an idea, we are grateful to Robert Ferréol, creator of the website “*Encyclopédie des formes mathématiques remarquables*,” <https://www.mathcurve.com/>.

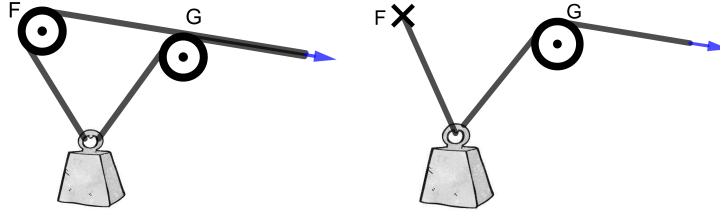


Figure 5: Vertical transposition of the proposed constructions.

[Left] The weight is attached to two strings (toward the pulleys F and G), and the two strings are pulled together. Even in this case the curve described by the weight is a branch of hyperbola of foci F and G (however, according to the gravity, the traceable branch is comprised in the strip under the segment FG). [Right] The weight can freely slide along the string that is attached in F and passes through the pulley G . The curve described by the weight is different from the horizontal counterpart: even though it is still a hyperbola, it has to be a rectangular one, and F and G are no longer the foci.

horizontal version, we can construct any hyperbola, not only rectangular, and F, G are the foci, not passage points). An analysis of the implementations shows that in the first methods, only distances are imposed (the length of the rays is directly given by the string, thus determining the locus of the weight). On the contrary, in the second methods, the constraint is about direction. However, it applies in quite a different way: in the vertical case, the rays bisector has to keep constant direction (that of the gravity) while in the horizontal case, the direction changes according to the friction on the plane.

5 Conclusions

In this work we introduced some new methods to construct hyperbolas by strings given the foci. It is interesting to compare them with the ones of Section 1: differently from the classic Ibn Sahl construction, in the newly proposed methods the total length of the string does not determine the shape of the hyperbola. Indeed, as in Kepler's method, the curve is defined by the passage through an initial point. Furthermore, the origin of the motion is the free pulling of a hand: our hand does not have to follow a predetermined path, we just have to take care that the string always has to pass through the focus (e.g. G of Fig. 2). These differences with the classic use of the string could elucidate why our constructions, although very simple, were unknown.

To conclude, we need a remark on string-and-weight constructions. On a horizontal plane, a draggable point (idealization of a weight making a great resistance with the plane) has 2 d.o.f. (degrees of freedom): if, as in our first method of Section 2, we attach to the dragged point two strings of well-defined length connected to two points, the weight imposes the strings to be taut under

the pulling and so its position is determined as the dynamic intersection of two circles (if at a certain point one of the strings is not taut, the weight simply moves in a straight line along the taut thread). This behavior does not change if we convert the construction from horizontal to vertical (Section 4).

However, if the length of the string imposes only one d.o.f., the second d.o.f. (the pulling of the string) is guided by the conditions on the direction of the motion. That means to solve an inverse tangent problem (to construct a curve given its tangent properties): this is at the basis of the construction of the tractrix (the curve defined by a fixed-length string with as its extremes a dragged point and a point moving along a line, e.g. cf. [2]). The motion determined by the traction of a dragged point got named *tractional motion*, a general method to construct curves beyond the algebraic limits. Out of the historical interest (cf. [1] or, for a more extended text, [10]), this topic is also studied from epistemological [4] and foundational [5] perspectives.

As for practical considerations, being generated by imposing tangent conditions, tractional constructions require more attention during the execution than constructions involving only distance conditions. However, even though the trace of the hyperbola obtained by the method of Section 3 is probably less precise than the one of Section 2, it is particularly interesting not only for its simplicity but also because (according to the authors' knowledge) this is a first example of tractional motion generated by a folded string: historically the tractional motion concerns the traction of a direct string, as for the tractrix. Future works are expected to deepen the geometry of planar constructions with a dragged point in general cases.

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