The Hodge realization functor on the derived category of relative motives

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March 3, 2021

Abstract

We give, for a complex algebraic variety S, a Hodge realization functor $\mathcal{F}_S^{Hdg}$ from the (unbounded) derived category of constructible motives $\mathrm{DA}_c(S)$ over S to the (undounded) derived category $D(MHM(S))$ of algebraic mixed Hodge modules over S. Moreover, for $f : T \to S$ a morphism of complex quasi-projective algebraic varieties, $\mathcal{F}_-^{Hdg}$ commutes with the four operations $f^*, f_*, f_!, f^!$ on $\mathrm{DA}_c(-)$ and $D(MHM(-))$, making in particular the Hodge realization functor a morphism of 2-functor on the category of complex quasi-projective algebraic varieties which for a given S sends $\mathrm{DA}_c(S)$ to $D(MHM(S))$, moreover $\mathcal{F}_S^{Hdg}$ commutes with tensor product. We also give an algebraic and analytic Gauss-Manin realization functor from which we obtain a base change theorem for algebraic De Rham cohomology and for all smooth morphisms a relative version of the comparaison theorem of Grothendieck between the algebraic De Rahm cohomology and the analytic De Rahm cohomology.

Contents

1 Introduction

Saito's theory of mixed Hodge modules associate to each complex algebraic variety S a category $MHM(S)$ which is a full subcategory of $\mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I C_{fil}(S)$ which extend variations of mixed Hodge structure and admits a canonical monoidal structure given by tensor product, and associate to each morphism of complex algebraic varieties $f:X\to S$, four functor $Rf_{Hdg!}, Rf_{Hdg*}, f^{\hat{*}Hdg}, f^{*Hdg}$. In the case of a smooth proper morphism $f:X\to S$ with S and X smooth, $H^nRf_{Hdg*}\mathbb{Z}_X^{Hdg}$ is the variation of Hodge structure given by the Gauss-Manin connexion and the local system $H^nRf_*\mathbb{Z}_X$. Moreover, these functors induce the six functor formalism of Grothendieck. We thus have, for a complex algebraic variety S a canonical functor

$$MH(/S):\mathrm{Var}(\mathbb{C})/S\to D(MHM(S)),\ (f:X\to S)\mapsto Rf_{!Hdg}\mathbb{Z}_X^{Hdg}$$

and

$$MH(/-):\mathrm{Var}(\mathbb{C})\to\mathrm{TriCat},\ S\mapsto(MH(S):\mathrm{Var}(\mathbb{C})/S\to D(MHM(S))),$$

is a morphism of 2-functor. In this work, we extend $MH(/-)$ to motives by constructing, for each complex algebraic variety S, a canonical functor $\mathcal{F}_S^{Hdg}:\mathrm{DA}(S)\to D(MHM(S))$ which is monoidal, that is commutes with tensor product, together with, for each morphism of complex algebraic varieties $g:T\to S$ a canonical transformation map $T(g,\mathcal{F}^{Hdg})$, which make

$$\mathcal{F}_-^{Hdg}:\mathrm{Var}(\mathbb{C})\to\mathrm{TriCat},\ S\mapsto(\mathcal{F}_S^{Hdg}:\mathrm{DA}(S)\to D(MHM(S))),$$

is a morphism of 2-functor : this is the contain of theorem 47. A partial result in this direction has been obtained by Ivorra in [17] using a different approach. We already have a Betti realization functor

$$\mathrm{Bti}_-:\mathrm{Var}(\mathbb{C})\to\mathrm{TriCat},\ S\mapsto(\mathrm{Bti}_S^*:\mathrm{DA}(S)\to D(S)),$$

which extend the Betti realization. The functor $\mathcal{F}_-^{Hdg}:=(\mathcal{F}_-^{FDR},\mathrm{Bti}_-)$ is obtained by constructing the De Rham part

$$\mathcal{F}_-^{FDR}:\mathrm{Var}(\mathbb{C})\to\mathrm{TriCat},\ S\mapsto(\mathcal{F}_S^{FDR}:\mathrm{DA}(S)\to D_{\mathcal{D}(1,0)fil,\infty}(S/(\tilde{S}_I))),$$

which takes values in the derived category of filtered algebraic D-modules obtained by inverting ∞-filtered Zariski local equivalence and then using the following key theorem (theorem 33)

Theorem 1. *(i) Let $S\in\mathrm{Var}(\mathbb{C})$. Let $S=\cup_{i\in I}S_i$ an open cover such that there exists closed embedding $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$. Then the full embedding*

$$\iota_S:MHM(S)\hookrightarrow C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I C_{fil}(S^{an})$$

induces a full embedding

$$\iota_S : D(MHM(S)) \hookrightarrow D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$$

whose image consists of $(((M_I, F, W), u_{IJ}), (K, W), \alpha) \in D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$ *such that*

$$((H^n(M_I, F, W), H^n(u_{IJ})), H^n(K, W), H^n\alpha) \in MHM(S)$$

for all $n \in \mathbb{Z}$ *and such that for all* $p \in \mathbb{Z}$*, the differential of* $(\mathrm{Gr}_W^p M_I, F)$ *are strict for the filtration* F *(in particular, the differentials of* (M_I, F, W) *are strict for the filtration* F*).*

(ii) Let $S \in \mathrm{Var}(\mathbb{C})$*. Let* $S = \cup_{i\in I} S_i$ *an open cover such that there exists closed embedding* $i_i : S_i \hookrightarrow \tilde{S}_i$ *with* $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$*. Then the full embedding*

$$\iota_S : MHM(S) \hookrightarrow C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I C_{fil}(S^{an})$$

induces a full embedding

$$\iota_S : D(MHM(S)) \hookrightarrow D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$$

whose image consists of $(((M_I, F, W), u_{IJ}), (K, W), \alpha) \in D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$ *such that*

$$((H^n(M_I, F, W), H^n(u_{IJ})), H^n(K, W), H^n\alpha) \in MHM(S)$$

for all $n \in \mathbb{Z}$ *and such that there exist* $r \in \mathbb{N}$ *and an* r*-homotopy equivalence* $((M_I, F, W), u_{IJ}) \to ((M'_I, F, W), u_{IJ})$ *such that for all* $p \in \mathbb{Z}$*, the differential of* $(\mathrm{Gr}_W^p M'_I, F)$ *are strict for the filtration* F *(in particular, the differentials of* (M'_I, F, W) *are strict for the filtration* F*).*

Note that the category $D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I))$ is NOT triangulated. More precisely the canonical triangles of $D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I))$ does NOT satisfy the 2 of 3 axiom of a triangulated category. Moreover there exist canonical triangles of $D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I))$ which are NOT the image of distinguish triangles of $\pi_S(D(MHM(S)))$. This method can be seen as a relative version of the construction of F.Lecompte and N.Wach in [20].

In section 6.1.1 and 6.2.1, we construct an algebraic and analytic Gauss-Manin realization functor, but this functor does NOT give a complex of filtered D-module, BUT a complex of filtered O-modules whose cohomology sheaves have a structure of filtered D_S modules. Hence, it does NOT get to the desired category. Moreover the Hodge filtration is NOT the right one : see proposition 116 and proposition 109 However this functor gives some interesting results. Let $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$ smooth. For $I \subset [1, \cdots l]$, denote by $S_I = \cap_{i\in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \Pi_{i\in I}\tilde{S}_i$. We define the filtered algebraic Gauss-Manin realization functor defined as

$$\mathcal{F}_S^{GM} : C(\mathrm{Var}(\mathbb{C})^{sm}/S) \to C_{Ofil,\mathcal{D}}(S/(\tilde{S}_I))^{\vee},\ M \mapsto$$
$$\mathcal{F}_S^{GM}(F) := ((e(\tilde{S}_I)_* \mathcal{H}om^{\bullet}(L(i_{I*}j_I^*F), E_{et}(\Omega^{\bullet}_{/\tilde{S}_I}), F_b))[-d_{\tilde{S}_I}], u_{IJ}^q(F)),$$

see definition 104 and corollary 4. Note that the canonical triangles of $D_{Ofil,\mathcal{D},\infty}(S/(\tilde{S}_I))$ does NOT satisfy the 2 of 3 axiom of a triangulated category. The filtered algebraic Gauss-Manin realization functor induces by proposition 105

$$\mathcal{F}_S^{GM} : \mathrm{DA}_c(S)^{op} \to D_{Ofil,\mathcal{D},\infty}(S/(\tilde{S}_I)),\ M \mapsto$$
$$\mathcal{F}_S^{GM}(M) := ((e(\tilde{S}_I)_* \mathcal{H}om^{\bullet}(L(i_{I*}j_I^*F), E_{et}(\Omega^{\bullet}_{/\tilde{S}_I}), F_b))[-d_{\tilde{S}_I}], u_{IJ}^q(F))$$

where $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$. We then prove (theorem 34):

Theorem 2. *(i) Let $g : T \to S$ is a morphism with $T, S \in \mathrm{Var}(\mathbb{C})$. Assume there exist a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S}$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. Then, for $M \in \mathrm{DA}_c(S)$*

$$T(g, \mathcal{F}^{GM})(M) : Rg^{*mod[-],\Gamma} \mathcal{F}_S^{GM}(M) \to \mathcal{F}_T^{GM}(g^* M)$$

is an isomorphism in $D_{O_T fil,\mathcal{D},\infty}(T/(Y \times \tilde{S}_I))$.

(ii) Let $g : T \to S$ is a morphism with $T, S \in \mathrm{SmVar}(\mathbb{C})$. Then, for $M \in \mathrm{DA}_c(S)$

$$T^O(g, \mathcal{F}^{GM})(M) : Lg^{*mod} \mathcal{F}_S^{GM}(M) \to \mathcal{F}_T^{GM}(g^* M)$$

is an isomorphism in $D_{O_T}(T)$.

(iii) A base change theorem for algebraic De Rham cohomology : Let $g : T \to S$ is a morphism with $T, S \in \mathrm{SmVar}(\mathbb{C})$. Let $h : U \to S$ a smooth morphism with $U \in \mathrm{Var}(\mathbb{C})$. Then the map (see definition 1)

$$T_w^O(g, h) : Lg^{*mod} Rh_*(\Omega^\bullet_{U/S}, F_b) \to Rh'_*(\Omega^\bullet_{U_T/T}, F_b)$$

is an isomorphism in $D_{O_T}(T)$.

Let $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. For $I \subset [1, \cdots l]$, denote by $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \Pi_{i \in I} \tilde{S}_i$. We define the filtered analytic Gauss-Manin realization functor defined as

$$\mathcal{F}_S^{GM} : \mathrm{DA}_c(S)^{op} \to D_{Ofil,\mathcal{D},\infty}(S/(\tilde{S}_I))^\vee, \; M \mapsto$$
$$\mathcal{F}_S^{GM}(M) := ((e(\tilde{S}_I)_* \mathcal{H}om^\bullet(\mathrm{An}^*_{\tilde{S}_I} L(i_{I*} j_I^* F), E_{et}(\Omega^\bullet_{/\tilde{S}_I}), F_b))[-d_{\tilde{S}_I}], u^q_{IJ}(F))$$

where $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$, see definition 128 and corollary 7. We then prove (theorem 38):

Theorem 3. *(i) Let $g : T \to S$ is a morphism with $T, S \in \mathrm{Var}(\mathbb{C})$. Assume there exist a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S}$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. Then, for $M \in \mathrm{DA}_c(S)$*

$$T(g, \mathcal{F}_{an}^{GM})(M) : Rg^{*mod[-],\Gamma} \mathcal{F}_{S,an}^{GM}(M) \to \mathcal{F}_{T,an}^{GM}(g^* M)$$

is an isomorphism in $D_{O_T fil,\mathcal{D}^\infty,\infty}(T/(Y \times \tilde{S}_I))$.

(ii) Let $g : T \to S$ is a morphism with $T, S \in \mathrm{SmVar}(\mathbb{C})$. Then, for $M \in \mathrm{DA}_c(S)$

$$T(g, \mathcal{F}_{an}^{GM})(M) : Lg^{*mod[-]} \mathcal{F}_{S,an}^{GM}(M) \to \mathcal{F}_{T,an}^{GM}(g^* M)$$

is an isomorphism in $D_{O_T}(T)$.

A consequence of the construction of the transformation map between the algebraic and analytic Gauss-Manin realization functor is the following (theorem 42)

Theorem 4. *(i) Let $S \in \mathrm{Var}(\mathbb{C})$. Then, for $M \in \mathrm{DA}_c(S)$*

$$\mathcal{J}_S(-) \circ H^n T(\mathrm{An}, \mathcal{F}_{an}^{GM})(M) : J_S(H^n(\mathcal{F}_S^{GM}(M))^{an}) \xrightarrow{\sim} H^n \mathcal{F}_{S,an}^{GM}(M)$$

is an isomorphism in $\mathrm{PSh}_{\mathcal{D}}(S^{an}/(\tilde{S}_I^{an}))$.

(ii) *A relative version of Grothendieck GAGA theorem for De Rham cohomology Let $h : U \to S$ a smooth morphism with $S, U \in \mathrm{SmVar}(\mathbb{C})$. Then,*

$$\mathcal{J}_S(-) \circ J_S T^O_\omega(an, h) : J_S((R^n h_* \Omega^\bullet_{U/S})^{an}) \xrightarrow{\sim} R^n h_* \Omega^\bullet_{U^{an}/S^{an}}$$

is an isomorphism in $\mathrm{PSh}_{\mathcal{D}}(S^{an})$.

In section 6.1.2, using results of sections 2, 4 and 5, we construct the algebraic filtered De Rham realization functor $\mathcal{F}^{FDR}_{-}$. We construct it via a larger category and use theorem 33(ii): Let $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. For $I \subset [1, \cdots l]$, denote by $S_I = \cap_{i\in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \Pi_{i\in I}\tilde{S}_i$. we define in definition 116(ii) which use definition 112 and definition 34, the filtered algebraic De Rahm realization functor defined as

$$\mathcal{F}^{FDR}_S : C(\mathrm{Var}(\mathbb{C})^{sm}/S) \to C_{\mathcal{D}fil}(S/(\tilde{S}_I)), F \mapsto \mathcal{F}^{FDR}_S(F) :=$$
$$(e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(L\rho_{\tilde{S}_I *}\mu_{\tilde{S}_I *} R^{CH}(\rho^*_{\tilde{S}_I} L(i_{I*} j_I^* F)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u^q_{IJ}(F)).$$

By proposition 112(ii) and corollary 5, it induces

$$\mathcal{F}^{FDR}_S : \mathrm{DA}_c(S) \to D_{\mathcal{D}(1,0)fil,\infty}(S/(\tilde{S}_I)), M \mapsto \mathcal{F}^{FDR}_S(M) :=$$
$$(e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(L\rho_{\tilde{S}_I *}\mu_{\tilde{S}_I *} R^{CH}(\rho^*_{\tilde{S}_I} L(i_{I*} j_I^* (F, W))), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u^q_{IJ}(F))$$

where $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$. We compute this functor for an homological motive and we get by proposition 114 and corollary 6, for $S \in \mathrm{Var}(\mathbb{C})$ and $M \in \mathrm{DA}_c(S)$, $\mathcal{F}^{FDR}_S(M) \in \pi_S(D(MHM(S))$, and the following (theorem 35, theorem 36 and theorem 37):

Theorem 5. *(i) Let $g : T \to S$ a morphism, with $S, T \in \mathrm{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $M \in \mathrm{DA}_c(S)$. Then map in $\pi_T(D(MHM(T)))$*

$$T(g, \mathcal{F}^{FDR})(M) : g^{\hat{*}mod}_{Hdg} \mathcal{F}^{FDR}_S(M) \xrightarrow{\sim} \mathcal{F}^{FDR}_T(g^* M)$$

given in definition 121 is an isomorphism.

(ii) *Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Then, for $M \in \mathrm{DA}_c(X)$, the map given in definition 122*

$$T_!(f, \mathcal{F}^{FDR})(M) : Rf^{Hdg}_! \mathcal{F}^{FDR}_X(M) \xrightarrow{\sim} \mathcal{F}^{FDR}_S(Rf_! M)$$

is an isomorphism in $\pi_S(D(MHM(S))$.

(iii) *Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$, S quasi-projective. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have, for $M \in \mathrm{DA}_c(X)$, the map given in definition 122*

$$T_*(f, \mathcal{F}^{FDR})(M) : \mathcal{F}^{FDR}_S(Rf_* M) \xrightarrow{\sim} Rf^{Hdg}_* \mathcal{F}^{FDR}_X(M)$$

is an isomorphism in $\pi_S(D(MHM(S))$.

(iv) *Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$, S quasi-projective. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Then, for $M \in \mathrm{DA}_c(S)$, the map given in definition 122*

$$T^!(f, \mathcal{F}^{FDR})(M) : \mathcal{F}^{FDR}_X(f^! M) \xrightarrow{\sim} f^{*mod}_{Hdg} \mathcal{F}^{FDR}_S(M)$$

is an isomorphism in $\pi_X(D(MHM(X))$.

(v) *Let $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^{l} S_i$ an open affine covering and denote, for $I \subset [1, \cdots l]$, $S_I = \cap_{i\in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. Then, for $M, N \in \mathrm{DA}_c(S)$, the map in $\pi_S(D(MHM(S)))$*

$$T(\mathcal{F}_S^{FDR}, \otimes)(M,N) : \mathcal{F}_S^{FDR}(M) \otimes_{O_S}^{L} \mathcal{F}_S^{FDR}(N) \xrightarrow{\sim} \mathcal{F}_S^{FDR}(M \otimes N)$$

given in definition 124 is an isomorphism.

We also have a canonical transformation map between the Gauss-Manin and the De Rham functor given in definition 118 wich satisfy (see proposition 116) :

Proposition 1. *Let $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^{l} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$.*

(i) *For $M \in \mathrm{DA}_c(S)$ the map in $D_{O_S,\mathcal{D}}(S/(\tilde{S}_I)) = D_{O_S,\mathcal{D}}(S)$*

$$o_{fil} T(\mathcal{F}_S^{GM}, \mathcal{F}_S^{FDR})(M) : o_{fil}\mathcal{F}_S^{GM}(L\mathbb{D}_S M) \xrightarrow{\sim} o_{fil}\mathcal{F}_S^{FDR}(M)$$

given in definition 118 is an isomorphism if we forgot the Hodge filtration F.

(ii) *For $M \in \mathrm{DA}_c(S)$ and all $n, p \in \mathbb{Z}$, the map in $\mathrm{PSh}_{O_S,\mathcal{D}}(S/(\tilde{S}_I))$*

$$F^p H^n T(\mathcal{F}_S^{GM}, \mathcal{F}_S^{FDR})(M) : F^p H^n \mathcal{F}_S^{GM}(L\mathbb{D}_S M) \hookrightarrow F^p H^n \mathcal{F}_S^{FDR}(M)$$

given in definition 118 is a monomorphism. Note that $F^p H^n T(\mathcal{F}_S^{GM}, \mathcal{F}_S^{FDR})(M)$ is NOT an isomorphism in general : take for example $M(S^o/S)^\vee = D(\mathbb{A}^1, et)(j_* E_{et}(\mathbb{Z}(S^o/S)))$ for an open embedding $j : S^o \hookrightarrow S$, then

$$\mathcal{F}_S^{GM}(L\mathbb{D}_S M(S^o/S)^\vee) = \mathcal{F}_S^{GM}(\mathbb{Z}(S^o/S)) = j_* E(O_{S^o}, F_b) \notin \pi_S(MHM(S))$$

and hence NOT isomorphic to $\mathcal{F}_S^{GM}(L\mathbb{D}_S M(S^o/S)^\vee) \in \pi_S(MHM(S))$, see remark 9. It is an isomorphism in the very particular cases where $M = D(\mathbb{A}^1, et)(\mathbb{Z}(X/S))$ or $M = D(\mathbb{A}^1, et)(\mathbb{Z}(X^o/S))$ for $f : X \to S$ is a smooth proper morphism and $n : X^o \hookrightarrow X$ is an open subset such that $X \backslash X^o = \cup D_i$ is a normal crossing divisor and such that $f_{|D_i} = f \circ i_i : D_i \to X$ are SMOOTH morphism with $i_i : D_i \hookrightarrow X$ the closed embedding and considering $f_{|X^o} = f \circ n : X^o \to S$ (see proposition 109).

Let $S \in \mathrm{Var}(\mathbb{C})$. Let $S = \cup_i S_i$ an open cover such that there exists closed embedding $i_i : S \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. The functor

$$\begin{aligned} \mathcal{F}_S^{Hdg} := (\mathcal{F}_S^{FDR}, \mathrm{Bti}_S^*) : C(\mathrm{Var}(\mathbb{C})^{sm}/S \to C_{\mathcal{D}fil}(S/(\tilde{S}_I)) \times_I C(S^{an}), \\ F \mapsto \mathcal{F}_S^{Hdg}(F) := (\mathcal{F}_S^{FDR}(F), \mathrm{Bti}_S^* F, \alpha(F)), \end{aligned}$$

where $\alpha(F)$ is given in definition 157, induces the functor

$$\begin{aligned} \mathcal{F}_S^{Hdg} := (\mathcal{F}_S^{FDR}, \mathrm{Bti}_S^*) : \mathrm{DA}(S) \to D_{\mathcal{D}(1,0)fil,\infty}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}), \\ M \mapsto \mathcal{F}_S^{Hdg}(M) := (\mathcal{F}_S^{FDR}(M), \mathrm{Bti}_S^* M, \alpha(M)). \end{aligned}$$

The main theorem of this article is the following (theorem 47):

Theorem 6. (i) *For $S \in \mathrm{Var}(\mathbb{C})$, we have $\mathcal{F}_S^{Hdg}(\mathrm{DA}_c(S)) \subset D(MHM(S))$.*

(ii) *The Hodge realization functor $\mathcal{F}_-^{Hdg}$ define a morphism of 2-functor on $\mathrm{Var}(\mathbb{C})$*

$$\mathcal{F}_-^{Hdg} : \mathrm{Var}(\mathbb{C}) \to (\mathrm{DA}_c(-) \to D(MHM(-)))$$

whose restriction to $\mathrm{QPVar}(\mathbb{C})$ is an homotopic 2-functor in sense of Ayoub. More precisely, *(ii0) for $g: T \to S$ a morphism, with $T, S \in \mathrm{QPVar}(\mathbb{C})$, and $M \in \mathrm{DA}_c(S)$, the the maps of definition 121 and of definition 152 induce an isomorphism in $D(MHM(T))$*

$$\begin{aligned} T(g, \mathcal{F}^{Hdg})(M) &:= (T(g, \mathcal{F}^{FDR})(M), T(g, bti)(M)) : \\ g^{\hat{*}Hdg}\mathcal{F}_S^{Hdg}(M) &:= (g_{Hdg}^{\hat{*}mod}\mathcal{F}_S^{FDR}(M), g^* \mathrm{Bti}_S(M), g^*(\alpha(M))) \\ &\xrightarrow{\sim} (\mathcal{F}_T^{FDR}(g^*M), \mathrm{Bti}_T^*(g^*M), \alpha(g^*M)) =: \mathcal{F}_T^{Hdg}(g^*M), \end{aligned}$$

(ii1) for $f: T \to S$ a morphism, with $T, S \in \mathrm{QPVar}(\mathbb{C})$, and $M \in \mathrm{DA}_c(T)$, the maps of definition 122 and of definition 153 induce an isomorphism in $D(MHM(S))$

$$\begin{aligned} T_*(f, \mathcal{F}^{Hdg})(M) &:= (T_*(f, \mathcal{F}^{FDR})(M), T_*(f, bti)(M)) : \\ Rf_{Hdg*}\mathcal{F}_T^{Hdg}(M) &:= (Rf_*^{Hdg}\mathcal{F}_T^{FDR}(M), Rf_* \mathrm{Bti}_S(M), f_*(\alpha(M))) \\ &\xrightarrow{\sim} (\mathcal{F}_S^{FDR}(Rf_*M), \mathrm{Bti}_S^*(Rf_*M), \alpha(Rf_*M)) =: \mathcal{F}_S^{Hdg}(Rf_*M), \end{aligned}$$

(ii2) for $f: T \to S$ a morphism, with $T, S \in \mathrm{QPVar}(\mathbb{C})$, and $M \in \mathrm{DA}_c(T)$, the maps of definition 122 and of definition 153 induce an isomorphism in $D(MHM(S))$

$$\begin{aligned} T_!(f, \mathcal{F}^{Hdg})(M) &:= (T_!(f, \mathcal{F}^{FDR})(M), T_!(f, bti)(M)) : \\ Rf_{!Hdg}\mathcal{F}_T^{Hdg}(M) &:= (Rf_!^{Hdg}\mathcal{F}_T^{FDR}(M), Rf_! \mathrm{Bti}_S^*(M), f_!(\alpha(M))) \\ &\xrightarrow{\sim} (\mathcal{F}_S^{FDR}(Rf_!M), \mathrm{Bti}_S^*(Rf_!M), \alpha(f_!M)) =: \mathcal{F}_T^{Hdg}(f_!M), \end{aligned}$$

(ii3) for $f: T \to S$ a morphism, with $T, S \in \mathrm{QPVar}(\mathbb{C})$, and $M \in \mathrm{DA}_c(S)$, the maps of definition 122 and of definition 153 induce an isomorphism in $D(MHM(T))$

$$\begin{aligned} T^!(f, \mathcal{F}^{Hdg})(M) &:= (T^!(f, \mathcal{F}^{FDR})(M), T^!(f, bti)(M)) : \\ f^{*Hdg}\mathcal{F}_S^{Hdg}(M) &:= (f_{Hdg}^{*mod}\mathcal{F}_S^{FDR}(M), f^! \mathrm{Bti}_S(M), f^!(\alpha(M))) \\ &\xrightarrow{\sim} (\mathcal{F}_T^{FDR}(f^!M), \mathrm{Bti}_T^*(f^!M), \alpha(f^!M)) =: \mathcal{F}_T^{Hdg}(f^!M), \end{aligned}$$

(ii4) for $S \in \mathrm{Var}(\mathbb{C})$, and $M, N \in \mathrm{DA}_c(S)$, the maps of definition 124 and of definition 154 induce an isomorphism in $D(MHM(S))$

$$\begin{aligned} T(\otimes, \mathcal{F}^{Hdg})(M, N) &:= (T(\otimes, \mathcal{F}_S^{FDR})(M, N), T(\otimes, bti)(M, N)) : \\ (\mathcal{F}_S^{FDR}(M) &\otimes_{O_S}^L \mathcal{F}_S^{FDR}(N), \mathrm{Bti}_S(M) \otimes \mathrm{Bti}_S(N), \alpha(M) \otimes \alpha(N)) \\ \xrightarrow{\sim} \mathcal{F}_S^{Hdg}(M \otimes N) &:= (\mathcal{F}_S^{FDR}(M \otimes N), \mathrm{Bti}_S(M \otimes N), \alpha(M \otimes N)). \end{aligned}$$

(iii) For $S \in \mathrm{Var}(\mathbb{C})$, the following diagram commutes :

$$\begin{tikzcd} \mathrm{Var}(\mathbb{C})/S \arrow[r, "MH(/S)"] \arrow[d, "M(/S)"'] & D(MHM(S)) \arrow[d, "\iota^S"] \\ \mathrm{DA}(S) \arrow[r, "\mathcal{F}_S^{Hdg}"] & D_{\mathcal{D}fil,\infty}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}) \end{tikzcd}$$

We obtain theorem 6 from theorem 5 and from the result on the Betti factor after checking the compatibility of these transformation maps with the isomorphisms $\alpha(M)$.

I am grateful to F.Mokrane for his help and support during the preparation of this work as well as J.Wildeshaus for the interest and remarks that he made on a first version of this text. I also thank J.Ayoub, C.Sabbah and M.Saito for the interest they have brought to this work.

2 Generalities and Notations

2.1 Notations

- After fixing a universe, we denote by
  - Set the category of sets,
  - Top the category of topological spaces,
  - Ring the category of rings and cRing $\subset$ Ring the full subscategory of commutative rings,
  - RTop the category of ringed spaces,
    - whose set of objects is $\mathrm{RTop} := \{(X, O_X),\ X \in \mathrm{Top},\ O_X \in \mathrm{PSh}(X, \mathrm{Ring})\}$
    - whose set of morphism is $\mathrm{Hom}((T, O_T), (S, O_S)) := \{((f : T \to S), (a_f : f^*O_S \to O_T))\}$

    and by $ts : \mathrm{RTop} \to \mathrm{Top}$ the forgetfull functor.
  - Cat the category of small categories which comes with the forgetful functor $o : \mathrm{Cat} \to \mathrm{Fun}(\Delta^1, \mathrm{Set})$, where $\mathrm{Fun}(\Delta^1, \mathrm{Set})$ is the category of simplicial sets,
  - RCat the category of ringed topos
    - whose set of objects is $\mathrm{RCat} := \{(\mathcal{X}, O_X),\ \mathcal{X} \in \mathrm{Cat},\ O_X \in \mathrm{PSh}(\mathcal{X}, \mathrm{Ring})\}$,
    - whose set of morphism is $\mathrm{Hom}((\mathcal{T}, O_T), (\mathcal{S}, O_S)) := \{((f : \mathcal{T} \to \mathcal{S}), (a_f : f^*O_S \to O_T)), \}$

    and by $tc : \mathrm{RCat} \to \mathrm{Cat}$ the forgetfull functor.
- Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor with $\mathcal{C}, \mathcal{C}' \in \mathrm{Cat}$. For $X \in \mathcal{C}$, we denote by $F(X) \in \mathcal{C}'$ the image of X, and for $X, Y \in \mathcal{C}$, we denote by $F^{X,Y} : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(F(X), F(Y))$ the corresponding map.
- Let $\mathcal{C} \in \mathrm{Cat}$. For $S \in \mathcal{C}$, we denote by $\mathcal{C}/S$ the category
  - whose set of objects $(\mathcal{C}/S)^0 = \{X/S = (X, h)\}$ consist of the morphisms $h : X \to S$ with $X \in \mathcal{C}$,
  - whose set of morphism $\mathrm{Hom}(X'/S, X/S)$ between $X'/S = (X', h'), X/S = (X, h) \in \mathcal{C}/S$ consits of the morphisms $(g : X' \to X) \in \mathrm{Hom}(X', X)$ such that $h \circ g = h'$.

  We have then, for $S \in \mathcal{C}$, the canonical forgetful functor

  $$r(S) : \mathcal{C}/S \to \mathcal{C},\ \ X/S \mapsto r(S)(X/S) = X,\ (g : X'/S \to X/S) \mapsto r(S)(g) = g$$

  and we denote again $r(S) : \mathcal{C} \to \mathcal{C}/S$ the corresponding morphism of (pre)sites.

  - Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor with $\mathcal{C}, \mathcal{C}' \in \mathrm{Cat}$. Then for $S \in \mathcal{C}$, we have the canonical functor

    $$F_S : \mathcal{C}/S \to \mathcal{C}'/F(S),\ \ X/S \mapsto F(X/S) = F(X)/F(S),$$
    $$(g : X'/S \to X/S) \mapsto (F(g) : F(X')/F(S) \to F(X)/F(S))$$

  - Let $\mathcal{S} \in \mathrm{Cat}$. Then, for a morphism $f : X' \to X$ with $X, X' \in \mathcal{S}$ we have the functor

    $$C(f) : \mathcal{S}/X' \to \mathcal{S}/X,\ \ Y/X' = (Y, f_1) \mapsto C(f)(Y/X') := (Y, f \circ f_1) \in \mathcal{S}/X,$$
    $$(g : Y_1/X' \to Y_2/X') \mapsto (C(f)(g) := g : Y_1/X \to Y_2/X)$$

  - Let $\mathcal{S} \in \mathrm{Cat}$ a category which admits fiber products. Then, for a morphism $f : X' \to X$ with $X, X' \in \mathcal{S}$, we have the pullback functor

    $$P(f) : \mathcal{S}/X \to \mathcal{S}/X',\ \ Y/X \mapsto P(f)(Y/X) := Y \times_X X'/X' \in \mathcal{S}/X',$$
    $$(g : Y_1/X \to Y_2/X) \mapsto (P(f)(g) := (g \times I) : Y_1 \times_X X' \to Y_2 \times_X X')$$

    which is right adjoint to $C(f) : \mathcal{S}/X' \to \mathcal{S}/X$, and we denote again $P(f) : \mathcal{S}/X' \to \mathcal{S}/X$ the corresponding morphism of (pre)sites.

- Let $\mathcal{C}, \mathcal{I} \in \mathrm{Cat}$. Assume that $\mathcal{C}$ admits fiber products. For $(S_\bullet) \in \mathrm{Fun}(\mathcal{I}^{op}, \mathcal{C})$, we denote by $\mathcal{C}/(S_\bullet) \in \mathrm{Fun}(\mathcal{I}, \mathrm{Cat})$ the diagram of category given by
  - for $I \in \mathcal{I}$, $\mathcal{C}/(S_\bullet)(I) := \mathcal{C}/S_I$,
  - for $r_{IJ} : I \to J$, $\mathcal{C}/(S_\bullet)(r_{IJ}) := P(r_{IJ}) : \mathcal{C}/S_I \to \mathcal{C}/S_J$, where we denoted again $r_{IJ} : S_J \to S_I$ the associated morphism in $\mathcal{C}$.
- Let $(F, G) : \mathcal{C} \leftrightarrows \mathcal{C}'$ an adjonction between two categories.
  - For $X \in C$ and $Y \in C'$, we consider the adjonction isomorphisms
    * $I(F,G)(X,Y) : \mathrm{Hom}(F(X), Y) \to \mathrm{Hom}(X, G(Y))$, $(u : F(X) \to Y) \mapsto (I(F,G)(X,Y)(u) : X \to G(Y))$
    * $I(F,G)(X,Y) : \mathrm{Hom}(X, G(Y)) \to \mathrm{Hom}(F(X), Y)$, $(v : X \to G(Y)) \mapsto (I(F,G)(X,Y)(v) : F(X) \to Y)$.
  - For $X \in \mathcal{C}$, we denote by $\mathrm{ad}(F,G)(X) := I(F,G)(X, F(X))(I_{F(X)}) : X \to G \circ F(X)$.
  - For $Y \in \mathcal{C}'$ we denote also by $\mathrm{ad}(F,G)(Y) := I(F,G)(G(Y), Y)(I_{G(Y)}) : F \circ G(Y) \to Y$.

  Hence,
  - for $u : F(X) \to Y$ a morphism with $X \in C$ and $Y \in C'$, we have $I(F,G)(X,Y)(u) = G(u) \circ \mathrm{ad}(F,G)(X)$,
  - for $v : X \to G(Y)$ a morphism with $X \in C$ and $Y \in C'$, we have $I(F,G)(X,Y)(v) = \mathrm{ad}(F,G)(Y) \circ F(v)$.
- Let $\mathcal{C}$ a category.
  - We denote by $(\mathcal{C}, F)$ the category of filtered objects : $(X, F) \in (\mathcal{C}, F)$ is a sequence $(F^\bullet X)_{\bullet \in \mathbb{Z}}$ indexed by $\mathbb{Z}$ with value in $\mathcal{C}$ together with monomorphisms $a_p : F^p X \hookrightarrow F^{p-1} X \hookrightarrow X$.
  - We denote by $(\mathcal{C}, F, W)$ the category of bifiltered objects : $(X, F, W) \in (\mathcal{C}, F, W)$ is a sequence $(W^\bullet F^\bullet X)_{\bullet, \bullet} \in \mathbb{Z}^2$ indexed by $\mathbb{Z}^2$ with value in $\mathcal{C}$ together with monomorphisms $W^q F^p X \hookrightarrow F^{p-1} X$, $W^q F^p X \hookrightarrow W^{q-1} F^p X$.
- For $\mathcal{C}$ a category and $\Sigma : \mathcal{C} \to \mathcal{C}$ an endofunctor, we denote by $(\mathcal{C}, \Sigma)$ the corresponding category of spectra, whose objects are sequence of objects of $\mathcal{C}$ $(T_i)_{i \in \mathbb{Z}} \in \mathrm{Fun}(\mathbb{Z}, \mathcal{C})$ together with morphisms $s_i : T_i \to \Sigma T_{i+1}$, and whose morphism from (T_i) to (T'_i) are sequence of morphisms $T_i \to T'_i$ which commutes with the s_i.
- Let $\mathcal{A}$ an additive category.
  - We denote by $C(\mathcal{A}) := \mathrm{Fun}(\mathbb{Z}, \mathcal{A})$ the category of (unbounded) complexes with value in $\mathcal{A}$, where we have denoted $\mathbb{Z}$ the category whose set of objects is $\mathbb{Z}$, and whose set of morphism between $m, n \in \mathbb{Z}$ consists of one element (identity) if $n = m$, of one elemement if $n = m + 1$ and is $\emptyset$ in the other cases.
  - We have the full subcategories $C^b(\mathcal{A})$, $C^-(\mathcal{A})$, $C^+(\mathcal{A})$ of $C(\mathcal{A})$ consisting of bounded, resp. bounded above, resp. bounded below complexes.
  - We denote by $K(\mathcal{A}) := \mathrm{Ho}(C(\mathcal{A}))$ the homotopy category of $C(\mathcal{A})$ whose morphisms are equivalent homotopic classes of morphism and by $Ho : C(\mathcal{A}) \to K(\mathcal{A})$ the full homotopy functor. The category $K(\mathcal{A})$ is in the standard way a triangulated category.
- Let $\mathcal{A}$ an additive category.
  - We denote by $C_{fil}(\mathcal{A}) \subset (C(\mathcal{A}), F) = C(\mathcal{A}, F)$ the full additive subcategory of filtered complexes of $\mathcal{A}$ such that the filtration is biregular : for $(A^\bullet, F) \in (C(\mathcal{A}), F)$, we say that F is biregular if $F^\bullet A^r$ is finite for all $r \in \mathbb{Z}$.

- We denote by $C_{2fil}(\mathcal{A}) \subset (C(\mathcal{A}), F, W) = C(\mathcal{A}, F, W)$ the full subcategory of bifiltered complexes of $\mathcal{A}$ such that the filtration is biregular.
- For $A^\bullet \in C(\mathcal{A})$, we denote by $(A^\bullet, F_b) \in (C(\mathcal{A}), F)$ the complex endowed with the trivial filtration (filtration bete) : $F^p A^n = 0$ if $p \geq n+1$ and $F^p A^n = A^n$ if $p \leq n$. Obviously, a morphism $\phi : A^\bullet \to B^\bullet$, with $A^\bullet, B^\bullet \in C(\mathcal{A})$ induces a morphism $\phi : (A^\bullet, F_b) \to (B^\bullet, F_b)$.
- For $(A^\bullet, F) \in C(\mathcal{A}, F)$, we denote by $(A^\bullet, F(r)) \in C(\mathcal{A}, F)$ the filtered complex where the filtration is given by $F^p(A^\bullet, F(r)) := F^{p+r}(A^\bullet, F)$.
- Two morphisms $\phi_1, \phi_2 : (M, F) \to (N, F)$ with $(M, F), (N, F) \in C(\mathcal{A}, F)$ are said to be r-filtered homotopic if there exist a morphism in $\mathrm{Fun}(\mathbb{Z}, (\mathcal{A}, F))$

  $$h : (M, F(r-1))[1] \to (N, F), h := (h^n : (M^{n+1}, F(r-1)) \to (N^n, F))_{n \in \mathbb{Z}},$$

  where $\mathbb{Z}$ have only trivial morphism (i.e. h is a graded morphism but not a morphism of complexes) such that $d'h + hd = \phi_1 - \phi_2$, where d is the differential of M and d' is the differential of N, and we have $h(F^p M^{n+1}) \subset F^{p-r+1} N^n$, note that by definition r does NOT depend on p and n ; we then say that

  $$(h, \phi_1, \phi_2) : (M, F)[1] \to (N, F)$$

  is an r-filtered homotopy. By definition, an r-filtered homotopy $(h, \phi_1, \phi_2) : (M, F)[1] \to (N, F)$ is an r'-filtered homotopy for all $r' \geq r$, and a 1-filtered homotopy is an homotopy of $C(\mathcal{A}, F)$. By definition, an r-filtered homotopy $(h, \phi_1, \phi_2) : (M, F)[1] \to (N, F)$ gives if we forgot filtration an homotopy $(h, \phi_1, \phi_2) : M[1] \to N$ in $C(\mathcal{A})$.
- Two morphisms $\phi_1, \phi_2 : (M, F) \to (N, F)$ with $(M, F), (N, F) \in C(\mathcal{A}, F)$ are said to be ∞-filtered homotopic if there exist $r \in \mathbb{N}$ such that $\phi_1, \phi_2 : (M, F) \to (N, F)$ are r-filtered homotopic. Hence, if $\phi_1, \phi_2 : (M, F) \to (N, F)$ are ∞-filtered homotopic, then $\phi_1, \phi_2 : M \to N$ are homotopic ; of course the converse is NOT true since r does NOT depend on $p, n \in \mathbb{Z}$.
- We will use the fact that by definition if $\phi : M \to N$ with $M, N \in C(\mathcal{A})$ is an homotopy equivalence, then $\phi : (M, F_b) \to \phi(N, F_b)$ is a 2-filtered homotopy equivalence.
- A morphism $\phi : (M, F) \to (N, F)$ with $(M, F), (N, F) \in C(\mathcal{A}, F)$ is said to be an r-filtered homotopy equivalence if there exist a morphism $\phi' : (N, F) \to (M, F)$ such that
  * $\phi' \circ \phi : (M, F) \to (M, F)$ is r-filtered homotopic to I_M and
  * $\phi \circ \phi' : (N, F) \to (N, F)$ is r-filtered homotopic to I_N.

  If $\phi : (M, F) \to (N, F)$ is an r-filtered homotopy equivalence, then it is an s-filtered homotopy equivalence for $s \geq r$. If $\phi : (M, F) \to (N, F)$ is an r-filtered homotopy equivalence, $\phi : M \to N$ is an homotopy equivalence
- A morphism $\phi : (M, F) \to (N, F)$ with $(M, F), (N, F) \in C(\mathcal{A}, F)$ is said to be an ∞-filtered homotopy equivalence if there exist $r \in \mathbb{Z}$ such that $\phi : (M, F) \to (N, F)$ with $(M, F), (N, F) \in C(\mathcal{A}, F)$ is an r-filtered homotopy equivalence. If $\phi : (M, F) \to (N, F)$ is an ∞-filtered homotopy equivalence, $\phi : M \to N$ is an homotopy equivalence ; the converse is NOT true since r does NOT depend on $p, n \in \mathbb{Z}$.
- We denote by $K_r(\mathcal{A}, F) := \mathrm{Ho}_r(C(\mathcal{A}, F))$ the homotopy category of $C(\mathcal{A}, F)$ whose morphisms are r-filtered homotopic equivalence classes of morphism of $C(\mathcal{A}, F)$ and by $\mathrm{Ho}_r : C(\mathcal{A}, F) \to K_r(\mathcal{A}, F)$ the full homotopy functor. However, for $r > 1$ the category $K_r(\mathcal{A}, F)$ with the canonical triangles the standard ones does NOT satify the 2 of 3 axiom of a triangulated category.
- We denote by $K_{fil,r}(\mathcal{A}) := \mathrm{Ho}_r(C_{fil}(\mathcal{A}))$ the homotopy category of $C_{fil}(\mathcal{A})$ whose morphisms are r-filtered homotopic equivalence classes of morphism of $C(\mathcal{A}, F)$ and by $\mathrm{Ho}_r : C_{fil}(\mathcal{A}) \to K_{fil,r}(\mathcal{A})$ the full homotopy functor. However, for $r > 1$ the category $K_{fil,r}(\mathcal{A})$ with the canonical triangles the standard ones does NOT satify the 2 of 3 axiom of a triangulated category.

– We denote by $K_{fil,\infty}(\mathcal{A}) := \mathrm{Ho}_\infty(C_{fil}(\mathcal{A}))$ the homotopy category of $C_{fil}(\mathcal{A})$ whose morphisms are ∞-filtered homotopic equivalence classes of morphism of $C(\mathcal{A}, F)$ and by $\mathrm{Ho}_\infty : C_{fil}(\mathcal{A}) \to K_{fil,\infty}(\mathcal{A})$ the full homotopy functor. However, the category $K_{fil,\infty}(\mathcal{A})$ with the canonical triangles the standard ones does NOT satify the 2 of 3 axiom of a triangulated category.

– We have the Deligne decalage functor

$$\begin{aligned}\mathrm{Dec} : C(\mathcal{A}, F) \to C(\mathcal{A}, F),\ (M, F) \mapsto \mathrm{Dec}(M, F) &:= (M, \mathrm{Dec}\, F), \\ \mathrm{Dec}\, F^p M^n &:= F^{p+n} M^n \cap d^{-1}(F^{p+n+1} M^{n+1})\end{aligned}$$

It is the right adjoint of the shift functor

$$S : C(\mathcal{A}, F) \to C(\mathcal{A}, F),\ (M, F) \mapsto S(M, F) := (M, SF),\ SF^p M^n := F^{p-n} M^n$$

The dual decalage functor

$$\begin{aligned}\mathrm{Dec}^\vee : C(\mathcal{A}, F) \to C(\mathcal{A}, F),\ (M, F) \mapsto \mathrm{Dec}^\vee(M, F) &:= (M, \mathrm{Dec}^\vee F), \\ \mathrm{Dec}^\vee F^p M^n &:= F^{p+n} M^n + d(F^{p+n-1} M^{n+1})\end{aligned}$$

is the left adjoint of the shift functor. Note that $\mathrm{Dec}((M, F)[1]) \neq (\mathrm{Dec}(M, F))[1]$, $\mathrm{Dec}^\vee((M, F)[1]) \neq (\mathrm{Dec}^\vee(M, F))[1]$ and $S((M, F)[1]) \neq (S(M, F))[1]$.

- Let $\mathcal{A}$ be an abelian category. Then the additive category $(\mathcal{A}, F)$ is an exact category which admits kernel and cokernel (but is NOT an abelian category). A morphism $\phi : (M, F) \to (N, F)$ with $(M, F) \in (\mathcal{A}, F)$ is strict if the inclusion $\phi(F^n M) \subset F^n N \cap \mathrm{Im}(\phi)$ is an equality, i.e. if $\phi(F^n M) = F^n N \cap \mathrm{Im}(\phi)$.

- Let $\mathcal{A}$ be an abelian category.

– For $(A^\bullet, F) \in C(\mathcal{A}, F)$, considering $a_p : F^p A^\bullet \hookrightarrow A^\bullet$ the structural monomorphism of of the filtration, we denote by, for $n \in \mathbb{N}$,

$$H^n(A^\bullet, F) \in (\mathcal{A}, F),\ F^p H^n(A^\bullet, F) := \mathrm{Im}(H^n(a_p) : H^n(F^p A^\bullet) \to H^n(A^\bullet)) \subset H^n(A^\bullet)$$

the filtration induced on the cohomology objects of the complex. In the case $(A^\bullet, F) \in C_{fil}(\mathcal{A})$, the spectral sequence $E_r^{p,q}(A^\bullet, F)$ associated to $(A^\bullet, F)$ converge to $\mathrm{Gr}_F^p H^{p+q}(A^\bullet, F)$, that is for all $p, q \in \mathbb{Z}$, there exist $r_{p+q} \in \mathbb{N}$, such that $E_s^{p,q}(A^\bullet, F) = \mathrm{Gr}_F^p H^{p+q}(A^\bullet, F)$ for all $s \leq r_{p+q}$.

– A morphism $m : (A^\bullet, F) \to (B^\bullet, F)$ with $(A^\bullet, F), (B^\bullet, F) \in C(\mathcal{A}, F)$ is said to be a filtered quasi-isomorphism if for all $n, p \in \mathbb{Z}$,

$$H^n \mathrm{Gr}_F^p(m) : H^n(\mathrm{Gr}_F^p A^\bullet) \xrightarrow{\sim} H^n(\mathrm{Gr}_F^p B^\bullet)$$

is an isomorphism in $\mathcal{A}$. Consider a commutative diagram in $C(\mathcal{A}, F)$

$$\begin{tikzcd}
(A^\bullet, F) \arrow[r, "m"] \arrow[d, "\phi"'] & (B^\bullet, F) \arrow[r] \arrow[d, "\psi"'] & \mathrm{Cone}^{i_2}(m) = ((A^\bullet, F)[1] \oplus (B^\bullet, F), d, d' \overset{p_1}{-} m) \arrow[r] \arrow[d, "{(\phi[1],\psi)}"] & (A^\bullet, F)[1] \arrow[d, "{\phi[1]}"] \\
(A'^\bullet, F) \arrow[r, "m'"] & (B'^\bullet, F) \arrow[r] & \mathrm{Cone}^{i_2}(m) = ((A'^\bullet, F)[1] \oplus (B'^\bullet, F), d, d' \overset{p_1}{-} m') \arrow[r] & (A'^\bullet, F)[1]
\end{tikzcd}$$

If ϕ and ψ are filtered quasi-isomorphisms, then $(\phi[1], \psi)$ is an filtered quasi-isomorphim.

– If two morphisms $\phi_1, \phi_2 : (M, F) \to (N, F)$ with $(M, F), (N, F) \in C(\mathcal{A}, F)$ are r-filtered homotopic, then for all $p, q \in \mathbb{Z}$ and $s \geq r$.

$$E_s^{p,q}(\phi_1) = E_s^{p,q}(\phi_2) : E_s^{p,q}(M, F) \to E_s^{p,q}(M, F).$$

Hence if $\phi : (M, F) \to (N, F)$ with $(M, F), (N, F) \in C(\mathcal{A}, F)$ is an r-filtered homotopy equivalence then for all $p, q \in \mathbb{Z}$ and $s \geq r$.

$$E_r^{p,q}(\phi) : E_r^{p,q}(M, F) \xrightarrow{\sim} E_r^{p,q}(N, F)$$

is an isomorphism in $\mathcal{A}$.

– Let $r \in \mathbb{N}$. A morphism $m : (A^\bullet, F) \to (B^\bullet, F)$ with $(A^\bullet, F), (B^\bullet, F) \in C(\mathcal{A}, F)$ is said to be an r-filtered quasi-isomorphism if there exist an r-filtered homotopy

$$(h, m, m') : (A^\bullet, F)[1] \to (B^\bullet, F)$$

such that $m' : (A^\bullet, F) \to (B^\bullet, F)$ is a filtered quasi-isomorphism Note that our definition is stronger then the one given in [9] in order to get a multiplicative system. Indeed, if $m : (A^\bullet, F) \to (B^\bullet, F)$ with $(A^\bullet, F), (B^\bullet, F) \in C(\mathcal{A}, F)$ is an r-filtered quasi-isomorphism then for all $p, q \in \mathbb{Z}$ and $s \geq r$,

$$E_r^{p,q}(m) : E_r^{p,q}(A^\bullet, F) \xrightarrow{\sim} E_r^{p,q}(B^\bullet, F)$$

is an isomorphism in $\mathcal{A}$, but the converse is NOT true. If a morphism $m : (A^\bullet, F) \to (B^\bullet, F)$, with $(A^\bullet, F), (B^\bullet, F) \in C_{fil}(\mathcal{A})$ is an r-filtered quasi-isomorphism, then for all $n \in \mathbb{Z}$

$$H^n(m) : H^n(A^\bullet, F) \xrightarrow{\sim} H^n(B^\bullet, F)$$

is a filtered isomorphism, i.e. an isomorphism in $(\mathcal{A}, F)$. The converse is true if there exist $N_1, N_2 \in \mathbb{Z}$ such that $H^n(A^\bullet) = H^n(B^\bullet) = 0$ for $n \leq N_1$ or $n \geq N_2$. A filtered quasi-isomorphism is obviously a 1-filtered quasi-isomorphism. However for $r > 1$, the r-filtered quasi-isomorphism does NOT satisfy the 2 of 3 property for morphisms of canonical triangles.

– A morphism $m : (A^\bullet, F) \to (B^\bullet, F)$ with $(A^\bullet, F), (B^\bullet, F) \in C(\mathcal{A}, F)$ is said to be an ∞-filtered quasi-isomorphism if there exist $r \in \mathbb{N}$ such that $m : (A^\bullet, F) \to (B^\bullet, F)$ an r-filtered quasi-isomorphism. If a morphism $m : (A^\bullet, F) \to (B^\bullet, F)$, with $(A^\bullet, F), (B^\bullet, F) \in C_{fil}(\mathcal{A})$ is an ∞-filtered quasi-isomorphism, then for all $n \in \mathbb{Z}$

$$H^n(m) : H^n(A^\bullet, F) \xrightarrow{\sim} H^n(B^\bullet, F)$$

is a filtered isomorphism. The converse is true if there exist $N_1, N_2 \in \mathbb{Z}$ such that $H^n(A^\bullet) = H^n(B^\bullet) = 0$ for $n \leq N_1$ or $n \geq N_2$. The ∞-filtered quasi-isomorphism does NOT satisfy the 2 of 3 property for morphisms of canonical triangles. If $m : (A^\bullet, F) \to (B^\bullet, F)$ is such that $m : A^\bullet \to B^\bullet$ is a quasi-isomorphism but $m : (A^\bullet, F) \to (B^\bullet, F)$ is not an ∞-filtered quasi-isomorphism, then it induces an isomorphisms $H^n(m) : H^n(A^\bullet) \xrightarrow{\sim} H^n(B^\bullet)$, hence injective maps

$$H^n(m) : F^p H^n(A^\bullet, F) \hookrightarrow F^p H^n(B^\bullet, F)$$

which are not isomorphism (the non strict case). If $m : (A^\bullet, F) \to (B^\bullet, F)$ is such that $m : A^\bullet \to B^\bullet$ is a quasi-isomorphism but $m : (A^\bullet, F) \to (B^\bullet, F)$ is not an ∞-filtered quasi-isomorphism (the non strict case), then $H^n \operatorname{Cone}(m) = 0$ for all $n \in \mathbb{N}$, hence $\operatorname{Cone}(m) \to 0$ is an ∞-filtered quasi-isomorphism ; this shows that the ∞-filtered quasi-isomorphism does NOT satisfy the 2 of 3 property for morphism of canonical triangles.

• Let $\mathcal{A}$ be an abelian category.

  – We denote by $D(\mathcal{A})$ the localization of $K(\mathcal{A})$ with respect to the quasi-isomorphisms and by $D : K(\mathcal{A}) \to D(\mathcal{A})$ the localization functor. The category $D(\mathcal{A})$ is a triangulated category in the unique way such that D a triangulated functor.

- We denote by $D_{fil}(\mathcal{A})$ the localization of $K_{fil}(\mathcal{A})$ with respect to the filtered quasi-isomorphisms and by $D : K_{fil}(\mathcal{A}) \to D_{fil}(\mathcal{A})$ the localization functor.

- Let $\mathcal{A}$ be an abelian category. We denote by $\mathrm{Inj}(A) \subset A$ the full subcategory of injective objects, and by $\mathrm{Proj}(A) \subset A$ the full subcategory of projective objects.

- For $\mathcal{S} \in \mathrm{Cat}$ a small category, we denote by
  - $\mathrm{PSh}(\mathcal{S}) := \mathrm{PSh}(\mathcal{S}, \mathrm{Ab}) := \mathrm{Fun}(\mathcal{S}, \mathrm{Ab})$ the category of presheaves on $\mathcal{S}$, i.e. the category of presheaves of abelian groups on $\mathcal{S}$,
  - $\mathrm{PSh}(\mathcal{S}, \mathrm{Ring}) := \mathrm{Fun}(\mathcal{S}, \mathrm{Ring})$ the category of presheaves of ring on $\mathcal{S}$, and $\mathrm{PSh}(\mathcal{S}, \mathrm{cRing}) \subset \mathrm{PSh}(\mathcal{S}, \mathrm{Ring})$ the full subcategory of presheaves of commutative ring.
  - for $F \in \mathrm{PSh}(\mathcal{S})$ and $X \in \mathcal{S}$, $F(X) = \Gamma(X, F)$ the sections on X and for $h : X' \to X$ a morphism with $X, X' \in \mathcal{S}$, $F(h) := F^{X,Y}(h) : F(X) \to F(X')$ the morphism of abelian groups,
  - $C(\mathcal{S}) = \mathrm{PSh}(\mathcal{S}, C(\mathbb{Z})) = C(\mathrm{PSh}(\mathcal{S})) = \mathrm{PSh}(\mathcal{S} \times \mathbb{Z})$ the big abelian category of complexes of presheaves on $\mathcal{S}$ with value in abelian groups,
  - $K(\mathcal{S}) := K(\mathrm{PSh}(\mathcal{S})) = \mathrm{Ho}(C(\mathcal{S}))$ In particular, we have the full homotopy functor $Ho : C(\mathcal{S}) \to K(\mathcal{S})$,
  - $C_{(2)fil}(\mathcal{S}) := C_{(2)fil}(\mathrm{PSh}(\mathcal{S})) \subset C(\mathrm{PSh}(\mathcal{S}), F, W)$ the big abelian category of (bi)filtered complexes of presheaves on $\mathcal{S}$ with value in abelian groups such that the filtration is biregular, and $\mathrm{PSh}_{(2)fil}(\mathcal{S}) = (\mathrm{PSh}(\mathcal{S}), F, W)$,
  - $K_{fil}(\mathcal{S}) := K(\mathrm{PSh}_{fil}(\mathcal{S})) = \mathrm{Ho}(C_{fil}(\mathcal{S}))$.

  For $f : \mathcal{T} \to \mathcal{S}$ a morphism a presite with $\mathcal{T}, \mathcal{S} \in \mathrm{Cat}$, given by the functor $P(f) : \mathcal{S} \to \mathcal{T}$, we will consider the adjonctions given by the direct and inverse image functors :

  - $(f^*, f_*) = (f^{-1}, f_*) : \mathrm{PSh}(\mathcal{S}) \leftrightarrows \mathrm{PSh}(\mathcal{T})$, which induces $(f^*, f_*) : C(\mathcal{S}) \leftrightarrows C(\mathcal{T})$, we denote, for $F \in C(\mathcal{S})$ and $G \in C(\mathcal{T})$ by

    $$\mathrm{ad}(f^*, f_*)(F) : F \to f_* f^* F \ , \ \mathrm{ad}(f^*, f_*)(G) : f^* f_* G \to G$$

    the adjonction maps,
  - $(f_*, f^{\perp}) : \mathrm{PSh}(\mathcal{T}) \leftrightarrows \mathrm{PSh}(\mathcal{S})$, which induces $(f_*, f^{\perp}) : C(\mathcal{T}) \leftrightarrows C(\mathcal{S})$, we denote for $F \in C(\mathcal{S})$ and $G \in C(\mathcal{T})$ by

    $$\mathrm{ad}(f_*, f^{\perp})(F) : G \to f^{\perp} f_* G \ , \ \mathrm{ad}(f_*, f^{\perp})(G) : f_* f^{\perp} F \to F$$

    the adjonction maps.

- For $(\mathcal{S}, O_S) \in \mathrm{RCat}$ a ringed topos, we denote by
  - $\mathrm{PSh}_{O_S}(\mathcal{S})$ the category of presheaves of O_S modules on $\mathcal{S}$, whose objects are $\mathrm{PSh}_{O_S}(\mathcal{S})^0 := \{(M, m), M \in \mathrm{PSh}(\mathcal{S}), m : M \otimes O_S \to M\}$, together with the forgetful functor $o : \mathrm{PSh}(\mathcal{S}) \to \mathrm{PSh}_{O_S}(\mathcal{S})$,
  - $C_{O_S}(\mathcal{S}) = C(\mathrm{PSh}_{O_S}(\mathcal{S}))$ the big abelian category of complexes of presheaves of O_S modules on $\mathcal{S}$,
  - $K_{O_S}(\mathcal{S}) := K(\mathrm{PSh}_{O_S}(\mathcal{S})) = \mathrm{Ho}(C_{O_S}(\mathcal{S}))$, in particular, we have the full homotopy functor $Ho : C_{O_S}(\mathcal{S}) \to K_{O_S}(\mathcal{S})$,
  - $C_{O_S(2)fil}(\mathcal{S}) := C_{(2)fil}(\mathrm{PSh}_{O_S}(\mathcal{S})) \subset C(\mathrm{PSh}_{O_S}(\mathcal{S}), F, W)$, the big abelian category of (bi)filtered complexes of presheaves of O_S modules on $\mathcal{S}$ such that the filtration is biregular and $\mathrm{PSh}_{O_S(2)fil}(\mathcal{S}) = (\mathrm{PSh}_{O_S}(\mathcal{S}), F, W)$.

- For $\mathcal{S}_\bullet \in \mathrm{Fun}(\mathcal{I},\mathrm{Cat})$ a diagram of (pre)sites, with $\mathcal{I} \in \mathrm{Cat}$ a small category, we denote by
  - $\Gamma\mathcal{S}_\bullet \in \mathrm{Cat}$ the associated diagram category
    - whose objects are $\Gamma\mathcal{S}_\bullet^0 := \{(X_I, u_{IJ})_{I\in\mathcal{I}}\}$, with $X_I \in \mathcal{S}_I$, and for $r_{IJ} : I \to J$ with $I, J \in \mathcal{I}$, $u_{IJ} : X_J \to r_{IJ}(X_I)$ are morphism in $\mathcal{S}_J$ noting again $r_{IJ} : \mathcal{S}_I \to \mathcal{S}_J$ the associated functor,
    - whose morphism are $m = (m_I) : (X_I, u_{IJ}) \to (X'_I, v_{IJ})$ satisfying $v_{IJ} \circ m_I = r_{IJ}(m_J) \circ u_{IJ}$ in $\mathcal{S}_J$,
  - $\mathrm{PSh}(\mathcal{S}_\bullet) := \mathrm{PSh}(\Gamma\mathcal{S}_\bullet, \mathrm{Ab})$ the category of presheaves on $\mathcal{S}_\bullet$,
    - whose objects are $\mathrm{PSh}(\mathcal{S}_\bullet)^0 := \{(F_I, u_{IJ})_{I\in\mathcal{I}}\}$, with $F_I \in \mathrm{PSh}(\mathcal{S}_I)$, and for $r_{IJ} : I \to J$ with $I, J \in \mathcal{I}$, $u_{IJ} : F_I \to r_{IJ*}F_J$ are morphism in $\mathrm{PSh}(\mathcal{S}_I)$, noting again $r_{IJ} : \mathcal{S}_J \to \mathcal{S}_I$ the associated morphism of presite,
    - whose morphism are $m = (m_I) : (F_I, u_{IJ}) \to (G_I, v_{IJ})$ satisfying $v_{IJ} \circ m_I = r_{IJ*}m_J \circ u_{IJ}$ in $\mathrm{PSh}(\mathcal{S}_I)$,
  - $\mathrm{PSh}(\mathcal{S}_\bullet, \mathrm{Ring}) := \mathrm{PSh}(\Gamma\mathcal{S}_\bullet, \mathrm{Ring})$ the category of presheaves of ring on $\mathcal{S}_\bullet$ given in the same way, and $\mathrm{PSh}(\mathcal{S}_\bullet, \mathrm{cRing}) \subset \mathrm{PSh}(\mathcal{S}_\bullet, \mathrm{Ring})$ the full subcategory of presheaves of commutative ring.
  - $C(\mathcal{S}_\bullet) := C(\mathrm{PSh}(\mathcal{S}_\bullet))$ the big abelian category of complexes of presheaves on $\mathcal{S}_\bullet$ with value in abelian groups,
  - $K(\mathcal{S}_\bullet) := K(\mathrm{PSh}(\mathcal{S}_\bullet)) = \mathrm{Ho}(C(\mathcal{S}_\bullet))$, in particular, we have the full homotopy functor $Ho : C(\mathcal{S}_\bullet) \to K(\mathcal{S}_\bullet)$,
  - $C_{(2)fil}(\mathcal{S}_\bullet) := C_{(2)fil}(\mathrm{PSh}(\mathcal{S}_\bullet)) \subset C(\mathrm{PSh}(\mathcal{S}_\bullet), F, W)$ the big abelian category of (bi)filtered complexes of presheaves on $\mathcal{S}_\bullet$ with value in abelian groups such that the filtration is biregular, and $\mathrm{PSh}_{(2)fil}(\mathcal{S}_\bullet) = (\mathrm{PSh}(\mathcal{S}_\bullet), F, W)$, by definition $C_{(2)fil}(\mathcal{S}_\bullet)$ is the category
    - whose objects are $C_{(2)fil}(\mathcal{S}_\bullet)^0 := \{((F_I, F, W), u_{IJ})_{I\in\mathcal{I}}\}$, with $(F_I, F, W) \in C_{(2)fil}(\mathcal{S}_I)$, and for $r_{IJ} : I \to J$ with $I, J \in \mathcal{I}$, $u_{IJ} : (F_I, F, W) \to r_{IJ*}(F_J, F, W)$ are morphism in $C_{(2)fil}(\mathcal{S}_I)$, noting again $r_{IJ} : \mathcal{S}_J \to \mathcal{S}_I$ the associated morphism of presite,
    - whose morphism are $m = (m_I) : ((F_I, F, W), u_{IJ}) \to ((G_I, F, W), v_{IJ})$ satisfying $v_{IJ} \circ m_I = r_{IJ*}m_J \circ u_{IJ}$ in $C_{(2)fil}(\mathcal{S}_I)$,
  - $K_{fil}(\mathcal{S}_\bullet) := K(\mathrm{PSh}_{fil}(\mathcal{S}_\bullet)) = \mathrm{Ho}(C_{fil}(\mathcal{S}_\bullet))$,
  - $K_{fil,r}(\mathcal{S}_\bullet) := K_r(\mathrm{PSh}_{fil}(\mathcal{S}_\bullet)) = \mathrm{Ho}_r(C_{fil}(\mathcal{S}_\bullet))$.

Let $\mathcal{I}, \mathcal{I}' \in \mathrm{Cat}$ be small categories. Let $(f_\bullet, s) : \mathcal{T}_\bullet \to \mathcal{S}_\bullet$ a morphism a diagrams of (pre)site with $\mathcal{T}_\bullet \in \mathrm{Fun}(\mathcal{I}, \mathrm{Cat}), \mathcal{S}_\bullet \in \mathrm{Fun}(\mathcal{I}', \mathrm{Cat})$, which is by definition given by a functor $s : \mathcal{I} \to \mathcal{I}'$ and morphism of functor $P(f_\bullet) : \mathcal{S}_{s(\bullet)} := \mathcal{S}_\bullet \circ s \to \mathcal{T}_\bullet$. Here, we denote for short, $\mathcal{S}_{s(\bullet)} := \mathcal{S}_\bullet \circ s \in \mathrm{Fun}(\mathcal{I}, \mathrm{Cat})$. We have then, for $r_{IJ} : I \to J$ a morphism, with $I, J \in \mathcal{I}$, a commutative diagram in Cat

$$D_{fIJ} := \begin{tikzcd} \mathcal{S}_{s(J)} \arrow[r, "r^s_{IJ}"] & \mathcal{S}_{s(I)} \\ \mathcal{T}_J \arrow[u, "f_J"] \arrow[r, "r^t_{IJ}"] & \mathcal{T}_I \arrow[u, "f_I"'] \end{tikzcd} .$$

We will consider the the adjonction given by the direct and inverse image functors :

$$\begin{aligned} ((f_\bullet, s)^*, (f_\bullet, s)^*) = ((f_\bullet, s)^{-1}, (f_\bullet, s)_*) : \mathrm{PSh}(\mathcal{S}_{s(\bullet)}) \leftrightarrows \mathrm{PSh}(\mathcal{T}_\bullet), \\ F = (F_I, u_{IJ}) \mapsto (f_\bullet, s)^* F := (f_I^* F_I, T(D_{fIJ})(F_J) \circ f_I^* u_{IJ}), \\ G = (G_I, v_{IJ}) \mapsto (f_\bullet, s)_* G := (f_{I*} G_I, f_{I*} v_{IJ}). \end{aligned}$$

It induces the adjonction $((f_\bullet,s)^*,(f_\bullet,s)_*):C(\mathcal{S}_{s(\bullet)})\leftrightarrows C(\mathcal{T}_\bullet)$. We denote, for $(F_I,u_{IJ})\in C(\mathcal{S}_{s(\bullet)})$ and $(G_I,v_{IJ})\in C(\mathcal{T}_\bullet)$ by

$$\mathrm{ad}((f_\bullet,s)^*,(f_\bullet,s)_*)((F_I,u_{IJ})):(F_I,u_{IJ})\to(f_\bullet,s)_*(f_\bullet,s)^*(F_I,u_{IJ}),$$
$$\mathrm{ad}((f_\bullet,s)^*,(f_\bullet,s)_*)((G_I,v_{IJ})):(f_\bullet,s)^*(f_\bullet,s)_*(G_I,v_{IJ})\to(G_I,v_{IJ})$$

the adjonction maps.

- Let $\mathcal{I}\in\mathrm{Cat}$ a small category. For $(\mathcal{S}_\bullet,O_{S_\bullet})\in\mathrm{Fun}(\mathcal{I},\mathrm{RCat})$ a diagram of ringed topos, we denote by
  - $\mathrm{PSh}_{O_{S_\bullet}}(\mathcal{S}_\bullet):=\mathrm{PSh}_{O_{\Gamma S_\bullet}}(\Gamma\mathcal{S}_\bullet)$ the category of presheaves of modules on $(\mathcal{S}_\bullet,O_{S_\bullet})$,
    * whose objects are $\mathrm{PSh}_{O_{S_\bullet}}(\mathcal{S}_\bullet)^0:=\{(F_I,u_{IJ})_{I\in\mathcal{I}}\}$, with $F_I\in\mathrm{PSh}_{O_{S_I}}(\mathcal{S}_I)$, and for $r_{IJ}:I\to J$ with $I,J\in\mathcal{I}$, $u_{IJ}:F_I\to r_{IJ*}F_J$ are morphism in $\mathrm{PSh}_{O_{S_I}}(\mathcal{S}_I)$, noting again $r_{IJ}:\mathcal{S}_J\to\mathcal{S}_I$ the associated morphism of presite,
    * whose morphism are $m=(m_I):(F_I,u_{IJ})\to(G_I,v_{IJ})$ satisfying $v_{IJ}\circ m_I=r_{IJ*}m_J\circ u_{IJ}$ in $\mathrm{PSh}_{O_{S_I}}(\mathcal{S}_I)$,
  - $C_{O_{S_\bullet}}(\mathcal{S}_\bullet):=C(\mathrm{PSh}_{O_{S_\bullet}}(\mathcal{S}_\bullet))$,
  - $K_{O_{S_\bullet}}(\mathcal{S}_\bullet):=K(\mathrm{PSh}_{O_{S_\bullet}}(\mathcal{S}_\bullet))=\mathrm{Ho}(C_{O_{S_\bullet}}(\mathcal{S}_\bullet))$, in particular, we have the full homotopy functor $Ho:C(\mathcal{S}_\bullet)\to K(\mathcal{S}_\bullet)$,
  - $C_{O_{S_\bullet}(2)fil}(\mathcal{S}_\bullet):=C_{O_{S_\bullet}(2)fil}(\mathrm{PSh}(\mathcal{S}_\bullet))\subset C(\mathrm{PSh}_{O_{S_\bullet}}(\mathcal{S}_\bullet),F,W)$ the big abelian category of (bi)filtered complexes of presheaves of modules on $(\mathcal{S}_\bullet,O_{S_\bullet})$ such that the filtration is biregular, and $\mathrm{PSh}_{O_{S_\bullet}(2)fil}(\mathcal{S}_\bullet)=(\mathrm{PSh}_{O_{S_\bullet}}(\mathcal{S}_\bullet),F,W)$, by definition $C_{O_{S_\bullet}(2)fil}(\mathcal{S}_\bullet)$ is the category
    * whose objects are $C_{O_{S_\bullet}(2)fil}(\mathcal{S}_\bullet)^0:=\{((F_I,F,W),u_{IJ})_{I\in\mathcal{I}}\}$, with $(F_I,F,W)\in C_{O_{S_I}(2)fil}(\mathcal{S}_I)$, and for $r_{IJ}:I\to J$ with $I,J\in\mathcal{I}$, $u_{IJ}:(F_I,F,W)\to r_{IJ*}(F_J,F,W)$ are morphism in $C_{O_{S_I}(2)fil}(\mathcal{S}_I)$, noting again $r_{IJ}:\mathcal{S}_J\to\mathcal{S}_I$ the associated morphism of presite,
    * whose morphism are $m=(m_I):((F_I,F,W),u_{IJ})\to((G_I,F,W),v_{IJ})$ satisfying $v_{IJ}\circ m_I=r_{IJ*}m_J\circ u_{IJ}$ in $C_{O_{S_I}(2)fil}(\mathcal{S}_I)$,
  - $K_{O_{S_\bullet}(2)fil}(\mathcal{S}_\bullet):=K(\mathrm{PSh}_{O_{S_\bullet}(2)fil}(\mathcal{S}_\bullet))=\mathrm{Ho}(C_{O_{S_\bullet}(2)fil}(\mathcal{S}_\bullet))$.
  - $K_{O_{S_\bullet}(2)fil,r}(\mathcal{S}_\bullet):=K_r(\mathrm{PSh}_{O_{S_\bullet}(2)fil}(\mathcal{S}_\bullet))=\mathrm{Ho}(C_{O_{S_\bullet}(2)fil,r}(\mathcal{S}_\bullet))$.
- Let $\mathcal{S}\in\mathrm{Cat}$. For $\Sigma:C(\mathcal{S})\to C(\mathcal{S})$ an endofunctor, we denote by $C_\Sigma(\mathcal{S})=(C(\mathcal{S}),\Sigma)$ the corresponding category of spectra.
- Denote by $\mathrm{Sch}\subset\mathrm{RTop}$ the full subcategory of schemes. For a field k, we consider $\mathrm{Sch}/k:=\mathrm{Sch}/\mathrm{Spec}\,k$ the category of schemes over $\mathrm{Spec}\,k$. We then denote by
  - $\mathrm{Var}(k)\subset\mathrm{Sch}/k$ the full subcategory consisting of algebraic varieties over k, i.e. schemes of finite type over k,
  - $\mathrm{PVar}(k)\subset\mathrm{QPVar}(k)\subset\mathrm{Var}(k)$ the full subcategories consisting of quasi-projective varieties and projective varieties respectively,
  - $\mathrm{PSmVar}(k)\subset\mathrm{SmVar}(k)\subset\mathrm{Var}(k)$ the full subcategories consisting of smooth varieties and smooth projective varieties respectively.

  A morphism $h:U\to S$ with $U,S\in\mathrm{Var}(\mathbb{C})$ is said to be smooth if it is flat with smooth fibers. A morphism $r:U\to X$ with $U,X\in\mathrm{Var}(\mathbb{C})$ is said to be etale if it is non ramified and flat. In particular an etale morphism $r:U\to X$ with $U,X\in\mathrm{Var}(\mathbb{C})$ is smooth and quasi-finite (i.e. the fibers are either the empty set or a finite subset of X)

- Denote by Top^2 the category whose set of objects is

$$(\mathrm{Top}^2)^0 := \{(X,Z),\ Z\subset X \text{ closed}\} \subset \mathrm{Top}\times\mathrm{Top}$$

and whose set of morphism between $(X_1,Z_1),(X_2,Z_2)\in\mathrm{Top}^2$ is

$$\mathrm{Hom}_{\mathrm{Top}^2}((X_1,Z_1),(X_2,Z_2)) := \left\{(f:X_1\to X_2),\ \text{s.t. } Z_1\subset f^{-1}(Z_2)\right\}\subset\mathrm{Hom}_{\mathrm{Top}}(X_1,X_2)$$

For $S\in\mathrm{Top}$, $\mathrm{Top}^2/S := \mathrm{Top}^2/(S,S)$ is then by definition the category whose set of objects is

$$(\mathrm{Top}^2/S)^0 := \{((X,Z),h), h:X\to S,\ Z\subset X \text{ closed }\}\subset\mathrm{Top}/S\times\mathrm{Top}$$

and whose set of morphisms between $(X_1,Z_1)/S=((X_1,Z_1),h_1),(X_2,Z_2)/S=((X_2,Z_2),h_2)\in\mathrm{Top}^2/S$ is the subset

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{Top}^2/S}((X_1,Z_1)/S,(X_2,Z_2)/S) := \\ &\left\{(f:X_1\to X_2),\ \text{s.t. } h_1\circ f=h_2 \text{ and } Z_1\subset f^{-1}(Z_2)\right\}\subset\mathrm{Hom}_{\mathrm{RTop}}(X_1,X_2)\end{aligned}$$

We denote by

$$\mu_S:\mathrm{Top}^{2,pr}/S := \{((Y\times S,Z),p), p:Y\times S\to S,\ Z\subset Y\times S \text{ closed }\}\hookrightarrow\mathrm{Top}^2/S$$

the full subcategory whose objects are those with $p:Y\times S\to S$ a projection, and again $\mu_S:\mathrm{Top}^2/S\to\mathrm{Top}^{2,pr}/S$ the corresponding morphism of sites. We denote by

$$\begin{aligned}&\mathrm{Gr}_S^{12}:\mathrm{Top}/S\to\mathrm{Top}^{2,pr}/S,\ X/S\mapsto\mathrm{Gr}_S^{12}(X/S):=(X\times S,\bar{X})/S,\\ &(g:X/S\to X'/S)\mapsto\mathrm{Gr}_S^{12}(g):=(g\times I_S:(X\times S,\bar{X})\to(X'\times S,\bar{X}'))\end{aligned}$$

the graph functor, $X\hookrightarrow X\times S$ being the graph embedding (which is a closed embedding if X is separated), and again $\mathrm{Gr}_S^{12}:\mathrm{Top}^{2,pr}/S\to\mathrm{Top}/S$ the corresponding morphism of sites.

- Denote by RTop^2 the category whose set of objects is

$$(\mathrm{RTop}^2)^0 := \{((X,O_X),Z),\ Z\subset X \text{ closed}\}\subset\mathrm{RTop}\times\mathrm{Top}$$

and whose set of morphism between $((X_1,O_{X_1}),Z_1),((X_2,O_{X_2}),Z_2)\in\mathrm{RTop}^2$ is

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{RTop}^2}(((X_1,O_{X_1}),Z_1),((X_2,O_{X_2}),Z_2)) := \\ &\left\{(f:(X_1,O_{X_1})\to(X_2,O_{X_2})),\ \text{s.t. } Z_1\subset f^{-1}(Z_2)\right\}\subset\mathrm{Hom}_{\mathrm{RTop}}((X_1,O_{X_1}),(X_2,O_{X_2}))\end{aligned}$$

For $(S,O_S)\in\mathrm{RTop}$, $\mathrm{RTop}^2/(S,O_S) := \mathrm{RTop}^2/((S,O_S),S)$ is then by definition the category whose set of objects is

$$\begin{aligned}&(\mathrm{RTop}^2/(S,O_S))^0 := \\ &\{(((X,O_X),Z),h), h:(X,O_X)\to(S,O_S),\ Z\subset X \text{ closed }\}\subset\mathrm{RTop}/(S,O_S)\times\mathrm{Top}\end{aligned}$$

and whose set of morphisms between $(((X_1,O_{X_1}),Z_1),h_1),(((X_2,O_{X_2}),Z_2),h_2)\in\mathrm{RTop}^2/(S,O_S)$ is the subset

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{RTop}^2/(S,O_S)}(((X_1,O_{X_1}),Z_1)/(S,O_S),((X_2,O_{X_2}),Z_2)/(S,O_S)) := \\ &\left\{(f:(X_1,O_{X_1})\to(X_2,O_{X_2})),\ \text{s.t. } h_1\circ f=h_2 \text{ and } Z_1\subset f^{-1}(Z_2)\right\}\\ &\subset\mathrm{Hom}_{\mathrm{RTop}}((X_1,O_{X_1}),(X_2,O_{X_2}))\end{aligned}$$

We denote by

$$\mu_S:\mathrm{RTop}^{2,pr}/S := \{(((Y\times S,q^*O_Y\otimes p^*O_S),Z),p), p:Y\times S\to S,\ Z\subset Y\times S \text{ closed }\}\hookrightarrow\mathrm{RTop}^2/S$$

the full subcategory whose objects are those with $p : Y \times S \to S$ is a projection, and again $\mu_S : \mathrm{RTop}^2/S \to \mathrm{RTop}^{2,pr}/S$ the corresponding morphism of sites. We denote by

$$\begin{aligned} \mathrm{Gr}^{12}_S : \mathrm{RTop}/S \to \mathrm{RTop}^{2,pr}/S, \\ (X, O_X)/(S, O_S) \mapsto \mathrm{Gr}^{12}_S((X, O_X)/(S, O_S)) := ((X \times S, q^*O_X \otimes p^*O_S), \bar{X})/(S, O_S), \\ (g : (X, O_X)/(S, O_S) \to (X', O_{X'})/(S, O_S)) \mapsto \\ \mathrm{Gr}^{12}_S(g) := (g \times I_S : ((X \times S, q^*O_X \otimes p^*O_S), \bar{X}) \to ((X' \times S, q^*O_X \otimes p^*O_S), \bar{X}')) \end{aligned}$$

the graph functor, $X \hookrightarrow X \times S$ being the graph embedding (which is a closed embedding if X is separated), $p : X \times S \to S$, $q : X \times S \to X$ the projections, and again $\mathrm{Gr}^{12}_S : \mathrm{RTop}^{2,pr}/S \to \mathrm{RTop}/S$ the corresponding morphism of sites.

- We denote by $\mathrm{Sch}^2 \subset \mathrm{RTop}^2$ the full subcategory such that the first factors are schemes. For a field k, we denote by $\mathrm{Sch}^2/k := \mathrm{Sch}^2/(\mathrm{Spec}\, k, \{\mathrm{pt}\})$ and by
  - $\mathrm{Var}(k)^2 \subset \mathrm{Sch}^2/k$ the full subcategory such that the first factors are algebraic varieties over k, i.e. schemes of finite type over k,
  - $\mathrm{PVar}(k)^2 \subset \mathrm{QPVar}(k)^2 \subset \mathrm{Var}(k)^2$ the full subcategories such that the first factors are quasi-projective varieties and projective varieties respectively,
  - $\mathrm{PSmVar}(k)^2 \subset \mathrm{SmVar}(k)^2 \subset \mathrm{Var}(k)^2$ the full subcategories such that the first factors are smooth varieties and smooth projective varieties respectively.

  In particular we have, for $S \in \mathrm{Var}(k)$, the graph functor

  $$\begin{aligned} \mathrm{Gr}^{12}_S : \mathrm{Var}(k)/S \to \mathrm{Var}(k)^{2,pr}/S,\ X/S \mapsto \mathrm{Gr}^{12}_S(X/S) := (X \times S, X)/S, \\ (g : X/S \to X'/S) \mapsto \mathrm{Gr}^{12}_S(g) := (g \times I_S : (X \times S, X) \to (X' \times S, X')) \end{aligned}$$

  the graph embedding $X \hookrightarrow X \times S$ is a closed embedding since X is separated in the subcategory of schemes $\mathrm{Sch} \subset \mathrm{RTop}$, and again $\mathrm{Gr}^{12}_S : \mathrm{Var}(k)^{2,pr}/S \to \mathrm{Var}(k)/S$ the corresponding morphism of sites.

- Denote by $\mathrm{CW} \subset \mathrm{Top}$ the full subcategory of CW complexes, by $\mathrm{CS} \subset \mathrm{CW}$ the full subcategory of Δ complexes, by $\mathrm{TM}(\mathbb{R}) \subset \mathrm{CW}$ the full subcategory of topological (real) manifolds which admits a CW structure (a topological manifold admits a CW structure if it admits a differential structure) and by $\mathrm{Diff}(\mathbb{R}) \subset \mathrm{RTop}$ the full subcategory of differentiable (real) manifold. We denote by $\mathrm{CW}^2 \subset \mathrm{Top}^2$ the full subcategory such that the first factors are CW complexes, by $\mathrm{TM}(\mathbb{R})^2 \subset \mathrm{CW}^2$ the full subcategory such that the first factors are topological (real) manifolds and by $\mathrm{Diff}(\mathbb{R})^2 \subset \mathrm{RTop}^2$ the full subcategory such that the first factors are differentiable (real) manifold.

- Denote by $\mathrm{AnSp}(\mathbb{C}) \subset \mathrm{RTop}$ the full subcategory of analytic spaces over $\mathbb{C}$, and by $\mathrm{AnSm}(\mathbb{C}) \subset \mathrm{AnSp}(\mathbb{C})$ the full subcategory of smooth analytic spaces (i.e. complex analytic manifold). A morphism $h : U \to S$ with $U, S \in \mathrm{AnSp}(\mathbb{C})$ is said to be smooth if it is flat with smooth fibers. A morphism $r : U \to X$ with $U, X \in \mathrm{AnSp}(\mathbb{C})$ is said to be etale if it is non ramified and flat. By the Weirstrass preparation theorem (or the implicit function theorem if U and X are smooth), a morphism $r : U \to X$ with $U, X \in \mathrm{AnSp}(\mathbb{C})$ is etale if and only if it is an isomorphism local.

  We denote by $\mathrm{AnSp}(\mathbb{C})^2 \subset \mathrm{RTop}^2$ the full subcategory such that the first factors are analytic spaces over $\mathbb{C}$, and by $\mathrm{AnSm}(\mathbb{C})^2 \subset \mathrm{AnSp}(\mathbb{C})^2$ the full subcategory such that the first factors are smooth analytic spaces (i.e. complex analytic manifold). In particular we have, for $S \in \mathrm{AnSp}(\mathbb{C})$, the graph functor

  $$\begin{aligned} \mathrm{Gr}^{12}_S : \mathrm{AnSp}(\mathbb{C})/S \to \mathrm{AnSp}(\mathbb{C})^{2,pr}/S,\ X/S \mapsto \mathrm{Gr}^{12}_S(X/S) := (X \times S, X)/S, \\ (g : X/S \to X'/S) \mapsto \mathrm{Gr}^{12}_S(g) := (g \times I_S : (X \times S, X) \to (X' \times S, X')) \end{aligned}$$

  the graph embedding $X \hookrightarrow X \times S$ is a closed embedding since X is separated in RTop, and again $\mathrm{Gr}^{12}_S : \mathrm{AnSp}(\mathbb{C})^{2,pr}/S \to \mathrm{AnSp}(\mathbb{C})/S$ the corresponding morphism of sites.

- For $V \in \mathrm{Var}(\mathbb{C})$, we denote by $V^{an} \in \mathrm{AnSp}(\mathbb{C})$ the complex analytic space associated to V with the usual topology induced by the usual topology of $\mathbb{C}^N$. For $W \in \mathrm{AnSp}(\mathbb{C})$, we denote by $W^{cw} \in \mathrm{AnSp}(\mathbb{C})$ the topological space given by W which is a CW complex. For simplicity, for $V \in \mathrm{Var}(\mathbb{C})$, we denote by $V^{cw} := (V^{an})^{cw} \in \mathrm{CW}$. We have then
  - the analytical functor $\mathrm{An} : \mathrm{Var}(\mathbb{C}) \to \mathrm{AnSp}(\mathbb{C})$, $\mathrm{An}(V) = V^{an}$,
  - the forgetful functor $\mathrm{Cw} = tp : \mathrm{AnSp}(\mathbb{C}) \to \mathrm{CW}$, $\mathrm{Cw}(W) = W^{cw}$,
  - the composite of these two functors $\widetilde{\mathrm{Cw}} = \mathrm{Cw} \circ \mathrm{An} : \mathrm{Var}(\mathbb{C}) \to \mathrm{CW}$, $\widetilde{\mathrm{Cw}}(V) = V^{cw}$.

  We have then
  - the analytical functor $\mathrm{An} : \mathrm{Var}(\mathbb{C})^2 \to \mathrm{AnSp}(\mathbb{C})^2$, $\mathrm{An}((V,Z)) = (V^{an}, Z^{an})$,
  - the forgetful functor $\mathrm{Cw} = tp : \mathrm{AnSp}(\mathbb{C})^2 \to \mathrm{CW}^2$, $\mathrm{Cw}((W,Z)) = (W^{cw}, Z^{cw})$,
  - the composite of these two functors $\widetilde{\mathrm{Cw}} = \mathrm{Cw} \circ \mathrm{An} : \mathrm{Var}(\mathbb{C})^2 \to \mathrm{CW}^2$, $\widetilde{\mathrm{Cw}}((V,Z)) = (V^{cw}, Z^{cw})$.

2.2 Additive categories, abelian categories and tensor triangulated categories

Let $\mathcal{A}$ an additive category.

- For $\phi : F^\bullet \to G^\bullet$ a morphism with $F^\bullet, G^\bullet \in C(\mathcal{A})$, we have the mapping cylinder $\mathrm{Cyl}(\phi) := ((F^n \oplus F^{n+1} \oplus G^{n+1}, (\partial^n_F, \partial^{n+1}_F, \phi^{n+1} + \partial^n G) \in C(\mathcal{A})$. and the mapping cone $\mathrm{Cone}(\phi) := ((F^n \oplus G^{n+1}, (\partial^n_F, \phi^{n+1} + \partial^n G) \in C(\mathcal{A})$.

- The category $K(\mathcal{A}) := \mathrm{Ho}(C(\mathcal{A}))$ is a triangulated category with distinguish triangles $F^\bullet \xrightarrow{i_F} \mathrm{Cyl}(\phi) \xrightarrow{q_F} \mathrm{Cone}(\phi) \xrightarrow{r_F} F^\bullet[1]$.

- The category $(\mathcal{A}, F)$ is obviously again an additive category.

- Let $\phi : F^\bullet \to G^\bullet$ a morphism with $F^\bullet, G^\bullet \in C(\mathcal{A})$. Then it is obviously a morphism of filtered complex $\phi : (F^\bullet, F_b) \to (G^\bullet, F_b)$, where we recall that F_b is the trivial filtration $(F^\bullet, F_b), (G^\bullet, F_b) \in C_{fil}(\mathcal{A})$.

We recall the following property of the internal hom functor if it exists of a tensor triangulated category and the definition of compact and cocompact object.

Proposition 2. *Let $(\mathcal{T}, \otimes)$ a tensor triangulated category admitting countable direct sum and product compatible with the triangulation. Assume that $\mathcal{T}$ has an internal hom (bi)functor $R\mathcal{H}om(.,.) : \mathcal{T}^2 \to \mathcal{T}$ which is by definition the right adjoint to $(\cdot \otimes \cdot) : \mathcal{T}^2 \to \mathcal{T}$. Then,*

- *for $N \in \mathcal{T}$, the functor $R\mathcal{H}om(\cdot, N) : \mathcal{T} \to \mathcal{T}$ commutes with homotopy colimits : for $M = \mathrm{ho}\lim_{\to i \in I} M_i$, where I is a countable category, we have*
$$R\mathcal{H}om(M, N) \xrightarrow{\sim} \mathrm{ho}\lim_{\leftarrow i \in I} R\mathcal{H}om(M_i, N).$$

- *dually, for $M \in \mathcal{T}$, the functor $R\mathcal{H}om(M, \cdot) : \mathcal{T} \to \mathcal{T}$ commutes with homotopy limits : for $N = \mathrm{ho}\lim_{\leftarrow i \in I} N_i$, where I is a countable category, we have*
$$R\mathcal{H}om(M, N) \xrightarrow{\sim} \mathrm{ho}\lim_{\leftarrow} R\mathcal{H}om(M, N_i).$$

Proof. Standard. □

Let $(\mathcal{T}, \otimes)$ a tensor triangulated category admitting countable direct sum and product compatible with the triangulation. Assume that $\mathcal{T}$ has an internal hom functor $R\mathcal{H}om(.,.) : \mathcal{T} \to \mathcal{T}$.

- For $N \in \mathcal{T}$, the functor $R\mathcal{H}om(\cdot, N) : \mathcal{T} \to \mathcal{T}$ does not commutes in general with homotopy limits : for $M = \operatorname{ho}\lim_{\leftarrow i \in I} M_i$, where I is a countable category, the canonical map

$$\operatorname{ho}\lim_{\to i \in I} R\mathcal{H}om(M_i, N) \to R\mathcal{H}om(M, N)$$

is not an isomorphism in general if I is infinite. It commutes if and only if N is compact.

- Dually, for $M \in \mathcal{T}$, the functor $R\mathcal{H}om(M, \bullet) : \mathcal{T} \to \mathcal{T}$ does not commutes in general with infinite homotopy colimits. It commutes if and only if M is cocompact.

Most triangulated category comes from the localization of the category of complexes of an abelian category with respect to quasi-isomorphisms. In the case where the abelian category have enough injective or projective object, the triangulated category is the homotopy category of the complexes of injective, resp. projective, objects.

Proposition 3. *Let $\mathcal{A}$ an abelian category with enough injective and projective.*

- *A quasi-isomorphism $\phi : Q^\bullet \to F^\bullet$, with $F^\bullet, Q^\bullet \in C^-(\mathcal{A})$ such that the Q^n are projective is an homotopy equivalence.*
- *Dually,a quasi-isomorphism $\phi : F^\bullet \to I^\bullet$, with $F^\bullet, I^\bullet \in C^+(\mathcal{A})$ such that the I^n are projective is an homotopy equivalence.*

Proof. Standard. □

Proposition 4. *Let $\mathcal{A}$ an abelian category with enough injective and projective satisfying AB3 (i.e. countable direct sum of exact sequences are exact sequence).*

- *Let $K(P) \subset K(\mathcal{A})$ be the thick subcategory generated by (unbounded) complexes of projective objects. Then, $K(P) \hookrightarrow K(\mathcal{A}) \xrightarrow{D} D(\mathcal{A})$ is an equivalence of triangulated categories.*
- *Similarly, let $K(I) \subset K(\mathcal{A})$ be the thick subcategory generated by (unbounded) complexes of injective objects. Then $K(I) \hookrightarrow K(\mathcal{A}) \xrightarrow{D} D(\mathcal{A})$ is an equivalence of triangulated categories.*

Proof. It follows from proposition 3 : see [22]. □

Let $\mathcal{A} \subset \operatorname{Cat}$ an abelian category. Let $\phi : (M, F) \to (N, F)$ a morphism with $(M, F), (N, F) \in C_{fil}(\mathcal{A})$. Then the distinguish triangle

$$(M, F) \xrightarrow{\phi} (N, F) \xrightarrow{i_1} \operatorname{Cone}(\phi) = ((M, F)[1] \oplus (N, F), (d, d' - \phi) \xrightarrow{p_1} (M, F)[1]$$

gives a sequence

$$\cdots \to H^n(M, F) \xrightarrow{H^n(\phi)} H^n(N, F) \xrightarrow{H^n(i_1)} H^n(\operatorname{Cone}(\phi)) \xrightarrow{H^n(p_1)} H^{n+1}(M, F) \to \cdots$$

which, if we forgot filtration is a long exact sequence in $\mathcal{A}$; however the morphism are NOT strict in general.

2.3 Presheaves on a site and on a ringed topos

2.3.1 Functorialities

Let $\mathcal{S} \in \operatorname{Cat}$ a small category. For $X \in \mathcal{S}$ we denote by $\mathbb{Z}(X) \in \operatorname{PSh}(\mathcal{S})$ the presheaf represented by X. By Yoneda lemma, a representable presheaf $\mathbb{Z}(X)$ is projective.

Proposition 5. • *Let $\mathcal{S} \in \operatorname{Cat}$ a small category. The projective presheaves $\operatorname{Proj}(\operatorname{PSh}(\mathcal{S})) \subset \operatorname{PSh}(\mathcal{S})$ are the direct summand of the representable presheaves $\mathbb{Z}(X)$ with $X \in \mathcal{S}$.*

- *More generally let* $(\mathcal{S}, O_S) \in \mathrm{RCat}$ *a ringed topos. The projective presheaves* $\mathrm{Proj}(\mathrm{PSh}_{O_S}(\mathcal{S})) \subset \mathrm{PSh}_{O_S}(\mathcal{S})$ *of* O_S *modules are the direct summand of the representable presheaves* $\mathbb{Z}(X) \otimes O_S$ *with* $X \in \mathcal{S}$.

Proof. Standard. $\square$

Let $f : \mathcal{T} \to \mathcal{S}$ a morphism of presite with $\mathcal{T}, \mathcal{S} \in \mathrm{Cat}$. For $h : U \to S$ a morphism with $U, S \in \mathcal{S}$, we have $f^*\mathbb{Z}(U/S) = \mathbb{Z}(P(f)(U/S))$.

We will consider in this article filtered complexes of presheaves on a site. Let $f : \mathcal{T} \to \mathcal{S}$ a morphism of presite with $\mathcal{T}, \mathcal{S} \in \mathrm{Cat}$.

- The functor $f_* : C(\mathcal{T}) \to C(\mathcal{S})$ gives, by functoriality, the functor

$$f_* : C_{(2)fil}(\mathcal{T}) \to C_{(2)fil}(\mathcal{S}),\ (G, F) \mapsto f_*(G, F) := (f_*G, f_*F,),$$

since f_* preserves monomorphisms.

- The functor $f^* : C(\mathcal{S}) \to C(\mathcal{T})$ gives, by functoriality, the functor

$$f^* : C_{(2)fil}(\mathcal{S}) \to C_{(2)fil}(\mathcal{T}),\ (G, F) \mapsto f^*(G, F),\ F^p(f^*(G, F)) := \mathrm{Im}(f^*F^pG \to f^*G).$$

In the particular case where $f^* : \mathrm{PSh}(\mathcal{S}) \to \mathrm{PSh}(\mathcal{T})$ preserves monomorphisms, we have $f^*(G, F) = (f^*G, f^*F)$.

- The functor $f^\perp : C(\mathcal{S}) \to C(\mathcal{T})$ gives, by functoriality, the functor

$$f^\perp : C_{(2)fil}(\mathcal{T}) \to C_{(2)fil}(\mathcal{S}),\ (G, F) \mapsto f^\perp(G, F) := (f^\perp G, f^\perp F),$$

since $f^\perp : C(\mathcal{S}) \to C(\mathcal{T})$ preserves monomorphisms.

Let $f : \mathcal{T} \to \mathcal{S}$ a morphism of presite with $\mathcal{T}, \mathcal{S} \in \mathrm{Cat}$.

- The adjonction $(f^*, f_*) = (f^{-1}, f_*) : C(\mathcal{S}) \leftrightarrows C(\mathcal{T})$, gives an adjonction

$$(f^*, f_*) : C_{(2)fil}(\mathcal{S}) \leftrightarrows C_{(2)fil}(\mathcal{T}),\ (G, F) \mapsto f^*(G, F)\ ,\ (G, F) \mapsto f_*(G, F),$$

with adjonction maps, for $(G_1, F) \in C_{(2)fil}(\mathcal{S})$ and $(G_2, F) \in C_{(2)fil}(\mathcal{T})$

$$\mathrm{ad}(f^*, f_*)(G_1, F) : (G_1, F) \to f_*f^*(G_1, F)\ ,\ \mathrm{ad}(f^*, f_*)(G_2, F) : f^*f_*(G_2, F) \to (G_2, F).$$

- The adjonction $(f_*, f^\perp) : C(\mathcal{S}) \leftrightarrows C(\mathcal{T})$, gives an adjonction

$$(f_*, f^\perp) : C_{(2)fil}(\mathcal{T}) \leftrightarrows C_{(2)fil}(\mathcal{S}),\ (G, F) \mapsto f_*(G, F)\ ,\ (G, F) \mapsto f^\perp(G, F),$$

with adjonction maps, for $(G_1, F) \in C_{(2)fil}(\mathcal{S})$ and $(G_2, F) \in C_{(2)fil}(\mathcal{S})$

$$\mathrm{ad}(f^*, f_*)(G_2, F) : (G_2, F) \to f^\perp f_*(G_2, F)\ ,\ \mathrm{ad}(f^*, f_*)(G_1, F) : f_*f^\perp(G_1, F) \to (G_1, F).$$

Remark 1. *Let* $\mathcal{T}, \mathcal{S} \in \mathrm{Cat}$ *small categories and* $f : \mathcal{T} \to \mathcal{S}$ *a morphism of presite. Then the functor* $f^* : \mathrm{PSh}(\mathcal{S}) \to \mathrm{PSh}(\mathcal{T})$ *preserve epimorphism but does NOT preserve monomorphism in general (the colimits involved are NOT filetered colimits). However it preserve monomorphism between projective presheaves by Yoneda and we thus set for* $(Q, F) \in C_{fil}(\mathrm{Proj}(\mathrm{PSh}(\mathcal{S})))$, *that is* $F^pQ^n \in \mathrm{Proj}(\mathrm{PSh}(\mathcal{S}))$ *for all* $p, n \in \mathbb{Z}$, $f^*(Q, F) := (f^*Q, f^*F)$.

For a commutative diagram of presite :

$$D = \begin{tikzcd} \mathcal{T}' \arrow[r, "g_2"] \arrow[d, "f_2"] & \mathcal{S}' \arrow[d, "f_1"] \\ \mathcal{T} \arrow[r, "g_1"] & \mathcal{S} \end{tikzcd} ,$$

with $\mathcal{T}, \mathcal{T}'\mathcal{S}, \mathcal{S}' \in \mathrm{Cat}$, we denote by, for $F \in C(\mathcal{S}')$,

$$T(D)(F) : g_1^* f_{1*} F \xrightarrow{g_1^* f_{1*} \operatorname{ad}(g_2^*, g_{2*})(F)} g_1^* f_{1*} g_{2*} g_2^* F = g_1^* g_{1*} f_{2*} g_2^* F \xrightarrow{\operatorname{ad}(g_1^* g_{1*})(f_{2*} g_2^* F)} f_{2*} g_2^* F$$

the canonical transformation map in $C(\mathcal{T})$, and for $(G, F) \in C_{fil}(\mathcal{S}')$,

$$T(D)(G,F) : g_1^* f_{1*}(G,F) \xrightarrow{g_1^* f_{1*} \operatorname{ad}(g_2^*, g_{2*})(G,F)} g_1^* f_{1*} g_{2*} g_2^*(G,F) = g_1^* g_{1*} f_{2*} g_2^*(G,F)$$
$$\xrightarrow{\operatorname{ad}(g_1^* g_{1*})(f_{2*} g_2^*(G,F))} f_{2*} g_2^*(G,F).$$

the canonical transformation map in $C_{fil}(\mathcal{T})$ given by the adjonction maps.

We will use the internal hom functor and the tensor product for presheaves on a site or for presheaves of modules on a ringed topos. We recall the definition in the filtrered case.

- Let $(\mathcal{S}, O_S) \in \mathrm{RCat}$. We have the tensor product bifunctor

  $$(\cdot) \otimes (\cdot) : \mathrm{PSh}(\mathcal{S})^2 \to \mathrm{PSh}(\mathcal{S}), (F,G) \longmapsto (X \in \mathcal{S} \mapsto (F \otimes G)(X) := F(X) \otimes G(X)$$

  It induces a bifunctor :

  $$(\cdot) \otimes (\cdot) : C(\mathcal{S}) \times C(\mathcal{S}) \to C(\mathcal{S}), (F,G) \mapsto F \otimes G := \mathrm{Tot}(F^\bullet \otimes G^\bullet), (F \otimes G)^n = \oplus_{r \in \mathbb{Z}} F^r \otimes G^{n-r}$$

  and a bifunctor

  $$(\cdot) \otimes (\cdot) : C(\mathcal{S}) \times C_{O_S}(\mathcal{S}) \to C_{O_S}(\mathcal{S}),\ \alpha.(F \otimes G) := F \otimes (\alpha.G)$$

  For $(G_1, F), (G_2, F) \in C_{fil}(\mathcal{S})$, $G_3 \in C(\mathcal{S})$, we define (note that tensor product preserve monomorphism only after tensoring with $\mathbb{Q}_S \in \mathrm{PSh}(\mathcal{S})$)

  - $F^p((G_1,F) \otimes G_3) := \mathrm{Im}(F^p G_1 \otimes G_3 \to G_1 \otimes G_3)$ and $F^p(G_3 \otimes (G_1,F)) := \mathrm{Im}(G_3 \otimes F^p G_3 \to G_3 \otimes G_1)$,
  - $F^p F^q((G_1,F) \otimes (G_2,F)) := \mathrm{Im}(F^p G_1 \otimes F^q G_2 \to G_1 \otimes G_2)$ and

    $$F^k((G_1,F) \otimes (G_2,F)) := F^k \mathrm{Tot}_{FF}((G_1,F) \otimes (G_2,F)) := \oplus_{p \in \mathbb{Z}} \mathrm{Im}(F^p G_1 \otimes F^{k-q} G_2 \to G_1 \otimes G_2)$$

  Note that in the case where $G_1^n = 0$ for $n < 0$, we have $(G_1, F_b) \otimes (G_2, F) = G_1 \otimes (G_2, F)$. We get the bifunctors

  $$(-) \otimes (-) : C_{fil}(\mathcal{S})^2 \to C_{fil}(\mathcal{S}),\ \ (-) \otimes (-) : C_{fil}(\mathcal{S}) \times C_{O_S fil}(\mathcal{S}) \to C_{O_S fil}(\mathcal{S}).$$

  We have the tensor product bifunctor

  $$(\cdot) \otimes_{O_S} (\cdot) : \mathrm{PSh}_{O_S}(\mathcal{S})^2 \to \mathrm{PSh}(\mathcal{S}), (F,G) \longmapsto (X \in \mathcal{S} \mapsto (F \otimes_{O_S} G)(X) := F(X) \otimes_{O_S(X)} G(X)$$

  It induces a bifunctor :

  $$(\cdot) \otimes_{O_S} (\cdot) : C_{O_S}(\mathcal{S}) \times C_{O_S}(\mathcal{S}) \to C(\mathcal{S}), (F,G) \mapsto F \otimes_{O_S} G := \mathrm{Tot}(F^\bullet \otimes_{O_S} G^\bullet)$$

  For $(G_1,F), (G_2,F) \in C_{O_S fil}(\mathcal{S})$, $G_3 \in C_{O_S}(\mathcal{S})$, we define similarly $(G_1,F) \otimes_{O_S} G_3$, $G_3 \otimes_{O_S} (G_1,F)$, and

  $$F^k((G_1,F) \otimes_{O_S} (G_2,F)) := F^k \mathrm{Tot}_{FF}((G_1,F) \otimes_{O_S} (G_2,F)) := \oplus_{p \in \mathbb{Z}} \mathrm{Im}(F^p G_1 \otimes_{O_S} F^{k-q} G_2 \to G_1 \otimes_{O_S} G_2)$$

  Note that in the case where $G_1^n = 0$ for $n < 0$, we have $(G_1, F_b) \otimes_{O_S} (G_2, F) = G_1 \otimes_{O_S} (G_2, F)$. This gives

- in all case it gives the bifunctor $(-) \otimes_{O_S} (-) : C_{O_S^{op}fil}(\mathcal{S}) \otimes C_{O_S fil}(\mathcal{S}) \to C_{fil}(\mathcal{S})$.
- in the case O_S is commutative, it gives the bifunctor $(-) \otimes_{O_S} (-) : C_{O_S fil}(\mathcal{S})^2 \to C_{O_S fil}(\mathcal{S})$.

- Let $(\mathcal{S}, O_S) \in \mathrm{RCat}$. We have the internal hom bifunctor

$$\mathcal{H}om(\cdot,\cdot) : \mathrm{PSh}(\mathcal{S})^2 \to \mathrm{PSh}(\mathcal{S}),$$
$$(F,G) \longmapsto (X \in \mathcal{S} \mapsto \mathcal{H}om(F,G)(X) := \mathrm{Hom}(r(X)_*F, r(X)_*G)$$

with $r(X) : \mathcal{S} \to \mathcal{S}/X$ (see subsection 2.1). It induces a bifunctors :

$$\mathcal{H}om(\cdot,\cdot) : C(\mathcal{S}) \times C(\mathcal{S}) \to C(\mathcal{S}),\ (F,G) \mapsto \mathcal{H}om^\bullet(F,G)$$

and a bifunctor

$$\mathcal{H}om(\cdot,\cdot) : C(\mathcal{S}) \times C_{O_S}(\mathcal{S}) \to C_{O_S}(\mathcal{S}),\ \alpha.\mathcal{H}om(F,G) := \mathcal{H}om(F, \alpha.G)$$

For $(G_1,F),(G_2,F) \in C_{fil}(\mathcal{S})$, $G_3 \in C(\mathcal{S})$, we define

- $F^p\mathrm{Hom}(G_3,(G_1,F)) := \mathcal{H}om(G_3, F^pG_1) \hookrightarrow \mathcal{H}om(G_3,G_1))$, note that the functor $G \mapsto \mathcal{H}om(F,G)$ preserve monomorphism,
- the dual filtration $F^{-p}\mathcal{H}om((G_1,F),G_3) := \ker(\mathcal{H}om(G_1,G_3) \to \mathrm{Hom}(F^pG_1,G_3))$
- $F^pF^q\mathcal{H}om((G_1,F),(G_2,F)) := \ker(\mathcal{H}om(G_1,F^pG_2) \to \mathrm{Hom}(F^qG_1,F^pG_2))$, and

$$F^k\mathcal{H}om^\bullet((G_1,F),(G_2,F)) := \mathrm{Tot}_{FF}\,\mathcal{H}om((G_1,F),(G_2,F)) :=$$
$$\oplus_{p\in\mathbb{Z}} \ker(\mathcal{H}om(G_1,F^{k+p}G_2) \to \mathcal{H}om(F^pG_1,F^{k+p}G_2))$$

We get the bifunctors

$$\mathcal{H}om(\cdot,\cdot) : C_{fil}(\mathcal{S}) \times C_{fil}(\mathcal{S}) \to C_{fil}(\mathcal{S}),\ \ \mathcal{H}om(\cdot,\cdot) : C_{fil}(\mathcal{S}) \times C_{O_S fil}(\mathcal{S}) \to C_{O_S fil}(\mathcal{S}).$$

We have the internal hom bifunctor

$$\mathcal{H}om_{O_S}(\cdot,\cdot) : \mathrm{PSh}_{O_S}(\mathcal{S}) \times \mathrm{PSh}_{O_S}(\mathcal{S}) \to \mathrm{PSh}(\mathcal{S})$$
$$(F,G) \longmapsto (X \in \mathcal{S} \mapsto \mathcal{H}om_{O_S}(F,G)(X) := \mathrm{Hom}_{O_S}(r(X)_*F, r(X)_*G).$$

It gives similarly

- in all case a bifunctor $\mathcal{H}om_{O_S}(\cdot,\cdot) : C_{filO_S}(\mathcal{S}) \times C_{filO_S}(\mathcal{S}) \to C_{fil}(\mathcal{S})$,
- the case O_S is commutative, a bifunctor $\mathcal{H}om_{O_S}(\cdot,\cdot) : C_{filO_S}(\mathcal{S}) \times C_{filO_S}(\mathcal{S}) \to C_{O_S fil}(\mathcal{S})$.

Let $\phi : A \to B$ of rings.

- Let M a A module. We say that M admits a B module structure if there exits a structure of B module on the abelian group M which is compatible with ϕ together with the A module structure on M.

- For N_1 a A-module and N_2 a B module. $I(A/B)(N_1,N_2) : \mathrm{Hom}_A(N_1,N_2) \xrightarrow{\sim} \mathrm{Hom}_B(N_1 \otimes_A B, N_2)$ is the adjonction between the restriction of scalars and the extension of scalars.

- For N', N'' a A-modules, $ev_A(\mathrm{hom},\otimes)(N',N'',B) : \mathrm{Hom}_A(N',N'') \otimes_A B \to \mathrm{Hom}_A(N', N'' \otimes_A B)$. is the evaluation classical map.

Let $\phi : (\mathcal{S}, O_1) \to (\mathcal{S}, O_2)$ a morphism of presheaves of ring on $\mathcal{S} \in \mathrm{Cat}$.

- Let $M \in \mathrm{PSh}_{O_1}(\mathcal{S})$. We say that M admits an O_2 module structure if there exits a structure of O_2 module on $M \in \mathrm{PSh}(\mathcal{S})$ which is compatible with ϕ together with the O_1 module structure on M.

- For $N_1 \in C_{O_1}(\mathcal{S})$ and $N_2 \in C_{O_2}(\mathcal{S})$,

$$I(O_1/O_2)(N_1, N_2) : \mathrm{Hom}_{O_1}(N_1, N_2) \xrightarrow{\sim} \mathrm{Hom}_{O_2}(N_1 \otimes_{O_1} B, N_2)$$

  is the adjonction between the restriction of scalars and the extension of scalars.

- For $N', N'' \in C_{O_1}(\mathcal{S})$,

$$ev_{O_1}(\mathrm{hom}, \otimes)(N', N'', O_2) : \mathcal{H}om_{O_1}(N', N'') \otimes_{O_1} O_2 \to \mathcal{H}om_{O_1}(N', N'' \otimes_{O_1} O_2).$$

  is the classical evaluation map.

Let $(\mathcal{S}, O_S) \in \mathrm{RCat}$.

- For $F_1, F_2, G_1, G_2 \in C(\mathcal{S})$, we denote by

$$T(\otimes, \mathcal{H}om)(F_1, F_2, G_1, G_2) : \mathcal{H}om(F_1, G_1) \otimes \mathcal{H}om(F_2, G_2) \to \mathcal{H}om(F_1 \otimes F_2, G_1 \otimes G_2)$$

  the canonical map.

- For $G_3 \in C(\mathcal{S})$ and $G_1, G_2 \in C_{O_S}(\mathcal{S})$, we denote by

$$\begin{aligned} ev(hom, \otimes)(G_3, G_1, G_2) : \mathcal{H}om(G_3, G_1) \otimes_{O_S} G_2 &\to \mathcal{H}om(G_3, G_1 \otimes_{O_S} G_2) \\ \phi \otimes s &\mapsto (s' \mapsto \phi(s') \otimes s) \end{aligned}$$

- Let $\mathcal{S} \in \mathrm{Cat}$ a small category. Let $(H_X : C(\mathcal{S}/X) \to C(\mathcal{S}/X))_{X \in \mathcal{S}}$ a familly of functors which is functorial in X. We have by definition, for $F_1, F_2 \in C(\mathcal{S})$, the canonical transformation map

$$T(H, hom)(F_1, F_2) : H(\mathcal{H}om^\bullet(F_1, F_2)) \to \mathcal{H}om^\bullet(H(F_1), H(F_2)), \tag{1}$$
$$\phi \in \mathrm{Hom}(F_{1|X}, F_{2|X}) \mapsto H^{F_{1|X}, F_{2|X}}(\phi) \in \mathrm{Hom}(H(F_{1|X}), H(F_{2|X})) \tag{2}$$

  in $C(\mathcal{S})$.

Let $\mathcal{T}, \mathcal{S} \in \mathrm{Cat}$ small categories and $f : \mathcal{T} \to \mathcal{S}$ a morphism of presite.

- For $F_1, F_2 \in C(\mathcal{T})$ we have by definition $f_*(F_1 \otimes F_2) = f_*F_1 \otimes f_*F_2$. For $G_1, G_2 \in C(\mathcal{S})$, we have a canonical isomorphism $f^*G_1 \otimes f^*G_2 \xrightarrow{\sim} f^*(G_1 \otimes G_2)$ since the tensor product is a right exact functor, and a canonical map $f^\perp G_1 \otimes f^\perp G_2 \to f^\perp(G_1 \otimes G_2)$.

- We have for $F \in C(\mathcal{S})$ and $G \in C(\mathcal{T})$ the adjonction isomorphim,

$$I(f^*, f_*)(F, G) : f_*\mathcal{H}om^\bullet(f^*F, G) \xrightarrow{\sim} \mathcal{H}om^\bullet(F, f_*G). \tag{3}$$

- Let $O_S \in \mathrm{PSh}(\mathcal{S}, \mathrm{Ring})$ by a presheaf of ring so that $(\mathcal{S}, O_S), (\mathcal{T}, f^*O_S) \in \mathrm{RCat}$. We have for $F \in C_{O_S}(\mathcal{S})$ and $G \in C_{f^*O_S}(\mathcal{T})$ the adjonction isomorphim,

$$I(f^*, f_*)(F, G) : f_*\mathcal{H}om^\bullet_{f^*O_S}(f^*F, G) \xrightarrow{\sim} \mathcal{H}om^\bullet_{O_S}(F, f_*G), \tag{4}$$

  and

  - the map $\mathrm{ad}(f^*, f_*)(F) : F \to f_*f^*F$ in $C(\mathcal{S})$ is O_S linear, that is is a map in $C_{O_S}(\mathcal{S})$,
  - the map $\mathrm{ad}(f^*, f_*)(G) : f^*f_*G \to G$ in $C(\mathcal{T})$ is f^*O_S linear, that is is a map in $C_{f^*O_S}(\mathcal{T})$.

- For $F_1, F_2 \in C(\mathcal{T})$, we have the canonical map

$$T_*(f, hom)(F_1, F_2) := T(f_*, hom) : f_*\mathcal{H}om^\bullet(F_1, F_2) \to \mathcal{H}om^\bullet(f_*F_1, f_*F_2), \tag{5}$$
$$\text{for } X \in \mathcal{S},\ \phi \in \mathrm{Hom}(F_{1|f^*(X)}, F_{2|f^*(X)}) \mapsto {f_*}^{F_{1|f^*(X)}, F_{2|f^*(X)}}(\phi) \in \mathrm{Hom}(f_*F_{1|f^*(X)}, f_*F_{2|f^*(X)}) \tag{6}$$

  given by evaluation.

- For $G_1, G_2 \in C(\mathcal{S})$, we have the following canonical transformation in $C(\mathcal{T})$

$$T(f, hom)(G_1, G_2) := T(f^*, hom)(G_1, G_2) : \quad (7)$$
$$f^*\mathcal{H}om^\bullet(G_1, G_2) \xrightarrow{f^*\mathcal{H}om(G_1, \mathrm{ad}(f^*, f_*)(G_2))} f^*\mathcal{H}om^\bullet(G_1, f_*f^*G_2) \xrightarrow{f^*I(f^*, f_*)(G_1, G_2)} \quad (8)$$
$$f^*f_*\mathcal{H}om^\bullet(f^*G_1, f^*G_2) \xrightarrow{\mathrm{ad}(f^*, f_*)(\mathcal{H}om(f^*G_1, f^*G_2))} \mathcal{H}om^\bullet(f^*G_1, f^*G_2), \quad (9)$$

- Let $O_S \in \mathrm{PSh}(\mathcal{S}, \mathrm{Ring})$ by a presheaf of ring so that $(\mathcal{S}, O_S), (\mathcal{T}, f^*O_S) \in \mathrm{RCat}$. For $G_1, G_2 \in C_{O_S}(\mathcal{S})$, we have the following canonical transformation in $C_{f^*O_S}(\mathcal{T})$

$$T(f, hom)(G_1, G_2) := T(f^*, hom)(G_1, G_2) : \quad (10)$$
$$f^*\mathcal{H}om^\bullet_{O_S}(G_1, G_2) \xrightarrow{f^*\mathcal{H}om_{O_S}(G_1, \mathrm{ad}(f^*, f_*)(G_2))} f^*\mathcal{H}om^\bullet_{O_S}(G_1, f_*f^*G_2) \xrightarrow{f^*I(f^*, f_*)(G_1, G_2)} \quad (11)$$
$$f^*f_*\mathcal{H}om^\bullet_{f^*O_S}(f^*G_1, f^*G_2) \xrightarrow{\mathrm{ad}(f^*, f_*)(\mathcal{H}om_{f^*O_S}(f^*G_1, f^*G_2))} \mathcal{H}om^\bullet_{f^*O_S}(f^*G_1, f^*G_2), \quad (12)$$

- Let $O_S \in \mathrm{PSh}(\mathcal{S}, \mathrm{Ring})$ by a presheaf of ring so that $(\mathcal{S}, O_S), (\mathcal{T}, f^*O_S) \in \mathrm{RCat}$. For $M \in C_{O_S}(\mathcal{S})$ and $N \in C_{f^*O_S}(\mathcal{T})$, we denote by

$$T(f, \otimes)(M, N) : M \otimes_{O_S} f_*N \xrightarrow{\mathrm{ad}(f^*, f_*)(M \otimes_{O_S} f_*N)} \quad (13)$$
$$f_*f^*(M \otimes_{O_S} f_*N) = f_*(f^*M \otimes_{f^*O_S} f^*f_*N) \xrightarrow{\mathrm{ad}(f^*, f_*)(N)} f_*(f^*M \otimes_{f^*O_S} N) \quad (14)$$

the canonical transformation map.

Let $f : (\mathcal{T}, O_T) \to (\mathcal{S}, O_S)$ a morphism with $(\mathcal{S}, O_S), (\mathcal{T}, O_T) \in \mathrm{RCat}$. We have the adjonction

$$(f^{*mod}, f_*) : C_{O_S}(\mathcal{S}) \leftrightarrows C_{O_T}(\mathcal{T})$$

with $f^{*mod}G := f^*G \otimes_{f^*O_S} O_T$. If $f^* : C(\mathcal{S}) \to C(\mathcal{T})$ preserve monomorphisms, it induces the adjonction

$$(f^{*mod}, f_*) : C_{O_S fil}(\mathcal{S}) \leftrightarrows C_{O_T fil}(\mathcal{T})$$

with $f^{*mod}(G, F) := f^*(G, F) \otimes_{f^*O_S} O_T$.

For a commutative diagram in RCat :

$$D = \begin{tikzcd} (\mathcal{T}', O'_2) \arrow[r, "g_2"] \arrow[d, "f_2"] & (\mathcal{S}', O'_1) \arrow[d, "f_1"] \\ (\mathcal{T}, O_2) \arrow[r, "g_1"] & (\mathcal{S}, O_1) \end{tikzcd},$$

we denote by, for $F \in C_{O'_1}(\mathcal{S}')$,

$$T^{mod}(D)(F) : g_1^{*mod}f_{1*}F \xrightarrow{g_1^{*mod}f_{1*}\,\mathrm{ad}(g_2^{*mod}, g_{2*})(F)} g_1^{*mod}f_{1*}g_{2*}g_2^{*mod}F = g_1^{*mod}g_{1*}f_{2*}g_2^{*mod}F \xrightarrow{\mathrm{ad}(g_1^{*mod}g_{1*})(f_{2*}g_2^{*mod}F)} f_{2*}g_2^{*mod}F$$

the canonical transformation map in $C_{O_2}(\mathcal{T})$ and, for $(G, F) \in C_{O'_1 fil}(\mathcal{S}')$,

$$T^{mod}(D)(G, F) : g_1^{*mod}f_{1*}(G, F) \xrightarrow{g_1^{*mod}f_{1*}\,\mathrm{ad}(g_2^{*mod}, g_{2*})(G, F)} g_1^{*mod}f_{1*}g_{2*}g_2^{*mod}(G, F) = g_1^{*mod}g_{1*}f_{2*}g_2^{*mod}(G, F) \xrightarrow{\mathrm{ad}(g_1^{*mod}g_{1*})(f_{2*}g_2^{*mod}(G, F))} f_{2*}g_2^{*mod}(G, F)$$

the canonical transformation map in $C_{O_2 fil}(\mathcal{T})$ given by the adjonction maps.

Let $f : (\mathcal{T}, O_T) \to (\mathcal{S}, O_S)$ a morphism with $(\mathcal{S}, O_S), (\mathcal{T}, O_T) \in \mathrm{RCat}$.

- We have, for $M, N \in C_{O_S}(\mathcal{S})$ the canonical transformation map in $C_{O_T}(\mathcal{T})$

$$T^{mod}(f, hom)(M,N) : f^{*mod}\mathrm{Hom}_{O_1}(M,N) \xrightarrow{T(f,hom)(M,N)\otimes_{f^*O_1}O_2} \mathcal{H}om_{f^*O_1}(f^*M, f^*N) \otimes_{f^*O_1} O_2$$
$$\xrightarrow{e(\mathrm{hom},\otimes)(f^*M,f^*N)} \mathcal{H}om_{f^*O_1}(f^*M, f^{*mod}N) \xrightarrow{I(f^*O_1/O_2)(f^*M,f^{*mod}N)} \mathcal{H}om_{O_2}(f^{*mod}M, f^{*mod}N)$$

- We have, for $M \in C_{O_S}(\mathcal{S})$ and $N \in C_{O_T}(\mathcal{T})$, the canonical transformation map in $C_{O_T}(\mathcal{T})$

$$T^{mod}(f,\otimes)(M,N) : M \otimes_{O_S} f_*N \xrightarrow{\mathrm{ad}(f^{*mod},f_*)(M\otimes_{O_S}f_*N)} \quad (15)$$
$$f_*f^{*mod}(M \otimes_{O_S} f_*N) = f_*(f^{*mod}M \otimes_{O_T} f^{*mod}f_*N) \xrightarrow{\mathrm{ad}(f^{*mod},f_*)(N)} f_*(f^{*mod}M \otimes_{O_T} N) \quad (16)$$

  the canonical transformation map.

We now give some properties of the tensor product functor and hom functor given above

Proposition 6. *Let $(\mathcal{S}, O_S) \in \mathrm{RCat}$. Then, the functors*

- $(-) \otimes (-) : C(\mathcal{S})^2 \to C(\mathcal{S}),\ C(\mathcal{S}) \times C_{O_S}(\mathcal{S}) \to C_{O_S}(\mathcal{S})$
- $(-) \otimes_{O_S} (-) : C_{O_S^{op}}(\mathcal{S}) \times C_{O_S}(\mathcal{S}) \to C(\mathcal{S})$ *and in case O_S is commutative* $(-) \otimes_{O_S} (-) : C_{O_S}(\mathcal{S})^2 \to C_{O_S}(\mathcal{S})$

are left Quillen functor for the projective model structure. In particular,

- *for $L \in C(\mathcal{S})$ is such that $L^n \in \mathrm{PSh}(\mathcal{S})$ are projective for all $n \in \mathbb{Z}$, and $\phi : F \to G$ is a quasi-isomorphism with $F, G \in C(\mathcal{S})$, then $\phi \otimes I : F \otimes L \to G \otimes L$ is a quasi-isomorphism,*
- *for $L \in C_{O_S}(\mathcal{S})$ is such that $L^n \in \mathrm{PSh}_{O_S}(\mathcal{S})$ are projective for all $n \in \mathbb{Z}$, and $\phi : F \to G$ is a quasi-isomorphism with $F, G \in C_{O_S}(\mathcal{S})$, then $\phi \otimes I : F \otimes_{O_S} L \to G \otimes_{O_S} L$ is a quasi-isomorphism.*

Proof. Standard. □

Proposition 7. *Let $(\mathcal{S}, O_S) \in \mathrm{RCat}$. Then, the functors*

- $\mathcal{H}om(\cdot,\cdot) : C(\mathcal{S}) \times C_{O_S}(\mathcal{S}) \to C_{O_S}(\mathcal{S}),\ C(\mathcal{S}) \times C_{O_S}(\mathcal{S}) \to C_{O_S}(\mathcal{S}),$
- $\mathcal{H}om_{O_S}(\cdot,\cdot) : C_{O_S^{op}}(\mathcal{S}) \times C_{O_S}(\mathcal{S}) \to C(\mathcal{S})$ *and in the case O_S is commutative* $\mathcal{H}om_{O_S}(\cdot,\cdot) : C_{O_S}(\mathcal{S}) \times C_{O_S}(\mathcal{S}) \to C_{O_S}(\mathcal{S})$,

are on the left hand side left Quillen functor for the projective model structure. In particular,

- *for $L \in C(\mathcal{S})$ is such that $L^n \in \mathrm{PSh}(\mathcal{S})$ are projective for all $n \in \mathbb{Z}$, and $\phi : F \to G$ is a quasi-isomorphism with $F, G \in C(\mathcal{S})$, then $\mathcal{H}om(L, \phi) : \mathcal{H}om^\bullet(L, F) \to \mathcal{H}om^\bullet(L, G)$ is a quasi-isomorphism,*
- *for $L \in C_{O_S}(\mathcal{S})$ is such that $L^n \in \mathrm{PSh}_{O_S}(\mathcal{S})$ are projective for all $n \in \mathbb{Z}$, and $\phi : F \to G$ is a quasi-isomorphism with $F, G \in C_{O_S}(\mathcal{S})$, then $\mathcal{H}om_{O_S}(L, \phi) : \mathcal{H}om^\bullet_{O_S}(L, F) \to \mathcal{H}om^\bullet_{O_S}(L, G)$ is a quasi-isomorphism.*

Proof. Standard. □

Let $\mathcal{S} \in \mathrm{Cat}$ a site endowed with topology τ. Denote by $a_\tau : \mathrm{PSh}(\mathcal{S}) \to \mathrm{Sh}(\mathcal{S})$ the sheaftification functor A morphism $\phi : F^\bullet \to G^\bullet$ with $F^\bullet, G^\bullet \in C(\mathcal{S}))$ is said to be a τ local equivalence if

$$a_\tau H^n(\phi) : a_\tau H^n(F^\bullet) \to a_\tau H^n(G^\bullet)$$

is an isomorphism for all $n \in \mathbb{Z}$, where a_τ is the sheaftification functor. Recall that $C_{fil}(\mathcal{S}) \subset (C(\mathcal{S}), F) = C(\mathrm{PSh}(\mathcal{S}), F)$ denotes the category of filtered complexes of abelian presheaves on $\mathcal{S}$ whose filtration is biregular.

- A morphism $\phi : (F^\bullet, F) \to (G^\bullet, F)$ with $(F^\bullet, F), (G^\bullet, F) \in C_{fil}(\mathcal{S})$ is said to be a filtered τ local equivalence or an 1-filtered τ local equivalence if
$$a_\tau H^n(\phi) : a_\tau H^n(\mathrm{Gr}_F^p F^\bullet) \xrightarrow{\sim} a_\tau H^n(\mathrm{Gr}_F^p G^\bullet)$$
is an isomorphism for all $n, p \in \mathbb{Z}$.

- Let $r \in \mathbb{N}$. A morphism $\phi : (F^\bullet, F) \to (G^\bullet, F)$ with $(F^\bullet, F), (G^\bullet, F) \in C_{fil}(\mathcal{S})$ is said to be an r-filtered τ local equivalence if there exist an r-filtered homotopy
$$(h, \phi, \phi') : (F^\bullet, F)[1] \to (G^\bullet, F)$$
such that $\phi' : (F^\bullet, F) \to (G^\bullet, F)$ is a filtered τ local equivalence If $\phi : (F^\bullet, F) \to (G^\bullet, F)$ with $(F^\bullet, F), (G^\bullet, F) \in C_{fil}(\mathcal{S})$ is an r-filtered τ local equivalence, then for all $p, q \in \mathbb{Z}$,
$$a_\tau E_r^{p,q}(\phi) : a_\tau E_r^{p,q}(F^\bullet, F) \xrightarrow{\sim} a_\tau E_r^{p,q}(G^\bullet, F)$$
is an isomorphism but the converse is NOT true. Note that if ϕ is an r-filtered τ local equivalence, that it is an s-filtered τ local equivalence for all $s \geq r$.

- A morphism $\phi : (F^\bullet, F) \to (G^\bullet, F)$ with $(F^\bullet, F), (G^\bullet, F) \in C_{fil}(\mathcal{S})$ is said to be a ∞-filtered τ local equivalence if there exists $r \in \mathbb{N}$ such that ϕ is an r-filtered τ local equivalence. If a morphism $\phi : (F^\bullet, F) \to (G^\bullet, F)$ with $(F^\bullet, F), (G^\bullet, F) \in C_{fil}(\mathcal{S})$ is an ∞-filtered τ local equivalence then, for all $n \in \mathbb{Z}$,
$$a_\tau H^n(\phi) : a_\tau H^n(F^\bullet, F) \to a_\tau H^n(G^\bullet, F)$$
is an isomorphism of filtered sheaves on $\mathcal{S}$. Recall the converse is true in the case there exist $N_1, N_2 \in \mathbb{Z}$, such that $H^n(F^\bullet, F) = H^n(G^\bullet, F) = 0$ for $n \leq N_1$ or $n \geq N_2$.

Let $(\mathcal{S}, O)$ a ringed topos where $\mathcal{S} \in \mathrm{Cat}$ is a site endowed with topology τ. A morphism $\phi : (F^\bullet, F) \to (G^\bullet, F)$ with $(F^\bullet, F), (G^\bullet, F) \in C_{O_S fil}(\mathcal{S})$ is said to be a filtered τ local equivalence or an 1-filtered τ local equivalence if $o\phi : (F^\bullet, F) \to (G^\bullet, F)$ is one in $C_{fil}(\mathcal{S})$, i.e.
$$a_\tau H^n(\phi) : a_\tau H^n(\mathrm{Gr}_F^p F^\bullet) \xrightarrow{\sim} a_\tau H^n(\mathrm{Gr}_F^p G^\bullet)$$
is an isomorphism for all $n, p \in \mathbb{Z}$. Let $r \in \mathbb{N}$. A morphism $\phi : (F^\bullet, F) \to (G^\bullet, F)$ with $(F^\bullet, F), (G^\bullet, F) \in C_{O_S fil}(\mathcal{S})$ is said to be an r-filtered τ local equivalence if there exist an r-filtered homotopy
$$(h, \phi, \phi') : (F^\bullet, F)[1] \to (G^\bullet, F)$$
such that $\phi' : (F^\bullet, F) \to (G^\bullet, F)$ is a filtered τ local equivalence.

Let $\mathcal{S} \in \mathrm{Cat}$ a site which admits fiber product, endowed with topology τ. A complex of presheaves $F^\bullet \in C(\mathcal{S})$ is said to be τ fibrant if it satisfy descent for covers in $\mathcal{S}$, i.e. if for all $X \in \mathcal{S}$ and all τ covers $(c_i : U_i \to X)_{i \in I}$ of X, denoting $U_J := (U_{i_0} \times_S U_{i_1} \times_S \cdots U_{i_r})_{i_k \in J}$ and for $I \subset J$, $p_{IJ} : U_J \to U_I$ is the projection,
$$F^\bullet(c_i) : F^\bullet(X) \to \mathrm{Tot}(\oplus_{cardI=\bullet} F^\bullet(U_I))$$
is a quasi-isomorphism of complexes of abelian groups.

- A complex of filtered presheaves $(F^\bullet, F) \in C_{fil}(\mathcal{S})$ is said to be filtered τ fibrant or 1-filtered τ fibrant if it satisfy descent for covers in $\mathcal{S}$, i.e. if for all $X \in \mathcal{S}$ and all τ covers $(c_i : U_i \to X)_{i \in I}$ of X,
$$(F^\bullet, F)(c_i) : (F^\bullet, F)(X) \to \mathrm{Tot}(\oplus_{cardI=\bullet} (F^\bullet, F)(U_I))$$
is a filtered quasi-isomorphism of filtered complexes of abelian groups.

- Let $r \in \mathbb{N}$. More generally, a complex of filtered presheaves $(F^\bullet, F) \in C_{fil}(\mathcal{S})$ is said to be r-filtered τ fibrant if there exist an r-filtered homotopy equivalence $m : (F^\bullet, F) \to (F^{'\bullet}, F)$ with $(F^{'\bullet}, F) \in C_{fil}(\mathcal{S})$ filtered τ fibrant. If $(F^\bullet, F) \in C_{fil}(\mathcal{S})$ is r-filtered τ fibrant, then for all $X \in \mathcal{S}$ and all τ covers $(c_i : U_i \to X)_{i \in I}$ of X,

$$E_r^{p,q}(F^\bullet, F)(c_i) : E_r^{p,q}(F^\bullet, F)(X) \to E_r^{p,q}(\mathrm{Tot}(\oplus_{cardI=\bullet}(F^\bullet, F)(U_I)))$$

is an isomorphism for all $n, p \in \mathbb{Z}$, but the converse is NOT true. Note that if $(F^\bullet, F)$ is r-filtered τ fibrant, then it is s-filtered τ fibrant for all $s \geq r$.

- A complex of filtered presheaves $(F^\bullet, F) \in C_{fil}(\mathcal{S})$ is said to be ∞-filtered τ fibrant if there exist $r \in \mathbb{N}$ such that $(F^\bullet, F)$ is r-filtered τ fibrant. If a complex of filtered presheaves $(F^\bullet, F) \in C_{fil}(\mathcal{S})$ is ∞-filtered τ fibrant, then for all $X \in \mathcal{S}$ and all τ covers $(c_i : U_i \to X)_{i \in I}$ of X,

$$H^n(F^\bullet, F)(c_i) : H^n(F^\bullet, F)(X) \to H^n \,\mathrm{Tot}(\oplus_{cardI=\bullet}(F^\bullet, F)(U_I))$$

is a filtered isomorphism for all $n \in \mathbb{Z}$.

Let $(\mathcal{S}, O)$ a ringed topos where $\mathcal{S} \in \mathrm{Cat}$ is a site endowed with topology τ. Let $r \in \mathbb{N}$.

- A complex of presheaves $F^\bullet \in C_{O_S}(\mathcal{S})$ is said to be τ fibrant if $F^\bullet = oF^\bullet \in C(\mathcal{S})$ is τ fibrant.
- A complex of presheaves $(F^\bullet, F) \in C_{O_S fil}(\mathcal{S})$ is said to be filtered τ fibrant if $(F^\bullet, F) = (oF^\bullet, F) \in C_{fil}(\mathcal{S})$ is filtered τ fibrant.
- A complex of presheaves $(F^\bullet, F) \in C_{O_S fil}(\mathcal{S})$ is said to be r-filtered τ fibrant if there exist an r-filtered homotopy $m : (F^\bullet, F) \to (F^{'\bullet}, F)$ with $(F^{'\bullet}, F) \in C_{O_S fil}(\mathcal{S})$ filtered τ fibrant.

2.3.2 Canonical flasque resolution of a presheaf on a site or a presheaf of module on a ringed topos

Let $\mathcal{S} \in \mathrm{Cat}$ a site with topology τ. Denote $a_\tau : \mathrm{PSh}(\mathcal{S}) \to \mathrm{Shv}(\mathcal{S})$ the sheaftification functor. There is for $F \in C(\mathcal{S})$ an explicit τ fibrant replacement :

- $k : F^\bullet \to E_\tau^\bullet(F^\bullet) := \mathrm{Tot}(E_\tau^\bullet(F^\bullet))$, if $F^\bullet \in C^+(\mathcal{S})$,
- $k : F^\bullet \to E_\tau^\bullet(F^\bullet) := \mathrm{holim}\,\mathrm{Tot}(E_\tau^\bullet(F^{\bullet \geq n}))$, if $F^\bullet \in C(\mathcal{S})$ is not bounded below.

The bicomplex $E^\bullet(F^\bullet) := E_\tau^\bullet(F^\bullet)$ together with the map $k : F^\bullet \to E^\bullet(F^\bullet)$ is given inductively by

- considering $p_{\mathcal{S}} : \mathcal{S}^\delta \to \mathcal{S}$ the morphism of site from the discrete category $\mathcal{S}^\tau$ whose objects are the points of the topos $\mathcal{S}$ and we take

$$k_0 := \mathrm{ad}(p_S^*, p_{S*})(F^\bullet) \to E^0(F^\bullet) := p_{S*}p_S^* F^\bullet := \bigoplus_{s \in \mathcal{S}^\tau} \lim_{X \in \mathcal{S}, s \in X} F^\bullet(X),$$

then $a_\tau k_0 : a_\tau F^\bullet \to E^0(F^\bullet)$ is injective and $E^0(F^\bullet)$ is τ fibrant,

- denote $Q^0(F^\bullet) := a_\tau \,\mathrm{coker}(k_0 : F^\bullet \to E^0(F^\bullet))$ and take the composite

$$E^0(F^\bullet) \to Q^0(F^\bullet) \to E^1(F^\bullet) := E^0(Q^0(F^\bullet)).$$

Note that $k : F^\bullet \to E^\bullet(F^\bullet)$ is a τ local equivalence and that $a_\tau k : a_\tau F^\bullet \to E^\bullet(F^\bullet)$ is injective by construction.

Since E^0 is functorial, E is functorial: for $m : F^\bullet \to G^\bullet$ a morphism, with $F^\bullet, G^\bullet \in C(\mathcal{S})$, we have a canonical morphism $E(m) : E(F) \to E(G)$ such that $E(m) \circ k = k' \circ m$, with $k : F \to E(F)$ and

$k' : G \to E(G)$. Note that E^0, hence E preserve monomorphisms. This gives, for $(F^\bullet, F) \in C_{fil}(\mathcal{S})$, a filtered τ local equivalence $k : (F^\bullet, F) \to E^\bullet(F^\bullet, F)$ with $E^\bullet(F^\bullet, F)$ filtered τ fibrant.

Moreover, we have a canonical morphism $E(F) \otimes E(G) \to E(F \otimes G)$.

There is, for $g : \mathcal{T} \to \mathcal{S}$ a morpism of presite with $\mathcal{T}, \mathcal{S} \in \mathrm{Cat}$ two site, and $F^\bullet \in C(\mathcal{S})$, a canonical transformation

$$T(g, E)(F^\bullet) : g^* E(F^\bullet) \to E(g^* F^\bullet) \tag{17}$$

given inductively by

- $T(g, E^0)(F) := \underline{T(g, p_S)(p_S^* F)} : g^* E^0(F) = g^* p_{S*} p_S^* F \to p_{T*} g^* p_S^* F = p_{T*} p_T^* g^* F = E^0(g^* F)$, $T(g, Q^0)(F) := \overline{T(g, E^0)(F)} : g^* Q^0(F) = \mathrm{coker}(g^* F \to g^* E^0(F)) \to Q^0(g^* F) = \mathrm{coker}(g^* F \to E^0(g^* F)$
- $T(g, Q^1)(F) : g^* E^1(F) = g^* E^0(Q^0(F)) \xrightarrow{T(g, E^0)(Q^0(F))} E^0(g^* Q^0(F)) \xrightarrow{E^0(T(g, Q^0)(F))} E^0(Q^0(g^* F)) = E^1(g^* F)$.

Let $(\mathcal{S}, O_S) \in \mathrm{RCat}$ with topology τ. Then, for $F^\bullet \in C_{O_S}(\mathcal{S})$, $E_\tau(F^\bullet)$ is naturally a complex of O_S modules such that $k : F^\bullet \to E_\tau(F^\bullet)$ is O_S linear, that is is a morphism in $C_{O_S}(\mathcal{S})$.

2.3.3 Canonical projective resolution of a presheaf of module on a ringed topos

Let $(\mathcal{S}, O_S) \in \mathrm{RCat}$. We recall that we denote by, for $U \in \mathcal{S}$, $\mathbb{Z}(U) \in \mathrm{PSh}(\mathcal{S})$ the presheaf represented by U : for $V \in \mathcal{S}$ $\mathbb{Z}(U)(V) = \mathbb{Z}\,\mathrm{Hom}(V, U)$, and for $h : V_1 \to V_2$ a morphism in $\mathcal{S}$, and $h_1 : V_1 \to U$ $\mathbb{Z}(U)(h) : h_1 \to h \circ h_1$, and s is the morphism of presheaf given by $s(V_1)(h_1) = F(h_1)(s) \in F(V_1)$. There is for $F \in C_{O_S}(\mathcal{S})$ a complex of O_S module an explicit projective replacement :

- $q : L_O^\bullet(F^\bullet) := \mathrm{Tot}(L_O^\bullet(F^\bullet)) \to F^\bullet$, if $F^\bullet \in C^-(\mathcal{S})$,
- $q : L_O^\bullet(F^\bullet) := \mathrm{holim}\,\mathrm{Tot}(L_O^\bullet(F^{\bullet \leq n}))$ if $F^\bullet \in C(\mathcal{S})$ is not bounded above.

For $O_S = \mathbb{Z}_\mathcal{S}$, we denote $L_{\mathbb{Z}_\mathcal{S}}^\bullet(F^\bullet) =: L(F^\bullet)$. The bicomplex $L_O^\bullet(F^\bullet)$ together with the map $q : L_O^\bullet(F^\bullet) \to F^\bullet$ is given inductively by

- considering the pairs $\{U \in \mathcal{S}, s \in F(U)\}$, where U is an object of $\mathcal{S}$ and s a section of F over U we take
$$q_0 : L_O^0(F) := \bigoplus_{(U \in \mathcal{S}, s \in F(U))} \mathbb{Z}(U) \otimes O_S \xrightarrow{s} F,$$
then q_0 is surjective and $L_O^0(F)$ is projective, this construction is functorial : for $m : F \to G$ a morphism in $\mathrm{PSh}(\mathcal{S})$ the following diagram commutes
$$\begin{tikzcd} \bigoplus_{(U \in \mathcal{S}, s \in F(U))} \mathbb{Z}(U) \otimes O_S \arrow[r, "q_0"] \arrow[d, "L_O(m)"'] & F \arrow[d, "m"] \\ \bigoplus_{(U \in \mathcal{S}, s' \in G(U))} \mathbb{Z}(U) \otimes O_S \arrow[r, "q_0"] & G \end{tikzcd}$$
where $(L_O(m)_{|(U,s)})_{(U, m(U)(s))} = I_{\mathbb{Z}(U)}$ and $(L_O(m)_{|(U,s)})_{(U,s')} = 0$ if $s' \neq m(U)(s)$,
- denote $K_O^0(F) := \ker(q_0 : L_O^0(F) \to F))$ and take the composite
$$q_1 : L_O^1(F^\bullet) := L_O^0(K_O^0(F^\bullet)) \xrightarrow{q_0(K_O^0(F))} K_O^0(F^\bullet) \hookrightarrow L_O^0(F^\bullet).$$

Note that $q = q(F) : L(F^\bullet) \to F^\bullet$ is a surjective quasi-isomorphism by construction. Since L^0_O is functorial, L_O is functorial : for $m : F^\bullet \to G^\bullet$ a morphism, with $F^\bullet, G^\bullet \in C(\mathcal{S})$, we have a canonical morphism $L_O(m) : L_O(F) \to L_O(G)$ such that $q' \circ L_O(m) = m \circ q'$, with $q : L_O(F) \to F$ and $q' : L_O(G) \to G$. Note that L^0_O and hence L_O preserve monomorphisms. In particular, it gives for $(F^\bullet, F) \in C_{O_S}(\mathcal{S})$, a filtered quasi-isomorphism $q : L_O(F^\bullet, F) \to (F^\bullet, F)$. Moreover, we have a canonical morphism $L_O(F) \otimes L_O(G) \to L_O(F \otimes G)$.

Let $g : \mathcal{T} \to \mathcal{S}$ a morphism of presite with $\mathcal{T}, \mathcal{S} \in \mathrm{Cat}$ two sites.

- Let $F^\bullet \in C(\mathcal{S})$. Since $g^*L(F^\bullet)$ is projective and $q(g^*F) : L(g^*F^\bullet) \to g^*F^\bullet$ is a surjective quasi-isomorphism, there is a canonical transformation

$$T(g, L)(F^\bullet) : g^*L(F^\bullet) \to L(g^*F^\bullet) \tag{18}$$

  unique up to homotopy such that $q(g^*F) \circ T(g, L)(F^\bullet) = g^*q(F)$.

- Let $F^\bullet \in C(\mathcal{S})$. Since $L(g^*F^\bullet)$ is projective and $g^*q(F) : g^*L(F^\bullet) \to g^*F^\bullet$ is a surjective quasi-isomorphism, there is a canonical transformation

$$T(g, L)(F^\bullet) : L(g^*F^\bullet) \to g^*L(F^\bullet) \tag{19}$$

  unique up to homotopy such that $g^*q(F) \circ T(g, L)(F^\bullet) = q(g^*F)$.

- Let $F^\bullet \in C(\mathcal{T})$. Since $L(g_*F^\bullet)$ is projective and $g_*q(F) : g_*L(F^\bullet) \to g_*F^\bullet$ is a surjective quasi-isomorphism, there is a canonical transformation

$$T_*(g, L)(F^\bullet) : L(g_*F^\bullet) \to g_*L(F^\bullet) \tag{20}$$

  unique up to homotopy such that $g_*q(F) \circ T_*(g, L)(F^\bullet) = q(g_*F)$.

Let $g : (\mathcal{T}, O_T) \to (\mathcal{S}, O_S)$ a morphism with $(\mathcal{T}, O_T), (\mathcal{S}, O_S) \in \mathrm{RCat}$. Let $F^\bullet \in C_{O_S}(\mathcal{S})$. Since $g^{*mod}L_O(F^\bullet)$ is projective and $q(g^{*mod}F) : L_O(g^{*mod}F^\bullet) \to g^{*mod}F^\bullet$ is a surjective quasi-isomorphism, there is a canonical transformation

$$T(g, L_O)(F^\bullet) : g^{*mod}L_O(F^\bullet) \to L_O(g^{*mod}F^\bullet) \tag{21}$$

unique up to homotopy such that $q(g^{*mod}F) \circ T(g, L_O)(F^\bullet) = g^{*mod}q(F)$.

Let $p : (\mathcal{S}_{12}, O_{S_{12}}) \to (\mathcal{S}_1, O_{S_1})$ a morphism with $(\mathcal{S}_{12}, O_{S_{12}}), (\mathcal{S}_1, O_{S_1}) \in \mathrm{RCat}$, such that the structural morphism $p^*O_{S_1} \to O_{S_{12}}$ is flat. Let $F^\bullet \in C_{O_S}(\mathcal{S})$. Since $L_O(p^{*mod}F^\bullet)$ is projective and $p^{*mod}q(F) : p^{*mod}L_O(F^\bullet) \to p^{*mod}F^\bullet$ is a surjective quasi-isomorphism, there is also in this case a canonical transformation

$$T(p, L_O)(F^\bullet) : L_O(p^{*mod}F^\bullet) \to p^{*mod}L_O(F^\bullet) \tag{22}$$

unique up to homotopy such that $p^{*mod}q(F) \circ T(p, L_O)(F^\bullet) = q(p^{*mod}F)$.

2.3.4 The De Rham complex of a ringed topos and functorialities

Let $A \in \mathrm{cRing}$ a commutative ring. For $M \in \mathrm{Mod}(A)$, we denote by

$$\mathrm{Der}_A(A, M) \subset \mathrm{Hom}(A, M) = \mathrm{Hom}_{\mathrm{Ab}}(A, M)$$

the abelian subgroup of derivation. Denote by $I_A = \ker(s_A : A \otimes A \to A) \subset A \otimes A$ the diagonal ideal with $s_A(a_1, a_2) = a_1 - a_2$. Let $\Omega_A := I_A/I_A^2 \in \mathrm{Mod}(A)$ together with its derivation map $d = d_A : A \to \Omega_A$. Then, for $M \in \mathrm{Mod}(A)$ the canonical map

$$w(M) : \mathrm{Hom}_A(\Omega_A, M) \xrightarrow{\sim} \mathrm{Der}_A(A, M), \ \psi \mapsto \phi \circ d$$

is an isomorphism, that is Ω_A is the universal derivation. In particular, its dual $T_A := \mathbb{D}^A(\Omega_A) = D^A(I_A/I_A^2)$ is isomorphic to the derivations group : $w(A) : T_A \xrightarrow{\sim} \mathrm{Der}_A(A,A)$. Also note that $\mathrm{Der}_A(A,A) \subset \mathrm{Hom}(A,A)$ is a Lie subalgebra. If $\phi : A \to B$ is a morphism of commutative ring, we have a canonical morphism of abelian group $\Omega_{(B/A)}\phi : \Omega_A \to \Omega_B$.

Let $(\mathcal{S}, O_S) \in \mathrm{RCat}$, with $O_S \in \mathrm{PSh}(\mathcal{S}, \mathrm{cRing})$ commutative. For $G \in \mathrm{PSh}_{O_S}(\mathcal{S})$, we denote by

$$\mathrm{Der}_{O_S}(O_S, G) \subset \mathcal{H}om(O_S, G) = \mathcal{H}om_{\mathrm{Ab}}(O_S, G)$$

the abelian subpresheaf of derivation. Denote by $\mathcal{I}_S = \ker(s_S : O_S \otimes O_S \to O_S) \in \mathrm{PSh}_{O_S \times O_S}(\mathcal{S})$ the diagonal ideal with $s_S(X) = s_{O_S(X)}$ for $X \in \mathcal{S}$. Then $\Omega_{O_S} := \mathcal{I}_S/\mathcal{I}_S^2 \in \mathrm{PSh}_{O_S}(\mathcal{S})$ together with its derivation map $d : O_S \to \Omega_{O_S}$ is the universal derivation O_S-module : the canonical map

$$w(G) : \mathcal{H}om_{O_S}(\Omega_{O_S}, G) \xrightarrow{\sim} \mathrm{Der}_{O_S}(O_S, G), \ \phi \mapsto \phi \circ d$$

is an isomorphism. In particular, its dual $T_{O_S} := \mathbb{D}^O_{O_S}(\Omega_{O_S}) = \mathbb{D}^O_{\mathcal{S}}(\mathcal{I}_S/\mathcal{I}_S^2)$ is isomorphic to the presheaf of derivations : $w(O_S) : T_{O_S} \xrightarrow{\sim} \mathrm{Der}_{O_S}(O_S, O_S)$ and $\mathrm{Der}_{O_S}(O_S, O_S) \subset \mathrm{Hom}(O_S, O_S)$ is a Lie subalgebra. The universal derivation $d = d_{O_S} : O_S \to \Omega_{O_S}$ induces the De Rham complex

$$DR(O_S) : \Omega^\bullet_{\mathcal{S}} := \wedge^\bullet \Omega_{O_S} \in C(\mathcal{S})$$

A morphism $\phi : O'_S \to O_S$ with $O_S, O'_S\,\mathrm{PSh}(\mathcal{S}, \mathrm{cRing})$ induces by the universal property canonical morphisms

$$\Omega_{O'_S/O_S} : \Omega_{O'_S} \to \Omega_{O_S} \ , \ \mathbb{D}^O_{O_S}\Omega_{O'_S/O_S} : T_{O_S} \to T_{O'_S}$$

in $\mathrm{PSh}_{O_S}(\mathcal{S})$.

- In the particular cases where $S = (S, O_S) \in \mathrm{Var}(\mathbb{C})$ or $S = (S, O_S) \in \mathrm{AnSp}(\mathbb{C})$, we denote as usual $\Omega_S := \Omega_{O_S/\mathbb{C}_S}$, $T_S := T_{O_S/\mathbb{C}_S}$ and $DR(S) := DR(O_S/\mathbb{C}_S) : \Omega^\bullet_S \in C(S)$.
- In the particular cases where $S = (S, O_S) \in \mathrm{Diff}(\mathbb{R})$ is a differential manifold, we denote as usual $\mathcal{A}_S := \Omega_{O_S/\mathbb{R}_S}$, $T_S := T_{O_S/\mathbb{R}_S}$ and $DR(S) := DR(O_S/\mathbb{R}_S) : \mathcal{A}^\bullet_S \in C(S)$.

For $f : (\mathcal{X}, O_X) \to (\mathcal{S}, O_S)$ with $(\mathcal{X}, O_X), (\mathcal{S}, O_S) \in \mathrm{RCat}$ such that O_X and O_S are commutative, we denote by

$$\Omega_{O_X/f^*O_S} := \mathrm{coker}(\Omega_{O_X/f^*O_S} : \Omega_{f^*O_S} \to \Omega_{O_X}) \in \mathrm{PSh}_{f^*O_S}(\mathcal{X})$$

the relative cotangent sheaf. The surjection $q = q_{O_X/f} : \Omega_{O_X} \to \Omega_{O_X/f^*O_S}$ gives the derivation $w(\Omega_{O_X/f^*O_S})(q) = d_{O_X/f} : O_X \to \Omega_{O_X/f^*O_S}$. It induces the surjections $q^p := \wedge^p q : \Omega^p_{O_X} \to \Omega^p_{O_X/f^*O_S}$. We then have the realtive De Rham complex

$$DR(O_X/f^*O_S) := \Omega^\bullet_{O_X/f^*O_S} := \wedge^\bullet \Omega_{O_X/f^*O_S} \in C_{f^*O_S}(\mathcal{X}).$$

whose differnetials are given by

$$\text{for } X^o \in \mathcal{X} \text{ and } \omega \in \Gamma(X^o, \Omega^p_{O_X}) \ d(q^p(\omega)) := q^{p+1}(d(\omega))$$

Note that $\Omega^\bullet_{O_X/f^*O_S} \in C_{f^*O_S}(\mathcal{S})$ is a complex of f^*O_S modules, but is NOT a complex of O_X module since the differential is a derivation hence NOT O_X linear. On the other hand, the canonical map in $\mathrm{PSh}_{f^*O_S}(\mathcal{S})$

$$T(f, hom)(O_S, O_S) : f^*\mathcal{H}om(O_S, O_S) \to \mathcal{H}om(f^*O_S, f^*O_S)$$

induces morphisms

$$T(f, hom)(O_S, O_S) : f^*T_{O_S} \to T_{f^*O_S} \text{ and } \mathbb{D}^O_{f^*O_S}T(f, hom)(O_S, O_S) : \Omega_{f^*O_S} \to f^*\Omega_{O_S}.$$

In this article, We will be interested in the following particular cases :

- In the particular case where O_S PSh($\mathcal{S}$, cRing) is a sheaf, $\Omega_{O_S}, T_{O_S} \in \mathrm{PSh}_{O_S}(\mathcal{S})$ are sheaves. Hence,

  $$T(f, hom)(O_S, O_S) : a_\tau f^* T_{O_S} \xrightarrow{\sim} T_{a_\tau f^* O_S} \text{ and } \mathbb{D}^O_{f^* O_S} T(f, hom)(O_S, O_S) : \Omega_{a_\tau f^* O_S} \xrightarrow{\sim} a_\tau f^* \Omega_{O_S}$$

  are isomorphisms,where $a_\tau : \mathrm{PSh}(\mathcal{S}) \to \mathrm{Shv}(\mathcal{S})$ is the sheaftification functor. We will note again in this case by abuse (as usual) $f^* O_S := a_\tau f^* O_S$, $f^* \Omega_{O_S} := a_\tau f^* \Omega_{O_S}$ and $f^* T_{O_S} := a_\tau f^* T_{O_S}$, so that

  $$\Omega_{f^* O_S} = f^* \Omega_{O_S} \text{ and } f^* T_{O_S} = T_{f^* O_S}$$

- In the particular cases where $S = (S, O_S), X = (X, O_X) \in \mathrm{Var}(\mathbb{C})$ or $S = (S, O_S), X = (X, O_X) \in \mathrm{AnSp}(\mathbb{C})$, we denote as usual $\Omega_{X/S} := \Omega_{O_X/f^* O_S}$, $q_{X/S} := q_{O_X/S} : \Omega_X \to \Omega_{X/S}$ and $DR(X/S) := DR(O_X/f^* O_S) : \Omega^\bullet_{X/S} \in C_{f^* O_S}(S)$.

- In the particular cases where $S = (S, O_S), X = (X, O_X) \in \mathrm{Diff}(\mathbb{R})$, we denote as usual $\mathcal{A}_{X/S} := \Omega_{O_X/f^* O_S}$, $q_{X/S} := q_{O_X/S} : \mathcal{A}_X \to \mathcal{A}_{X/S}$ and $DR(X/S) := DR(O_X/f^* O_S) : \mathcal{A}^\bullet_{X/S} \in C_{f^* O_S}(S)$.

Definition 1. *For a commutative diagram in* RCat

$$\begin{tikzcd} D = & (\mathcal{X}, O_X) \arrow[r, "f"] & (\mathcal{S}, O_S) \ , \\ & (\mathcal{X}', O_{X'}) \arrow[u, "g'"] \arrow[r, "f'"] & (\mathcal{T}, O_T) \arrow[u, "g"'] \end{tikzcd}$$

whose structural presheaves are commutative sheaves, the map in $C_{g'^* O_X fil}(\mathcal{X}')$

$$\Omega_{O_{X'}/g'^* O_X} : g'^*(\Omega^\bullet_{O_X}, F_b) = (\Omega^\bullet_{g'^* O_X}, F_b) \to (\Omega^\bullet_{O_{X'}}, F_b)$$

pass to quotient to give the map in $C_{g'^* O_X fil}(\mathcal{X}')$

$$\Omega_{(O_Y/g'^* O_X)/(O_T/g^* O_S)} := (\Omega_{O_Y/g'^* O_X})^q :$$
$$g'^*(\Omega^\bullet_{O_X/f^* O_S}, F_b) = (\Omega^\bullet_{g'^* O_X/g'^* f^* O_S}, F_b) \to (\Omega^\bullet_{O_Y/f'^* O_T}, F_b)$$

It is in particular given for $X'^o \in \mathcal{X}'$, $g'^*(X^o) \leftarrow X'^o$ *and* $\hat{\omega} \in \Gamma(X^o, \Omega^p_{O_X/f^* O_S})$,

$$\Omega_{(O_{X'}/g'^* O_X)/(O_T/g^* O_S)}(X'^o)(\omega) := q_{O_{X'}/f'}(\Omega_{O_{X'}/g'^* O_X}(\omega)) \in \Gamma(X'^o, \Omega^p_{O_{X'}/f'^* O_T}).$$

where $\omega \in \Gamma(X^o, \Omega^p_{O_X})$ *such that* $q_{O_X/f}(\omega) = \hat{\omega}$. *We then have the following canonical transformation map in* $C_{O_T fil}(\mathcal{T})$

$$T^O_\omega(D) : g^{*mod} L_O f_* E(\Omega^\bullet_{O_X/f^* O_S}, F_b) \xrightarrow{q} g^* f_* E(\Omega^\bullet_{O_X/f^* O_S}, F_b) \otimes_{g^* O_S} O_T$$
$$\xrightarrow{T(g', E)(-) \circ T(D)(E(\Omega^\bullet_{O_X/f^* O_S}))} (f'_* E(g'^*(\Omega^\bullet_{O_X/f^* O_S}, F_b))) \otimes_{g^* O_S} O_T$$
$$\xrightarrow{E(\Omega_{(O_{X'}/g'^* O_X)/(O_T/g^* O_S)})} f'_* E(\Omega^\bullet_{O_{X'}/f'^* O_T}, F_b) \otimes_{g^* O_S} O_T \xrightarrow{m} f'_* E(\Omega^\bullet_{O_{X'}/f'^* O_T}, F_b),$$

with $m(n \otimes s) = s.n$.

2.4 Presheaves on diagrams of sites or on diagrams of ringed topos

Let $\mathcal{I}, \mathcal{I}' \in \mathrm{Cat}$ and $(f_\bullet, s) : \mathcal{T}_\bullet \to \mathcal{S}_\bullet$ a morphism of diagrams of presites with $\mathcal{T}_\bullet \in \mathrm{Fun}(\mathcal{I}, \mathrm{Cat})$, $\mathcal{S}_\bullet \in \mathrm{Fun}(\mathcal{I}', \mathrm{Cat})$. Recall it is by definition given by a functor $s : \mathcal{I} \to I'$ and morphism of functor $P(f_\bullet)$:

$\mathcal{S}_{s(\bullet)} := \mathcal{S}_\bullet \circ s \to \mathcal{T}_\bullet$. and that we denote for short, $\mathcal{S}_{s(\bullet)} := \mathcal{S}_\bullet \circ s \in \mathrm{Fun}(\mathcal{I}, \mathrm{Cat})$. Recall that, for $r_{IJ} : I \to J$ a morphism, with $I, J \in \mathcal{I}$, D_{fIJ} is the commutative diagram in Cat

$$D_{fIJ} := \begin{tikzcd} \mathcal{S}_{s(J)} \arrow[r, "r^s_{IJ}"] & \mathcal{S}_{s(I)} \\ \mathcal{T}_J \arrow[u, "f_J"] \arrow[r, "r^t_{IJ}"] & \mathcal{T}_I \arrow[u, "f_I"'] \end{tikzcd} .$$

The adjonction

$$\begin{aligned} ((f_\bullet, s)^*, (f_\bullet, s)_*) = ((f_\bullet, s)^{-1}, (f_\bullet, s)_*) : C(\mathcal{S}_{s(\bullet)}) \leftrightarrows C(\mathcal{T}_\bullet), \\ G = (G_I, u_{IJ}) \mapsto (f_\bullet, s)^*(G) := (f_I^*(G_I), T(D_{fIJ})(G_J) \circ f_I^* u_{IJ}) \\ G = (G_I, u_{IJ}) \mapsto (f_\bullet, s)_*(G) := (f_{I*}(G_I), f_{I*} u_{IJ}) \end{aligned}$$

gives an adjonction

$$\begin{aligned} ((f_\bullet, s)^*, (f_\bullet, s)_*) : C_{(2)fil}(\mathcal{S}_{s(\bullet)}) \leftrightarrows C_{(2)fil}(\mathcal{T}_\bullet), \\ (G, F) = ((G_I, F), u_{IJ}) \mapsto (f_\bullet, s)^*(G, F) := (f_I^*(G_I, F), T(D_{fIJ})(G_J, F) \circ f_I^* u_{IJ}) \\ (G, F) = ((G_I, F), u_{IJ}) \mapsto (f_\bullet, s)_*(G, F) := (f_{I*}(G_I, F), f_{I*} u_{IJ}). \end{aligned}$$

For a commutative diagram of diagrams of presite :

$$D = \begin{tikzcd} \mathcal{T}'_\bullet \arrow[r, "(g_2, s'_2)"] \arrow[d, "(f_2, s_2)"] & \mathcal{S}'_{s'_2(\bullet)} \arrow[d, "(f_1, s_1)"] \\ \mathcal{T}_{s_2(\bullet)} \arrow[r, "(g_1, s'_1)"] & \mathcal{S}_{s(\bullet)} \end{tikzcd} ,$$

with $\mathcal{I}, \mathcal{I}', \mathcal{J}, \mathcal{J}' \in \mathrm{Cat}$ and $\mathcal{T}_\bullet \in \mathrm{Fun}(\mathcal{I}, \mathrm{Cat}), \mathcal{T}'_\bullet \in \mathrm{Fun}(\mathcal{I}', \mathrm{Cat}), \mathcal{S}_\bullet \in \mathrm{Fun}(\mathcal{J}, \mathrm{Cat}), \mathcal{S}'_\bullet \in \mathrm{Fun}(\mathcal{J}', \mathrm{Cat})$, and $s = s_1 \circ s'_2 = s_2 \circ s'_1 : \mathcal{I}' \to J$, we denote by, for $F = (F_I, u_{IJ}) \in C(\mathcal{S}'_{s'_2(\bullet)})$,

$$T(D)(F) : g_1^* f_{1*} F \xrightarrow{g_1^* f_{1*} \operatorname{ad}(g_2^*, g_{2*})(F)} g_1^* f_{1*} g_{2*} g_2^* F = g_1^* g_{1*} f_{2*} g_2^* F \xrightarrow{\operatorname{ad}(g_1^* g_{1*})(f_{2*} g_2^* F)} f_{2*} g_2^* F$$

the canonical transformation map in $C(\mathcal{T}_{s_2(\bullet)})$, and for $(G, F) = ((G_I, F), u_{IJ}) \in C_{fil}(\mathcal{S}'_{s'_2(\bullet)})$,

$$\begin{aligned} T(D)(G, F) : g_1^* f_{1*}(G, F) \xrightarrow{g_1^* f_{1*} \operatorname{ad}(g_2^*, g_{2*})(G, F)} g_1^* f_{1*} g_{2*} g_2^*(G, F) = g_1^* g_{1*} f_{2*} g_2^*(G, F) \\ \xrightarrow{\operatorname{ad}(g_1^* g_{1*})(f_{2*} g_2^*(G, F))} f_{2*} g_2^*(G, F) \end{aligned}$$

the canonical transformation map in $C_{fil}(\mathcal{T}_{s_2(\bullet)})$ given by the adjonction maps.

Let $\mathcal{S}_\bullet \in \mathrm{Fun}(\mathcal{I}, \mathrm{RCat})$ a diagram of ringed topos with $\mathcal{I} \in \mathrm{Cat}$. We have the tensor product bifunctor

$$\begin{aligned} (\cdot) \otimes (\cdot) : \mathrm{PSh}(\mathcal{S}_\bullet)^2 \to \mathrm{PSh}(\mathcal{S}_\bullet), \\ ((F_I, u_{IJ}, (G_I, u_{IJ})) \mapsto (F_I, u_{IJ}) \otimes (G_I, v_{IJ}) := (F_I \otimes G_I, u_{IJ} \otimes v_{IJ}) \end{aligned}$$

We get the bifunctors

$$(-) \otimes (-) : C_{fil}(\mathcal{S}_\bullet)^2 \to C_{fil}(\mathcal{S}_\bullet), \;\; (-) \otimes (-) : C_{fil}(\mathcal{S}_\bullet) \times C_{O_S fil}(\mathcal{S}_\bullet) \to C_{O_S fil}(\mathcal{S}_\bullet).$$

We have the tensor product bifunctor

$$\begin{aligned} (\cdot) \otimes_{O_S} (\cdot) : \mathrm{PSh}_{O_S}(\mathcal{S}_\bullet)^2 \to \mathrm{PSh}(\mathcal{S}_\bullet), \\ ((F_I, u_{IJ}, (G_I, u_{IJ})) \mapsto (F_I, u_{IJ}) \otimes_{O_{S_I}} (G_I, v_{IJ}) := (F_I \otimes_{O_{S_I}} G_I, u_{IJ} \otimes v_{IJ}) \end{aligned}$$

which gives,

- in all case it gives the bifunctor $(-)\otimes_{O_S}(-) : C_{O_S^{op}fil}(\mathcal{S}_\bullet)\otimes C_{O_Sfil}(\mathcal{S}_\bullet)\to C_{fil}(\mathcal{S}_\bullet)$.
- in the case O_S is commutative, it gives the bifunctor $(-)\otimes_{O_S}(-) : C_{O_Sfil}(\mathcal{S}_\bullet)^2\to C_{O_Sfil}(\mathcal{S}_\bullet)$.

Let $(f_\bullet,s):(\mathcal{T}_\bullet,O_T)\to(\mathcal{S}_\bullet,O_S)$ a morphism with $(\mathcal{S}_\bullet,O_S)\in\mathrm{Fun}(\mathcal{I}',\mathrm{RCat}),(\mathcal{T}_\bullet,O_T)\in\mathrm{Fun}(\mathcal{I},\mathrm{RCat})$ and $\mathcal{I},\mathcal{I}'\in\mathrm{Cat}$. which is by definition given by a functor $s:\mathcal{I}\to\mathcal{I}'$ and morphism of ringed topos $f_\bullet:(\mathcal{T}_\bullet,O_T)\to(\mathcal{S}_{s(\bullet)},O_S)$. As before, we denote for short, $(\mathcal{S}_{s(\bullet)},O_S):=(\mathcal{S}_\bullet,O_S)\circ s\in\mathrm{Fun}(\mathcal{I},\mathrm{RCat})$. Denote as before, for $r_{IJ}:I\to J$ a morphism, with $I,J\in\mathcal{I}$, D_{fIJ} the commutative diagram in RCat

$$D_{fIJ}:=\begin{tikzcd} \mathcal{S}_{s(J)} \arrow[r, "r^s_{IJ}"] & \mathcal{S}_{s(I)} \\ \mathcal{T}_J \arrow[u, "f_J"] \arrow[r, "r^t_{IJ}"] & \mathcal{T}_I \arrow[u, "f_I"'] \end{tikzcd}.$$

We have then the adjonction

$$((f_\bullet,s)^{*mod},(f_\bullet,s)_*):C_{O_S}(\mathcal{S}_{s(\bullet)})\leftrightarrows C_{O_T}(\mathcal{T}_\bullet),$$
$$(G_I,u_{IJ})\mapsto(f_\bullet,s)^{*mod}(G_I,u_{IJ}):=(f_I^{*mod}G_I,T^{mod}(D_{fIJ})(G_J)\circ f_I^{*mod}u_{IJ}),$$
$$(G_I,u_{IJ})\mapsto(f_\bullet,s)_*(G_I,u_{IJ}):=(f_{I*}G_I,f_{I*}u_{IJ}).$$

which induces the adjonction

$$((f_\bullet,s)^{*mod},(f_\bullet,s)*):C_{O_Sfil}(\mathcal{S}_{s(\bullet)})\leftrightarrows C_{O_Tfil}(\mathcal{T}_\bullet),$$
$$((G_I,F),u_{IJ})\mapsto(f_\bullet,s)^{*mod}((G_I,F),u_{IJ}):=(f_I^{*mod}(G_I,F),T^{mod}(D_{fIJ})(G_J)\circ f_I^{*mod}u_{IJ}),$$
$$((G_I,F),u_{IJ})\mapsto(f_\bullet,s)_*((G_I,F),u_{IJ}):=(f_{I*}(G_I,F),f_{I*}u_{IJ}).$$

For a commutative diagram of diagrams of ringed topos, :

$$D=\begin{tikzcd} (\mathcal{T}'_\bullet,O'_2) \arrow[r, "(g_2,s'_2)"] \arrow[d, "(f_2,s_2)"] & (\mathcal{S}'_{s'_2(\bullet)},O'_1) \arrow[d, "(f_1,s_1)"] \\ (\mathcal{T}_\bullet,O_2) \arrow[r, "(g_1,s'_1)"] & (\mathcal{S}_{s(\bullet)},O_1) \end{tikzcd},$$

with $\mathcal{I},\mathcal{I}',\mathcal{J},\mathcal{J}'\in\mathrm{Cat}$ and $\mathcal{T}_\bullet\in\mathrm{Fun}(\mathcal{I},\mathrm{Cat}),\mathcal{T}'_\bullet\in\mathrm{Fun}(\mathcal{I}',\mathrm{Cat}),\mathcal{S}_\bullet\in\mathrm{Fun}(\mathcal{J},\mathrm{Cat}),\mathcal{S}'_\bullet\in\mathrm{Fun}(\mathcal{J}',\mathrm{Cat})$, and $s=s_1\circ s'_2=s'_1\circ s_2:\mathcal{I}'\to\mathcal{J}$, we denote by, for $F=(F_I,u_{IJ})\in C_{O'_1}(\mathcal{S}'_{s'_2(\bullet)})$,

$$T^{mod}(D)(F):g_1^{*mod}f_{1*}F\xrightarrow{g_1^{*mod}f_{1*}\,\mathrm{ad}(g_2^{*mod},g_{2*})(F)}g_1^{*mod}f_{1*}g_{2*}g_2^{*mod}F=g_1^{*mod}g_{1*}f_{2*}g_2^{*mod}F$$
$$\xrightarrow{\mathrm{ad}(g_1^{*mod}g_{1*})(f_{2*}g_2^{*mod}F)}f_{2*}g_2^{*mod}F$$

the canonical transformation map in $C_{O_2}(\mathcal{T}_{s_2(\bullet)})$, and for $G=((G_I,F),u_{IJ})\in C_{O'_1fil}(\mathcal{S}'_{s'_2(\bullet)})$,

$$T^{mod}(D)(G,F):g_1^{*mod}f_{1*}(G,F)\xrightarrow{g_1^{*mod}f_{1*}\,\mathrm{ad}(g_2^{*mod},g_{2*})(G,F)}g_1^{*mod}f_{1*}g_{2*}g_2^{*mod}(G,F)=g_1^{*mod}g_{1*}f_{2*}g_2^{*mod}(G,F)$$
$$\xrightarrow{\mathrm{ad}(g_1^{*mod}g_{1*})(f_{2*}g_2^{*mod}(G,F))}f_{2*}g_2^{*mod}(G,F)$$

the canonical transformation map in $C_{O_2fil}(\mathcal{T}_{s_2(\bullet)})$ given by the adjonction maps.

Let $(\mathcal{S}_\bullet,O_S)\in\mathrm{Fun}(\mathcal{I},\mathrm{RCat})$ a diagram of ringed topos with $\mathcal{I}\in\mathrm{Cat}$ and, for $I\in\mathcal{I}$, $\mathcal{S}_I$ is endowed with topology τ_I and for $r:I\to J$ a morphism with $I,J\in\mathcal{I}$, $r_{IJ}:\tilde{S}_J\to\tilde{S}_I$ is continous. Then the diagram category $(\Gamma\mathcal{S}_\bullet,O_S)\in\mathrm{RCat}$ is endowed with the associated canonical topology τ, and then

- A morphism $\phi = (\phi_I) : ((F_I, F), u_{IJ}) \to ((G_I, F), u_{IJ})$ with $((F_I, F), u_{IJ}), ((G_I, F), u_{IJ}) \in C_{O_S fil}(\mathcal{S}_\bullet)$ is a filtered τ local equivalence if and only if the ϕ_I are filtered τ local equivalences for all $I \in \mathcal{I}$.

- Let $r \in \mathbb{N}$. A morphism $\phi = (\phi_I) : ((F_I, F), u_{IJ}) \to ((G_I, F), u_{IJ})$ with $((F_I, F), u_{IJ}), ((G_I, F), u_{IJ}) \in C_{O_S fil}(\mathcal{S}_\bullet)$ is an r-filtered τ local equivalence if and only if the ϕ_I are r-filtered τ local equivalences for all $I \in \mathcal{I}$.

- A complex of presheaves $((G_I, F), u_{IJ}) \in C_{O_S fil}(\mathcal{S}_\bullet)$ is filtered τ fibrant if and only if the $(G_I, F) \in C_{O_S fil}(\mathcal{S}_I)$ are filtered τ fibrant for all $I \in \mathcal{I}$.

- Let $r \in \mathbb{N}$. A complex of presheaves $((G_I, F), u_{IJ}) \in C_{O_S fil}(\mathcal{S}_\bullet)$ is r-filtered τ fibrant if and only if $(G_I, F) \in C_{O_S fil}(\mathcal{S}_I)$ are r-filtered τ fibrant for all $I \in \mathcal{I}$.

2.5 Presheaves on topological spaces and presheaves of modules on a ringed spaces

In this subsection, we will consider the particular case of presheaves on topological spaces.

Let $f : T \to S$ a continous map with $S, T \in \mathrm{Top}$. We denote as usual the adjonction

$$(f^*, f_*) := (P(f)^*, P(f)_*) : \mathrm{PSh}(S) \leftrightarrows \mathrm{PSh}(T)$$

induced by the morphism of site given by the pullback functor

$$P(f) : \mathrm{Ouv}(S) \to \mathrm{Ouv}(T),\ (S^o \subset S) \mapsto P(f)(S^o) := S^o \times_S T \xrightarrow{\sim} f^{-1}(S^o) \subset T$$

Since the colimits involved in the definition of $f^* = P(f)^*$ are filtered, f^* also preserve monomorphism. Hence, we get an adjonction

$$(f^*, f_*) : \mathrm{PSh}_{fil}(S) \leftrightarrows \mathrm{PSh}_{fil}(T),\ f^*(G, F) := (f^*G, f^*F)$$

Let $f : (T, O_T) \to (S, O_S)$ a morphism with $(S, O_S), (T, O_T) \in \mathrm{Top}$. We have then the adjonction

$$(f^{*mod}, f_*) := (P(f)^{*mod}, P(f)_*) : \mathrm{PSh}_{O_S fil}(S) \leftrightarrows \mathrm{PSh}_{O_T fil}(T),\ f^{*mod}(G, F) := f^*(G, F) \otimes_{f^*O_S} O_T$$

Recall $\mathrm{CW} \subset \mathrm{Top}$ is the full subcategory whose objects consists of CW complexes. Denote, for $n \in \mathbb{N}$, $\mathbb{I}^n := [0, 1]^n, S^n := \mathbb{I}^n/\partial\mathbb{I}^n \in \mathrm{CW}$ and $\Delta^n \subset I^n$ the n dimensional simplex. We get $\mathbb{I}^*, \Delta^* \in \mathrm{Fun}(\Delta, \mathrm{CW})$ Denote for $S \in \mathrm{Top}$, $\Sigma_1 S := S \times \mathbb{I}^1/((\{0\} \times S) \cup (\{1\} \times S)) \in \mathrm{Top}$.

- Let $f : T \to S$ a morphism with $T, S \in \mathrm{Top}$. We have the mapping cylinder $\mathrm{Cyl}(f) := (T \times I^1) \sqcup_f S \in \mathrm{Top}$ and the mapping cone $\mathrm{Cone}(f) := (T \times \mathbb{I}^1) \sqcup_f S \in \mathrm{Top}$. We have then the quotient map $q_f : \mathrm{Cyl}(f) \to \mathrm{Cone}(f)$ and a canonical retraction $r_f : \mathrm{Cone}(f) \to \Sigma^1 T$

- Recall two morphisms $f, g : T \to S$ with $T, S \in \mathrm{Top}$ are homotopic if there exist $H : T \times I^1 \to S$ continous such that $H \circ (I \times i_0) = f$ and $H \circ (I \times i_1) = g$. Then $K(\mathrm{Top}) := \mathrm{Ho}_{I^1}(\mathrm{Top})$ is a triangulated category with distinguish triangle

$$T \xrightarrow{i_T} \mathrm{Cyl}(f) \xrightarrow{q_f} \mathrm{Cone}(f) \xrightarrow{r_f} \Sigma^1 T.$$

- For $X \in \mathrm{Top}$, denote for $n \in \mathbb{N}$, $\pi_n(X) : \mathrm{Hom}_{K(\mathrm{Top})}(S^n, X)$ the homotopy groups. For $f : T \to S$ a morphism with $T, S \in \mathrm{Top}$, we have for $n \in \mathbb{N}$ the morphisms of abelian groups

$$f_* : \pi_n(T) \to \pi_n(S), h \mapsto f \circ h$$

Recall two morphisms $f, g : T \to S$ with $T, S \in \mathrm{Top}$ are weakly homotopic if $f_* = g_* : \pi_n(T) \to \pi_n(S)$ for all $n \in \mathbb{N}$.

- For $X \in \mathrm{Top}$, denote by $C_*^{\mathrm{sing}}(X) := \mathbb{Z}\operatorname{Hom}(\Delta^*, X) \in C^-(\mathbb{Z})$ the complex of singular chains and by $C^*_{\mathrm{sing}}(X) := \mathbb{D}^{\mathbb{Z}} C_*^{\mathrm{sing}}(X) := \mathbb{D}\mathbb{Z}\operatorname{Hom}(\Delta^*, X) \in C^-(\mathbb{Z})$ the complex of singular cochains. For $f : T \to S$ a morphism with $T, S \in \mathrm{Top}$, we have

  - the morphism of complexes of abelian groups
$$f_* : C_*^{\mathrm{sing}}(T) \to C_*^{\mathrm{sing}}(S), \sigma \mapsto f \circ \sigma,$$
  - the morphism of complexes of abelian groups
$$f^* := \mathbb{D}^{\mathbb{Z}} f_* : C^*_{\mathrm{sing}}(T) \to C^*_{\mathrm{sing}}(S),\ \alpha \longmapsto f^*\alpha : (\sigma \mapsto f^*\alpha(\sigma) := \alpha(f \circ \sigma))$$

  We denote by $C^*_{X,\mathrm{sing}} \in C^+(X)$ the complex of presheaves of singular cochains given by,
$$\begin{aligned}(U \subset X) \mapsto C^*_{X,\mathrm{sing}}(U) := C^*_{X,\mathrm{sing}}(U) := C^*_{\mathrm{sing}}(U) := \mathbb{D}^{\mathbb{Z}}\mathbb{Z}\operatorname{Hom}(\Delta^*, U),\\ (j : U_2 \hookrightarrow U_1) \mapsto (j^* : C^*_{\mathrm{sing}}(U_1) \to C^*_{\mathrm{sing}}(U_2)\end{aligned}$$
  and by $c_X : \mathbb{Z}_X \to C^*_{X,\mathrm{sing}}$ the inclusion map. For $f : T \to S$ a morphism with $T, S \in \mathrm{Top}$, we have the morphism of complexes of presheaves
$$f^* : C^*_{S,\mathrm{sing}} \to f_* C^*_{T,\mathrm{sing}}$$
  in $C(S)$.

Theorem 7. *(i) If two morphisms $f, g : T \to S$ with $T, S \in \mathrm{Top}$ are weakly homotopic, then*
$$H^n(f_*) = H^n(g_*) : H_{n,\mathrm{sing}}(T, \mathbb{Z}) := H^n C_*^{\mathrm{sing}}(T) \to H_{n,\mathrm{sing}}(S, \mathbb{Z}) := H^n C_*^{\mathrm{sing}}(S).$$

(ii) For $S \in \mathrm{Top}$ there exists $CW(S) \in \mathrm{CW}$ together with a morphism $L_S : CW(S) \to S$ which is a weakly homotopic equivalence, that is $L_{S*} : \pi_n(CW(S)) \xrightarrow{\sim} \pi_n(S)$ are isomorphisms of abelian groups for all $n \in \mathbb{N}$.

(ii)' For $f : T \to S$ a morphism, with $T, S \in \mathrm{Top}$, and $L_S : CW(S) \to S$, $L_S : CW(T) \to T$ weakly homotopy equivalence with $CW(S), CW(T) \in \mathrm{CW}$ there exist a morphism $L(f) : CW(T) \to CW(S)$ unique up to homotopy such that the following diagram in Top commutes
$$\begin{tikzcd} CW(S) \arrow[r, "L_S"] & S \\ CW(T) \arrow[u, "L(f)"] \arrow[r, "L_T"] & T \arrow[u, "f"'] \end{tikzcd}.$$

In particular, for $S \in \mathrm{Top}$, $CW(S)$ is unique up to homotopy.

Proof. See [14]. □

We have Kunneth formula for singular cohomology :

Proposition 8. *Let $X_1, X_2 \in \mathrm{Top}$. Denote by $p_1 : X_1 \times X_2 \to X_1$ and $p_2 : X_1 \times X_2 \to X_2$ the projections. Then*
$$p_1^* \otimes p_2^* : C^*_{\mathrm{sing}}(X_1) \otimes C^*_{\mathrm{sing}}(X_2) \to C^*_{\mathrm{sing}}(X_1 \times X_2)$$
is a quasi-isomorphism.

Proof. Standard (see [14] for example): follows from the fact that for all $p \in \mathbb{N}$, $H^n C^*_{\mathrm{sing}}(\Delta^p) = 0$ for all $n \in \mathbb{Z}$. □

Remark 2. *By definition, $X \in \mathrm{Top}$ is locally contractile if an only if the inclusion map $c_X : \mathbb{Z}_X \to C^*_{X,\mathrm{sing}}$ is an equivalence top local. In this case it induce, by taking injective resolutions, for $n \in \mathbb{Z}$ isomorphisms*

$$H^n c^k_X : H^n(X, \mathbb{Z}_X) \xrightarrow{\sim} \mathbb{H}^n(X, C^*_{X,\mathrm{sing}}) = H^n C^*_{\mathrm{sing}}(X) =: H^n_{\mathrm{sing}}(X, \mathbb{Z}).$$

We will use the following easy propositions :

Proposition 9. *(i) Let $(S, O_S) \in \mathrm{RTop}$. Then, if $K^\bullet \in C^-_{O_S}(S)$ is a bounded above complex such that $K^n \in \mathrm{PSh}_{O_S}(S)$ are locally free for all $n \in \mathbb{Z}$, and $\phi : F^\bullet \to G^\bullet$ is a top local equivalence with $F, G \in C_{O_S}(\mathcal{S})$, then $\phi \otimes I : F^\bullet \otimes_{O_S} L^\bullet \to G^\bullet \otimes_{O_S} L^\bullet$ is an equivalence top local.*

(ii) Let $f : (T, O_T) \to (S, O_S)$ a morphism with $(T, O_T), (S, O_S) \in \mathrm{RTop}$. Then, if $K \in C^b_{O_S}(S)$ is a bounded complex such that $K^n \in \mathrm{PSh}_{O_S}(S)$ are locally free for all $n \in \mathbb{Z}$, and $N \in C_{O_T}(T)$

$$k \circ T^{mod}(f, \otimes)(M, E(N)) : K \otimes_{O_S} f_* E(N) \to f_*((f^{*mod} K) \otimes_{O_T} E(N)) \to f_* E((f^{*mod} K) \otimes_{O_T} E(N))$$

is an equivalence top local.

Proof. Standard. $\square$

Proposition 10. *Let $i : (Z, O_Z) \hookrightarrow (S, O_S)$ a closed embedding of ringed spaces, with $Z, S \in \mathrm{Top}$. Then for $M \in C_{O_S}(S)$ and $M \in C_{i^* O_S}(Z)$,*

$$T(i, \otimes)(M, N) : M \otimes_{O_S} i_* N \to i_*(i^* M \otimes_{i^* O_S} N)$$

is an equivalence top local.

Proof. Standard. Follows form the fact that $j^* i_* N = 0$. $\square$

We note the following :

Proposition 11. *Let $(S, O_S) \in \mathrm{Sch}$ such that $O_{S,s}$ are reduced local rings for all $s \in S$. For $s \in S$ consider $q : L_{O_{S,s}}(k(s)) \to k(s)$ the canonical projective resolution of the $O_{S,s}$ module $k(s) := O_{S,s}/m_s$ (the residual field) of $s \in S$. For $s \in S$ denote by $i_s : \{s\} \hookrightarrow S$ the embedding. Let $\phi : F \to G$ a morphism with $F, G \in C_{O_S,c}(S)$ i.e. such that $a_{zar} H^n F, a_{zar} H^n G \in \mathrm{Coh}(S)$. If*

$$i_s^* \phi \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s)) : i_s^* F \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s)) \to i_s^* G \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s))$$

is a quasi-isomorphism for all $s \in S$, then $\phi : F \to G$ is an equivalence top local.

Proof. Let $s \in S$. Since tensorizing with $L_{i_s^* O_S}(k(s))$ is an exact functor, we have canonical isomorphism $\alpha(F)$,$\alpha(G)$ fiting in a commutative diagram

$$\begin{tikzcd}
H^n(i_s^* F \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s))) \arrow[r, "H^n(i_s^* \phi \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s)))"] \arrow[d, "\alpha(F)"] & H^n(i_s^* G \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s))) \arrow[d, "\alpha(G)"] \\
i_s^*(H^n F) \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s)) \arrow[r, "i_s^*(H^n \phi) \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s))"] & i_s^*(H^n G) \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s))
\end{tikzcd}$$

Let $n \in \mathbb{Z}$. By hypothesis

$$H^n(i_s^* \phi \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s))) : H^n(i_s^* F \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s))) \xrightarrow{\sim} H^n(i_s^* G \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s)))$$

is an isomorphism. Hence, the diagram 2.5 implies that

$$i_s^*(H^n \phi) \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s)) : i_s^*(H^n F) \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s)) \xrightarrow{\sim} i_s^*(H^n G) \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s))$$

is an isomorphism. We conclude on the one hand that $i_s^*H^n\phi : i_s^*H^nF \to i_s^*H^nG$ is surjective by Nakayama lemma since $i_s^*H^nF$, $i_s^*H^nG$ are $O_{S,s}$ modules of finite type as $F, G \in C_{O_S,c}(S)$ has coherent cohomology sheaves, and on the other hand that the rows of the following commutative diagram are isomorphism

$$\begin{tikzcd}
H^0(i_s^*(H^nF)\otimes_{i_s^*O_S} L_{i_s^*O_S}(k(s))) \arrow[r, "H^0(i_s^*(H^n\phi)\otimes_{i_s^*O_S} L_{i_s^*O_S}(k(s)))\sim"] \arrow[d, "="] & H^0(i_s^*(H^nG)\otimes_{i_s^*O_S} L_{i_s^*O_S}(k(s))) \arrow[d, "="] \\
i_s^*(H^nF)\otimes_{i_s^*O_S} k(s) \arrow[r, "i_s^*(H^n\phi)\otimes_{i_s^*O_S} k(s)\sim"] & i_s^*(H^nG)\otimes_{i_s^*O_S} k(s)
\end{tikzcd}.$$

Since

$$i_s^*(H^n\phi)\otimes_{i_s^*O_S} k(s) : i_s^*(H^nF)\otimes_{i_s^*O_S} k(s) \xrightarrow{\sim} i_s^*(H^nF)\otimes_{i_s^*O_S} k(s)$$

is an isomorphism for all $s \in S$, $O_{S,s} =: i_s^*O_S$ are reduced, and $a_{zar}H^nF, a_{zar}H^nG$ are coherent, $i_s^*H^n\phi : i_s^*H^nF \to i_s^*H^nG$ are injective. $\square$

Let $i : Z \hookrightarrow S$ a closed embedding, with $S, Z \in \mathrm{Top}$. Denote by $j : S\backslash Z \hookrightarrow S$ the open embedding of the complementary subset. We have the adjonction

$$(i_*, i^!) := (i_*, i^\perp) : C(Z) \to C(S), \text{ with in this case } i^!F := \ker(F \to j_*j^*F).$$

It induces the adjonction $(i_*, i^!) : C_{(2)fil}(Z) \to C_{(2)fil}(S)$ (we recall that $i^! := i^\perp$ preserve monomorphisms).

Let $i : Z \hookrightarrow S$ a closed embedding, with $S, Z \in \mathrm{Top}$. Denote by $j : S\backslash Z \hookrightarrow S$ the open embedding of the complementary subset. We have the support section functors :

- We have the functor

  $$\Gamma_Z : C(S) \to C(S),\ F \mapsto \Gamma_Z(F) := \mathrm{Cone}(\mathrm{ad}(j^*, j_*)(F) : F \to j_*j^*F)[-1],$$

  together with the canonical map $\gamma_Z(F) : \Gamma_ZF \to F$. We have the factorization

  $$\mathrm{ad}(i_*, i^!)(F) : i_*i^!F \xrightarrow{\mathrm{ad}(i_*,i^!)(F)^\gamma} \Gamma_ZF \xrightarrow{\gamma_Z(F)} F,$$

  and $\mathrm{ad}(i_*, i^!)(F)^\gamma : i_*i^!F \to \Gamma_ZF$ is an homotopy equivalence. Since Γ_Z preserve monomorphisms, it induce the functor

  $$\Gamma_Z : C_{fil}(S) \to C_{fil}(S),\ (G, F) \mapsto \Gamma_Z(G, F) := (\Gamma_ZG, \Gamma_ZF),$$

  together with the canonical map $\gamma_Z((G, F) : \Gamma_Z(G, F) \to (G, F)$.

- We have also the functor

  $$\Gamma_Z^\vee : C(S) \to C(S),\ F \mapsto \Gamma_Z^\vee F := \mathrm{Cone}(\mathrm{ad}(j_!, j^*)(F) : j_!j^*F \to F),$$

  together with the canonical map $\gamma_Z^\vee(F) : F \to \Gamma_Z^\vee F$. We have the factorization

  $$\mathrm{ad}(i^*, i_*)(F) : F \xrightarrow{\gamma_Z^\vee(F)} \Gamma_Z^\vee F \xrightarrow{\mathrm{ad}(i^*,i_*)(F)^\gamma} i_*i^*F,$$

  and $\mathrm{ad}(i^*, i_*)(F)^\gamma : \Gamma_Z^\vee F \to i_*i^*F$ is an homotopy equivalence. Since $\Gamma_Z^\vee$ preserve monomorphisms, it induce the functor

  $$\Gamma_Z : C_{fil}(S) \to C_{fil}(S),\ (G, F) \mapsto \Gamma_Z^\vee(G, F) := (\Gamma_Z^\vee G, \Gamma_Z^\vee F),$$

  together with the canonical map $\gamma_Z^\vee(G, F) : (G, F) \to \Gamma_Z^\vee(G, F)$.

Definition-Proposition 1. *(i) Let $g : S' \to S$ a morphism and $i : Z \hookrightarrow S$ a closed embedding with $S', S, Z \in \mathrm{Top}$. Then, for $(G,F) \in C_{fil}(S)$, there is a canonical map in $C_{fil}(S')$*

$$T(g,\gamma)(G,F) : g^*\Gamma_Z(G,F) \to \Gamma_{Z\times_S S'}g^*(G,F)$$

unique up to homotopy such that $\gamma_{Z\times_S S'}(g^*(G,F)) \circ T(g,\gamma)(G,F) = g^*\gamma_Z(G,F)$.

(ii) Let $i_1 : Z_1 \hookrightarrow S$, $i_2 : Z_2 \hookrightarrow Z_1$ be closed embeddings with $S, Z_1, Z_2 \in \mathrm{Top}$. Then, for $(G,F) \in C_{fil}(S)$,

- *there is a canonical map $T(Z_2/Z_1,\gamma)(G,F) : \Gamma_{Z_2}(G,F) \to \Gamma_{Z_1}(G,F)$ in $C_{fil}(S)$ unique up to homotopy such that $\gamma_{Z_1}(G,F) \circ T(Z_2/Z_1,\gamma)(G,F) = \gamma_{Z_2}(G,F)$ together with a distinguish triangle in $K_{fil}(S) := K(\mathrm{PSh}_{fil}(S))$*

$$\Gamma_{Z_2}(G,F) \xrightarrow{T(Z_2/Z_1,\gamma)(G,F)} \Gamma_{Z_1}(G,F) \xrightarrow{\mathrm{ad}(j_2^*,j_{2*})(\Gamma_{Z_1}(G,F))} \Gamma_{Z_1/\backslash Z_2}(G,F) \to \Gamma_{Z_2}(G,F)[1]$$

- *there is a canonical map $T(Z_2/Z_1,\gamma^\vee)(G,F) : \Gamma^\vee_{Z_1}(G,F) \to \Gamma^\vee_{Z_2}(G,F)$ in $C_{fil}(S)$ unique up to homotopy such that $\gamma^\vee_{Z_2}(G,F) = T(Z_2/Z_1,\gamma^\vee)(G,F) \circ \gamma^\vee_{Z_1}(G,F)$. together with a distinguish triangle in $K_{fil}(S)$*

$$\Gamma^\vee_{Z_1\backslash Z_2}(G,F) \xrightarrow{\mathrm{ad}(j_{2!},j_2^*)(G,F)} \Gamma^\vee_{Z_1}(G,F) \xrightarrow{T(Z_2/Z_1,\gamma^\vee)(G,F))} \Gamma^\vee_{Z_2}(G,F) \to \Gamma^\vee_{Z_2\backslash Z_1}(G,F)[1]$$

(iii) Consider a morphism $g : (S',Z') \to (S,Z)$ with $(S',Z'),(S,Z) \in \mathrm{Top}^2$. We denote, for $G \in C(S)$ the composite

$$T(D,\gamma^\vee)(G) : g^*\Gamma^\vee_Z G \xrightarrow{\sim} \Gamma^\vee_{Z\times_S S'}g^*G \xrightarrow{T(Z'/Z\times_S S',\gamma^\vee)(G)} \Gamma^\vee_{Z'}g^*G$$

and we have then the factorization $\gamma^\vee_{Z'}(g^*G) : g^*G \xrightarrow{g^*\gamma^\vee_Z(G)} g^*\Gamma^\vee_Z G \xrightarrow{T(D,\gamma^\vee)(G)} \Gamma^\vee_{Z'}g^*G$.

Proof. (i): We have the cartesian square

$$\begin{tikzcd} S\backslash Z \arrow[r, "j"] & S \\ S'\backslash Z\times_S S' \arrow[u, "g'"] \arrow[r, "j'"] & S' \arrow[u, "g"'] \end{tikzcd}$$

and the map is given by

$$(I, T(g,j)(j^*G)) : \mathrm{Cone}(g^*G \to g^*j_*j^*G) \to \mathrm{Cone}(g^*G \to j'_*j'^*g^*G = j'_*g'^*j^*G).$$

(ii): Follows from the fact that $j_1^*\Gamma_{Z_2}G = 0$ and $j_1^*\Gamma^\vee_{Z_2}G = 0$, with $j_1 : S\backslash Z_1 \hookrightarrow S$ the closed embedding.
(iii): Obvious. ∎

Let $(S, O_S) \in \mathrm{RTop}$. Let $Z \subset S$ a closed subset. Denote by $j : S\backslash Z \hookrightarrow S$ the open complementary embedding,

- For $G \in C_{O_S}(S)$, $\Gamma_Z G := \mathrm{Cone}(\mathrm{ad}(j^*,j_*)(G) : F \to j_*j^*G)[-1]$ has a (unique) structure of O_S module such that $\gamma_Z(G) : \Gamma_Z G \to G$ is a map in $C_{O_S}(S)$. This gives the functor

$$\Gamma_Z : C_{O_S fil}(S) \to C_{filO_S}(S),\ (G,F) \mapsto \Gamma_Z(G,F) := (\Gamma_Z G, \Gamma_Z F),$$

  together with the canonical map $\gamma_Z((G,F) : \Gamma_Z(G,F) \to (G,F)$. Let $Z_2 \subset Z$ a closed subset. Then, for $G \in C_{O_S}(S)$, $T(Z_2/Z,\gamma)(G) : \Gamma_{Z_2}G \to \Gamma_Z G$ is a map in $C_{O_S}(S)$ (i.e. is O_S linear).

- For $G \in C_{O_S}(S)$, $\Gamma^{\vee}_Z G := \mathrm{Cone}(\mathrm{ad}(j_!, j^*)(G) : j_! j^* G \to G)$ has a unique structure of O_S module, such that $\gamma^{\vee}_Z(G) : G \to \Gamma^{\vee}_Z G$ is a map in $C_{O_S}(S)$. This gives the functor

$$\Gamma^{\vee}_Z : C_{O_S fil}(S) \to C_{filO_S}(S),\ (G, F) \mapsto \Gamma^{\vee}_Z(G, F) := (\Gamma^{\vee}_Z G, \Gamma^{\vee}_Z F),$$

together with the canonical map $\gamma^{\vee}_Z((G, F) : (G, F) \to \Gamma^{\vee}_Z(G, F)$. Let $Z_2 \subset Z$ a closed subset. Then, for $G \in C_{O_S}(S)$, $T(Z_2/Z, \gamma^{\vee})(G) : \Gamma^{\vee}_Z G \to \Gamma^{\vee}_{Z_2} G$ is a map in $C_{O_S}(S)$ (i.e. is O_S linear).

- For $G \in C_{O_S}(S)$, we will use

$$\begin{aligned}\Gamma^{\vee,h}_Z G : &= \mathbb{D}^O_S L_O \Gamma_Z E(\mathbb{D}^O_S G) \\ : &= \mathrm{Cone}(\mathbb{D}^O_S L_O \,\mathrm{ad}(j_*, j^*)(E(\mathbb{D}^O_S G)) : \mathbb{D}^O_S L_O j_* j^* E(\mathbb{D}^O_S G) \to \mathbb{D}^O_S L_O E(\mathbb{D}^O_S G))\end{aligned}$$

and we have the canonical map $\gamma^{\vee,h}_Z(G) : M \to \Gamma^{\vee,h}_Z G$ of O_S module. The factorization

$$\mathrm{ad}(j_!, j^*)(L_O M) : j_! j^* L_O G \xrightarrow{(k \circ \mathbb{D}^O I(j_!, j^*)(\mathbb{D}^O j^* L_O G) \circ d(j_! j^* L_O G))^q}$$
$$\mathbb{D}^O_S L_O j_* j^* E(\mathbb{D}^O_S L_O G) \xrightarrow{\mathrm{ad}(j_*, j^*)(E(\mathbb{D}^O_S L_O G))} \mathbb{D}^O_S L_O E(\mathbb{D}^O_S L_O G)$$

gives the factorization $\gamma^{\vee,h}_Z(L_O G) : L_O G \xrightarrow{\gamma^{\vee}_Z(L_O G)} \Gamma^{\vee}_Z L_O G \xrightarrow{(k \circ \mathbb{D}^O I(j_!, j^*)(\mathbb{D}^O j^* L_O G) \circ d(j_! j^* L_O G))^q} \Gamma^{\vee,h}_Z L_O G$. We get the functor

$$\Gamma^{\vee,h}_Z : C_{O_S fil}(S) \to C_{O_S fil}(S),\ (G, F) \mapsto \Gamma^{\vee,h}_Z(G, F) := \mathbb{D}^O_S L_O \Gamma_Z E(\mathbb{D}^O_S(G, F)),$$

together with the factorization

$$\gamma^{\vee,h}_Z(L_O(G, F)) : L_O(G, F) \xrightarrow{\gamma^{\vee}_Z(L_O(G,F))} \Gamma^{\vee}_Z L_O(G, F)$$
$$\xrightarrow{(k \circ \mathbb{D}^O I(j_!, j^*)(\mathbb{D}^O_S j^* L_O(G,F)) \circ d(j_! j^* L_O(G,F)))^q} \Gamma^{\vee,h}_Z L_O(G, F),$$

- Consider $\mathcal{I} \subset O_S$ a right ideal of O_S such that $\mathcal{I}^o_Z \subset \mathcal{I}$, where $\mathcal{I}^o_Z \subset O_S$ is the left and right ideal consisting of section which vanish on Z.
  - For $G \in \mathrm{PSh}_{O_S}(S)$, we consider, $S^o \subset S$ being an open subset,

    $$\mathcal{I}G(S^o) =< \{f.m, m \in G(S^o), f \in \mathcal{I}(S^o)\} >\subset G(S^o)$$

    since $\mathcal{I}$ is a right ideal, and we denote by $b_I(G) : \mathcal{I}G \to G$ the injective morphism of O_S modules and by $c_Z(G) : G \to G/\mathcal{I}G$ the quotient map. The adjonction map $\mathrm{ad}(j_!, j^*)(G) : j_! j^* G \to G$ factors trough $b_I(G)$:

    $$ad(j_!, j^*)(G) : j_! j^* G \xrightarrow{b^I_{Z/S}(G)} \mathcal{I}G \xrightarrow{b_I(G)} G$$

    We have then the support section functor,

    $$\Gamma^{\vee,O,I}_Z : C_{O_S}(S) \to C_{O_S}(S),\ G \mapsto \Gamma^{\vee,O,I}_Z G := \mathrm{Cone}(b_I(G) : \mathcal{I}G \to G)$$

    together with the canonical map $\gamma^{\vee,O}_Z(G) : G \to \Gamma^{\vee,O}_Z G$ which factors through

    $$\gamma^{\vee,O,I}_Z(G) : G \xrightarrow{\gamma^{\vee}_Z(G)} \Gamma^{\vee}_Z G \xrightarrow{b^I_{S/Z}(G)} \Gamma^{\vee,O}_Z G.$$

    By the exact sequence $0 \to \mathcal{I}G \xrightarrow{b_I(G)} G \xrightarrow{c_I(G)} G/\mathcal{I}G \to 0$, we have an homotopy equivalence $c_I(G) : \Gamma^{\vee,O,I}_Z G \to G/\mathcal{I}G$.

– For $G \in \mathrm{PSh}_{O_S}(S)$, we consider

$$b'_I(G) : G \to G \otimes_{O_S} \mathbb{D}^O_S(\mathcal{I}) := G \otimes_{O_S} \mathcal{H}om(\mathcal{I}, O_S)$$

The adjonction map $\mathrm{ad}(j^*, j_*)(G) : G \to j_*j^*G$ factors trough $b'_I(G)$:

$$ad(j^*, j_*)(G) : G \xrightarrow{b'_I(G)} G \otimes_{O_S} \mathbb{D}^O_S(\mathcal{I}) \xrightarrow{b'^I_{Z/S}(G)} j_*j^*G$$

We have then the support section functor,

$$\Gamma^{O,I}_Z : C_{O_S}(S) \to C_{O_S}(S),\ G \mapsto \Gamma^{O,I}_Z G := \mathrm{Cone}(b'_I(G) : G \to G \otimes_{O_S} \mathbb{D}^O_S(\mathcal{I}))[-1]$$

together with the canonical map $\gamma^O_Z(G) : \Gamma^O_Z G \to G$ which factors through

$$\gamma^{O,I}_Z(G) : \Gamma^O_Z G \xrightarrow{b^I_{S/Z}(G)} \Gamma_Z G \xrightarrow{\gamma_Z(G)} G.$$

– By definition, we have for a canonical isomorphism

$$I(D, \gamma^O)(G) : \mathbb{D}^O_S \Gamma^{\vee,O,I} G \xrightarrow{\sim} \Gamma^{O,I}_Z \mathbb{D}^O_S G$$

which gives the transformation map in $C_{O_S}(S)$

$$T(D, \gamma^O)(G) : \Gamma^{\vee,O,I}\mathbb{D}^O_S G \xrightarrow{d(-)} \mathbb{D}^{O,2}_S \Gamma^{\vee,O,I}\mathbb{D}^O_S G \xrightarrow{\mathbb{D}^O_S I(D,\gamma^O)(\mathbb{D}^O_S G)^{-1}}$$
$$\mathbb{D}^O_S \Gamma^{O,I}_Z \mathbb{D}^{O,2}_S G \xrightarrow{\mathbb{D}^O_S \Gamma^{O,I}_Z d(G)} \mathbb{D}^O_S \Gamma^{O,I}_Z G$$

Definition-Proposition 2. *(i) Let $g : (S', O_{S'}) \to (S, O_S)$ a morphism and $i : Z \hookrightarrow S$ a closed embedding with $(S', O_{S'}, (S, O_S) \in \mathrm{RTop}$. Then, for $(G, F) \in C_{O_S fil}(S)$, there is a canonical map in $C_{O_{S'} fil}(S')$*

$$T^{mod}(g, \gamma)(G, F) : g^{*mod}\Gamma_Z(G, F) \to \Gamma_{Z \times_S S'} g^{*mod}(G, F)$$

unique up to homotopy, such that $\gamma_{Z \times_S S'}(g^{*mod}G) \circ T^{mod}(g, \gamma)(G) = g^{*mod}\gamma_Z G$.

(ii) Let $i_1 : (Z_1, O_{Z_1}) \hookrightarrow (S, O_S)$, $i_2 : (Z_2, O_{Z_2}) \hookrightarrow (Z_1, O_{Z_1})$ be closed embeddings with $S, Z_1, Z_2 \in \mathrm{Top}$. Then, for $(G, F) \in C_{O_S fil}(S)$, there is a canonical map in $C_{O_S fil}(S)$

$$T(Z_2/Z_1, \gamma^{\vee,O})(G, F) : \Gamma^{\vee,O}_{Z_1}(G, F) \to \Gamma^{\vee,O}_{Z_2}(G, F)$$

unique up to homotopy such that $\gamma^{\vee,O}_{Z_2}(G, F) = T(Z_2/Z_1, \gamma^{\vee,O})(G, F) \circ \gamma^{\vee,O}_{Z_1}(G, F)$.

(iii) Consider a morphism $g : ((S', O_{S'}), Z') \to ((S, O_S), Z)$ with $((S', O_{S'}), Z') \to ((S, O_S), Z) \in \mathrm{RTop}^2$. We denote, for $M \in C_{O_S}(S)$ the composite

$$T^{mod}(D, \gamma^{\vee,O})(G) : g^{*mod}\Gamma^{\vee,O}_Z G \xrightarrow{\sim} \Gamma^{\vee,O}_{Z \times_S S'} g^{*mod} G \xrightarrow{T(Z'/Z \times_S S', \gamma^{\vee,O})(G)} \Gamma^{\vee,O}_{Z'} g^{*mod} G$$

and we have then the factorization

$$\gamma^{\vee,O}_{Z'}(g^{*mod}M) : g^{*mod}G \xrightarrow{g^{*mod}\gamma^{\vee,O}_Z(G)} g^{*mod}\Gamma^{\vee,O}_Z G \xrightarrow{T^{mod}(D,\gamma^{\vee,O})(G)} \Gamma^{\vee,O}_{Z'} g^{*mod} G$$

Proof. (i): We have the cartesian square

$$\begin{tikzcd} S\backslash Z \arrow[r, "j"] & S \\ S'\backslash Z \times_S S' \arrow[u, "g'"] \arrow[r, "j'"] & S' \arrow[u, "g"] \end{tikzcd}$$

and the map is given by

$$(I,T^{mod}(g,j)(j^*G)) : \mathrm{Cone}(g^{*mod}G \to g^{*mod}j_*j^*G) \to \mathrm{Cone}(g^{*mod}G \to j'_*j^{'*}g^{*mod}G = j'_*g^{'*mod}j^*G).$$

(ii):Obvious.
(iii):Obvious. □

Definition-Proposition 3. *Consider a commutative diagram in* RTop

$$\begin{tikzcd} D_0 = & f:(X,O_X) \arrow[r, "i"] & (Y,O_Y) \arrow[r, "p"] & (S,O_S) \\ & f':(X',O_{X'}) \arrow[r, "i'"] \arrow[u, "g'"] & (Y',O_{Y'}) \arrow[r, "p'"] \arrow[u, "g''"] & (T,O_T) \arrow[u, "g"] \end{tikzcd}$$

with i, i' being closed embeddings. Denote by D the right square of D_0. The closed embedding $i' : X' \hookrightarrow Y'$ factors through $i' : X' \xrightarrow{i'_1} X \times_Y Y' \xrightarrow{i'_0} Y'$ where i'_1, i'_0 are closed embeddings.

(i) We have the canonical map,

$$E(\Omega_{((O_{Y'}/g''^*O_Y))/(O_T/g^*O_S)}) \circ T(g'',E)(-) \circ T(g'',\gamma)(-) : \\ g^{''*}\Gamma_X E(\Omega^\bullet_{O_Y/p^*O_S},F_b) \to \Gamma_{X\times_Y Y'}E(\Omega^\bullet_{O_{Y'}/p'^*O_T},F_b)$$

unique up to homotopy such that the following diagram in $C_{g''^*p^*O_Sfil}(Y') = C_{p'^*g^*O_Sfil}(Y')$ commutes

$$\begin{tikzcd} g^{''*}\Gamma_X E(\Omega^\bullet_{O_Y/p^*O_S},F_b) \arrow[rr, "E(\Omega_{((O_{Y'}/Y))/(O_T/S)})\circ T(g'',E)(-)\circ T(g'',\gamma)(-)"] \arrow[d, "\gamma_X(-)"] & & \Gamma_{X\times_Y Y'}E(\Omega^\bullet_{O_{Y'}/p'^*O_T},F_b) \arrow[d, "\gamma_{X\times_Y Y'}(-)"] \\ g^{''*}E(\Omega^\bullet_{O_Y/p^*O_S},F_b) \arrow[rr, "E(\Omega_{((O_{Y'}/g''^*O_Y)/(O_T/S))}\circ T(g'',E)(-)"] & & E(\Omega^\bullet_{O'_Y/p'^*O_T},F_b) \end{tikzcd} .$$

(ii) There is a canonical map,

$$T^O_\omega(D)^\gamma : g^{*mod}L_O p_*\Gamma_X E(\Omega^\bullet_{O_Y/p^*O_S},F_b) \to p'_*\Gamma_{X\times_Y Y'}E(\Omega^\bullet_{O_{Y'}/p'^*O_T},F_b)$$

unique up to homotopy such that the following diagram in $C_{O_Tfil}(T)$ commutes

$$\begin{tikzcd} g^{*mod}L_O p_*\Gamma_X E(\Omega^\bullet_{O_Y/p^*O_S},F_b) \arrow[r, "T^O_\omega(D)^\gamma"] \arrow[d, "\gamma_X(-)"] & p'_*\Gamma_{X\times_Y Y'}E(\Omega^\bullet_{O_{Y'}/p'^*O_T},F_b) \arrow[d, "\gamma_{X\times_Y Y'}(-)"] \\ g^{*mod}L_O p_*E(\Omega^\bullet_{O_Y/p^*O_S},F_b) \arrow[r, "T^O_\omega(D)"] & p'_*E(\Omega^\bullet_{O_{Y'}/p'^*O_T},F_b) \end{tikzcd} .$$

(iii) We have the canonical map in $C_{f'^*O_T}(Y')$

$$T(X'/X\times_Y Y',\gamma)(E(\Omega^\bullet_{O_{Y'}/p'^*O_T},F_b)) : \Gamma_{X'}E(\Omega^\bullet_{O_{Y'}/p'^*O_T},F_b) \to \Gamma_{X\times_Y Y'}E(\Omega^\bullet_{O_{Y'}/p'^*O_T},F_b)$$

unique up to homotopy such that $\gamma_{X\times_Y Y'}(-) \circ T(X'/X\times_Y Y',\gamma)(-) = \gamma_{X'}(-)$.

Proof. Immediate from definition. We take for the map of point (ii) the composite

$$T^O_\omega(D)^\gamma : g^{*mod}L_O p_*\Gamma_X E(\Omega^\bullet_{O_Y/p^*O_S}, F_b) \xrightarrow{q} g^*p_*\Gamma_X E(\Omega^\bullet_{O_Y/p^*O_S}, F_b) \otimes_{g^*O_S} O_T$$
$$\xrightarrow{T(g'',E)(-)\circ T(g'',\gamma)(-)\circ T(D)(E(\Omega^\bullet_{O_X/p^*O_S}))} (p'_*\Gamma_{X\times_Y Y'}E(g''\Omega^\bullet_{O_Y/p^*O_S}, F_b)) \otimes_{g^*O_S} O_T$$
$$\xrightarrow{E(\Omega_{(O_{Y'}/g''^*O_Y)/(O_T/g^*O_S)}))} p'_*\Gamma_{X\times_Y Y'}E(\Omega^\bullet_{O_{Y'}/p'^*O_T}, F_b) \otimes_{g^*O_S} O_T \xrightarrow{m} p'_*\Gamma_{X\times_Y Y'}E(\Omega^\bullet_{O_{Y'}/p'^*O_T}, F_b),$$

with $m(n\otimes s) = s.n$. □

Definition 2. *(i) Let $S \in \mathrm{Top}$. For $Z \subset S$ a closed subset, we denote by $C_Z(S) \subset C(S)$ the full subcategory consisting of complexes of presheaves $F \in C(S)$ such that $a_{top}H^n(j^*F) = 0$ for all $n \in \mathbb{Z}$, where $j : S\backslash Z \hookrightarrow S$ is the complementary open embedding and a_{top} is the sheaftification functor.*

(i)' More generally, let $(S, O_S) \in \mathrm{RTop}$. For $Z \subset S$ a closed subset, we denote by

$$C_{O_S,Z}(S) \subset C_{O_S}(S)\ ,\ \mathcal{Q}Coh_Z(S) \subset \mathcal{Q}Coh(S)$$

the full subcategories consisting of complexes of presheaves $G \in C_{O_S}(S)$ such that $a_{top}H^n(j^*F) = 0$ for all $n \in \mathbb{Z}$, resp. quasi-coherent sheaves $G \in \mathcal{Q}Coh(S)$ such that $j^*F = 0$.

(ii) Let $S \in \mathrm{Top}$. For $Z \subset S$ a closed subset, we denote by $C_{fil,Z}(S) \subset C_{fil}(S)$ the full subcategory consisting of filtered complexes of presheaves $(G, F) \in C_{fil}(S)$ such that there exist $r \in \mathbb{N}$ and an r-filtered homotopy equivalence $\phi : (G, F) \to (G', F)$ with $(G', F) \in C_{fil}(S)$ such that $a_{top}j^*H^n \operatorname{Gr}^p_F(G', F) = 0$ for all $n, p \in \mathbb{Z}$, where $j : S\backslash Z \hookrightarrow S$ is the complementary open embedding and a_{top} is the sheaftification functor. Note that this definition say that this r does NOT depend on n and p.

(ii)' More generally, let $(S, O_S) \in \mathrm{RTop}$. For $Z \subset S$ a closed subset, we denote by

$$C_{O_S,fil,Z}(S) \subset C_{O_S,fil}(S), \mathcal{Q}Coh_{fil,Z}(S) \subset \mathcal{Q}Coh(S)$$

the full subcategories consisting of filtered complexes of presheaves $(G, F) \in C_{O_S fil}(S)$ such that there exist $r \in \mathbb{N}$ and an r-filtered homotopy equivalence $\phi : (G, F) \to (G', F)$ with $(G', F) \in C_{fil}(S)$ such that $a_{top}j^*H^n \operatorname{Gr}^p_F(G', F) = 0$ for all $n, p \in \mathbb{Z}$, resp. filtered quasi-coherent sheaves $(G, F) \in \mathcal{Q}Coh(S)$ and an r-filtered homotopy equivalence $\phi : (G, F) \to (G', F)$ with $(G', F) \in C_{fil}(S)$ such that there exist $r \in \mathbb{N}$ such that $j^*H^n \operatorname{Gr}^p_F(G, F) = 0$ for all $n, p \in \mathbb{Z}$. Note that this definition say that this r does NOT depend on n and p.

Let $(S, O_S) \in \mathrm{RTop}$ and $Z \subset S$ a closed subset.

- For $(G, F) \in C_{fil}(S)$, we have $\Gamma_Z(G, F), \Gamma^\vee_Z(G, F) \in C_{fil,Z}(S)$.
- For $(G, F) \in C_{O_S fil}(S)$, we have $\Gamma_Z(G, F), \Gamma^\vee_Z(G, F), \Gamma^{\vee,h}_Z(G, F), \Gamma^{\vee,O}_Z(G, F) \in C_{O_S fil,Z}(S)$.

Proposition 12. *Let $S \in \mathrm{Top}$ and $Z \subset S$ a closed subspace. Denote by $i : Z \hookrightarrow S$ the closed embedding.*

(i) The functor $i^* : \mathrm{Shv}_Z(S) \to \mathrm{Shv}(Z)$ is an equivalence of category whose inverse is $i_* : \mathrm{Shv}(Z) \to \mathrm{Shv}_Z(S)$. More precisely $\mathrm{ad}(i_*, i^*)(H) : i^*i_*H \to H$ is an isomorphism if $H \in \mathrm{Shv}(Z)$ and $\mathrm{ad}(i_*, i^*)(G) : G \to i_*i^*G$ is an isomorphism if $G \in \mathrm{Shv}_Z(S)$.

(ii) : The functor $i^* : \mathrm{Shv}_{fil,Z}(S) \to \mathrm{Shv}_{fil}(Z)$ is an equivalence of category whose inverse is $i_* : \mathrm{Shv}_{fil}(Z) \to \mathrm{Shv}_{fil,Z}(S)$. More precisely $\mathrm{ad}(i_*, i^*)(H, F) : i^*i_*(H, F) \to (H, F)$ is an isomorphism if $(H, F) \in \mathrm{Shv}(Z)$ and $\mathrm{ad}(i_*, i^*)(G, F) : (G, F) \to i_*i^*(G, F)$ is an isomorphism if $(G, F) \in \mathrm{Shv}_Z(S)$.

(iii) : The functor $i^*: D_{\tau,fil,Z}(S) \to D_{\tau,fil}(Z)$ is an equivalence of category whose inverse is $i_*: D_{\tau,fil}(Z) \to D_{\tau,fil,Z}(S)$. More precisely $\mathrm{ad}(i_*,i^*)(H,F): i^*i_*(H,F) \to (H,F)$ is an equivalence top local if $(H,F) \in C_{fil}(Z)$ and $\mathrm{ad}(i_*,i^*)(G,F): (G,F) \to i_*i^*(G,F)$ is an equivalence top local if $(G,F) \in C_{fil,Z}(S)$.

Proof. (i):Standard.
(ii): Follows from (i).
(iii): Follows from (ii). □

Let $S \in \mathrm{Top}$ and $Z \subset S$ a closed subspace. By proposition 12, if $G \in C(S)$, $\mathrm{ad}(i_*,i^*)(\Gamma_Z G): \Gamma_Z G \to i_*i^*\Gamma_Z G$ is an equivalence top local since $\Gamma_Z G \in C_Z(S)$.

Let $(S,O_S) \in \mathrm{RTop}$. Let $D = \cup_i D_i \subset X$ a normal crossing divisor, denote by $j: S\backslash D \hookrightarrow S$ the open embedding, and consider $\mathcal{I}_D \subset O_S$ the ideal of vanishing function on D which is invertible. We set, for $M \in C_{O_S}(S)$,

$$M(*D) := \lim_n \mathcal{H}om_{O_S}(\mathcal{I}^n, M),$$

and we denote by $a_D(F): F \to F(*D)$ the surjective morphism of presheaves. The adjonction map $\mathrm{ad}(j^*,j_*)(F): F \to j_*j^*F$ factors trough $a_D(F)$:

$$ad(j^*,j_*)(F): F \xrightarrow{a_D(F)} F(*D) \xrightarrow{a_{S/D}(F)} j_*j^*F$$

Remark 3.
- *Let $j: U \hookrightarrow X$ an open embedding, with $(X,O_X) \in \mathrm{RTop}$. Then if $F \in \mathrm{Coh}_{O_U}(U)$ is a coherent sheaf of O_U module, j_*F is quasi-coherent but NOT coherent in general. In particular for $F \in C_{O_U}(U)$ whose cohomology sheaves $a_{tau}H^nF$ are coherent for all $n \in \mathbb{Z}$, the cohomology sheaves $R^nj_*F := a_\tau H^n j_*E(F)$ of $Rj_*F = j_*E(F)$ are quasi-coherent but NOT coherent.*
- *Let $j: U \hookrightarrow X$ an open embedding, with $X \in \mathrm{Sch}$. Then if $F \in \mathrm{Coh}(U)$ is a coherent sheaf of O_U module, j_*F is quasi-coherent but NOT coherent. However, there exist an O_X submodule $\tilde{F} \subset j_*F$ such that $j^*\tilde{F} = F$ and $\tilde{F} \in \mathrm{Coh}(X)$.*

The following propositions are true for schemes but NOT for arbitrary ringed spaces like analytic spaces :

Proposition 13. *(i) Let $X = (X,O_X) \in \mathrm{Sch}$ a noetherien scheme and $D \subset X$ a closed subset. Denote by $j: U = X\backslash D \hookrightarrow X$ an open embedding. Then for $F \in \mathcal{Q}Coh_{O_U}(U)$ a quasi coherent sheaf, $j_*F \in \mathcal{Q}Coh_{O_X}(X)$ is quasi-coherent and is the direct limit of its coherent subsheaves.*

(ii) Let $X = (X,O_X)$ a noetherien scheme and $D = \cup D_i \subset X$ a normal crossing divisor. Denote by $j: U = X\backslash D \hookrightarrow X$ an open embedding. Then for $F \in \mathcal{Q}Coh_{O_U}(U)$ a quasi coherent sheaf, the canonical map $a_{X/D}(F): F(*D) \xrightarrow{\sim} j_*F$ is an isomorphism.

Proof. Standard. □

Proposition 14. *Let $S = (S,O_S) \in \mathrm{Sch}$ and $Z \subset S$ a closed subscheme. Denote by $i: Z \hookrightarrow S$ the closed embedding.*

(i) For $G \in \mathcal{Q}Coh_Z(S)$, i^*G has a canonical structure of O_Z module. Moreover, the functor $i^*: \mathcal{Q}Coh_Z(S) \to \mathcal{Q}Coh(Z)$ is an equivalence of category whose inverse is $i_*: \mathcal{Q}Coh(Z) \to \mathcal{Q}Coh_Z(S)$.

(ii) : The functor $i^*: \mathcal{Q}Coh_{fil,Z}(S) \to \mathcal{Q}Coh_{fil}(Z)$ is an equivalence of category whose inverse is $i_*: \mathcal{Q}Coh_{fil}(Z) \to \mathcal{Q}Coh_{fil,Z}(S)$.

(iii) : The functor $i^*: D_{O_S fil,Z,qc}(S) \to D_{O_Z fil,qc}(Z)$ is an equivalence of category whose inverse is $i_*: D_{O_Z fil,qc}(Z) \to D_{O_S fil,Z,qc}(S)$.

Proof. (i):Standard.
(ii): Follows from (i).
(iii): Follows from (ii) since i^* and i_* are exact functors. □

Definition 3. *Let $(S,O_S)\in\mathrm{RTop}$ a locally ringed space with O_S commutative. Consider an $\kappa_S\in C_{O_S}(S)$. Let $\mathcal{I}\subset O_S$ an ideal subsheaf and $Z=V(\mathcal{I})\subset S$ the associated closed subset. For $G\in\mathrm{PSh}_{O_S}(S)$, we denote by $\hat{G}_Z:=\hat{G}_I:=\lim_k G/\mathcal{I}^kG$ the completion with respect to the ideal $\mathcal{I}$ and by $c_Z^{\infty}(G):G\to\hat{G}_Z$ the quotient map. Then, the canonical map*

$$d_{\kappa_S,Z}(G):G\xrightarrow{d(G)}\mathbb{D}_S^{O,2}G\xrightarrow{T^{mod}(\otimes\kappa_S,hom)(\mathbb{D}_S^{O}G,O_S)}$$
$$\mathcal{H}om_{O_S}(\mathbb{D}_S^{O}G\otimes_{O_S}\kappa_S,\kappa_S)\xrightarrow{T^{mod}(\Gamma_ZE,hom)(-,-)}\mathcal{H}om_{O_S}(\Gamma_ZE(\mathbb{D}_S^{O}G\otimes_{O_S}\kappa_S),\Gamma_ZE(\kappa_S))$$

factors through

$$d_{\kappa_S,Z}(G):G\xrightarrow{c_Z^{\infty}(G)}\hat{G}_Z\xrightarrow{d_{\kappa_S,Z}(G)}\mathcal{H}om_{O_S}(\Gamma_ZE(\mathbb{D}_S^{O}G\otimes_{O_S}\kappa_S),\Gamma_ZE(\kappa_S))$$

Clearly if $G\in C_{O_S}(S)$ then $d_{\kappa_S,Z}(G)$ is a map in $C_{O_S}(S)$. On the other hand, we have a commutative diagram

$$\begin{tikzcd}
\Omega^p_{O_S} \arrow[r, "d_{\kappa_S,Z}(\Omega^p_{O_S})"] \arrow[d, "d"] & \mathcal{H}om_{O_S}(\Gamma_ZE(\mathbb{D}_S^{O}\Omega^p_{O_S}\otimes_{O_S}\kappa_S),\Gamma_ZE(\kappa_S)) \arrow[d, "d^{\kappa_S,\gamma}"] \\
\Omega^{p+1}_S \arrow[r, "d_{\kappa_S,Z}(\Omega^{p+1}_{O_S})"] & \mathcal{H}om_{O_S}(\Gamma_ZE(\mathbb{D}_S^{O}\Omega^{p+1}_{O_S}\otimes_{O_S}\kappa_S),\Gamma_ZE(\kappa_S))
\end{tikzcd}$$

so that $d_{\kappa_S,Z}(\Omega^{\bullet}_S)\in C(S)$.

The following theorem is the from [13]

Theorem 8. *Let $S\in\mathrm{Var}(\mathbb{C})$. Let $Z=V(\mathcal{I})\subset S$ a closed subset. Denote by $K_S\in\mathrm{PSh}_{O_S}(S)$ the canonical bundle. Then, for $G\in C_{O_S,c}(S)$,*

$$d_{K_S,Z}(G):\hat{G}_Z\to\mathcal{H}om_{O_S}(\Gamma_ZE(\mathbb{D}_S^{O}G\otimes_{O_S}K_S),\Gamma_ZE(K_S))$$

is an equivalence Zariski local.

Let $f:(X,O_X)\to(S,O_S)$ a morphism with $(S,O_S)\in\mathrm{RTop}$. In the particular case where O_S is a commutative sheaf of ring, $T_{O_S}\in\mathrm{PSh}_{O_S}(S)$ and $\Omega_{O_S}=\mathbb{D}_{O_S}T_{O_S}\in\mathrm{PSh}_{O_S}(S)$ are sheaves and the morphism in $\mathrm{PSh}(X)$

$$T(f,\hom)(O_S,O_S):f^*\mathcal{H}om(O_S,O_S)\to\mathcal{H}om(f^*O_S,f^*O_S)$$

induces isomorphisms $T(f,\hom)(O_S,O_S):f^*T_{O_S}\xrightarrow{\sim}T_{f^*O_S}$ and $\mathbb{D}_{f^*O_S}T(f,\hom)(O_S,O_S):\Omega_{f^*O_S}\to f^*\Omega_{O_S}$ where for $F\in\mathrm{Shv}(S)$, we denote again (as usual) by abuse $f^*F:=a_\tau f^*F\in\mathrm{Shv}(S)$, $a_{tau}:\mathrm{PSh}(S)\to\mathrm{Shv}(S)$ being the sheaftification functor.

Definition 4. *(i) Let $(X,O_X)\in\mathrm{RTop}$. A foliation $(X,O_X)/\mathcal{F}$ on (X,O_X) is an O_X module $\Omega_{O_X/\mathcal{F}}\in\mathrm{PSh}_{O_X}(X)$ together with a derivation map $d:=d_{\mathcal{F}}:O_X\to\Omega_{O_X/\mathcal{F}}$ such that*

- *the associated map $q:=q_{\mathcal{F}}:=\omega_X(d):\Omega_{O_X}\to\Omega_{O_X/\mathcal{F}}$ is surjective*
- *satisfy the integrability condition $d(\ker q)\subset\ker q$ which implies that the map $d:\Omega^p_{O_X}\to\Omega^{p+1}_{O_X}$ induce factors trough*

$$\begin{tikzcd}
\Omega^p_{O_X} \arrow[r, "d"] \arrow[d, "q^p:=\wedge^p q"'] & \Omega^{p+1}_{O_X} \arrow[d, "q^{p+1}:=\wedge^p q"] \\
\Omega^p_{O_X/\mathcal{F}} \arrow[r, "d"] & \Omega^{p+1}_{O_X/\mathcal{F}}
\end{tikzcd}$$

and $d:\Omega^p_{O_X/\mathcal{F}}\to\Omega^{p+1}_{O_X/\mathcal{F}}$ is neccessary unique by the surjectivity of $q^p:\Omega^p_{O_X}\to\Omega^p_{O_X/\mathcal{F}}$.

In the particular case where $\Omega_{O_X/\mathcal{F}} \in \mathrm{PSh}_{O_X}(X)$ is a locally free sheaf of O_X module, $\mathbb{D}_{O_X}q : T_{O_X/\mathcal{F}} := \mathbb{D}_{O_X}\Omega_{O_X/\mathcal{F}} \to T_{O_X}$ is injective and the second condition is then equivalent to the fact that the sub O_X module $T_{O_X/\mathcal{F}} \subset T_{O_X}$ is a Lie subalgebra, that is $[T_{O_X/\mathcal{F}}, T_{O_X/\mathcal{F}}] \subset T_{O_X/\mathcal{F}}$.

(ii) *A piece of leaf a foliation $(X, O_X)/\mathcal{F}$ with $(X, O_X) \in \mathrm{RTop}$ such that O_X is a commutative sheaf of ring is an injective morphism of ringed spaces $l : (Z, O_Z) \hookrightarrow (X, O_X)$ such that $\Omega_{i^*O_X/O_Z} :: \Omega_{i^*O_X} \to \Omega_{O_Z}$ factors trough an isomorphism*

$$\Omega_{i^*O_X/O_Z} : \Omega_{i^*O_X} \xrightarrow{\mathbb{D}_{i^*O_X}T(i,hom)(O_X,O_X)} i^*\Omega_{O_X} \xrightarrow{i^*q} i^*\Omega_{O_X/\mathcal{F}} \to \Omega_{O_Z}.$$

(iii) *If $f : (X, O_X) \to (S, O_S)$ is a morphism with $(X, O_X), (S, O_S) \in \mathrm{RTop}$, we have the foliation $(X, O_X)/(S, O_S) := ((X, O_X), f)$ on (X, O_X) given by the surjection*

$$q : \Omega_{O_X} \to \Omega_{O_X/f^*O_S} := \mathrm{coker}(\Omega_{O_X/f^*O_S} : \Omega_{f^*O_S} \to \Omega_{O_X}).$$

The fibers $i_{X_s} : (X_s, O_{X_s}) \hookrightarrow (X, O_X)$ for each $s \in S$ are the leaves of the foliation.

(iv) *We have the category $FolRTop$*

- *whose objects are foliated ringed spaces $(X, O_X)/\mathcal{F}$ with O_X a commutatif sheaf of ring and*
- *whose morphisms $f : (X, O_X)/\mathcal{F} \to (S, O_S)/\mathcal{G}$ are morphisms of ringed spaces $f : (X, O_X) \to (S, O_S)$ such that $\Omega_{O_X/f^*O_S} : \Omega_{f^*O_S} \to \Omega_{O_X}$ factors through*

$$\begin{tikzcd} f^*\Omega_{O_S} \arrow[r, "\mathbb{D}_{f^*O_S}T(f,\mathrm{hom})(O_S,O_S)^{-1}"] \arrow[d, "f^*q_{\mathcal{G}}"'] & \Omega_{f^*O_S} \arrow[r, "\Omega_{O_X/f^*O_S}"] & \Omega_{O_X} \arrow[d, "q_{\mathcal{F}}"] \\ f^*\Omega_{O_S/\mathcal{G}} \arrow[rr, "\Omega^q_{O_X/f^*O_S}"] & & \Omega_{O_X/\mathcal{F}} \end{tikzcd} .$$

This category admits inverse limits with $(X, O_X)/\mathcal{F} \times (Y, O_Y)/\mathcal{G} = (X \times Y, p_X^*O_X \otimes p_Y^*O_Y)/p_X^*\mathcal{F} \otimes p_Y^*\mathcal{G}$ and

$$(X, O_X)/\mathcal{F} \times_{(S,O_S)/\mathcal{H}} (Y, O_Y)/\mathcal{G} = (X \times_S Y, \delta_S^*(p_X^*O_X \otimes p_Y^*O_Y))/p_X^*\mathcal{F} \otimes p_Y^*\mathcal{G}$$

with $\delta_S : X \times_S Y \hookrightarrow X \times Y$ the embedding given by the diagonal $\delta_S : S \hookrightarrow S \times S$.

Let $S \in \mathrm{Top}$. Let $S = \cup_{i=1}^l S_i$ an open cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{Top}$. For $I \subset [1, \cdots l]$, denote by $\tilde{S}_I = \Pi_{i \in I}\tilde{S}_i$. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I$ and, for $J \subset I$, the following commutative diagram

$$D_{IJ} = \quad \begin{tikzcd} S_I \arrow[r, "i_I"] & \tilde{S}_I \\ S_J \arrow[u, "j_{IJ}"] \arrow[r, "i_J"] & \tilde{S}_J \arrow[u, "p_{IJ}"'] \end{tikzcd}$$

where $j_{IJ} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. and $p_{IJ} : \tilde{S}_J \to \tilde{S}_I$ the projection. This gives the diagram of topological spaces $(\tilde{S}_I) \in \mathrm{Fun}(\mathcal{P}(\mathbb{N}), \mathrm{Top})$ which which gives the diagram $(\tilde{S}_I) \in (\mathrm{Ouv}(\tilde{S}_I)) \in \mathrm{Fun}(\mathcal{P}(\mathbb{N}), \mathrm{Cat})$ Denote $m : \tilde{S}_I \backslash (S_I \backslash S_J) \hookrightarrow \tilde{S}_I$ the open embedding.

Definition 5. *Let $S \in \mathrm{Top}$. Let $S = \cup_{i=1}^l S_i$ an open cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{Top}$. We denote by $C_{fil}(S/(\tilde{S}_I)) \subset C_{fil}(\tilde{S}_I)$ the full subcategory*

- *whose objects are $(G, F) = ((G_I, F)_{I \subset [1,\cdots l]}, u_{IJ})$, with $(G_I, F) \in C_{fil,S_I}(\tilde{S}_I)$, and $u_{IJ} : m^*(G_I, F) \to m^*p_{IJ*}(G_J, F)$ are ∞-filtered top local equivalences satisfying for $I \subset J \subset K$, $p_{IJ*}u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{fil}(\tilde{S}_I)$,*

- *the morphisms $m : ((G_I, F), u_{IJ}) \to ((H_I, F), v_{IJ})$ being (see section 2.1) a family of morphism of complexes,*
$$m = (m_I : (G_I, F) \to (H_I, F))_{I\subset[1,\cdots l]}$$
such that $v_{IJ} \circ m_I = p_{IJ*} m_J \circ u_{IJ}$ in $C_{fil}(\tilde{S}_I)$.

A morphism $m : ((G_I, F), u_{IJ}) \to ((H_I, F), v_{IJ})$ is said to be an r-filtered top local equivalence if all the m_I are r-filtered top local equivalences.

Denote $L = [1, \ldots, l]$ and for $I \subset L$, $p_{0(0I)} : S \times \tilde{S}_I \to S$, $p_{I(0I)} : S \times \tilde{S}_I \to S_I$ the projections.By definition, we have functors

- $T(S/(\tilde{S}_I)) : C_{fil}(S) \to C_{fil}(S/(\tilde{S}_I))$, $(G, F) \mapsto (i_{I*} j_I^*(G, F), I)$
- $T((\tilde{S}_I)/S) : C_{fil}(S/(\tilde{S}_I)) \to C_{fil}(S)$, $((G_I, F), u_{IJ}) \mapsto \operatorname{ho}\lim_{I\subset L} p_{0(0I)*} \Gamma^\vee_{S_I} p^*_{I(0I)}(G_I, F)$.

Note that the functors $T(S/(\tilde{S}_I)$ are embedding, since
$$\operatorname{ad}(i_I^*, i_{I*})(j_I^* F) : i_I^* i_{I*} j_I^* F \to j_I^* F$$
are top local equivalence.

Let $f : X \to S$ a morphism, with $X, S \in \mathrm{Top}$. Let $S = \cup_{i=1}^l S_i$ and $X = \cup_{i=1}^l X_i$ be open covers and $i_i : S_i \hookrightarrow \tilde{S}_i$, $i'_i : X_i \hookrightarrow \tilde{X}_i$ be closed embeddings, such that, for each $i \in [1, l]$, $f_i := f_{|X_i} : X_i \to S_i$ lift to a morphism $\tilde{f}_i : \tilde{X}_i \to \tilde{S}_i$. Then, $f_I = f_{|X_I} : X_I = \cap_{i\in I} X_i \to S_I = \cap_{i\in I} S_i$ lift to the morphism
$$\tilde{f}_I = \Pi_{i\in I} \tilde{f}_i : \tilde{X}_I = \Pi_{i\in I} \tilde{X}_i \to \tilde{S}_I = \Pi_{i\in I} \tilde{S}_i$$

Denote by $p_{IJ} : \tilde{S}_J \to \tilde{S}_I$ and $p'_{IJ} : \tilde{X}_J \to \tilde{X}_I$ the projections. Consider for $J \subset I$ the following commutative diagrams

$$D_{IJ} = \begin{tikzcd} S_I \arrow[r, "i_I"] & \tilde{S}_I \\ S_J \arrow[u, "j_{IJ}"] \arrow[r, "i_J"] & \tilde{S}_J \arrow[u, "p_{IJ}"'] \end{tikzcd}, \quad D'_{IJ} = \begin{tikzcd} X_I \arrow[r, "i'_I"] & \tilde{X}_I \\ X_J \arrow[u, "j'_{IJ}"] \arrow[r, "i'_J"] & \tilde{X}_J \arrow[u, "p'_{IJ}"'] \end{tikzcd}, \quad D_{fI} = \begin{tikzcd} S_I \arrow[r, "i_I"] & \tilde{S}_I \\ X_I \arrow[u, "f_I"] \arrow[r, "i'_I"] & \tilde{X}_I \arrow[u, "\tilde{f}_I"'] \end{tikzcd}$$

We have then following commutative diagram

$$\begin{tikzcd} & X_I \arrow[r, "n'_I"] & \tilde{X}_I & \tilde{X}_I \backslash X_I \arrow[l, "n'_I"] \\ i'_J : X_J \arrow[ur, "j'_{IJ}"] \arrow[r, "l_{IJ}"] & X_I \times X_I \times \tilde{X}_{J\backslash I} \arrow[u, "p'_{IJ}"] \arrow[r, "n'_I \times I"] & \tilde{X}_J \arrow[u, "p'_{IJ}"] & \tilde{X}_J \backslash X_J \arrow[l, "n'_J"] \arrow[u, "p'_{IJ}"] \end{tikzcd} .$$

whose square are cartesian. We then have the pullback functor
$$f^* : C_{(2)fil}(S/(\tilde{S}_I)) \to C_{(2)fil}(X/(\tilde{X}_I)),$$
$$((G_I, F), u_{IJ}) \mapsto f^*((G_I, F), u_{IJ}) := (\Gamma^\vee_{X_I} \tilde{f}_I^*(G_I, F), \tilde{f}_J^* u_{IJ})$$

with

$$\tilde{f}_J^* u_{IJ} : \Gamma^\vee_{X_I} \tilde{f}_I^*(G_I, F) \xrightarrow{\operatorname{ad}(p'^*_{IJ}, p'_{IJ*})(-)} p'_{IJ*} p'^*_{IJ} \Gamma^\vee_{X_I} \tilde{f}_I^*(G_I, F) \xrightarrow{T_\sharp(p_{IJ}, n'_I)(-)^{-1})} p'_{IJ*} \Gamma^\vee_{X_I \times \tilde{X}_{J\backslash I}} p'^*_{IJ} \tilde{f}_I^*(G_I, F)$$
$$\xrightarrow{p'_{IJ*} \gamma^\vee_{X_J}(-)} p'_{IJ*} \Gamma^\vee_{X_J} p'^*_{IJ} \tilde{f}_I^*(G_I, F) = p'_{IJ*} \Gamma^\vee_{X_J} \tilde{f}_J^* p^*_{IJ}(G_I, F) \xrightarrow{\Gamma^\vee_{X_J} \tilde{f}_J^* I(p^*_{IJ}, p_{IJ*})(-,-)(u_{IJ})} \Gamma^\vee_{X_J} \tilde{f}_J^*(G_J, F)$$

Let $(G,F) \in C_{fil}(S)$. Since, $j_I^{'*} i'_{I*} j_I^{'*} f^*(G,F) = 0$, the morphism $T(D_{fI})(j_I^*(G,F)) : \tilde{f}_I^* i_{I*} j_I^*(G,F) \to i'_{I*} j_I^{'*} f^*(G,F)$ factors trough

$$T(D_{fI})(j_I^*(G,F)) : \tilde{f}_I^* i_{I*} j_I^*(G,F) \xrightarrow{\gamma^\vee_{X_I}(-)} \Gamma^\vee_{X_I} \tilde{f}_I^* i_{I*} j_I^*(G,F) \xrightarrow{T^\gamma(D_{fI})(j_I^*(G,F))} i'_{I*} j_I^{'*} f^*(G,F)$$

We have then, for $(G,F) \in C_{fil}(S)$, the canonical transformation map

$$\begin{tikzcd} f^*T(S/(\tilde{S}_I))(G,F) \arrow[rr, "T(f,T(0/I))(G,F)"] \arrow[d, "="'] & & T(X/(\tilde{X}_I))(f^*(G,F)) \arrow[d, "="] \\ (\Gamma^\vee_{X_I} \tilde{f}_I^* i_{I*} j_I^*(G,F), \tilde{f}_J^* I) \arrow[rr, "T^\gamma(D_{fI})(j_I^*(G,F))"] & & (i'_{I*} j_I^{'*} f^*(G,F), I) \end{tikzcd}$$

Proposition 15. *Let $S \in \mathrm{Top}$. Let $S = \cup_{i=1}^l S_i$ an open cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{Top}$. Denote by $D_{(2)fil,\infty}((S/(\tilde{S}_I))) = \mathrm{Ho}_{top,\infty}(C_{(2)fil}((S/(\tilde{S}_I))))$ the localization of $C_{(2)fil}((S/(\tilde{S}_I)))$ with respect to top local equivalences. The functor $T(S/(\tilde{S}_I))$ induces an equivalence of category*

$$T(S/(\tilde{S}_I)) : D_{(2)fil,\infty}(S) \xrightarrow{\sim} D_{(2)fil,\infty}((S/(\tilde{S}_I)))$$

with inverse $T((\tilde{S}_I)/S)$

Proof. Follows from the fact that for $(G,F) \in C_{fil}(S)$,

$$\mathrm{ho}\lim_{I \subset L} p_{0(0I)*} \Gamma^\vee_{S_I} p^*_{I(0I)}(i_{I*} j_I^*(G,F)) \to p_{0(0I)*} \Gamma^\vee_{S_I} j_I^*(G,F)$$

is an equivalence top local. □

For $f : T \to S$ a morphism with $T, S \in \mathrm{Top}$ locally compact (in particular Hausdorf), e.g. $T, S \in \mathrm{CW}$, there is also a functor $f_! : C(T) \to C(S)$ given by the section which have compact support over f, and, for $K_1, K_2 \in C(T)$, we have a canonical map

$$T_!(f, hom) : f_* \mathcal{H}om(K_1, K_2) \to \mathcal{H}om(f_! K_1, f_! K_2)$$

The main result on presheaves on locally compact spaces is the following :

Theorem 9. *Let $f : T \to S$ a morphism with $T, S \in \mathrm{Top}$ locally compact.*

(i) The derived functor $Rf_! : D(T) \to D(S)$ has a right adjoint $f^!$ (Verdier duality) and, for $K_1, K_2 \in D(T)$ and $K_3, K_4 \in D(S)$, we have canonical isomorphisms

- $Rf_* R\mathcal{H}om^\bullet(Rf_! K_1, K_3) \xrightarrow{\sim} R\mathcal{H}om^\bullet(K_1, f^! K_3)$
- $f^! R\mathcal{H}om^\bullet(K_3, K_4) \xrightarrow{\sim} R\mathcal{H}om^\bullet(f^* K_3, f^! K_4)$

(ii) Denote, for $K \in D(S)$, $D(K) = R\mathcal{H}om^\bullet(K, a_S^! \mathbb{Z}) \in D(S)$ the Verdier dual of K. Then, if $K \in D_c(S)$, the evaluation map $ev^c(S)(K) : K \to D(D(K)$ is an isomorphism.

(iii) Assume we have a factorization $f : T \xrightarrow{l} Y \xrightarrow{p} S$ of f with l a closed embedding and p a smooth morphism of relative dimension d. Then $f^! K = i^! p^* K[d]$

Proof. (i):Standard, the proof is formal (see [30]).
(ii): See [30].
(iii): The fact that $p^! K = p^* K[d]$ follows by Poincare duality for topological manifold. □

We have by theorem 9 a pair of adjoint functor

$$(Rf_!, f^!) : D(T) \leftrightarrows D(S)$$

- with $f_! = f_*$ if f is proper,
- with $f^! = f^*[d]$ if f is smooth of relative dimension d.

2.6 Presheaves on the big Zariski site or on the big etale site

For $S \in \mathrm{Var}(\mathbb{C})$, we denote by $\rho_S : \mathrm{Var}(\mathbb{C})^{sm}/S \hookrightarrow \mathrm{Var}(\mathbb{C})/S$ be the full subcategory consisting of the objects $U/S = (U,h) \in \mathrm{Var}(\mathbb{C})/S$ such that the morphism $h : U \to S$ is smooth. That is, $\mathrm{Var}(\mathbb{C})^{sm}/S$ is the category

- whose objects are smooth morphisms $U/S = (U,h)$, $h : U \to S$ with $U \in \mathrm{Var}(\mathbb{C})$,
- whose morphisms $g : U/S = (U,h_1) \to V/S = (V,h_2)$ is a morphism $g : U \to V$ of complex algebraic varieties such that $h_2 \circ g = h_1$.

We denote again $\rho_S : \mathrm{Var}(\mathbb{C})/S \to \mathrm{Var}(\mathbb{C})^{sm}/S$ the associated morphism of site. We will consider

$$r^s(S) : \mathrm{Var}(\mathbb{C}) \xrightarrow{r(S)} \mathrm{Var}(\mathbb{C})/S \xrightarrow{\rho_S} \mathrm{Var}(\mathbb{C})^{sm}/S$$

the composite morphism of site. For $S \in \mathrm{Var}(\mathbb{C})$, we denote by $\mathbb{Z}_S := \mathbb{Z}(S/S) \in \mathrm{PSh}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ the constant presheaf By Yoneda lemma, we have for $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$, $\mathcal{H}om(\mathbb{Z}_S, F) = F$. For $f : T \to S$ a morphism, with $T, S \in \mathrm{Var}(\mathbb{C})$, we have the following commutative diagram of sites

$$\begin{tikzcd} \mathrm{Var}(\mathbb{C})/T \arrow[r, "\rho_T"] \arrow[d, "P(f)"] & \mathrm{Var}(\mathbb{C})^{sm}/T \arrow[d, "P(f)"] \\ \mathrm{Var}(\mathbb{C})/S \arrow[r, "\rho_S"] & \mathrm{Var}(\mathbb{C})^{sm}/S \end{tikzcd} \tag{23}$$

We denote, for $S \in \mathrm{Var}(\mathbb{C})$, the obvious morphism of sites

$$\tilde{e}(S) : \mathrm{Var}(\mathbb{C})/S \xrightarrow{\rho_S} \mathrm{Var}(\mathbb{C})^{sm}/S \xrightarrow{e(S)} \mathrm{Ouv}(S)$$

where $\mathrm{Ouv}(S)$ is the set of the Zariski open subsets of S, given by the inclusion functors $\tilde{e}(S) : \mathrm{Ouv}(S) \hookrightarrow \mathrm{Var}(\mathbb{C})^{sm}/S \hookrightarrow \mathrm{Var}(\mathbb{C})/S$. By definition, for $f : T \to S$ a morphism with $S, T \in \mathrm{Var}(\mathbb{C})$, the commutative diagram of sites (23) extend a commutative diagram of sites :

$$\begin{tikzcd} \tilde{e}(T) : \mathrm{Var}(\mathbb{C})/T \arrow[r, "\rho_T"] \arrow[d, "P(f)"] & \mathrm{Var}(\mathbb{C})^{sm}/T \arrow[r, "e(T)"] \arrow[d, "P(f)"] & \mathrm{Ouv}(T) \arrow[d, "P(f)"] \\ \tilde{e}(S) : \mathrm{Var}(\mathbb{C})/S \arrow[r, "\rho_S"] & \mathrm{Var}(\mathbb{C})^{sm}/S \arrow[r, "e(S)"] & \mathrm{Ouv}(S) \end{tikzcd} \tag{24}$$

- As usual, we denote by

$$(f^*, f_*) := (P(f)^*, P(f)_*) : C(\mathrm{Var}(\mathbb{C})^{sm}/S) \to C(\mathrm{Var}(\mathbb{C})^{sm}/T)$$

  the adjonction induced by $P(f) : \mathrm{Var}(\mathbb{C})^{sm}/T \to \mathrm{Var}(\mathbb{C})^{sm}/S$. Since the colimits involved in the definition of $f^* = P(f)^*$ are filtered, f^* also preserve monomorphism. Hence, we get an adjonction

$$(f^*, f_*) : C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S) \leftrightarrows C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/T),\ f^*(G,F) := (f^*G, f^*F)$$

- As usual, we denote by

$$(f^*, f_*) := (P(f)^*, P(f)_*) : C(\mathrm{Var}(\mathbb{C})/S) \to C(\mathrm{Var}(\mathbb{C})/T)$$

  the adjonction induced by $P(f) : \mathrm{Var}(\mathbb{C})/T \to \mathrm{Var}(\mathbb{C})/S$. Since the colimits involved in the definition of $f^* = P(f)^*$ are filtered, f^* also preserve monomorphism. Hence, we get an adjonction

$$(f^*, f_*) : C_{fil}(\mathrm{Var}(\mathbb{C})/S) \leftrightarrows C_{fil}(\mathrm{Var}(\mathbb{C})/T),\ f^*(G,F) := (f^*G, f^*F)$$

- For $h : U \to S$ a smooth morphism with $U, S \in \mathrm{Var}(\mathbb{C})$, the pullback functor $P(h) : \mathrm{Var}(\mathbb{C})^{sm}/S \to \mathrm{Var}(\mathbb{C})^{sm}/U$ admits a left adjoint $C(h)(X \to U) = (X \to U \to S)$. Hence, $h^* : C(\mathrm{Var}(\mathbb{C})^{sm}/S) \to C(\mathrm{Var}(\mathbb{C})^{sm}/U)$ admits a left adjoint

$$h_\sharp : C(\mathrm{Var}(\mathbb{C})^{sm}/U) \to C(\mathrm{Var}(\mathbb{C})^{sm}/S),\ F \mapsto ((V, h_0) \mapsto \lim_{(V', h\circ h') \to (V, h_0)} F(V', h'))$$

Note that we have for $V/U = (V, h')$ with $h' : V \to U$ a smooth morphism we have $h_\sharp(\mathbb{Z}(V/U)) = \mathbb{Z}(V'/S)$ with $V'/S = (V', h \circ h')$. Hence, since projective presheaves are the direct summands of the representable presheaves, $h_\sharp$ sends projective presheaves to projective presheaves. For $F^\bullet \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ and $G^\bullet \in C(\mathrm{Var}(\mathbb{C})^{sm}/U)$, we have the adjonction maps

$$\mathrm{ad}(h_\sharp, h^*)(G^\bullet) : G^\bullet \to h^* h_\sharp G^\bullet\ ,\ \mathrm{ad}(h_\sharp, h^*)(F^\bullet) : h_\sharp h^* F^\bullet \to F^\bullet.$$

For a smooth morphism $h : U \to S$, with $U, S \in \mathrm{Var}(\mathbb{C})$, we have the adjonction isomorphism, for $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/U)$ and $G \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$,

$$I(h_\sharp, h^*)(F, G) : \mathcal{H}om^\bullet(h_\sharp F, G) \xrightarrow{\sim} h_* \mathcal{H}om^\bullet(F, h^* G). \tag{25}$$

- For $f : T \to S$ any morphism with $T, S \in \mathrm{Var}(\mathbb{C})$, the pullback functor $P(f) : \mathrm{Var}(\mathbb{C})/T \to \mathrm{Var}(\mathbb{C})/S$ admits a left adjoint $C(f)(X \to T) = (X \to T \to S)$. Hence, $f^* : C(\mathrm{Var}(\mathbb{C})/S) \to C(\mathrm{Var}(\mathbb{C})/T)$ admits a left adjoint

$$f_\sharp : C(\mathrm{Var}(\mathbb{C})/T) \to C(\mathrm{Var}(\mathbb{C})/S),\ F \mapsto ((V, h_0) \mapsto \lim_{(V', f\circ h') \to (V, h_0)} F(V', h'))$$

Note that we have, for $(V/T) = (V, h)$, $f_\sharp \mathbb{Z}(V/T) = \mathbb{Z}(V/S)$ with $V/S = (V, f \circ h)$. Hence, since projective presheaves are the direct summands of the representable presheaves, $h_\sharp$ sends projective presheaves to projective presheaves. For $F^\bullet \in C(\mathrm{Var}(\mathbb{C})/S)$ and $G^\bullet \in C(\mathrm{Var}(\mathbb{C})/T)$, we have the adjonction maps

$$\mathrm{ad}(f_\sharp, f^*)(G^\bullet) : G^\bullet \to f^* f_\sharp G^\bullet\ ,\ \mathrm{ad}(f_\sharp, f^*)(F^\bullet) : f_\sharp f^* F^\bullet \to F^\bullet.$$

For a morphism $f : T \to S$, with $T, S \in \mathrm{Var}(\mathbb{C})$, we have the adjonction isomorphism, for $F \in C(\mathrm{Var}(\mathbb{C})/T)$ and $G \in C(\mathrm{Var}(\mathbb{C})/S)$,

$$I(f_\sharp, f^*)(F, G) : \mathcal{H}om^\bullet(f_\sharp F, G) \xrightarrow{\sim} f_* \mathcal{H}om^\bullet(F, f^* G). \tag{26}$$

- For a commutative diagram in $\mathrm{Var}(\mathbb{C})$:

$$D = \begin{tikzcd} V \arrow[r, "g_2"] \arrow[d, "h_2"] & U \arrow[d, "h_1"] \\ T \arrow[r, "g_1"] & S \end{tikzcd} \quad ,$$

where h_1 and h_2 are smooth, we denote by, for $F^\bullet \in C(\mathrm{Var}(\mathbb{C})^{sm}/U)$,

$$T_\sharp(D)(F^\bullet) : h_{2\sharp} g_2^* F^\bullet \to g_1^* h_{1\sharp} F^\bullet$$

the canonical map in $C(\mathrm{Var}(\mathbb{C})^{sm}/T)$ given by adjonction. If D is cartesian with $h_1 = h$, $g_1 = g$ $f_2 = h' : U_T \to T$, $g' : U_T \to U$,

$$T_\sharp(D)(F^\bullet) =: T_\sharp(g, h)(F^\bullet) : h'_\sharp g'^* F^\bullet \xrightarrow{\sim} g^* h_\sharp F^\bullet$$

is an isomorphism and for $G^\bullet \in C(\mathrm{Var}(\mathbb{C})^{sm}/T)$

$$T(D)(G^\bullet) =: T(g, h)(G^\bullet) : g^* h_* G^\bullet \xrightarrow{\sim} h'_* g'^* G^\bullet$$

is an isomorphism.

- For a commutative diagram in $\mathrm{Var}(\mathbb{C})$:

$$D = \begin{tikzcd} V \arrow[r, "g_2"] \arrow[d, "f_2"] & X \arrow[d, "f_1"] \\ T \arrow[r, "g_1"] & S \end{tikzcd} ,$$

  we denote by, for $F^\bullet \in C(\mathrm{Var}(\mathbb{C})/X)$,

$$T_\sharp(D)(F^\bullet) : f_{2\sharp}g_2^*F^\bullet \to g_1^*f_{1\sharp}F^\bullet$$

  the canonical map in $C(\mathrm{Var}(\mathbb{C})/T)$ given by adjonction. If D is cartesian with $h_1 = h$, $g_1 = g$ $f_2 = h' : X_T \to T$, $g' : X_T \to X$,

$$T_\sharp(D)(F^\bullet) =: T_\sharp(g,f)(F^\bullet) : f'_\sharp g'^*F^\bullet \xrightarrow{\sim} g^*f_\sharp F^\bullet$$

  is an isomorphism and for $G^\bullet \in C(\mathrm{Var}(\mathbb{C})/T)$

$$T(D)(G^\bullet) =: T(g,h)(G^\bullet) : f^*g_*G^\bullet \xrightarrow{\sim} g'_*f'^*G^\bullet$$

  is an isomorphism.

For $f : T \to S$ a morphism with $S, T \in \mathrm{Var}(\mathbb{C})$,

- we get for $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ from the a commutative diagram of sites (24) the following canonical transformation

$$T(e,f)(F^\bullet) : f^*e(S)_*F^\bullet \to e(T)_*f^*F^\bullet,$$

  which is NOT a quasi-isomorphism in general. However, for $h : U \to S$ a smooth morphism with $S, U \in \mathrm{Var}(\mathbb{C})$, $T(e,h)(F^\bullet) : h^*e(S)_*F^\bullet \xrightarrow{\sim} e(T)_*h^*F^\bullet$ is an isomorphism.

- we get for $F \in C(\mathrm{Var}(\mathbb{C})/S)$ from the a commutative diagram of sites (24) the following canonical transformation

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Let $S \in \mathrm{Var}(\mathbb{C})$,

- We have for $F, G \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$,
  - $e(S)_*(F \otimes G) = (e(S)_*F) \otimes (e(S)_*G)$ by definition
  - the canonical forgetfull map

$$T(S, hom)(F,G) : e(S)_*\mathcal{H}om^\bullet(F,G) \to \mathcal{H}om^\bullet(e(S)_*F, e(S)_*G).$$

    which is NOT a quasi-isomorphism in general.

  By definition, we have for $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$, $e(S)_*E_{zar}(F) = E_{zar}(e(S)_*F)$.

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By definition, we have for $F \in C(\mathrm{Var}(\mathbb{C})/S)$, $e(S)_*E_{zar}(F) = E_{zar}(e(S)_*F)$.

Let $S \in \mathrm{Var}(\mathbb{C})$.

- We have the dual functor

$$\mathbb{D}_S : C(\mathrm{Var}(\mathbb{C})^{sm}/S) \to C(\mathrm{Var}(\mathbb{C})^{sm}/S),\ F \mapsto \mathbb{D}_S(F) := \mathcal{H}om(F, E_{et}(\mathbb{Z}_S))$$

  It induces the functor

$$L\mathbb{D}_S : C(\mathrm{Var}(\mathbb{C})^{sm}/S) \to C(\mathrm{Var}(\mathbb{C})^{sm}/S),\ F \mapsto L\mathbb{D}_S(F) := \mathbb{D}_S(LF) := \mathcal{H}om(LF, E_{et}(\mathbb{Z}_S))$$

- We have the dual functor

$$\mathbb{D}_S : C(\mathrm{Var}(\mathbb{C})/S) \to C(\mathrm{Var}(\mathbb{C})/S),\ F \mapsto \mathbb{D}_S(F) := \mathcal{H}om(F, E_{et}(\mathbb{Z}_S))$$

  It induces the functor

$$L\mathbb{D}_S : C(\mathrm{Var}(\mathbb{C})/S) \to C(\mathrm{Var}(\mathbb{C})/S),\ F \mapsto L\mathbb{D}_S(F) := \mathbb{D}_S(LF) := \mathcal{H}om(LF, E_{et}(\mathbb{Z}_S))$$

The adjonctions

$$(\tilde{e}(S)^*, \tilde{e}(S)_*) : C(\mathrm{Var}(\mathbb{C})/S) \leftrightarrows C(S)\ ,\ (e(S)^*, e(S)_*) : C(\mathrm{Var}(\mathbb{C})^{sm}/S) \leftrightarrows C(S)$$

induce adjonctions

$$(\tilde{e}(S)^*, \tilde{e}(S)_*) : C_{fil}(\mathrm{Var}(\mathbb{C})/S) \leftrightarrows C_{fil}(S)\ ,\ (e(S)^*, e(S)_*) : C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S) \leftrightarrows C_{fil}(S)$$

given by $e(S)_*(G, F) := (e(S)_*G, e(S)_*F)$, since $e(S)_*$ and $e(S)^*$ preserve monomorphisms. Note that

- for $F \in \mathrm{PSh}(\mathrm{Var}(\mathbb{C})^{sm}/S)$, $e(S)_*F$ is simply the restriction of F to the small Zariski site of X,
- for $F \in \mathrm{PSh}(\mathrm{Var}(\mathbb{C})/S)$, $\tilde{e}(S)_*F = e(S)_*\rho_{S*}F$ is simply the restriction of F to the small Zariski site of X, $\rho_{S*}F$ being the restriction of F to $\mathrm{Var}(\mathbb{C})^{sm}/S$.

Together with the internal hom functor, we get the bifunctor,

$$e(S)_*\mathcal{H}om(\cdot, \cdot) : C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S) \times C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S) \to C_{2fil}(S), \tag{27}$$

$$((F, W), (G, F)) \mapsto e(S)_*\mathcal{H}om^\bullet((F^\bullet, W), (G^\bullet, F)). \tag{28}$$

For $i : Z \hookrightarrow S$ a closed embedding, with $Z, S \in \mathrm{Var}(\mathbb{C})$, we denote by

$$(i_*, i^!) := (P(i)_*, P(i)^\perp) : C(\mathrm{Var}(\mathbb{C})^{sm}/Z) \leftrightarrows C(\mathrm{Var}(\mathbb{C})^{sm}/S)$$

the adjonction induced by the morphism of site $P(i) : \mathrm{Var}(\mathbb{C})^{sm}/Z \to \mathrm{Var}(\mathbb{C})^{sm}/S$ For $i : Z \hookrightarrow S$ a closed embedding, $Z, S \in \mathrm{Var}(\mathbb{C})$, we denote

$$\mathbb{Z}_{Z,S} := \mathrm{Cone}(\mathrm{ad}(i^*, i_*)(\mathbb{Z}_S) : \mathbb{Z}_S \to i_*\mathbb{Z}_Z)$$

We have the support section functors of a closed embedding $i : Z \hookrightarrow S$ for presheaves on the big Zariski site.

Definition 6. *Let $i : Z \hookrightarrow S$ be a closed embedding with $S, Z \in \mathrm{Var}(\mathbb{C})$ and $j : S \backslash Z \hookrightarrow S$ be the open complementary subset.*

(i) We define the functor

$$\Gamma_Z : C(\mathrm{Var}(\mathbb{C})^{sm}/S) \to C(\mathrm{Var}(\mathbb{C})^{sm}/S),\ G^\bullet \mapsto \Gamma_Z G^\bullet := \mathrm{Cone}(\mathrm{ad}(j^*, j_*)(G^\bullet) : G^\bullet \to j_*j^*G^\bullet)[-1],$$

so that there is then a canonical map $\gamma_Z(G^\bullet) : \Gamma_Z G^\bullet \to G^\bullet$.

(ii) We have the dual functor of (i) :

$$\Gamma^{\vee}_Z : C(\mathrm{Var}(\mathbb{C})^{sm}/S) \to C(\mathrm{Var}(\mathbb{C})^{sm}/S),\ F \mapsto \Gamma^{\vee}_Z(F^{\bullet}) := \mathrm{Cone}(\mathrm{ad}(j_{\sharp}, j^*)(G^{\bullet}) : j_{\sharp}j^*G^{\bullet} \to G^{\bullet}),$$

together with the canonical map $\gamma^{\vee}_Z(G) : F \to \Gamma^{\vee}_Z(G)$.

(iii) For $F, G \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$*, we denote by*

$$I(\gamma, hom)(F,G) := (I, I(j_{\sharp}, j^*)(F,G)^{-1}) : \Gamma_Z \mathcal{H}om(F,G) \xrightarrow{\sim} \mathcal{H}om(\Gamma^{\vee}_Z F, G)$$

the canonical isomorphism given by adjonction.

Let $i : Z \hookrightarrow S$ be a closed embedding with $S, Z \in \mathrm{Var}(\mathbb{C})$ and $j : S\backslash Z \hookrightarrow S$ be the open complementary.

- For $G \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$, the adjonction map $\mathrm{ad}(i_*, i^!)(G) : i_*i^!G \to G$ factor through $\gamma_Z(G)$:

  $$\mathrm{ad}(i_*, i^!)(G) : i_*i^!G \xrightarrow{\mathrm{ad}(i_*,i^!)(G)^{\gamma}} \Gamma_Z(G) \xrightarrow{\gamma_Z(G)} G.$$

  However, note that when dealing with the big sites $P(i) : \mathrm{Var}(\mathbb{C})^{sm}/Z \to \mathrm{Var}(\mathbb{C})^{sm}/S$, if $G \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is not $\mathbb{A}^1_S$ local and Zariski fibrant,

  $$\mathrm{ad}(i_*, i^!)(G)^{\gamma} : i_*i^!G \to \Gamma_Z(G)$$

  is NOT and homotopy equivalence, and $\Gamma_Z G \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is NOT in general in the image of the functor $i_* : C(\mathrm{Var}(\mathbb{C})^{sm}/Z) \to C(\mathrm{Var}(\mathbb{C})^{sm}/S)$.

- For $G \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$, the adjonction map $\mathrm{ad}(i^*, i_*)(G) : G \to i_*i^*G$ factor through $\gamma^{\vee}_Z(G)$:

  $$\mathrm{ad}(i^*, i_*)(G) : G \xrightarrow{\gamma^{\vee}_Z(G)} \Gamma^{\vee}_Z G \xrightarrow{\mathrm{ad}(i^*,i_*)(G)^{\gamma}} i_*i^*G,$$

  and as in (i), $\mathrm{ad}(i^*, i_*)(G)^{\gamma} : \Gamma^{\vee}_Z(G) \to i_*i^*G$ is NOT an homotopy equivalence but

Let $i : Z \hookrightarrow S$ be a closed embedding with $S, Z \in \mathrm{Var}(\mathbb{C})$.

- Since $\Gamma_Z : C(\mathrm{Var}(\mathbb{C})^{sm}/S) \to C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ preserve monomorphism, it induces a functor

  $$\Gamma_Z : C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S) \to C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S),\ (G,F) \mapsto \Gamma_Z(G,F) := (\Gamma_Z G, \Gamma_Z F)$$

- Since $\Gamma^{\vee}_Z : C(\mathrm{Var}(\mathbb{C})^{sm}/S) \to C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ preserve monomorphism, it induces a functor

  $$\Gamma^{\vee}_Z : C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S) \to C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S),\ (G,F) \mapsto \Gamma^{\vee}_Z(G,F) := (\Gamma^{\vee}_Z G, \Gamma^{\vee}_Z F)$$

Definition-Proposition 4. *(i) Let* $g : S' \to S$ *a morphism and* $i : Z \hookrightarrow S$ *a closed embedding with* $S', S, Z \in \mathrm{Var}(\mathbb{C})$*. Then, for* $(G,F) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$*, there exist a map in* $C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S')$

$$T(g,\gamma)(G,F) : g^*\Gamma_Z(G,F) \to \Gamma_{Z\times_S S'}g^*(G,F)$$

unique up to homotopy such that $\gamma_{Z\times_S S'}(g^*(G,F)) \circ T(g,\gamma)(G,F) = g^*\gamma_Z(G,F)$.

(ii) Let $i_1 : Z_1 \hookrightarrow S$*,* $i_2 : Z_2 \hookrightarrow Z_1$ *be closed embeddings with* $S, Z_1, Z_2 \in \mathrm{Var}(\mathbb{C})$*. Then, for* $(G,F) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$*,*

- *there exist a canonical map* $T(Z_2/Z_1,\gamma)(G,F) : \Gamma_{Z_2}(G,F) \to \Gamma_{Z_1}(G,F)$ *in* $C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ *unique up to homotopy such that* $\gamma_{Z_1}(G,F) \circ T(Z_2/Z_1,\gamma)(G,F) = \gamma_{Z_2}(G,F)$*, together with a distinguish triangle*

  $$\Gamma_{Z_2}(G,F) \xrightarrow{T(Z_2/Z_1,\gamma)(G,F)} \Gamma_{Z_1}(G,F) \xrightarrow{\mathrm{ad}(j_2^*, j_{2*})(\Gamma_{Z_1}(G,F))} \Gamma_{Z_1\backslash Z_2}(G,F) \to \Gamma_{Z_2}(G,F)[1]$$

  in $K_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S) := K(\mathrm{PSh}_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S))$*,*

– *there exist a map* $T(Z_2/Z_1,\gamma^\vee)(G,F):\Gamma^\vee_{Z_1}(G,F)\to\Gamma^\vee_{Z_2}(G,F)$ *in* $C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ *unique up to homotopy such that* $\gamma^\vee_{Z_2}(G,F)=T(Z_2/Z_1,\gamma^\vee)(G,F)\circ\gamma^\vee_{Z_1}(G,F)$*, together with a distinguish triangle*

$$\Gamma^\vee_{Z_1\backslash Z_2}(G,F)\xrightarrow{\mathrm{ad}(j_{2\sharp},j_2^*)(\Gamma^\vee_{Z_1}(G,F))}\Gamma^\vee_{Z_1}(G,F)\xrightarrow{T(Z_2/Z_1,\gamma^\vee)(G,F)}\Gamma^\vee_{Z_2}(G,F)\to\Gamma^\vee_{Z_1\backslash Z_2}(G,F)[1]$$

in $K_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$.

(iii) Consider a morphism $g:(S',Z')\to(S,Z)$ *with* $(S',Z'),(S,Z)\in\mathrm{Var}^2(\mathbb{C})$ *We denote, for* $G\in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ *the composite*

$$T(D,\gamma^\vee)(G):g^*\Gamma^\vee_Z G\xrightarrow{\sim}\Gamma^\vee_{Z\times_S S'}g^*G\xrightarrow{T(Z'/Z\times_S S',\gamma^\vee)(G)}\Gamma^\vee_{Z'}g^*G$$

and we have then the factorization $\gamma^\vee_{Z'}(g^*G):g^*G\xrightarrow{g^*\gamma^\vee_Z(G)}g^*\Gamma^\vee_Z G\xrightarrow{T(D,\gamma^\vee)(G)}\Gamma^\vee_{Z'}g^*G.$

Proof. (i): We have the cartesian square

$$\begin{tikzcd} S\backslash Z \arrow[r, "j"] & S \\ S'\backslash Z\times_S S' \arrow[u, "g'"] \arrow[r, "j'"] & S' \arrow[u, "g"] \end{tikzcd}$$

and the map is given by

$$(I,T(g,j)(j^*G)):\mathrm{Cone}(g^*G\to g^*j_*j^*G)\to\mathrm{Cone}(g^*G\to j'_*j'^*g^*G=j'_*g'^*j^*G).$$

(ii): Follows from the fact that $j_1^*\Gamma_{Z_2}G=0$ and $j_1^*\Gamma^\vee_{Z_2}G=0$, with $j_1:S\backslash Z_1\hookrightarrow S$ the closed embedding.
(iii): Obvious. □

The following easy proposition concern the restriction from the big Zariski site to the small site Zariski site :

Proposition 16. *For* $f:T\to S$ *a morphism and* $i:Z\hookrightarrow S$ *a closed embedding, with* $Z,S,T\in\mathrm{Var}(\mathbb{C})$*, we have*

(i) $e(S)_*f_*=f_*e(T)_*$ *and* $e(S)^*f_*=f_*e(T)^*$

(ii) $e(S)_*\Gamma_Z=\Gamma_Z e(S)_*$.

Proof. (i):The first equality $e(S)_*f_*=f_*e(T)_*$ is given by the diagram (24). The second equality is immediate from definition after a direct computation.
(ii) For $G^\bullet\in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$, we have the canonical equality

$$\begin{aligned} e(S)_*\Gamma_Z(G^\bullet)=e(S)_*\mathrm{Cone}(G\to j_*j^*G^\bullet)[-1] &= \mathrm{Cone}(e(S)_*G^\bullet\to e(S)_*j_*j^*G^\bullet)[-1]\\ &= \mathrm{Cone}(e(S)_*G^\bullet\to j_*j^*e(S)_*G^\bullet)[-1]\\ &= \Gamma_Z e(S)_*G^\bullet \end{aligned}$$

by (i) and since $j:S\backslash Z\hookrightarrow S$ is a smooth morphism. □

Definition 7. *For* $S\in\mathrm{Var}(\mathbb{C})$*, we denote by*

$$C_{O_S}(\mathrm{Var}(\mathbb{C})^{sm}/S):=C_{e(S)^*O_S}(\mathrm{Var}(\mathbb{C})^{sm}/S)$$

the category of complexes of presheaves on $\mathrm{Var}(\mathbb{C})^{sm}/S$ *endowed with a structure of* $e(S)^*O_S$ *module, and by*

$$C_{O_Sfil}(\mathrm{Var}(\mathbb{C})^{sm}/S):=C_{e(S)^*O_Sfil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$$

the category of filtered complexes of presheaves on $\mathrm{Var}(\mathbb{C})^{sm}/S$*endowed with a structure of* $e(S)^*O_S$ *module.*

Let $S \in \mathrm{Var}(\mathbb{C})$. Let $Z \subset S$ a closed subset. Denote by $j : S\backslash Z \hookrightarrow S$ the open complementary embedding,

- For $G \in C_{O_S}(\mathrm{Var}(\mathbb{C})^{sm}/S)$, $\Gamma_Z G := \mathrm{Cone}(\mathrm{ad}(j^*, j_*)(G) : F \to j_* j^* G)[-1]$ has a (unique) structure of $e(S)^* O_S$ module such that $\gamma_Z(G) : \Gamma_Z G \to G$ is a map in $C_{O_S}(\mathrm{Var}(\mathbb{C})^{sm}/S)$. This gives the functor
$$\Gamma_Z : C_{O_S fil}(\mathrm{Var}(\mathbb{C})^{sm}/S) \to C_{fil O_S}(\mathrm{Var}(\mathbb{C})^{sm}/S),\ (G, F) \mapsto \Gamma_Z(G, F) := (\Gamma_Z G, \Gamma_Z F),$$
together with the canonical map $\gamma_Z((G, F) : \Gamma_Z(G, F) \to (G, F)$. Let $Z_2 \subset Z$ a closed subset. Then, for $G \in C_{O_S}(\mathrm{Var}(\mathbb{C})^{sm}/S)$, $T(Z_2/Z, \gamma)(G) : \Gamma_{Z_2} G \to \Gamma_Z G$ is a map in $C_{O_S}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ (i.e. is $e(S)^* O_S$ linear).
- For $G \in C_{O_S}(\mathrm{Var}(\mathbb{C})^{sm}/S)$, $\Gamma^\vee_Z G := \mathrm{Cone}(\mathrm{ad}(j_\sharp, j^*)(G) : j_\sharp j^* G \to G)$ has a unique structure of $e(S)^* O_S$ module, such that $\gamma^\vee_Z(G) : G \to \Gamma^\vee_Z G$ is a map in $C_{O_S}(\mathrm{Var}(\mathbb{C})^{sm}/S)$. This gives the the functor
$$\Gamma^\vee_Z : C_{O_S fil}(S) \to C_{fil O_S}(S),\ (G, F) \mapsto \Gamma^\vee_Z(G, F) := (\Gamma^\vee_Z G, \Gamma^\vee_Z F),$$
together with the canonical map $\gamma^\vee_Z((G, F) : (G, F) \to \Gamma^\vee_Z(G, F)$. Let $Z_2 \subset Z$ a closed subset. Then, for $G \in C_{O_S}(\mathrm{Var}(\mathbb{C})^{sm}/S)$, $T(Z_2/Z, \gamma^\vee)(G) : \Gamma^\vee_Z G \to \Gamma^\vee_{Z_2} G$ is a map in $C_{O_S}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ (i.e. is $e(S)^* O_S$ linear).

Definition 8. *Let $S \in \mathrm{Var}(\mathbb{C})$. Let $Z \subset S$ a closed subset.*

(i) We denote by
$$C_Z(\mathrm{Var}(\mathbb{C})^{sm}/S) \subset C(\mathrm{Var}(\mathbb{C})^{sm}/S)$$
the full subcategory consisting of complexes of presheaves $F^\bullet \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $a_{et} H^n(j^* F^\bullet) = 0$ for all $n \in \mathbb{Z}$, where $j : S\backslash Z \hookrightarrow S$ is the complementary open embedding and a_{et} is the sheaftification functor.

(i)' We denote by
$$C_{O_S, Z}(\mathrm{Var}(\mathbb{C})^{sm}/S) \subset C_{O_S}(\mathrm{Var}(\mathbb{C})^{sm}/S),$$
the full subcategory consisting of complexes of presheaves $F^\bullet \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $a_{et} H^n(j^* F^\bullet) = 0$ for all $n \in \mathbb{Z}$, where $j : S\backslash Z \hookrightarrow S$ is the complementary open embedding and a_{et} is the sheaftification functor.

(ii) We denote by
$$C_{fil, Z}(\mathrm{Var}(\mathbb{C})^{sm}/S) \subset C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$$
the full subcategory consisting of filtered complexes of presheaves $(F^\bullet, F) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that there exist $r \in \mathbb{N}$ and an r-filtered homotopy equivalence $\phi : (F^\bullet, F) \to (F^{'\bullet}, F)$ with $(F^{'\bullet}, F) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $a_{et} j^* H^n \mathrm{Gr}^p_F(F^{'\bullet}, F) = 0$ for all $n, p \in \mathbb{Z}$, where $j : S\backslash Z \hookrightarrow S$ is the complementary open embedding and a_{et} is the sheaftification functor. Note that by definition this r does NOT depend on n and p.

(ii)' We denote by
$$C_{O_S fil, Z}(\mathrm{Var}(\mathbb{C})^{sm}/S) \subset C_{O_S fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$$
the full subcategory consisting of filtered complexes of presheaves $(F^\bullet, F) \in C_{O_S fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that there exist $r \in \mathbb{N}$ and an r-filtered homotopy equivalence $\phi : (F^\bullet, F) \to (F^{'\bullet}, F)$ with $(F^{'\bullet}, F) \in C_{O_S fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $a_{et} j^* H^n \mathrm{Gr}^p_F(F^{'\bullet}, F) = 0$ for all $p, q \in \mathbb{Z}$, where $j : S\backslash Z \hookrightarrow S$ is the complementary open embedding and a_{et} is the sheaftification functor. Note that by definition this r does NOT depend on n and p.

Let $S \in \mathrm{Var}(\mathbb{C})$ and $Z \subset S$ a closed subset.

- For $(G,F)\in C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$, we have $\Gamma_Z(G,F),\Gamma^{\vee}_Z(G,F)\in C_{fil,Z}(\mathrm{Var}(\mathbb{C})^{sm}/S)$.
- For $(G,F)\in C_{O_Sfil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$, we have $\Gamma_Z(G,F),\Gamma^{\vee}_Z(G,F)\in C_{O_Sfil,Z}(\mathrm{Var}(\mathbb{C})^{sm}/S)$.

Let $S\in\mathrm{Var}(\mathbb{C})$. Let $S=\cup_{i=1}^{l}S_i$ an open affine cover and denote by $S_I=\cap_{i\in I}S_i$. Let $i_i:S_i\hookrightarrow\tilde{S}_i$ closed embeddings, with $\tilde{S}_i\in\mathrm{Var}(\mathbb{C})$. For $I\subset[1,\cdots l]$, denote by $\tilde{S}_I=\Pi_{i\in I}\tilde{S}_i$. We then have closed embeddings $i_I:S_I\hookrightarrow\tilde{S}_I$ and for $J\subset I$ the following commutative diagram

$$D_{IJ}=\quad\begin{tikzcd} S_I \arrow[r, "i_I"] & \tilde{S}_I \\ S_J \arrow[u, "j_{IJ}"] \arrow[r, "i_J"] & \tilde{S}_J \arrow[u, "p_{IJ}"'] \end{tikzcd}$$

where $p_{IJ}:\tilde{S}_J\to\tilde{S}_I$ is the projection and $j_{IJ}:S_J\hookrightarrow S_I$ is the open embedding so that $j_I\circ j_{IJ}=j_J$. This gives the diagram of algebraic varieties $(\tilde{S}_I)\in\mathrm{Fun}(\mathcal{P}(\mathbb{N}),\mathrm{Var}(\mathbb{C}))$ which the diagram of sites $\mathrm{Var}(\mathbb{C})^{sm}/(\tilde{S}_I)\in\mathrm{Fun}(\mathcal{P}(\mathbb{N}),\mathrm{Cat})$. Denote by $m:\tilde{S}_I\backslash(S_I\backslash S_J)\hookrightarrow\tilde{S}_I$ the open embedding.

Definition 9. *Let $S\in\mathrm{Var}(\mathbb{C})$. Let $S=\cup_{i=1}^{l}S_i$ an open cover and denote by $S_I=\cap_{i\in I}S_i$. Let $i_i:S_i\hookrightarrow\tilde{S}_i$ closed embeddings, with $\tilde{S}_i\in\mathrm{Var}(\mathbb{C})$. We will denote by $C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/(S/(\tilde{S}_I)))\subset C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/(\tilde{S}_I))$ the full subcategory*

- *whose objects $(G,F)=((G_I,F)_{I\subset[1,\cdots l]},u_{IJ})$, with $(G_I,F)\in C_{fil,S_I}(\mathrm{Var}(\mathbb{C})^{sm}/\tilde{S}_I)$, and $u_{IJ}:m^*(G_I,F)\to m^*p_{IJ*}(G_J,F)$ for $I\subset J$, are ∞-filtered Zariski local equivalence, satisfying for $I\subset J\subset K$, $p_{IJ*}u_{JK}\circ u_{IJ}=u_{IK}$ in $C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/\tilde{S}_I)$,*
- *the morphisms $m:((G,F),u_{IJ})\to((H,F),v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,*
$$m=(m_I:(G_I,F)\to(H_I,F))_{I\subset[1,\cdots l]}$$
such that $v_{IJ}\circ m_I=p_{IJ*}m_J\circ u_{IJ}$ in $C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/\tilde{S}_I)$.

A morphism $m:((G_I,F),u_{IJ})\to((H_I,F),v_{IJ})$ is said to an r-filtered Zariski, resp. etale local, equivalence, if all the m_I are r-filtered Zariski, resp. etale, local equivalences.

Denote $L=[1,\ldots,l]$ and for $I\subset L$, $p_{0(0I)}:S\times\tilde{S}_I\to S$, $p_{I(0I)}:S\times\tilde{S}_I\to S_I$ the projections. By definition, we have functors

- $T(S/(\tilde{S}_I)):C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)\to C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/(S/(\tilde{S}_I)))$, $(G,F)\mapsto(i_{I*}j_I^*(G,F),T(D_{IJ})(j_I^*(G,F)))$,
- $T((\tilde{S}_I)/S):C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/(S/(\tilde{S}_I)))\to C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$, $((G_I,F),u_{IJ})\mapsto\mathrm{ho}\lim_{I\subset L}p_{0(0I)*}\Gamma^{\vee}_{S_I}p_{I(0I)}^*(G_I,F)$.

Note that the functors $T(S/(\tilde{S}_I)$ are NOT embedding, since

$$\mathrm{ad}(i_I^*,i_{I*})(j_I^*F):i_I^*i_{I*}j_I^*F\to j_I^*F$$

are Zariski local equivalence but NOT isomorphism since we are dealing with the morphism of big sites $P(i_I):\mathrm{Var}(\mathbb{C})^{sm}/S_I\to\mathrm{Var}(\mathbb{C})^{sm}/\tilde{S}_I$. However, by theorem 16, these functors induce full embeddings

$$T(S/(\tilde{S}_I)):D_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)\to D_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/(S/(\tilde{S}_I)))$$

since for $F\in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$,

$$\mathrm{ho}\lim_{I\subset L}p_{0(0I)*}\Gamma^{\vee}_{S_I}p_{I(0I)}^*(i_{I*}j_I^*F)\to p_{0(0I)*}\Gamma^{\vee}_{S_I}j_I^*F$$

is an equivalence Zariski local.

Let $f : X \to S$ a morphism, with $X, S \in \mathrm{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^{l} S_i$ and $X = \cup_{i=1}^{l} X_i$ be affine open covers and $i_i : S_i \hookrightarrow \tilde{S}_i$, $i'_i : X_i \hookrightarrow \tilde{X}_i$ be closed embeddings. Let $\tilde{f}_i : \tilde{X}_i \to \tilde{S}_i$ be a lift of the morphism $f_i = f_{|X_i} : X_i \to S_i$. Then, $f_I = f_{|X_I} : X_I = \cap_{i \in I} X_i \to S_I = \cap_{i \in I} S_i$ lift to the morphism

$$\tilde{f}_I = \Pi_{i \in I} \tilde{f}_i : \tilde{X}_I = \Pi_{i \in I} \tilde{X}_i \to \tilde{S}_I = \Pi_{i \in I} \tilde{S}_i$$

Denote by $p_{IJ} : \tilde{S}_J \to \tilde{S}_I$ and $p'_{IJ} : \tilde{X}_J \to \tilde{X}_I$ the projections. Consider for $J \subset I$ the following commutative diagrams

$$D_{IJ} = \begin{tikzcd} S_I \arrow[r, "i_I"] & \tilde{S}_I \\ S_J \arrow[u, "j_{IJ}"] \arrow[r, "i_J"] & \tilde{S}_J \arrow[u, "p_{IJ}"'] \end{tikzcd}, \quad D'_{IJ} = \begin{tikzcd} X_I \arrow[r, "i'_I"] & \tilde{X}_I \\ X_J \arrow[u, "j'_{IJ}"] \arrow[r, "i'_J"] & \tilde{X}_J \arrow[u, "p'_{IJ}"'] \end{tikzcd}, \quad D_{fI} = \begin{tikzcd} S_I \arrow[r, "i_I"] & \tilde{S}_I \\ X_I \arrow[u, "f_I"] \arrow[r, "i'_I"] & \tilde{X}_I \arrow[u, "\tilde{f}_I"'] \end{tikzcd}$$

We have then following commutative diagram

$$\begin{tikzcd} & X_I \arrow[rr, "n'_I"] & & \tilde{X}_I & \tilde{X}_I \backslash X_I \arrow[l, "n'_I"] \\ i'_J : X_J \arrow[ur, "j'_{IJ}"] \arrow[r, "l_{IJ}"] & X_I \times X_I \times \tilde{X}_{J \backslash I} \arrow[u, "p'_{IJ}"] \arrow[rr, "n'_I \times I"] & & \tilde{X}_J \arrow[u, "p'_{IJ}"] & \tilde{X}_J \backslash X_J \arrow[l, "n'_J"] \arrow[u, "p'_{IJ}"] \end{tikzcd} .$$

whose square are cartesian. We then have the pullback functor

$$f^* : C_{(2)fil}(\mathrm{Var}(\mathbb{C})^{sm}/S/(\tilde{S}_I)) \to C_{(2)fil}(\mathrm{Var}(\mathbb{C})^{sm}/X/(\tilde{X}_I)),$$
$$((G_I, F), u_{IJ}) \mapsto f^*((G_I, F), u_{IJ}) := (\Gamma^{\vee}_{X_I} \tilde{f}_I^*(G_I, F), \tilde{f}_J^* u_{IJ})$$

with

$$\tilde{f}_J^* u_{IJ} : \Gamma^{\vee}_{X_I} \tilde{f}_I^*(G_I, F) \xrightarrow{\mathrm{ad}(p'^*_{IJ}, p'_{IJ*})(-)} p'_{IJ*} p'^*_{IJ} \Gamma^{\vee}_{X_I} \tilde{f}_I^*(G_I, F) \xrightarrow{T_\sharp(p_{IJ}, n'_I)(-)^{-1})} p'_{IJ*} \Gamma^{\vee}_{X_I \times \tilde{X}_{J \backslash I}} p'^*_{IJ} \tilde{f}_I^*(G_I, F)$$
$$\xrightarrow{p'_{IJ*} \gamma^{\vee}_{X_J}(-)} p'_{IJ*} \Gamma^{\vee}_{X_J} p'^*_{IJ} \tilde{f}_I^*(G_I, F) = p'_{IJ*} \Gamma^{\vee}_{X_J} \tilde{f}_J^* p^*_{IJ}(G_I, F) \xrightarrow{\Gamma^{\vee}_{X_J} \tilde{f}_J^* I(p^*_{IJ}, p_{IJ*})(-,-)(u_{IJ})} \Gamma^{\vee}_{X_J} \tilde{f}_J^*(G_J, F)$$

Let $(G, F) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$. Since, $j_I'^* i'_{I*} j_I'^* f^*(G, F) = 0$, the morphism $T(D_{fI})(j_I^*(G, F)) : \tilde{f}_I^* i_{I*} j_I^*(G, F) \to i'_{I*} j_I'^* f^*(G, F)$ factors trough

$$T(D_{fI})(j_I^*(G, F)) : \tilde{f}_I^* i_{I*} j_I^*(G, F) \xrightarrow{\gamma^{\vee}_{X_I}(-)} \Gamma^{\vee}_{X_I} \tilde{f}_I^* i_{I*} j_I^*(G, F) \xrightarrow{T^{\gamma}(D_{fI})(j_I^*(G, F))} i'_{I*} j_I'^* f^*(G, F)$$

We have then, for $(G, F) \in C_{fil}(S)$, the canonical transformation map

$$\begin{tikzcd} f^* T(S/(\tilde{S}_I))(G, F) \arrow[r, "T(f, T(0/I))(G, F)"] \arrow[d, "="] & T(X/(\tilde{X}_I))(f^*(G, F)) \arrow[d, "="] \\ (\Gamma^{\vee}_{X_I} \tilde{f}_I^* i_{I*} j_I^*(G, F), \tilde{f}_J^* I) \arrow[r, "T^{\gamma}(D_{fI})(j_I^*(G, F))"] & (i'_{I*} j_I'^* f^*(G, F), I) \end{tikzcd}$$

To show that the cohomology sheaves of the filtered De Rham realization functor of constructible motives are mixed hodge modules, we will need to take presheaves of the following form

Definition 10. *(i) Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$. Assume that there exist a factorization $f : X \xrightarrow{i} Y \times S \xrightarrow{p} S$, with $Y \in \mathrm{SmVar}(\mathbb{C})$, $i : X \hookrightarrow Y$ is a closed embedding and p the projection. We then consider*

$$Q(X/S) := p_\sharp \Gamma^{\vee}_X \mathbb{Z}_{Y \times S} \in C(\mathrm{Var}(\mathbb{C})^{sm}/S).$$

By definition $Q(X/S)$ is projective.

(ii) *Let $f : X \to S$ and $g : T \to S$ two morphism with $X, S, T \in \mathrm{Var}(\mathbb{C})$. Assume that there exist a factorization $f : X \xrightarrow{i} Y \times S \xrightarrow{p} S$, with $Y \in \mathrm{SmVar}(\mathbb{C})$, $i : X \hookrightarrow Y$ is a closed embedding and p the projection. We then have the following commutative diagram whose squares are cartesian*

$$\begin{tikzcd} f : X \arrow[r, "i"] & Y \times S \arrow[r, "p"] & S \\ f' : X_T \arrow[u, "g'"] \arrow[r, "i'"] & Y \times T \arrow[u, "g'' := (I \times g)"] \arrow[r, "p'"] & T \arrow[u, "g"] \end{tikzcd}$$

We then have the canonical isomorphism in $C(\mathrm{Var}(\mathbb{C})^{sm}/T)$

$$T(f, g, Q) : g^* Q(X/S) := g^* p_\sharp \Gamma^\vee_X \mathbb{Z}_{Y \times S} \xrightarrow{T_\sharp(g,p)(-)^{-1}} p'_\sharp g''^* \Gamma^\vee_X \mathbb{Z}_{Y \times S}$$
$$\xrightarrow{p'_\sharp T(g'', \gamma^\vee)(-)^{-1}} p'_\sharp \Gamma^\vee_{X_T} \mathbb{Z}_{Y \times T} =: Q(X_T/T).$$

(iii) *Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$. Assume that there exist a factorization $f : X \xrightarrow{i} Y \times S \xrightarrow{p} S$, with $Y \in \mathrm{SmVar}(\mathbb{C})$, $i : X \hookrightarrow Y$ is a closed embedding and p the projection. We then consider*

$$Q^h(X/S) := p_* \Gamma_X E_{et}(\mathbb{Z}_{Y \times S}) \in C(\mathrm{Var}(\mathbb{C})^{sm}/S).$$

(iv) *Let $f : X \to S$ and $g : T \to S$ two morphism with $X, S, T \in \mathrm{Var}(\mathbb{C})$. Assume that there exist a factorization $f : X \xrightarrow{i} Y \times S \xrightarrow{p} S$, with $Y \in \mathrm{SmVar}(\mathbb{C})$, $i : X \hookrightarrow Y$ is a closed embedding and p the projection. We then have the following commutative diagram whose squares are cartesian*

$$\begin{tikzcd} f : X \arrow[r, "i"] & Y \times S \arrow[r, "p"] & S \\ f' : X_T \arrow[u, "g'"] \arrow[r, "i'"] & Y \times T \arrow[u, "g'' := (I \times g)"] \arrow[r, "p'"] & T \arrow[u, "g"] \end{tikzcd}$$

We then have the canonical morphism in $C(\mathrm{Var}(\mathbb{C})^{sm}/T)$

$$T(f, g, Q^h) : g^* Q^h(X/S) := g^* p_* \Gamma_X E_{et}(\mathbb{Z}_{Y \times S}) \xrightarrow{T(g,p)(-)} p'_* g''^* \Gamma_X E_{et}(\mathbb{Z}_{Y \times S})$$
$$\xrightarrow{p'_* T(g'', \gamma)(-)} p'_* \Gamma_{X_T} E_{et}(\mathbb{Z}_{Y \times T}) =: Q^h(X_T/T).$$

We now give the definition of the $\mathbb{A}^1$ local property :
Denote by

$$p_a : \mathrm{Var}(\mathbb{C})^{(sm)}/S \to \mathrm{Var}(\mathbb{C})^{(sm)}/S, \; X/S = (X, h) \mapsto (X \times \mathbb{A}^1)/S = (X \times \mathbb{A}^1, h \circ p_X),$$
$$(g : X/S \to X'/S) \mapsto ((g \times I_{\mathbb{A}^1}) : X \times \mathbb{A}^1/S \to X' \times \mathbb{A}^1/S)$$

the projection functor and again by $p_a : \mathrm{Var}(\mathbb{C})^{(sm)}/S \to \mathrm{Var}(\mathbb{C})^{(sm)}/S$ the corresponding morphism of site.

Definition 11. *Let $S \in \mathrm{Var}(\mathbb{C})$. Denote for short $\mathrm{Var}(\mathbb{C})^{(sm)}/S$ either the category $\mathrm{Var}(\mathbb{C})/S$ or the category $\mathrm{Var}(\mathbb{C})^{sm}/S$.*

(i0) *A complex $F \in C(\mathrm{Var}(\mathbb{C})^{(sm)}/S)$ is said to be $\mathbb{A}^1$ homotopic if $\mathrm{ad}(p_a^*, p_{a*})(F) : F \to p_{a*} p_a^* F$ is an homotopy equivalence.*

(i) *A complex $F \in C(\mathrm{Var}(\mathbb{C})^{(sm)}/S)$ is said to be $\mathbb{A}^1$ invariant if for all $U/S \in \mathrm{Var}(\mathbb{C})^{(sm)}/S$,*

$$F(p_U) : F(U/S) \to F(U \times \mathbb{A}^1/S)$$

is a quasi-isomorphism, where $p_U : U \times \mathbb{A}^1 \to U$ is the projection. Obviously, if a complex $F \in C(\mathrm{Var}(\mathbb{C})^{(sm)}/S)$ is $\mathbb{A}^1$ homotopic then it is $\mathbb{A}^1$ invariant.

(ii) Let τ a topology on $\mathrm{Var}(\mathbb{C})$. A complex $F \in C(\mathrm{Var}(\mathbb{C})^{(sm)}/S)$ is said to be $\mathbb{A}^1$ local for the topology τ, if for a (hence every) τ local equivalence $k : F \to G$ with k injective and $G \in C(\mathrm{Var}(\mathbb{C})^{(sm)}/S)$ τ fibrant, e.g. $k : F \to E_\tau(F)$, G is $\mathbb{A}^1$ invariant for all $n \in \mathbb{Z}$.

(iii) A morphism $m : F \to G$ with $F, G \in C(\mathrm{Var}(\mathbb{C})^{(sm)}/S)$ is said to an $(\mathbb{A}^1, et)$ local equivalence if for all $H \in C(\mathrm{Var}(\mathbb{C})^{(sm)}/S)$ which is $\mathbb{A}^1$ local for the etale topology

$$\mathrm{Hom}(L(m), E_{et}(H)) : \mathrm{Hom}(L(G), E_{et}(H)) \to \mathrm{Hom}(L(F), E_{et}(H))$$

is a quasi-isomorphism.

Denote $\square^* := \mathbb{P}^* \backslash \{1\}$

- Let $S \in \mathrm{Var}(\mathbb{C})$. For $U/S = (U, h) \in \mathrm{Var}(\mathbb{C})^{sm}/S$, we consider

$$\square^* \times U/S = (\square^* \times U, h \circ p) \in \mathrm{Fun}(\Delta, \mathrm{Var}(\mathbb{C})^{sm}/S).$$

For $F \in C^-(\mathrm{Var}(\mathbb{C})^{sm}/S)$, it gives the complex

$$C_*F \in C^-(\mathrm{Var}(\mathbb{C})^{sm}/S), U/S = (U, h) \mapsto C_*F(U/S) := \mathrm{Tot}\, F(\square^* \times U/S)$$

together with the canonical map $c_F := (0, I_F) : C_*F \to F$. For $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$, we get

$$C_*F := \mathrm{holim}_n\, C_*F^{\leq n} \in C(\mathrm{Var}(\mathbb{C})^{sm}/S),$$

together with the canonical map $c_F := (0, I_F) : C_*F \to F$. For $m : F \to G$ a morphism, with $F, G \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$, we get by functoriality the morphism $C_*m : C_*F \to C_*G$.

- Let $S \in \mathrm{Var}(\mathbb{C})$. For $U/S = (U, h) \in \mathrm{Var}(\mathbb{C})/S$, we consider

$$\square^* \times U/S = (\mathbb{A}^* \times U, h \circ p) \in \mathrm{Fun}(\Delta, \mathrm{Var}(\mathbb{C})/S).$$

For $F \in C^-(\mathrm{Var}(\mathbb{C})/S)$, it gives the complex

$$C_*F \in C^-(\mathrm{Var}(\mathbb{C})/S), U/S = (U, h) \mapsto C_*F(U/S) := \mathrm{Tot}\, F(\square^* \times U/S)$$

together with the canonical map $c = c(F) := (0, I_F) : F \to C_*F$. For $F \in C(\mathrm{Var}(\mathbb{C})/S)$, we get

$$C_*F := \mathrm{holim}_n\, C_*F^{\leq n} \in C(\mathrm{Var}(\mathbb{C})/S),$$

together with the canonical map $c_F := (0, I_F) : C_*F \to F$. For $m : F \to G$ a morphism, with $F, G \in C(\mathrm{Var}(\mathbb{C})/S)$, we get by functoriality the morphism $C_*m : C_*F \to C_*G$.

Proposition 17. *(i) Let $S \in \mathrm{Var}(\mathbb{C})$. Then for $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$, C_*F is $\mathbb{A}^1$ local for the etale topology and $c(F) : F \to C_*F$ is an equivalence $(\mathbb{A}^1, et)$ local.*

(ii) A morphism $m : F \to G$ with $F, G \in C(\mathrm{Var}(\mathbb{C})^{(sm)}/S)$ is an $(\mathbb{A}^1, et)$ local equivalence if and only if there exists

$$\{X_{1,\alpha}/S, \alpha \in \Lambda_1\}, \ldots, \{X_{r,\alpha}/S, \alpha \in \Lambda_r\} \subset \mathrm{Var}(\mathbb{C})^{(sm)}/S$$

such that we have in $\mathrm{Ho}_{et}(C(\mathrm{Var}(\mathbb{C})^{(sm)}/S))$

$$\begin{aligned}\mathrm{Cone}(m) \xrightarrow{\sim} \mathrm{Cone}(&\oplus_{\alpha \in \Lambda_1} \mathrm{Cone}(\mathbb{Z}(X_{1,\alpha} \times \mathbb{A}^1/S) \to \mathbb{Z}(X_{1,\alpha}/S)) \\ &\to \cdots \to \oplus_{\alpha \in \Lambda_r} \mathrm{Cone}(\mathbb{Z}(X_{r,\alpha} \times \mathbb{A}^1/S) \to \mathbb{Z}(X_{r,\alpha}/S)))\end{aligned}$$

Proof. Standard : see Ayoub's thesis for example. $\square$

Definition-Proposition 5. *Let $S \in \mathrm{Var}(\mathbb{C})$.*

(i) *With the weak equivalence the* $(\mathbb{A}^1, et)$ *local equivalence and the fibration the epimorphism with* $\mathbb{A}^1_S$ *local and etale fibrant kernels gives a model structure on* $C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ *: the left bousfield localization of the projective model structure of* $C(\mathrm{Var}(\mathbb{C})^{sm}/S)$*. We call it the projective* $(\mathbb{A}^1, et)$ *model structure.*

(ii) *With the weak equivalence the* $(\mathbb{A}^1, et)$ *local equivalence and the fibration the epimorphism with* $\mathbb{A}^1_S$ *local and etale fibrant kernels gives a model structure on* $C(\mathrm{Var}(\mathbb{C})/S)$ *: the left bousfield localization of the projective model structure of* $C(\mathrm{Var}(\mathbb{C})/S)$*. We call it the projective* $(\mathbb{A}^1, et)$ *model structure.*

Proof. See [10]. □

Proposition 18. *Let* $g : T \to S$ *a morphism with* $T, S \in \mathrm{Var}(\mathbb{C})$*.*

(i) *The adjonction* $(g^*, g_*) : C(\mathrm{Var}(\mathbb{C})^{sm}/S) \leftrightarrows C(\mathrm{Var}(\mathbb{C})^{sm}/T)$ *is a Quillen adjonction for the* $(\mathbb{A}^1, et)$ *projective model structure (see definition-proposition 5).*

(i)' *Let* $h : U \to S$ *a smooth morphism with* $U, S \in \mathrm{Var}(\mathbb{C})$*. The adjonction* $(h_\sharp, h^*) : C(\mathrm{Var}(\mathbb{C})^{sm}/U) \leftrightarrows C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ *is a Quillen adjonction for the* $(\mathbb{A}^1, et)$ *projective model structure.*

(i)" *The functor* $g^* : C(\mathrm{Var}(\mathbb{C})^{sm}/S) \to C(\mathrm{Var}(\mathbb{C})^{sm}/T)$ *sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends* $(\mathbb{A}^1, et)$ *local equivalence to* $(\mathbb{A}^1, et)$ *local equivalence.*

(ii) *The adjonction* $(g^*, g_*) : C(\mathrm{Var}(\mathbb{C})/S) \leftrightarrows C(\mathrm{Var}(\mathbb{C})/T)$ *is a Quillen adjonction for the* $(\mathbb{A}^1, et)$ *projective model structure (see definition-proposition 5).*

(ii)' *The adjonction* $(g_\sharp, g^*) : C(\mathrm{Var}(\mathbb{C})/T) \leftrightarrows C(\mathrm{Var}(\mathbb{C})/S)$ *is a Quillen adjonction for the* $(\mathbb{A}^1, et)$ *projective model structure.*

(ii)" *The functor* $g^* : C(\mathrm{Var}(\mathbb{C})/S) \to C(\mathrm{Var}(\mathbb{C})/T)$ *sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends* $(\mathbb{A}^1, et)$ *local equivalence to* $(\mathbb{A}^1, et)$ *local equivalence.*

Proof. Standard : see [10] for example. □

Proposition 19. *Let* $S \in \mathrm{Var}(\mathbb{C})$*.*

(i) *The adjonction* $(\rho_S^*, \rho_{S*}) : C(\mathrm{Var}(\mathbb{C})^{sm}/S) \leftrightarrows C(\mathrm{Var}(\mathbb{C})/S)$ *is a Quillen adjonction for the* $(\mathbb{A}^1, et)$ *projective model structure.*

(ii) *The functor* $\rho_{S*} : C(\mathrm{Var}(\mathbb{C})/S) \to C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ *sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends* $(\mathbb{A}^1, et)$ *local equivalence to* $(\mathbb{A}^1, et)$ *local equivalence.*

Proof. Standard : see [10] for example. □

- For $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$, we denote as usual (see [10] for example), $\mathbb{Z}^{tr}(X/S) \in \mathrm{PSh}(\mathrm{Var}(\mathbb{C})/S)$ the presheaf given by
  - for $X'/S \in \mathrm{Var}(\mathbb{C})/S$, with X' irreducible, $\mathbb{Z}^{tr}(X/S)(X'/S) := \mathcal{Z}^{fs/X}(X' \times_S X) \subset \mathcal{Z}_{d_{X'}}(X' \times_S X)$ which consist of algebraic cycles $\alpha = \sum_i n_i \alpha_i \in \mathcal{Z}_{d_{X'}}(X' \times_S X)$ such that, denoting $\mathrm{supp}(\alpha) = \cup_i \alpha_i \subset X' \times_S X$ its support and $f' : X' \times_S X \to X'$ the projection, $f'_{|\mathrm{supp}(\alpha)} : \mathrm{supp}(\alpha) \to X'$ is finite surjective,
  - for $g : X_2/S \to X_1/S$ a morphism, with $X_1/S, X_2/S \in \mathrm{Var}(\mathbb{C})/S$,

    $$\mathbb{Z}^{tr}(X/S)(g) : \mathbb{Z}^{tr}(X/S)(X_1/S) \to \mathbb{Z}^{tr}(X/S)(X_2/S),\ \alpha \mapsto (g \times I)^{-1}(\alpha)$$

    with $g \times I : X_2 \times_S X \to X_1 \times_S X$, noting that, by base change, $f_{2|\mathrm{supp}((g\times I)^{-1}(\alpha))} : \mathrm{supp}((g \times I)^{-1}(\alpha)) \to X_2$ is finite surjective, $f_2 : X_2 \times_S X \to X_2$ being the projection.

- For $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$ and $r \in \mathbb{N}$, we denote as usual (see [10] for example), $\mathbb{Z}^{equir}(X/S) \in \mathrm{PSh}(\mathrm{Var}(\mathbb{C})/S)$ the presheaf given by
  - for $X'/S \in \mathrm{Var}(\mathbb{C})/S$, with X' irreducible, $\mathbb{Z}^{equir}(X/S)(X'/S) := \mathcal{Z}^{equir/X}(X' \times_S X) \subset \mathcal{Z}_{d_{X'}}(X' \times_S X)$ which consist of algebraic cycles $\alpha = \sum_i n_i \alpha_i \in \mathcal{Z}_{d_{X'}}(X' \times_S X)$ such that, denoting $\mathrm{supp}(\alpha) = \cup_i \alpha_i \subset X' \times_S X$ its support and $f' : X' \times_S X \to X'$ the projection, $f'_{|\mathrm{supp}(\alpha)} : \mathrm{supp}(\alpha) \to X'$ is dominant, with fibers either empty or of dimension r,
  - for $g : X_2/S \to X_1/S$ a morphism, with $X_1/S, X_2/S \in \mathrm{Var}(\mathbb{C})/S$,

    $$\mathbb{Z}^{equir}(X/S)(g) : \mathbb{Z}^{equir}(X/S)(X_1/S) \to \mathbb{Z}^{equir}(X/S)(X_2/S),\ \alpha \mapsto (g \times I)^{-1}(\alpha)$$

    with $g \times I : X_2 \times_S X \to X_1 \times_S X$, noting that, by base change, $f_{2|\mathrm{supp}((g\times I)^{-1}(\alpha))} : \mathrm{supp}((g \times I)^{-1}(\alpha)) \to X_2$ is obviously dominant, with fibers either empty or of dimension r, $f_2 : X_2 \times_S X \to X_2$ being the projection.
- Let $S \in \mathrm{Var}(\mathbb{C})$. We denote by $\mathbb{Z}_S(d) := \mathbb{Z}^{equi0}(S \times \mathbb{A}^d/S)[-2d]$ the Tate twist. For $F \in C(\mathrm{Var}(\mathbb{C})/S)$, we denote by $F(d) := F \otimes \mathbb{Z}_S(d)$.

For $S \in \mathrm{Var}(\mathbb{C})$, let $\mathrm{Cor}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ be the category

- whose objects are smooth morphisms $U/S = (U, h)$, $h : U \to S$ with $U \in \mathrm{Var}(\mathbb{C})$,
- whose morphisms $\alpha : U/S = (U, h_1) \to V/S = (V, h_2)$ is finite correspondence that is $\alpha \in \oplus_i \mathcal{Z}^{fs}(U_i \times_S V)$, where $U = \sqcup_i U_i$, with U_i connected (hence irreducible by smoothness), and $\mathcal{Z}^{fs}(U_i \times_S V)$ is the abelian group of cycle finite and surjective over U_i.

We denote by $\mathrm{Tr}(S) : \mathrm{Cor}(\mathrm{Var}(\mathbb{C})^{sm}/S) \to \mathrm{Var}(\mathbb{C})^{sm}/S$ the morphism of site given by the inclusion functor $\mathrm{Tr}(S) : \mathrm{Var}(\mathbb{C})^{sm}/S \hookrightarrow \mathrm{Cor}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ It induces an adjonction

$$(\mathrm{Tr}(S)^* \,\mathrm{Tr}(S)_*) : C(\mathrm{Var}(\mathbb{C})^{sm}/S) \leftrightarrows C(\mathrm{Cor}(\mathrm{Var}(\mathbb{C})^{sm}/S))$$

A complex of preheaves $G \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is said to admit transferts if it is in the image of the embedding

$$\mathrm{Tr}(S)_* : C(\mathrm{Cor}(\mathrm{Var}(\mathbb{C})^{sm}/S) \hookrightarrow C(\mathrm{Var}(\mathbb{C})^{sm}/S),$$

that is $G = \mathrm{Tr}(S)_* \,\mathrm{Tr}(S)^* G$.

We will use to compute the algebraic Gauss-Manin realization functor the following

Theorem 10. *Let $\phi : F^\bullet \to G^\bullet$ an etale local equivalence with $F^\bullet, G^\bullet \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$. If $F^\bullet$ and $G^\bullet$ are $\mathbb{A}^1$ local and admit tranferts then $\phi : F^\bullet \to G^\bullet$ is a Zariski local equivalence. Hence if $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is $\mathbb{A}^1$ local and admits transfert*

$$k : E_{zar}(F) \to E_{et}(E_{zar}(F)) = E_{et}(F)$$

is a Zariski local equivalence.

Proof. See [10]. □

2.7 Presheaves on the big Zariski site or the big etale site of pairs

We recall the definition given in subsection 5.1 : For $S \in \mathrm{Var}(\mathbb{C})$, $\mathrm{Var}(\mathbb{C})^2/S := \mathrm{Var}(\mathbb{C})^2/(S,S)$ is by definition (see subsection 2.1) the category whose set of objects is

$$(\mathrm{Var}(\mathbb{C})^2/S)^0 := \{((X, Z), h), h : X \to S,\ Z \subset X \text{ closed } \} \subset \mathrm{Var}(\mathbb{C})/S \times \mathrm{Top}$$

and whose set of morphisms between $(X_1,Z_1)/S=((X_1,Z_1),h_1),(X_1,Z_1)/S=((X_2,Z_2),h_2)\in\mathrm{Var}(\mathbb{C})^2/S$ is the subset

$$\mathrm{Hom}_{\mathrm{Var}(\mathbb{C})^2/S}((X_1,Z_1)/S,(X_2,Z_2)/S):=$$
$$\left\{(f:X_2\to X_2),\ \text{s.t. } h_1\circ f=h_2 \text{ and } Z_1\subset f^{-1}(Z_2)\right\}\subset\mathrm{Hom}_{\mathrm{Var}(\mathbb{C})}(X_1,X_2)$$

The category $\mathrm{Var}(\mathbb{C})^2$ admits fiber products : $(X_1,Z_1)\times_{(S,Z)}(X_2,Z_2)=(X_1\times_S X_2,Z_1\times_Z Z_2)$. In particular, for $f:T\to S$ a morphism with $S,T\in\mathrm{Var}(\mathbb{C})$, we have the pullback functor

$$P(f):\mathrm{Var}(\mathbb{C})^2/S\to\mathrm{Var}(\mathbb{C})^2/T, P(f)((X,Z)/S):=(X_T,Z_T)/T, P(f)(g):=(g\times_S f)$$

and we note again $P(f):\mathrm{Var}(\mathbb{C})^2/T\to\mathrm{Var}(\mathbb{C})^2/S$ the corresponding morphism of sites.

We will consider in the construction of the filtered De Rham realization functor the full subcategory $\mathrm{Var}(\mathbb{C})^{2,sm}/S\subset\mathrm{Var}(\mathbb{C})^2/S$ such that the first factor is a smooth morphism : We will also consider, in order to obtain a complex of D modules in the construction of the filtered De Rham realization functor, the restriction to the full subcategory $\mathrm{Var}(\mathbb{C})^{2,pr}/S\subset\mathrm{Var}(\mathbb{C})^2/S$ such that the first factor is a projection :

Definition 12. *(i) Let $S\in\mathrm{Var}(\mathbb{C})$. We denote by*

$$\rho_S:\mathrm{Var}(\mathbb{C})^{2,sm}/S\hookrightarrow\mathrm{Var}(\mathbb{C})^2/S$$

the full subcategory consisting of the objects $(U,Z)/S=((U,Z),h)\in\mathrm{Var}(\mathbb{C})^2/S$ such that the morphism $h:U\to S$ is smooth. That is, $\mathrm{Var}(\mathbb{C})^{2,sm}/S$ is the category

- *whose objects are $(U,Z)/S=((U,Z),h)$, with $U\in\mathrm{Var}(\mathbb{C})$, $Z\subset U$ a closed subset, and $h:U\to S$ a smooth morphism,*
- *whose morphisms $g:(U,Z)/S=((U,Z),h_1)\to(U',Z')/S=((U',Z'),h_2)$ is a morphism $g:U\to U'$ of complex algebraic varieties such that $Z\subset g^{-1}(Z')$ and $h_2\circ g=h_1$.*

We denote again $\rho_S:\mathrm{Var}(\mathbb{C})^2/S\to\mathrm{Var}(\mathbb{C})^{2,sm}/S$ the associated morphism of site.We have

$$r^s(S):\mathrm{Var}(\mathbb{C})^2\xrightarrow{r(S):=r(S,S)}\mathrm{Var}(\mathbb{C})^2/S\xrightarrow{\rho_S}\mathrm{Var}(\mathbb{C})^{2,sm}/S$$

the composite morphism of site.

(ii) Let $S\in\mathrm{Var}(\mathbb{C})$. We will consider the full subcategory

$$\mu_S:\mathrm{Var}(\mathbb{C})^{2,pr}/S\hookrightarrow\mathrm{Var}(\mathbb{C})^2/S$$

whose subset of object consist of those whose morphism is a projection to S :

$$(\mathrm{Var}(\mathbb{C})^{2,pr}/S)^0:=\{((Y\times S,X),p),\ Y\in\mathrm{Var}(\mathbb{C}),\ p:Y\times S\to S\ \textit{the projection}\}\subset(\mathrm{Var}(\mathbb{C})^2/S)^0.$$

(iii) We will consider the full subcategory

$$\mu_S:(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)\hookrightarrow\mathrm{Var}(\mathbb{C})^{2,sm}/S$$

whose subset of object consist of those whose morphism is a smooth projection to S :

$$(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)^0:=\{((Y\times S,X),p),\ Y\in\mathrm{SmVar}(\mathbb{C}),\ p:Y\times S\to S\ \textit{the projection}\}\subset(\mathrm{Var}(\mathbb{C})^2/S)^0$$

For $f : T \to S$ a morphism with $T, S \in \mathrm{Var}(\mathbb{C})$, we have by definition, the following commutative diagram of sites

$$\begin{tikzcd}
\mathrm{Var}(\mathbb{C})^2/T \arrow[rr, "\mu_T"] \arrow[dr, "\rho_T"] \arrow[dd, "P(f)"'] & & \mathrm{Var}(\mathbb{C})^{2,pr}/T \arrow[dr, "\rho_T"] \arrow[dd, "P(f)"] & \\
& \mathrm{Var}(\mathbb{C})^{2,sm}/T \arrow[rr, "\mu_T"] \arrow[dd, "P(f)"] & & \mathrm{Var}(\mathbb{C})^{2,smpr}/T \arrow[dd, "P(f)"] \\
\mathrm{Var}(\mathbb{C})^2/S \arrow[rr, "\mu_S"] \arrow[dr, "\rho_S"] & & \mathrm{Var}(\mathbb{C})^{2,pr}/S \arrow[dr, "\rho_S"] & \\
& \mathrm{Var}(\mathbb{C})^{2,sm}/S \arrow[rr, "\mu_S"] & & \mathrm{Var}(\mathbb{C})^{2,smpr}/S
\end{tikzcd} . \tag{29}$$

Recall we have (see subsection 2.1), for $S \in \mathrm{Var}(\mathbb{C})$, the graph functor

$$\mathrm{Gr}_S^{12} : \mathrm{Var}(\mathbb{C})/S \to \mathrm{Var}(\mathbb{C})^{2,pr}/S, \; X/S \mapsto \mathrm{Gr}_S^{12}(X/S) := (X \times S, X)/S,$$
$$(g : X/S \to X'/S) \mapsto \mathrm{Gr}_S^{12}(g) := (g \times I_S : (X \times S, X) \to (X' \times S, X'))$$

Note that Gr_S^{12} is fully faithfull. For $f : T \to S$ a morphism with $T, S \in \mathrm{Var}(\mathbb{C})$, we have by definition, the following commutative diagram of sites

$$\begin{tikzcd}
\mathrm{Var}(\mathbb{C})^{2,pr}/T \arrow[rr, "\mathrm{Gr}_T^{12}"] \arrow[dr, "\rho_T"] \arrow[dd, "P(f)"'] & & \mathrm{Var}(\mathbb{C})/T \arrow[dr, "\rho_T"] \arrow[dd, "P(f)"] & \\
& \mathrm{Var}(\mathbb{C})^{2,smpr}/T \arrow[rr, "\mathrm{Gr}_T^{12}"] \arrow[dd, "P(f)"] & & \mathrm{Var}(\mathbb{C})^{sm}/T \arrow[dd, "P(f)"] \\
\mathrm{Var}(\mathbb{C})^{2,pr}/S \arrow[rr, "\mathrm{Gr}_S^{12}"] \arrow[dr, "\rho_S"] & & \mathrm{Var}(\mathbb{C})/S \arrow[dr, "\rho_S"] & \\
& \mathrm{Var}(\mathbb{C})^{2,sm}/S \arrow[rr, "\mathrm{Gr}_S^{12}"] & & \mathrm{Var}(\mathbb{C})^{sm}/S
\end{tikzcd} . \tag{30}$$

where we recall that $P(f)((X, Z)/S) := ((X_T, Z_T)/T)$, since smooth morphisms are preserved by base change.

- As usual, we denote by

  $$(f^*, f_*) := (P(f)^*, P(f)_*) : C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S) \to C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/T)$$

  the adjonction induced by $P(f) : \mathrm{Var}(\mathbb{C})^{2,(sm)}/T \to \mathrm{Var}(\mathbb{C})^{2,(sm)}/S$. Since the colimits involved in the definition of $f^* = P(f)^*$ are filtered, f^* also preserve monomorphism. Hence, we get an adjonction

  $$(f^*, f_*) : C_{fil}(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S) \leftrightarrows C_{fil}(\mathrm{Var}(\mathbb{C})^{2,(sm)}/T), \; f^*(G, F) := (f^*G, f^*F)$$

  For $S \in \mathrm{Var}(\mathbb{C})$, we denote by $\mathbb{Z}_S := \mathbb{Z}((S, S)/(S, S)) \in \mathrm{PSh}(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S)$ the constant presheaf. By Yoneda lemma, we have for $F \in C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S)$, $\mathcal{H}om(\mathbb{Z}_S, F) = F$.

- As usual, we denote by

  $$(f^*, f_*) := (P(f)^*, P(f)_*) : C(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S) \to C(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/T)$$

the adjonction induced by $P(f):\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/T\to\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S$. Since the colimits involved in the definition of $f^*=P(f)^*$ are filtered, f^* also preserve monomorphism. Hence, we get an adjonction

$$(f^*,f_*):C_{fil}(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S)\leftrightarrows C_{fil}(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/T),\ f^*(G,F):=(f^*G,f^*F)$$

For $S\in\mathrm{Var}(\mathbb{C})$, we denote by $\mathbb{Z}_S:=\mathbb{Z}((S,S)/(S,S))\in\mathrm{PSh}(\mathrm{Var}(\mathbb{C})^{2,sm}/S)$ the constant presheaf. By Yoneda lemma, we have for $F\in C(\mathrm{Var}(\mathbb{C})^{2,sm}/S)$, $\mathcal{H}om(\mathbb{Z}_S,F)=F$.

- For $h:U\to S$ a smooth morphism with $U,S\in\mathrm{Var}(\mathbb{C})$, $P(h):\mathrm{Var}(\mathbb{C})^{2,sm}/S\to\mathrm{Var}(\mathbb{C})^{2,sm}/U$ admits a left adjoint

$$C(h):\mathrm{Var}(\mathbb{C})^{2,sm}/U\to\mathrm{Var}(\mathbb{C})^{2,sm}/S,\ C(h)((U',Z'),h')=((U',Z'),h\circ h').$$

Hence $h^*:C(\mathrm{Var}(\mathbb{C})^{2,sm}/S)\to C(\mathrm{Var}(\mathbb{C})^{2,sm}/U)$ admits a left adjoint

$$h_\sharp:C(\mathrm{Var}(\mathbb{C})^{2,sm}/U)\to C(\mathrm{Var}(\mathbb{C})^{2,sm}/S),$$
$$F\mapsto(h_\sharp F:((U,Z),h_0)\mapsto\lim_{((U',Z'),h\circ h')\to((U,Z),h_0)}F((U',Z')/U))$$

- For $h:X\to S$ a morphism with $X,S\in\mathrm{Var}(\mathbb{C})$, $P(h):\mathrm{Var}(\mathbb{C})^2/S\to\mathrm{Var}(\mathbb{C})^2/X$ admits a left adjoint

$$C(h):\mathrm{Var}(\mathbb{C})^2/X\to\mathrm{Var}(\mathbb{C})^2/S,\ C(h)((X',Z'),h')=((X',Z'),h\circ h').$$

Hence $h^*:C(\mathrm{Var}(\mathbb{C})^2/S)\to C(\mathrm{Var}(\mathbb{C})^2/X)$ admits a left adjoint

$$h_\sharp:C(\mathrm{Var}(\mathbb{C})^2/X)\to C(\mathrm{Var}(\mathbb{C})^{2,sm}/S),$$
$$F\mapsto(h_\sharp F:((X,Z),h_0)\mapsto\lim_{((X',Z'),h\circ h')\to((X,Z),h_0)}F((X',Z')/X))$$

- For $p:Y\times S\to S$ a projection with $Y,S\in\mathrm{Var}(\mathbb{C})$ with Y smooth, $P(p):\mathrm{Var}(\mathbb{C})^{2,smpr}/S\to\mathrm{Var}(\mathbb{C})^{2,smpr}/Y\times S$ admits a left adjoint

$$C(p):\mathrm{Var}(\mathbb{C})^{2,smpr}/Y\times S\to\mathrm{Var}(\mathbb{C})^{2,smpr}/S,$$
$$C(p)((Y'\times S,Z'),p')=((Y'\times S,Z'),p\circ p').$$

Hence $p^*:C(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)\to C(\mathrm{Var}(\mathbb{C})^{2,smpr}/Y\times S)$ admits a left adjoint

$$p_\sharp:C(\mathrm{Var}(\mathbb{C})^{2,smpr}/Y\times S)\to C(\mathrm{Var}(\mathbb{C})^{2,smpr}/S),$$
$$F\mapsto(p_\sharp F:((Y_0\times S,Z),p_0)\mapsto\lim_{((Y'\times Y\times S,Z'),p\circ p')\to((Y_0\times S,Z),p_0)}F((Y'\times Y\times S,Z')/Y\times S))$$

- For $p:Y\times S\to S$ a projection with $Y,S\in\mathrm{Var}(\mathbb{C})$, $P(p):\mathrm{Var}(\mathbb{C})^{2,pr}/S\to\mathrm{Var}(\mathbb{C})^{2,pr}/Y\times S$ admits a left adjoint

$$C(p):\mathrm{Var}(\mathbb{C})^{2,pr}/Y\times S\to\mathrm{Var}(\mathbb{C})^{2,pr}/S,\ C(p)((Y'\times S,Z'),p')=((Y'\times S,Z'),p\circ p').$$

Hence $p^*:C(\mathrm{Var}(\mathbb{C})^{2,pr}/S)\to C(\mathrm{Var}(\mathbb{C})^{2,pr}/Y\times S)$ admits a left adjoint

$$p_\sharp:C(\mathrm{Var}(\mathbb{C})^{2,pr}/Y\times S)\to C(\mathrm{Var}(\mathbb{C})^{2,pr}/S),$$
$$F\mapsto(p_\sharp F:((Y_0\times S,Z),p_0)\mapsto\lim_{((Y'\times Y\times S,Z'),p\circ p')\to((Y_0\times S,Z),p_0)}F((Y'\times Y\times S,Z')/Y\times S))$$

Let $S\in\mathrm{Var}(\mathbb{C})$.

- We have the dual functor

$$\mathbb{D}_S : C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S) \to C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S),\ F \mapsto \mathbb{D}_S(F) := \mathcal{H}om(F, E_{et}(\mathbb{Z}((S,S)/S)))$$

  It induces the functor

$$L\mathbb{D}_S : C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S) \to C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S),\ F \mapsto L\mathbb{D}_S(F) := \mathbb{D}_S(LF) := \mathcal{H}om(LF, E_{et}(\mathbb{Z}((S,S)/S)))$$

- We have the dual functor

$$\mathbb{D}_S : C(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S) \to C(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S),\ F \mapsto \mathbb{D}_S(F) := \mathcal{H}om(F, E_{et}(\mathbb{Z}((S,S)/S)))$$

  It induces the functor

$$L\mathbb{D}_S : C(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S) \to C(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S),\ F \mapsto L\mathbb{D}_S(F) := \mathbb{D}_S(LF) := \mathcal{H}om(LF, E_{et}(\mathbb{Z}((S,S)/S)))$$

Proposition 20. *(i) Let $S \in \mathrm{Var}(\mathbb{C})$. Let $h : U \to S$ a smooth morphism with $U \in \mathrm{Var}(\mathbb{C})$. Then for $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$, the canonical map in $C(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$*

$$T(\mathrm{Gr}_S^{12}, hom)(\mathbb{Z}(U/S), F) : \mathrm{Gr}_S^{12*} \mathcal{H}om(\mathbb{Z}(U/S), F) \xrightarrow{\sim} \mathcal{H}om(\mathrm{Gr}_S^{12*} \mathbb{Z}(U/S), \mathrm{Gr}_S^{12*} F)$$

is an isomorphism.

(ii) Let $S \in \mathrm{Var}(\mathbb{C})$. Let $h : U \to S$ a morphism with $U \in \mathrm{Var}(\mathbb{C})$. Then for $F \in C(\mathrm{Var}(\mathbb{C})/S)$, the canonical map in $C(\mathrm{Var}(\mathbb{C})^{2,pr}/S)$

$$T(\mathrm{Gr}_S^{12}, hom)(\mathbb{Z}(U/S), F) : \mathrm{Gr}_S^{12*} \mathcal{H}om(\mathbb{Z}(U/S), F) \xrightarrow{\sim} \mathcal{H}om(\mathrm{Gr}_S^{12*} \mathbb{Z}(U/S), \mathrm{Gr}_S^{12*} F)$$

is an isomorphism.

Proof. (i): We have, for $(X \times S, Z)/S \in \mathrm{Var}(\mathbb{C})^{2,smpr}/S$ the following commutative diagram

$$\begin{tikzcd}
\mathrm{Gr}_S^{12*} \mathcal{H}om(\mathbb{Z}(U/S), F)((X \times S, Z)/S) \arrow[r, "T(\mathrm{Gr}_S^{12}, hom)(\mathbb{Z}(U/S),F)((X\times S,Z)/S)"] \arrow[d, "="] & \mathcal{H}om(\mathrm{Gr}_S^{12*} \mathbb{Z}(U/S), \mathrm{Gr}_S^{12*} F)((X \times S, Z)/S) \arrow[d, "="] \\
\lim_{((X\times S,Z)/S)\to \mathrm{Gr}_S^{12}(V/S)} F(U \times_S V) \arrow[r] & \lim_{((X\times U,Z)/Z\times_S U)\to \mathrm{Gr}_S^{12}(W/S)} F(W)
\end{tikzcd}$$

We then note that the map $\{(((X \times S, Z)/S) \to \mathrm{Gr}_S^{12}(V/S))\} \to \{((X \times U, Z)/Z \times_S U) \to \mathrm{Gr}_S^{12}(W/S)\}$ obviously admits an inverse since a map $(X \times U, Z \times_S U)/S \to (W \times S, W)/S)$ is uniquely determined by a map $g : X \to W$ such that $(g \times I_S)(Z) \subset W$. (ii):Similar to (i). $\square$

We have the support section functors of a closed embedding $i : Z \hookrightarrow S$ for presheaves on the big Zariski site of pairs.

Definition 13. *Let $i : Z \hookrightarrow S$ be a closed embedding with $S, Z \in \mathrm{Var}(\mathbb{C})$ and $j : S\backslash Z \hookrightarrow S$ be the open complementary subset.*

(i) We define the functor

$$\Gamma_Z : C(\mathrm{Var}(\mathbb{C})^{2,sm}/S) \to C(\mathrm{Var}(\mathbb{C})^{2,sm}/S),\ G^\bullet \mapsto \Gamma_Z G^\bullet := \mathrm{Cone}(\mathrm{ad}(j^*, j_*)(G^\bullet) : G^\bullet \to j_* j^* G^\bullet)[-1],$$

so that there is then a canonical map $\gamma_Z(G^\bullet) : \Gamma_Z G^\bullet \to G^\bullet$.

(ii) We have the dual functor of (i) :

$$\Gamma_Z^\vee : C(\mathrm{Var}(\mathbb{C})^{2,sm}/S) \to C(\mathrm{Var}(\mathbb{C})^{2,sm}/S),\ F \mapsto \Gamma_Z^\vee(F^\bullet) := \mathrm{Cone}(\mathrm{ad}(j_\sharp, j^*)(G^\bullet) : j_\sharp j^* G^\bullet \to G^\bullet),$$

together with the canonical map $\gamma_Z^\vee(G) : F \to \Gamma_Z^\vee(G)$.

(iii) For $F,G\in C(\mathrm{Var}(\mathbb{C})^{2,sm}/S)$, we denote by

$$I(\gamma,hom)(F,G):=(I,I(j_\sharp,j^*)(F,G)^{-1}):\Gamma_Z\mathcal{H}om(F,G)\xrightarrow{\sim}\mathcal{H}om(\Gamma^\vee_ZF,G)$$

the canonical isomorphism given by adjonction.

Note that we have similarly for $i:Z\hookrightarrow S$, $i':Z'\hookrightarrow Z$ closed embeddings, $g:T\to S$ a morphism with $T,S,Z\in\mathrm{Var}(\mathbb{C})$ and $F\in C(\mathrm{Var}(\mathbb{C})^{2,sm}/S)$, the canonical maps in $C(\mathrm{Var}(\mathbb{C})^{2,sm}/S)$

- $T(g,\gamma)(F):g^*\Gamma_ZF\xrightarrow{\sim}\Gamma_{Z\times_ST}g^*F$, $T(g,\gamma^\vee)(F):\Gamma^\vee_{Z\times_ST}g^*F\xrightarrow{\sim}g^*\Gamma_ZF$
- $T(Z'/Z,\gamma)(F):\Gamma_{Z'}F\to\Gamma_ZF$, $T(Z'/Z,\gamma^\vee)(F):\Gamma^\vee_ZF\to\Gamma^\vee_{Z'}F$

but we will not use them in this article.

We now define the Zariski and the etale topology on $\mathrm{Var}(\mathbb{C})^2/S$.

Definition 14. *Let $S\in\mathrm{Var}(\mathbb{C})$.*

(i) Denote by τ a topology on $\mathrm{Var}(\mathbb{C})$, e.g. the Zariski or the etale topology. The τ covers in $\mathrm{Var}(\mathbb{C})^2/S$ of $(X,Z)/S$ are the families of morphisms

$$\{(c_i:(U_i,Z\times_XU_i)/S\to(X,Z)/S)_{i\in I},\ \textit{with}\ (c_i:U_i\to X)_{i\in I}\,\tau\ \textit{cover of}X\ \textit{in}\ \mathrm{Var}(\mathbb{C})\}$$

(ii) Denote by τ the Zariski or the etale topology on $\mathrm{Var}(\mathbb{C})$. The τ covers in $\mathrm{Var}(\mathbb{C})^{2,sm}/S$ of $(U,Z)/S$ are the families of morphisms

$$\{(c_i:(U_i,Z\times_UU_i)/S\to(U,Z)/S)_{i\in I},\ \textit{with}\ (c_i:U_i\to U)_{i\in I}\,\tau\ \textit{cover of}U\ \textit{in}\ \mathrm{Var}(\mathbb{C})\}$$

(iii) Denote by τ the Zariski or the etale topology on $\mathrm{Var}(\mathbb{C})$. The τ covers in $\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S$ of $(Y\times S,Z)/S$ are the families of morphisms

$$\{(c_i\times I_S:(U_i\times S,Z\times_{Y\times S}U_i\times S)/S\to(Y\times S,Z)/S)_{i\in I},\ \textit{with}\ (c_i:U_i\to Y)_{i\in I}\,\tau\ \textit{cover of}Y\ \textit{in}\ \mathrm{Var}(\mathbb{C})\}$$

Let $S\in\mathrm{Var}(\mathbb{C})$. Denote by τ the Zariski or the etale topology on $\mathrm{Var}(\mathbb{C})$. In particular, denoting $a_\tau:\mathrm{PSh}(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S)\to\mathrm{Shv}(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S)$ and $a_\tau:\mathrm{PSh}(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S)\to\mathrm{Shv}(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S)$ the sheaftification functors,

- a morphism $\phi:F\to G$, with $F,G\in C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S)$, is a τ local equivalence if $a_\tau H^n\phi:a_\tau H^nF\to a_\tau H^nG$ is an isomorphism, a morphism $\phi:F\to G$, with $F,G\in C(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S)$, is a τ local equivalence if $a_\tau H^n\phi:a_\tau H^nF\to a_\tau H^nG$ is an isomorphism ;
- $F^\bullet\in C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S)$ is τ fibrant if for all $(U,Z)/S\in\mathrm{Var}(\mathbb{C})^{2,(sm)}/S$ and all τ covers $(c_i:(U_i,Z\times_UU_i)/S\to(U,Z)/S)_{i\in I}$ of $(U,Z)/S$,

  $$F^\bullet(c_i):F^\bullet((U,Z)/S)\to\mathrm{Tot}(\oplus_{cardI=\bullet}F^\bullet((U_I,Z\times_UU_I)/S))$$

  is a quasi-isomorphism of complexes of abelian groups, $F^\bullet\in C(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S)$ is τ fibrant if for all $(Y\times S,Z)/S\in\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S$ and all τ covers $(c_i\times I_S:(U_i\times S,Z\times_{Y\times S}U_i\times S)/S\to(Y\times S,Z)/S)_{i\in I}$ of $(Y\times S,Z)/S$,

  $$F^\bullet(c_i\times I_S):F^\bullet((Y\times S,Z)/S)\to\mathrm{Tot}(\oplus_{cardI=\bullet}F^\bullet((U_I\times S,Z_I\times_YU_J)/S))$$

  is a quasi-isomorphism of complexes of abelian groups ;

- a morphism $\phi : (G_1, F) \to (G_2, F)$, with $(G_1, F), (G_2, F) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S)$, is an filtered τ local equivalence if for all $n, p \in \mathbb{Z}$,
$$a_\tau H^n \operatorname{Gr}_F^p(\phi) : a_\tau H^n \operatorname{Gr}_F^p(G_1, F) \to a_\tau H^n \operatorname{Gr}_F^p(G_2, F)$$
is an isomorphism of sheaves on $\mathrm{Var}(\mathbb{C})^{2,(sm)}/S$, a morphism $\phi : (G_1, F) \to (G_2, F)$, with $(G_1, F), (G_2, F) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S)$, is an filtered τ local equivalence if for all $n, p \in \mathbb{Z}$,
$$a_\tau H^n \operatorname{Gr}_F^p(\phi) : a_\tau H^n \operatorname{Gr}_F^p(G_1, F) \to a_\tau H^n \operatorname{Gr}_F^p(G_2, F)$$
is an isomorphism of sheaves on $\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S$;
- let $r \in \mathbb{N}$, a morphism $\phi : (G_1, F) \to (G_2, F)$, with $(G_1, F), (G_2, F) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S)$, is an r-filtered τ local equivalence if there exists an r-filtered homotopy
$$(h, \phi, \phi') : (G_1, F)[1] \to (G_2, F)$$
such that $\phi' : (G_1, F) \to (G_2, F)$ is a filtered τ local equivalence, a morphism $\phi : (G_1, F) \to (G_2, F)$, with $(G_1, F), (G_2, F) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S)$, is an r-filtered τ local equivalence if there exists an r-filtered homotopy
$$(h, \phi, \phi') : (G_1, F)[1] \to (G_2, F)$$
such that $\phi' : (G_1, F) \to (G_2, F)$ is a filtered τ local equivalence ;
- $(F^\bullet, F) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S)$ is filtered τ fibrant if for all $(U, Z)/S \in \mathrm{Var}(\mathbb{C})^{2,(sm)}/S$ and all τ covers $(c_i : (U_i, Z \times_U U_i)/S \to (U, Z)/S)_{i \in I}$ of $(U, Z)/S$,
$$H^n \operatorname{Gr}_F^p(F^\bullet, F)(c_i) : (F^\bullet, F)((U, Z)/S) \to$$
$$H^n \operatorname{Gr}_F^p(\mathrm{Tot}(\oplus_{cardI=\bullet}(F^\bullet, F)((U_I, Z \times_U U_I)/S)))$$
is an isomorphism of of abelian groups for all $n, p \in \mathbb{Z}$; $(F^\bullet, F) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S)$ is filtered τ fibrant if for all $(Y \times S, Z)/S \in \mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S$ and all τ covers $(c_i \times I_S : (U_i \times S, Z \times_{Y \times S} U_i \times S)/S \to (Y \times S, Z)/S)_{i \in I}$ of $(Y \times S, Z)/S$,
$$H^n \operatorname{Gr}_F^p(F^\bullet, F)(c_i \times I_S) : H^n \operatorname{Gr}_F^p(F^\bullet, F)((Y \times S, Z)/S) \xrightarrow{\sim}$$
$$H^n \operatorname{Gr}_F^p(\mathrm{Tot}(\oplus_{cardI=\bullet}(F^\bullet, F)((U_I \times S, Z \times_Y U_I)/S)))$$
is an isomorphism of abelian groups for all $n, p \in \mathbb{Z}$.

Will now define the $\mathbb{A}^1$ local property on $\mathrm{Var}(\mathbb{C})^2/S$.

Let $S \in \mathrm{Var}(\mathbb{C})$. Denote for short $\mathrm{Var}(\mathbb{C})^{2,(sm)}/S$ either the category $\mathrm{Var}(\mathbb{C})^2/S$ or the category $\mathrm{Var}(\mathbb{C})^{2,sm}/S$. Denote by
$$p_a : \mathrm{Var}(\mathbb{C})^{2,(sm)}/S \to \mathrm{Var}(\mathbb{C})^{2,(sm)}/S,$$
$$(X, Z)/S = ((X, Z), h) \mapsto (X \times \mathbb{A}^1, Z \times \mathbb{A}^1)/S = ((X \times \mathbb{A}^1, Z \times \mathbb{A}^1, h \circ p_X),$$
$$(g : (X, Z)/S \to (X', Z')/S) \mapsto ((g \times I_{\mathbb{A}^1}) : (X \times \mathbb{A}^1, Z \times \mathbb{A}^1)/S \to (X' \times \mathbb{A}^1, Z' \times \mathbb{A}^1)/S)$$
the projection functor and again by $p_a : \mathrm{Var}(\mathbb{C})^{2,(sm)}/S \to \mathrm{Var}(\mathbb{C})^{2,(sm)}/S$ the corresponding morphism of site. Let $S \in \mathrm{Var}(\mathbb{C})$.Denote for short $\mathrm{Var}(\mathbb{C})^{2,(sm)}/S$ either the category $\mathrm{Var}(\mathbb{C})^2/S$ or the category $\mathrm{Var}(\mathbb{C})^{2,sm}/S$. Denote for short $\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S$ either the category $\mathrm{Var}(\mathbb{C})^{2,pr}/S$ or the category $\mathrm{Var}(\mathbb{C})^{2,smpr}/S$. Denote by
$$p_a : \mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S \to \mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S,$$
$$(Y \times S, Z)/S = ((Y \times S, Z), p_S) \mapsto (Y \times S \times \mathbb{A}^1, Z \times \mathbb{A}^1)/S = ((Y \times S \times \mathbb{A}^1, Z \times \mathbb{A}^1, p_S \circ p_{Y \times S}),$$
$$(g : (Y \times S, Z)/S \to (Y' \times S, Z')/S) \mapsto ((g \times I_{\mathbb{A}^1}) : (Y \times S \times \mathbb{A}^1, Z \times \mathbb{A}^1)/S \to (Y' \times S \times \mathbb{A}^1, Z' \times \mathbb{A}^1)/S)$$
the projection functor and again by $p_a : \mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S \to \mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S$ the corresponding morphism of site.

Definition 15. *(i0) A complex $F\in C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S)$ is said to be $\mathbb{A}^1$ homotopic if $\mathrm{ad}(p_a^*,p_{a*})(F):F\to p_{a*}p_a^*F$ is an homotopy equivalence.*

- (i0)' *A complex $F\in C(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S)$ is said to be $\mathbb{A}^1$ homotopic if $\mathrm{ad}(p_a^*,p_{a*})(F):F\to p_{a*}p_a^*F$ is an homotopy equivalence.*

- (i) *A complex $F\in C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S)$, is said to be $\mathbb{A}^1$ invariant if for all $(X,Z)/S\in\mathrm{Var}(\mathbb{C})^{2,(sm)}/S$*

$$F(p_X):F((X,Z)/S)\to F((X\times\mathbb{A}^1,(Z\times\mathbb{A}^1))/S)$$

is a quasi-isomorphism, where $p_X:(X\times\mathbb{A}^1,(Z\times\mathbb{A}^1))\to(X,Z)$ is the projection. Obviously, if a complex $F\in C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S)$ is $\mathbb{A}^1$ homotopic, then it is $\mathbb{A}^1$ invariant.

- (i)' *A complex $G\in C(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S)$, is said to be $\mathbb{A}^1$ invariant if for all $(Y\times S,Z)/S\in\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S$*

$$G(p_{Y\times S}):G((Y\times S,Z)/S)\to G((Y\times\mathbb{A}^1\times S,(Z\times\mathbb{A}^1))/S)$$

is a quasi-isomorphism of abelian group. Obviously, if a complex $F\in C(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S)$ is $\mathbb{A}^1$ homotopic, then it is $\mathbb{A}^1$ invariant.

- (ii) *Let τ a topology on $\mathrm{Var}(\mathbb{C})$. A complex $F\in C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S)$ is said to be $\mathbb{A}^1$ local for the τ topology induced on $\mathrm{Var}(\mathbb{C})^2/S$, if for an (hence every) τ local equivalence $k:F\to G$ with k injective and $G\in C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S)$ τ fibrant, e.g. $k:F\to E_\tau(F)$, G is $\mathbb{A}^1$ invariant.*

- (ii)' *Let τ a topology on $\mathrm{Var}(\mathbb{C})$. A complex $F\in C(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S)$ is said to be $\mathbb{A}^1$ local for the τ topology induced on $\mathrm{Var}(\mathbb{C})^{2,pr}/S$, if for an (hence every) τ local equivalence $k:F\to G$ with k injective and $G\in C(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S)$ τ fibrant, e.g. $k:F\to E_\tau(F)$, G is $\mathbb{A}^1$ invariant.*

- (iii) *A morphism $m:F\to G$ with $F,G\in C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S)$ is said to an $(\mathbb{A}^1,et)$ local equivalence if for all $H\in C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S)$ which is $\mathbb{A}^1$ local for the etale topology*

$$\mathrm{Hom}(L(m),E_{et}(H)):\mathrm{Hom}(L(G),E_{et}(H))\to\mathrm{Hom}(L(F),E_{et}(H))$$

is a quasi-isomorphism.

- (iii)' *A morphism $m:F\to G$ with $F,G\in C(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S)$ is said to an $(\mathbb{A}^1,et)$ local equivalence if for all $H\in C(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S)$ which is $\mathbb{A}^1$ local for the etale topology*

$$\mathrm{Hom}(L(m),E_{et}(H)):\mathrm{Hom}(L(G),E_{et}(H))\to\mathrm{Hom}(L(F),E_{et}(H))$$

is a quasi-isomorphism.

Denote $\square^*:=\mathbb{P}^*_{\mathbb{C}}\backslash\{1\}$

- Let $S\in\mathrm{Var}(\mathbb{C})$. For $(X,Z)/S=((X,Z),h)\in\mathrm{Var}(\mathbb{C})^{2,(sm)}/S$, we consider

$$(\square^*\times X,\square^*\times Z)/S=((\square^*\times X,\square^*\times Z,h\circ p)\in\mathrm{Fun}(\Delta,\mathrm{Var}(\mathbb{C})^{2,(sm)}/S).$$

For $F\in C^-(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S)$, it gives the complex

$C_*F\in C^-(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S),(X,Z)/S=((X,Z),h)\mapsto C_*F((X,Z)/S):=\mathrm{Tot}\,F((\square^*\times X,\square^*\times Z/S)$

together with the canonical map $c_F:=(0,I_F):C_*F\to F$. For $F\in C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S)$, we get

$$C_*F:=\mathrm{holim}_n\,C_*F^{\leq n}\in C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S),$$

together with the canonical map $c_F:=(0,I_F):C_*F\to F$. For $m:F\to G$ a morphism, with $F,G\in C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S)$, we get by functoriality the morphism $C_*m:C_*F\to C_*G$.

- Let $S \in \mathrm{Var}(\mathbb{C})$. For $(Y \times S, Z)/S = ((Y \times S, Z), h) \in \mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S$, we consider
$$(\square^* \times Y \times S, \square^* \times Z)/S = (\square^* \times Y \times S, \square^* \times Z, h \circ p) \in \mathrm{Fun}(\Delta, \mathrm{Var}(\mathbb{C})/S).$$
For $F \in C^-(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S)$, it gives the complex
$$C_*F \in C^-(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S),$$
$$(Y \times S, Z)/S = ((Y \times S, Z), h) \mapsto C_*F((Y \times S, Z)/S) := \mathrm{Tot}\, F(\square^* \times Y \times S, \square^* \times Z)/S)$$
together with the canonical map $c = c(F) := (0, I_F) : F \to C_*F$. For $F \in C(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S)$, we get
$$C_*F := \mathrm{holim}_n\, C_*F^{\leq n} \in C(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S),$$
together with the canonical map $c = c(F) := (0, I_F) : F \to C_*F$. For $m : F \to G$ a morphism, with $F, G \in C(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S)$, we get by functoriality the morphism $C_*m : C_*F \to C_*G$.

Proposition 21. *(i) Let $S \in \mathrm{Var}(\mathbb{C})$. Then for $F \in C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S)$, C_*F is $\mathbb{A}^1$ local for the etale topology and $c(F) : F \to C_*F$ is an equivalence $(\mathbb{A}^1, et)$ local.*

(i)' Let $S \in \mathrm{Var}(\mathbb{C})$. Then for $F \in C(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S)$, C_*F is $\mathbb{A}^1$ local for the etale topology and $c(F) : F \to C_*F$ is an equivalence $(\mathbb{A}^1, et)$ local.

(ii) A morphism $m : F \to G$ with $F, G \in C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S)$ is an $(\mathbb{A}^1, et)$ local equivalence if and only if $a_{et}H^nC_*\,\mathrm{Cone}(m) = 0$ for all $n \in \mathbb{Z}$.

(ii)' A morphism $m : F \to G$ with $F, G \in C(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S)$ is an $(\mathbb{A}^1, et)$ local equivalence if and only if $a_{et}H^nC_*\,\mathrm{Cone}(m) = 0$ for all $n \in \mathbb{Z}$.

(iii) A morphism $m : F \to G$ with $F, G \in C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S)$ is an $(\mathbb{A}^1, et)$ local equivalence if and only if there exists
$$\{(X_{1,\alpha}, Z_{1,\alpha})/S, \alpha \in \Lambda_1\}, \ldots, \{(X_{r,\alpha}, Z_{r,\alpha})/S, \alpha \in \Lambda_r\} \subset \mathrm{Var}(\mathbb{C})^{2,(sm)}/S$$
such that we have in $\mathrm{Ho}_{et}(C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S))$
$$\begin{aligned}\mathrm{Cone}(m) \xrightarrow{\sim} \mathrm{Cone}(&\oplus_{\alpha \in \Lambda_1} \mathrm{Cone}(\mathbb{Z}((X_{1,\alpha} \times \mathbb{A}^1, Z_{1,\alpha} \times \mathbb{A}^1)/S) \to \mathbb{Z}((X_{1,\alpha}, Z_{1,\alpha})/S)) \\ &\to \cdots \to \oplus_{\alpha \in \Lambda_r} \mathrm{Cone}(\mathbb{Z}((X_{r,\alpha} \times \mathbb{A}^1, Z_{r,\alpha} \times \mathbb{A}^1)/S) \to \mathbb{Z}((X_{r,\alpha}, Z_{r,\alpha})/S)))\end{aligned}$$

(iii)' A morphism $m : F \to G$ with $F, G \in C(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S)$ is an $(\mathbb{A}^1, et)$ local equivalence if and only if there exists
$$\{(Y_{1,\alpha} \times S, Z_{1,\alpha})/S, \alpha \in \Lambda_1\}, \ldots, \{(Y_{r,\alpha} \times S, Z_{r,\alpha})/S, \alpha \in \Lambda_r\} \subset \mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S$$
such that we have in $\mathrm{Ho}_{et}(C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S))$
$$\begin{aligned}\mathrm{Cone}(m) \xrightarrow{\sim} \mathrm{Cone}(&\oplus_{\alpha \in \Lambda_1} \mathrm{Cone}(\mathbb{Z}((Y_{1,\alpha} \times \mathbb{A}^1 \times S, Z_{1,\alpha} \times \mathbb{A}^1)/S) \to \mathbb{Z}((Y_{1,\alpha} \times S, Z_{1,\alpha})/S)) \\ &\to \cdots \to \oplus_{\alpha \in \Lambda_r} \mathrm{Cone}(\mathbb{Z}((Y_{r,\alpha} \times \mathbb{A}^1 \times S, Z_{r,\alpha} \times \mathbb{A}^1)/S) \to \mathbb{Z}((Y_{r,\alpha} \times S, Z_{r,\alpha})/S)))\end{aligned}$$

Proof. Standard : see Ayoub's thesis section 4 for example. Indeed, for (iii), by definition, if $\mathrm{Cone}(m)$ is of the given form, then it is an equivalence $(\mathbb{A}^1, et)$ local, on the other hand if m is an equivalence $(\mathbb{A}^1, et)$ local, we consider the commutative diagram
$$\begin{tikzcd} F \arrow[r, "c(F)"] \arrow[d, "m"] & C_*F \arrow[d, "c_*m"] \\ G \arrow[r, "c(G)"] & C_*G \end{tikzcd}$$
to deduce that $\mathrm{Cone}(m)$ is of the given form. □

Definition-Proposition 6. *Let $S \in \mathrm{Var}(\mathbb{C})$.*

(i) With the weak equivalence the $(\mathbb{A}^1, et)$ local equivalence and the fibration the epimorphism with $\mathbb{A}^1_S$ local and etale fibrant kernels gives a model structure on $C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S)$: the left bousfield localization of the projective model structure of $C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S)$. We call it the projective $(\mathbb{A}^1, et)$ model structure.

(ii) With the weak equivalence the $(\mathbb{A}^1, et)$ local equivalence and the fibration the epimorphism with $\mathbb{A}^1_S$ local and etale fibrant kernels gives a model structure on $C(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S)$: the left bousfield localization of the projective model structure of $C(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S)$. We call it the projective $(\mathbb{A}^1, et)$ model structure.

Proof. Similar to the proof of proposition 5. □

We have, similarly to the case of single varieties the following :

Proposition 22. *Let $g : T \to S$ a morphism with $T, S \in \mathrm{Var}(\mathbb{C})$.*

(i) The adjonction $(g^*, g_*) : C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S) \leftrightarrows C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/T)$ is a Quillen adjonction for the projective $(\mathbb{A}^1, et)$ model structure (see definition-proposition 6)

(i)' The functor $g^* : C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S) \to C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/T)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends $(\mathbb{A}^1, et)$ local equivalence to $(\mathbb{A}^1, et)$ local equivalence.

(ii) The adjonction $(g^*, g_*) : C(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S) \leftrightarrows C(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/T)$ is a Quillen adjonction for the projective $(\mathbb{A}^1, et)$ model structure (see definition-proposition 6)

(ii)' The functor $g^* : C(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S) \to C(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/T)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends $(\mathbb{A}^1, et)$ local equivalence to $(\mathbb{A}^1, et)$ local equivalence.

Proof. (i):Follows immediately from definition. (i)': Since the functor g^* preserve epimorphism and also monomorphism (the colimits involved being filetered), g^* sends quasi-isomorphism to quasi-isomorphism. Hence it preserve Zariski and etale local equivalence. The fact that it preserve $(\mathbb{A}^1, et)$ local equivalence then follows similarly to the single case by the fact that g_* preserve by definition $\mathbb{A}^1$ equivariant presheaves. (ii) and (ii)': Similar to (i) and (i)'. □

Proposition 23. *Let $S \in \mathrm{Var}(\mathbb{C})$.*

(i) The adjonction $(\rho_S^*, \rho_{S*}) : C(\mathrm{Var}(\mathbb{C})^{2,sm}/S) \leftrightarrows C(\mathrm{Var}(\mathbb{C})^2/S)$ is a Quillen adjonction for the $(\mathbb{A}^1, et)$ projective model structure.

(i)' The functor $\rho_{S*} : C(\mathrm{Var}(\mathbb{C})^2/S) \to C(\mathrm{Var}(\mathbb{C})^{2,sm}/S)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends $(\mathbb{A}^1, et)$ local equivalence to $(\mathbb{A}^1, et)$ local equivalence.

(ii) The adjonction $(\rho_S^*, \rho_{S*}) : C(\mathrm{Var}(\mathbb{C})^{2,smpr}/S) \leftrightarrows C(\mathrm{Var}(\mathbb{C})^{2,pr}/S)$ is a Quillen adjonction for the $(\mathbb{A}^1, et)$ projective model structure.

(ii)' The functor $\rho_{S*} : C(\mathrm{Var}(\mathbb{C})^{2,pr}/S) \to C(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends $(\mathbb{A}^1, et)$ local equivalence to $(\mathbb{A}^1, et)$ local equivalence.

Proof. Similar to the proof of proposition 19. □

Proposition 24. *Let $S \in \mathrm{Var}(\mathbb{C})$.*

(i) *The adjonction* $(\mu_S^*, \mu_{S*}) : C(\mathrm{Var}(\mathbb{C})^{2,pr}/S) \leftrightarrows C(\mathrm{Var}(\mathbb{C})^2/S)$ *is a Quillen adjonction for the* $(\mathbb{A}^1, et)$ *projective model structure.*

(i)' *The functor* $\mu_{S*} : C(\mathrm{Var}(\mathbb{C})^2/S) \to C(\mathrm{Var}(\mathbb{C})^{2,pr}/S)$ *sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends* $(\mathbb{A}^1, et)$ *local equivalence to* $(\mathbb{A}^1, et)$ *local equivalence.*

(ii) *The adjonction* $(\mu_S^*, \mu_{S*}) : C(\mathrm{Var}(\mathbb{C})^{2,smpr}/S) \leftrightarrows C(\mathrm{Var}(\mathbb{C})^{2,pr}/S)$ *is a Quillen adjonction for the* $(\mathbb{A}^1, et)$ *projective model structure.*

(ii)' *The functor* $\mu_{S*} : C(\mathrm{Var}(\mathbb{C})^{2,sm}/S) \to C(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$ *sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends* $(\mathbb{A}^1, et)$ *local equivalence to* $(\mathbb{A}^1, et)$ *local equivalence.*

Proof. Similar to the proof of proposition 19. Indeed, for (i)' or (ii)', if $m : F \to G$ with $F, G \in C(\mathrm{Var}(\mathbb{C})^{2,(sm)})$ is an equivalence $(\mathbb{A}^1, et)$ local then (see proposition 21), there exists

$$\{(X_{1,\alpha}, Z_{1,\alpha})/S, \alpha \in \Lambda_1\}, \ldots, \{(X_{r,\alpha}, Z_{r,\alpha})/S, \alpha \in \Lambda_r\} \subset \mathrm{Var}(\mathbb{C})^{2,(sm)}/S$$

such that we have in $\mathrm{Ho}_{et}(C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S))$

$$\begin{aligned}\mathrm{Cone}(m) &\xrightarrow{\sim} \mathrm{Cone}(\oplus_{\alpha\in\Lambda_1}\mathrm{Cone}(\mathbb{Z}((X_{1,\alpha}\times\mathbb{A}^1, Z_{1,\alpha}\times\mathbb{A}^1)/S) \to \mathbb{Z}((X_{1,\alpha}, Z_{1,\alpha})/S)) \\ &\to \cdots \to \oplus_{\alpha\in\Lambda_r}\mathrm{Cone}(\mathbb{Z}((X_{r,\alpha}\times\mathbb{A}^1, Z_{r,\alpha}\times\mathbb{A}^1)/S) \to \mathbb{Z}((X_{r,\alpha}, Z_{r,\alpha})/S))) \\ &\xrightarrow{\sim} \mathrm{Cone}(\mathrm{Cone}(\oplus_{\alpha\in\Lambda_1}\mathbb{Z}((X_{1,\alpha}, Z_{1,\alpha})/S)\otimes\mathbb{Z}(\mathbb{A}^1,\mathbb{A}^1)/S \to \oplus_{\alpha\in\Lambda_1}\mathbb{Z}((X_{1,\alpha}, Z_{1,\alpha})/S)) \\ &\to \cdots \to \mathrm{Cone}(\oplus_{\alpha\in\Lambda_r}\mathbb{Z}((X_{r,\alpha}, Z_{r,\alpha})/S)\otimes\mathbb{Z}((\mathbb{A}^1,\mathbb{A}^1)/S) \to \oplus_{\alpha\in\Lambda_r}\mathbb{Z}((X_{r,\alpha}, Z_{r,\alpha})/S)),\end{aligned}$$

this gives in $\mathrm{Ho}_{et}(C(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S))$

$$\begin{aligned}&\mathrm{Cone}(\mu_{S*}m) \xrightarrow{\sim} \mathrm{Cone}(\\ &\mathrm{Cone}((L\mu_{S*}\oplus_{\alpha\in\Lambda_1}\mathbb{Z}((X_{1,\alpha}, Z_{1,\alpha})/S))\otimes\mathbb{Z}((\mathbb{A}^1,\mathbb{A}^1)/S) \to (L\mu_{S*}\oplus_{\alpha\in\Lambda_1}\mathbb{Z}((X_{1,\alpha}, Z_{1,\alpha})/S)) \\ &\to \cdots \to \mathrm{Cone}((L\mu_{S*}\oplus_{\alpha\in\Lambda_r}\mathbb{Z}((X_{r,\alpha}, Z_{r,\alpha})/S))\otimes\mathbb{Z}((\mathbb{A}^1,\mathbb{A}^1)/S) \to (L\mu_{S*}\oplus_{\alpha\in\Lambda_r}\mathbb{Z}((X_{1,\alpha}, Z_{1,\alpha})/S))))\end{aligned}$$

hence $\mu_{S*}m : \mu_{S*}F \to \mu_{S*}G$ is an equivalence $(\mathbb{A}^1, et)$ local. □

We also have

Proposition 25. *Let* $S \in \mathrm{Var}(\mathbb{C})$.

(i) *The adjonction* $(\mathrm{Gr}_S^{12*}, \mathrm{Gr}_{S*}^{12}) : C(\mathrm{Var}(\mathbb{C})/S) \leftrightarrows C(\mathrm{Var}(\mathbb{C})^{2,pr}/S)$ *is a Quillen adjonction for the* $(\mathbb{A}^1, et)$ *projective model structure.*

(ii) *The adjonction* $(\mathrm{Gr}_S^{12*}\,\mathrm{Gr}_{S*}^{12} : C(\mathrm{Var}(\mathbb{C})^{sm}/S) \leftrightarrows C(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$ *is a Quillen adjonction for the* $(\mathbb{A}^1, et)$ *projective model structure.*

Proof. Immediate from definition. □

- For $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$ and $Z \subset X$ a closed subset, we denote $\mathbb{Z}^{tr}((X, Z)/S) \in \mathrm{PSh}(\mathrm{Var}(\mathbb{C})^2/S)$ the presheaf given by
  - for $(X', Z')/S \in \mathrm{Var}(\mathbb{C})^2/S$, with X' irreducible,

    $$\mathbb{Z}^{tr}((X,Z)/S)((X',Z')/S) := \left\{\alpha \in \mathcal{Z}^{fs/X}(X'\times_S X), s.t. p_X(p_{X'}^{-1}(Z')) \subset Z\right\} \subset \mathcal{Z}_{d_{X'}}(X'\times_S X)$$

  - for $g : (X_2, Z_2)/S \to (X_1, Z_1)/S$ a morphism, with $(X_1, Z_1)/S, (X_2, Z_2)/S \in \mathrm{Var}(\mathbb{C})^2/S$,

    $$\mathbb{Z}^{tr}((X,Z)/S)(g) : \mathbb{Z}^{tr}((X,Z)/S)((X_1,Z_1)/S) \to \mathbb{Z}^{tr}((X,Z)/S)((X_2,Z_2)/S),\ \alpha \mapsto (g\times I)^{-1}(\alpha)$$

    with $g \times I : X_2 \times_S X \to X_1 \times_S X$.

- For $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$, $Z \subset X$ a closed subset and $r \in \mathbb{N}$, we denote $\mathbb{Z}^{equir}((X,Z)/S) \in \mathrm{PSh}(\mathrm{Var}(\mathbb{C})^2/S)$ the presheaf given by
  - for $(X',Z')/S \in \mathrm{Var}(\mathbb{C})^2/S$, with X' irreducible,

    $$\mathbb{Z}^{equir}((X,Z)/S)((X',Z')/S) := \left\{\alpha \in \mathcal{Z}^{equir/X}(X' \times_S X), s.t.p_X(p_{X'}^{-1}(Z'))\right\} \subset \mathcal{Z}_{d_{X'}}(X' \times_S X)$$
  - for $g : (X_2,Z_2)/S \to (X_1,Z_1)/S$ a morphism, with $(X_1,Z_1)/S, (X_2,Z_2)/S \in \mathrm{Var}(\mathbb{C})^2/S$,

    $$\mathbb{Z}^{equir}((X,Z)/S)(g) : \mathbb{Z}^{equir}((X,Z)/S)((X_1,Z_1)/S) \to \mathbb{Z}^{equir}((X,Z)/S)((X_2,Z_2)/S),\ \alpha \mapsto (g \times I)^{-1}(\alpha)$$

    with $g \times I : X_2 \times_S X \to X_1 \times_S X$.
- Let $S \in \mathrm{Var}(\mathbb{C})$. We denote by $\mathbb{Z}_S(d) := \mathbb{Z}^{equi0}((S \times \mathbb{A}^d, S \times \mathbb{A}^d)/S)[-2d]$ the Tate twist. For $F \in C(\mathrm{Var}(\mathbb{C})^2/S)$, we denote by $F(d) := F \otimes \mathbb{Z}_S(d)$.

For $S \in \mathrm{Var}(\mathbb{C})$, let $\mathrm{Cor}(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S)$ be the category

- whose objects are those of $\mathrm{Var}(\mathbb{C})^{2,(sm)}/S$, i.e. $(X,Z)/S = ((X,Z),h)$, $h : X \to S$ with $X \in \mathrm{Var}(\mathbb{C})$, $Z \subset X$ a closed subset,
- whose morphisms $\alpha : (X',Z)/S = ((X',Z),h_1) \to (X,Z)/S = ((X,Z),h_2)$ is finite correspondence that is $\alpha \in \oplus_i \mathbb{Z}^{tr}((X_i,Z)/S)((X',Z')/S)$, where $X' = \sqcup_i X'_i$, with X'_i connected, the composition being defined in the same way as the morphism $\mathrm{Cor}(\mathrm{Var}(\mathbb{C})^{(sm)}/S)$.

We denote by $\mathrm{Tr}(S) : \mathrm{Cor}(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S) \to \mathrm{Var}(\mathbb{C})^{2,(sm)}/S$ the morphism of site given by the inclusion functor $\mathrm{Tr}(S) : \mathrm{Var}(\mathbb{C})^{2,(sm)}/S \hookrightarrow \mathrm{Cor}(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S)$ It induces an adjonction

$$(\mathrm{Tr}(S)^* \mathrm{Tr}(S)_*) : C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S) \leftrightarrows C(\mathrm{Cor}(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S))$$

A complex of preheaves $G \in C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S)$ is said to admit transferts if it is in the image of the embedding

$$\mathrm{Tr}(S)_* : C(\mathrm{Cor}(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S) \hookrightarrow C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S),$$

that is $G = \mathrm{Tr}(S)_* \mathrm{Tr}(S)^* G$. We then have the full subcategory $\mathrm{Cor}(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S) \subset \mathrm{Cor}(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S)$ consisting of the objects of $\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S)$. We have the adjonction

$$(\mathrm{Tr}(S)^* \mathrm{Tr}(S)_*) : C(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S) \leftrightarrows C(\mathrm{Cor}(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S))$$

A complex of preheaves $G \in C(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S)$ is said to admit transferts if it is in the image of the embedding

$$\mathrm{Tr}(S)_* : C(\mathrm{Cor}(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S) \hookrightarrow C(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S),$$

that is $G = \mathrm{Tr}(S)_* \mathrm{Tr}(S)^* G$.

In the filtered case, we also define

Definition 16. *(i) A filtered complex* $(G,F) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S)$ *is said to be r-filtered* $\mathbb{A}^1$ *homotopic if* $\mathrm{ad}(p_a^*, p_{a*})(G,F) : (G,F) \to p_{a*}p_a^*(G,F)$ *is an r-filtered homotopy equivalence.*

(ii) A filtered complex $(G,F) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S)$ *is said to be r-filtered* $\mathbb{A}^1$ *homotopic if* $\mathrm{ad}(p_a^*, p_{a*})(G,F) : (G,F) \to p_{a*}p_a^*(G,F)$ *is an r-filtered homotopy equivalence.*

We will use to compute the algebraic De Rahm realization functor the following

Theorem 11. *(i) Let* $\phi : F^\bullet \to G^\bullet$ *an etale local equivalence with* $F^\bullet, G^\bullet \in C(\mathrm{Var}(\mathbb{C})^{2,sm}/S)$*. If* $F^\bullet$ *and* $G^\bullet$ *are* $\mathbb{A}^1$ *local and admit tranferts then* $\phi : F^\bullet \to G^\bullet$ *is a Zariski local equivalence. Hence if* $F \in C(\mathrm{Var}(\mathbb{C})^{2,sm}/S)$ *is* $\mathbb{A}^1$ *local and admits transfert*

$$k : E_{zar}(F) \to E_{et}(E_{zar}(F)) = E_{et}(F)$$

is a Zariski local equivalence.

(ii) *Let $\phi : F^\bullet \to G^\bullet$ an etale local equivalence with $F^\bullet, G^\bullet \in C(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$. If $F^\bullet$ and $G^\bullet$ are $\mathbb{A}^1$ local and admit tranferts then $\phi : F^\bullet \to G^\bullet$ is a Zariski local equivalence. Hence if $F \in C(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$ is $\mathbb{A}^1$ local and admits transfert*

$$k : E_{zar}(F) \to E_{et}(E_{zar}(F)) = E_{et}(F)$$

is a Zariski local equivalence.

Proof. Similar to the proof of theorem 10. □

Theorem 12. (i) *Let $\phi : (F^\bullet, F) \to (G^\bullet, F)$ a filtered etale local equivalence with $(F^\bullet, F), (G^\bullet, F) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{2,sm}/S)$. If $(F^\bullet, F)$ and $(G^\bullet, F)$ are r-filtered $\mathbb{A}^1$ homotopic and admit tranferts then $\phi : (F^\bullet, F) \to (G^\bullet, F)$ is an r-filtered Zariski local equivalence. Hence if $(G, F) \in C(\mathrm{Var}(\mathbb{C})^{2,sm}/S)$ is r-filtered $\mathbb{A}^1$ homotopic and admits transfert*

$$k : E_{zar}(G, F) \to E_{et}(E_{zar}(G, F)) = E_{et}(G, F)$$

is an r-filtered Zariski local equivalence.

(ii) *Let $\phi : (F^\bullet, F) \to (G^\bullet, F)$ a filtered etale local equivalence with $(F^\bullet, F), (G^\bullet, F) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$. If $F^\bullet$ and $G^\bullet$ are r-filtered $\mathbb{A}^1$ homotopic and admit tranferts then $\phi : (F^\bullet, F) \to (G^\bullet, F)$ is an r-filtered Zariski local equivalence. Hence if $(G, F) \in C(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$ is r-filtered $\mathbb{A}^1$ homotopic and admits transfert*

$$k : E_{zar}(F) \to E_{et}(E_{zar}(F)) = E_{et}(F)$$

is an r-filtered Zariski local equivalence.

Proof. Follows from theorem 11. □

We have the following canonical functor :

Definition 17. (i) *For $S \in \mathrm{Var}(\mathbb{C})$, we have the functor*

$$\begin{aligned}
&(-)^\Gamma : C(\mathrm{Var}(\mathbb{C})^{sm}/S) \to C(\mathrm{Var}(\mathbb{C})^{2,sm}/S),\\
F \longmapsto F^\Gamma : (((U,Z)/S) = ((U,Z),h) \mapsto F^\Gamma((U,Z)/S) &:= (\Gamma^\vee_Z h^* LF)(U/U),\\
&(g : ((U',Z'),h') \to ((U,Z),h)) \mapsto\\
(F^\Gamma(g) : (\Gamma^\vee_Z h^* LF)(U/U) &\xrightarrow{i_{(\Gamma^\vee_Z h^* LF)(U/U)}} (g^*(\Gamma^\vee_Z h^* LF))(U'/U')\\
&\xrightarrow{T(g,\gamma^\vee)(h^* LF)(U'/U')} (\Gamma^\vee_{Z\times_U U'} g^* h^* LF)(U'/U')\\
&\xrightarrow{T(Z'/Z\times_U U',\gamma^\vee)(g^* h^* LF)(U'/U')} (\Gamma^\vee_{Z'} g^* h^* LF)(U'/U')))
\end{aligned}$$

where $i_{(\Gamma^\vee_Z h^* LF)(U/U)}$ is the canonical arrow of the inductive limit. Similarly, we have, for $S \in \mathrm{Var}(\mathbb{C})$, the functor

$$\begin{aligned}
&(-)^\Gamma : C(\mathrm{Var}(\mathbb{C})/S) \to C(\mathrm{Var}(\mathbb{C})^2/S),\\
F \longmapsto F^\Gamma : (((X,Z)/S) = ((X,Z),h) &\mapsto F^\Gamma((X,Z)/S) := (\Gamma^\vee_Z h^* F)(X/X),\\
(g : ((X',Z'),h') \to ((X,Z),h)) &\mapsto (F^\Gamma(g) : (\Gamma^\vee_Z h^* LF)(X/X) \to (\Gamma^\vee_{Z'} h^{'*} LF)(X'/X')))
\end{aligned}$$

Note that for $S \in \mathrm{Var}(\mathbb{C})$, $I(S/S) : \mathbb{Z}((S,S)/S) \to \mathbb{Z}(S/S)^\Gamma$ given by

$$I(S/S)((U,Z),h) : \mathbb{Z}((S,S)/S)(((U,Z),h)) \xrightarrow{\gamma^\vee_Z(\mathbb{Z}(U/U))(U/U)} \mathbb{Z}(S/S)^\Gamma((U,Z),h) := (\Gamma^\vee_Z \mathbb{Z}(U/U))(U/U)$$

is an isomorphism.

(ii) *Let $f : T \to S$ a morphism with $T, S \in \mathrm{Var}(\mathbb{C})$. For $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$, we have the canonical morphism in $C(\mathrm{Var}(\mathbb{C})^{2,sm}/T)$*

$$\begin{aligned}
T(f,\Gamma)(F) := T^*(f,\Gamma)(F) : f^*(F^\Gamma) \to (f^*F)^\Gamma, \\
T(f,\Gamma)(F)((U',Z')/T = ((U',Z'),h')) : \\
f^*(F^\Gamma)((U',Z'),h') := \lim_{((U',Z'),h') \xrightarrow{l} ((U_T,Z_T),h_T) \xrightarrow{f_U} ((U,Z),h)} (\Gamma^\vee_Z h^* LF)(U/U) \\
\xrightarrow{F^\Gamma(f_U \circ l)} (\Gamma^\vee_{Z'} l^* f_U^* h^* LF)(U'/U') = (\Gamma^\vee_{Z'} h^{'*} f^* LF)(U'/U') \\
\xrightarrow{(\Gamma^\vee_{Z'} h^{'*} T(f,L)(F))(U'/U')} (\Gamma^\vee_{Z'} h^{'*} L f^* F)(U'/U') =: (f^*F)^\Gamma((U',Z'),h')
\end{aligned}$$

where $f_U : U_T : U \times_S T \to U$ and $h_T : U_T := U \times_S T \to T$ are the base change maps, the equality following from the fact that $h \circ f_U \circ l = f \circ h_T \circ l = f \circ h'$. For $F \in C(\mathrm{Var}(\mathbb{C})/S)$, we have similarly the canonical morphism in $C(\mathrm{Var}(\mathbb{C})^2/T)$

$$T(f,\Gamma)(F) : f^*(F^\Gamma) \to (f^*F)^\Gamma.$$

(iii) *Let $h : U \to S$ a smooth morphism with $U, S \in \mathrm{Var}(\mathbb{C})$. We have, for $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/U)$, the canonical morphism in $C(\mathrm{Var}(\mathbb{C})^{2,sm}/S)$*

$$\begin{aligned}
T_\sharp(h,\Gamma)(F) : h_\sharp(F^\Gamma) \to (h_\sharp LF)^\Gamma, \\
T_\sharp(h,\Gamma)(F)((U',Z'),h') : h_\sharp(F^\Gamma)((U',Z'),h') := \lim_{((U',Z'),h') \xrightarrow{l} ((U,U),h)} (\Gamma^\vee_{Z'} l^* LF)(U'/U') \\
\xrightarrow{(\Gamma^\vee_{Z'} l^* \operatorname{ad}(h_\sharp,h^*)(LF))(U'/U')} (\Gamma^\vee_{Z'} l^* h^* h_\sharp LF)(U'/U') =: (h_\sharp LF)^\Gamma((U',Z')/h')
\end{aligned}$$

(iv) *Let $i : Z_0 \hookrightarrow S$ a closed embedding with $Z_0, S \in \mathrm{Var}(\mathbb{C})$. We have the canonical morphism in $C(\mathrm{Var}(\mathbb{C})^{2,sm}/S)$*

$$\begin{aligned}
T_*(i,\Gamma)(\mathbb{Z}(Z_0/Z_0)) : i_*((\mathbb{Z}(Z_0/Z_0))^\Gamma \to (i_*\mathbb{Z}(Z/Z))^\Gamma, \\
T_*(i,\Gamma)(\mathbb{Z}(Z_0/Z_0))((U,Z),h) : i_*((\mathbb{Z}(Z_0/Z_0))^\Gamma((U,Z),h) := (\Gamma^\vee_{Z \times_S Z_0} \mathbb{Z}(Z_0/Z_0))(U \times_S Z_0) \\
\xrightarrow{T(i_*,\gamma^\vee)(\mathbb{Z}(Z_0/Z_0))(U \times_S Z_0)} (\Gamma^\vee_Z i_* \mathbb{Z}(Z_0/Z_0))(U \times_S Z_0) =: (i_*\mathbb{Z}(Z/Z))^\Gamma((U,Z),h)
\end{aligned}$$

Definition 18. *Let $S \in \mathrm{Var}(\mathbb{C})$. We have for $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ the canonical map in $C(\mathrm{Var}(\mathbb{C})^{sm}/S)$*

$$\begin{aligned}
\mathrm{Gr}(F) : \mathrm{Gr}^{12}_{S*} \mu_{S*} F^\Gamma \to F, \\
\mathrm{Gr}(F)(U/S) : \Gamma^\vee_U p^* F(U \times S/U \times S) \xrightarrow{\operatorname{ad}(l^*,l_*)(p^*F)(U \times S/U \times S)} h^*F(U/U) = F(U/S)
\end{aligned}$$

where $h : U \to S$ is a smooth morphism with $U \in \mathrm{Var}(\mathbb{C})$ and $h : U \xrightarrow{l} U \times S \xrightarrow{p} S$ is the graph factorization with l the graph embedding and p the projection.

Proposition 26. *Let $S \in \mathrm{Var}(\mathbb{C})$.*

(i) *Then,*

- *if $m : F \to G$ with $F, G \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is a quasi-isomorphism, $m^\Gamma : F^\Gamma \to G^\Gamma$ is a quasi-isomorphism in $C(\mathrm{Var}(\mathbb{C})^{2,sm}/S)$,*
- *if $m : F \to G$ with $F, G \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is a Zariski local equivalence, $m^\Gamma : F^\Gamma \to G^\Gamma$ is a Zariski local equivalence in $C(\mathrm{Var}(\mathbb{C})^{2,sm}/S)$, if $m : F \to G$ with $F, G \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is an etale local equivalence, $m^\Gamma : F^\Gamma \to G^\Gamma$ is an etale local equivalence in $C(\mathrm{Var}(\mathbb{C})^{2,sm}/S)$,*
- *if $m : F \to G$ with $F, G \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is an $(\mathbb{A}^1, et)$ local equivalence, $m^\Gamma : F^\Gamma \to G^\Gamma$ is an $(\mathbb{A}^1, et)$ local equivalence in $C(\mathrm{Var}(\mathbb{C})^{2,sm}/S)$.*

(ii) Then,

- *if $m : F \to G$ with $F, G \in C(\mathrm{Var}(\mathbb{C})/S)$ is a quasi-isomorphism, $m^\Gamma : F^\Gamma \to G^\Gamma$ is a quasi-isomorphism in $C(\mathrm{Var}(\mathbb{C})^2/S)$,*
- *if $m : F \to G$ with $F, G \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is a Zariski local equivalence, $m^\Gamma : F^\Gamma \to G^\Gamma$ is a Zariski local equivalence in $C(\mathrm{Var}(\mathbb{C})^{2,sm}/S)$, if $m : F \to G$ with $F, G \in C(\mathrm{Var}(\mathbb{C})/S)$ is an etale local equivalence, $m^\Gamma : F^\Gamma \to G^\Gamma$ is an etale local equivalence in $C(\mathrm{Var}(\mathbb{C})^2/S)$,*
- *if $m : F \to G$ with $F, G \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is an $(\mathbb{A}^1, et)$ local equivalence, $m^\Gamma : F^\Gamma \to G^\Gamma$ is an $(\mathbb{A}^1, et)$ local equivalence in $C(\mathrm{Var}(\mathbb{C})^2/S)$.*

Proof. (i): Follows immediately from the fact that for $((U, Z), h) \in \mathrm{Var}(\mathbb{C})^{2,sm}/S$,

- if $m : F \to G$ with $F, G \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is a quasi-isomorphism, $\Gamma^\vee_Z h^* LF(m) : \Gamma^\vee_Z h^* LF \to \Gamma^\vee_Z h^* LG$ is a quasi-isomorphism
- if $m : F \to G$ with $F, G \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is a is a Zariski or etale local equivalence, $\Gamma^\vee_Z h^* LF(m) : \Gamma^\vee_Z h^* LF \to \Gamma^\vee_Z h^* LG$ is a Zariski, resp. etale, local equivalence,
- if $m : F \to G$ with $F, G \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is an $(\mathbb{A}^1, et)$ local equivalence, $\Gamma^\vee_Z h^* LF(m) : \Gamma^\vee_Z h^* LF \to \Gamma^\vee_Z h^* LG$ is an $(\mathbb{A}^1, et)$ local equivalence.

(ii): Similar to (i). □

2.8 Presheaves on the big analytical site

For $S \in \mathrm{AnSp}(\mathbb{C})$, we denote by $\rho_S : \mathrm{AnSp}(\mathbb{C})^{sm}/S \hookrightarrow \mathrm{AnSp}(\mathbb{C})/S$ be the full subcategory consisting of the objects $U/S = (U, h) \in \mathrm{AnSp}(\mathbb{C})/S$ such that the morphism $h : U \to S$ is smooth. That is, $\mathrm{AnSp}(\mathbb{C})^{sm}/S$ is the category

- whose objects are smooth morphisms $U/S = (U, h)$, $h : U \to S$ with $U \in \mathrm{AnSp}(\mathbb{C})$,
- whose morphisms $g : U/S = (U, h_1) \to V/S = (V, h_2)$ is a morphism $g : U \to V$ of complex algebraic varieties such that $h_2 \circ g = h_1$.

We denote again $\rho_S : \mathrm{AnSp}(\mathbb{C})/S \to \mathrm{AnSp}(\mathbb{C})^{sm}/S$ the associated morphism of site. We will consider

$$r^s(S) : \mathrm{AnSp}(\mathbb{C}) \xrightarrow{r(S)} \mathrm{AnSp}(\mathbb{C})/S \xrightarrow{\rho_S} \mathrm{AnSp}(\mathbb{C})^{sm}/S$$

the composite morphism of site. For $S \in \mathrm{AnSp}(\mathbb{C})$, we denote by $\mathbb{Z}_S := \mathbb{Z}(S/S) \in \mathrm{PSh}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$ the constant presheaf By Yoneda lemma, we have for $F \in C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$, $\mathcal{H}om(\mathbb{Z}_S, F) = F$. For $f : T \to S$ a morphism, with $T, S \in \mathrm{AnSp}(\mathbb{C})$, we have the following commutative diagram of sites

$$\begin{tikzcd} \mathrm{AnSp}(\mathbb{C})/T \arrow[r, "\rho_T"] \arrow[d, "P(f)"] & \mathrm{AnSp}(\mathbb{C})^{sm}/T \arrow[d, "P(f)"] \\ \mathrm{AnSp}(\mathbb{C})/S \arrow[r, "\rho_S"] & \mathrm{AnSp}(\mathbb{C})^{sm}/S \end{tikzcd} \tag{31}$$

We denote, for $S \in \mathrm{AnSp}(\mathbb{C})$, the obvious morphism of sites

$$\tilde{e}(S) : \mathrm{AnSp}(\mathbb{C})/S \xrightarrow{\rho_S} \mathrm{AnSp}(\mathbb{C})^{sm}/S \xrightarrow{e(S)} \mathrm{Ouv}(S)$$

where $\mathrm{Ouv}(S)$ is the set of the open subsets of S, given by the inclusion functors $\tilde{e}(S) : \mathrm{Ouv}(S) \hookrightarrow \mathrm{AnSp}(\mathbb{C})^{sm}/S \hookrightarrow \mathrm{AnSp}(\mathbb{C})/S$. By definition, for $f : T \to S$ a morphism with $S, T \in \mathrm{AnSp}(\mathbb{C})$, the commutative diagram of sites (31) extend a commutative diagram of sites :

$$\begin{tikzcd}
\tilde{e}(T) : \mathrm{AnSp}(\mathbb{C})/T \arrow[r, "\rho_T"] \arrow[d, "P(f)"] & \mathrm{AnSp}(\mathbb{C})^{sm}/T \arrow[r, "e(T)"] \arrow[d, "P(f)"] & \mathrm{Ouv}(T) \arrow[d, "P(f)"] \\
\tilde{e}(S) : \mathrm{AnSp}(\mathbb{C})/S \arrow[r, "\rho_S"] & \mathrm{AnSp}(\mathbb{C})^{sm}/S \arrow[r, "e(S)"] & \mathrm{Ouv}(S)
\end{tikzcd} \tag{32}$$

- As usual, we denote by

$$(f^*, f_*) := (P(f)^*, P(f)_*) : C(\mathrm{AnSp}(\mathbb{C})^{sm}/S) \to C(\mathrm{AnSp}(\mathbb{C})^{sm}/T)$$

the adjonction induced by $P(f) : \mathrm{AnSp}(\mathbb{C})^{sm}/T \to \mathrm{AnSp}(\mathbb{C})^{sm}/S$. Since the colimits involved in the definition of $f^* = P(f)^*$ are filtered, f^* also preserve monomorphism. Hence, we get an adjonction

$$(f^*, f_*) : C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S) \leftrightarrows C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/T),\ f^*(G, F) := (f^*G, f^*F)$$

- As usual, we denote by

$$(f^*, f_*) := (P(f)^*, P(f)_*) : C(\mathrm{AnSp}(\mathbb{C})/S) \to C(\mathrm{AnSp}(\mathbb{C})/T)$$

the adjonction induced by $P(f) : \mathrm{AnSp}(\mathbb{C})/T \to \mathrm{AnSp}(\mathbb{C})/S$. Since the colimits involved in the definition of $f^* = P(f)^*$ are filtered, f^* also preserve monomorphism. Hence, we get an adjonction

$$(f^*, f_*) : C_{fil}(\mathrm{AnSp}(\mathbb{C})/S) \leftrightarrows C_{fil}(\mathrm{AnSp}(\mathbb{C})/T),\ f^*(G, F) := (f^*G, f^*F)$$

- For $h : U \to S$ a smooth morphism with $U, S \in \mathrm{AnSp}(\mathbb{C})$, the pullback functor $P(h) : \mathrm{AnSp}(\mathbb{C})^{sm}/S \to \mathrm{AnSp}(\mathbb{C})^{sm}/U$ admits a left adjoint $C(h)(X \to U) = (X \to U \to S)$. Hence, $h^* : C(\mathrm{AnSp}(\mathbb{C})^{sm}/S) \to C(\mathrm{AnSp}(\mathbb{C})^{sm}/U)$ admits a left adjoint

$$h_\sharp : C(\mathrm{AnSp}(\mathbb{C})^{sm}/U) \to C(\mathrm{AnSp}(\mathbb{C})^{sm}/S),\ F \mapsto ((V, h_0) \mapsto \lim_{(V', h\circ h') \to (V, h_0)} F(V', h'))$$

Note that for $h' : V' \to V$ a smooth morphism, $V', V \in \mathrm{AnSp}(\mathbb{C})$, we have $h_\sharp(\mathbb{Z}(V'/V)) = \mathbb{Z}(V'/S)$ with $V'/S = (V', h \circ h')$. Hence, since projective presheaves are the direct summands of the representable presheaves, $h_\sharp$ sends projective presheaves to projective presheaves. For $F^\bullet \in C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$ and $G^\bullet \in C(\mathrm{AnSp}(\mathbb{C})^{sm}/U)$, we have the adjonction maps

$$\mathrm{ad}(h_\sharp, h^*)(G^\bullet) : G^\bullet \to h^* h_\sharp G^\bullet\ ,\ \mathrm{ad}(h_\sharp, h^*)(F^\bullet) : h_\sharp h^* F^\bullet \to F^\bullet.$$

For a smooth morphism $h : U \to S$, with $U, S \in \mathrm{AnSp}(\mathbb{C})$, we have the adjonction isomorphism, for $F \in C(\mathrm{AnSp}(\mathbb{C})^{sm}/U)$ and $G \in C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$,

$$I(h_\sharp, h^*)(F, G) : \mathcal{H}om^\bullet(h_\sharp F, G) \xrightarrow{\sim} h_* \mathcal{H}om^\bullet(F, h^* G). \tag{33}$$

- For $f : T \to S$ any morphism with $T, S \in \mathrm{AnSp}(\mathbb{C})$, the pullback functor $P(f) : \mathrm{AnSp}(\mathbb{C})/T \to \mathrm{AnSp}(\mathbb{C})/S$ admits a left adjoint $C(f)(X \to T) = (X \to T \to S)$. Hence, $f^* : C(\mathrm{AnSp}(\mathbb{C})/S) \to C(\mathrm{AnSp}(\mathbb{C})/T)$ admits a left adjoint

$$f_\sharp : C(\mathrm{AnSp}(\mathbb{C})/T) \to C(\mathrm{AnSp}(\mathbb{C})/S),\ F \mapsto ((V, h_0) \mapsto \lim_{(V', h\circ h') \to (V, h_0)} F(V', h'))$$

Note that we have for $h' : V' \to V$ a morphism, $V', V \in \mathrm{AnSp}(\mathbb{C})$, $h_\sharp(\mathbb{Z}(V'/V)) = \mathbb{Z}(V'/S)$ with $V'/S = (V', h \circ h')$. Hence, since projective presheaves are the direct summands of the representable presheaves, $h_\sharp$ sends projective presheaves to projective presheaves. For $F^\bullet \in C(\mathrm{AnSp}(\mathbb{C})/S)$ and $G^\bullet \in C(\mathrm{AnSp}(\mathbb{C})/T)$, we have the adjonction maps

$$\mathrm{ad}(f_\sharp, f^*)(G^\bullet) : G^\bullet \to f^* f_\sharp G^\bullet \ , \ \mathrm{ad}(f_\sharp, f^*)(F^\bullet) : f_\sharp f^* F^\bullet \to F^\bullet.$$

For a morphism $f : T \to S$, with $T, S \in \mathrm{AnSp}(\mathbb{C})$, we have the adjonction isomorphism, for $F \in C(\mathrm{AnSp}(\mathbb{C})/T)$ and $G \in C(\mathrm{AnSp}(\mathbb{C})/S)$,

$$I(f_\sharp, f^*)(F, G) : \mathcal{H}om^\bullet(f_\sharp F, G) \xrightarrow{\sim} f_*\mathcal{H}om^\bullet(F, f^*G). \tag{34}$$

- For a commutative diagram in $\mathrm{AnSp}(\mathbb{C})$:

$$D = \begin{tikzcd} V \arrow[r, "g_2"] \arrow[d, "h_2"] & U \arrow[d, "h_1"] \\ T \arrow[r, "g_1"] & S \end{tikzcd} ,$$

  where h_1 and h_2 are smooth, we denote by, for $F^\bullet \in C(\mathrm{AnSp}(\mathbb{C})^{sm}/U)$,

$$T_\sharp(D)(F^\bullet) : h_{2\sharp} g_2^* F^\bullet \to g_1^* h_{1\sharp} F^\bullet$$

  the canonical map in $C(\mathrm{AnSp}(\mathbb{C})^{sm}/T)$ given by adjonction. If D is cartesian with $h_1 = h$, $g_1 = g$ $f_2 = h' : U_T \to T$, $g' : U_T \to U$,

$$T_\sharp(D)(F^\bullet) =: T_\sharp(g, h)(F^\bullet) : h'_\sharp g'^* F^\bullet \xrightarrow{\sim} g^* h_\sharp F^\bullet$$

  is an isomorphism and for $G^\bullet \in C(\mathrm{AnSp}(\mathbb{C})^{sm}/T)$

$$T(D)(G^\bullet) =: T(g, h)(G^\bullet) : g^* h_* G^\bullet \xrightarrow{\sim} h'_* g'^* G^\bullet$$

  is an isomorphism.

- For a commutative diagram in $\mathrm{AnSp}(\mathbb{C})$:

$$D = \begin{tikzcd} V \arrow[r, "g_2"] \arrow[d, "f_2"] & X \arrow[d, "f_1"] \\ T \arrow[r, "g_1"] & S \end{tikzcd} ,$$

  we denote by, for $F^\bullet \in C(\mathrm{AnSp}(\mathbb{C})/X)$,

$$T_\sharp(D)(F^\bullet) : f_{2\sharp} g_2^* F^\bullet \to g_1^* f_{1\sharp} F^\bullet$$

  the canonical map in $C(\mathrm{AnSp}(\mathbb{C})/T)$ given by adjonction. If D is cartesian with $h_1 = h$, $g_1 = g$ $f_2 = h' : X_T \to T$, $g' : X_T \to X$,

$$T_\sharp(D)(F^\bullet) =: T_\sharp(g, f)(F^\bullet) : f'_\sharp g'^* F^\bullet \xrightarrow{\sim} g^* f_\sharp F^\bullet$$

  is an isomorphism and for $G^\bullet \in C(\mathrm{AnSp}(\mathbb{C})/T)$

$$T(D)(G^\bullet) =: T(g, h)(G^\bullet) : f^* g_* G^\bullet \xrightarrow{\sim} g'_* f'^* G^\bullet$$

  is an isomorphism.

For $f : T \to S$ a morphism with $S, T \in \mathrm{Var}\,\mathrm{AnSp}(\mathbb{C})$,

- we get for $F \in C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$ from the a commutative diagram of sites (32) the following canonical transformation
$$T(e,f)(F^\bullet) : f^*e(S)_*F^\bullet \to e(T)_*f^*F^\bullet,$$
which is NOT a quasi-isomorphism in general. However, for $h : U \to S$ a smooth morphism with $S, U \in \mathrm{AnSp}(\mathbb{C})$, $T(e,h)(F^\bullet) : h^*e(S)_*F^\bullet \xrightarrow{\sim} e(T)_*h^*F^\bullet$ is an isomorphism.
- we get for $F \in C(\mathrm{AnSp}(\mathbb{C})/S)$ from the a commutative diagram of sites (32) the following canonical transformation
$$T(e,f)(F^\bullet) : f^*e(S)_*F^\bullet \to e(T)_*f^*F^\bullet,$$
which is NOT a quasi-isomorphism in general. However, for $h : U \to S$ a smooth morphism with $S, U \in \mathrm{AnSp}(\mathbb{C})$, $T(e,h)(F^\bullet) : h^*e(S)_*F^\bullet \xrightarrow{\sim} e(T)_*h^*F^\bullet$ is an isomorphism.

Let $S \in \mathrm{AnSp}(\mathbb{C})$,

- We have for $F, G \in C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$,
  - $e(S)_*(F \otimes G) = (e(S)_*F) \otimes (e(S)_*G)$ by definition
  - the canonical forgetfull map
  $$T(S, hom)(F,G) : e(S)_*\mathcal{H}om^\bullet(F,G) \to \mathcal{H}om^\bullet(e(S)_*F, e(S)_*G).$$
  which is NOT a quasi-isomorphism in general.

  By definition, we have for $F \in C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$, $e(S)_*E_{usu}(F) = E_{usu}(e(S)_*F)$.
- We have for $F, G \in C(\mathrm{AnSp}(\mathbb{C})/S)$,
  - $e(S)_*(F \otimes G) = (e(S)_*F) \otimes (e(S)_*G)$ by definition
  - the canonical forgetfull map
  $$T(S, hom)(F,G) : e(S)_*\mathcal{H}om^\bullet(F,G) \to \mathcal{H}om^\bullet(e(S)_*F, e(S)_*G).$$
  which is NOT a quasi-isomorphism in general.

  By definition, we have for $F \in C(\mathrm{AnSp}(\mathbb{C})/S)$, $e(S)_*E_{usu}(F) = E_{usu}(e(S)_*F)$.

Let $S \in \mathrm{AnSp}(\mathbb{C})$. We have the support section functor of a closed subset $Z \subset S$ for presheaves on the big analytical site.

Definition 19. *Let $S \in \mathrm{AnSp}(\mathbb{C})$. Let $Z \subset S$ a closed subset. Denote by $j : S \backslash Z \hookrightarrow S$ be the open complementary subset.*

(i) We define the functor

$$\Gamma_Z : C(\mathrm{AnSp}(\mathbb{C})^{sm}/S) \to C(\mathrm{AnSp}(\mathbb{C})^{sm}/S),\ G^\bullet \mapsto \Gamma_Z G^\bullet := \mathrm{Cone}(\mathrm{ad}(j^*, j_*)(G^\bullet) : G^\bullet \to j_*j^*G^\bullet)[-1],$$

so that there is then a canonical map $\gamma_Z(G^\bullet) : \Gamma_Z G^\bullet \to G^\bullet$.

(ii) We have the dual functor of (i) :

$$\Gamma^\vee_Z : C(\mathrm{AnSp}(\mathbb{C})^{sm}/S) \to C(\mathrm{AnSp}(\mathbb{C})^{sm}/S),\ F \mapsto \Gamma^\vee_Z(F^\bullet) := \mathrm{Cone}(\mathrm{ad}(j_\sharp, j^*)(G^\bullet) : j_\sharp j^*G^\bullet \to G^\bullet),$$

together with the canonical map $\gamma^\vee_Z(G) : F \to \Gamma^\vee_Z(G)$.

(iii) For $F, G \in C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$, we denote by

$$I(\gamma, hom)(F,G) := (I, I(j_\sharp, j^*)(F,G)) : \Gamma_Z \mathcal{H}om(F,G) \xrightarrow{\sim} \mathcal{H}om(\Gamma^\vee_Z F, G)$$

the canonical isomorphism given by adjonction.

Let $S \in \mathrm{AnSp}(\mathbb{C})$ and $Z \subset S$ a closed subset.

- Since $\Gamma_Z : C(\mathrm{AnSp}(\mathbb{C})^{sm}/S) \to C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$ preserve monomorphism, it induces a functor
$$\Gamma_Z : C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S) \to C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S), \; (G,F) \mapsto \Gamma_Z(G,F) := (\Gamma_Z G, \Gamma_Z F)$$
- Since $\Gamma^{\vee}_Z : C(\mathrm{AnSp}(\mathbb{C})^{sm}/S) \to C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$ preserve monomorphism, it induces a functor
$$\Gamma^{\vee}_Z : C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S) \to C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S), \; (G,F) \mapsto \Gamma^{\vee}_Z(G,F) := (\Gamma^{\vee}_Z G, \Gamma^{\vee}_Z F)$$

Definition-Proposition 7. *(i) Let $g : S' \to S$ a morphism and $i : Z \hookrightarrow S$ a closed embedding with $S', S, Z \in \mathrm{AnSp}(\mathbb{C})$. Then, for $(G,F) \in C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$, there exist a map in $C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S')$*
$$T(g,\gamma)(G,F) : g^*\Gamma_Z(G,F) \to \Gamma_{Z\times_S S'} g^*(G,F)$$
unique up to homotopy, such that $\gamma_{Z\times_S S'}(g^*(G,F)) \circ T(g,\gamma)(G,F) = g^*\gamma_Z(G,F)$.

(ii) Let $i_1 : Z_1 \hookrightarrow S$, $i_2 : Z_2 \hookrightarrow Z_1$ be closed embeddings with $S, Z_1, Z_2 \in \mathrm{AnSp}(\mathbb{C})$. Then, for $(G,F) \in C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$,

- *there exist a canonical map $T(Z_2/Z_1,\gamma)(G,F) : \Gamma_{Z_2}(G,F) \to \Gamma_{Z_1}(G,F)$ in $C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$ unique up to homotopy such that $\gamma_{Z_1}(G,F) \circ T(Z_2/Z_1,\gamma)(G,F) = \gamma_{Z_2}(G,F)$, together with a distinguish triangle*
$$\Gamma_{Z_2}(G,F) \xrightarrow{T(Z_2/Z_1,\gamma)(G,F)} \Gamma_{Z_1}(G,F) \xrightarrow{\mathrm{ad}(j_2^*,j_{2*})(\Gamma_{Z_1}(G,F))} \Gamma_{Z_1\setminus Z_2}(G,F) \to \Gamma_{Z_2}(G,F)[1]$$
in $K_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$,
- *there exist a map $T(Z_2/Z_1,\gamma^{\vee})(G,F) : \Gamma^{\vee}_{Z_1}(G,F) \to \Gamma^{\vee}_{Z_2}(G,F)$ in $C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$ unique up to homotopy such that $\gamma^{\vee}_{Z_2}(G,F) = T(Z_2/Z_1,\gamma^{\vee})(G,F) \circ \gamma^{\vee}_{Z_1}(G,F)$, together with a distinguish triangle*
$$\Gamma^{\vee}_{Z_1\setminus Z_2}(G,F) \xrightarrow{\mathrm{ad}(j_{2\sharp},j_2^*)(\Gamma^{\vee}_{Z_1}G)} \Gamma^{\vee}_{Z_1}(G,F) \xrightarrow{T(Z_2/Z_1,\gamma^{\vee})(G,F)} \Gamma^{\vee}_{Z_2}(G,F) \to \Gamma^{\vee}_{Z_1\setminus Z_2}(G,F)[1]$$
in $K_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$.

(iii) Consider a morphism $g : (S',Z) \to (S,Z)$ with $(S',Z) \to (S,Z) \in \mathrm{AnSp}(\mathbb{C})$. We denote, for $G \in C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$ the composite
$$T(D,\gamma^{\vee})(G) : g^*\Gamma^{\vee}_Z G \xrightarrow{\sim} \Gamma^{\vee}_{Z\times_S S'} g^*G \xrightarrow{T(Z'/Z\times_S S',\gamma^{\vee})(G)} \Gamma^{\vee}_{Z'} g^*G$$
and we have then the factorization $\gamma^{\vee}_{Z'}(g^*G) : g^*G \xrightarrow{g^*\gamma^{\vee}_Z(G)} g^*\Gamma^{\vee}_Z G \xrightarrow{T(D,\gamma^{\vee})(G)} \Gamma^{\vee}_{Z'} g^*G$.

Proof. Similar to definition-proposition 1 or definition-proposition 4. □

Definition 20. *For $S \in \mathrm{AnSp}(\mathbb{C})$, we denote by*
$$C_{O_S}(\mathrm{AnSp}(\mathbb{C})^{sm}/S) := C_{e(S)^*O_S}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$$
the category of complexes of presheaves on $\mathrm{AnSp}(\mathbb{C})^{sm}/S$ endowed with a structure of $e(S)^*O_S$ module, and by
$$C_{O_S fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S) := C_{e(S)^*O_S fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$$
the category of filtered complexes of presheaves on $\mathrm{Var}(\mathbb{C})^{sm}/S$endowed with a structure of $e(S)^*O_S$ module.

Let $S \in \mathrm{AnSp}(\mathbb{C})$. Let $Z \subset S$ a closed subset. Denote by $j : S\backslash Z \hookrightarrow S$ the open complementary embedding,

- For $G \in C_{O_S}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$, $\Gamma_Z G := \mathrm{Cone}(\mathrm{ad}(j^*, j_*)(G) : F \to j_* j^* G)[-1]$ has a (unique) structure of $e(S)^* O_S$ module such that $\gamma_Z(G) : \Gamma_Z G \to G$ is a map in $C_{O_S}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$. This gives the functor

$$\Gamma_Z : C_{O_S fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S) \to C_{filO_S}(\mathrm{AnSp}(\mathbb{C})^{sm}/S),\ (G,F) \mapsto \Gamma_Z(G,F) := (\Gamma_Z G, \Gamma_Z F),$$

  together with the canonical map $\gamma_Z((G,F) : \Gamma_Z(G,F) \to (G,F)$. Let $Z_2 \subset Z$ a closed subset. Then, for $G \in C_{O_S}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$, $T(Z_2/Z,\gamma)(G) : \Gamma_{Z_2} G \to \Gamma_Z G$ is a map in $C_{O_S}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$ (i.e. is $e(S)^* O_S$ linear).

- For $G \in C_{O_S}(\mathrm{Var}(\mathbb{C})^{sm}/S)$, $\Gamma^{\vee}_Z G := \mathrm{Cone}(\mathrm{ad}(j_\sharp, j^*)(G) : j_\sharp j^* G \to G)$ has a unique structure of $e(S)^* O_S$ module, such that $\gamma^{\vee}_Z(G) : G \to \Gamma^{\vee}_Z G$ is a map in $C_{O_S}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$. This gives the the functor

$$\Gamma^{\vee}_Z : C_{O_S fil}(S) \to C_{filO_S}(S),\ (G,F) \mapsto \Gamma^{\vee}_Z(G,F) := (\Gamma^{\vee}_Z G, \Gamma^{\vee}_Z F),$$

  together with the canonical map $\gamma^{\vee}_Z((G,F) : (G,F) \to \Gamma^{\vee}_Z(G,F)$. Let $Z_2 \subset Z$ a closed subset. Then, for $G \in C_{O_S}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$, $T(Z_2/Z,\gamma^{\vee})(G) : \Gamma^{\vee}_Z G \to \Gamma^{\vee}_{Z_2} G$ is a map in $C_{O_S}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$ (i.e. is $e(S)^* O_S$ linear).

Definition 21. *Let $S \in \mathrm{AnSp}(\mathbb{C})$. Let $Z \subset S$ a closed subset.*

(i) We denote by

$$C_Z(\mathrm{AnSp}(\mathbb{C})^{sm}/S) \subset C(\mathrm{AnSp}(\mathbb{C})^{sm}/S),$$

the full subcategory consisting of complexes of presheaves $F^\bullet \in C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$ such that $a_{usu} H^n(j^* F^\bullet) = 0$ for all $n \in \mathbb{Z}$, where $j : S\backslash Z \hookrightarrow S$ is the complementary open embedding and a_{usu} is the sheaftification functor.

(i)' We denote by

$$C_{O_S,Z}(\mathrm{AnSp}(\mathbb{C})^{sm}/S) \subset C_{O_S}(\mathrm{AnSp}(\mathbb{C})^{sm}/S),$$

the full subcategory consisting of complexes of presheaves $F^\bullet \in C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$ such that $a_{usu} H^n(j^* F^\bullet) = 0$ for all $n \in \mathbb{Z}$, where $j : S\backslash Z \hookrightarrow S$ is the complementary open embedding and a_{usu} is the sheaftification functor.

(ii) We denote by

$$C_{fil,Z}(\mathrm{AnSp}(\mathbb{C})^{sm}/S) \subset C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$$

the full subcategory consisting of filtered complexes of presheaves $(F^\bullet, F) \in C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$ such that there exist $r \in \mathbb{N}$ and an r-filtered homotopy equivalence $\phi : (F^\bullet, F) \to (F^{'\bullet}, F)$ with $(F^{'\bullet}, F) \in C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$ such that $a_{usu} j^* H^n \mathrm{Gr}^p_F(F^{'\bullet}, F) = 0$ for all $n, p \in \mathbb{Z}$, where $j : S\backslash Z \hookrightarrow S$ is the complementary open embedding and a_{usu} is the sheaftification functor.

(ii)' We denote by

$$C_{O_S fil,Z}(\mathrm{AnSp}(\mathbb{C})^{sm}/S) \subset C_{O_S fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$$

the full subcategory consisting of filtered complexes of presheaves $(F^\bullet, F) \in C_{O_S fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$ such that there exist $r \in \mathbb{N}$ and an r-filtered homotopy equivalence $\phi : (F^\bullet, F) \to (F^{'\bullet}, F)$ with $(F^{'\bullet}, F) \in C_{O_S fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$ such that $a_{usu} j^* H^n \mathrm{Gr}^p_F(F^\bullet, F) = 0$ for all $n, p \in \mathbb{Z}$, where $j : S\backslash Z \hookrightarrow S$ is the complementary open embedding and a_{usu} is the sheaftification functor.

Let $S \in \mathrm{AnSp}(\mathbb{C})$ and $Z \subset S$ a closed subset.

- For $(G,F) \in C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$, we have $\Gamma_Z(G,F), \Gamma^{\vee}_Z(G,F) \in C_{fil,Z}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$.
- For $(G,F) \in C_{O_S fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$, we have $\Gamma_Z(G,F), \Gamma^{\vee}_Z(G,F) \in C_{O_S fil,Z}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$.

Let $S \in \mathrm{AnSp}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open cover and denote by $S_I = \cap_{i\in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{AnSp}(\mathbb{C})$. For $I \subset [1, \cdots l]$, denote by $\tilde{S}_I = \Pi_{i\in I}\tilde{S}_i$. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I$, and for $J \subset I$ the following commutative diagram

$$D_{IJ} = \quad \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ {\scriptstyle j_{IJ}}\uparrow & & \uparrow{\scriptstyle p_{IJ}} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}$$

where $p_{IJ} : \tilde{S}_J \to \tilde{S}_I$ is the projection and $j_{IJ} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. This gives the diagram of analytic spaces $(\tilde{S}_I) \in \mathrm{Fun}(\mathcal{P}(\mathbb{N}), \mathrm{AnSp}(\mathbb{C}))$ which which gives the diagram of sites $\mathrm{AnSp}(\mathbb{C})^{sm}/(\tilde{S}_I) \in \mathrm{Fun}(\mathcal{P}(\mathbb{N}), \mathrm{Cat})$. Denote by $m : \tilde{S}_I \backslash (S_I \backslash S_J) \hookrightarrow \tilde{S}_I$ the open embedding.

Definition 22. *Let $S \in \mathrm{AnSp}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open cover and denote by $S_I = \cap_{i\in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{AnSp}(\mathbb{C})$. We denote by the full subcategory $C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/(S/(\tilde{S}_I))) \subset C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/(\tilde{S}_I))$ the full subcategory*

- *whose objects $(G, F) = ((G_I, F)_{I\subset[1,\cdots l]}, u_{IJ})$, with $(G_I, F) \in C_{fil,S_I}(\mathrm{AnSp}(\mathbb{C})^{sm}/\tilde{S}_I)$, and $u_{IJ} : m^*(G_I, F) \to m^* p_{IJ*}(G_J, F)$ for $I \subset J$, are ∞-filtered usu local equivalence, satisfying for $I \subset J \subset K$, $p_{IJ*}u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/\tilde{S}_I)$,*
- *the morphisms $m : ((G, F), u_{IJ}) \to ((H, F), v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,*
$$m = (m_I : (G_I, F) \to (H_I, F))_{I\subset[1,\cdots l]}$$
such that $v_{IJ} \circ m_I = p_{IJ*}m_J \circ u_{IJ}$ in $C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/\tilde{S}_I)$.

A morphism $m : ((G_I, F), u_{IJ}) \to ((H_I, F), v_{IJ})$ is said to an r-filtered usu local, equivalence, if all the m_I are r-filtered usu local equivalences.

Denote $L = [1, \ldots, l]$ and for $I \subset L$, $p_{0(0I)} : S \times \tilde{S}_I \to S$, $p_{I(0I)} : S \times \tilde{S}_I \to S_I$ the projections. By definition, we have functors

- $T(S/(\tilde{S}_I)) : C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S) \to C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/(S/(\tilde{S}_I)))$, $(G, F) \mapsto (i_{I*}j_I^*F, T(D_{IJ})(j_I^*(G, F)))$,
- $T((\tilde{S}_I)/S) : C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/(S/(\tilde{S}_I))) \to C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$, $((G_I, F), u_{IJ}) \mapsto \mathrm{ho}\lim_{I\subset L} p_{0(0I)*}\Gamma^\vee_{S_I} p^*_{I(0I)}(G_I, F)$.

Note that the functors $T(S/(\tilde{S}_I)$ are NOT embedding, since

$$\mathrm{ad}(i_I^*, i_{I*})(j_I^*F) : i_I^* i_{I*} j_I^* F \to j_I^* F$$

are usu local equivalence but NOT isomorphism since we are dealing with the morphism of big sites $P(i_I) : \mathrm{AnSp}(\mathbb{C})^{sm}/S_I \to \mathrm{AnSp}(\mathbb{C})^{sm}/\tilde{S}_I$. However, these functors induces full embeddings

$$T(S/(\tilde{S}_I)) : D_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S) \to D_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/(S/(\tilde{S}_I)))$$

since for $F \in C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$,

$$\mathrm{ho}\lim_{I\subset L} p_{0(0I)*}\Gamma_{S_I} p^*_{I(0I)}(i_{I*}j_I^*F) \to p_{0(0I)*}\Gamma_{S_I} j_I^* F$$

is an equivalence usu local.

Let $f : X \to S$ a morphism, with $X, S \in \mathrm{AnSp}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ and $X = \cup_{i=1}^l X_i$ be affine open covers and $i_i : S_i \hookrightarrow \tilde{S}_i$, $i'_i : X_i \hookrightarrow \tilde{X}_i$ be closed embeddings. Let $\tilde{f}_i : \tilde{X}_i \to \tilde{S}_i$ be a lift of the morphism $f_i = f_{|X_i} : X_i \to S_i$. Then, $f_I = f_{|X_I} : X_I = \cap_{i\in I} X_i \to S_I = \cap_{i\in I} S_i$ lift to the morphism

$$\tilde{f}_I = \Pi_{i\in I}\tilde{f}_i : \tilde{X}_I = \Pi_{i\in I}\tilde{X}_i \to \tilde{S}_I = \Pi_{i\in I}\tilde{S}_i$$

Denote by $p_{IJ} : \tilde{S}_J \to \tilde{S}_I$ and $p'_{IJ} : \tilde{X}_J \to \tilde{X}_I$ the projections. Consider for $J \subset I$ the following commutative diagrams

$$D_{IJ} = \begin{tikzcd} S_I \arrow[r, "i_I"] & \tilde{S}_I \\ S_J \arrow[u, "j_{IJ}"] \arrow[r, "i_J"] & \tilde{S}_J \arrow[u, "p_{IJ}"'] \end{tikzcd}, \; D'_{IJ} = \begin{tikzcd} X_I \arrow[r, "i'_I"] & \tilde{X}_I \\ X_J \arrow[u, "j'_{IJ}"] \arrow[r, "i'_J"] & \tilde{X}_J \arrow[u, "p'_{IJ}"'] \end{tikzcd}, \; D_{fI} = \begin{tikzcd} S_I \arrow[r, "i_I"] & \tilde{S}_I \\ X_I \arrow[u, "f_I"] \arrow[r, "i'_I"] & \tilde{X}_I \arrow[u, "\tilde{f}_I"'] \end{tikzcd}$$

We have then following commutative diagram

$$\begin{tikzcd} & X_I \arrow[r, "n'_I"] & \tilde{X}_I & \tilde{X}_I\backslash X_I \arrow[l, "n'_I"'] \\ i'_J : X_J \arrow[ur, "j'_{IJ}"] \arrow[r, "l_{IJ}"] & X_I \times X_I \times \tilde{X}_{J\backslash I} \arrow[u, "p'_{IJ}"] \arrow[r, "n'_I \times I"] & \tilde{X}_J \arrow[u, "p'_{IJ}"] & \tilde{X}_J\backslash X_J \arrow[l, "n'_J"'] \arrow[u, "p'_{IJ}"] \end{tikzcd} .$$

whose square are cartesian. We then have the pullback functor

$$f^* : C_{(2)fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S/(\tilde{S}_I)) \to C_{(2)fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/X/(\tilde{X}_I)),$$
$$((G_I, F), u_{IJ}) \mapsto f^*((G_I, F), u_{IJ}) := (\Gamma^{\vee}_{X_I} \tilde{f}_I^*(G_I, F), \tilde{f}_J^* u_{IJ})$$

with

$$\tilde{f}_J^* u_{IJ} : \Gamma^{\vee}_{X_I} \tilde{f}_I^*(G_I, F) \xrightarrow{\mathrm{ad}(p'^*_{IJ}, p'_{IJ*})(-)} p'_{IJ*} p'^*_{IJ} \Gamma^{\vee}_{X_I} \tilde{f}_I^*(G_I, F) \xrightarrow{T_\sharp(p_{IJ}, n'_I)(-)^{-1})} p'_{IJ*} \Gamma^{\vee}_{X_I \times \tilde{X}_{J\backslash I}} p'^*_{IJ} \tilde{f}_I^*(G_I, F)$$
$$\xrightarrow{p'_{IJ*}\gamma^{\vee}_{X_J}(-)} p'_{IJ*} \Gamma^{\vee}_{X_J} p'^*_{IJ} \tilde{f}_I^*(G_I, F) = p'_{IJ*} \Gamma^{\vee}_{X_J} \tilde{f}_J^* p^*_{IJ}(G_I, F) \xrightarrow{\Gamma^{\vee}_{X_J} \tilde{f}_J^* I(p^*_{IJ}, p_{IJ*})(-,-)(u_{IJ})} \Gamma^{\vee}_{X_J} \tilde{f}_J^*(G_J, F)$$

Let $(G, F) \in C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$. Since, $j_I'^{*} i'_{I*} j_I'^{*} f^*(G, F) = 0$, the morphism $T(D_{fI})(j_I^*(G, F)) : \tilde{f}_I^* i_{I*} j_I^*(G, F) \to i'_{I*} j_I'^{*} f^*(G, F)$ factors trough

$$T(D_{fI})(j_I^*(G, F)) : \tilde{f}_I^* i_{I*} j_I^*(G, F) \xrightarrow{\gamma^{\vee}_{X_I}(-)} \Gamma^{\vee}_{X_I} \tilde{f}_I^* i_{I*} j_I^*(G, F) \xrightarrow{T^{\gamma}(D_{fI})(j_I^*(G,F))} i'_{I*} j_I'^{*} f^*(G, F)$$

We have then, for $(G, F) \in C_{fil}(S)$, the canonical transformation map

$$\begin{tikzcd} f^* T(S/(\tilde{S}_I))(G, F) \arrow[r, "T(f, T(0/I))(G,F)"] \arrow[d, "="] & T(X/(\tilde{X}_I))(f^*(G, F)) \arrow[d, "="] \\ (\Gamma^{\vee}_{X_I} \tilde{f}_I^* i_{I*} j_I^*(G, F), \tilde{f}_J^* I) \arrow[r, "T^{\gamma}(D_{fI})(j_I^*(G,F))"] & (i'_{I*} j_I'^{*} f^*(G, F), I) \end{tikzcd}$$

We have similarly to the algebraic case, we have:

Definition 23. *(i) Let $f : X \to S$ a morphism with $X, S \in \mathrm{AnSp}(\mathbb{C})$. Assume that there exist a factorization $f : X \xrightarrow{i} Y \times S \xrightarrow{p} S$, with $Y \in \mathrm{AnSm}(\mathbb{C})$, $i : X \hookrightarrow Y$ is a closed embedding and p the projection. We then consider*

$$Q(X/S) := p_\sharp \Gamma^{\vee}_X \mathbb{Z}_{Y \times S} := \mathrm{Cone}(\mathbb{Z}((Y \times S)\backslash X/S) \to \mathbb{Z}(Y \times S/S)) \in C(\mathrm{AnSp}(\mathbb{C})^{sm}/S).$$

By definition $Q(X/S)$ is projective since it is a complex of two representative presheaves.

(ii) *Let $f : X \to S$ and $g : T \to S$ two morphism with $X, S, T \in \mathrm{AnSp}(\mathbb{C})$. Assume that there exist a factorization $f : X \xrightarrow{i} Y \times S \xrightarrow{p} S$, with $Y \in \mathrm{AnSm}(\mathbb{C})$, $i : X \hookrightarrow Y$ is a closed embedding and p the projection. We then have the following commutative diagram whose squares are cartesian*

$$\begin{tikzcd} f : X \arrow[r, "i"] & Y \times S \arrow[r, "p"] & S \\ f' : X_T \arrow[u, "g'"] \arrow[r, "i'"] & Y \times T \arrow[u, "g'':=(I\times g)"] \arrow[r, "p'"] & T \arrow[u, "g"] \end{tikzcd}$$

We then have the canonical isomorphism in $C(\mathrm{AnSp}(\mathbb{C})^{sm}/T)$

$$T(f, g, Q) := T_\sharp(g, p)(-)^{-1} \circ T_\sharp(g'', j)(-)^{-1} :$$
$$g^* Q(X/S) := g^* p_\sharp \Gamma^\vee_X \mathbb{Z}_{Y\times S} \xrightarrow{\sim} p'_\sharp \Gamma^\vee_{X_T} \mathbb{Z}_{Y\times T} =: Q(X_T/T)$$

with $j : Y \times S \backslash X \hookrightarrow Y \times S$ the closed embedding.

Let $S \in \mathrm{AnSp}(\mathbb{C})$. Denote for short $\mathrm{AnSp}(\mathbb{C})^{(sm)}/S$ either the category $\mathrm{AnSp}(\mathbb{C})/S$ or the category $\mathrm{AnSp}(\mathbb{C})^{sm}/S$. Denote by

$$p_a : \mathrm{AnSp}(\mathbb{C})^{(sm)}/S \to \mathrm{AnSp}(\mathbb{C})^{(sm)}/S,\ X/S = (X, h) \mapsto (X \times \mathbb{D}^1)/S = (X \times \mathbb{D}^1, h \circ p_X),$$
$$(g : X/S \to X'/S) \mapsto ((g \times I_{\mathbb{D}^1}) : X \times \mathbb{D}^1/S \to X' \times \mathbb{D}^1/S)$$

the projection functor and again by $p_a : \mathrm{AnSp}(\mathbb{C})^{(sm)}/S \to \mathrm{AnSp}(\mathbb{C})^{(sm)}/S$ the corresponding morphism of site.

We now define the $\mathbb{D}^1$ localization property :

Definition 24. *Let $S \in \mathrm{Var}(\mathbb{C})$.*

(i0) *A complex $F \in C(\mathrm{AnSp}(\mathbb{C})^{(sm)}/S)$ is said to be $\mathbb{D}^1$ homotopic if $\mathrm{ad}(p_a^*, p_{a*})(F) : F \to p_{a*}p_a^* F$ is an homotopy equivalence.*

(i) *A complex $F \in C(\mathrm{AnSp}(\mathbb{C})^{(sm)}/S)$ is said to be $\mathbb{D}^1$ invariant if for all $U/S \in \mathrm{AnSp}(\mathbb{C})^{(sm)}/S$,*

$$F(p_U) : F(U/S) \to F(U \times \mathbb{D}^1/S)$$

is a quasi-isomorphism, where $p_U : U \times \mathbb{D}^1 \to U$ is the projection. Obviously, a $\mathbb{D}^1$ homotopic complex is $\mathbb{D}^1$ invariant.

(ii) *A complex $F \in C(\mathrm{AnSp}(\mathbb{C})^{(sm)}/S)$ is said to be $\mathbb{D}^1$ local for the usual topology, if for a (hence every) usu local equivalence $k : F \to G$ with k injective and $G \in C(\mathrm{AnSp}(\mathbb{C})^{(sm)}/S)$ usu fibrant, e.g. $k : F \to E_{usu}(F)$, G is $\mathbb{D}^1$ invariant for all $n \in \mathbb{Z}$.*

(iii) *A morphism $m : F \to G$ with $F, G \in C(\mathrm{AnSp}(\mathbb{C})^{(sm)}/S)$ is said to an $(\mathbb{D}^1, usu)$ local equivalence if for all $H \in C(\mathrm{AnSp}(\mathbb{C})^{(sm)}/S)$ which is $\mathbb{A}^1$ local for the etale topology*

$$\mathrm{Hom}(L(m), E_{usu}(H)) : \mathrm{Hom}(L(G), E_{usu}(H)) \to \mathrm{Hom}(L(F), E_{usu}(H))$$

is a quasi-isomorphism.

Proposition 27. *A morphism $m : F \to G$ with $F, G \in C(\mathrm{AnSp}(\mathbb{C})^{(sm)}/S)$ is an $(\mathbb{D}^1, et)$ local equivalence if and only if there exists*

$$\{X_{1,\alpha}/S, \alpha \in \Lambda_1\}, \ldots, \{X_{r,\alpha}/S, \alpha \in \Lambda_r\} \subset \mathrm{AnSp}(\mathbb{C})^{(sm)}/S$$

such that we have in $\mathrm{Ho}_{et}(C(\mathrm{Var}(\mathbb{C})^{(sm)}/S))$

$$\mathrm{Cone}(m) \xrightarrow{\sim} \mathrm{Cone}(\oplus_{\alpha\in\Lambda_1} \mathrm{Cone}(\mathbb{Z}(X_{1,\alpha}\times\mathbb{D}^1/S) \to \mathbb{Z}(X_{1,\alpha}/S)) \to \cdots \to \oplus_{\alpha\in\Lambda_r} \mathrm{Cone}(\mathbb{Z}(X_{r,\alpha}\times\mathbb{D}^1/S) \to \mathbb{Z}(X_{r,\alpha}/S)))$$

Proof. Standard. □

Definition-Proposition 8. *Let $S \in \mathrm{AnSp}(\mathbb{C})$*

(i) With the weak equivalence the $(\mathbb{D}^1, usu)$ local equivalence and the fibration the epimorphism with $\mathbb{D}^1_S$ local and usu fibrant kernels gives a model structure on $C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$: the left bousfield localization of the projective model structure of $C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$. We call it the $(\mathbb{D}^1, usu)$ projective model structure.

(ii) With the weak equivalence the $(\mathbb{D}^1, usu)$ local equivalence and the fibration the epimorphism with $\mathbb{D}^1_S$ local and usu fibrant kernels gives a model structure on $C(\mathrm{AnSp}(\mathbb{C})/S)$: the left bousfield localization of the projective model structure of $C(\mathrm{AnSp}(\mathbb{C})/S)$. We call it the $(\mathbb{D}^1, usu)$ projective model structure.

Proof. Similar to the proof of definition-proposition 5. □

Proposition 28. *Let $g : T \to S$ a morphism with $T, S \in \mathrm{AnSp}(\mathbb{C})$.*

(i) The adjonction $(g^*, g_*) : C(\mathrm{AnSp}(\mathbb{C})^{sm}/S) \leftrightarrows C(\mathrm{AnSp}(\mathbb{C})^{sm}/T)$ is a Quillen adjonction for the $(\mathbb{D}^1, et)$ projective model structure.

(i)' Let $h : U \to S$ a smooth morphism with $U, S \in \mathrm{AnSp}(\mathbb{C})$. The adjonction $(h_\sharp, h^*) : C(\mathrm{AnSp}(\mathbb{C})^{sm}/U) \leftrightarrows C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$ is a Quillen adjonction for the $(\mathbb{D}^1, et)$ projective model structure.

(i)" The functor $g^* : C(\mathrm{AnSp}(\mathbb{C})^{sm}/S) \to C(\mathrm{AnSp}(\mathbb{C})^{sm}/T)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence usu local to equivalence usu local, sends $(\mathbb{D}^1, et)$ local equivalence to $(\mathbb{D}^1, et)$ local equivalence.

(ii) The adjonction $(g^*, g_*) : C(\mathrm{AnSp}(\mathbb{C})/S) \leftrightarrows C(\mathrm{AnSp}(\mathbb{C})/T)$ is a Quillen adjonction for the $(\mathbb{D}^1, et)$ projective model structure (see definition 5).

(ii)' The adjonction $(g_\sharp, g^*) : C(\mathrm{AnSp}(\mathbb{C})/T) \leftrightarrows C(\mathrm{AnSp}(\mathbb{C})/S)$ is a Quillen adjonction for the $(\mathbb{D}^1, et)$ projective model structure (see definition 5).

(ii)" The functor $g^* : C(\mathrm{AnSp}(\mathbb{C})/S) \to C(\mathrm{AnSp}(\mathbb{C})/T)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence usu local to equivalence usu local, sends $(\mathbb{D}^1, et)$ local equivalence to $(\mathbb{D}^1, et)$ local equivalence.

Proof. Similar to the proof of proposition 18. □

Proposition 29. *Let $S \in \mathrm{AnSp}(\mathbb{C})$.*

(i) The adjonction $(\rho_S^*, \rho_{S*}) : C(\mathrm{AnSp}(\mathbb{C})^{sm}/S) \leftrightarrows C(\mathrm{AnSp}(\mathbb{C})/S)$ is a Quillen adjonction for the $(\mathbb{D}^1, et)$ projective model structure.

(ii) The functor $\rho_{S*} : C(\mathrm{AnSp}(\mathbb{C})/S) \to C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence usu local to equivalence usu local, sends $(\mathbb{D}^1, usu)$ local equivalence to $(\mathbb{D}^1, usu)$ local equivalence.

Proof. Similar to the proof of proposition 19. □

2.9 Presheaves on the big analytical site of pairs

We recall the definition given in subsection 5.1 : For $S \in \mathrm{AnSp}(\mathbb{C})$, $\mathrm{AnSp}(\mathbb{C})^2/S := \mathrm{AnSp}(\mathbb{C})^2/(S,S)$ is by definition (see subsection 2.1) the category whose set of objects is

$$(\mathrm{AnSp}(\mathbb{C})^2/S)^0 := \{((X,Z),h), h : X \to S,\ Z \subset X \text{ closed }\} \subset \mathrm{AnSp}(\mathbb{C})/S \times \mathrm{Top}$$

and whose set of morphisms between $(X_1,Z_1)/S = ((X_1,Z_1),h_1), (X_1,Z_1)/S = ((X_2,Z_2),h_2) \in \mathrm{AnSp}(\mathbb{C})^2/S$ is the subset

$$\mathrm{Hom}_{\mathrm{AnSp}(\mathbb{C})^2/S}((X_1,Z_1)/S,(X_2,Z_2)/S) := \\ \left\{(f : X_2 \to X_2),\ \text{s.t. } h_1 \circ f = h_2 \text{ and } Z_1 \subset f^{-1}(Z_2)\right\} \subset \mathrm{Hom}_{\mathrm{AnSp}(\mathbb{C})}(X_1,X_2)$$

The category $\mathrm{AnSp}(\mathbb{C})^2$ admits fiber products : $(X_1,Z_1)\times_{(S,Z)}(X_2,Z_2) = (X_1 \times_S X_2, Z_1 \times_Z Z_2)$. In particular, for $f : T \to S$ a morphism with $S,T \in \mathrm{AnSp}(\mathbb{C})$, we have the pullback functor

$$P(f) : \mathrm{AnSp}(\mathbb{C})^2/S \to \mathrm{AnSp}(\mathbb{C})^2/T, P(f)((X,Z)/S) := (X_T,Z_T)/T, P(f)(g) := (g \times_S f)$$

and we note again $P(f) : \mathrm{AnSp}(\mathbb{C})^2/T \to \mathrm{AnSp}(\mathbb{C})^2/S$ the corresponding morphism of sites.

We will consider in the construction of the filtered De Rham realization functor the full subcategory $\mathrm{AnSp}(\mathbb{C})^{2,sm}/S \subset \mathrm{AnSp}(\mathbb{C})^2/S$ such that the first factor is a smooth morphism : We will also consider, in order to obtain a complex of D modules in the construction of the filtered De Rham realization functor, the restriction to the full subcategory $\mathrm{AnSp}(\mathbb{C})^{2,pr}/S \subset \mathrm{AnSp}(\mathbb{C})^2/S$ such that the first factor is a projection :

Definition 25. *(i) Let $S \in \mathrm{AnSp}(\mathbb{C})$. We denote by*

$$\rho_S : \mathrm{AnSp}(\mathbb{C})^{2,sm}/S \hookrightarrow \mathrm{AnSp}(\mathbb{C})^2/S$$

the full subcategory consisting of the objects $(U,Z)/S = ((U,Z),h) \in \mathrm{AnSp}(\mathbb{C})^2/S$ such that the morphism $h : U \to S$ is smooth. That is, $\mathrm{AnSp}(\mathbb{C})^{2,sm}/S$ is the category

- *whose objects are $(U,Z)/S = ((U,Z),h)$, with $U \in \mathrm{AnSp}(\mathbb{C})$, $Z \subset U$ a closed subset, and $h : U \to S$ a smooth morphism,*
- *whose morphisms $g : (U,Z)/S = ((U,Z),h_1) \to (U',Z')/S = ((U',Z'),h_2)$ is a morphism $g : U \to U'$ of complex algebraic varieties such that $Z \subset g^{-1}(Z')$ and $h_2 \circ g = h_1$.*

We denote again $\rho_S : \mathrm{AnSp}(\mathbb{C})^2/S \to \mathrm{AnSp}(\mathbb{C})^{2,sm}/S$ the associated morphism of site. We have

$$r^s(S) : \mathrm{AnSp}(\mathbb{C})^2 \xrightarrow{r(S):=r(S,S)} \mathrm{AnSp}(\mathbb{C})^2/S \xrightarrow{\rho_S} \mathrm{AnSp}(\mathbb{C})^{2,sm}/S$$

the composite morphism of site.

(ii) Let $S \in \mathrm{AnSp}(\mathbb{C})$. We will consider the full subcategory

$$\mu_S : \mathrm{AnSp}(\mathbb{C})^{2,pr}/S \hookrightarrow \mathrm{AnSp}(\mathbb{C})^2/S$$

whose subset of object consist of those whose morphism is a projection to S :

$$(\mathrm{AnSp}(\mathbb{C})^{2,pr}/S)^0 := \{((Y \times S, X),p),\ Y \in \mathrm{AnSp}(\mathbb{C}),\ p : Y \times S \to S \text{ the projection}\} \subset (\mathrm{AnSp}(\mathbb{C})^2/S)^0.$$

(iii) We will consider the full subcategory

$$\mu_S : (\mathrm{AnSp}(\mathbb{C})^{2,smpr}/S) \hookrightarrow \mathrm{AnSp}(\mathbb{C})^{2,sm}/S$$

whose subset of object consist of those whose morphism is a smooth projection to S :

$$(\mathrm{AnSp}(\mathbb{C})^{2,smpr}/S)^0 := \{((Y \times S, X),p),\ Y \in \mathrm{SmVar}(\mathbb{C}),\ p : Y \times S \to S \text{ the projection}\} \subset (\mathrm{AnSp}(\mathbb{C})^2/S)^0$$

For $f : T \to S$ a morphism with $T, S \in \mathrm{AnSp}(\mathbb{C})$, we have by definition, the following commutative diagram of sites

$$\begin{tikzcd}
\mathrm{AnSp}(\mathbb{C})^2/T \arrow[rr, "\mu_T"] \arrow[dr, "\rho_T"] \arrow[dd, "P(f)"'] & & \mathrm{AnSp}(\mathbb{C})^{2,pr}/T \arrow[dr, "\rho_T"] \arrow[dd, "P(f)"] & \\
& \mathrm{AnSp}(\mathbb{C})^{2,sm}/T \arrow[rr, "\mu_T"] \arrow[dd, "P(f)"] & & \mathrm{AnSp}(\mathbb{C})^{2,smpr}/T \arrow[dd, "P(f)"] \\
\mathrm{AnSp}(\mathbb{C})^2/S \arrow[rr, "\mu_S"] \arrow[dr, "\rho_S"] & & \mathrm{AnSp}(\mathbb{C})^{2,pr}/S \arrow[dr, "\rho_S"] & \\
& \mathrm{AnSp}(\mathbb{C})^{2,sm}/S \arrow[rr, "\mu_S"] & & \mathrm{AnSp}(\mathbb{C})^{2,smpr}/S
\end{tikzcd} . \quad (35)$$

Recall we have (see subsection 2.1), for $S \in \mathrm{Var}(\mathbb{C})$, the graph functor

$$\mathrm{Gr}_S^{12} : \mathrm{AnSp}(\mathbb{C})/S \to \mathrm{AnSp}(\mathbb{C})^{2,pr}/S, \; X/S \mapsto \mathrm{Gr}_S^{12}(X/S) := (X \times S, X)/S,$$
$$(g : X/S \to X'/S) \mapsto \mathrm{Gr}_S^{12}(g) := (g \times I_S : (X \times S, X) \to (X' \times S, X'))$$

For $f : T \to S$ a morphism with $T, S \in \mathrm{AnSp}(\mathbb{C})$, we have by definition, the following commutative diagram of sites

$$\begin{tikzcd}
\mathrm{AnSp}(\mathbb{C})^{2,pr}/T \arrow[rr, "\mathrm{Gr}_T^{12}"] \arrow[dr, "\rho_T"] \arrow[dd, "P(f)"'] & & \mathrm{AnSp}(\mathbb{C})/T \arrow[dr, "\rho_T"] \arrow[dd, "P(f)"] & \\
& \mathrm{AnSp}(\mathbb{C})^{2,smpr}/T \arrow[rr, "\mathrm{Gr}_T^{12}"] \arrow[dd, "P(f)"] & & \mathrm{AnSp}(\mathbb{C})^{sm}/T \arrow[dd, "P(f)"] \\
\mathrm{AnSp}(\mathbb{C})^{2,pr}/S \arrow[rr, "\mathrm{Gr}_S^{12}"] \arrow[dr, "\rho_S"] & & \mathrm{AnSp}(\mathbb{C})/S \arrow[dr, "\rho_S"] & \\
& \mathrm{AnSp}(\mathbb{C})^{2,sm}/S \arrow[rr, "\mathrm{Gr}_S^{12}"] & & \mathrm{AnSp}(\mathbb{C})^{sm}/S
\end{tikzcd} . \quad (36)$$

where we recall that $P(f)((X, Z)/S) := ((X_T, Z_T)/T)$, since smooth morphisms are preserved by base change.

As usual, we denote by

$$(f^*, f_*) := (P(f)^*, P(f)_*) : C(\mathrm{AnSp}(\mathbb{C})^{2,sm}/S) \to C(\mathrm{AnSp}(\mathbb{C})^{2,sm}/T)$$

the adjonction induced by $P(f) : \mathrm{AnSp}(\mathbb{C})^{2,sm}/T \to \mathrm{AnSp}(\mathbb{C})^{2,sm}/S$. Since the colimits involved in the definition of $f^* = P(f)^*$ are filtered, f^* also preserve monomorphism. Hence, we get an adjonction

$$(f^*, f_*) : C_{fil}(\mathrm{AnSp}(\mathbb{C})^{2,sm}/S) \leftrightarrows C_{fil}(\mathrm{AnSp}(\mathbb{C})^{2,sm}/T), \; f^*(G, F) := (f^*G, f^*F)$$

For $S \in \mathrm{AnSp}(\mathbb{C})$, we denote by $\mathbb{Z}_S := \mathbb{Z}((S, S)/(S, S)) \in \mathrm{PSh}(\mathrm{AnSp}(\mathbb{C})^{2,sm}/S)$ the constant presheaf By Yoneda lemma, we have for $F \in C(\mathrm{AnSp}(\mathbb{C})^{2,sm}/S)$, $\mathcal{H}om(\mathbb{Z}_S, F) = F$.

For $h : U \to S$ a smooth morphism with $U, S \in \mathrm{AnSp}(\mathbb{C})$, $P(h) : \mathrm{AnSp}(\mathbb{C})^{2,sm}/S \to \mathrm{AnSp}(\mathbb{C})^{2,sm}/U$ admits a left adjoint

$$C(h) : \mathrm{AnSp}(\mathbb{C})^{2,sm}/U \to \mathrm{AnSp}(\mathbb{C})^{2,sm}/S, \; C(h)((U', Z'), h') = ((U', Z'), h \circ h').$$

Hence $h^* : C(\mathrm{AnSp}(\mathbb{C})^{2,sm}/S) \to C(\mathrm{AnSp}(\mathbb{C})^{2,sm}/U)$ admits a left adjoint

$$h_\sharp : C(\mathrm{AnSp}(\mathbb{C})^{2,sm}/U) \to C(\mathrm{AnSp}(\mathbb{C})^{2,sm}/S), \ F \mapsto (h_\sharp F : ((U,Z),h_0) \mapsto \lim_{((U',Z'),h\circ h')\to((U,Z),h_0)} F((U',Z')/U))$$

For $F^\bullet \in C(\mathrm{AnSp}(\mathbb{C})^{2,sm}/S)$ and $G^\bullet \in C(\mathrm{AnSp}(\mathbb{C})^{2,sm}/U)$, we have the adjonction maps

$$\mathrm{ad}(h_\sharp, h^*)(G^\bullet) : G^\bullet \to h^* h_\sharp G^\bullet \ , \ \mathrm{ad}(h_\sharp, h^*)(F^\bullet) : h_\sharp h^* F^\bullet \to F^\bullet.$$

For a smooth morphism $h : U \to S$, with $U, S \in \mathrm{AnSp}(\mathbb{C})$, we have the adjonction isomorphism, for $F \in C(\mathrm{AnSp}(\mathbb{C})^{2,sm}/U)$ and $G \in C(\mathrm{AnSp}(\mathbb{C})^{2,sm}/S)$,

$$I(h_\sharp, h^*)(F,G) : \mathcal{H}om^\bullet(h_\sharp F, G) \xrightarrow{\sim} h_* \mathcal{H}om^\bullet(F, h^* G). \tag{37}$$

For a commutative diagram in $\mathrm{AnSp}(\mathbb{C})$:

$$D = \begin{tikzcd} V \arrow[r, "g_2"] \arrow[d, "h_2"] & U \arrow[d, "h_1"] \\ T \arrow[r, "g_1"] & S \end{tikzcd} ,$$

where h_1 and h_2 are smooth, we denote by, for $F^\bullet \in C(\mathrm{AnSp}(\mathbb{C})^{2,sm}/U)$,

$$T_\sharp(D)(F^\bullet) : h_{2\sharp} g_2^* F^\bullet \to g_1^* h_{1\sharp} F^\bullet$$

the canonical map given by adjonction. If D is cartesian with $h_1 = h$, $g_1 = g$ $f_2 = h' : U_T \to T$, $g' : U_T \to U$,

$$T_\sharp(D)(F^\bullet) =: T_\sharp(g,h)(F) : h'_\sharp g'^* F^\bullet \xrightarrow{\sim} g^* h_\sharp F^\bullet$$

is an isomorphism.

We have the support section functors of a closed embedding $i : Z \hookrightarrow S$ for presheaves on the big analytical site of pairs.

Definition 26. *Let $i : Z \hookrightarrow S$ be a closed embedding with $S, Z \in \mathrm{AnSp}(\mathbb{C})$ and $j : S\backslash Z \hookrightarrow S$ be the open complementary subset.*

(i) We define the functor

$$\Gamma_Z : C(\mathrm{AnSp}(\mathbb{C})^{2,sm}/S) \to C(\mathrm{AnSp}(\mathbb{C})^{2,sm}/S), \ G^\bullet \mapsto \Gamma_Z G^\bullet := \mathrm{Cone}(\mathrm{ad}(j^*, j_*)(G^\bullet) : G^\bullet \to j_* j^* G^\bullet)[-1],$$

so that there is then a canonical map $\gamma_Z(G^\bullet) : \Gamma_Z G^\bullet \to G^\bullet$.

(ii) We have the dual functor of (i) :

$$\Gamma_Z^\vee : C(\mathrm{AnSp}(\mathbb{C})^{2,sm}/S) \to C(\mathrm{AnSp}(\mathbb{C})^{2,sm}/S), \ F \mapsto \Gamma_Z^\vee(F^\bullet) := \mathrm{Cone}(\mathrm{ad}(j_\sharp, j^*)(G^\bullet) : j_\sharp j^* G^\bullet \to G^\bullet),$$

together with the canonical map $\gamma_Z^\vee(G) : F \to \Gamma_Z^\vee(G)$.

(iii) For $F, G \in C(\mathrm{AnSp}(\mathbb{C})^{2,sm}/S)$, we denote by

$$I(\gamma, hom)(F,G) := (I, I(j_\sharp, j^*)(F,G)^{-1}) : \Gamma_Z \mathcal{H}om(F,G) \xrightarrow{\sim} \mathcal{H}om(\Gamma_Z^\vee F, G)$$

the canonical isomorphism given by adjonction.

Note that we have similarly for $i : Z \hookrightarrow S$, $i' : Z' \hookrightarrow Z$ closed embeddings, $g : T \to S$ a morphism with $T, S, Z \in \mathrm{AnSp}(\mathbb{C})$ and $F \in C(\mathrm{AnSp}(\mathbb{C})^{2,sm}/S)$, the canonical maps in $C(\mathrm{AnSp}(\mathbb{C})^{2,sm}/S)$

- $T(g,\gamma)(F) : g^* \Gamma_Z F \xrightarrow{\sim} \Gamma_{Z \times_S T} g^* F$, $T(g, \gamma^\vee)(F) : \Gamma^\vee_{Z \times_S T} g^* F \xrightarrow{\sim} g^* \Gamma_Z F$

- $T(Z'/Z,\gamma)(F) : \Gamma_{Z'}F \to \Gamma_Z F$, $T(Z'/Z,\gamma^\vee)(F) : \Gamma^\vee_Z F \to \Gamma^\vee_{Z'} F$

but we will not use them in this article.

We now define the usual topology on $\mathrm{AnSp}(\mathbb{C})^2/S$.

Definition 27. *Let $S \in \mathrm{AnSp}(\mathbb{C})$.*

(i) Denote by τ a topology on $\mathrm{AnSp}(\mathbb{C})$, e.g. the usual topology. The τ covers in $\mathrm{AnSp}(\mathbb{C})^2/S$ of $(X,Z)/S$ are the families of morphisms

$$\{(c_i : (U_i, Z\times_X U_i)/S \to (X,Z)/S)_{i\in I}, \text{ with } (c_i : U_i \to X)_{i\in I}\, \tau \text{ cover of} X \text{ in } \mathrm{AnSp}(\mathbb{C})\}$$

(ii) Denote by τ the usual or the etale topology on $\mathrm{AnSp}(\mathbb{C})$. The τ covers in $\mathrm{AnSp}(\mathbb{C})^{2,sm}/S$ of $(U,Z)/S$ are the families of morphisms

$$\{(c_i : (U_i, Z\times_U U_i)/S \to (U,Z)/S)_{i\in I}, \text{ with } (c_i : U_i \to U)_{i\in I}\, \tau \text{ cover of} U \text{ in } \mathrm{AnSp}(\mathbb{C})\}$$

(iii) Denote by τ the usual or the etale topology on $\mathrm{AnSp}(\mathbb{C})$. The τ covers in $\mathrm{AnSp}(\mathbb{C})^{2,(sm)pr}/S$ of $(Y\times S,Z)/S$ are the families of morphisms

$$\{(c_i\times I_S : (U_i\times S, Z\times_{Y\times S} U_i\times S)/S \to (Y\times S,Z)/S)_{i\in I}, \text{ with } (c_i : U_i \to Y)_{i\in I}\, \tau \text{ cover of} Y \text{ in } \mathrm{AnSp}(\mathbb{C})\}$$

Let $S \in \mathrm{AnSp}(\mathbb{C})$. Denote by τ the usual topology on $\mathrm{AnSp}(\mathbb{C})$. In particular, denoting $a_\tau : \mathrm{PSh}(\mathrm{AnSp}(\mathbb{C})^{2,(sm)}/S) \to \mathrm{Shv}(\mathrm{AnSp}(\mathbb{C})^{2,(sm)}/S)$ and $a_\tau : \mathrm{PSh}(\mathrm{AnSp}(\mathbb{C})^{2,(sm)pr}/S) \to \mathrm{Shv}(\mathrm{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ the sheaftification functors,

- a morphism $\phi : F \to G$, with $F,G \in C(\mathrm{AnSp}(\mathbb{C})^{2(sm)}/S)$, is a τ local equivalence if $a_\tau H^n\phi : a_\tau H^n F \to a_\tau H^n G$ is an isomorphism, a morphism $\phi : F \to G$, with $F,G \in C(\mathrm{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$, is a τ local equivalence if $a_\tau H^n\phi : a_\tau H^n F \to a_\tau H^n G$ is an isomorphism,

- $F^\bullet \in C(\mathrm{AnSp}(\mathbb{C})^{2,(sm)}/S)$ is τ fibrant if for all $(U,Z)/S \in \mathrm{AnSp}(\mathbb{C})^{2,(sm)}/S$ and all τ covers $(c_i : (U_i, Z\times_U U_i)/S \to (U,Z)/S)_{i\in I}$ of $(U,Z)/S$,

  $$F^\bullet(c_i) : F^\bullet((U,Z)/S) \to \mathrm{Tot}(\oplus_{cardI=\bullet} F^\bullet((U_I, Z\times_U U_I)/S))$$

  is a quasi-isomorphism of complexes of abelian groups ; $F^\bullet \in C(\mathrm{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ is τ fibrant if for all $(Y\times S,Z)/S \in \mathrm{AnSp}(\mathbb{C})^{2,smpr}/S$ and all τ covers $(c_i\times I_S : (U_i\times S, Z\times_{Y\times S} U_i\times S)/S \to (Y\times S,Z)/S)_{i\in I}$ of $(Y\times S,Z)/S$,

  $$F^\bullet(c_i\times I_S) : F^\bullet((Y\times S,Z)/S) \to \mathrm{Tot}(\oplus_{cardI=\bullet} F^\bullet((U_I\times S, Z_I\times_Y U_I)/S))$$

  is a quasi-isomorphism of complexes of abelian groups,

- a morphism $\phi : (G_1,F) \to (G_2,F)$, with $(G_1,F),(G_2,F) \in C_{fil}(\mathrm{AnSp}(\mathbb{C})^{2,(sm)}/S)$, is a filtered τ local equivalence if for all $n,p \in \mathbb{Z}$,

  $$a_\tau H^n \mathrm{Gr}^p_F \phi : a_\tau H^n \mathrm{Gr}^p_F(G_1,F) \to a_\tau H^n \mathrm{Gr}^p_F(G_2,F)$$

  is an isomorphism of sheaves on $\mathrm{AnSp}(\mathbb{C})^{2,(sm)}/S$; a morphism $\phi : (G_1,F) \to (G_2,F)$, with $(G_1,F),(G_2,F) \in C_{fil}(\mathrm{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$, is an filtered τ local equivalence if for all $n,p \in \mathbb{Z}$

  $$a_\tau H^n \mathrm{Gr}^p_F \phi : a_\tau H^n \mathrm{Gr}^p_F(G_1,F) \to a_\tau H^n \mathrm{Gr}^p_F(G_2,F)$$

  is an isomorphism of sheaves on $\mathrm{AnSp}(\mathbb{C})^{2,(sm)pr}/S$,

- a morphism $\phi : (G_1, F) \to (G_2, F)$, with $(G_1, F), (G_2, F) \in C_{fil}(\mathrm{AnSp}(\mathbb{C})^{2,(sm)}/S)$, is an r-filtered τ local equivalence if there exists an r-filtered homotopy equivalence

$$(h, \phi, \phi') : (G_1, F)[1] \to (G_2, F)$$

such that $\phi' : (G_1, F) \to (G_2, F)$ is a filtered τ local equivalence ; a morphism $\phi : (G_1, F) \to (G_2, F)$, with $(G_1, F), (G_2, F) \in C_{fil}(\mathrm{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$, is an r-filtered τ local equivalence if there exists an r-filtered homotopy equivalence

$$(h, \phi, \phi') : (G_1, F)[1] \to (G_2, F)$$

such that $\phi' : (G_1, F) \to (G_2, F)$ is a filtered τ local equivalence,

- $(F^\bullet, F) \in C_{fil}(\mathrm{AnSp}(\mathbb{C})^{2,(sm)}/S)$ is filtered τ fibrant for all $(U, Z)/S \in \mathrm{AnSp}(\mathbb{C})^{2,(sm)}/S$ and all τ covers $(c_i : (U_i, Z \times_U U_i)/S \to (U, Z)/S)_{i \in I}$ of $(U, Z)/S$,

$$H^n \operatorname{Gr}_F^p(F^\bullet, F)(c_i) : H^n \operatorname{Gr}_F^p(F^\bullet, F)((U, Z)/S) \to \\ H^n \operatorname{Gr}_F^p(\mathrm{Tot}(\oplus_{cardI=\bullet}(F^\bullet, F)((U_I, Z \times_U U_I)/S)))$$

is an isomorphism of abelian groups for all $n, p \in \mathbb{Z}$; $(F^\bullet, F) \in C_{fil}(\mathrm{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ is filtered τ fibrant for all $(Y \times S, Z)/S \in \mathrm{AnSp}(\mathbb{C})^{2,(sm)pr}/S$ and all τ covers $(c_i \times I_S : (U_i \times S, Z \times_{Y \times S} U_i \times S)/S \to (Y \times S, Z)/S)_{i \in I}$ of $(Y \times S, Z)/S$,

$$H^n \operatorname{Gr}_F^p(F^\bullet, F)(c_i \times I_S) : H^n \operatorname{Gr}_F^p(F^\bullet, F)((Y \times S, Z)/S) \to \\ H^n \operatorname{Gr}_F^p(\mathrm{Tot}(\oplus_{cardI=\bullet}(F^\bullet, F)((U_I \times S, Z \times_Y U_I)/S)))$$

is an isomorphism of abelian groups for all $n, p \in \mathbb{Z}$.

Will now define the $\mathbb{D}^1$ local property on $\mathrm{AnSp}(\mathbb{C})^2/S$. Let $S \in \mathrm{AnSp}(\mathbb{C})$. Denote for short $\mathrm{AnSp}(\mathbb{C})^{2,(sm)}/S$ either the category $\mathrm{AnSp}(\mathbb{C})^2/S$ or the category $\mathrm{AnSp}(\mathbb{C})^{2,sm}/S$. Denote by

$$p_a : \mathrm{AnSp}(\mathbb{C})^{2,(sm)}/S \to \mathrm{AnSp}(\mathbb{C})^{2,(sm)}/S, \\ (X, Z)/S = ((X, Z), h) \mapsto (X \times \mathbb{D}^1, Z \times \mathbb{D}^1)/S = ((X \times \mathbb{D}^1, Z \times \mathbb{D}^1, h \circ p_X), \\ (g : (X, Z)/S \to (X', Z')/S) \mapsto ((g \times I_{\mathbb{D}^1}) : (X \times \mathbb{D}^1, Z \times \mathbb{D}^1)/S \to (X' \times \mathbb{D}^1, Z' \times \mathbb{D}^1)/S)$$

the projection functor and again by $p_a : \mathrm{AnSp}(\mathbb{C})^{2,(sm)}/S \to \mathrm{AnSp}(\mathbb{C})^{2,(sm)}/S$ the corresponding morphism of site. Denote for short $\mathrm{AnSp}(\mathbb{C})^{2,(sm)pr}/S$ either the category $\mathrm{AnSp}(\mathbb{C})^{2,pr}/S$ or the category $\mathrm{AnSp}(\mathbb{C})^{2,smpr}/S$. Denote by

$$p_a : \mathrm{AnSp}(\mathbb{C})^{2,(sm)pr}/S \to \mathrm{AnSp}(\mathbb{C})^{2,(sm)pr}/S, \\ (Y \times S, Z)/S = ((Y \times S, Z), p_S) \mapsto (Y \times S \times \mathbb{D}^1, Z \times \mathbb{D}^1)/S = ((Y \times S \times \mathbb{D}^1, Z \times \mathbb{D}^1, p_S \circ p_{Y \times S}), \\ (g : (Y \times S, Z)/S \to (Y' \times S, Z')/S) \mapsto ((g \times I_{\mathbb{D}^1}) : (Y \times S \times \mathbb{D}^1, Z \times \mathbb{D}^1)/S \to (Y' \times S \times \mathbb{D}^1, Z' \times \mathbb{D}^1)/S)$$

the projection functor and again by $p_a : \mathrm{AnSp}(\mathbb{C})^{2,(sm)pr}/S \to \mathrm{AnSp}(\mathbb{C})^{2,(sm)pr}/S$ the corresponding morphism of site.

Definition 28. *Let $S \in \mathrm{AnSp}(\mathbb{C})$.*

(i0) A complex $F \in C(\mathrm{AnSp}(\mathbb{C})^{2,(sm)}/S)$, is said to be $\mathbb{D}^1$ homotopic if $\mathrm{ad}(p_a^*, p_{a*})(F) : F \to F$ is an homotopy equivalence.

(i) A complex $F \in C(\mathrm{AnSp}(\mathbb{C})^{2,(sm)}/S)$, is said to be $\mathbb{D}^1$ invariant if for all $(X, Z)/S \in \mathrm{AnSp}(\mathbb{C})^{2,(sm)}/S$

$$F(p_X) : F((X, Z)/S) \to F((X \times \mathbb{D}^1, (Z \times \mathbb{D}^1))/S)$$

is a quasi-isomorphism, where $p_X : (X \times \mathbb{D}^1, (Z \times \mathbb{D}^1)) \to (X, Z)$ is the projection.

(i0)' Similarly, a complex $F \in C(\mathrm{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$, is said to be $\mathbb{D}^1$ homotopic if $\mathrm{ad}(p_a^*, p_{a*})(F) : F \to F$ is an homotopy equivalence.

(i)' Similarly, a complex $F \in C(\mathrm{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ is said to be $\mathbb{D}^1$ invariant if for all $(Y \times S, Z)/S \in \mathrm{AnSp}(\mathbb{C})^{2,(sm)pr}/S$

$$F(p_{Y\times S}) : F((Y \times S, Z)/S) \to F((Y \times S \times \mathbb{D}^1, (Z \times \mathbb{D}^1))/S)$$

is a quasi-isomorphism

(ii) A complex $F \in C(\mathrm{AnSp}(\mathbb{C})^{2,(sm)}/S)$ is said to be $\mathbb{D}^1$ local for the τ topology induced on $\mathrm{AnSp}(\mathbb{C})^2/S$, if for an (hence every) τ local equivalence $k : F \to G$ with k injective and $G \in C(\mathrm{AnSp}(\mathbb{C})^{2,(sm)}/S)$ τ fibrant, (e.g. $k : F \to E_\tau(F)$), G is $\mathbb{D}^1$ invariant.

(ii)' Similarly, a complex $F \in C(\mathrm{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ is said to be $\mathbb{D}^1$ local for the τ topology induced on $\mathrm{AnSp}(\mathbb{C})^2/S$, if for an (hence every) τ local equivalence $k : F \to G$ with k injective and $G \in C(\mathrm{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ τ fibrant, e.g. $k : F \to E_\tau(F)$, G is $\mathbb{D}^1$ invariant.

(iii) A morphism $m : F \to G$ with $F, G \in C(\mathrm{AnSp}(\mathbb{C})^{2,(sm)}/S)$ is said to an $(\mathbb{D}^1, usu)$ local equivalence if for all $H \in C(\mathrm{AnSp}(\mathbb{C})^{2,(sm)}/S)$ which is $\mathbb{D}^1$ local for the usual topology

$$\mathrm{Hom}(L(m), E_{usu}(H)) : \mathrm{Hom}(L(G), E_{usu}(H)) \to \mathrm{Hom}(L(F), E_{usu}(H))$$

is a quasi-isomorphism.

(iii)' Similarly, a morphism $m : F \to G$ with $F, G \in C(\mathrm{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ is said to be an $(\mathbb{D}^1, usu)$ local equivalence if for all $H \in C(\mathrm{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ which is $\mathbb{D}^1$ local for the usual topology

$$\mathrm{Hom}(L(m), E_{usu}(H)) : \mathrm{Hom}(L(G), E_{usu}(H)) \to \mathrm{Hom}(L(F), E_{usu}(H))$$

is a quasi-isomorphism.

Proposition 30. *(i) A morphism $m : F \to G$ with $F, G \in C(\mathrm{AnSp}(\mathbb{C})^{2,(sm)}/S)$ is an $(\mathbb{D}^1, usu)$ local equivalence if and only if there exists*

$$\{(X_{1,\alpha}, Z_{1,\alpha})/S, \alpha \in \Lambda_1\}, \ldots, \{(X_{r,\alpha}, Z_{r,\alpha})/S, \alpha \in \Lambda_r\} \subset \mathrm{AnSp}(\mathbb{C})^{2,(sm)}/S$$

such that we have in $\mathrm{Ho}_{et}(C(\mathrm{AnSp}(\mathbb{C})^{2,(sm)}/S))$

$$\begin{aligned}\mathrm{Cone}(m) \xrightarrow{\sim} &\mathrm{Cone}(\oplus_{\alpha\in\Lambda_1} \mathrm{Cone}(\mathbb{Z}((X_{1,\alpha} \times \mathbb{D}^1, Z_{1,\alpha} \times \mathbb{D}^1)/S) \to \mathbb{Z}((X_{1,\alpha}, Z_{1,\alpha})/S)) \\ &\to \cdots \to \oplus_{\alpha\in\Lambda_1} \mathrm{Cone}(\mathbb{Z}((X_{1,\alpha} \times \mathbb{D}^1, Z_{1,\alpha} \times \mathbb{D}^1)/S) \to \mathbb{Z}((X_{1,\alpha}, Z_{1,\alpha})/S)))\end{aligned}$$

(ii) A morphism $m : F \to G$ with $F, G \in C(\mathrm{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ is an $(\mathbb{D}^1, usu)$ local equivalence if and only if there exists

$$\{(Y_{1,\alpha} \times S, Z_{1,\alpha})/S, \alpha \in \Lambda_1\}, \ldots, \{(Y_{r,\alpha} \times S, Z_{r,\alpha})/S, \alpha \in \Lambda_r\} \subset \mathrm{AnSp}(\mathbb{C})^{2,(sm)}/S$$

such that we have in $\mathrm{Ho}_{et}(C(\mathrm{AnSp}(\mathbb{C})^{2,(sm)pr}/S))$

$$\begin{aligned}\mathrm{Cone}(m) \xrightarrow{\sim} &\mathrm{Cone}(\oplus_{\alpha\in\Lambda_1} \mathrm{Cone}(\mathbb{Z}((Y_{1,\alpha} \times S \times \mathbb{D}^1, Z_{1,\alpha} \times \mathbb{D}^1)/S) \to \mathbb{Z}((Y_{1,\alpha} \times S, Z_{1,\alpha})/S)) \\ &\to \cdots \to \oplus_{\alpha\in\Lambda_1} \mathrm{Cone}(\mathbb{Z}((Y_{1,\alpha} \times S \times \mathbb{D}^1, Z_{1,\alpha} \times \mathbb{D}^1)/S) \to \mathbb{Z}((Y_{1,\alpha} \times S, Z_{1,\alpha})/S)))\end{aligned}$$

Proof. Standard. ∎

Definition-Proposition 9. *Let $S \in \mathrm{AnSp}(\mathbb{C})$.*

(i) *With the weak equivalence the* $(\mathbb{D}^1, et)$ *equivalence and the fibration the epimorphism with* $\mathbb{D}^1$ *local and etale fibrant kernels gives a model structure on* $C(\mathrm{AnSp}(\mathbb{C})^{2,(sm)}/S)$ *: the left bousfield localization of the projective model structure of* $C(\mathrm{AnSp}(\mathbb{C})^{2,(sm)}/S)$*. We call it the projective* $(\mathbb{D}^1, et)$ *model structure.*

(ii) *With the weak equivalence the* $(\mathbb{D}^1, et)$ *equivalence and the fibration the epimorphism with* $\mathbb{D}^1$ *local and etale fibrant kernels gives a model structure on* $C(\mathrm{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ *: the left bousfield localization of the projective model structure of* $C(\mathrm{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$*. We call it the projective* $(\mathbb{D}^1, et)$ *model structure.*

Proof. Similar to the proof of proposition 5. □

We have, similarly to the case of single varieties the following :

Proposition 31. *Let* $g : T \to S$ *a morphism with* $T, S \in \mathrm{AnSp}(\mathbb{C})$*.*

(i) *The adjonction* $(g^*, g_*) : C(\mathrm{AnSp}(\mathbb{C})^{2,(sm)}/S) \leftrightarrows C(\mathrm{AnSp}(\mathbb{C})^{2,(sm)}/T)$ *is a Quillen adjonction for the* $(\mathbb{D}^1, usu)$ *model structure.*

(i)' *The functor* $g^* : C(\mathrm{AnSp}(\mathbb{C})^{2,(sm)}/S) \to C(\mathrm{AnSp}(\mathbb{C})^{2,(sm)}/T)$ *sends quasi-isomorphism to quasi-isomorphism and equivalence usu local to equivalence usu local, sends* $(\mathbb{D}^1, usu)$ *local equivalence to* $(\mathbb{D}^1, usu)$ *local equivalence.*

(ii) *The adjonction* $(g^*, g_*) : C(\mathrm{AnSp}(\mathbb{C})^{2,(sm)pr}/S) \leftrightarrows C(\mathrm{AnSp}(\mathbb{C})^{2,(sm)pr}/T)$ *is a Quillen adjonction for the* $(\mathbb{D}^1, usu)$ *model structure.*

(ii)' *The functor* $g^* : C(\mathrm{AnSp}(\mathbb{C})^{2,(sm)pr}/S) \to C(\mathrm{AnSp}(\mathbb{C})^{2,(sm)pr}/T)$ *sends quasi-isomorphism to quasi-isomorphism and equivalence usu local to equivalence usu local, sends* $(\mathbb{D}^1, usu)$ *local equivalence to* $(\mathbb{D}^1, usu)$ *local equivalence.*

Proof. Similar to the proof of proposition 22. □

Proposition 32. *Let* $S \in \mathrm{Var}(\mathbb{C})$*.*

(i) *The adjonction* $(\rho_S^*, \rho_{S*}) : C(\mathrm{AnSp}(\mathbb{C})^{2,sm}/S) \leftrightarrows C(\mathrm{AnSp}(\mathbb{C})^2/S)$ *is a Quillen adjonction for the* $(\mathbb{A}^1, et)$ *projective model structure.*

(i)' *The functor* $\rho_{S*} : C(\mathrm{AnSp}(\mathbb{C})^2/S) \to C(\mathrm{AnSp}(\mathbb{C})^{2,sm}/S)$ *sends quasi-isomorphism to quasi-isomorphism, sends equivalence usu local to equivalence usu local, sends* $(\mathbb{D}^1, usu)$ *local equivalence to* $(\mathbb{D}^1, usu)$ *local equivalence.*

(ii) *The adjonction* $(\rho_S^*, \rho_{S*}) : C(\mathrm{AnSp}(\mathbb{C})^{2,smpr}/S) \leftrightarrows C(\mathrm{AnSp}(\mathbb{C})^{2,pr}/S)$ *is a Quillen adjonction for the* $(\mathbb{A}^1, et)$ *projective model structure.*

(ii)' *The functor* $\rho_{S*} : C(\mathrm{AnSp}(\mathbb{C})^{2,pr}/S) \to C(\mathrm{AnSp}(\mathbb{C})^{2,smpr}/S)$ *sends quasi-isomorphism to quasi-isomorphism, sends equivalence usu local to equivalence usu local, sends* $(\mathbb{D}^1, usu)$ *local equivalence to* $(\mathbb{D}^1, usu)$ *local equivalence.*

Proof. Similar to the proof of proposition 19. □

We also have

Proposition 33. *Let* $S \in \mathrm{AnSp}(\mathbb{C})$*.*

(i) *The adjonction* $(\mathrm{Gr}_S^{12*}, \mathrm{Gr}_{S*}^{12}) : C(\mathrm{AnSp}(\mathbb{C})/S) \leftrightarrows C(\mathrm{AnSp}(\mathbb{C})^{2,pr}/S)$ *is a Quillen adjonction for the* $(\mathbb{D}^1, usu)$ *projective model structure.*

(ii) *The adjonction* $(\mathrm{Gr}_S^{12*}\,\mathrm{Gr}_{S*}^{12} : C(\mathrm{AnSp}(\mathbb{C})^{sm}/S) \leftrightarrows C(\mathrm{AnSp}(\mathbb{C})^{2,smpr}/S)$ *is a Quillen adjonction for the* $(\mathbb{D}^1, usu)$ *projective model structure.*

Proof. Immediate from definition. $\square$

In the filtered case we also define

Definition 29. *(i) A filtered complex $(G,F)\in C_{fil}(\mathrm{AnSp}(\mathbb{C})^{2,(sm)}/S)$ is said to be r-filtered $\mathbb{D}^1$ homotopic if $\mathrm{ad}(p_a^*,p_{a*})(G,F):(G,F)\to p_{a*}p_a^*(G,F)$ is an r-filtered homotopy equivalence.*

(ii) A filtered complex $(G,F)\in C_{fil}(\mathrm{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ is said to be r-filtered $\mathbb{D}^1$ homotopic if $\mathrm{ad}(p_a^*,p_{a*})(G,F):(G,F)\to p_{a*}p_a^*(G,F)$ is an r-filtered homotopy equivalence.

We have the following canonical functor :

Definition 30. *(i) For $S\in\mathrm{AnSp}(\mathbb{C})$, we have the functor*

$$\begin{aligned}(-)^\Gamma:C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)\to C(\mathrm{AnSp}(\mathbb{C})^{2,sm}/S),\\ F\longmapsto F^\Gamma:(((U,Z)/S)=((U,Z),h)\mapsto F^\Gamma((U,Z)/S):=(\Gamma^\vee_Z h^*LF)(U/U),\\ (g:((U',Z'),h')\to((U,Z),h))\mapsto\\ (F^\Gamma(g):(\Gamma^\vee_Z h^*LF)(U/U)\xrightarrow{i_{(\Gamma^\vee_Z h^*LF)(U/U)}}(g^*(\Gamma^\vee_Z h^*LF))(U'/U')\\ \xrightarrow{T(g,\gamma^\vee)(h^*LF)(U'/U')}(\Gamma^\vee_{Z\times_U U'}g^*h^*LF)(U'/U')\\ \xrightarrow{T(Z'/Z\times_U U',\gamma^\vee)(g^*h^*LF)(U'/U')}(\Gamma^\vee_{Z'}g^*h^*LF)(U'/U')))\end{aligned}$$

where $i_{(\Gamma^\vee_Z h^*LF)(U/U)}$ is the canonical arrow of the inductive limit. Similarly, we have, for $S\in\mathrm{AnSp}(\mathbb{C})$, the functor

$$\begin{aligned}(-)^\Gamma:C(\mathrm{AnSp}(\mathbb{C})/S)\to C(\mathrm{AnSp}(\mathbb{C})^2/S),\\ F\longmapsto F^\Gamma:(((X,Z)/S)=((X,Z),h)\mapsto F^\Gamma((X,Z)/S):=(\Gamma^\vee_Z h^*F)(X/X),\\ (g:((X',Z'),h')\to((X,Z),h))\mapsto(F^\Gamma(g):(\Gamma^\vee_Z h^*LF)(X/X)\to(\Gamma^\vee_{Z'}h'^*LF)(X'/X')))\end{aligned}$$

Note that for $S\in\mathrm{AnSp}(\mathbb{C})$, $I(S/S):\mathbb{Z}((S,S)/S)\to\mathbb{Z}(S/S)^\Gamma$ given by

$$I(S/S)((U,Z),h):\mathbb{Z}((S,S)/S)(((U,Z),h))\xrightarrow{\gamma^\vee_Z(\mathbb{Z}(U/U))(U/U)}\mathbb{Z}(S/S)^\Gamma((U,Z),h):=(\Gamma^\vee_Z\mathbb{Z}(U/U))(U/U)$$

is an isomorphism.

(ii) Let $f:T\to S$ a morphism with $T,S\in\mathrm{AnSp}(\mathbb{C})$. For $F\in C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$, we have the canonical morphism in $C(\mathrm{AnSp}(\mathbb{C})^{2,sm}/T)$

$$\begin{aligned}T(f,\Gamma)(F):=T^*(f,\Gamma)(F):f^*(F^\Gamma)\to(f^*F)^\Gamma,\\ T(f,\Gamma)(F)((U',Z')/T=((U',Z'),h')):\\ f^*(F^\Gamma)((U',Z'),h'):=\lim_{((U',Z'),h')\xrightarrow{l}((U_T,Z_T),h_T)\xrightarrow{f_U}((U,Z),h)}(\Gamma^\vee_Z h^*LF)(U/U)\\ \xrightarrow{F^\Gamma(f_U\circ l)}(\Gamma^\vee_{Z'}l^*f_U^*h^*LF)(U'/U')=(\Gamma^\vee_{Z'}h'^*f^*LF)(U'/U')\\ \xrightarrow{(\Gamma^\vee_{Z'}h'^*T(f,L)(F))(U'/U')}(\Gamma^\vee_{Z'}h'^*Lf^*F)(U'/U')=:(f^*F)^\Gamma((U',Z'),h')\end{aligned}$$

where $f_U:U_T:U\times_S T\to U$ and $h_T:U_T:=U\times_S T\to T$ are the base change maps, the equality following from the fact that $h\circ f_U\circ l=f\circ h_T\circ l=f\circ h'$. For $F\in C(\mathrm{AnSp}(\mathbb{C})/S)$, we have similarly the canonical morphism in $C(\mathrm{AnSp}(\mathbb{C})^2/T)$

$$T(f,\Gamma)(F):f^*(F^\Gamma)\to(f^*F)^\Gamma.$$

(iii) *Let $h : U \to S$ a smooth morphism with $U, S \in \mathrm{AnSp}(\mathbb{C})$. We have, for $F \in C(\mathrm{AnSp}(\mathbb{C})^{sm}/U)$, the canonical morphism in $C(\mathrm{AnSp}(\mathbb{C})^{2,sm}/S)$*

$$T_\sharp(h,\Gamma)(F) : h_\sharp(F^\Gamma) \to (h_\sharp LF)^\Gamma,$$
$$T_\sharp(h,\Gamma)(F)((U',Z'),h') : h_\sharp(F^\Gamma)((U',Z'),h') := \lim_{((U',Z'),h')\xrightarrow{l}((U,U),h)} (\Gamma^\vee_{Z'} l^* LF)(U'/U')$$
$$\xrightarrow{(\Gamma^\vee_{Z'} l^* \operatorname{ad}(h_\sharp,h^*)(LF))(U'/U')} (\Gamma^\vee_{Z'} l^* h^* h_\sharp LF)(U'/U') =: (h_\sharp LF)^\Gamma((U',Z')/h')$$

(iv) *Let $i : Z_0 \hookrightarrow S$ a closed embedding with $Z_0, S \in \mathrm{AnSp}(\mathbb{C})$. We have the canonical morphism in $C(\mathrm{AnSp}(\mathbb{C})^{2,sm}/S)$*

$$T_*(i,\Gamma)(\mathbb{Z}(Z_0/Z_0)) : i_*((\mathbb{Z}(Z_0/Z_0))^\Gamma \to (i_*\mathbb{Z}(Z/Z))^\Gamma,$$
$$T_*(i,\Gamma)(\mathbb{Z}(Z_0/Z_0))((U,Z),h) : i_*((\mathbb{Z}(Z_0/Z_0))^\Gamma((U,Z),h) := (\Gamma^\vee_{Z\times_S Z_0}\mathbb{Z}(Z_0/Z_0))(U\times_S Z_0)$$
$$\xrightarrow{T(i_*,\gamma^\vee)(\mathbb{Z}(Z_0/Z_0))(U\times_S Z_0)} (\Gamma^\vee_Z i_*\mathbb{Z}(Z_0/Z_0))(U\times_S Z_0) =: (i_*\mathbb{Z}(Z/Z))^\Gamma((U,Z),h)$$

Definition 31. *Let $S \in \mathrm{AnSp}(\mathbb{C})$. We have for $F \in C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$ the canonical map in $C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$*

$$\mathrm{Gr}(F) : \mathrm{Gr}^{12}_{S*}\mu_{S*}F^\Gamma \to F,$$
$$\mathrm{Gr}(F)(U/S) : \Gamma^\vee_U p^* F(U\times S/U\times S) \xrightarrow{\operatorname{ad}(l^*,l_*)(p^*F)(U\times S/U\times S)} h^*F(U/U) = F(U/S)$$

where $h : U \to S$ is a smooth morphism with $U \in \mathrm{AnSp}(\mathbb{C})$ and $h : U \xrightarrow{l} U \times S \xrightarrow{p} S$ is the graph factorization with l the graph embedding and p the projection.

Proposition 34. *Let $S \in \mathrm{AnSp}(\mathbb{C})$.*

(i) *Then,*

- *if $m : F \to G$ with $F, G \in C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$ is a quasi-isomorphism, $m^\Gamma : F^\Gamma \to G^\Gamma$ is a quasi-isomorphism in $C(\mathrm{AnSp}(\mathbb{C})^{2,sm}/S)$,*
- *if $m : F \to G$ with $F, G \in C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$ is an usu local equivalence, $m^\Gamma : F^\Gamma \to G^\Gamma$ is an usu local equivalence in $C(\mathrm{AnSp}(\mathbb{C})^{2,sm}/S)$,*
- *if $m : F \to G$ with $F, G \in C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$ is an $(\mathbb{D}^1, usu)$ local equivalence, $m^\Gamma : F^\Gamma \to G^\Gamma$ is an $(\mathbb{D}^1, usu)$ local equivalence in $C(\mathrm{AnSp}(\mathbb{C})^{2,sm}/S)$.*

(ii) *Then,*

- *if $m : F \to G$ with $F, G \in C(\mathrm{AnSp}(\mathbb{C})/S)$ is a quasi-isomorphism, $m^\Gamma : F^\Gamma \to G^\Gamma$ is a quasi-isomorphism in $C(\mathrm{AnSp}(\mathbb{C})^2/S)$,*
- *if $m : F \to G$ with $F, G \in C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$ is an usu local equivalence, $m^\Gamma : F^\Gamma \to G^\Gamma$ is an usu local equivalence in $C(\mathrm{AnSp}(\mathbb{C})^{2,sm}/S)$,*
- *if $m : F \to G$ with $F, G \in C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$ is an $(\mathbb{D}^1, usu)$ local equivalence, $m^\Gamma : F^\Gamma \to G^\Gamma$ is an $(\mathbb{D}^1, usu)$ local equivalence in $C(\mathrm{AnSp}(\mathbb{C})^2/S)$.*

Proof. Similar to the proof of proposition 26. □

2.10 The analytical functor for presheaves on the big Zariski or etale site and on the big Zariski or etale site of pairs

We have for $f : T \to S$ a morphism with $T, S \in \mathrm{Var}(\mathbb{C})$ the following commutative diagram of sites

$$
\begin{tikzcd}
Dia(S) := \mathrm{AnSp}(\mathbb{C})/T^{an} \arrow[rr, "\mathrm{An}_T"] \arrow[dd, "P(f)"] \arrow[dr, "\rho_T"] & & \mathrm{Var}(\mathbb{C})/T \arrow[dd, "P(f)"] \arrow[dr, "\rho_T"] & \\
& \mathrm{AnSp}(\mathbb{C})^{sm}/T^{an} \arrow[rr, "\mathrm{An}_T"] \arrow[dd, "P(f)"] & & \mathrm{Var}(\mathbb{C})^{sm}/T \arrow[dd, "P(f)"] \\
\mathrm{AnSp}(\mathbb{C})/S^{an} \arrow[rr, "\mathrm{An}_S"] \arrow[dr, "\rho_S"] & & \mathrm{Var}(\mathbb{C})/S \arrow[dr, "\rho_S"] & \\
& \mathrm{AnSp}(\mathbb{C})^{sm}/S^{an} \arrow[rr, "\mathrm{An}_S"] & & \mathrm{Var}(\mathbb{C})^{sm}/S
\end{tikzcd}
\tag{38}
$$

and

$$
\begin{tikzcd}
Dia^{12}(S) := \mathrm{AnSp}(\mathbb{C})^2/T^{an} \arrow[rr, "\mathrm{An}_T"] \arrow[dd, "P(f)"] \arrow[dr, "\rho_T"] & & \mathrm{Var}(\mathbb{C})^2/T \arrow[dd, "P(f)"] \arrow[dr, "\rho_T"] & \\
& \mathrm{AnSp}(\mathbb{C})^{2,sm}/T^{an} \arrow[rr, "\mathrm{An}_T"] \arrow[dd, "P(f)"] & & \mathrm{Var}(\mathbb{C})^{2,sm}/T \arrow[dd, "P(f)"] \\
\mathrm{AnSp}(\mathbb{C})^2/S^{an} \arrow[rr, "\mathrm{An}_S"] \arrow[dr, "\rho_S"] & & \mathrm{Var}(\mathbb{C})^2/S \arrow[dr, "\rho_S"] & \\
& \mathrm{AnSp}(\mathbb{C})^{2,sm}/S^{an} \arrow[rr, "\mathrm{An}_S"] & & \mathrm{Var}(\mathbb{C})^{2,sm}/S
\end{tikzcd}
\tag{39}
$$

For $S \in \mathrm{Var}(\mathbb{C})$ we have the following commutative diagrams of sites

$$
\begin{tikzcd}
\mathrm{AnSp}(\mathbb{C})^2/S \arrow[rr, "\mu_S"] \arrow[dd, "\mathrm{An}_S"] \arrow[dr, "\rho_S"] & & \mathrm{AnSp}(\mathbb{C})^{2,pr}/S \arrow[dd, "\mathrm{An}_S"] \arrow[dr, "\rho_S"] & \\
& \mathrm{AnSp}(\mathbb{C})^{2,sm}/S \arrow[rr, "\mu_S"] \arrow[dd, "\mathrm{An}_S"] & & \mathrm{AnSp}(\mathbb{C})^{2,smpr}/S \arrow[dd, "\mathrm{An}_S"] \\
\mathrm{Var}(\mathbb{C})^2/S \arrow[rr, "\mu_S"] \arrow[dr, "\rho_S"] & & \mathrm{Var}(\mathbb{C})^{2,smpr}/S \arrow[dr, "\rho_S"] & \\
& \mathrm{Var}(\mathbb{C})^2/S \arrow[rr, "\mu_S"] & & \mathrm{Var}(\mathbb{C})^{2,smpr}/S
\end{tikzcd}
$$

and

$$\begin{tikzcd}
\mathrm{AnSp}(\mathbb{C})^{2,pr}/S \arrow[rr, "\mathrm{Gr}^{12}_S"] \arrow[dr, "\rho_S"] \arrow[dd, "\mathrm{An}_S"'] & & \mathrm{AnSp}(\mathbb{C})/S \arrow[dr, "\rho_S"] \arrow[dd, "\mathrm{An}_S"] & \\
& \mathrm{AnSp}(\mathbb{C})^{2,smpr}/S \arrow[rr, "\mathrm{Gr}^{12}_S"] \arrow[dd, "\mathrm{An}_S"] & & \mathrm{AnSp}(\mathbb{C})^{sm}/S \arrow[dd, "\mathrm{An}_S"] \\
\mathrm{Var}(\mathbb{C})^{2,pr}/S \arrow[rr, "\mathrm{Gr}^{12}_S"] \arrow[dr, "\rho_S"] & & \mathrm{Var}(\mathbb{C})/S \arrow[dr, "\rho_S"] & \\
& \mathrm{Var}(\mathbb{C})^{2,sm}/S \arrow[rr, "\mathrm{Gr}^{12}_S"] & & \mathrm{Var}(\mathbb{C})^{sm}/S
\end{tikzcd} \quad . \tag{40}$$

For $f : T \to S$ a morphism in $\mathrm{Var}(\mathbb{C})$ the diagramm $Dia(S)$ and $Dia(T)$ commutes with the pullback functors : we have $e(S) \circ P(f) = P(f) \circ e(T)$.

For $S \in \mathrm{Var}(\mathbb{C})$, the analytical functor is

$$(-)^{an} : C_{O_S}(S) \to C_{O_{S^{an}}}, G \mapsto G^{an} := \mathrm{an}_S^{*mod} G := \mathrm{an}_S^* G \otimes_{\mathrm{an}_S^* O_S} O_{S^{an}}$$

Let $S \in \mathrm{Var}(\mathbb{C})$.

- As $\mathrm{an}_S^* : \mathrm{PSh}(S) \to \mathrm{PSh}(S^{an})$ preserve monomorphisms (the colimits involved being filtered colimits), we define, for $(G, F) \in C_{(2)fil}(S)$, $\mathrm{an}_S^*(G, F) := (\mathrm{an}_S^* G, \mathrm{an}_S^* F) \in C_{(2)fil}(S^{an})$.
- As $(-)^{an} := \mathrm{an}_S^{*mod} : \mathrm{PSh}_{O_S}(S) \to \mathrm{PSh}(S^{an})$ preserve monomorphisms (an_S^* preserve monomorphism and $(-) \otimes_{O_S} O_{S^{an}}$ preserve monomorphism since $O_{S^{an}}$ is a flat O_S module), we define, for $(G, F) \in C_{(2)fil}(S)$, $(G, F)^{an} := (G^{an}, \mathrm{an}_S^* F \otimes_{O_S} O_{S^{an}}) \in C_{(2)fil}(S^{an})$.

Let $f : T \to S$ a morphism with $T, S \in \mathrm{Var}(\mathbb{C})$. Then,

- the commutative diagrams of sites $D(\mathrm{An}, f) := (\mathrm{An}_S, f, \mathrm{An}_T, f = f^{an})$ gives, for $G \in C(\mathrm{Var}(\mathbb{C})^{sm}/T)$, the canonical map in $C(\mathrm{AnSp}(\mathbb{C})^{am}/T)$

$$T(\mathrm{An}, f)(G) : \mathrm{An}_S^* f_* G \xrightarrow{\mathrm{ad}(\mathrm{An}_T^*, \mathrm{An}_{T*})(G)} \mathrm{An}_S^* f_* \mathrm{An}_{T*} \mathrm{An}_T^* G = \mathrm{An}_S^* \mathrm{An}_{S*} f_* \mathrm{An}_T^* G \xrightarrow{\mathrm{ad}(\mathrm{An}_S^* \mathrm{An}_{S*})(f_* \mathrm{An}_T^* G)} f_* \mathrm{An}_T^* G.$$

- the commutative diagrams of sites $D(an, f) := (\mathrm{an}_S, f\, \mathrm{an}_T, f)$ gives, for $G \in C(T)$, the canonical map in $C(T^{an})$

$$T(an, f)(G) : \mathrm{an}_S^* f_* G \xrightarrow{\mathrm{ad}(\mathrm{an}_T^*, \mathrm{an}_{T*})(G)} \mathrm{an}_S^* f_* \mathrm{an}_{T*} \mathrm{an}_T^* G = \mathrm{an}_S^* \mathrm{an}_{S*} f_* \mathrm{an}_T^* G \xrightarrow{\mathrm{ad}(\mathrm{an}_S^*, \mathrm{an}_{S*})(f_* \mathrm{an}_T^* G)} f_* \mathrm{an}_T^* G$$

  and for $G \in C_{O_T}(T)$, the canonical map in $C_{O_{T^{an}}}(T^{an})$

$$T^{mod}(an, f)(G) : (f_* G)^{an} := \mathrm{an}_S^{*mod} f_* G \xrightarrow{\mathrm{ad}(\mathrm{an}_T^{*mod}, \mathrm{an}_{T*})(G)}$$
$$\mathrm{an}_S^{*mod} f_* \mathrm{an}_{T*} \mathrm{an}_T^{*mod} G = \mathrm{an}_S^{*mod} \mathrm{an}_{S*} f_* \mathrm{an}_T^{*mod} G \xrightarrow{\mathrm{an}_S^{*mod}, \mathrm{an}_{S*})(f_* \mathrm{an}_T^* G)} f_* \mathrm{an}_T^{*mod} G =: f_* G^{an}$$

Definition-Proposition 10. *Consider a closed embedding $i : Z \hookrightarrow S$ with $S, Z \in \mathrm{Var}(\mathbb{C})$. Then, for $G^\bullet \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$, there exist a map in $C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$*

$$T(\mathrm{An}, \gamma)(G) : \mathrm{An}_S^* \Gamma_Z G \to \Gamma_Z \mathrm{An}_S^* G$$

unique up to homotopy, such that $\gamma_Z(\mathrm{An}_S^* G) \circ T(\mathrm{An}, \gamma)(G) = \mathrm{An}_S^* \gamma_Z G$.

Proof. Denote by $j : S\backslash Z \hookrightarrow S$ the open complementary embedding. The map is given by $(I, T(\mathrm{An}, j)(j^*G)) : \mathrm{Cone}(\mathrm{An}_S^* G \to \mathrm{An}_S^* j_*j^*G) \to (\mathrm{An}_S^* G \to j_*j^* \mathrm{An}_S^* G)$. $\square$

Definition 32. *Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$. Assume that there exist a factorization $f : X \xrightarrow{i} Y \times S \xrightarrow{p} S$, with $Y \in \mathrm{SmVar}(\mathbb{C})$, $i : X \hookrightarrow Y$ is a closed embedding and p the projection. We then have the canonical isomorphism in $C(\mathrm{AnSp}(\mathbb{C})^{sm}/S^{an})$*

$$T(f, g, Q) := T_\sharp(\mathrm{An}, p)(-)^{-1} \circ T_\sharp(\mathrm{An}, j)(-)^{-1} :$$
$$\mathrm{An}_S^* Q(X/S) := An_S^* p_\sharp \Gamma_X^\vee \mathbb{Z}_{Y\times S}[d_Y] \xrightarrow{=} p_\sharp \Gamma_{X^{an}}^\vee \mathbb{Z}_{Y\times S}[d_Y] =: Q(X^{an}/S^{an})$$

with $j : Y \times S\backslash X \hookrightarrow Y \times S$ the closed embedding.

Definition-Proposition 11. *Consider a closed embedding $i : Z \hookrightarrow S$ with $S \in \mathrm{Var}(\mathbb{C})$. Then, for $G \in C_{O_S}(S)$, there is a canonical map in $C_{O_{S^{an}}}(S^{an})$*

$$T^{mod}(an, \gamma)(G) : (\Gamma_Z G)^{an} \to \Gamma_{Z^{an}} G^{an}$$

unique up to homotopy, such that $\gamma_{Z^{an}}(G^{an}) \circ T^{mod}(an, \gamma)(G) = g^* \gamma_Z G$.

Proof. It is a particular case of definition-proposition 2(i). $\square$

We recall the first GAGA theorem for coherent sheaf on the projective spaces :

Theorem 13. *For $X \in \mathrm{Var}(\mathbb{C})$ and $F \in C_{O_X}(X)$ denote by*

$$a(F) : \mathrm{ad}(\mathrm{an}_X^{*mod}, \mathrm{an}(X)_*)(E(F)) : E(F) \to \mathrm{an}_{X*}(E(F))^{an} = \mathrm{an}_{X*} E(F^{an}),$$

the canonical morphism.

(i) Let $X \in \mathrm{PVar}(\mathbb{C})$ a proper complex algebraic variety. For $F \in \mathrm{Coh}_{O_X}(X)$ a coherent sheaf, the morphism

$$H^n\Gamma(X, a(F)) : H^n(X, F) = H^n\Gamma(X, E(F)) \to H^n(X, F^{an}) = H^n\Gamma(X, E(F^{an}))$$

is an isomorphism for all $n \in \mathbb{Z}$.

(ii) Let $f : X \to S$ a proper morphism with $X, S \in \mathrm{Var}(\mathbb{C})$. For $F \in \mathrm{Coh}_{O_X}(X)$ a coherent sheaf, the morphism

$$H^n f_* a(F) : R^n f_* F = H^n f_*(E(F)) \to R^n f_* F^{an} = H^n f_* E(F^{an})$$

is an isomorphism for all $n \in \mathbb{Z}$.

Proof. See [29]. (i) reduces to the case where X is projective and (ii) to the case where f is projective. Hence, the theorem reduce to the case of a coherent sheaf $F \in \mathrm{Coh}_{O_{\mathbb{P}^N}}(\mathbb{P}^N)$ on $\mathbb{P}^N$. $\square$

2.11 The De Rahm complexes of algebraic varieties and analytical spaces

For $X \in \mathrm{Var}(\mathbb{C})$, we denote by $\iota_X : \mathbb{C}_X \to \Omega_X^\bullet =: DR(X)$ the canonical inclusion map. More generaly, for $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$, we denote by $\iota_{X/S} : f^*O_S \to \Omega_{X/S}^\bullet =: DR(X/S)$ the canonical inclusion map.

For $X \in \mathrm{AnSp}(\mathbb{C})$, we denote by $\iota_X : \mathbb{C}_X \to \Omega_X^\bullet =: DR(X)$ the canonical inclusion map. More generaly, for $f : X \to S$ a morphism with $X, S \in \mathrm{AnSp}(\mathbb{C})$, we denote by $\iota_{X/S} : f^*O_S \to \Omega_{X/S}^\bullet =: DR(X/S)$ the canonical inclusion map.

Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$. Then, the commutative diagram of site $(an, f) := (f, \mathrm{An}_S, f = f^{an}, \mathrm{an}(X))$ gives the transformation map in $C_{O_{S^{an}}}(S^{an})$ (definition 1)

$$T_\omega^O(an, f) : (f_* E(\Omega_{X/S}^\bullet, F_b))^{an} := \mathrm{an}_S^{*mod} f_* E(\Omega_{X/S}^\bullet, F_b) \xrightarrow{T(an(X), E)(-) \circ T(an, f)(E(\Omega_{X/S}^\bullet))}$$
$$(f_* E(\mathrm{an}(X)^*(\Omega_{X/S}^\bullet, F_b))) \otimes_{\mathrm{an}_S^* O_S} O_{S^{an}} \xrightarrow{m \otimes E(\Omega_{(X^{an}/X)/(S^{an}/S)})} f_* E(\Omega_{X^{an}/S^{an}}^\bullet, F_b)$$

We will give is this paper a relative version for all smooth morphisms of the following theorem of Grothendieck

Theorem 14. *Let $U \in \mathrm{SmVar}(\mathbb{C})$. Denote by $a_U : U \to \{\mathrm{pt}\}$ the terminal map. Then the map*

$$T^O_\omega(a_U, an) : \Gamma(U, E(\Omega^\bullet_U)) \to \Gamma(U^{an}, E(\Omega^\bullet_U))$$

is a quasi-isomorphism of complexes.

Proof. Take a compactification (X, D) of U, with $X \in \mathrm{PSmVar}(\mathbb{C})$ and $D = X\backslash U$ a normal crossing divisor. The proof then use proposition 13, the first GAGA theorem (theorem 13 (i)) for the coherent sheaves $\Omega^p_U(nD)$ on X, and the fact (which is specific of the De Rahm complex) that $\Omega^\bullet_{U^{an}}(*D^{an}) \to j_*E(\Omega_{U^{an}})$ is a quasi-isomorphism. □

We recall Poincare lemma for smooth morphisms of complex analytic spaces and in particular complex analytic manifold :

Proposition 35. *(i) For $h : U \to S$ a smooth morphism with $U, S \in \mathrm{AnSp}(\mathbb{C})$, the inclusion map $\iota_{X/S} : h^*O_S \to \Omega^\bullet_{U/S} =: DR(U/S)$ is a quasi-isomorphism.*

(ii) For $X \in \mathrm{AnSm}(\mathbb{C})$, the inclusion map $\iota_X : \mathbb{C}_X \to \Omega^\bullet_X$ is a quasi-isomorphism.

Proof. Standard. (ii) is a particular case of (i) (the absolute case $S = \{\mathrm{pt}\}$). □

Remark 4. *We do NOT have poincare lemma in general if $h : U \to S$ is not a smooth morphism. Already in the absolute case, we can find $X \in \mathrm{Var}(\mathbb{C})$ singular such that the inclusion map $\iota_X : \mathbb{C}_{X^{an}} \to \Omega^\bullet_{X^{an}}$ is not a quasi-isomorphism. Indeed, we can find exemple of $X \in \mathrm{PVar}(\mathbb{C})$ projective singular where*

$$H^p(c_X) : H^p(X^{an}, \mathbb{C}_{X^{an}}) \xrightarrow{\sim} H^pC^\bullet_{sing}(X^{an})$$

X^{an} being locally contractible since $X^{an} \in \mathrm{CW}$, have not the same dimension then the De Rham cohomology

$$H^p(T^O_\omega(an, a_X)) : \mathbb{H}^p(X, E(\Omega^\bullet_X)) \xrightarrow{\sim} \mathbb{H}^p(X^{an}, E(\Omega^\bullet_{X^{an})})$$

X being projective, that is are not isomorphic as vector spaces. Hence, in particular, the canonical map

$$H^p\iota_X : H^p(X^{an}, \mathbb{C}_{X^{an}}) \to \mathbb{H}^p(X^{an}, E(\Omega^\bullet_{X^{an}}))$$

is not an isomorphism.

Consider a commutative diagram

$$D_0 = \begin{tikzcd} f : X \arrow[r, "i"] & Y \arrow[r, "p"] & S \\ f' : X' \arrow[u, "g'"] \arrow[r, "i'"] & Y' \arrow[u, "g''"] \arrow[r, "p'"] & T \arrow[u, "g"] \end{tikzcd}$$

with $X, X', Y, Y', S, T \in \mathrm{Var}(\mathbb{C})$ or $X, X', Y, Y', S, T \in \mathrm{AnSp}(\mathbb{C})$, i, i' being closed embeddings. Denote by D the right square of D_0. The closed embedding $i' : X' \hookrightarrow Y'$ factors through $i' : X' \xrightarrow{i'_1} X \times_Y Y' \xrightarrow{i'_0} Y'$ where i'_1, i'_0 are closed embeddings. Then, definition-proposition 3 say that

- there is a canonical map,

$$E(\Omega_{((Y'/Y))/(T/S)}) \circ T(g'', E)(-) \circ T(g'', \gamma)(-) : g''^*\Gamma_X E(\Omega^\bullet_{Y/S}, F_b) \to \Gamma_{X \times_Y Y'} E(\Omega^\bullet_{Y'/T}, F_b)$$

unique up to homotopy such that the following diagram in $C_{g''^*p^*O_Sfil}(Y') = C_{p'^*g^*O_Sfil}(Y')$ commutes

$$\begin{tikzcd}
g''^*\Gamma_X E(\Omega^\bullet_{Y/S},F_b) \arrow[r, "E(\Omega_{((Y'/Y))/(T/S)})\circ T(g'',E)(-)\circ T(g'',\gamma)(-)"] \arrow[d, "\gamma_X(-)"'] & \Gamma_{X\times_Y Y'}E(\Omega^\bullet_{Y'/T},F_b) \arrow[d, "\gamma_{X\times_Y Y'}(-)"] \\
g''^*E(\Omega^\bullet_{Y/S},F_b) \arrow[r, "E(\Omega_{((Y'/Y)/(T/S))}\circ T(g'',E)(-)"] & E(\Omega^\bullet_{Y'/T},F_b)
\end{tikzcd},$$

- there is a canonical map,

$$T^O_\omega(D)^\gamma : g^{*mod}L_Op_*\Gamma_X E(\Omega^\bullet_{Y/S},F_b) \to p'_*\Gamma_{X\times_Y Y'}E(\Omega^\bullet_{Y'/T},F_b)$$

unique up to homotopy such that the following diagram in $C_{O_Tfil}(T)$ commutes

$$\begin{tikzcd}
g^{*mod}L_Op_*\Gamma_X E(\Omega^\bullet_{Y/S}) \arrow[r, "T^O_\omega(D)^\gamma"] \arrow[d, "\gamma_X(-)"'] & p'_*\Gamma_{X\times_Y Y'}E(\Omega^\bullet_{Y'/T}) \arrow[d, "\gamma_{X\times_Y Y'}(-)"] \\
g^{*mod}L_Op_*E(\Omega^\bullet_{Y/S}) \arrow[r, "T^O_\omega(D)"] & p'_*E(\Omega^\bullet_{Y'/T})
\end{tikzcd}.$$

(iii) there is a map in $C_{f'^*O_Tfil}(Y')$

$$T(X'/X\times_Y Y',\gamma)(E(\Omega^\bullet_{Y'/T},F_b)) : \Gamma_{X'}E(\Omega^\bullet_{Y'/T},F_b) \to \Gamma_{X\times_Y Y'}E(\Omega^\bullet_{Y'/T},F_b)$$

unique up to homotopy such that $\gamma_{X\times_Y Y'}(-)\circ T^(X'/X\times_Y Y',\gamma)(-) = \gamma_{X'}(-)$.

Let $h : Y \to S$ a morphism and $i : X \hookrightarrow Y$ a closed embedding with $S,Y,X \in \mathrm{Var}(\mathbb{C})$. Then, definition-proposition 3 say that

- there is a canonical map

$$E(\Omega_{(Y^{an}/Y)/(S^{an}/S)})\circ T(an,\gamma)(-) : \mathrm{an}(Y)^*\Gamma_X E(\Omega^\bullet_{Y/S},F_b) \to \Gamma_{X^{an}}E(\Omega^\bullet_{Y/S},F_b)$$

unique up to homotopy such that the following diagram in $C_{h^*O_Sfil}(Y^{an})$ commutes

$$\begin{tikzcd}
\mathrm{an}(Y)^*\Gamma_X E(\Omega^\bullet_{Y/S},F_b) \arrow[r, "E(\Omega_{(Y^{an}/Y)/(S^{an}/S)})\circ T(an,\gamma)(-)"] \arrow[d, "\gamma_X(-)"'] & \Gamma_{X^{an}}E(\Omega^\bullet_{Y/S},F_b) \arrow[d, "\gamma_{X^{an}}(-)"] \\
\mathrm{an}(Y)^*E(\Omega^\bullet_{Y/S},F_b) \arrow[r, "E(\Omega_{(Y^{an}/Y)/(S^{an}/S)})"] & E(\Omega^\bullet_{Y/S},F_b)
\end{tikzcd}$$

- there is a canonical map

$$T^O_\omega(an,h)^\gamma : (h_*\Gamma_X E(\Omega^\bullet_{Y/S},F_b))^{an} \to h_*\Gamma_{X^{an}}E(\Omega^\bullet_{Y/S},F_b)$$

unique up to homotopy such that the following diagram in $C(Y)$commutes

$$\begin{tikzcd}
(h_*\Gamma_X E(\Omega^\bullet_{Y/S},F_b))^{an} \arrow[r, "T^O_\omega(an,h)^\gamma"] \arrow[d, "\gamma_X(-)"'] & h_*\Gamma_{X^{an}}E(\Omega^\bullet_{Y/S},F_b) \arrow[d, "\gamma_{X^{an}}(-)"] \\
(h_*E(\Omega^\bullet_{Y/S},F_b))^{an} \arrow[r, "T^O_\omega(an,h)"] & h_*E(\Omega^\bullet_{Y/S},F_b)
\end{tikzcd}$$

2.12 The key functor R^{CH} from complexes of representable presheaves on $\mathrm{Var}(\mathbb{C})^{sm}/S$ with S smooth by a Borel-Moore Corti-Hanamura resolution complex of presheaves on $\mathrm{Var}(\mathbb{C})^2/S$, and the functorialities of these resolutions

Definition 33. *(i) Let $X_0 \in \mathrm{Var}(\mathbb{C})$ and $Z \subset X_0$ a closed subset. A desingularization of (X_0, Z) is a pair of complex varieties $(X, D) \in \mathrm{Var}^2(\mathbb{C}))$, together with a morphism of pair of varieties $\epsilon : (X, D) \to (X_0, \Delta)$ with $Z \subset \Delta$ such that*

- *$X \in \mathrm{SmVar}(\mathbb{C})$ and $D := \epsilon^{-1}(\Delta) = \epsilon^{-1}(Z) \cup (\cup_i E_i) \subset X$ is a normal crossing divisor*
- *$\epsilon : X \to X_0$ is a proper modification with discriminant Δ, that is $\epsilon : X \to X_0$ is proper and $\epsilon : X \backslash D \xrightarrow{\sim} X \backslash \Delta$ is an isomorphism.*

(ii) Let $X_0 \in \mathrm{Var}(\mathbb{C})$ and $Z \subset X_0$ a closed subset such that $X_0 \backslash Z$ is smooth. A strict desingularization of (X_0, Z) is a pair of complex varieties $(X, D) \in \mathrm{Var}^2(\mathbb{C}))$, together with a morphism of pair of varieties $\epsilon : (X, D) \to (X_0, Z)$ such that

- *$X \in \mathrm{SmVar}(\mathbb{C})$ and $D := \epsilon^{-1}(Z) \subset X$ is a normal crossing divisor*
- *$\epsilon : X \to X_0$ is a proper modification with discriminant Z, that is $\epsilon : X \to X_0$ is proper and $\epsilon : X \backslash D \xrightarrow{\sim} X \backslash Z$ is an isomorphism.*

We have the following well known resolution of singularities of complex algebraic varieties and their functorialities :

Theorem 15. *(i) Let $X_0 \in \mathrm{Var}(\mathbb{C})$ and $Z \subset X_0$ a closed subset. There exists a desingularization of (X_0, Z), that is a pair of complex varieties $(X, D) \in \mathrm{Var}^2(\mathbb{C}))$, together with a morphism of pair of varieties $\epsilon : (X, D) \to (X_0, \Delta)$ with $Z \subset \Delta$ such that*

- *$X \in \mathrm{SmVar}(\mathbb{C})$ and $D := \epsilon^{-1}(\Delta) = \epsilon^{-1}(Z) \cup (\cup_i E_i) \subset X$ is a normal crossing divisor*
- *$\epsilon : X \to X_0$ is a proper modification with discriminant Δ, that is $\epsilon : X \to X_0$ is proper and $\epsilon : X \backslash D \xrightarrow{\sim} X \backslash \Delta$ is an isomorphism.*

(ii) Let $X_0 \in \mathrm{PVar}(\mathbb{C})$ and $Z \subset X_0$ a closed subset such that $X_0 \backslash Z$ is smooth. There exists a strict desingularization of (X_0, Z), that is a pair of complex varieties $(X, D) \in \mathrm{PVar}^2(\mathbb{C}))$, together with a morphism of pair of varieties $\epsilon : (X, D) \to (X_0, Z)$ such that

- *$X \in \mathrm{PSmVar}(\mathbb{C})$ and $D := \epsilon^{-1}(Z) \subset X$ is a normal crossing divisor*
- *$\epsilon : X \to X_0$ is a proper modification with discriminant Z, that is $\epsilon : X \to X_0$ is proper and $\epsilon : X \backslash D \xrightarrow{\sim} X \backslash Z$ is an isomorphism.*

Proof. (i):Standard. See [24] for example.
(ii):Follows immediately from (i). □

We use this theorem to construct a resolution of a morphism by Corti-Hanamura morphisms, we will need these resolution in the definition of the filtered De Rham realization functor :

Definition-Proposition 12. *(i) Let $h : V \to S$ a morphism, with $V, S \in \mathrm{Var}(\mathbb{C})$. Let $\bar{S} \in \mathrm{PVar}(\mathbb{C})$ be a compactification of S.*

- *There exist a compactification $\bar{X}_0 \in \mathrm{PVar}(\mathbb{C})$ of V such that $h : V \to S$ extend to a morphism $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \to \bar{S}$. Denote by $\bar{Z} = \bar{X}_0 \backslash V$. We denote by $j : V \hookrightarrow \bar{X}_0$ the open embedding and by $i_0 : \bar{Z} \hookrightarrow \bar{X}_0$ the complementary closed embedding. We then consider $X_0 := \bar{f}_0^{-1}(S) \subset \bar{X}_0$ the open subset, $f_0 := \bar{f}_{0|X_0} : X_0 \to S$, $Z = \bar{Z} \cap X_0$, and we denote again $j : V \hookrightarrow X_0$ the open embedding and by $i_0 : Z \hookrightarrow X_0$ the complementary closed embedding.*

- *In the case V is smooth, we take, using theorem 15(ii), a strict desingularization* $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ *of the pair* $(\bar{X}_0, \bar{Z})$, *with* $\bar{X} \in \mathrm{PSmVar}(\mathbb{C})$ *and* $\bar{D} = \cup_{i=1}^{s} \bar{D}_i \subset \bar{X}$ *a normal crossing divisor. We denote by* $i_\bullet : \bar{D}_\bullet \hookrightarrow \bar{X} = \bar{X}_{c(\bullet)}$ *the morphism of simplicial varieties given by the closed embeddings* $i_I : \bar{D}_I = \cap_{i \in I} \bar{D}_i \hookrightarrow \bar{X}$. *Then the morphisms* $\bar{f} := \bar{f}_0 \circ \bar{\epsilon} : \bar{X} \to \bar{S}$ *and* $\bar{f}_{D_\bullet} := \bar{f} \circ i_\bullet : \bar{D}_\bullet \to \bar{S}$ *are projective since* $\bar{X}$ *and* $\bar{D}_I$ *are projective varieties. We then consider* $(X, D) := \bar{\epsilon}^{-1}(X_0, Z)$, $\epsilon := \bar{\epsilon}_{|X} : (X, D) \to (X_0, Z)$ *We denote again by* $i_\bullet : D_\bullet \hookrightarrow X = X_{c(\bullet)}$ *the morphism of simplicial varieties given by the closed embeddings* $i_I : D_I = \cap_{i \in I} D_i \hookrightarrow X$. *Then the morphisms* $f := f_0 \circ \epsilon : X \to S$ *and* $f_{D_\bullet} := f \circ i_\bullet : D_\bullet \to S$ *are projective since* $\bar{f} : \bar{X}_0 \to \bar{S}$ *is projective.*

(ii) Let $g : V'/S \to V/S$ *a morphism, with* $V'/S = (V', h'), V/S = (V, h) \in \mathrm{Var}(\mathbb{C})/S$

- *Take (see (i)) a compactification* $\bar{X}_0 \in \mathrm{PVar}(\mathbb{C})$ *of V such that* $h : V \to S$ *extend to a morphism* $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \to \bar{S}$. *Denote by* $\bar{Z} = \bar{X}_0 \backslash V$. *Then, there exist a compactification* $\bar{X}'_0 \in \mathrm{PVar}(\mathbb{C})$ *of V' such that* $h' : V' \to S$ *extend to a morphism* $\bar{f}'_0 = \bar{h}'_0 : \bar{X}'_0 \to \bar{S}$, $g : V' \to V$ *extend to a morphism* $\bar{g}_0 : \bar{X}'_0 \to \bar{X}_0$ *and* $\bar{f}_0 \circ \bar{g}_0 = \bar{f}'_0$ *that is* $\bar{g}_0$ *is gives a morphism* $\bar{g}_0 : \bar{X}'_0/\bar{S} \to \bar{X}_0/\bar{S}$. *Denote by* $\bar{Z}' = \bar{X}'_0 \backslash V'$. *We then have the following commutative diagram*

$$\begin{tikzcd}
V \arrow[r, "j"] & \bar{X}_0 & & \bar{Z} \arrow[ll, "i"'] \\
V' \arrow[u, "\bar{g}"] \arrow[r, "j'"] & \bar{X}'_0 \arrow[u, "\bar{g}_0"] & \bar{Z}' \arrow[l, "i'"'] & \bar{g}_0^{-1}(\bar{Z}) : i'_{g,0} \arrow[l, "i''_{g,0}"'] \arrow[u, "\bar{g}'"]
\end{tikzcd}$$

It gives the following commutative diagram

$$\begin{tikzcd}
V \arrow[r, "j"] & X_0 := \bar{f}_0^{-1}(S) & & Z \arrow[ll, "i"'] \\
V' \arrow[u, "g"] \arrow[r, "j'"] & X'_0 := \bar{f}_0'^{-1}(S) \arrow[u, "\bar{g}_0"] & Z' \arrow[l, "i'"'] & \bar{g}_0^{-1}(Z) : i'_{g,0} \arrow[l, "i''_{g,0}"'] \arrow[u, "\bar{g}'_0"]
\end{tikzcd}$$

- *In the case V and V' are smooth, we take using theorem 15 a strict desingularization* $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ *of* $(\bar{X}_0, \bar{Z})$. *Then there exist a strict desingularization* $\bar{\epsilon}'_\bullet : (\bar{X}', \bar{D}') \to (\bar{X}'_0, \bar{Z}')$ *of* $(\bar{X}'_0, \bar{Z}')$ *and a morphism* $\bar{g} : \bar{X}' \to \bar{X}$ *such that the following diagram commutes*

$$\begin{tikzcd}
\bar{X}'_0 \arrow[r, "\bar{g}_0"] & \bar{X}_0 . \\
\bar{X}' \arrow[u, "\bar{\epsilon}'"] \arrow[r, "\bar{g}"] & \bar{X} \arrow[u, "\bar{\epsilon}"]
\end{tikzcd}$$

We then have the following commutative diagram in $\mathrm{Fun}(\Delta, \mathrm{Var}(\mathbb{C}))$

$$\begin{tikzcd}
V = V_{c(\bullet)} \arrow[r, "j"] & \bar{X} = \bar{X}_{c(\bullet)} & & \bar{D}_{s_g(\bullet)} \arrow[ll, "i_\bullet"'] \\
V' = V'_{c(\bullet)} \arrow[u, "g"] \arrow[r, "j'"] & \bar{X}' = \bar{X}'_{c(\bullet)} \arrow[u, "\bar{g}"] & \bar{D}'_\bullet \arrow[l, "i'_\bullet"'] & \bar{g}^{-1}(\bar{D}_{s_g(\bullet)}) : i'_{g\bullet} \arrow[l, "i''_{g\bullet}"'] \arrow[u, "\bar{g}'_\bullet"]
\end{tikzcd}$$

where $i_\bullet : \bar{D}_\bullet \hookrightarrow \bar{X}_\bullet$ *the morphism of simplicial varieties given by the closed embeddings* $i_n : \bar{D}_n \hookrightarrow \bar{X}_n$, *and* $i'_\bullet : \bar{D}'_\bullet \hookrightarrow \bar{X}'_\bullet$ *the morphism of simplicial varieties given by the closed* *embeddings* $i'_n : \bar{D}'_n \hookrightarrow \bar{X}'_n$. *It gives the commutative diagram in* $\mathrm{Fun}(\Delta, \mathrm{Var}(\mathbb{C}))$

$$\begin{tikzcd}
V = V_{c(\bullet)} \arrow[r, "j"] & X := \bar{\epsilon}^{-1}(X_0) = X_{c(\bullet)} & & D_{s_g(\bullet)} \arrow[ll, "i_\bullet"'] \\
V' = V'_{c(\bullet)} \arrow[u, "g"] \arrow[r, "j'"] & X' := \bar{\epsilon}'^{,-1}(X'_0) = X'_{c(\bullet)} \arrow[u, "\bar{g}"] & D'_\bullet \arrow[l, "i'_\bullet"'] & \bar{g}^{-1}(D_{s_g(\bullet)}) : i'_{g\bullet} \arrow[l, "i''_{g\bullet}"'] \arrow[u, "\bar{g}'_\bullet"]
\end{tikzcd}$$

Proof. (i): Let $\bar{X}_{00} \in \mathrm{PVar}(\mathbb{C})$ be a compactification of V. Let $l_0 : \bar{X}_0 = \bar{\Gamma}_h \hookrightarrow \bar{X}_{00} \times \bar{S}$ be the closure of the graph of h and $\bar{f}_0 := p_{\bar{S}} \circ l_0 : \bar{X}_0 \hookrightarrow \bar{X}_{00} \times \bar{S} \to \bar{S}$, $\epsilon_{\bar{X}_0} := p_{\bar{X}_{00}} \circ l_0 : \bar{X}_0 \hookrightarrow \bar{X}_{00} \times \bar{S} \to \bar{X}_{00}$ be the restriction to $\bar{X}_0$ of the projections. Then, $\bar{X} \in \mathrm{PVar}(\mathbb{C})$, $\epsilon_{\bar{X}_0} : \bar{X}_0 \to \bar{X}_{00}$ is a proper modification which does not affect the open subset $V \subset \bar{X}_0$, and $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \to \bar{S}$ is a compactification of h.
(ii): There is two things to prove:

- Let $\bar{f}_0 : \bar{X}_0 \to \bar{S}$ a compactification of $h : V \to S$ and $\bar{f}'_{00} : \bar{X}'_{00} \to \bar{S}$ a compactification of $h' : V' \to S$ (see (i)). Let $l_0 : \bar{X}'_0 \hookrightarrow \bar{\Gamma}_g \subset \bar{X}'_{00} \times_{\bar{S}} \bar{X}_0$ be the closure of the graph of g, $\bar{f}'_0 := (\bar{f}'_{00}, \bar{f}_0) \circ l_0 : \bar{X}'_0 \hookrightarrow \bar{X}'_{00} \times_S \bar{X}_0 \to \bar{S}$ and $\bar{g}_0 := p_{\bar{X}_0} \circ l_0 : \bar{X}'_0 \hookrightarrow \bar{X}'_{00} \times_{\bar{S}} \bar{X}_0 \to \bar{X}_0$, $\epsilon_{\bar{X}'_{00}} := p_{\bar{X}'_0} \circ i : \bar{X}'_0 \hookrightarrow \bar{X}'_{00} \times_{\bar{S}} \bar{X}_0 \to \bar{X}'_{00}$ be the restriction to X of the projections. Then $\epsilon_{\bar{X}'_{00}} : \bar{X}'_0 \to \bar{X}'_{00}$ is a proper modification which does not affect the open subset $V' \subset \bar{X}'_0$, $\bar{f}'_0 : \bar{X}'_0 \to \bar{S}$ is an other compactification of $h' : V' \to S$ and $\bar{g}_0 : \bar{X}'_0 \to \bar{X}_0$ is a compactification of g.
- In the case V and V' are smooth, we take, using theorem 15, a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ of the pair $(\bar{X}_0, \bar{Z})$. Take then, using theorem 15, a strict desingularization $\bar{\epsilon}'_1 : (\bar{X}', \bar{D}') \to (\bar{X} \times_{\bar{X}_0} \bar{X}'_0, \bar{X} \times_{\bar{X}_0} \bar{Z}')$ of the pair $(\bar{X} \times_{\bar{X}_0} \bar{X}'_0, \bar{X} \times_{\bar{X}_0} \bar{Z}')$. We consider then following commutative diagram whose square is cartesian :

$$\begin{tikzcd}
 & \bar{X}'_0 \arrow[r, "\bar{g}_0"] & X_0 \\
 & \bar{X} \times_{\bar{X}_0} \bar{X}'_0 \arrow[u, "\bar{\epsilon}'_0"] \arrow[r, "\bar{g}'_0"] & \bar{X} \arrow[u, "\epsilon"'] \\
\bar{X}' \arrow[uur, "\bar{\epsilon}'"] \arrow[ur, "\bar{\epsilon}'_1"] \arrow[urr, "\bar{g}"'] & &
\end{tikzcd}$$

  and $\bar{\epsilon}' := \bar{\epsilon}'_0 \circ \bar{\epsilon}'_1 : (\bar{X}', \bar{D}') \to (\bar{X}'_0, \bar{Z}')$ is a strict desingularization of the pair $(\bar{X} \times_{\bar{X}_0} \bar{X}'_0, \bar{X} \times_{\bar{X}_0} \bar{Z}')$.

□

Let $S \in \mathrm{Var}(\mathbb{C})$. Recall we have the dual functor

$$\mathbb{D}_S : C(\mathrm{Var}(\mathbb{C})/S) \to C(\mathrm{Var}(\mathbb{C})/S),\ F \mapsto \mathbb{D}_S(F) := \mathcal{H}om(F, E_{et}(\mathbb{Z}(S/S)))$$

which induces the functor

$$L\mathbb{D}_S : C(\mathrm{Var}(\mathbb{C})/S) \to C(\mathrm{Var}(\mathbb{C})/S),\ F \mapsto L\mathbb{D}_S(F) := \mathbb{D}_S(LF) := \mathcal{H}om(LF, E_{et}(\mathbb{Z}(S/S))).$$

We will use the following resolutions of representable presheaves by Corti-Hanamura presheaves and their the functorialities.

Definition 34. *(i) Let $h : U \to S$ a morphism, with $U, S \in \mathrm{Var}(\mathbb{C})$ and U smooth. Take, see definition-proposition 12, $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \to \bar{S}$ a compactification of $h : U \to S$ and denote by $\bar{Z} = \bar{X}_0 \backslash U$. Take, using theorem 15(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ of the pair $(\bar{X}_0, \bar{Z})$, with $\bar{X} \in \mathrm{PSmVar}(\mathbb{C})$ and $\bar{D} := \epsilon^{-1}(\bar{Z}) = \cup_{i=1}^s \bar{D}_i \subset \bar{X}$ a normal crossing divisor. We denote by $i_\bullet : \bar{D}_\bullet \hookrightarrow \bar{X} = \bar{X}_{c(\bullet)}$ the morphism of simplicial varieties given by the closed embeddings*

$i_I : \bar{D}_I = \cap_{i\in I}\bar{D}_i \hookrightarrow \bar{X}$ *We denote by* $j : U \hookrightarrow \bar{X}$ *the open embedding and by* $p_S : \bar{X}\times S \to S$ *and* $p_S : U \times S \to S$ *the projections. Considering the graph factorization* $\bar{f} : \bar{X} \xrightarrow{\bar{l}} \bar{X}\times\bar{S} \xrightarrow{p_{\bar{S}}} \bar{S}$ *of* $\bar{f} : \bar{X} \to \bar{S}$, *where* $\bar{l}$ *is the graph embedding and* $p_{\bar{S}}$ *the projection, we get closed embeddings* $l := \bar{l}\times_{\bar{S}} S : X \hookrightarrow \bar{X}\times S$ *and* $l_{D_I} := \bar{D}_I \times_{\bar{X}} l : D_I \hookrightarrow \bar{D}_I \times S$. *We then consider the following map in* $C(\mathrm{Var}(\mathbb{C})^2/S)$

$$r_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S)) : R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S))$$
$$\xrightarrow{:=} p_{S*}E_{et}\,\mathrm{Cone}(\mathbb{Z}(i_\bullet\times I) : p_{S*}E_{et}((\mathbb{Z}((\bar{D}_\bullet\times S, D_\bullet)/\bar{X}\times S), u_{IJ}) \to \mathbb{Z}((\bar{X}\times S, X)/\bar{X}\times S)))$$
$$\xrightarrow{(0,k\circ\mathrm{ad}((j\times I)^*,(j\times I)_*)(\mathbb{Z}((\bar{X}\times S,X)/\bar{X}\times S)))} p_{S*}E_{et}(\mathbb{Z}((U\times S,U)/U\times S))) =: \mathbb{D}^{12}_S(\mathbb{Z}(U/S)).$$

Note that $\mathbb{Z}((\bar{D}_I\times S, D_I)/\bar{X}\times S)$ *and* $\mathbb{Z}((\bar{X}\times S, X)/\bar{X}\times S))$ *are obviously* $\mathbb{A}^1$ *invariant. Note that* $r_{(X,D)/S}$ *is NOT an equivalence* $(\mathbb{A}^1, et)$ *local by proposition 23 since* $\rho_{\bar{X}\times S*}\mathbb{Z}((\bar{D}_\bullet\times S, D_\bullet)/\bar{X}\times S) = 0$, *whereas* $\rho_{\bar{X}\times S*}\mathrm{ad}((j\times I)^*,(j\times I)_*)(\mathbb{Z}((\bar{X}\times S, X)/\bar{X}\times S)))$ *is not an equivalence* $(\mathbb{A}^1, et)$ *local.*

(ii) *Let* $g : U'/S \to U/S$ *a morphism, with* $U'/S = (U', h'), U/S = (U, h) \in \mathrm{Var}(\mathbb{C})/S$, *with* U *and* U' *smooth. Take, see definition-proposition 12(ii),a compactification* $\bar{f}_0 = \bar{h} : \bar{X}_0 \to \bar{S}$ *of* $h : U \to S$ *and a compactification* $\bar{f}'_0 = \bar{h}' : \bar{X}'_0 \to \bar{S}$ *of* $h' : U' \to S$ *such that* $g : U'/S \to U/S$ *extend to a morphism* $\bar{g}_0 : \bar{X}'_0/\bar{S} \to \bar{X}_0/\bar{S}$. *Denote* $\bar{Z} = \bar{X}_0\backslash U$ *and* $\bar{Z}' = \bar{X}'_0\backslash U'$. *Take, see definition-proposition 12(ii), a strict desingularization* $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ *of* $(\bar{X}_0, \bar{Z})$, *a strict desingularization* $\bar{\epsilon}'_\bullet : (\bar{X}', \bar{D}') \to (\bar{X}'_0, \bar{Z}')$ *of* $(\bar{X}'_0, \bar{Z}')$ *and a morphism* $\bar{g} : \bar{X}' \to \bar{X}$ *such that the following diagram commutes*

$$\begin{tikzcd} \bar{X}'_0 \arrow[r, "\bar{g}_0"] & \bar{X}_0 \\ \bar{X}' \arrow[u, "\bar{\epsilon}'"] \arrow[r, "\bar{g}"] & \bar{X} \arrow[u, "\bar{\epsilon}"] \end{tikzcd}.$$

We then have, see definition-proposition 12(ii), the following commutative diagram in $\mathrm{Fun}(\Delta, \mathrm{Var}(\mathbb{C}))$

$$\begin{tikzcd} U = U_{c(\bullet)} \arrow[r, "j"] & \bar{X} = \bar{X}_{c(\bullet)} & & \bar{D}_{s_g(\bullet)} \arrow[ll, "i_\bullet"'] \\ U' = U'_{c(\bullet)} \arrow[u, "g"] \arrow[r, "j'"] & \bar{X}' = \bar{X}'_{c(\bullet)} \arrow[u, "\bar{g}"] & \bar{D}'_\bullet \arrow[l, "i'_\bullet"'] & \bar{g}^{-1}(\bar{D}_{s_g(\bullet)}) : i'_{g\bullet} \arrow[l, "i''_{g\bullet}"'] \arrow[u, "\bar{g}'_\bullet"] \end{tikzcd} \tag{41}$$

Denote by $p_S : \bar{X}\times S \to S$ *and* $p'_S : \bar{X}'\times S \to S$ *the projections We then consider the following map in* $C(\mathrm{Var}(\mathbb{C})^2/S)$

$$R^{CH}_S(g) : R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S))$$
$$p_{S*}E_{et}\,\mathrm{Cone}(\mathbb{Z}(i_\bullet\times I) : p_{S*}E_{et}((\mathbb{Z}((\bar{D}_{s_g(\bullet)}\times S, D_{s_g(\bullet)})/\bar{X}\times S), u_{IJ}) \to \mathbb{Z}((\bar{X}\times S, X)/\bar{X}\times S)))$$
$$\xrightarrow{T(\bar{g},E)(-)\circ p_{\bar{X}*}\mathrm{ad}((\bar{g}\times I)^*,(\bar{g}\times I)_*)(-)}$$
$$p'_{S*}E_{et}\,\mathrm{Cone}(\mathbb{Z}(i'_{g\bullet}\times I) :$$
$$p'_{S*}E_{et}((\mathbb{Z}((\bar{g}^{-1}(\bar{D}_{s_g(\bullet)})\times S, \bar{g}^{-1}(D_{s_g(\bullet)})/\bar{X}'\times S), u_{IJ}) \to \mathbb{Z}((\bar{X}'\times S, X')/\bar{X}'\times S)))$$
$$\xrightarrow{p'_{S*}E_{et}(\mathbb{Z}(i''_{g\bullet}\times I),I)}$$
$$p'_{S*}E_{et}\,\mathrm{Cone}(\mathbb{Z}(i'_\bullet\times I) : p'_{S*}E_{et}((\mathbb{Z}((\bar{D}'_\bullet\times S, D'_\bullet))/\bar{X}'\times S), u_{IJ}) \to \mathbb{Z}((\bar{X}'\times S, X')/\bar{X}'\times S)))$$
$$\xrightarrow{=:} R_{(\bar{X}',\bar{D}')/S}(\mathbb{Z}(U'/S))$$

Then by the diagram (41) and adjonction, the following diagram in $C(\mathrm{Var}(\mathbb{C})^2/S)$ *obviously com-*

mutes

$$\begin{tikzcd} R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S)) \arrow[r, "r_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S))"] \arrow[d, "R^{CH}_S(g)"'] & p_{S*}E_{et}(\mathbb{Z}((U\times S,U)/U\times S)) =: \mathbb{D}^{12}_S(\mathbb{Z}(U/S)) \arrow[d, "D^{12}_S(g):=T(g\times I,E)(-)\circ\mathrm{ad}((g\times I)^*,(g\times I)_*)(E_{et}(\mathbb{Z}((U\times S,U)/U\times S)))"] \\ R_{(\bar{X}',\bar{D}')/S}(\mathbb{Z}(U'/S)) \arrow[r, "r_{(\bar{X}',\bar{D}')/S}(\mathbb{Z}(U'/S))"] & p'_{S*}E_{et}(\mathbb{Z}((U'\times S,U')/U'\times S)) =: \mathbb{D}^{12}_S(\mathbb{Z}(U'/S)) \end{tikzcd}.$$

(iii0) For

$$Q^* = (\mathbb{Z}(U^*/S)) = (\cdots \to \mathbb{Z}(U^n/S) \xrightarrow{\mathbb{Z}(g_n)} \mathbb{Z}(U^{n-1}/S) \to \cdots) \in C(\mathrm{Var}(\mathbb{C})/S)$$

a complex of representable presheaves with U^* *smooth, we get from (i) and (ii)* $(\bar{X}^*,\bar{D}^*)/S \in\in \mathrm{Var}(\mathbb{C})^2/S$ *with* $\bar{X}^* \in \mathrm{PSmVar}(\mathbb{C})$, $\bar{D}^* \subset \bar{X}^*$ *a normal crossing divisor, inducing a complex together with a map in* $C(\mathrm{Var}(\mathbb{C})^2/S)$

$$r_(\bar{X}^*,\bar{D}^*)/S(Q^*) : R_{(\bar{X}^*,\bar{D}^*)/S}(Q^*) = (\cdots \to R_{(\bar{X}^{n-1},\bar{D}^{n-1})/S}(\mathbb{Z}(U^{n-1}/S)) \xrightarrow{R^{CH}_S(g_n)} R_{(\bar{X}^n,\bar{D}^n)/S}(\mathbb{Z}(U^n/S)) \to \cdots) \to \mathbb{D}^{12}_S(Q^*)$$

(iii) For

$$Q^* := (\cdots \to \oplus_{\alpha\in\Lambda^n}\mathbb{Z}(U^n_\alpha/S) \xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))} \oplus_{\beta\in\Lambda^{n-1}}\mathbb{Z}(U^{n-1}_\beta/S) \to \cdots) \in C(\mathrm{Var}(\mathbb{C})/S)$$

a complex of (maybe infinite) direct sum of representable presheaves with U^*_α *smooth, we get from (i) and (ii) the map in* $C(\mathrm{Var}(\mathbb{C})^2/S)$

$$r^{CH}_S(Q^*) : R^{CH}(Q^*) := (\cdots \to \oplus_{\beta\in\Lambda^{n-1}} \varinjlim_{(\bar{X}^{n-1}_\beta,\bar{D}^{n-1}_\beta)/S} R_{(\bar{X}^{n-1}_\beta,\bar{D}^{n-1}_\beta)/S}(\mathbb{Z}(U^{n-1}_\beta/S)) \xrightarrow{(R^{CH}_S(g^n_{\alpha,\beta}))} \oplus_{\alpha\in\Lambda^n} \varinjlim_{(\bar{X}^n_\alpha,\bar{D}^n_\alpha)/S} R_{(\bar{X}^n_\alpha,\bar{D}^n_\alpha)/S}(\mathbb{Z}(U^n_\alpha/S)) \to \cdots) \to \mathbb{D}^{12}_S(Q^*),$$

where for $(U^n_\alpha,h^n_\alpha) \in \mathrm{Var}(\mathbb{C})/S$, *the inductive limit run over all the compactifications* $\bar{f}_\alpha : \bar{X}_\alpha \to \bar{S}$ *of* $h_\alpha : U_\alpha \to S$ *with* $\bar{X}_\alpha \in \mathrm{PSmVar}(\mathbb{C})$ *and* $\bar{D}_\alpha := \bar{X}_\alpha\backslash U_\alpha$ *a normal crossing divisor. For* $m = (m^*) : Q^*_1 \to Q^*_2$ *a morphism with*

$$Q^*_1 := (\cdots \to \oplus_{\alpha\in\Lambda^n}\mathbb{Z}(U^n_{1,\alpha}/S) \xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))} \oplus_{\beta\in\Lambda^{n-1}}\mathbb{Z}(U^{n-1}_{1,\beta}/S) \to \cdots),$$
$$Q^*_2 := (\cdots \to \oplus_{\alpha\in\Lambda^n}\mathbb{Z}(U^n_{2,\alpha}/S) \xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))} \oplus_{\beta\in\Lambda^{n-1}}\mathbb{Z}(U^{n-1}_{2,\beta}/S) \to \cdots) \in C(\mathrm{Var}(\mathbb{C})/S)$$

complexes of (maybe infinite) direct sum of representable presheaves with $U^*_{1,\alpha}$ *and* $U^*_{2,\alpha}$ *smooth, we get again from (i) and (ii) a commutative diagram in* $C(\mathrm{Var}(\mathbb{C})^2/S)$

$$\begin{tikzcd} R^{CH}(Q^*_2) \arrow[r, "r^{CH}_S(Q^*_2)"] \arrow[d, "R^{CH}_S(m):=(R^{CH}_S(m^*))"'] & \mathbb{D}^{12}_S(Q^*_2) \arrow[d, "\mathbb{D}^{12}_S(m):=(\mathbb{D}^{12}_S(m^*))"] \\ R^{CH}(Q^*_1) \arrow[r, "r^{CH}_S(Q^*_1)"] & \mathbb{D}^{12}_S(Q^*_1) \end{tikzcd}.$$

- Let $S \in \mathrm{SmVar}(\mathbb{C})$. Let $F \in \mathrm{PSh}(\mathrm{Var}(\mathbb{C})^{sm}/S)$. Consider

$$q : LF := (\cdots \to \oplus_{(U_\alpha,h_\alpha)\in\mathrm{Var}(\mathbb{C})^{sm}/S}\mathbb{Z}(U_\alpha/S) \xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))} \oplus_{(U_\alpha,h_\alpha)\in\mathrm{Var}(\mathbb{C})^{sm}/S}\mathbb{Z}(U_\alpha/S) \to \cdots) \to F$$

the canonical projective resolution given in subsection 2.3.3. Note that the U_α are smooth since S is smooth and h_α are smooth morphism. Definition 34(iii) gives in this particular case the map in $C(\mathrm{Var}(\mathbb{C})^2/S)$

$$r_S^{CH}(\rho_S^* LF) : R^{CH}(\rho_S^* LF) := (\cdots \to \oplus_{(U_\alpha,h_\alpha)\in\mathrm{Var}(\mathbb{C})^{sm}/S} \varinjlim_{(\bar{X}_\alpha,\bar{D}_\alpha)/S} R_{(\bar{X}_\alpha,\bar{D}_\alpha)/S}(\mathbb{Z}(U_\alpha/S))$$
$$\xrightarrow{(R_S^{CH}(g^n_{\alpha,\beta}))} \oplus_{(U_\alpha,h_\alpha)\in\mathrm{Var}(\mathbb{C})^{sm}/S} \varinjlim_{(\bar{X}_\alpha,\bar{D}_\alpha)/S} R_{(\bar{X}_\alpha,\bar{D}_\alpha)/S}(\mathbb{Z}(U_\alpha/S)) \to \cdots) \to \mathbb{D}_S^{12}(\rho_S^* LF),$$

where for $(U_\alpha, h_\alpha) \in \mathrm{Var}(\mathbb{C})^{sm}/S$, the inductive limit run over all the compactifications $\bar{f}_\alpha : \bar{X}_\alpha \to \bar{S}$ of $h_\alpha : U_\alpha \to S$ with $\bar{X}_\alpha \in \mathrm{PSmVar}(\mathbb{C})$ and $\bar{D}_\alpha := \bar{X}_\alpha \backslash U_\alpha$ a normal crossing divisor. Definition 34(iii) gives then by functoriality in particular, for $F = F^\bullet \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$, the map in $C(\mathrm{Var}(\mathbb{C})^2/S)$

$$r_S^{CH}(\rho_S^* LF) = (r_S^{CH}(\rho_S^* LF^*)) : R^{CH}(\rho_S^* LF) \to \mathbb{D}_S^{12}(\rho_S^* LF).$$

- Let $g : T \to S$ a morphism with $T, S \in \mathrm{SmVar}(\mathbb{C})$. Let $h : U \to S$ a smooth morphism with $U \in \mathrm{Var}(\mathbb{C})$. Consider the cartesian square

$$\begin{tikzcd} U_T \arrow[r, "h'"] \arrow[d, "g'"] & T \arrow[d, "g"] \\ U \arrow[r, "h"] & S \end{tikzcd}$$

Note that U is smooth since S and h are smooth, and U_T is smooth since T and h' are smooth. Take, see definition-proposition 12(ii),a compactification $\bar{f}_0 = \bar{h} : \bar{X}_0 \to \bar{S}$ of $h : U \to S$ and a compactification $\bar{f}'_0 = g \bar{\circ} h' : \bar{X}'_0 \to \bar{S}$ of $g \circ h' : U' \to S$ such that $g' : U_T/S \to U/S$ extend to a morphism $\bar{g}'_0 : \bar{X}'_0/\bar{S} \to \bar{X}_0/\bar{S}$. Denote $\bar{Z} = \bar{X}_0 \backslash U$ and $\bar{Z}' = \bar{X}'_0 \backslash U_T$. Take, see definition-proposition 12(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ of $(\bar{X}_0, \bar{Z})$, a desingularization $\bar{\epsilon}'_\bullet : (\bar{X}', \bar{D}') \to (\bar{X}'_0, \bar{Z}')$ of $(\bar{X}'_0, \bar{Z}')$ and a morphism $\bar{g}' : \bar{X}' \to \bar{X}$ such that the following diagram commutes

$$\begin{tikzcd} \bar{X}'_0 \arrow[r, "\bar{g}'_0"] & \bar{X}_0 \\ \bar{X}' \arrow[u, "\bar{\epsilon}'"] \arrow[r, "\bar{g}'"] & \bar{X} \arrow[u, "\bar{\epsilon}"] \end{tikzcd} .$$

We then have, see definition-proposition 12(ii), the following commutative diagram in $\mathrm{Fun}(\Delta, \mathrm{Var}(\mathbb{C}))$

$$\begin{tikzcd} U = U_{c(\bullet)} \arrow[r, "j"] & \bar{X} = \bar{X}_{c(\bullet)} & & \bar{D}_{s_{g'}(\bullet)} \arrow[ll, "i_\bullet"'] \\ U_T = U_{T,c(\bullet)} \arrow[u, "g'"] \arrow[r, "j'"] & \bar{X}' = X'_{c(\bullet)} \arrow[u, "\bar{g}'"] & \bar{D}'_\bullet \arrow[l, "i'_\bullet"'] & \bar{g}^{-1}(\bar{D}_{s_{g'}(\bullet)}) : i'_{g\bullet} \arrow[l, "i''_{g'\bullet}"'] \arrow[u, "(\bar{g}')'_\bullet"] \end{tikzcd}$$

We then consider the following map in $C(\mathrm{Var}(\mathbb{C})^2/T)$, see definition 34(ii)

$$T(g, R^{CH})(\mathbb{Z}(U/S)) : g^* R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S))$$
$$\xrightarrow{g^* R_S^{CH}(g')} g^* R_{(\bar{X}',\bar{D}')/S}(\mathbb{Z}(U_T/S)) = g^* g_* R_{(\bar{X}',\bar{D}')/T}(\mathbb{Z}(U_T/T))$$
$$\xrightarrow{\mathrm{ad}(g^*,g_*)(R_{(\bar{X}',\bar{D}')/T}(\mathbb{Z}(U_T/T)))} R_{(\bar{X}',\bar{D}')/T}(\mathbb{Z}(U_T/T))$$

For

$$Q^* := (\cdots \to \oplus_{\alpha\in\Lambda^n}\mathbb{Z}(U^n_\alpha/S) \xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))} \oplus_{\beta\in\Lambda^{n-1}}\mathbb{Z}(U^{n-1}_\beta/S) \to \cdots) \in C(\mathrm{Var}(\mathbb{C})/S)$$

a complex of (maybe infinite) direct sum of representable presheaves with $h^n_\alpha : U^n_\alpha \to S$ smooth, we get the map in $C(\mathrm{Var}(\mathbb{C})^2/T)$

$$T(g,R^{CH})(Q^*) : g^*R^{CH}(Q^*) = (\cdots \to \oplus_{\alpha\in\Lambda^n} \varinjlim_{(\bar{X}^n_\alpha,\bar{D}^n_\alpha)/S} g^*R_{(\bar{X}^n_\alpha,\bar{D}^n_\alpha)/S}(\mathbb{Z}(U^n_\alpha/S)) \to \cdots)$$

$$\xrightarrow{(T(g,R^{CH})(\mathbb{Z}(U^n_\alpha/S)))} (\cdots \to \oplus_{\alpha\in\Lambda^n} \varinjlim_{(\bar{X}^{n\prime}_\alpha,\bar{D}^{n\prime}_\alpha)/T} R_{(\bar{X}^{n\prime}_\alpha,\bar{D}^{n\prime}_\alpha)/T}(\mathbb{Z}(U^n_{\alpha,T}/S)) \to \cdots) =: R^{CH}(g^*Q^*).$$

Let $F \in \mathrm{PSh}(\mathrm{Var}(\mathbb{C})^{sm}/S)$. Consider

$$q : LF := (\cdots \to \oplus_{(U_\alpha,h_\alpha)\in\mathrm{Var}(\mathbb{C})^{sm}/S}\mathbb{Z}(U_\alpha/S) \to \cdots) \to F$$

the canonical projective resolution given in subsection 2.3.3. We then get in particular the map in $C(\mathrm{Var}(\mathbb{C})^2/T)$

$$T(g,R^{CH})(\rho_S^*LF) : g^*R^{CH}(\rho_S^*LF) =$$

$$(\cdots \to \oplus_{(U_\alpha,h_\alpha)\in\mathrm{Var}(\mathbb{C})^{sm}/S} \varinjlim_{(\bar{X}_\alpha,\bar{D}_\alpha)/S} g^*R_{(\bar{X}_\alpha,\bar{D}_\alpha)/S}(\mathbb{Z}(U_\alpha/S)) \to \cdots) \xrightarrow{(T(g,R^{CH})(\mathbb{Z}(U_\alpha/S)))}$$

$$(\cdots \to \oplus_{(U_\alpha,h_\alpha)\in\mathrm{Var}(\mathbb{C})^{sm}/S} \varinjlim_{(\bar{X}'_\alpha,\bar{D}'_\alpha)/T} R_{(\bar{X}'_\alpha,\bar{D}'_\alpha)/T}(\mathbb{Z}(U_{\alpha,T}/S)) \to \cdots) =: R^{CH}(\rho_T^*g^*LF).$$

By functoriality, we get in particular for $F = F^\bullet \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$, the map in $C(\mathrm{Var}(\mathbb{C})^2/T)$

$$T(g,R^{CH})(\rho_S^*LF) : g^*R^{CH}(\rho_S^*LF) \to R^{CH}(\rho_T^*g^*LF).$$

- Let $S_1, S_2 \in \mathrm{SmVar}(\mathbb{C})$ and $p : S_1 \times S_2 \to S_1$ the projection. Let $h : U \to S_1$ a smooth morphism with $U \in \mathrm{Var}(\mathbb{C})$. Consider the cartesian square

$$\begin{tikzcd} U \times S_2 \arrow[r, "h\times I"] \arrow[d, "p'"] & S_1 \times S_2 \arrow[d, "p"] \\ U \arrow[r, "h"] & S_1 \end{tikzcd}$$

Take, see definition-proposition 12(i),a compactification $\bar{f}_0 = \bar{h} : \bar{X}_0 \to \bar{S}_1$ of $h : U \to S_1$. Then $\bar{f}_0\times I : \bar{X}_0\times S_2 \to \bar{S}_1\times S_2$ is a compactification of $h\times I : U\times S_2 \to S_1\times S_2$ and $p' : U\times S_2 \to U$ extend to $\bar{p}'_0 := p_{X_0} : \bar{X}_0 \times S_2 \to \bar{X}_0$. Denote $Z = X_0\backslash U$. Take see theorem 15(i), a strict desingularization $\bar{\epsilon} : (\bar{X},\bar{D}) \to (\bar{X}_0,\bar{Z})$ of the pair $(\bar{X}_0,\bar{Z})$. We then have the following commutative diagram in $\mathrm{Fun}(\Delta,\mathrm{Var}(\mathbb{C}))$ whose squares are cartesian

$$\begin{tikzcd} U = U_{c(\bullet)} \arrow[r, "j"] & \bar{X} & \bar{D}_\bullet \arrow[l, "i_\bullet"'] \\ U \times S_2 = (U\times S_2)_{c(\bullet)} \arrow[u, "g"] \arrow[r, "j\times I"] & \bar{X}\times S_2 \arrow[u, "\bar{p}':=p_{\bar{X}}"'] & \bar{D}_\bullet \times S_2 \arrow[l, "i'_\bullet"'] \arrow[u, "\bar{p}'_\bullet"'] \end{tikzcd} \tag{42}$$

Then the map in $C(\mathrm{Var}(\mathbb{C})^2/S_1\times S_2)$

$$T(p,R^{CH})(\mathbb{Z}(U/S_1)) : p^*R_{(\bar{X},\bar{D})/S_1}(\mathbb{Z}(U/S_1)) \xrightarrow{\sim} R_{(\bar{X}\times S_2,\bar{D}_\bullet\times S_2)/S_1\times S_2}(\mathbb{Z}(U\times S_2/S_1\times S_2))$$

is an isomorphism. Hence, for $Q^* \in C(\mathrm{Var}(\mathbb{C})/S_1)$ a complex of (maybe infinite) direct sum of representable presheaves of smooth morphism, the map in $C(\mathrm{Var}(\mathbb{C})^2/S_1 \times S_2)$

$$T(p, R^{CH})(Q^*) : p^* R^{CH}(Q^*) \xrightarrow{\sim} R^{CH}(p^* Q^*)$$

is an isomorphism. In particular, for $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S_1)$ the map in $C(\mathrm{Var}(\mathbb{C})^2/S_1 \times S_2)$

$$T(p, R^{CH})(\rho_{S_1}^* LF) : p^* R^{CH}(\rho_{S_1}^* LF) \xrightarrow{\sim} R^{CH}(\rho_{S_1 \times S_2}^* p^* LF)$$

is an isomorphism.

- Let $h_1 : U_1 \to S$, $h_2 : U_2 \to S$ two morphisms with $U_1, U_2, S \in \mathrm{Var}(\mathbb{C})$, U_1, U_2 smooth. Denote by $p_1 : U_1 \times_S U_2 \to U_1$ and $p_2 : U_1 \times_S U_2 \to U_2$ the projections. Take, see definition-proposition 12(i)), a compactification $\bar{f}_{10} = \bar{h}_1 : \bar{X}_{10} \to \bar{S}$ of $h_1 : U_1 \to S$ and a compactification $\bar{f}_{20} = \bar{h}_2 : \bar{X}_{20} \to \bar{S}$ of $h_2 : U_2 \to S$. Then,

  - $\bar{f}_{10} \times \bar{f}_{20} : \bar{X}_{10} \times_{\bar{S}} \bar{X}_{20} \to S$ is a compactification of $h_1 \times h_2 : U_1 \times_S U_2 \to S$.
  - $\bar{p}_{10} := p_{X_{10}} : \bar{X}_{10} \times_{\bar{S}} \bar{X}_{20} \to \bar{X}_{10}$ is a compactification of $p_1 : U_1 \times_S U_2 \to U_1$.
  - $\bar{p}_{20} := p_{X_{20}} : \bar{X}_{10} \times_{\bar{S}} \bar{X}_{20} \to \bar{X}_{20}$ is a compactification of $p_2 : U_1 \times_S U_2 \to U_2$.

  Denote $\bar{Z}_1 = \bar{X}_{10} \backslash U_1$ and $\bar{Z}_2 = \bar{X}_{20} \backslash U_2$. Take, see theorem 15(i), a strict desingularization $\bar{\epsilon}_1 : (\bar{X}_1, \bar{D}) \to (\bar{X}_{10}, Z_1)$ of the pair $(\bar{X}_{10}, \bar{Z}_1)$ and a strictdesingularization $\bar{\epsilon}_2 : (\bar{X}_2, \bar{E}) \to (\bar{X}_{20}, Z_2)$ of the pair $(\bar{X}_{20}, \bar{Z}_2)$. Take then a strict desingularization

$$\bar{\epsilon}_{12} : ((\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N, \bar{F}) \to (\bar{X}_1 \times_{\bar{S}} \bar{X}_2, (D \times_{\bar{S}} \bar{X}_2) \cup (\bar{X}_1 \times_{\bar{S}} \bar{E}))$$

  of the pair $(\bar{X}_1 \times_{\bar{S}} \bar{X}_2, (\bar{D} \times_{\bar{S}} \bar{X}_2) \cup (\bar{X}_1 \times_{\bar{S}} \bar{E}))$. We have then the following commutative diagram

$$\begin{tikzcd}
 & \bar{X}_1 \arrow[r, "\bar{f}_1"] & \bar{S} \\
 & \bar{X}_1 \times_{\bar{S}} \bar{X}_2 \arrow[u, "\bar{p}_2"] \arrow[r, "\bar{p}_1"] & \bar{X}_2 \arrow[u, "\bar{f}_2"'] \\
(\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N \arrow[uur, "(\bar{p}_2)^N"] \arrow[ur, "\bar{\epsilon}_{12}"] \arrow[urr, "(\bar{p}_1)^N"'] & &
\end{tikzcd}$$

  and

  - $\bar{f}_1 \times \bar{f}_2 : \bar{X}_1 \times_{\bar{S}} \bar{X}_2 \to \bar{S}$ is a compactification of $h_1 \times h_2 : U_1 \times_S U_2 \to S$.
  - $(\bar{p}_1)^N := \bar{p}_1 \circ \epsilon_{12} : (\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N \to \bar{X}_1$ is a compactification of $p_1 : U_1 \times_S U_2 \to U_1$.
  - $(\bar{p}_2)^N := \bar{p}_2 \circ \epsilon_{12} : (\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N \to \bar{X}_2$ is a compactification of $p_2 : U_1 \times_S U_2 \to U_2$.

  We have then the isomorphism in $C(\mathrm{Var}(\mathbb{C})^2/S)$

$$\begin{aligned} T(\otimes, R_S^{CH})(\mathbb{Z}(U_1/S), \mathbb{Z}(U_2/S)) := R_S^{CH}(p_1) \otimes R_S^{CH}(p_2) : \\ R_{(\bar{X}_1, \bar{D})/S}(\mathbb{Z}(U_1/S)) \otimes R_{(X_2, E))/S}(\mathbb{Z}(U_2/S)) \xrightarrow{\sim} R_{(\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N, \bar{F})/S}(\mathbb{Z}(U_1 \times_S U_2/S)) \end{aligned}$$

  For

$$\begin{aligned} Q_1^* := (\cdots \to \oplus_{\alpha \in \Lambda^n} \mathbb{Z}(U_{1,\alpha}^n/S) \xrightarrow{(\mathbb{Z}(g_{\alpha,\beta}^n))} \oplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U_{1,\beta}^{n-1}/S) \to \cdots), \\ Q_2^* := (\cdots \to \oplus_{\alpha \in \Lambda^n} \mathbb{Z}(U_{2,\alpha}^n/S) \xrightarrow{(\mathbb{Z}(g_{\alpha,\beta}^n))} \oplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U_{2,\beta}^{n-1}/S) \to \cdots) \in C(\mathrm{Var}(\mathbb{C})/S) \end{aligned}$$

complexes of (maybe infinite) direct sum of representable presheaves with U^*_α smooth, we get the morphism in $C(\mathrm{Var}(\mathbb{C})^2/S)$

$$T(\otimes,R^{CH}_S)(Q^*_1,Q^*_2):R^{CH}(Q^*_1)\otimes R^{CH}(Q^*_2)\xrightarrow{(T(\otimes,R^{CH}_S)(\mathbb{Z}(U^m_{1,\alpha}),\mathbb{Z}(U^n_{2,\beta}))}R^{CH}(Q^*_1\otimes Q^*_2)).$$

For $F_1,F_2\in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$, we get in particular the morphism in $C(\mathrm{Var}(\mathbb{C})^2/S)$

$$T(\otimes,R^{CH}_S)(\rho^*_S LF_1,\rho^*_S LF_2):R^{CH}(\rho^*_S LF_1)\otimes R^{CH}(\rho^*_S LF_2)\to R^{CH}(\rho^*_S(LF_1\otimes LF_2)).$$

Definition 35. *Let $h:U\to S$ a morphism, with $U,S\in\mathrm{Var}(\mathbb{C})$. Take, see definition-proposition 12, $\bar{f}_0=\bar{h}_0:\bar{X}_0\to\bar{S}$ a compactification of $h:U\to S$ and denote by $\bar{Z}=\bar{X}_0\backslash U$. Take, using theorem 15(ii), a desingularization $\bar{\epsilon}:(\bar{X},\bar{D})\to(\bar{X}_0,\Delta)$ of the pair $(\bar{X}_0,\Delta)$, $\bar{Z}\subset\Delta$, with $\bar{X}\in\mathrm{PSmVar}(\mathbb{C})$ and $\bar{D}:=\bar{\epsilon}^{-1}(\Delta)=\cup^s_{i=1}\bar{D}_i\subset\bar{X}$ a normal crossing divisor.*

(i) The cycle $(\Delta_{\bar{D}_\bullet}\times S)\subset\bar{D}_\bullet\times\bar{D}_\bullet\times S$ induces by the diagonal $\Delta_{\bar{D}_\bullet}\subset\bar{D}_\bullet\times\bar{D}_\bullet$ gives the morphism in $C(\mathrm{Var}(\mathbb{C})^2/S)$

$$\begin{aligned}&[\Delta_{\bar{D}_\bullet}]\in\mathrm{Hom}(\mathbb{Z}((\bar{D}_\bullet\times S,D_\bullet)/S),p_{S*}E_{et}(\mathbb{Z}((\bar{D}_\bullet\times S,D_\bullet)/\bar{X}\times S)(d_X)[2d_X]))\xrightarrow{\sim}\\&\mathrm{Hom}(\mathbb{Z}((\bar{D}_\bullet\times S\times\bar{X},D_\bullet)/\bar{X}\times S),C_*\mathbb{Z}^{equi0}((\bar{D}_\bullet\times S\times\mathbb{A}^{d_X},D_\bullet\times\mathbb{A}^{d_X})/\bar{X}\times S))\\&\subset H^0(\mathcal{Z}_{d_{D_\bullet}+d_S}(\square^*\times\bar{D}_\bullet\times\bar{D}_\bullet\times S),s.t.\alpha_*(\square^*\times D_\bullet)=D_\bullet)\end{aligned}$$

(ii) The cycle $(\Delta_{\bar{X}}\times S)\subset\bar{X}\times\bar{X}\times S$ induces by the diagonal $\Delta_{\bar{X}}\subset\bar{X}\times\bar{X}$ gives the morphism in $C(\mathrm{Var}(\mathbb{C})^2/S)$

$$\begin{aligned}&[\Delta_{\bar{X}}]\in\mathrm{Hom}(\mathbb{Z}((\bar{X}\times S,X)/S),p_{S*}E_{et}(\mathbb{Z}((\bar{X}\times S,X)/\bar{X}\times S)(d_X)[2d_X]))\xrightarrow{\sim}\\&\mathrm{Hom}(\mathbb{Z}((\bar{X}\times S\times\bar{X},X)/\bar{X}\times S),C_*\mathbb{Z}^{equi0}((\bar{X}\times S\times\mathbb{A}^{d_X},X\times\mathbb{A}^{d_X})/\bar{X}\times S))\\&\subset H^0(\mathcal{Z}_{d_X+d_S}(\square^*\times\bar{X}\times\bar{X}\times S),s.t.\alpha_*(\square^*\times X)=X)\end{aligned}$$

Let $h:U\to S$ a morphism, with $U,S\in\mathrm{Var}(\mathbb{C})$, U smooth. Take, see definition-proposition 12, $\bar{f}_0=\bar{h}_0:\bar{X}_0\to\bar{S}$ a compactification of $h:U\to S$ and denote by $\bar{Z}=\bar{X}_0\backslash U$. Take, using theorem 15(ii), a strict desingularization $\bar{\epsilon}:(\bar{X},\bar{D})\to(\bar{X}_0,\bar{Z})$ of the pair $(\bar{X}_0,\bar{Z})$, with $\bar{X}\in\mathrm{PSmVar}(\mathbb{C})$ and $\bar{D}:=\bar{\epsilon}^{-1}(\bar{Z})=\cup^s_{i=1}\bar{D}_i\subset\bar{X}$ a normal crossing divisor. We get from (i) and (ii) the morphism in $C(\mathrm{Var}(\mathbb{C})^2/S)$

$$\begin{aligned}&T(p_{S\sharp},p_{S*})(\mathbb{Z}((\bar{D}_\bullet\times S,D_\bullet)/\bar{X}\times S),\mathbb{Z}((\bar{X}\times S,X)/\bar{X}\times S)):=([\Delta_{\bar{D}_\bullet}],[\Delta_{\bar{X}}]):\\&\mathrm{Cone}(\mathbb{Z}(i_\bullet\times I):(\mathbb{Z}((\bar{D}_\bullet\times S,D_\bullet)/S),u_{IJ})\to\mathbb{Z}((\bar{X}\times S,X)/S))\to\\&p_{S*}E_{et}\,\mathrm{Cone}(\mathbb{Z}(i_\bullet\times I):p_{S*}E_{et}((\mathbb{Z}((\bar{D}_\bullet\times S,D_\bullet)/\bar{X}\times S),u_{IJ})\to\\&\mathbb{Z}((\bar{X}\times S,X)/\bar{X}\times S)))(d_X)[2d_X]=:R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S))(d_X)[2d_X]\end{aligned}$$

We then consider the factorization in $C(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$

$$\begin{aligned}&\rho_{S*}\mu_{S*}T(p_{S\sharp},p_{S*})(\mathbb{Z}((\bar{D}_\bullet\times S,D_\bullet)/\bar{X}\times S),\mathbb{Z}((\bar{X}\times S,X)/\bar{X}\times S)):\\&\mathrm{Cone}(\mathbb{Z}(i_\bullet\times I):(\mathbb{Z}((\bar{D}_\bullet\times S,D_\bullet)/S),u_{IJ})\to\mathbb{Z}((\bar{X}\times S,X)/S))=\\&\rho_{S*}\mu_{S*}\,\mathrm{Cone}(\mathbb{Z}(i_\bullet\times I):(\mathbb{Z}((\bar{D}_\bullet\times S,D_\bullet)/S),u_{IJ})\to\mathbb{Z}((\bar{X}\times S,X)/S))\\&\xrightarrow{T^{\mu,q}(p_{S\sharp},p_{S*})(\mathbb{Z}((\bar{D}_\bullet\times S,D_\bullet)/\bar{X}\times S),\mathbb{Z}((\bar{X}\times S,X)/\bar{X}\times S))}\\&L\rho_{S*}\mu_{S*}R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S))(d_X)[2d_X]\xrightarrow{q}\rho_{S*}\mu_{S*}R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S))(d_X)[2d_X]\end{aligned}$$

Proposition 36. *Let $h:U\to S$ a morphism, with $U,S\in\mathrm{Var}(\mathbb{C})$. Take, see definition-proposition 12, $\bar{f}_0=\bar{h}_0:\bar{X}_0\to\bar{S}$ a compactification of $h:U\to S$ and denote by $\bar{Z}=\bar{X}_0\backslash U$. Take, using theorem 15(ii), a desingularization $\bar{\epsilon}:(\bar{X},\bar{D})\to(\bar{X}_0,\Delta)$ of the pair $(\bar{X}_0,\Delta)$, $\bar{Z}\subset\Delta$ with $\bar{X}\in\mathrm{PSmVar}(\mathbb{C})$ and $\bar{D}:=\bar{\epsilon}^{-1}(\Delta)=\cup^s_{i=1}\bar{D}_i\subset\bar{X}$ a normal crossing divisor.*

(i) The morphism

$$[\Delta_{\bar{D}_\bullet}] : \mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/S), \to p_{S*}E_{et}(\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/\bar{X} \times S)(d_X)[2d_X])$$

given in definition 35(i) is an equivalence $(\mathbb{A}^1, et)$ *local.*

(ii) The morphism

$$[\Delta_{\bar{X}}] : \mathbb{Z}((\bar{X} \times S, X)/S), \to p_{S*}E_{et}(\mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S)(d_X)[2d_X])$$

given in definition 35(ii) is an equivalence $(\mathbb{A}^1, et)$ *local.*

Let $h : U \to S$ *a morphism, with* $U, S \in \mathrm{Var}(\mathbb{C})$*,* U *smooth. Take, see definition-proposition 12,* $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \to \bar{S}$ *a compactification of* $h : U \to S$ *and denote by* $\bar{Z} = \bar{X}_0 \backslash U$*. Take, using theorem 15(ii), a strict desingularization* $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ *of the pair* $(\bar{X}_0, \bar{Z})$*, with* $\bar{X} \in \mathrm{PSmVar}(\mathbb{C})$ *and* $\bar{D} := \bar{\epsilon}^{-1}(Z) = \cup_{i=1}^s \bar{D}_i \subset \bar{X}$ *a normal crossing divisor.*

(iii) The morphism

$$\begin{aligned} T(p_{S\sharp}, p_{S*})(\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/\bar{X} \times S), \mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S)) := ([\Delta_{\bar{D}_\bullet}], [\Delta_{\bar{X}}]) : \\ \mathrm{Cone}(\mathbb{Z}(i_\bullet \times I) : (\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/S), u_{IJ}) \to \mathbb{Z}((\bar{X} \times S, X)/S)) \to \\ p_{S*}E_{et}\,\mathrm{Cone}(\mathbb{Z}(i_\bullet \times I) : p_{S*}E_{et}((\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/\bar{X} \times S), u_{IJ}) \to \\ \mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S)))(d_X)[2d_X] =: R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S))(d_X)[2d_X] \end{aligned}$$

given in definition 35(iii) is an equivalence $(\mathbb{A}^1, et)$ *local.*

(iii)' The morphism

$$\begin{aligned} T^{\mu,q}(p_{S\sharp}, p_{S*})(\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/\bar{X} \times S), \mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S)) : \\ \mathrm{Cone}(\mathbb{Z}(i_\bullet \times I) : (\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/S), u_{IJ}) \to \mathbb{Z}((\bar{X} \times S, X)/S)) \\ \to L\rho_{S*}\mu_{S*}R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S))(d_X)[2d_X] \end{aligned}$$

given in definition 35(iii) is an equivalence $(\mathbb{A}^1, et)$ *local.*

Proof. (i): By Yoneda lemma, it is equivalent to show that for every morphism $g : T \to S$ with $T \in \mathrm{Var}(\mathbb{C})$ and every closed subset $E \subset T$, the composition morphism

$$\begin{aligned} [\Delta_{\bar{D}_\bullet}] : \mathrm{Hom}^\bullet(\mathbb{Z}((T, E)/S), C_*\mathbb{Z}^{tr}((\bar{D}_\bullet \times S, D_\bullet)/S)) \xrightarrow{\mathrm{Hom}^\bullet(\mathbb{Z}((T,E)/S), C_*\Delta_{\bar{D}_\bullet})} \\ \mathrm{Hom}^\bullet(\mathbb{Z}((T, E)/S), p_{S*}E_{et}(\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/\bar{X} \times S)(d_X)[2d_X])) \end{aligned}$$

is a quasi-isomorphism of abelian groups. But this map is the composite

$$\begin{aligned} \mathrm{Hom}^\bullet(\mathbb{Z}((T, E)/S), C_*\mathbb{Z}^{tr}((\bar{D}_\bullet \times S, D_\bullet)/S)) \xrightarrow{[\Delta_{\bar{D}_\bullet}]} \\ \mathrm{Hom}^\bullet(\mathbb{Z}((T, E)/S), p_{S*}E_{et}(\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/\bar{X} \times S)(d_X)[2d_X])) \xrightarrow{\sim} \\ \mathrm{Hom}^\bullet(\mathbb{Z}((T \times \bar{X}, E)/S \times \bar{X}), C_*\mathbb{Z}^{equi0}((\bar{D}_\bullet \times S \times \mathbb{A}^{d_X}, D_\bullet \times \mathbb{A}^{d_X})/\bar{X} \times S)) \end{aligned}$$

which is clearly a quasi-isomorphism.
(ii): Similar to (i).
(iii):Follows from (i) and (ii).
(iii)':Follows from (iii) and the fact that μ_{S*} preserve $(\mathbb{A}^1, et)$ local equivalence (see proposition 24) and the fact that ρ_{S*} preserve $(\mathbb{A}^1, et)$ local equivalence (see proposition 23). $\square$

Definition 36. *(i) Let $h : U \to S$ a morphism, with $U, S \in \mathrm{Var}(\mathbb{C})$, U smooth. Take, see definition-proposition 12, $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \to \bar{S}$ a compactification of $h : U \to S$ and denote by $\bar{Z} = \bar{X}_0 \backslash U$. Take, using theorem 15(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ of the pair $(\bar{X}_0, \bar{Z})$, with $\bar{X} \in \mathrm{PSmVar}(\mathbb{C})$ and $\bar{D} := \bar{\epsilon}^{-1}(\bar{Z}) = \cup_{i=1}^s \bar{D}_i \subset \bar{X}$ a normal crossing divisor. We will consider the following canonical map in $C(\mathrm{Var}(\mathbb{C})^{sm}/S)$*

$$T_{(\bar{X},\bar{D})/S}(U/S) : \mathrm{Gr}_{S*}^{12} L\rho_{S*}\mu_{S*}R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S)) \xrightarrow{q} \mathrm{Gr}_{S*}^{12} \rho_{S*}\mu_{S*}R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S))$$
$$\xrightarrow{r_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S))} \mathrm{Gr}_{S*}^{12} \rho_{S*}\mu_{S*}p_{S*}E_{et}(\mathbb{Z}((U \times S, U)/U \times S)) \xrightarrow{l(U/S)} h_*E_{et}(\mathbb{Z}(U/U)) =: \mathbb{D}_S^0(\mathbb{Z}(U/S))$$

where, for $h' : V \to S$ a smooth morphism with $V \in \mathrm{Var}(\mathbb{C})$,

$$l^{00}(U/S)(V/S) : \mathbb{Z}((U \times S, U)/U \times S)(V \times U \times S, V \times_S U/U \times S) \to \mathbb{Z}(U/U)(V \times_S U), \alpha \mapsto \alpha_{|V \times_S U}$$

which gives

$$l^0(U/S)(V/S) : E_{et}^0(\mathbb{Z}((U \times S, U)/U \times S))(V \times U \times S, V \times_S U/U \times S) \to E_{et}^0(\mathbb{Z}(U/U))(V \times_S U),$$

and by induction

$$\tau^{\leq i} l(U/S) : \mathrm{Gr}_{S*}^{12} \rho_{S*}\mu_{S*}p_{S*}E_{et}^{\leq i}(\mathbb{Z}((U \times S, U)/U \times S)) \to h_*E_{et}^{\leq i}(\mathbb{Z}(U/U))$$

where $\tau^{\leq i}$ is the cohomological truncation.

(ii) Let $g : U'/S \to U/S$ a morphism, with $U'/S = (U', h'), U/S = (U, h) \in \mathrm{Var}(\mathbb{C})/S$, U,U' smooth. Take, see definition-proposition 12(ii),a compactification $\bar{f}_0 = \bar{h} : \bar{X}_0 \to \bar{S}$ of $h : U \to S$ and a compactification $\bar{f}'_0 = \bar{h}' : \bar{X}'_0 \to S$ of $h' : U' \to S$ such that $g : U'/S \to U/S$ extend to a morphism $\bar{g}_0 : \bar{X}'_0/\bar{S} \to \bar{X}_0/\bar{S}$. Denote $\bar{Z} = \bar{X}_0 \backslash U$ and $\bar{Z}' = \bar{X}'_0 \backslash U'$. Take, see definition-proposition 12(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ of $(\bar{X}_0, \bar{Z})$, a strict desingularization $\bar{\epsilon}'_\bullet : (\bar{X}', \bar{D}') \to (\bar{X}'_0, \bar{Z}')$ of $(\bar{X}'_0, \bar{Z}')$ and a morphism $\bar{g} : \bar{X}' \to \bar{X}$ such that the following diagram commutes

$$\begin{tikzcd} \bar{X}'_0 \arrow[r, "\bar{g}_0"] & \bar{X}_0 \\ \bar{X}' \arrow[u, "\bar{\epsilon}'"] \arrow[r, "\bar{g}"] & \bar{X} \arrow[u, "\bar{\epsilon}"'] \end{tikzcd} .$$

Then by the diagram given in definition 34(ii), the following diagram in $C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ obviously commutes

$$\begin{tikzcd} \mathrm{Gr}_{S*}^{12} L\rho_{S*}\mu_{S*}R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S)) \arrow[r, "T_{(\bar{X},\bar{D})/S}(U/S)"] \arrow[d, "R_S^{CH}(g)"'] & h_*E_{et}(\mathbb{Z}(U/U)) := \mathbb{D}_S^0(\mathbb{Z}(U/S)) \arrow[d, "T(g,E)(-)\circ\mathrm{ad}(g^*,g_*)(E_{et}(\mathbb{Z}(U/U))):=\mathbb{D}_S^0(g)"] \\ \mathrm{Gr}_{S*}^{12} L\rho_{S*}\mu_{S*}R_{(\bar{X}',\bar{D}')/S}(\mathbb{Z}(U'/S)) \arrow[r, "T_{(\bar{X}',\bar{D}')/S}(U'/S)"] & h'_*E_{et}(\mathbb{Z}(U'/U')) := \mathbb{D}_S^0(\mathbb{Z}(U'/S)) \end{tikzcd}$$

where $l(U/S)$ are $l(U'/S)$ are the maps given in (i).

(iii) Let $S \in \mathrm{SmVar}(\mathbb{C})$. Let $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$. We get from (i) and (ii) morphisms in $C(\mathrm{Var}(\mathbb{C})^{sm}/S)$

$$T_S^{CH}(LF) : \mathrm{Gr}_{S*}^{12} L\rho_{S*}\mu_{S*}R_{(\bar{X}^*,\bar{D}^*)/S}(\rho_S^* LF)$$
$$\xrightarrow{r_S^{CH}(LF)} \mathrm{Gr}_{S*}^{12} L\rho_{S*}\mu_{S*}\mathbb{D}_S^{12}(\rho_S^* LF) \xrightarrow{l(L(F)} \mathbb{D}_S^0(L(F))$$

We will also need the following lemma

Lemma 1. *(i) Let $h : U \to S$ a morphism, with $U, S \in \mathrm{Var}(\mathbb{C})$, U smooth. Take, see definition-proposition 12, $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \to \bar{S}$ a compactification of $h : U \to S$ and denote by $\bar{Z} = \bar{X}_0 \backslash U$. Take, using theorem 15(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ of the pair $(\bar{X}_0, \bar{Z})$, with $\bar{X} \in \mathrm{PSmVar}(\mathbb{C})$ and $\bar{D} := \bar{\epsilon}^{-1}(\bar{Z}) = \cup_{i=1}^{s} \bar{D}_i \subset \bar{X}$ a normal crossing divisor. Then the map in $C(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$*

$$\mathrm{ad}(\mathrm{Gr}_S^{12*}, \mathrm{Gr}_{S*}^{12})(L\rho_{S*}\mu_{S*}R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S))) \circ q :$$
$$\mathrm{Gr}_S^{12*} L \, \mathrm{Gr}_{S*}^{12} L\rho_{S*}\mu_{S*}R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S)) \to L\rho_{S*}\mu_{S*}R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S))$$

is an equivalence $(\mathbb{A}^1, et)$ local.

(ii) Let $S \in \mathrm{SmVar}(\mathbb{C})$. Let $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$. Then the map in $C(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$

$$\mathrm{ad}(\mathrm{Gr}_S^{12*}, \mathrm{Gr}_{S*}^{12})(L\rho_{S*}\mu_{S*}R_{(\bar{X}^*,\bar{D}^*)/S}(\rho_S^* LF)) \circ q :$$
$$\mathrm{Gr}_S^{12*} L \, \mathrm{Gr}_{S*}^{12} L\rho_{S*}\mu_{S*}R_{(\bar{X}^*,\bar{D}^*)/S}(\rho_S^* LF) \to L\rho_{S*}\mu_{S*}R_{(\bar{X}^*,\bar{D}^*)/S}(\rho_S^* LF)$$

is an equivalence $(\mathbb{A}^1, et)$ local.

Proof. (i): Follows from proposition 36.
(ii): Follows from (i). □

2.13 The derived categories of filtered complexes of presheaves on a site or of filtered complexes of presheaves of modules on a ringed topos

Definition 37. *Let $\mathcal{S} \in \mathrm{Cat}$ a site endowed with topology τ.*

(i) We denote by $D(\mathcal{S}) := \mathrm{Ho}_{\mathrm{Top}} C(\mathcal{S})$ the localization of the category of complexes of presheaves on S with respect to top local equivalence and by $D(\tau) : C(\mathcal{S}) \to D(\mathcal{S})$ the localization functor.

(ii) We denote for $r = 1, \ldots \infty$, resp. $r = (1, \ldots \infty)^2$,

$$D_{fil,r}(\mathcal{S}) := \mathrm{Ho}_{Fr\tau} C_{fil}(\mathcal{S}) \, , \; D_{2fil,r}(\mathcal{S}) := \mathrm{Ho}_{Fr\tau} C_{2fil}(\mathcal{S}),$$

the localizations of the category of filtered complexes of presheaves on $\mathcal{S}$ whose filtration is biregular with respect to r-filtered τ local equivalence. By definition, we have sequences of functors

$$C_{fil}(\mathcal{S}) \to K_{fil}(\mathcal{S}) \to D_{fil}(\mathcal{S}) \to D_{fil,2}(\mathcal{S}) \to \cdots \to D_{fil,\infty}(\mathcal{S}).$$

and commutative diagrams of functors

$$\begin{tikzcd} K_{fil}(\mathcal{S}) \arrow[r] \arrow[d] & D_{fil}(\mathcal{S}) \arrow[d] \\ K_{fil,2}(\mathcal{S}) \arrow[r] & D_{fil,2}(\mathcal{S}) \end{tikzcd} \quad , \quad \begin{tikzcd} K_{fil,r}(\mathcal{S}) \arrow[r] \arrow[d] & D_{fil,r}(\mathcal{S}) \arrow[d] \\ K_{fil,r+1}(\mathcal{S}) \arrow[r] & D_{fil,r}(\mathcal{S}) \end{tikzcd}$$

where $K_{fil}(\mathcal{S}) := K(\mathrm{PSh}_{fil}(\mathcal{S}))$ and $K_{fil,r}(\mathcal{S}) := K_r(\mathrm{PSh}_{fil}(\mathcal{S}))$. are the categories where the morphisms are r-filtered homotopy classes of morphisms. Then, for $r = 1$, $K_{fil}(\mathcal{S})$ and $D_{fil}(\mathcal{S})$ are in the canonical way triangulated categories. However, for $r > 1$, the categories $K_{fil,r}(\mathcal{S})$ and $D_{fil,r}(\mathcal{S})$ together with the canonical triangles does NOT satisfy the 2 of 3 axiom of triangulated categories.

Definition 38. *Let $(\mathcal{S}, O_S) \in \mathrm{RCat}$ where $\mathcal{S} \in \mathrm{Cat}$ is a site endowed with topology τ.*

(i) We denote by $D_{O_S}(\mathcal{S}) := \mathrm{Ho}_{\mathrm{Top}} C_{O_S}(\mathcal{S})$ the localization of the category of complexes of presheaves on S with respect to top local equivalence and by $D(\tau) : C_{O_S}(\mathcal{S}) \to D_{O_S}(\mathcal{S})$ the localization functor.

(ii) We denote for $r = 1, \ldots \infty$, resp. $r = (1, \ldots \infty)^2$,

$$D_{O_S fil,r}(\mathcal{S}) := \mathrm{Ho}_{Fr\tau} C_{O_S fil}(\mathcal{S}) \ , \ D_{O_S 2fil,r}(\mathcal{S}) := \mathrm{Ho}_{Fr\tau} C_{O2fil}(\mathcal{S}),$$

the localizations of the category of filtered complexes of presheaves on $\mathcal{S}$ whose filtration is biregular with respect to r-filtered τ local equivalence (see section 2.1 and 2.3). By definition, we have sequences of functors

$$C_{O_S fil}(\mathcal{S}) \to K_{O_S fil}(\mathcal{S}) \to D_{O_S fil}(\mathcal{S}) \to D_{O_S fil,2}(\mathcal{S}) \to \cdots \to D_{O_S fil,\infty}(\mathcal{S}).$$

and commutative diagrams of functors

$$\begin{tikzcd} K_{O_S fil}(\mathcal{S}) \arrow[r] \arrow[d] & D_{O_S fil}(\mathcal{S}) \arrow[d] \\ K_{O_S fil,2}(\mathcal{S}) \arrow[r] & D_{fil,2}(\mathcal{S}) \end{tikzcd} \quad , \quad \begin{tikzcd} K_{fil,r}(\mathcal{S}) \arrow[r] \arrow[d] & D_{O_S fil,r}(\mathcal{S}) \arrow[d] \\ K_{O_S fil,r+1}(\mathcal{S}) \arrow[r] & D_{O_S fil,r}(\mathcal{S}) \end{tikzcd}$$

where $K_{O_S fil}(\mathcal{S}) := K(\mathrm{PSh}_{O_S fil}(\mathcal{S}))$ and $K_{O_S fil,r}(\mathcal{S}) := K_r(\mathrm{PSh}_{O_S fil}(\mathcal{S}))$ (see section 2.1) are the categories where the morphisms are r-filtered homotopy classes of morphisms. Then, for $r = 1$, $K_{O_S fil}(\mathcal{S})$ and $D_{O_S fil}(\mathcal{S})$ are in the canonical way triangulated categories. However, for $r > 1$, the categories $K_{O_S fil,r}(\mathcal{S})$ and $D_{O_S fil,r}(\mathcal{S})$ together with the canonical triangles does NOT satisfy the 2 of 3 axiom of triangulated categories.

Let $f : \mathcal{T} \to \mathcal{S}$ a morphism of presite, with $\mathcal{S}, \mathcal{T} \in \mathrm{Cat}$ endowed with a topology τ. If f is a morphism of site, the adjonctions

$$(f^*, f_*) = (f^{-1}, f_*) : C(\mathcal{S}) \leftrightarrows C(\mathcal{T}), \ (f^*, f_*) = (f^{-1}, f_*) : C_{(2)fil}(\mathcal{S}) \leftrightarrows C_{(2)fil}(\mathcal{T}).$$

are Quillen adjonctions. They induces respectively in the derived categories, for $r = (1, \ldots, \infty)$, resp. $r = (1, \ldots, \infty)$ (note that f^* derive trivially)

$$(f^*, Rf_*) : D(\mathcal{S}) \leftrightarrows D(\mathcal{T}) \ , \ (f^*, Rf_*) : D_{fil,r}(\mathcal{S}) \leftrightarrows D_{fil,r}(\mathcal{T}).$$

For $F^\bullet \in C(\mathcal{S})$, we have the adjonction maps

$$\mathrm{ad}(f^*, f_*)(F^\bullet) : F^\bullet \to f_* f^* F^\bullet \ , \ \mathrm{ad}(f^*, f_*)(F^\bullet) : f^* f_* F^\bullet \to F^\bullet,$$

induces in the derived categories, for $(M, F) \in D_{fil}(\mathcal{S})$ and $(N, F) \in D_{fil}(\mathcal{T})$, the adjonction maps

$$\mathrm{ad}(f^*, Rf_*)(M) : (M, F) \to Rf_* f^*(M, F) \ , \ \mathrm{ad}(f^*, Rf_*)(N, F) : f^* Rf_*(N, F) \to (N, F).$$

For a commutative diagram of sites :

$$D = \begin{tikzcd} \mathcal{Y} \arrow[r, "g_2"] \arrow[d, "f_2"] & \mathcal{X} \arrow[d, "f_1"] \\ \mathcal{T} \arrow[r, "g_1"] & \mathcal{S} \end{tikzcd} \quad ,$$

with $\mathcal{Y}, \mathcal{T}, \mathcal{S}, \mathcal{X} \in \mathrm{Cat}$ with topology $\tau_Y, \tau_T, \tau_S, \tau X$, the maps, for $F \in C(\mathcal{X})$,

$$T(D)(F) : g_1^* f_{1*} F \to f_{2*} g_2^* F$$

induce in the derived category the maps in $D_{fil,r}(\mathcal{T})$, given by, for $(G, F) \in D_{fil,r}(\mathcal{X})$ with $(G, F) = D(\tau_X, r)((G, F))$,

$$\begin{tikzcd} g_1^* Rf_{1*}(M, F) \arrow[r, "T(D)(M,F)"] & Rf_{2*} g_2^*(M, F) \\ g_1^* f_{1*}(E(G, F)) \arrow[u, "="] \arrow[r, "k \circ T(D)(E(G,F))"] & f_{2*} E(g_2^*(E(G, F))) \arrow[u, "="] \end{tikzcd} \quad .$$

Let $\mathcal{S} \in \mathrm{Cat}$ a site with topology τ. The tensor product of complexes of abelian groups and the internal hom of presheaves on $\mathcal{S}$

$$((\cdot \otimes \cdot), \mathcal{H}om^{\bullet}(\cdot, \cdot)) : C(\mathcal{S})^2 \to C(\mathcal{S}),$$

is a Quillen adjonction which induces in the derived category

$$((\cdot \otimes^L \cdot), R\mathcal{H}om^{\bullet}(\cdot, \cdot)) : D_{fil,r}(\mathcal{S})^2 \to D_{fil,r}(\mathcal{S}), \ R\mathcal{H}om^{\bullet}((M, W), (N, W)) = \mathcal{H}om^{\bullet}((Q, W), E(G, F)),$$

where, Q is projectively cofibrant such that $M = D(\tau)(Q^{\bullet})$ and G such that $N = D(\tau)(G)$.

Let $i : Z \hookrightarrow S$ a closed embedding, with $S, Z \in \mathrm{Top}$. Denote by $j : S \backslash Z \hookrightarrow S$ the open embedding of the complementary subset. The adjonction

$$(i_*, i^!) := (i_*, i^{\perp}) : C(Z) \to C(S), \text{ with in this case } i^! F := \ker(F \to j_* j^* F)$$

is a Quillen adjonction. Since $i^!$ preserve monomorphisms, we have also Quillen adjonctions

$$(i_*, i^!) : C_{(2)fil}(Z) \to C_{(2)fil}(S), \text{ with } i^!(G, F) = (i^! G, F).$$

which induces in the derived category (i_* derive trivially)

$$(i_*, Ri^!) : D_{(2)fil}(Z) \to D_{(2)fil}(S), \text{ with } Ri^!(G, F) = i^! E(G, F).$$

The 2-functor $S \in \mathrm{Top} \mapsto D(S)$ obviously satisfy the localization property, that is for $i : Z \hookrightarrow S$ a closed embedding with $Z, S \in \mathrm{Top}$, denote by $j : S \backslash Z \hookrightarrow S$ the open complementary subset, we have for $K \in D(S)$ a distinguish triangle in $D(S)$

$$j_{\sharp} j^* K \xrightarrow{\mathrm{ad}(j_{\sharp}, j^*)(K)} K \xrightarrow{\mathrm{ad}(i^*, i_*)(K)} i_* i^* K \to j_{\sharp} j^* K[1]$$

equivalently,

- the functor
$$(i^*, j^*) : D(S) \xrightarrow{\sim} D(Z) \times D(S \backslash Z)$$
is conservative,
- and for $K \in C(Z)$, the adjonction map $\mathrm{ad}(i^*, i_*)(K) : i^* i_* K \to K$ is an equivalence top local, hence for $K \in D(S)$, the induced map in the derived category
$$\mathrm{ad}(i^*, i_*)(K) : i^* i_* K \xrightarrow{\sim} K$$
is an isomorphism.

3 Triangulated category of motives

3.1 Definition and the six functor formalism

The category of motives is obtained by inverting the $(\mathbb{A}^1_S, et)$ equivalence. Hence the $\mathbb{A}^1_S$ local complexes of presheaves plays a key role.

Definition 39. *The derived category of motives of complex algebraic varieties over S is the category*

$$\mathrm{DA}(S) := \mathrm{Ho}_{\mathbb{A}^1_S, et}(C(\mathrm{Var}(\mathbb{C})^{sm}/S)),$$

which is the localization of the category of complexes of presheaves on $\mathrm{Var}(\mathbb{C})^{sm}/S$ with respect to $(\mathbb{A}^1_S, et)$ local equivalence and we denote by

$$D(\mathbb{A}^1_S, et) := D(\mathbb{A}^1_S) \circ D(et) : C(\mathrm{Var}(\mathbb{C})^{sm}/S) \to \mathrm{DA}(S)$$

the localization functor. We have $\mathrm{DA}^-(S) := D(\mathbb{A}^1_S, et)(\mathrm{PSh}(\mathrm{Var}(\mathbb{C})^{sm}/S, C^-(\mathbb{Z}))) \subset \mathrm{DA}(S)$ the full subcategory consisting of bounded above complexes.

Definition 40. *The stable derived category of motives of complex algebraic varieties over S is the category*

$$\mathrm{DA}_{st}(S) := \mathrm{Ho}_{\mathbb{A}^1_S,et}(C_\Sigma(\mathrm{Var}(\mathbb{C})^{sm}/S)),$$

which is the localization of the category of $\mathbb{G}_{mS}$-spectra ($\Sigma F^\bullet = F^\bullet \otimes \mathbb{G}_{mS}$) of complexes of presheaves on $\mathrm{Var}(\mathbb{C})^{sm}/S$ with respect to $(\mathbb{A}^1_S, et)$ local equivalence. The functor

$$\Sigma^\infty : C(\mathrm{Var}(\mathbb{C})^{sm}/S) \hookrightarrow C_\Sigma(\mathrm{Var}(\mathbb{C})^{sm}/S)$$

induces the functor $\Sigma^\infty : \mathrm{DA}(S) \to \mathrm{DA}_{st}(S)$.

We have all the six functor formalism by [10]. We give a list of the operation we will use :

- For $f : T \to S$ a morphism with $S, T \in \mathrm{Var}(\mathbb{C})$, the adjonction

  $$(f^*, f_*) : C(\mathrm{Var}(\mathbb{C})^{sm}/S) \leftrightarrows C(\mathrm{Var}(\mathbb{C})^{sm}/T)$$

  is a Quillen adjonction which induces in the derived categories (f^* derives trivially), (f^*, Rf_*) : $\mathrm{DA}(S) \leftrightarrows \mathrm{DA}(T)$.

- For $h : V \to S$ a smooth morphism with $V, S \in \mathrm{Var}(\mathbb{C})$, the adjonction

  $$(h_\sharp, h^*) : C(\mathrm{Var}(\mathbb{C})^{sm}/V) \leftrightarrows C(\mathrm{Var}(\mathbb{C})^{sm}/S)$$

  is a Quillen adjonction which induces in the derived categories (h^* derive trivially) $(Lh_\sharp, h^*) =:$ $\mathrm{DA}(V) \leftrightarrows \mathrm{DA}(S)$.

- For $i : Z \hookrightarrow S$ a closed embedding, with $Z, S \in \mathrm{Var}(\mathbb{C})$,

  $$(i_*, i^!) := (i_*, i^\perp) : C(\mathrm{Var}(\mathbb{C})^{sm}/Z) \leftrightarrows C(\mathrm{Var}(\mathbb{C})^{sm}/S)$$

  is a Quillen adjonction, which induces in the derived categories (i_* derive trivially) $(i_*, Ri^!)$: $\mathrm{DA}(Z) \leftrightarrows \mathrm{DA}(S)$. The fact that i_* derive trivially (i.e. send $(\mathbb{A}^1, et)$ local equivalence to $(\mathbb{A}^1, et)$ local equivalence is proved in [4].

- For $S \in \mathrm{Var}(\mathbb{C})$, the adjonction given by the tensor product of complexes of abelian groups and the internal hom of presheaves

  $$((\cdot \otimes \cdot), \mathcal{H}om^\bullet(\cdot, \cdot)) : C(\mathrm{Var}(\mathbb{C})^{sm}/S)^2 \to C(\mathrm{Var}(\mathbb{C})^{sm}/S),$$

  is a Quillen adjonction, which induces in the derived category

  $$,((\cdot \otimes^L \cdot), R\mathcal{H}om^\bullet(\cdot, \cdot)) : \mathrm{DA}(S)^2 \to \mathrm{DA}(S),$$

  - Let $M, N \in \mathrm{DA}(S)$, $Q^\bullet$ projectively cofibrant such that $M = D(\mathbb{A}^1, et)(Q^\bullet)$, and $G^\bullet$ be $\mathbb{A}^1$ local for the etale topology such that $N = D(\mathbb{A}^1, et)(G^\bullet)$. Then,

    $$R\mathcal{H}om^\bullet(M, N) = \mathcal{H}om^\bullet(Q^\bullet, E(G^\bullet)) \in \mathrm{DA}(S). \tag{43}$$

    This is well defined since if $s : Q_1 \to Q_2$ is a etale local equivalence,

    $$\mathcal{H}om(s, E(G)) : \mathcal{H}om(Q_1, E(G)) \to \mathcal{H}om(Q_2, E(G))$$

    is a etale local equivalence for $1 \leq i \leq l$.

- For a commutative diagram in $\mathrm{Var}(\mathbb{C})$:

$$D = \begin{tikzcd} Y \arrow[r, "g_2"] \arrow[d, "f_2"] & X \arrow[d, "f_1"] \\ T \arrow[r, "g_1"] & S \end{tikzcd} ,$$

  and $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/X)$, the transformation map $T(D)(F) : g_1^* f_{1*}F \to f_{2*}g_2^*F$ induces in the derived category, for $M \in \mathrm{DA}(X)$, $M = D(\mathbb{A}^1, et)(F)$ with F $\mathbb{A}^1$ local for the etale topology,

$$\begin{tikzcd} g_1^* Rf_{1*}M \arrow[rr, "T(D)(M)"] & & Rf_{2*}g_2^*M \\ g_1^* f_{1*}E(F) \arrow[u, "="] \arrow[rr, "k\circ c\circ T(D)(E(F))"] & & f_{2*}E(C_*(g_2^*E(F))) \arrow[u, "="'] \end{tikzcd}$$

  If D is cartesian with $f_1 = f$, $g_1 = g$ $f_2 = f' : X_T \to T$, $g' : X_T \to X$, we denote

  - $T(D)(F) =: T(f,g)(F) : g^*f_*F \to f'_*g^{'*}F$,
  - $T(D)(M) =: T(f,g)(M) : g^*Rf_*M \to Rf'_*g^{*'}M$.

We get from the first point 2 functors :

- The 2-functor $C(\mathrm{Var}(\mathbb{C})^{sm}/\cdot) : \mathrm{Var}(\mathbb{C}) \to Ab\,\mathrm{Cat}$, given by

$$S \mapsto C(\mathrm{Var}(\mathbb{C})^{sm}/S) \ , \ (f : T \to S) \mapsto (f^* : C(\mathrm{Var}(\mathbb{C})^{sm}/S) \to C(\mathrm{Var}(\mathbb{C})^{sm}/T)).$$

- The 2-functor $\mathrm{DA}(\cdot) : \mathrm{Var}(\mathbb{C}) \to \mathrm{TriCat}$, given by

$$S \mapsto \mathrm{DA}(S) \ , \ (f : T \to S) \mapsto (f^* : \mathrm{DA}(S) \to \mathrm{DA}(T)).$$

The main theorem is the following :

Theorem 16. *[4][10] The 2-functor* $\mathrm{DA}(\cdot) : \mathrm{Var}(\mathbb{C}) \to \mathrm{TriCat}$*, given by*

$$S \mapsto \mathrm{DA}(S) \ , \ (f : T \to S) \mapsto (f^* : \mathrm{DA}(S) \to \mathrm{DA}(T))$$

is a 2-homotopic functor ([4])

From theorem 16, we get in particular

- For $f : T \to S$ a morphism with $T, S \in \mathrm{Var}(\mathbb{C})$, there by theorem 16 is also a pair of adjoint functor

$$(f_!, f^!) : \mathrm{DA}(S) \leftrightarrows \mathrm{DA}(T)$$

  - with $f_! = Rf_*$ if f is proper,
  - with $f^! = f^*[d]$ if f is smooth of relative dimension d.

  For $h : U \to S$ a smooth morphism with $U, S \in \mathrm{Var}(\mathbb{C})$ irreducible, have, for $M \in \mathrm{DA}(U)$, an isomorphism

$$Lh_\sharp M \to h_! M[d], \tag{44}$$

  in $\mathrm{DA}(S)$.

- The 2-functor $S \in \mathrm{Var}(\mathbb{C}) \mapsto \mathrm{DA}(S)$ satisfy the localization property, that is for $i : Z \hookrightarrow X$ a closed embedding with $Z, X \in \mathrm{Var}(\mathbb{C})$, denote by $j : S\backslash Z \hookrightarrow S$ the open complementary subset, we have for $M \in \mathrm{DA}(S)$ a distinguish triangle in $\mathrm{DA}(S)$

$$j_\sharp j^* M \xrightarrow{\mathrm{ad}(j_\sharp, j^*)(M)} M \xrightarrow{\mathrm{ad}(i^*, i_*)(M)} i_* i^* M \to j_\sharp j^* M[1]$$

  equivalently,

  - the functor
$$(i^*, j^*) : \mathrm{DA}(S) \xrightarrow{\sim} \mathrm{DA}(Z) \times \mathrm{DA}(S\backslash Z)$$
    is conservative,
  - and for $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/Z)$, the adjonction map $\mathrm{ad}(i^*, i_*)(F) : i^* i_* F \to F$ is an equivalence Zariski local, hence for $M \in \mathrm{DA}(S)$, the induced map in the derived category
$$\mathrm{ad}(i^*, i_*)(M) : i^* i_* M \xrightarrow{\sim} M$$
    is an isomorphism.

- For $f : X \to S$ a proper map, $g : T \to S$ a morphism, with $T, X, S \in \mathrm{Var}(\mathbb{C})$, and $M \in \mathrm{DA}(X)$,

$$T(f, g)(M) : g^* R f_* M \to R f'_* \tilde{g}^{'*} M$$

  is an isomorphism in $\mathrm{DA}(T)$ if f is proper.

Definition 41. *The derived category of extended motives of complex algebraic varieties over S is the category*

$$\underline{\mathrm{DA}}(S) := \mathrm{Ho}_{\mathbb{A}^1_S, et}(C(\mathrm{Var}(\mathbb{C})/S)),$$

which is the localization of the category of complexes of presheaves on $\mathrm{Var}(\mathbb{C})/S$ with respect to $(\mathbb{A}^1_S, et)$ local equivalence and we denote by

$$D(\mathbb{A}^1_S, et) := D(\mathbb{A}^1_S) \circ D(et) : C(\mathrm{Var}(\mathbb{C})/S) \to \underline{\mathrm{DA}}(S)$$

the localization functor. We have $\underline{\mathrm{DA}}^-(S) := D(\mathbb{A}^1_S, et)(\mathrm{PSh}(\mathrm{Var}(\mathbb{C})/S, C^-(\mathbb{Z}))) \subset \underline{\mathrm{DA}}(S)$ the full subcategory consisting of bounded above complexes.

Remark 5. *Let $i : Z \hookrightarrow S$ a closed embedding, with $Z, S \in \mathrm{Var}(\mathbb{C})$.*

(i) By theorem 16, for $X/S = (X, h) \in \mathrm{Var}(\mathbb{C})^{sm}/S)$,

$$(0, \mathrm{ad}(i^*, i_*)(\mathbb{Z}(X/S))) : \Gamma^\vee_Z \mathbb{Z}(X/S) \to i_* \mathbb{Z}(X_Z/Z)$$

is an equivalence $(\mathbb{A}^1, et)$ local.

(ii) For $X/S = (X, f) \in \mathrm{Var}(\mathbb{C})/S)$,

$$(0, \mathrm{ad}(i^*, i_*)(\mathbb{Z}(X/S))) : \Gamma^\vee_Z \mathbb{Z}(X/S) \to i_* \mathbb{Z}(X_Z/Z)$$

is NOT an equivalence $(\mathbb{A}^1, et)$ local in general, since for example if $f(X) = Z \subset S$, $\rho_{S*}\mathbb{Z}(X/S) = 0$ but $D(\mathbb{A}^1, et)(\rho_{S*} i_* \mathbb{Z}(X_Z/Z) = i_* \rho_{S*} \mathbb{Z}(X_Z/Z)) \neq 0 \in \underline{\mathrm{DA}}(S)$, hence it is NOT an equivalence $(\mathbb{A}^1, et)$ local in this case by proposition 19. In particular $\underline{\mathrm{DA}}(S)$ dos NOT satisfy the localization property.

(ii)' For $X/Z = (X, f) \in \mathrm{Var}(\mathbb{C})/Z)$, the inclusion

$$T(i_\sharp, i_*) : i_\sharp \mathbb{Z}(X/Z) \hookrightarrow i_* \mathbb{Z}(X/Z)$$

is NOT an equivalence $(\mathbb{A}^1, et)$ local by proposition 19 since $\rho_{S*} i_\sharp \mathbb{Z}(X/Z) = 0$ but $D(\mathbb{A}^1, et)(\rho_{S*} i_* \mathbb{Z}(X/Z) = i_* \rho_{S*} \mathbb{Z}(X/Z)) \neq 0 \in \underline{\mathrm{DA}}(S)$.

(iii) *Let $f : X \to S$ a smooth proper morphism with $X, S \in \mathrm{Var}(\mathbb{C})$ of relative dimension $d = d_X - d_S$ and X smooth. Then, we have then by proposition 36(i) the equivalence $(\mathbb{A}^1, et)$ local in $C(\mathrm{Var}(\mathbb{C})^{sm}/S)$*

$$T_0(f_\sharp, f_*)(\mathbb{Z}(X/X)) := [\Delta_X] : f_\sharp \mathbb{Z}(X/X) = \mathbb{Z}(X/S) \to f_* E_{et}(\mathbb{Z}(X/X))(d)[2d]$$

given by the class of the diagonal $[\Delta_X] \in \mathrm{Hom}(f_\sharp \mathbb{Z}(X/X), f_* E_{et}(\mathbb{Z}(X/X))(d)[2d])$.

(iii)' *Let $f : X \to S$ a proper surjective morphism with $X, S \in \mathrm{Var}(\mathbb{C})$ with equidimensional fiber of relative dimension $d = d_X - d_S$. Assume X smooth. Then, we have then by proposition 36(i) the equivalence $(\mathbb{A}^1, et)$ local in $C(\mathrm{Var}(\mathbb{C})/S)$*

$$T_0(f_\sharp, f_*)(\mathbb{Z}(X/X)) := [\Delta_X] : f_\sharp \mathbb{Z}(X/X) = \mathbb{Z}(X/S) \to f_* E_{et}(\mathbb{Z}(X/X))(d)[2d]$$

given by the class of the diagonal $[\Delta_X] \in \mathrm{Hom}(f_\sharp \mathbb{Z}(X/X), f_* E_{et}(\mathbb{Z}(X/X))(d)[2d])$.

3.2 Constructible motives and resolution of a motive by Corti-Hanamura motives

We now give the definition of the motives of morphisms $f : X \to S$ which are constructible motives and the definition of the category of Corti-Hanamura motives.

Definition 42. *Let $S \in \mathrm{Var}(\mathbb{C})$,*

- *the homological motive functor is $M(/S) : \mathrm{Var}(\mathbb{C})/S \to \mathrm{DA}(S)$, $(f : X \to S) \mapsto M(X/S) := f_! f^! M(S/S)$,*
- *the cohomological motive functor is $M^\vee(/S) : \mathrm{Var}(\mathbb{C})/S \to \mathrm{DA}(S)$, $(f : X \to S) \mapsto M(X/S)^\vee := Rf_* M(X/X) = f_* E(\mathbb{Z}_X)$,*
- *the Borel-Moore motive functor is $M^{BM}(/S) : \mathrm{Var}(\mathbb{C})/S \to \mathrm{DA}(S)$, $(f : X \to S) \mapsto M^{BM}(X/S) := f_! M(X/X)$,*
- *the (homological) motive with compact support functor is $M_c(/S) : \mathrm{Var}(\mathbb{C})/S \to \mathrm{DA}(S)$, $(f : X \to S) \mapsto M_c(X/S) := Rf_* f^! M(S/S)$.*

Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$. Assume that there exist a factorization $f : X \xrightarrow{i} Y \times S \xrightarrow{p} S$, with $Y \in \mathrm{SmVar}(\mathbb{C})$, $i : X \hookrightarrow Y$ is a closed embedding and p the projection. Then,

$$Q(X/S) := p_\sharp \Gamma_X^\vee \mathbb{Z}_{Y \times S} \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$$

(see definition 10)is projective, admits transfert, and satisfy $D(\mathbb{A}^1_S, et)(Q(X/S)) = M(X/S)$.

Definition 43. *(i) Let $S \in \mathrm{Var}(\mathbb{C})$. We define the full subcategory $\mathrm{DA}_c(S) \subset \mathrm{DA}(S)$ whose objects are constructible motives to be the thick triangulated category generated by the motives of the form $M(X/S)$, with $f : X \to S$ a morphism, $X \in \mathrm{Var}(\mathbb{C})$.*

(ii) *Let $X, S \in \mathrm{Var}(\mathbb{C})$. If $f : X \to S$ is proper (but not necessary smooth) and X is smooth, $M(X/S)$ is said to be a Corti-Hanamura motive and we have by above in this case $M(X/S) = M^{BM}(X/S)[c] = M(X/S)^\vee[c]$, with $c = \mathrm{codim}(X, X \times S)$ where $f : X \hookrightarrow X \times S \to S$. We denote by*

$$\mathcal{CH}(S) = \{M(X/S)\}^{pa}_{\{X/S=(X,f), f pr, X sm\}} \subset DM(S)$$

the full subcategory which is the pseudo-abelian completion of the full subcategory whose objects are Corti-Hanamura motives.

(iii) *We denote by*

$$\mathcal{CH}^0(S) \subset \mathcal{CH}(S)$$

the full subcategory which is the pseudo-abelian completion of the full subcategory whose objects are Corti-Hanamura motives $M(X/S)$ such that the morphism $f : X \to S$ is projective.

For bounded above motives, we have

Theorem 17. *Let $S \in \mathrm{Var}(\mathbb{C})$.*

(i) There exists a unique weight structure ω on $\mathrm{DA}^-(S)$ such that $\mathrm{DA}^-(S)^{\omega=0} = \mathcal{CH}(S)$

(ii) There exist a well defined functor

$$W(S) : \mathrm{DA}^-(S) \to K^-(\mathcal{CH}(S)) \ , \ W(S)(M) = [M^{(\bullet)}]$$

where $M^{(\bullet)} \in C^-(\mathcal{CH}(S))$ is a bounded above weight complex, such that if $m \in \mathbb{Z}$ is the highest weight, we have a generalized distinguish triangle for all $i \leq m$

$$T_i : M^{(i)}[i] \to M^{(i+1)}[(i+1)] \to \cdots \to M^{(m)}[m] \to M^{w \geq i} \tag{45}$$

Moreover the maps $w(M)^{(\geq i)} : M^{\geq i} \to M$ induce an isomorphism $w(M) : holim_i M^{\geq i} \xrightarrow{\sim} M$ in $\mathrm{DA}^-(S)$.

(iii) Denote by $Chow(S)$ the category of Chow motives, which is the pseudo-abelian completion of the category

- *whose set of objects consist of the $X/S = (X, f) \in \mathrm{Var}(\mathbb{C})/S$ such that f is proper and X is smooth,*
- *whose set of morphisms between X_1/S and X_2/S is $\mathrm{CH}^{d_1}(X_1 \times_S X_2)$, and the composition law is given in [11].*

We have then a canonical functor $CH_S : Chow(S) \hookrightarrow \mathrm{DA}(S)$, with $CH_S(X/S) := M(X/S) := Rf_*\mathbb{Z}(X/X)$, which is a full embedding whose image is the category $\mathcal{CH}(S)$.

Proof. (i): The category $\mathrm{DA}(S)$ is clearly weakly generated by $\mathcal{CH}(S)$. Moreover $\mathcal{CH}(S) \subset \mathrm{DA}(S)$ is negative. Hence, the result follows from [6] theorem 4.3.2 III.
(ii): Follows from (i) by standard fact of weight structure on triangulated categories. See [6] theorem 3.2.2 and theorem 4.3.2 V for example.
(iii): See [12]. □

Theorem 18. *Let $S \in \mathrm{Var}(\mathbb{C})$.*

(i) There exists a unique weight structure ω on $\mathrm{DA}^-(S)$ such that $\mathrm{DA}^-(S)^{\omega=0} = \mathcal{CH}^0(S)$

(ii) There exist a well defined functor

$$W(S) : \mathrm{DA}^-(S) \to K^-(\mathcal{CH}^0(S)) \ , \ W(S)(M) = [M^{(\bullet)}]$$

where $M^{(\bullet)} \in C^-(\mathcal{CH}^0(S))$ is a bounded above weight complex, such that if $m \in \mathbb{Z}$ is the highest weight, we have a generalized distinguish triangle for all $i \leq m$

$$T_i : M^{(i)}[i] \to M^{(i+1)}[(i+1)] \to \cdots \to M^{(m)}[m] \to M^{w \geq i} \tag{46}$$

Moreover the maps $w(M)^{(\geq i)} : M^{\geq i} \to M$ induce an isomorphism $w(M) : holim_i M^{\geq i} \xrightarrow{\sim} M$ in $\mathrm{DA}^-(S)$.

Proof. Similar to the proof of theorem 17. □

Corollary 1. *Let $S \in \mathrm{Var}(\mathbb{C})$. Let $M \in \mathrm{DA}(S)$. Then there exist $(F, W) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $D(\mathbb{A}^1, et)(F) = M$ and $D(\mathbb{A}^1, et)(\mathrm{Gr}_p^W F) \in \mathcal{CH}^0(S)$.*

Proof. By theorem 18, there exist, by induction, for $i \in \mathbb{Z}$, a distinguish triangle in $\mathrm{DA}(S)$

$$T_i : M^{(i)}[i] \xrightarrow{m_i} M^{(i+1)} \xrightarrow{m_{i+1}} \cdots \xrightarrow{m_{m-1}} M^{(m)}[m] \to M^{w\geq i} \tag{47}$$

with $M^{(j)}[j] \in \mathcal{CH}^0(S)$ and $w(M) : holim_i M^{\geq i} \xrightarrow{\sim} M$ in $\mathrm{DA}^-(S)$. For $i \in \mathbb{Z}$, take $(F_j)_{j\geq i}, F_{w\geq i} \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $D(\mathbb{A}^1, et)(F_j) = M^{(j)}[j]$, $D(\mathbb{A}^1, et)(F_{w\geq i}) = M^{w\geq i}$ and such that we have in $C(\mathrm{Var}(\mathbb{C})^{sm}/S)$,

$$F_{w\geq i} = \mathrm{Cone}(F_i \xrightarrow{m_i} F_{i+1} \xrightarrow{m_{i+1}} \cdots \xrightarrow{m_{m-1}} F_m) \tag{48}$$

where $m_j : F_j \to F_{j+1}$ are morphisms in $C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $D(\mathbb{A}^1, et)(m_j) = m_j$. Now set $F = \mathrm{holim}_i F_{w\geq i} \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ and $W_i F := F_{w\geq i} \hookrightarrow F$, so that $(F, W) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ satisfy $D(\mathbb{A}^1, et)(\mathrm{Gr}_p^W F) = M^{(p)}[p] \in \mathcal{CH}^0(S)$. □

3.3 The restriction of relative motives to their Zariski sites

Let $S \in \mathrm{Var}(\mathbb{C})$. The adjonction

$$(e(S)^*, e(S)_*) : C(\mathrm{Var}(\mathbb{C})^{sm}/S) \leftrightarrows C(S)$$

is a Quillen adjonction and induces in the derived category

- $(e(S)^*, e(S)_*) : \mathrm{Ho}_{zar}(\mathrm{Var}(\mathbb{C})^{sm}/S) \leftrightarrows D(T) := \mathrm{Ho}_{zar}\, C(S)$, since $e(S)_*$ sends Zariski local equivalence on the big site $\mathrm{Var}(\mathbb{C})^{sm}/S$ to Zariski local equivalence in the small Zariski site of S,
- $(e(S)^*, Re(S)_*) : \mathrm{DA}(S) \leftrightarrows D(T) := \mathrm{Ho}_{zar}\, C(S)$.

We will use in the defintion of the De Rahm realization functor on $\mathrm{DA}(S)$ the following proposition concerning the restriction of the derived internal hom functor to the Zariski site :

Proposition 37. *Let $M, N \in \mathrm{DA}(S)$ and $m : M \to N$ be a morphism. Let $F^\bullet, G^\bullet \in \mathrm{PSh}(\mathrm{Var}(\mathcal{C})^{sm}/S, C(\mathbb{Z}))$ such that $M = D(\mathbb{A}^1_S, et)(F^\bullet)$ and $N = D(\mathbb{A}^1_S, et)(G^\bullet)$. If we take $G^\bullet$ $(\mathbb{A}^1_S, et)$ fibrant and admitting transfert, and $F^\bullet$ cofibrant for the projective model structure, we have*

$$Re(S)_* R\mathcal{H}om^\bullet(M, N) = e(S)_* \mathcal{H}om^\bullet(F^\bullet, G^\bullet)$$

in D(S).

Proof. Since $F^\bullet$ is projectively cofibrant and $G^\bullet$ is (projectively) $(\mathbb{A}^1_S, et)$ fibrant, we have $R\mathcal{H}om^\bullet(M, N) = \mathcal{H}om^\bullet(F^\bullet, G^\bullet)$. Then, $\mathcal{H}om^\bullet(F^\bullet, G^\bullet)$ is $\mathbb{A}^1_S$ local and admits transfert. On the other hand, we have

$$\mathcal{L}_{\mathbb{A}^1_S} D_{et}(\mathrm{Cor}\,\mathrm{Var}(\mathbb{C})^{sm}/S) = \mathcal{L}_{\mathbb{A}^1_S} D_{zar}(\mathrm{Cor}\,\mathrm{Var}(\mathbb{C})^{sm}/S) \subset D(\mathrm{Var}(\mathbb{C})^{sm}/S)$$

by theorem 10 (ii). This gives the equality of the proposition. □

We will also have :

Proposition 38. *For $f : T \to S$ a morphism and $i : Z \hookrightarrow S$ a closed embedding, with $Z, S, T \in \mathrm{Var}(\mathbb{C})$, we have*

(i) $Re(S)_* Rf_* = Rf_* Re(T)_*$ *and* $e(S)^* Rf_* = Rf_* e(T)^*$

(ii) $Re(S)_* R\Gamma_Z = R\Gamma_Z Re(S)_*$.

Proof. (i):Follows from proposition 16 (i) and the fact that f_* preserve $(\mathbb{A}^1, et)$ fibrant complex of presheaves.
(ii):Follows from proposition 16 (ii) and the fact that Γ_Z preserve $(\mathbb{A}^1, et)$ fibrant complex of presheaves. □

3.4 Motives of complex analytic spaces

The category of motives is obtained by inverting the $(\mathbb{D}^1_S, usu)$ local equivalence. Hence the $\mathbb{D}^1_S$ local complexes of presheaves plays a key role.

Definition 44. *The derived category of motives of complex algebraic varieties over S is the category*

$$\mathrm{AnDA}(S) := \mathrm{Ho}_{\mathbb{D}^1_S, usu}(C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)),$$

which is the localization of the category of complexes of presheaves on $\mathrm{AnSp}(\mathbb{C})^{sm}/S$ with respect to $(\mathbb{D}^1_S, usu)$ local equivalence and we denote by

$$D(\mathbb{D}^1_S, usu) := D(\mathbb{A}^1_S) \circ D(et) : C(\mathrm{AnSp}(\mathbb{C})^{sm}/S) \to \mathrm{AnDA}(S)$$

the localization functor. We have $\mathrm{DA}^-(S) := D(\mathbb{D}^1_S, usu)(\mathrm{PSh}(\mathrm{AnSp}(\mathbb{C})^{sm}/S, C^-(\mathbb{Z}))) \subset \mathrm{DA}(S)$ the full subcategory consisting of bounded above complexes.

Theorem 19. *Let $S \in \mathrm{AnSp}(\mathbb{C})$. The adjonction $(e(S)^*, e(S)_*) : C(\mathrm{AnSp}(\mathbb{C})^{sm}/S) \to C(S)$ induces an equivalence of categories*

$$(e(S)^*, Re(S)_*) : \mathrm{AnDM}(S) \xrightarrow{\sim} D(S).$$

In particular, for $F \in C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$, the adjonction map $\mathrm{ad}(e(S)^*, e(S)_*)(F) : e(S)^* e(S)_* F \to F$ is an equivalence $(\mathbb{D}^1, usu)$ local.

Proof. See [1]. □

We deduce from this theorem the following :

Proposition 39. *Let $S \in \mathrm{AnSp}(\mathbb{C})$. Let $F, G \in C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$. If G is $\mathbb{D}^1$ local, then the canonical map*

$$T(e, hom)(F, G) : e(S)_* \mathcal{H}om(F, G) \to \mathcal{H}om(e(S)_* F, e(S)_* G)$$

is an equivalence usu local.

Proof. The map $T(e, hom)(F, G)$ is the composite

$$T(e, hom)(F, G) : e(S)_* \mathcal{H}om(F, G) \xrightarrow{\mathcal{H}om(\mathrm{ad}(e(S)^*, e(S)_*)(F), G)} e(S)_* \mathcal{H}om(e(S)^* e(S)_* F, G)$$
$$\xrightarrow{I(e(S)^*, e(S)_*)(F, G)} \mathcal{H}om(e(S)_* F, e(S)_* G)$$

where the last map is the adjonction isomorphism. The first map is an isomorphism by theorem 19 since G is $\mathbb{D}^1$ local. □

4 The category of filtered D modules on commutative ringed topos, on commutative ringed spaces, complex algebraic varieties complex analytic spaces and the functorialities

4.1 The The category of filtered D modules on commutative ringed topos, on commutative ringed spaces, and the functorialities

4.1.1 Definitions et functorialities

Let $(\mathcal{S}, O_S) \in \mathrm{RCat}$ with O_S commutative. Recall that $\Omega_{O_S} := \mathbb{D}^O_S(\mathcal{I}_S / \mathcal{I}_S^2) \in \mathrm{PSh}_{O_S}(S)$ is the universal derivation O_S-module together with its derivation map $d : O_S \to \Omega_{O_S}$, where $\mathcal{I}_S = \ker(s_S : O_S \otimes O_S \to O_S) \in \mathrm{PSh}_{O_S \times O_S}(\mathcal{S})$ the diagonal ideal.

In the particular case of a ringed space $(S,O_S)\in\mathrm{RTop}$, $s_S:O_S\otimes O_S=\Delta_S^*(p_1^*O_S\otimes p_2^*O_S)\to O_S$ is the structural morphism of diagonal embedding $\Delta_S:(S,O_S)\hookrightarrow(S\times S,p_1^*O_S\otimes p_2^*O_S)$, $p_1:S\times S\to S$ and $p_2:S\times S\to S$ being the projections. More generally, for $k\in\mathbb{N}$, $k\geq 1$ we have the sheaf of k-jets $J^k(O_S):=\Delta_S^*\mathcal{I}_S/\mathcal{I}_S^{k+1}$ with in particular $J^1(O_S)=T_S$. We have, for $s\in S$, $J^k(O_S)_s=m_s/m_s^k$ where $m_s\subset O_{S,s}$ is the maximal ideal if $O_{S,s}$ is a local ring.

Definition 45. *(i) Let $(\mathcal{S},O_S)\in\mathrm{RCat}$ with O_S a commutative sheaf of ring and $\mathcal{S}$ is endowed with a topology τ. We denote by*

$$D(O_S)=<O_S,\mathrm{Der}_{O_S}(O_S,O_S)>\subset a_\tau\mathcal{H}om(O_S,O_S)$$

the subsheaf of ring generated by O_S and the subsheaf of derivations $\mathrm{Der}_{O_S}(O_S,O_S)=T_S:=\mathbb{D}_S^O\Omega_{O_S}$, $a_\tau:\mathrm{PSh}(\mathcal{S})\to\mathrm{Shv}(\mathcal{S})$ being the sheaftification functor.

(ii) Let $f:\mathcal{X}\to\mathcal{S}$ be a morphism of site, with $\mathcal{X},\mathcal{S}\in\mathrm{Cat}$ endowed with topology τ, resp. τ', and $O_S\in\mathrm{PSh}(\mathcal{S},\mathrm{cRing})$ a commutative sheaf of ring. We will note in this case by abuse $f^*O_S:=a_\tau f^*O_S$ and $f^*D(O_S):=a_\tau f^*D(O_S)$, $a_\tau:\mathrm{PSh}(\mathcal{X})\to\mathrm{Shv}(\mathcal{X})$ being the sheaftification functor.

Let $f:\mathcal{X}\to\mathcal{S}$ a morphism of site, with $\mathcal{X},\mathcal{S}\in\mathrm{Cat}$ endowed with topology τ, resp. τ', and $O_S\in\mathrm{PSh}(\mathcal{S},\mathrm{cRing})$ a commutative sheaf of ring. Consider the ringed space $(\mathcal{X},f^*O_S):=(\mathcal{X},a_\tau f^*O_S)\in\mathrm{RCat}$, $a_\tau:\mathrm{PSh}(\mathcal{X})\to\mathrm{Shv}(\mathcal{X})$ being the sheaftification functor. Then, the map in $\mathrm{PSh}(\mathcal{X})$

$$T(f,hom)(O_S,O_S):f^*\mathcal{H}om(O_S,O_S)\to\mathcal{H}om(f^*O_S,f^*O_S)$$

induces a canonical morphism of sheaf of rings

$$T(f,hom)(O_S,O_S):a_\tau f^*D(O_S)=:f^*D(O_S)\to D(a_\tau f^*O_S)=:D(f^*O_S).$$

In the special case of ringed spaces, we have then :

Proposition 40. *Let $f:X\to S$ is a continous map, with $X,S\in\mathrm{Top}$ and $O_S\in\mathrm{PSh}(S,\mathrm{cRing})$ a commutative sheaf of ring. Consider the ringed space $(X,f^*O_S):=(X,a_\tau f^*O_S)\in\mathrm{RTop}$, $a_\tau:\mathrm{PSh}(X)\to\mathrm{Shv}(X)$ being the sheaftification functor. Then, the map in $\mathrm{PSh}(X)$*

$$T(f,hom)(O_S,O_S):f^*\mathcal{H}om(O_S,O_S)\to\mathcal{H}om(f^*O_S,f^*O_S)$$

induces a canonical isomorphism of sheaf of rings

$$T(f,hom)(O_S,O_S):f^*D(O_S):=a_\tau f^*D(O_S)\xrightarrow{\sim}D(a_\tau f^*O_S)=:D(f^*O_S).$$

Proof. For all $x\in X$,

$$T(f,hom)(O_S,O_S)_x:(f^*D(O_S))_x\xrightarrow{\sim}D(O_{S,f(x)})\xrightarrow{\sim}(D(f^*O_S))_x.$$

Hence, since $a_\tau f^*D(O_S)$ and $D(a_\tau f^*O_S)$ are sheaves,

$$T(f,hom)(O_S,O_S):f^*D(O_S):=a_\tau f^*D(O_S)\xrightarrow{\sim}D(a_\tau f^*O_S)=:D(f^*O_S).$$

is an isomorphism $\square$

We will consider presheaves of $D(O_S)$ modules on a ringed topos $(\mathcal{S},O_S)$:

Definition 46. *Let $(\mathcal{S},O_S)\in\mathrm{RCat}$ with O_S a commutative sheaf of ring.*

(i) We will consider $\mathrm{PSh}_{D(O_S)}(\mathcal{S})$ the category of presheaves of (left) $D(O_S)$ modules on S and $C_{D(O_S)}(\mathcal{S}):=C(\mathrm{PSh}_{D(O_S)}(\mathcal{S}))$ its category of complexes. We will consider $\mathrm{PSh}_{D(O_S)^{op}}(\mathcal{S})$ the category of presheaves of right $D(O_S)$ modules on S and $C_{D(O_S)^{op}}(\mathcal{S}):=C(\mathrm{PSh}_{D(O_S)^{op}}(\mathcal{S}))$ its category of complexes. We denote again by abuse

$$\mathrm{PSh}_{D(O_S)fil}(\mathcal{S})=(\mathrm{PSh}_{D(O_S)}(\mathcal{S}),F):=(\mathrm{PSh}_{(D(O_S),F^{ord})}(\mathcal{S}),F)$$

the category of filtered $(D(O_S),F^{ord})$-module, with, for $-p\leq 0$, $F^{ord,-p}D(O_S)=\{P\in D(O_S),ord(P)\leq p\}$ and $F^{ord,p}D(O_S)=0$ for $p>0$,

- *whose objects are* $(M,F)\in(\mathrm{PSh}_{O_S}(\mathcal{S}),F)$ *such that* (M,F) *is compatible with* $(D(O_S),F^{ord})$ *that is* $F^{ord,-p}D(O_S)\cdot F^qM\subset F^{q-p}M$ *(Griffitz transversality), this is to say that the structural map* $md:M\otimes_{O_S}D(O_S)\to M$ *induces a filtered map of presheaves (i.e a map in* $(\mathrm{PSh}_{O_S}(\mathcal{S}),F)$*)* $md:(M,F)\otimes_{O_S}(D(O_S),F^{ord})\to(M,F)$,
- *whose morphism* $\phi:(M_1,F)\to(M_2,F)$ *are as usual the morphisms of presheaves* $\phi:M_1\to M_2$ *which are morphism of filtered presheaves (i.e.* $\phi(F^pM_1)\subset F^pM_2$*) and which are* $D(O_S)$ *linear (in particular* O_S *linear).*

Note that this a NOT the category of filtered $D(O_S)$ *modules in the usual sense, that is the* $(M,F)\in(\mathrm{PSh}_{O_S}(\mathcal{S}),F)$ *together with a map* $md:(M,F)\otimes_{O_S}D(O_S)\to(M,F)$ *in* $(\mathrm{PSh}_{O_S}(\mathcal{S}),F)$*, since* F^{ord} *is NOT the trivial filtration. More precisely the* O_S *submodules* $F^pM\subset M$ *are NOT* $D(O_S)$ *submodules but satisfy Griffitz transversality. We denote by*

$$\mathrm{PSh}_{D(O_S)0fil}(\mathcal{S})\subset\mathrm{PSh}_{D(O_S)fil}(\mathcal{S}),\ \mathrm{PSh}_{D(O_S)(1,0)fil}(\mathcal{S})\subset\mathrm{PSh}_{D(O_S)2fil}(\mathcal{S}):=(\mathrm{PSh}_{(D(O_S),F^{ord})}(\mathcal{S}),F,W)$$

the full subcategory consisting of filtered $D(O_S)$ *module in the usual sense, resp. the full subcategory such that* W *is a filtration by* $D(O_S)$ *submodules.*

(ii) *We denote again by*

$$C_{D(O_S)fil}(\mathcal{S})\subset C(\mathrm{PSh}_{D(O_S)}(\mathcal{S}),F)\ ,\ C_{D(O_S)2fil}(\mathcal{S})\subset C(\mathrm{PSh}_{D(O_S)}(\mathcal{S}),F,W)$$

the full subcategory of complexes such that the filtration(s) is (are) regular. We will consider also

$$C_{D(O_S)0fil}(\mathcal{S})\subset C_{D(O_S)fil}(\mathcal{S}),C_{D(O_S)(1,0)fil}(\mathcal{S})\subset C_{D(O_S)2fil}(\mathcal{S})$$

the full subcategory consisting of complexes of filtered $D(O_S)$ *modules in the usual sense (i.e. by* $D(O_S)$ *submodule), respectively the full subcategory consisting of complexes of bifilterd* $D(O_S)$ *modules such that* $W^pM\subset M$ *are* $D(O_S)$ *submodules i.e. the filtration* W *is a filtration in the usual sense, but NOT* F *wich satify only Griffitz transversality.*

Proposition 41. *Let* $(\mathcal{S},O_S)\in\mathrm{RCat}$ *with a* O_S *commutative sheaf of ring.*

(i) *Let* $M\in\mathrm{PSh}_{O_S}(\mathcal{S})$*. Then, there is a one to one correspondence between*

- *the* $D(O_S)$ *module structure on* M *compatible with the* O_S *module structure, that is the maps* $md:D(O_S)\otimes_{O_S}M\to M$ *in* $\mathrm{PSh}_{O_S}(\mathcal{S})$ *and*
- *the integrable connexions on* M*, that is the maps* $\nabla:M\to\Omega_{O_S}\otimes_{O_S}M$ *satisfying* $\nabla\circ\nabla=0$ *with* $\nabla:\Omega_{O_S}\otimes_{O_S}M\to\Omega^2_{O_S}\otimes_{O_S}M$ *given by* $\nabla(\omega\otimes m)=(d\omega)\otimes m+\omega\wedge\nabla(m)$

(ii) *Let* $(M,F)\in\mathrm{PSh}_{O_Sfil}(\mathcal{S})$*. Then, there is a one to one correspondence between*

- *the* $D(O_S)$ *module structure on* (M,F) *compatible with the* O_S *module structure, that is the maps* $md:(D(O_S),F^{ord})\otimes_{O_S}(M,F)\to(M,F)$ *in* $\mathrm{PSh}_{O_Sfil}(\mathcal{S})$ *and*
- *the integrable connexions on* M*, that is the maps* $\nabla:(M,F)\to\Omega_{O_S}\otimes_{O_S}(M,F)$ *satisfying* $\nabla\circ\nabla=0$ *and Griffith transversality (i.e.* $\nabla(F^pM)\subset\Omega_{O_S}\otimes_{O_S}F^{p-1}M$*).*

Proof. Standard. □

The following proposition tells that the O-tensor product of D modules has a canonical structure of D module.

Definition-Proposition 13. *(i) Let* $f:(\mathcal{X},O_X)\to(\mathcal{S},O_S)$ *a morphism with* $(\mathcal{X},O_X),(\mathcal{S},O_S)\in\mathrm{RCat}$ *with commutative structural sheaf of ring. For* $N\in\mathrm{PSh}_{O_X,D(f^*O_S)}(\mathcal{X})$ *and* $M\in\mathrm{PSh}_{O_X,D(f^*O_S)}(\mathcal{X})$*,* $N\otimes_{O_X}M$ *has the canonical* $D(f^*O_S)$ *module structure given by, for* $X^o\in\mathcal{X}$*,*

$$\gamma\in\Gamma(X^o,D(f^*O_S)),m\in\Gamma(X^o,M),n\in\Gamma(X^o,N),\ \gamma.(n\otimes m)=(\gamma.n)\otimes m+n\otimes(\gamma.m).$$

This gives the functor

$$\mathrm{PSh}_{O_X,D(f^*O_S)fil}(\mathcal{X}) \times \mathrm{PSh}_{O_X,D(f^*O_S)fil}(\mathcal{X}) \to \mathrm{PSh}_{O_X,D(f^*O_S)fil}(\mathcal{X}), ((M,F),(N,F)) \mapsto (M,F) \otimes_{O_X} (N,F), F^p(M,F) \otimes_{O_X} (N,F) := \mathrm{Im}(\oplus_{q\in\mathbb{Z}} F^q M \otimes_{O_X} F^{p-q}N \to M \otimes_{O_X} N)$$

(ii) *Let $f : (\mathcal{X}, O_X) \to (\mathcal{S}, O_S)$ a morphism with $(\mathcal{X}, O_X), (\mathcal{S}, O_S) \in \mathrm{RCat}$ with commutative structural sheaf of ring. For $N \in C_{D(O_X),D(f^*O_S)}(\mathcal{X})$ and $M \in C_{D(O_X)^{op}}(\mathcal{X})$, $N \otimes_{D(O_X)} M$ has the canonical $f^*D(O_S)$ module structure given by, for $X^o \in \mathcal{X}$,*

$$\gamma \in \Gamma(X^o, D(f^*O_S)), m \in \Gamma(X^o, M), n \in \Gamma(X^o, N),\ \gamma.(n \otimes m) = (\gamma.n) \otimes m.$$

This gives the functor

$$C_{D(O_X),D(f^*O_S)fil}(\mathcal{X}) \times C_{D(O_X)fil}(X) \to C_{D(f^*O_S)fil}(X), ((M,F),(N,F)) \mapsto (M,F) \otimes_{D(O_X)} (N,F), F^p(M,F) \otimes_{D(O_X)} (N,F) := \mathrm{Im}(\oplus_{q\in\mathbb{Z}} F^q M \otimes_{D(O_X)} F^{p-q}N \to M \otimes_{D(O_X)} N)$$

Note that, by definition, we have for $(M,F) \in (\mathrm{PSh}_{D(O_S)fil}(\mathcal{S}))$, the canonical isomorphism

$$(M,F) \otimes_{D(O_S)} (D(O_S), F^{ord}) \xrightarrow{\sim} (M,F),\ m \otimes P \mapsto Pm,\ m \mapsto (m \otimes 1)$$

Proof. Immediate from definition. □

We now look at the functorialites for morphisms of ringed spaces, using proposition 40. First note that for $f : (X, O_X) \to (S, O_S)$ a morphism, with $(X, O_X), (S, O_S) \in \mathrm{RTop}$ with structural presheaves commutative sheaves of rings, there is NO canonical morphism between $D(f^*O_S) = f^*D(O_S)$ (see proposition 40) and $D(O_X)$.

We have the pullback functor for (left) D-modules :

Definition-Proposition 14. (i) *Let $f : (X, O_X) \to (S, O_S)$ a morphism with $(X, O_X), (S, O_S) \in \mathrm{RTop}$ with structural presheaves commutative sheaves of rings. Recall that $f^*D(O_S) = D(f^*O_S)$ in this case. Then for $(M,F) \in \mathrm{PSh}_{D(O_S)fil}(S)$,*

$$f^{*mod}(M,F) := (O_X, F_b) \otimes_{f^*O_S} f^*(M,F) \in \mathrm{PSh}_{O_X fil}(X)$$

has a canonical structure of filtered $D(O_X)$ module given by

for $\gamma \in \Gamma(X^o, T_{O_X})$, $n \otimes m \in \Gamma(X^o, O_X \otimes_{f^*O_S} f^*M)$, $\gamma.(n \otimes m) := (\gamma.n) \otimes m + n \otimes df(\gamma)(m)$

with $df := \mathbb{D}^O_S \Omega_{O_X/f^*O_S} : T_{O_X} \to T_{f^*O_S} = f^*T_{O_S}$ *and* $f^*(M,F) \in \mathrm{PSh}_{f^*D(O_S)fil}(X) = \mathrm{PSh}_{D(f^*O_S)fil}(X)$.

(ii) *More generally, let $f : (\mathcal{X}, O_X) \to (\mathcal{S}, O_S)$ a morphism with $(\mathcal{X}, O_X), (\mathcal{S}, O_S) \in \mathrm{RCat}$ with structural presheaves commutative sheaves of rings. Assume that the canonical morphism $T(f, \hom)(O_S, O_S) : f^*D(O_S) \to D(f^*O_S)$ is an isomorphism of sheaves. Then for $(M,F) \in \mathrm{PSh}_{D(O_S)fil}(\mathcal{S})$,*

$$f^{*mod}(M,F) := (O_X, F_b) \otimes_{f^*O_S} f^*(M,F) \in \mathrm{PSh}_{O_X fil}(\mathcal{X})$$

has a canonical structure of filtered $D(O_X)$ module given by

for $\gamma \in \Gamma(X^o, T_{O_X})$, $n \otimes m \in \Gamma(X^o, O_X \otimes_{f^*O_S} f^*M)$, $\gamma.(n \otimes m) := (\gamma.n) \otimes m + n \otimes df(\gamma)(m)$

with $df := \mathbb{D}^O_S \Omega_{O_X/f^*O_S} : T_{O_X} \to T_{f^*O_S} = f^*T_{O_S}$ *and* $f^*(M,F) \in \mathrm{PSh}_{f^*D(O_S)fil}(\mathcal{X}) = \mathrm{PSh}_{D(f^*O_S)fil}(\mathcal{X})$.

Proof. Standard. □

Remark 6. • *Let $f : (X, O_X) \to (S, O_S)$ a morphism with $(X, O_X), (S, O_S) \in \mathrm{RTop}$ with structural presheaves commutative sheaves of rings. Recall that $f^*D(O_S) = D(f^*O_S)$.Then by definition $f^{*mod}(O_S, F_b) = (O_X, F_b)$.*

- *More generally, let $f:(\mathcal{X},O_X)\to(\mathcal{S},O_S)$ a morphism with $(\mathcal{X},O_X),(\mathcal{S},O_S)\in\mathrm{RCat}$ with structural presheaves commutative sheaves of rings. Assume that the canonical morphism $T(f,\hom)(O_S,O_S):f^*D(O_S)\to D(f^*O_S)$ is an isomorphism of sheaves. Then by definition $f^{*mod}(O_S,F_b)=(O_X,F_b)$.*

For the definition of a push-forward functor for a right D module we use the transfert module

Let $f:(X,O_X)\to(S,O_S)$ be a morphism with $(X,O_X),(S,O_S)\in\mathrm{RTop}$ with structural presheaves commutative sheaves of rings. Then, the transfer module is

$$(D(O_X\to f^*O_S),F^{ord}):=f^{*mod}(D(O_S),F^{ord}):=f^*(D(O_S),F^{ord})\otimes_{f^*O_S}(O_X,F_b)$$

which is a left $D(O_X)$ module and a left and right $f^*D(O_S)=D(f^*O_S)$ module.

Lemma 2. *Let $f_1:(X,O_X)\to(Y,O_Y)$, $f_2:(Y,O_Y)\to(S,O_S)$ be two morphism with $(X,O_X),(Y,O_Y)(S,O_S)\in\mathrm{RTop}$. We have in $C_{D(O_X),(f_2\circ f_1)^*D(O_S)fil}(X)$*

$$(D_{O_X\to(f_2\circ f_1)^*O_S},F^{ord})=f_1^*(D_{O_Y\to f_2^*O_S},F^{ord})\otimes_{f_1^*D(O_Y)}(D_{O_X\to f_1^*O_Y},F^{ord})$$

Proof. Follows immediately from definition. □

For right D module, we have the direct image functor :

Definition 47. *Let $f:(X,O_X)\to(S,O_S)$ a morphism with $(X,O_X),(S,O_S)\in\mathrm{RTop}$ with structural presheaves commutative sheaves of rings. Then for $(M,F)\in C_{D(O_X)^{op}fil}(X)$, we define*

$$f^{00}_{*mod}(M,F)=f_*((D_{O_X\to f^*O_S},F^{ord})\otimes_{D(O_X)}(M,F))\in C_{D(O_S)fil}(S)$$

For a closed embedding of topological spaces, there is the V-filtration on the structural sheaf, it will play an important role in this article

Definition 48. *(i) Let $(S,O_S)\in\mathrm{RTop}$ a locally ringed space. Let $Z=V(\mathcal{I}_Z)\subset S$ a Zariski closed subset. We set, for $S^o\subset S$ an open subset, $p\in\mathbb{Z}$,*

- *$V_{Z,p}O_S(S^o):=O_S(S^o)$ if $p>0$,*
- *$V_{Z,-q}O_S(S^o):=\mathcal{I}_Z^q(S^o)\subset O_S(S^o)$ for $p=-q\leq 0$.*

We immediately check that, by definition, this filtration satisfy Griffitz transversality, that is $(O_S,V_Z)\in\mathrm{PSh}_{D(O_S)fil}(S)$. For a morphism $g:((T,O_T),Z')\to((S,O_S),Z)$ with $((T,O_T),Z),((S,O_S),Z)\in\mathrm{RTop}^2$ locally ringed spaces, where Z and Z' are Zariski closed subsets, the structural morphism $a_g:g^*O_S\to O_T$ is a filtered morphism :

$$a_g:g^*(O_S,V_Z)\to(O_T,V_{Z'}),h\mapsto a_g(h)$$

(ii) Let $(S,O_S)\in\mathrm{RTop}$. Let $i:Z\hookrightarrow S$ a closed embedding. The V_Z-filtration on O_S (see (i)) gives the filtration, given by for $p\in\mathbb{Z}$,

$$V_{Z,p}\operatorname{Hom}(O_S,O_S):=\left\{P\in\operatorname{Hom}(O_S,O_S),\ s.t.P\mathcal{I}_Z^k\subset\mathcal{I}_Z^{k-p}\right\}$$

on $\operatorname{Hom}(O_S,O_S)$, which induces the filtration $V_{Z,p}D(O_S):=D(O_S)\cap V_{Z,p}\operatorname{Hom}(O_S,O_S)$ on $D(O_S)\subset\operatorname{Hom}(O_S,O_S)$. We get $(D(O_S),V_Z)\in\mathrm{PSh}_{fil}(S,\mathrm{Ring})$ and we call it the V_Z-filtration on $D(O_S)$.

(iii) Let $(S,O_S)\in\mathrm{RTop}$ a locally ringed space. Let $i:Z=V(\mathcal{I}_Z)\hookrightarrow S$ a Zariski closed embedding and $O_Z:=i^*O_S/\mathcal{I}_Z$. We say that $M\in\mathrm{PSh}_{D(O_S)}(S)$ is specializable on Z if it admits an (increasing) filtration (called a V_Z-filtration) $(M,V)\in\mathrm{PSh}_{O_Sfil}(S)$ compatible with (D_S,V_Z), that is $V_{Z,p}D_S\cdot V_qM\subset V_{p+q}M$, this is to say that the structural map $md:M\otimes_{O_S}D(O_S)\to M$ induces filtered map of presheaves $md:(M,V)\otimes_{O_S}(D(O_S),V_Z)\to(M,V)$. For $(M,F)\in\mathrm{PSh}_{D(O_S)fil}(S)$ such that M is specializable on Z, we thus get a filtered morphism $md:(M,F,V)\otimes_{O_S}(D(O_S),F^{ord},V_Z)\to(M,F,V)$.

(iii)' Consider an injective morphism $m : M_1 \hookrightarrow M_2$ with $M_1, M_2 \in \mathrm{PSh}_{D(O_S)}(S)$. If M_2 admits a V_Z filtration V_2, then the filtration V_{21} induced on M_1 (recall $V_{21,p}M_1 := V_{2,p}M_2 \cap M_1$) is a V_Z filtration. Consider a surjective morphism $n : M_1 \to M_2$ with $M_1, M_2 \in \mathrm{PSh}_{D(O_S)}(S)$. If M_1 admits a V_Z filtration V_1, then the filtration V_{12} induced on M_2 (recall $V_{12,p}M_2 := n(V_{1,p}M_1)$) is a V_Z filtration.

(iv) Let $(S, O_S) \in \mathrm{RTop}$ a locally ringed space. Let $i : Z = V(\mathcal{I}_Z) \hookrightarrow S$ a Zariski closed embedding and $O_Z := i^*O_S/\mathcal{I}_Z$. For $(M, F) \in \mathrm{PSh}_{D(O_S)fil}(S)$ such that M admits a V_Z filtration V so that $(M, F, V) \in \mathrm{PSh}_{O_S2fil}(S)$, we will consider the quotient map in $\mathrm{PSh}_{O_Sfil}(S)$

$$q_{V0} : (M, F) \to (M, F)/V_{-1}(M, F) =: Q_{V,0}(M, F).$$

The quotient $i^*Q_{V,0}(M, F)$ has an action of T_{O_Z} since for $S^o \subset S$ an open subset and $\partial_z \in \Gamma(Z \cap S^o, T_{O_Z}) \subset \Gamma(S^o, T_{O_S})$, we have $\partial_z \in \Gamma(S, V_{Z,0}D(O_S))$ since for $f = \sum_{i=1}^r t_ih_i \in \Gamma(S^o, \mathcal{I}_Z)$, where $(t_i) = \mathcal{I}_Z(S^o)$ are generators of the ideal $I_Z(S^o) \subset O_S(S^o)$ and $h_i \in \Gamma(S^o, O_S)$, we have

$$\partial_z(\sum_{i=1}^r t_ih_i) = \sum_{i=1}^r(\partial_z(t_i)h_i + t_i\partial_z(h_i)) = \sum_{i=1}^r t_i(\partial_z(h_i)) \in \Gamma(S, I_Z)$$

as $\partial_z(t_i) = 0$ (only the vector field of T_{O_S} which are transversal to $T_{O_Z} \subset T_{O_S}$ increase the grading), Then, obviously, by definition, the map in $\mathrm{PSh}_{i^*O_Sfil}(Z)$

$$i^*q_{V0} : i^*(M, F)/V_{-1}(M, F) =: i^*Q_{V,0}(M, F)$$

commutes with the action of $T_{O_Z} \subset i^*T_{O_S}$ and we call it the specialization map.

Definition-Proposition 15. *Let $(S, O_S) \in \mathrm{RTop}$ a locally ringed space. Consider a commutative diagram*

$$\begin{tikzcd} Z_1 = V(\mathcal{I}_1) \arrow[r, "i_1"] & (S, O_S) \\ Z = V(\mathcal{I}) \arrow[u, "i'_2"] \arrow[r, "i'_1"] & Z_2 = V(\mathcal{I}_2) \arrow[u, "i_2"'] \end{tikzcd}$$

where the maps are Zariski closed embeddings and which is cartesian (i.e. $\mathcal{I} = (\mathcal{I}_1, \mathcal{I}_2)$, in particular $Z = Z_1 \cap Z_2$).

(i) Let $(M, F) \in \mathrm{PSh}_{D(O_S)fil}(S)$ such that M admits a V_{Z_1}-filtration V_1 and a V_{Z_2}-filtration V_2 (see definition 48). Let $p, q \in \mathbb{Z}$. Then, we consider

- *the filtration V_{21} on $V_{1,p}(M, F) := (V_{1,p}M, F)$ induced by V_2,*
- *the filtration V_{12} on $Q_{V_2,p}(M, F) := (M/V_{2,p-1}M, F)$ induced by V_1.*

The quotient map in $\mathrm{PSh}_{i'_{1*}O_{Z_1}fil}(S)$

$$q_{V_2,p} : V_{1,q}(M, F) \to V_{12,q}Q_{V_2,p}(M, F)$$

factors trough

$$q_{V_2,p} : V_{1,q}(M, F) \xrightarrow{q_{V_{21},p}} Q_{V_{21},p}V_{1,q}(M, F) \xrightarrow{Q^{p,q}_{V_1,V_2}(M,F)} V_{12,q}Q_{V_2,p}(M, F),$$

and the map $Q^{p,q}_{V_1,V_2}(M, F)$ in $\mathrm{PSh}_{i'_{1*}O_{Z_1}fil}(S)$ commute with the action of T_{O_Z}.

(ii) If $m : (M,F) \to (M',F)$ is a morphism with $(M,F),(M',F) \in \mathrm{PSh}_{D(O_S)fil}(S)$ admitting V_{Z_1}-filtration V_1 and V'_1 respectively such that $m(V_{1,p}M) \subset V'_{1,p}M'$ and V_{Z_2}-filtration V_2 and V'_2 respectively such that $m(V_{1,p}M) \subset V'_{1,p}M'$. Then for all $p,q \in \mathbb{Z}$ the following diagram commutes

$$\begin{tikzcd}
Q_{V_{21},p}V_{1,q}(M,F) \arrow[r, "Q^{p,q}_{V_1,V_2}(M,F)"] \arrow[d, "m"] & V_{12,q}Q_{V_2,p}(M,F) \arrow[d, "m"] \\
Q_{V'_{21},p}V'_{1,q}(M',F) \arrow[r, "Q^{p,q}_{V'_1,V'_2}(M',F)"] & V_{12,q}Q_{V_2,p}(M',F)
\end{tikzcd}$$

Let $(S,O_S) \in \mathrm{RTop}$ a locally ringed space. Consider a commutative diagram

$$\begin{tikzcd}
Z'_1 = V(\mathcal{I}'_1) \arrow[r, "l_1"] & Z_1 = V(\mathcal{I}_1) \arrow[r, "i_1"] & (S,O_S) \\
Z' = V(\mathcal{I}') \arrow[r, "l'_1"] \arrow[u, "i''_2"] & Z = V(\mathcal{I}) \arrow[r, "i'_1"] \arrow[u, "i'_2"] & Z_2 = V(\mathcal{I}_2) \arrow[u, "i_2"]
\end{tikzcd}$$

where the maps are Zariski closed embeddings and whose squares are cartesian (i.e. $\mathcal{I} = (\mathcal{I}_1,\mathcal{I}_2)$ and $\mathcal{I}' = (\mathcal{I}'_1,\mathcal{I})$, in particular $Z = Z_1 \cap Z_2$ and $Z' = Z'_1 \cap Z$). Let $(M,F) \in \mathrm{PSh}_{D(O_S)fil}(S)$ such that M admits a V_{Z_1}-filtration V_1, a $V_{Z'_1}$-filtration V'_1, and a V_{Z_2}-filtration V_2(see definition 48). Then for all $p,q \in \mathbb{Z}$, denoting again V'_{12} the filtration induced by V'_1 on $Q_{V_{21},p}V_{1,q}M$ and V'_{21} the filtration induced by V'_2 on $V_{12,q}Q_{V_2,p}$

- $Q_{V'_{21},p}V'_{12,q}Q_{V_{21},p}V_{1,q}(M,F) = Q_{V'_{21},p}V'_{1,q}(M,F)$,
- $V'_{12,q}Q_{V'_{21},p}V_{12,q}Q_{V_2,p}(M,F) = V'_{12,q}Q_{V_2,p}(M,F)$

and

$$Q^{p,q}_{V'_{12},V_{21}}(Q^{p,q}_{V_1,V_2}(M,F)) = Q^{p,q}_{V'_1,V_2}(M_2,F).$$

Proof. Obvious. □

We will also consider the following categories

Definition 49. *Let $(\mathcal{X},O_X) \in \mathrm{RCat}$. We denote by $C_{O_Xfil,D(O_X)}(\mathcal{X})$ the category*

- *whose objects $(M,F) \in C_{O_Xfil,D(O_X)}(\mathcal{X})$ are filtered complexes of presheaves of O_X modules $(M,F) \in C_{O_Xfil}(\mathcal{X})$ whose cohomology presheaves $H^n(M,F) \in \mathrm{PSh}_{O_Xfil}(\mathcal{X})$ are emdowed with a structure of filtered $D(O_X)$ modules for all $n \in \mathbb{Z}$.*
- *whose set of morphisms $\mathrm{Hom}_{C_{O_Xfil,D(O_X)}(\mathcal{X})}((M,F),(N,F)) \subset \mathrm{Hom}_{C_{O_Xfil}(\mathcal{X})}((M,F),(N,F))$ between $(M,F),(N,F) \in C_{O_Xfil,D(O_X)}(\mathcal{X})$ are the morphisms of filtered complexes of O_X modules $m : (M,F) \to (N,F)$ such that $H^n m : H^n(M,F) \to H^n(N,F)$ is $D(O_X)$ linear, i.e. is a morphism of (filtered) $D(O_X)$ modules, for all $n \in \mathbb{Z}$.*

4.1.2 The De Rham complex of a (left) filtered D-module and the Spencer complex of a right filtered D-module

Using proposition 41, we define the filtered De Rham complex of a complex of filtered (left) D-modules :

Definition 50. *(i) Let $(\mathcal{S},O_S) \in \mathrm{RCat}$ with O_S commutative. Let $(M,F) \in C_{D(O_S)fil}(\mathcal{S})$. By proposition 41, we have the complex*

$$DR(O_S)(M,F) := (\Omega^{\bullet}_{O_S},F_b) \otimes_{O_S} (M,F) \in C_{fil}(\mathcal{S})$$

whose differentials are $d(\omega \otimes m) = (d\omega) \otimes m + \omega \wedge (\nabla m)$.

(ii) More generally, let $f : (\mathcal{X}, O_X) \to (\mathcal{S}, O_S)$ with $(\mathcal{X}, O_X), (\mathcal{S}, O_S) \in \mathrm{RCat}$. The quotient map $q : \Omega_{O_X} \to \Omega_{O_X/f^*O_S}$ induce, for $G \in \mathrm{PSh}_{O_X}(\mathcal{X})$ the quotient map

$$q^p(G) := \wedge^p q \otimes I\Omega^q_{O_X} \otimes_{O_X} G \to \Omega^q_{O_X/f^*O_S} \otimes_{O_X} G.$$

Let $(M, F) \in C_{D(O_X)fil}(\mathcal{X})$. By proposition 41, we have the relative De Rham complex

$$DR(O_X/f^*O_S)(M, F) := (\Omega^\bullet_{X/S}, F_b) \otimes_{O_X} (M, F) \in C_{f^*O_S fil}(\mathcal{X})$$

whose differentials are $d(q^p(M)(\omega \otimes m)) := q^{p+1}(M)((d\omega) \otimes m) + q^{p+1}(M)(\omega \otimes (\nabla m))$.

(iii) Let $(X, O_X)/\mathcal{F} \in FolRTop$, that is $(X, O_X) \in \mathrm{RTop}$ endowed with a foliation with quotient map $q : \Omega_{O_X} \to \Omega_{O_X/\mathcal{F}}$. Let $(M, F) \in C_{D(O_X)fil}(X)$. By proposition 41, we have the foliated De Rham complex

$$DR(O_X/\mathcal{F})(M, F) := (\Omega^\bullet_{O_X/\mathcal{F}}, F_b) \otimes_{O_X} (M, F) \in C_{fil}(X)$$

whose differentials are $d(q(M)(\omega \otimes m)) := q(M)((d\omega) \otimes m) + q(M)(\omega \otimes (\nabla m))$.

By definition,

- *with the notation of (ii) if $\phi : (M_1, F) \to (M_2, F)$ is a morphism in $C_{D(O_X)fil}(\mathcal{X})$,*

$$DR(O_X/f^*O_S)(\phi) := (I \otimes \phi) : (\Omega^\bullet_{X/S}, F) \otimes_{O_X} (M_1, F) \to (\Omega^\bullet_{X/S}, F) \otimes_{O_X} (M_2, F)$$

  is a morphism in $C_{f^*O_S fil}(\mathcal{X})$,

- *with the notation of (ii) $DR(O_X)(O_X) = DR(O_X)$ and more generally in the relative case $DR(O_X/f^*O_S)(O_X) = DR(O_X/f^*O_S)$, and with the notation of (iii) $DR(O_X/\mathcal{F})(O_X) = DR(O_X/\mathcal{F})$.*

Dually, we have the filtered Spencer complex of a complex of filtered right D-module :

Definition 51. *(i) Let $(\mathcal{S}, O_S) \in \mathrm{RCat}$ with O_S commutative. Let $(M, F) \in C_{D(O_S)^{op}fil}(\mathcal{S})$. By proposition 41, we have the complex*

$$SP(O_S)(M, F) := (T^\bullet_{O_S}, F_b) \otimes_{O_S} (M, F) \in C_{fil}(\mathcal{S})$$

whose differentials are, for $X \in \mathcal{S}$, and $\partial_1 \wedge \cdots \wedge \partial_r \otimes m \in \Gamma(X, T^{r-1}_{O_S} \otimes_{O_S} M)$,

$$d(\partial_1 \wedge \cdots \wedge \partial_r \otimes m) : (\omega \in \Gamma(X, \Omega^{r-1}_{O_S}) \mapsto \sum_i \omega(\partial_1 \wedge \cdots \wedge \partial_{\hat{i}} \cdots \partial_r)m - \sum_{i<j} \omega([\partial_i, \partial_j])m).$$

(ii) More generally, let $f : (\mathcal{X}, O_X) \to (\mathcal{S}, O_S)$ with $(\mathcal{X}, O_X), (\mathcal{S}, O_S) \in \mathrm{RCat}$. The quotient map $q : \Omega_{O_X} \to \Omega_{O_X/f^*O_S}$ induce, for $G \in \mathrm{PSh}_{O_X}(\mathcal{X})$ the injective map

$$q^{\vee,p}(G) := \wedge^p q^\vee \otimes I : T^q_{O_X/f^*O_S} \otimes_{O_X} G \to T^q_{O_X} \otimes_{O_X} G.$$

Let $(M, F) \in C_{D(O_X)^{op}fil}(\mathcal{X})$. By proposition 41, we have the relative Spencer complex

$$SP(O_X/f^*O_S)(M, F) := (T^\bullet_{X/S}, F_b) \otimes_{O_X} (M, F) \in C_{f^*O_S fil}(\mathcal{X})$$

whose differentials are the one of $SP(O_X)(M, F)$ given in (i) by the embedding $q^\vee : SP(O_X/f^*O_S)(M, F) \hookrightarrow SP(O_X)(M, F)$.

(iii) Let $(X, O_X)/\mathcal{F} \in FolRTop$, that is $(X, O_X) \in \mathrm{RTop}$ endowed with a foliation with quotient map $q : \Omega_{O_X} \to \Omega_{O_X/\mathcal{F}}$. Let $(M, F) \in C_{D(O_X)^{op}fil}(X)$. By proposition 41, we have the foliated Spencer complex

$$SP(O_X/\mathcal{F})(M, F) := (T^\bullet_{O_X/\mathcal{F}}, F_b) \otimes_{O_X} (M, F) \in C_{fil}(X)$$

whose differentials are of $SP(O_X)(M, F)$ given in (i) by the embedding $q^\vee : SP(O_X/\mathcal{F})(M, F) \hookrightarrow SP(O_X)(M, F)$.

By definition, with the notation of (ii) if $\phi : (M_1, F) \to (M_2, F)$ is a morphism in $C_{D(O_X)^{op}fil}(\mathcal{X})$,

$$SP(O_X/f^*O_S)(\phi) := (I \otimes \phi) : (T^\bullet_{X/S}, F_b) \otimes_{O_X} (M_1, F) \to (T^\bullet_{X/S}, F_b) \otimes_{O_X} (M_2, F)$$

is a morphism in $C_{f^*O_S fil}(\mathcal{X})$.

Proposition 42. *(i) Let $f : (X, O_X) \to (S, O_S)$ a morphism with $(S, O_S), (X, O_X) \in \mathrm{RTop}$. Assume that the canonical map $T(f, \hom)(O_X, O_X) : f^* D(O_X) \to D(f^* O_X)$ is an isomorphism of sheaves. For $(M, F) \in C_{D(O_X)^{op}, f^* D(O_S) fil}(X)$ and $(M', F), (N, F) \in C_{D(O_X) fil}(X)$, we have canonical isomorphisms in $C_{f^* D(O_S) fil}(X)$:*

$$\begin{aligned}(M', F) \otimes_{O_X} (N, F) \otimes_{D(O_X)} (M, F) &= (M', F) \otimes_{D(O_X)} ((M, F) \otimes_{O_X} (N, F)) \\ &= ((M', F) \otimes_{O_X} (M, F)) \otimes_{D(O_X)} (N, F)\end{aligned}$$

(ii) Let $f : (X, O_X) \to (S, O_S)$ a morphism with $(S, O_S), (X, O_X) \in \mathrm{RTop}$. For $(M, F) \in C_{D(O_X) fil}(X)$, we have a canonical isomorphisms of filtered $f^* O_S$ modules, i.e. isomorphisms in $C_{f^* O_S fil}(X)$,

$$(\Omega^{\bullet}_{O_X / f^* O_S}, F_b) \otimes_{O_X} (M, F) = ((\Omega^{\bullet}_{O_X / f^* O_S}, F_b) \otimes_{O_X} D(O_X)) \otimes_{D(O_X)} (M, F)$$

Proof. These are standard fact of algebra. □

Definition-Proposition 16. *Consider a commutative diagram in* RCat

$$\begin{tikzcd} D = & (\mathcal{X}, O_X) \arrow[r, "f"] & (\mathcal{S}, O_S) \\ & (\mathcal{X}', O_{X'}) \arrow[u, "g'"] \arrow[r, "f'"] & (\mathcal{T}, O_T) \arrow[u, "g"] \end{tikzcd} .$$

with commutative structural sheaf of rings. Assume that the canonical map $T(g', \hom)(O_X, O_X) : g^{'*} D(O_X) \to D(g^{'*} O_X)$ is an isomorphism of sheaves.

(i) For $(M, F) \in \mathrm{PSh}_{D(O_X) fil}(\mathcal{X})$, the graded map in $(\mathrm{PSh}_{g'^* O_X}(\mathbb{N} \times \mathcal{X}'), F)$

$$\begin{aligned}&\Omega_{(O_{X'}/g'^* O_X)/(O_T/g^* O_S)}(M, F) := m' \circ (\Omega_{(O_{X'}/g'^* O_X)/(O_T/g^* O_S)} \otimes I) : \\ &g^{'*}((\Omega^{\bullet}_{O_X/f^* O_S}, F_b) \otimes_{O_X} (M, F)) \to (\Omega^{\bullet}_{O_{X'}/f'^* O_T}, F_b) \otimes_{O_{X'}} g^{'*mod}(M, F)\end{aligned}$$

given in degree $p \in \mathbb{N}$ by, for $X^{'o} \in \mathcal{X}'$ and $X^o \in \mathcal{X}$ such that $g^{'*}(X^o) \leftarrow X^{'o}$,

$$\begin{aligned}&\Omega^p_{(O_{X'}/g'^* O_X)/(O_T/g^* O_S)}(M)(X'^o) := m' \circ (\Omega^p_{(O_{X'}/g'^* O_X)/(O_T/g^* O_S)} \otimes I)(X'^o) : \\ &\qquad \omega \otimes m \in \Gamma(X^o, \Omega^p_{O_X} \otimes_{O_X} M) \mapsto \Omega_{O_{X'}/g'^* O_X}(\omega) \otimes (m \otimes 1)\end{aligned}$$

is a map of complexes, that is a map in $C_{(f \circ g')^* O_S fil}(\mathcal{X}')$.

(ii) For $(M, F) \in C_{D(O_X) fil}(\mathcal{X})$, we get from (i) by functoriality, the map in $C_{(f \circ g')^* O_S fil}(\mathcal{X}')$

$$\begin{aligned}&\Omega_{(O_{X'}/g'^* O_X)/(O_T/g^* O_S)}(M, F) := m' \circ (\Omega_{(O_{X'}/g'^* O_X)/(O_T/g^* O_S)} \otimes I) : \\ &g^{'*}((\Omega^{\bullet}_{O_X/f^* O_S}, F_b) \otimes_{O_X} (M, F)) \to (\Omega^{\bullet}_{O_{X'}/f'^* O_T}, F_b) \otimes_{O_{X'}} g^{'*mod}(M, F)\end{aligned}$$

(iii) For $(M, F) \in C_{D(O_X) fil}(\mathcal{X})$, we get from (ii) the canonical transformation map in $C_{O_T fil}(\mathcal{T})$

$$\begin{aligned}&T^O_\omega(D)(M, F) : g^{*mod} L_O(f_* E((\Omega^{\bullet}_{O_X/f^* O_S}, F_b) \otimes_{O_X} (M, F))) \xrightarrow{q} \\ &(g^* f_* E((\Omega^{\bullet}_{O_X/f^* O_S}, F_b) \otimes_{O_X} (M, F))) \otimes_{g^* O_S} O_T \xrightarrow{T(g', E)(-) \circ T(D)(E(\Omega^{\bullet}_{O_X/f^* O_S} \otimes_{O_X} (M, F)))} \\ &(f'_* E(g'^*((\Omega^{\bullet}_{O_X/f^* O_S}, F_b) \otimes_{O_X} (M, F)))) \otimes_{g^* O_S} O_T \xrightarrow{E(\Omega_{(O_{X'}/g'^* O_X)/(O_T/g^* O_S)}(M, F))} \\ &(f'_* E((\Omega^{\bullet}_{O_{X'}/f'^* O_T}, F_b) \otimes_{O_X} (M, F))) \otimes_{g^* O_S} O_T \xrightarrow{m} f'_* E((\Omega^{\bullet}_{O_{X'}/f'^* O_T}, F_b) \otimes_{O_{X'}} g^{'*mod}(M, F))\end{aligned}$$

with $m(n \otimes s) = s.n$.

Proof. (i): We check that the map in $(\mathrm{PSh}_{g'^*O_X}(\mathbb{N}\times\mathcal{X}'),F)$

$$\begin{gathered}\Omega_{(O_{X'}/g'^*O_X)/(O_T/g^*O_S)}(M,F):=m'\circ(\Omega_{(O_{X'}/g'^*O_X)/(O_T/g^*O_S)}\otimes I):\\ g'^*((\Omega^\bullet_{O_X/f^*O_S},F_b)\otimes_{O_X}(M,F))\to(\Omega^\bullet_{O_{X'}/f'^*O_T},F_b)\otimes_{O_{X'}}g'^{*mod}(M,F)\end{gathered}$$

is a map in $C_{(f\circ g')^*O_Sfil}(\mathcal{X}')$. But we have, for $X'^o\in\mathcal{X}'$ the following equality in $\Gamma(X'^o,\Omega^{p+1}_{O_{X'}}\otimes_{O_{X'}}g'^{*mod}M)$

$$\begin{aligned}d(\Omega^p_{(O_{X'}/g'^*O_X)(O_T/g^*O_S)}(M)(\omega\otimes m)):&=d(\Omega^p_{O_{X'}/g'^*O_X}(\omega)\otimes(m\otimes 1))\\ :&=d(\Omega^p_{O_{X'}/g'^*O_X}(\omega))\otimes(m\otimes 1)+\Omega^p_{O_{X'}/g'^*O_X}(\omega)\otimes\nabla(m\otimes 1)\\ &=\Omega^{p+1}_{O_{X'}/g'^*O_X}(d\omega)\otimes(m\otimes 1)+\Omega^p_{O_{X'}/g'^*O_X}(\omega)\otimes\nabla(m)\otimes 1\\ &=\Omega^{p+1}_{O_{X'}/g'^*O_X}(d\omega)\otimes(m\otimes 1)+\Omega^{p+1}_{O_{X'}/g'^*O_X}(\omega\otimes\nabla(m))\otimes 1\\ &=:\Omega^{p+1}_{(O_{X'}/g'^*O_X)(O_T/g^*O_S)}(M)(d(\omega)\otimes m+\omega\otimes\nabla(m))\\ &=:\Omega^{p+1}_{(O_{X'}/g'^*O_X)(O_T/g^*O_S)}(M)(d(\omega\otimes m))\end{aligned}$$

since for $\partial'\in T_{O_{X'}}(X'^o)$,

$$\nabla_{\partial'}(m\otimes 1)=\nabla_{dg'(\partial')}(m)\otimes 1+m\otimes\nabla_{\partial'}1=\nabla_{dg'(\partial')}(m)\otimes 1:$$

see in definition-proposition 14 the definition of the $D(O_{X'})$ module structure on the $O_{X'}$ module $g'^{*mod}M:=g'^*M\otimes_{g'^*O_X}O_{X'}$.
(ii) and (iii): There is nothing to prove. □

Remark 7. *Consider a commutative diagram in* RCat

$$\begin{tikzcd}D= & (\mathcal{X},O_X)\arrow[r,"f"] & (\mathcal{S},O_S)\ .\\ & (\mathcal{X}',O_{X'})\arrow[u,"g'"]\arrow[r,"f'"] & (\mathcal{T},O_T)\arrow[u,"g"]\end{tikzcd}$$

Assume that the canonical map $T(g',\hom)(O_X,O_X):g'^*D(O_X)\to D(g'^*O_X)$ is an isomorphism of sheaves. Under the canonical isomophism $(-)\otimes 1:(\Omega^\bullet_{O_X/f^*O_S},F_b)\xrightarrow{\sim}(\Omega^\bullet_{O_X/f^*O_S},F_b)\otimes_{O_X}(O_X,F_b)$, we have (see definition-proposition 16 and definition 1)

- $\Omega_{(O_{X'}/g'^*O_X)/(O_T/g^*O_S)}(O_X)=\Omega_{(O_{X'}/g'^*O_X)/(O_T/g^*O_S)}:g'^*\Omega^\bullet_{O_X/f^*O_S}\to\Omega^\bullet_{O_{X'}/f'^*O_T}$
- $T^O_\omega(D)(O_X)=T^O_\omega(D):g^{*mod}L_O(f_*E(\Omega^\bullet_{O_X/f^*O_S},F_b))\to f'_*E(\Omega^\bullet_{O_{X'}/f'^*O_T},F_b)$.

Definition 52. *Consider a commutative diagram in* RCat

$$\begin{tikzcd}D= & (\mathcal{X},O_X)\arrow[r,"f"] & (\mathcal{S},O_S)\ .\\ & (\mathcal{X}',O_{X'})\arrow[u,"g'"]\arrow[r,"f'"] & (\mathcal{T},O_T)\arrow[u,"g"]\end{tikzcd}$$

with commutative structural sheaf of rings. Assume that the canonical map $T(g',\hom)(O_X,O_X):g'^*D(O_X)\to D(g'^*O_X)$ is an isomorphism of sheaves. For $(N,F)\in C_{D(O_{X'}),g'^*D(O_X)fil}(\mathcal{X}')$, we have by definition-

proposition 16 the map in $C_{f^*O_Sfil}(\mathcal{X})$

$$T^O_\omega(g',\otimes)(N,F):\Omega^\bullet_{O_X/f^*O_S}\otimes_{O_X}g'_*(N,F)\xrightarrow{\mathrm{ad}(g'^{*mod},g'_*)(-)}$$
$$g'_*(g'^*(\Omega^\bullet_{O_X/f^*O_S}\otimes_{O_X}g'_*N)\otimes_{g'^*O_X}O_{X'}\xrightarrow{m\circ\Omega_{(O_{X'}/g'^*O_X)/(O_T/g^*O_S)}(g'_*(N,F))}$$
$$g'_*(\Omega^\bullet_{O_X/f^*O_S}\otimes_{O_X}g'^{*mod}g'_*(N,F))\xrightarrow{\mathrm{ad}(g'^{*mod},g'_*)(N,F)}g'_*(\Omega^\bullet_{O_{X'}/f^*O_T}\otimes_{O_{X'}}(N,F))$$

with $m(n\otimes s)=s.n$ *and* $g'_*N\in C_{D(O_X)}(\mathcal{X})$*, the structure of* $D(O_X)$ *module being given by the canonical morphism* $\mathrm{ad}(g'^*,g'_*)(D(O_X)):D(O_X)\to g'_*g'^*D(O_X)$ *applied to* $g'_*N\in C_{g'_*g'^*D(O_X)}(\mathcal{X})$*.*

We finish this subsection by a proposition for ringed spaces similar to proposition 9

Proposition 43. *Let* $f:(X,O_X)\to(S,O_S)$ *a morphism with* $(X,O_X),(S,O_S)\in\mathrm{RTop}$ *with commutative sheaves of rings. Assume that* $\Omega_{O_X/f^*O_S}\in\mathrm{PSh}_{O_X}(X)$ *is a locally free* O_X *module of finite rank.*

(i) If $\phi:(M,F)\to(N,F)$ *is an r-filtered top local equivalence with* $(M,F),(N,F)\in C_{D(O_X)fil}(X)$*, then*

$$DR(O_X/f^*O_S)(\phi):(\Omega^\bullet_{O_X/f^*O_S},F_b)\otimes_{O_X}(M,F)\to(\Omega^\bullet_{O_X/f^*O_S},F_b)\otimes_{O_T}(N,F)$$

is an r-filtered top local equivalence.

(ii) Consider a commutative diagram in RTop

$$D=\begin{array}{ccc}(X,O_X)&\xrightarrow{f}&(S,O_S)\\ {\scriptstyle g'}\uparrow&&\uparrow{\scriptstyle g}\\ (X',O_{X'})&\xrightarrow{f'}&(T,O_T)\end{array}.$$

with commutative structural sheaf of rings. For $(N,F)\in C_{D(O_{X_T})fil}(X')$*, the map in* $C_{f^*O_Sfil}(X)$

$$k\circ T^O_\omega(g',\otimes)(E(N,F)):(\Omega^\bullet_{O_X/f^*O_S},F_b)\otimes_{O_X}g'_*E(N,F)\to g'_*E((\Omega^\bullet_{O_{X'}/f^*O_T},F_b)\otimes_{O_{X'}}E(N,F))$$

is a filtered top local equivalence (see definition 52).

Proof. (i):Follows from proposition 9 (i) since $\Omega^\bullet_{O_X/f^*O_S}\in C^b(X)$ is then a bounded complex with $\Omega^n_{O_X/f^*O_S}\in\mathrm{PSh}_{O_X}(X)$ a locally free O_X module of finite rank.
(ii):Follows from proposition 9 (ii) since $\Omega^\bullet_{O_X/f^*O_S}\in C^b(X)$ is then a bounded complex with $\Omega^n_{O_X/f^*O_S}\in\mathrm{PSh}_{O_X}(X)$ a locally free O_X module of finite rank. □

4.1.3 The support section functor for D module on ringed spaces

Let $(S,O_S)\in\mathrm{RTop}$ with O_S commutative. Let $Z\subset S$ a closed subset. Denote by $j:S\backslash Z\hookrightarrow S$ the open complementary embedding,

- For $G\in C_{D(O_S)}(S)$, $\Gamma_ZG:=\mathrm{Cone}(\mathrm{ad}(j^*,j_*)(G):F\to j_*j^*G)[-1]$ has a (unique) structure of $D(O_S)$ module such that $\gamma_Z(G):\Gamma_ZG\to G$ is a map in $C_{D(O_S)}(S)$. This gives the functor

$$\Gamma_Z:C_{D(O_S)fil}(S)\to C_{D(O_S)fil}(S),\ (G,F)\mapsto\Gamma_Z(G,F)$$

  together with the canonical map $\gamma_Z(G,F):\Gamma_Z(G,F)\to(G,F)$. Let $Z_2\subset Z$ a closed subset, then for $G\in C_{D(O_S)}(S)$, $T(Z_2/Z,\gamma)(G):\Gamma_{Z_2}G\to\Gamma_ZG$ is a map in $C_{D(O_S)}(S)$.

- For $G\in C_{O_S}(S)$, $\Gamma^{\vee}_Z G:=\mathrm{Cone}(\mathrm{ad}(j_!,j^*)(G):j_!j^*G\to G)$ has a unique structure of $D(O_S)$ module, such that $\gamma^{\vee}_Z(G):G\to\Gamma^{\vee}_Z G$ is a map in $C_{D(O_S)}(S)$. This gives the functor

$$\Gamma^{\vee}_Z:C_{D(O_S)fil}(S)\to C_{D(O_S)fil}(S),\ (G,F)\mapsto\Gamma^{\vee}_Z(G,F),$$

together with the canonical map $\gamma^{\vee}_Z(G,F):(G,F)\to\Gamma^{\vee}_Z(G,F)$. Let $Z_2\subset Z$ a closed subset, then for $G\in C_{D(O_S)}(S)$, $T(Z_2/Z,\gamma^{\vee})(G):\Gamma^{\vee}_Z G\to\Gamma^{\vee}_{Z_2}G$ is a map in $C_{D(O_S)}(S)$.

- For $G\in C_{D(O_S)}(S)$,

$$\begin{array}{rcl}\Gamma^{\vee,h}_Z G: & = & \mathbb{D}^O_S L_O\Gamma_Z E(\mathbb{D}^O_S G)\\ : & = & \mathrm{Cone}(\mathbb{D}^O_S L_O\,\mathrm{ad}(j_*,j^*)(E(\mathbb{D}^O_S G)):\mathbb{D}^O_S L_O j_*j^*E(\mathbb{D}^O_S G)\to\mathbb{D}^O_S L_O E(\mathbb{D}^O_S G))\end{array}$$

has also canonical $D(O_S)$-module structure, and $\gamma^{\vee,h}_Z(G):G\to\Gamma^{\vee,h}_Z G$ is a map in $C_{D(O_S)}$. This gives the functor

$$\Gamma^{\vee,h}_Z:C_{D(O_S)fil}(S)\to C_{D(O_S)fil}(S),\ (G,F)\mapsto\Gamma^{\vee,h}_Z(G,F),$$

together with the canonical map $\gamma^{\vee,h}_Z(G,F):(G,F)\to\Gamma^{\vee,h}_Z(G,F)$.

- Consider $\mathcal{I}^o_Z\subset O_S$ the ideal of vanishing function on Z and $\mathcal{I}_Z\subset D_S$ the right ideal of D_S generated by $\mathcal{I}^o_Z$. We have then $\mathcal{I}^D_Z\subset\mathcal{I}_Z$, where $\mathcal{I}^D_Z\subset D_S$ is the left and right ideal consisting of sections which vanish on Z. For $G\in\mathrm{PSh}_{D(O_S)}(S)$, we consider, $S^o\subset S$ being an open subset,

$$\mathcal{I}_Z G(S^o)=<\{f.m,m\in G(S^o),f\in\mathcal{I}_Z(S^o)\}>\subset G(S^o)$$

the $D(O_S)$-submodule generated by the functions which vanish on Z ($\mathcal{I}_Z$ is a right $D(O_S)$ ideal), This gives the functor,

$$\Gamma^{\vee,O}_Z:=\Gamma^{\vee,O,I_Z}_Z:C_{D(O_S)fil}(S)\to C_{D(O_S)fil}(S),$$
$$(G,F)\mapsto\Gamma^{\vee,O}_Z(G,F):=\mathrm{Cone}(b_Z(G,F):\mathcal{I}_Z(G,F)\to(G,F)),\ b_Z(-):=b_{I_Z}(-)$$

together with the canonical map $\gamma^{\vee,O}_Z(G,F):(G,F)\to\Gamma^{\vee,O}_Z(G,F)$. which factors through

$$\gamma^{\vee,O}_Z(G):G\xrightarrow{\gamma^{\vee}_Z(G)}\Gamma^{\vee}_Z G\xrightarrow{b_{S/Z}(G)}\Gamma^{\vee,O}_Z G.$$

with $b_{S/Z}(-)=b^I_{S/Z}$and we have an homotopy equivalence $c_Z(G):=c_{I_Z}(G):\Gamma^{\vee,O}_Z G\to G/\mathcal{I}_Z G$.

Lemma 3. *Let $(Y,O_Y)\in\mathrm{RTop}$ and $i:X\hookrightarrow Y$ a closed embedding.*

(i) For $(M,F)\in C_{D(O_Y)fil}(Y)$ and $(N,F)\in\mathrm{PSh}_{D(O_Y)^{op}fil}(Y)$ such that $a_\tau N$ is a locally free $D(O_Y)$) module of finite rank, the canonical map

$$T(\gamma,\otimes)(E(M,F),(N,F)):=(I,T(j,\otimes)(E(M,F),(N,F))):$$
$$(\Gamma_X E(M,F))\otimes_{D(O_Y)}(N,F)\to\Gamma_X E((M,F)\otimes_{D(O_Y)}(N,F))$$

is an equivalence top local.

(ii) For $(M,F)\in C_{D(O_Y)^{(op)}fil}(Y)$ and $(N,F)\in\mathrm{PSh}_{D(O_Y)^{(op)}fil}(Y)$ such that $a_\tau N$ is a locally free O_Y module of finite rank, the canonical map

$$T(\gamma,\otimes)(E(M,F),(N,F)):=(I,T(j,\otimes)(E(M,F),(N,F))):$$
$$(\Gamma_X E(M,F))\otimes_{O_Y}(N,F)\to\Gamma_X E((M,F)\otimes_{O_Y}(N,F))$$

is a filtered top local equivalence.

Proof. Follows from proposition 9. Also note that $T(j,\otimes)(-,-) = T^{mod}(j,\otimes)(-,-)$. $\square$

We now look at the pullback map and the transformation map of De Rahm complexes together with the support section functor. The follwoing is a generalization of definition-proposition 3 :

Definition-Proposition 17. *Consider a commutative diagram in* RTop

$$D_0 = \begin{tikzcd} f:(X,O_X) \arrow[r, "i"] & (Y,O_Y) \arrow[r, "p"] & (S,O_S) \\ f':(X',O_{X'}) \arrow[r, "i'"] \arrow[u, "g'"] & (Y',O_{Y'}) \arrow[r, "p'"] \arrow[u, "g''"] & (T,O_T) \arrow[u, "g"] \end{tikzcd}$$

with i, i' being closed embeddings. Denote by D the right square of D. We have a factorization $i' : X' \xrightarrow{i'_1} X \times YY' \xrightarrow{i'_0} Y'$, where i'_0, i'_1 are closed embedding.

(i) For $(M,F) \in C_{\mathcal{D}(O_Y)fil}(Y)$, the canonical map,

$$E(\Omega_{(O_{Y'}/g''^*O_Y)/(O_T/g^*O_S)}(M,F)) \circ T(g'',E)(-) \circ T(g'',\gamma)(-) :$$
$$g''^*\Gamma_X E((\Omega^\bullet_{O_Y/p^*O_S}, F_b) \otimes_{O_Y} (M,F)) \to \Gamma_{X\times_Y Y'} E((\Omega^\bullet_{O_{Y'}/p'^*O_T}, F_b) \otimes_{O_{Y'}} g''^{*mod}(M,F))$$

unique up to homotopy such that the following diagram in $C_{g''^*p^*O_S fil}(Y') = C_{p'^*g^*O_S fil}(Y')$ commutes

$$\begin{tikzcd} g''^*\Gamma_X E((\Omega^\bullet_{O_Y/p^*O_S}, F_b) \otimes_{O_Y} (M,F)) \arrow[r, "E(\Omega_{(-)/(-)}(M,F))\circ T(g'',E)(-)\circ T(g'',\gamma)(-)"] \arrow[d, "\gamma_X(-)"] & \Gamma_{X\times_Y Y'} E((\Omega^\bullet_{O_{Y'}/p'^*O_T}, F_b) \otimes_{O_{Y'}} (g''^{*mod}(M,F))) \arrow[d, "\gamma_{X\times_Y Y'}(-)"] \\ g''^* E((\Omega^\bullet_{O_Y/p^*O_S}, F_b) \otimes_{O_Y} (M,F)) \arrow[r, "E(\Omega_{(-)/(-)}(M,F)\circ T(g'',E)(-)"] & E((\Omega^\bullet_{O_{Y'}/p'^*O_T}, F_b) \otimes_{O_{Y'}} g''^{*mod}(M,F)) \end{tikzcd} .$$

(ii) For $M \in C_{\mathcal{D}}(Y)$, there is a canonical map

$$T^O_\omega(D)(M,F)^\gamma : g^{*mod} L_O p_* \Gamma_X E((\Omega^\bullet_{O_Y/p^*O_S}, F_b) \otimes_{O_Y} (M,F)) \to$$
$$p'_*\Gamma_{X\times_Y Y'} E((\Omega^\bullet_{O_{Y'}/p'^*O_T}, F_b) \otimes_{O_{Y'}} g''^{*mod}(M,F))$$

unique up to homotopy such that the following diagram in $C_{O_T fil}(T)$ commutes

$$\begin{tikzcd} g^{*mod} L_O p_* \Gamma_X E((\Omega^\bullet_{O_Y/p^*O_S}, F_b) \otimes_{O_Y} (M,F)) \arrow[r, "T^O_\omega(D)(M,F)^\gamma"] \arrow[d, "\gamma_X(-)"] & p'_*\Gamma_{X\times_Y Y'} E((\Omega^\bullet_{O_{Y'}/p'^*O_T}, F_b) \otimes_{O_{Y'}} (g''^{*mod}(M,F))) \arrow[d, "\gamma_{X\times_Y Y'}(-)"] \\ g^{*mod} L_O p_* E((\Omega^\bullet_{O_Y/p^*O_S}, F_b) \otimes_{O_Y} (M,F)) \arrow[r, "T^O_\omega(D)(M,F)"] & p'_* E((\Omega^\bullet_{O_{Y'}/p'^*O_T}, F_b) \otimes_{O_{Y'}} g''^{*mod}(M,F)) \end{tikzcd} .$$

(iii) For $N \in C_{\mathcal{D}}(Y \times T)$, the canonical map in $C_{h'^*O_T fil}(Y')$

$$T(X'/X \times_Y Y', \gamma)(-) : \Gamma_{X'} E((\Omega^\bullet_{Y'/T}, F_b) \otimes_{O_{Y'}} (N,F)) \to \Gamma_{X\times_Y Y'} E((\Omega^\bullet_{O_{Y'}/O_T}, F_b) \otimes_{O_{Y'}} (N,F))$$

is unique up to homotopy such that $\gamma_{X\times_Y Y'}(-) \circ T(X'/X \times_Y Y', \gamma)(-) = \gamma_{X'}(-)$.

(iv) For $M = O_Y$, we have $T^O_\omega(D)(O_{Y\times S})^\gamma = T^O_\omega(D)^\gamma$ and $T^O_\omega(X \times_Y Y'/Y')(O_{Y'})^\gamma = T^O_\omega(X \times_Y Y'/Y')^\gamma$ (see definition-proposition 3).

Proof. Immediate from definition. We take for the map of point (ii) the composite

$$T^O_\omega(D)(M,F)^\gamma : g^{*mod}L_Op_*\Gamma_X E((\Omega^\bullet_{O_Y/p^*O_S},F_b)\otimes_{O_Y}(M,F)) \xrightarrow{q}$$
$$g^*p_*\Gamma_X E((\Omega^\bullet_{O_Y/p^*O_S},F_b)\otimes_{O_Y}(M,F))\otimes_{g^*O_S}O_T \xrightarrow{T(g'',E)(-)\circ T(g'',\gamma)(-)\circ T(D)(E(\Omega^\bullet_{O_Y/p^*O_S},F_b))}$$
$$(p'_*\Gamma_{X\times_Y Y'}E(g^{''}((\Omega^\bullet_{O_Y/p^*O_S},F_b)\otimes_{O_Y}(M,F))))\otimes_{g^*O_S}O_T \xrightarrow{E(\Omega_{(O_{Y'}/g''^*O_Y)/(O_T/g^*O_S}(M,F)))}$$
$$p'_*\Gamma_{X\times_Y Y'}E((\Omega^\bullet_{O_{Y'}/p'^*O_T},F_b)\otimes_{O_{Y'}}g^{''*mod}(M,F))\otimes_{g^*O_S}O_T \xrightarrow{m}$$
$$p'_*\Gamma_{X\times_Y Y'}E((\Omega^\bullet_{O_{Y'}/p'^*O_T},F_b)\otimes_{O_{Y'}}g^{''*mod}(M,F)),$$

with $m(n\otimes s)=s.n$. □

Let $p:(Y,O_Y)\to(S,O_S)$ a morphism with $(Y,O_Y),(S,O_S)\in \mathrm{RTop}$. Let $i:X\hookrightarrow Y$ a closed embedding. Denote by $j:Y\backslash X\hookrightarrow Y$ the complementary open embedding. Consider, for $(M,F)\in C_{D(O_Y)fil}(Y)$, the map in $C_{p^*O_Sfil}(Y)$ (see definition 52):

$$k\circ T^O_w(j,\otimes)(E(M,F)):(\Omega^\bullet_{O_Y/p^*O_S},F_b)\otimes_{O_Y}j_*j^*E(M,F))$$
$$\xrightarrow{DR(O_Y/p^*O_S)(\mathrm{ad}(j^*,j_*)(-))}$$
$$j_*j^*((\Omega^\bullet_{O_Y/p^*O_S},F_b)\otimes_{O_Y}j_*j^*E(M,F))=j_*j^*(\Omega^\bullet_{O_Y/p^*O_S},F_b)\otimes_{O_Y}j^*j_*j^*E(M,F)$$
$$\xrightarrow{k\circ DR(O_Y/p^*O_S)(\mathrm{ad}(j^*,j_*)(j^*E(M)))}$$
$$j_*E(j^*(\Omega^\bullet_{O_Y/p^*O_S},F_b)\otimes_{O_Y}j^*E(M,F))=j_*E(j^*((\Omega^\bullet_{O_Y/p^*O_S},F_b)\otimes_{O_Y}E(M,F)))$$

Definition 53. *Let $p:(Y,O_Y)\to(S,O_S)$ a morphism with $(Y,O_Y),(S,O_S)\in\mathrm{RTop}$. Let $i:X\hookrightarrow Y$ a closed embedding. Denote by $j:Y\backslash X\hookrightarrow Y$ the complementary open embedding. We consider, for $(M,F)\in C_{D(O_Y)fil}(Y)$ the canonical map in $C_{p^*O_Sfil}(Y)$*

$$T^O_w(\gamma,\otimes)(M,F):=(I,k\circ T^O_w(j,\otimes)(E(M,F))):$$
$$(\Omega^\bullet_{O_Y/p^*O_S},F_b)\otimes_{O_Y}\Gamma_X E(M,F)\to\Gamma_X E((\Omega^\bullet_{O_Y/p^*O_S},F_b)\otimes_{O_Y}E(M,F)).$$

Proposition 44. *Let $p:(Y,O_Y)\to(S,O_S)$ a morphism with $(Y,O_Y),(S,O_S)\in\mathrm{RTop}$. Let $i:X\hookrightarrow Y$ a closed embedding. Then, if Ω_{O_Y/p^*O_S} is a locally free O_Y module, for $(M,F)\in C_{D(O_Y)fil}(Y)$*

(i) the map

$$T^O_w(\gamma,\otimes)(M,F):(\Omega^\bullet_{O_Y/p^*O_S},F_b)\otimes_{O_Y}\Gamma_X E(M,F)\to\Gamma_X E((\Omega^\bullet_{O_Y/p^*O_S},F_b)\otimes_{O_Y}E(M,F))$$

is a 1-filtered top local equivalence,

(ii) the map in $D_{p^*O_Sfil}(Y)$

$$T^O_w(\gamma,\otimes):=DR(O_Y/p^*O_S)(k)^{-1}\circ T^O_w(\gamma,\otimes)(M,F):$$
$$(\Omega^\bullet_{O_Y/p^*O_S},F_b)\otimes_{O_Y}\Gamma_X E(M,F)\to\Gamma_X E((\Omega^\bullet_{O_Y/p^*O_S},F_b)\otimes_{O_Y}(M,F))$$

is an isomorphism.

Proof. By proposition 43,

- $\mathrm{Gr}^p_F(k\circ T^O_w(j,\otimes)(E(M,F))):\Omega^\bullet_{O_Y/p^*O_S}\otimes_{O_Y}j_*j^*F^{p-\bullet}E(M)\to j_*E(j^*(\Omega^\bullet_{O_Y/p^*O_S}\otimes_{O_Y}F^{p-\bullet}E(M)))$ is a top local equivalence and
- $DR(O_Y/p^*O_S)(k):\Omega^\bullet_{O_Y/p^*O_S}\otimes_{O_Y}(M,F)\to\Omega^\bullet_{O_Y/p^*O_S}\otimes_{O_Y}E(M,F)$ is a filtered top local equivalence.

□

4.2 The D-modules on smooth complex algebraic varieties and on complex analytic maninfold and their functorialities in the filtered case

For convenience, we will work with and state the results for presheaves of D-modules. In this section, it is possible to assume that all the presheaves are sheaves and take the sheaftification functor after the pullback functor f^* for a morphism $f : X \to S$, $X, S \in \mathrm{Var}(\mathbb{C})$ or $X, S \in \mathrm{AnSp}(\mathbb{C})$, and after the internal hom functors and tensor products of presheaves of modules on $S \in \mathrm{Var}(\mathbb{C})$ or $S \in \mathrm{AnSp}(\mathbb{C})$.

For $S = (S, O_S) \in \mathrm{SmVar}(\mathbb{C})$, resp. $S = (S, O_S) \in \mathrm{AnSm}(\mathbb{C})$, we denote by

- $D_S := D(O_S) \subset \mathcal{H}om_{\mathbb{C}_S}(O_S, O_S)$ the subsheaf consisting of differential operators. By a D_S module, we mean a left D_S module.
- we denote by
  - $\mathrm{PSh}_{\mathcal{D}}(S)$ the abelian category of Zariski (resp. usu) presheaves on S with a structure of left D_S module, and by $\mathrm{PSh}_{\mathcal{D},h}(S) \subset \mathrm{PSh}_{\mathcal{D},c}(S) \subset \mathrm{PSh}_{\mathcal{D}}(S)$ the full subcategories whose objects are coherent, resp. holonomic, sheaves of left D_S modules, and by $\mathrm{PSh}_{\mathcal{D},rh}(S) \subset \mathrm{PSh}_{\mathcal{D},h}(S)$ the full subcategory of regular holonomic sheaves of left D_S modules,
  - $\mathrm{PSh}_{\mathcal{D}^{op}}(S)$ the abelian category of Zariski (resp. usu) presheaves on S with a structure of right D_S module, and by $\mathrm{PSh}_{\mathcal{D}^{op},h}(S) \subset \mathrm{PSh}_{\mathcal{D}^{op},c}(S) \subset \mathrm{PSh}_{\mathcal{D}^{op}}(S)$ the full subcategories whose objects are coherent, resp. holonomic, sheaves of right D_S modules, and by $\mathrm{PSh}_{\mathcal{D}^{op},rh}(S) \subset \mathrm{PSh}_{\mathcal{D}^{op},h}(S)$ the full subcategory of regular holonomic sheaves of right D_S modules,
- we denote by
  - $C_{\mathcal{D}}(S) = C(\mathrm{PSh}_{\mathcal{D}}(S))$ the category of complexes of Zariski presheaves on S with a structure of D_S module,
  $$C_{\mathcal{D},rh}(S) \subset C_{\mathcal{D},h}(S) \subset C_{\mathcal{D},c}(S) \subset C_{\mathcal{D}}(S)$$
  the full subcategories consisting of complexes of presheaves M such that $a_\tau H^n(M)$ are coherent, resp. holonomic, resp. regular holonomic, sheaves of D_S modules, a_τ being the sheaftification functor for the Zariski, resp. usual, topology,
  - $C_{\mathcal{D}^{op}}(S) = C(\mathrm{PSh}_{\mathcal{D}^{op}}(S))$ the category of complexes of Zariski presheaves on S with a structure of right D_S module,
  $$C_{\mathcal{D}^{op},rh}(S) \subset C_{\mathcal{D}^{op},h}(S) \subset C_{\mathcal{D}^{op},c}(S) \subset C_{\mathcal{D}^{op}}(S)$$
  the full subcategories consisting of complexes of presheaves M such that $a_\tau H^n(M)$ are coherent, resp. holonomic, resp. regular holonomic, sheaves of right D_S modules,
- in the filtered case we have
  - $C_{\mathcal{D}(2)fil}(S) \subset C(\mathrm{PSh}_{\mathcal{D}}(S), F, W) := C(\mathrm{PSh}_{D(O_S)}(S), F, W)$ the category of (bi)filtered complexes of algebraic (resp. analytic) D_S modules such that the filtration is biregular (see definition 46,
  $$C_{\mathcal{D}(2)fil,rh}(S) \subset C_{\mathcal{D}(2)fil,h}(S) \subset C_{\mathcal{D}(2)fil,c}(S) \subset C_{\mathcal{D}(2)fil}(S),$$
  the full subcategories consisting of filtered complexes of presheaves (M, F) such that $a_\tau H^n(M, F)$ are filtered coherent, resp. filtered holonomic, resp filtered regular holonomic, sheaves of D_S modules, that is $a_\tau H^n(M)$ are coherent, resp. holonomic, resp. regular holonomic, sheaves of D_S modules and F induces a good filtration on $a_\tau H^n(M)$ (in particular $F^p a_\tau H^n(M) \subset a_\tau H^n(M)$ are coherent sub O_S modules),
  - $C_{\mathcal{D}0fil}(S) \subset C_{\mathcal{D}fil}(S)$ the full subcategory such that the filtration is a filtration by D_S submodule (which is stronger then Griffitz transversality), $C_{\mathcal{D}(1,0)fil}(S) \subset C_{\mathcal{D}2fil}(S)$ the full subcategory such that W is a filtration by D_S submodules (see definition 46),
  $$C_{\mathcal{D}(1,0)fil,h}(S) = C_{\mathcal{D}2fil,h}(S) \cap C_{\mathcal{D}(1,0)fil}(S) \subset C_{\mathcal{D}2fil,h}(S),$$

the full subcategory consisting of filtered complexes of presheaves (M,F,W) such that $a_\tau H^n(M,F)$ are filtered holonomic sheaves of D_S modules and such that $W^pM \subset M$ are D_S submodules (recall that the O_S submodules $F^pM \subset M$ are NOT D_S submodules but satisfy by definition $md : F^rD_S \otimes F^pM \subset F^{p+r}M$),

$$C_{\mathcal{D}(1,0)fil,rh}(S) = C_{\mathcal{D}2fil,rh}(S) \cap C_{\mathcal{D}(1,0)fil}(S) \subset C_{\mathcal{D}2fil,rh}(S),$$

the full subcategory consisting of filtered complexes of presheaves (M,F,W) such that $a_\tau H^n(M,F)$ are filtered regular holonomic sheaves of D_S modules and such that $W^pM \subset M$ are D_S submodules

  - $C_{\mathcal{D}^{op}(2)fil}(S) \subset C(\mathrm{PSh}_{\mathcal{D}^{op}}(S),F,W) := C(\mathrm{PSh}_{D(O_S)^{op}}(S),F,W)$ the category of (bi)filtered complexes of algebraic (resp. analytic) right D_S modules such that the filtration is biregular, as in the left case we consider the subcategories

$$C_{\mathcal{D}^{op}(2)fil,rh}(S) \subset C_{\mathcal{D}^{op}(2)fil,h}(S) \subset C_{\mathcal{D}^{op}(2)fil,c}(S) \subset C_{\mathcal{D}^{op}(2)fil}(S),$$

the full subcategories consisting of filtered complexes of presheaves (M,F) such that $a_\tau H^n(M,F)$ are filtered coherent, resp. filtered holonomic, resp. filtered regular holonomic, sheaves of right D_S modules.

For $S = (S,O_S) \in \mathrm{AnSm}(\mathbb{C})$, we have the natural extension $D_S \subset D_S^\infty \subset \mathcal{H}om_{\mathbb{C}_S}(O_S,O_S)$ where $D_S^\infty \subset \mathcal{H}om_{\mathbb{C}_S}(O_S,O_S)$ is the subsheaf of differential operators of possibly infinite order (see [18]) for the definition of the action of a differential operator of infinite order on O_S) Similarly, we have

- $C_{\mathcal{D}^\infty(2)fil}(S) \subset C(\mathrm{PSh}_{\mathcal{D}^\infty}(S),F,W) := C(\mathrm{PSh}_{D_S^\infty}(S),F,W)$ the category of (bi)filtered complexes of D_S^∞ modules such that the filtration is biregular,

$$C_{\mathcal{D}^\infty(2)fil,h}(S) \subset C_{\mathcal{D}^\infty(2)fil,c}(S) \subset C_{\mathcal{D}^\infty(2)fil}(S),$$

  the full subcategories consisting of filtered complexes of presheaves (M,F) such that $a_\tau H^n(M,F)$ are filtered coherent (resp. holonomic) sheaves of D_S^∞ modules that is $a_\tau H^n(M)$ are coherent (resp. holonomic) sheaves of D_S^∞ modules and F induces a good filtration on $a_\tau H^n(M)$.

- $C_{\mathcal{D}^\infty 0fil}(S) \subset C_{\mathcal{D}^\infty fil}(S)$ the full subcategory such that the filtration is a filtration by D_S^∞ submodule, $C_{\mathcal{D}^\infty(1,0)fil}(S) \subset C_{\mathcal{D}^\infty 2fil}(S)$ the full subcategory such that W is a filtarion by D_S^∞ submodules,

$$C_{\mathcal{D}^\infty(1,0)fil,h}(S) = C_{\mathcal{D}^\infty 2fil,h}(S) \cap C_{\mathcal{D}^\infty(1,0)fil}(S) \subset C_{\mathcal{D}^\infty 2fil,h}(S),$$

  the full subcategory consisting of filtered complexes of presheaves (M,F,W) such that $a_\tau H^n(M,F)$ are filtered holonomic sheaves of D_S^∞ modules and such that $W^pM \subset M$ are D_S submodules

- $C_{\mathcal{D}^{\infty,op}(2)fil}(S) \subset C(\mathrm{PSh}_{\mathcal{D}^{\infty,op}}(S),F,W) := C(\mathrm{PSh}_{D_S^{\infty,op}}(S),F,W)$ the category of (bi)filtered complexes of right D_S^∞ modules such that the filtration is biregular,

$$C_{\mathcal{D}^{\infty,op}(2)fil,h}(S) \subset C_{\mathcal{D}^{\infty,op}(2)fil,c}(S) \subset C_{\mathcal{D}^{\infty,op}(2)fil}(S),$$

  the full subcategories consisting of filtered complexes of presheaves (M,F) such that $a_\tau H^n(M,F)$ are filtered coherent (resp. holonomic) sheaves of D_S modules.

For $f : X \to S$ a morphism with $X,S \in \mathrm{SmVar}(\mathbb{C})$ or with $(X,S) \in \mathrm{AnSm}(\mathbb{C})$,

- we denote by
  - $\mathrm{PSh}_{f^*\mathcal{D}}(X)$ the abelian category of Zariski (resp. usu) presheaves on S with a structure of left f^*D_S module, and $C_{f^*\mathcal{D}}(X) = C(\mathrm{PSh}_{f^*\mathcal{D}}(X))$,
  - $\mathrm{PSh}_{\mathcal{D},f^*\mathcal{D}}(X)$ the abelian category of Zariski (resp. usu) presheaves on S with a structure of left f^*D_S module and left D_X module, and $C_{\mathcal{D},f^*\mathcal{D}}(X) = C(\mathrm{PSh}_{\mathcal{D},f^*\mathcal{D}}(X))$,

– $\mathrm{PSh}_{\mathcal{D}^{op},f^*\mathcal{D}}(X)$ the abelian category of Zariski (resp. usu) presheaves on S with a structure of left f^*D_S module and right D_X module and $C_{\mathcal{D}^{op},f^*\mathcal{D}}(X) = C(\mathrm{PSh}_{\mathcal{D}^{op},f^*\mathcal{D}}(X))$,

- we denote by
  – $C_{f^*\mathcal{D}fil}(X) \subset C(\mathrm{PSh}_{f^*\mathcal{D}}(X), F) := C(\mathrm{PSh}_{f^*D(O_S)}(X), F)$ the category of filtered complexes of algebraic (resp. analytic) f^*D_S modules such that the filtration is biregular,
  – $C_{\mathcal{D},f^*\mathcal{D}fil}(X) \subset C(\mathrm{PSh}_{\mathcal{D},f^*\mathcal{D}}(X), F)$ the category of filtered complexes of algebraic (resp. analytic) (f^*D_S, D_X) modules such that the filtration is biregular,
  – $C_{\mathcal{D}^{op},f^*\mathcal{D}fil}(X) \subset C(\mathrm{PSh}_{\mathcal{D}^{op},f^*\mathcal{D}}(X), F)$ the category of filtered complexes of algebraic (resp. analytic) (f^*D_S, D_X^{op}) modules such that the filtration is biregular.

For $f : X \to S$ a morphism with $X, S \in \mathrm{AnSm}(\mathbb{C})$, we denote by

- $C_{f^*\mathcal{D}^\infty fil}(X) \subset C(\mathrm{PSh}_{f^*\mathcal{D}^\infty}(X), F) := C(\mathrm{PSh}_{f^*D_S^\infty}(X), F)$ the category of filtered complexes of $f^*D_S^\infty$ modules such that the filtration is biregular,
- $C_{\mathcal{D}^\infty,f^*\mathcal{D}^\infty fil}(X) \subset C(\mathrm{PSh}_{\mathcal{D}^\infty,f^*\mathcal{D}^\infty}(X), F)$ the category of filtered complexes of $(f^*D_S^\infty, D_X^\infty)$ modules such that the filtration is biregular,
- $C_{\mathcal{D}^{\infty,op},f^*\mathcal{D}^\infty fil}(X) \subset C(\mathrm{PSh}_{\mathcal{D}^{\infty,op},f^*\mathcal{D}^\infty}(X), F)$ the category of filtered complexes of $(f^*D_S^\infty, D_X^{\infty,op})$ modules such that the filtration is biregular.

For $S \in \mathrm{AnSm}(\mathbb{C})$, we denote by

$$J_S : C_{\mathcal{D}(2)fil}(S) \to C_{\mathcal{D}^\infty(2)fil}(S),\ (M, F) \mapsto J_S(M, F) := (M, F) \otimes_{D_S} (D_S^\infty, F^{ord})$$

the natural functor. For $(M, F) \in C_{\mathcal{D}^\infty fil}(S)$, we will consider the map

$$\mathcal{J}_S(M, F) : J_S(M, F) := (M, F) \otimes_{D_S} (D_S^\infty, F^{ord}) \to (M, F), m \otimes P \mapsto Pm$$

Of course $J_S(C_{\mathcal{D}(1,0)fil}(S)) \subset C_{\mathcal{D}^\infty(1,0)fil}(S)$. More generally, for $f : X \to S$ a morphism with $X, S \in \mathrm{AnSm}(\mathbb{C})$, we denote by

$$J_{X/S} : C_{f^*\mathcal{D}(2)fil}(X) \to C_{f^*\mathcal{D}^\infty(2)fil}(X),\ (M, F) \mapsto J_{X/S}(M, F) := (M, F) \otimes_{f^*(D_S,F)} f^*(D_S^\infty, F)$$

the natural functor, together with, for $(M, F) \in C_{f^*\mathcal{D}^\infty fil}(X)$, the map $\mathcal{J}_S(M, F) : J_S(M, F) \to (M, F)$.

Definition 54. *Let $S \in \mathrm{SmVar}(\mathbb{C})$, resp. $S \in \mathrm{AnSm}(\mathbb{C})$. Let $Z \subset S$ a closed subset and denote by $j : S\backslash Z \hookrightarrow S$ the open embedding.*

(i) We denote by

  – $\mathrm{PSh}_{\mathcal{D},Z}(S) \subset \mathrm{PSh}_{\mathcal{D}}(S)$, *the full subcategory consisting of presheaves $M \in \mathrm{PSh}_{\mathcal{D}}(S)$, such that $j^*M = 0$,*
  – $C_{\mathcal{D},Z}(S) \subset C_{\mathcal{D}}(S)$, *the full subcategory consisting of complexes presheaves $M \in C_{\mathcal{D}}(S)$ such that $a_\tau j^* H^n M = 0$ for all $n \in \mathbb{Z}$,*
  – $C_{\mathcal{D},Z,h}(S) := C_{\mathcal{D},Z}(S) \cap C_{\mathcal{D},h}(S) \subset C_{\mathcal{D}}(S)$ *the full subcategory consising of $M \in C_{\mathcal{D}}(S)$ such that $a_\tau H^n(M)$ are holonomic and $a_\tau j^* H^n M = 0$ for all $n \in \mathbb{Z}$,*
  – $C_{\mathcal{D},Z,c}(S) := C_{\mathcal{D},Z}(S) \cap C_{\mathcal{D},c}(S) \subset C_{\mathcal{D}}(S)$ *the full subcategory consising of $M \in C_{\mathcal{D}}(S)$ such that $a_\tau H^n(M)$ are coherent and $a_\tau j^* H^n M = 0$ for all $n \in \mathbb{Z}$.*

(ii) We denote by

  – $C_{\mathcal{D}(2)fil,Z}(S) \subset C_{\mathcal{D}(2)fil}(S)$, *the full subcategory consisting of $(M, F) \in C_{\mathcal{D}(2)fil}(S)$ such that there exists $r \in \mathbb{N}$ and an r-filtered homotopy equivalence $\phi : (M, F) \to (N, F)$ with $(N, F) \in C_{\mathcal{D}(2)fil}(S)$ such that $a_\tau j^* H^n \mathrm{Gr}_F^p(M, F) = 0$ for all $n, p \in \mathbb{Z}$, note that by definition this r does NOT depend on n and p,*

– $C_{\mathcal{D}(2)fil,Z,rh}(S) := C_{\mathcal{D}(2)fil,Z}(S) \cap C_{\mathcal{D}(2)fil,rh}(S) \subset C_{\mathcal{D}(2)fil}(S)$ *the full subcategory consising of* (M,F) *such that* $a_\tau H^n(M,F)$ *are filtered regular holonomic for all* $n \in \mathbb{Z}$ *and such that there exists* $r \in \mathbb{N}$ *and an* r*-filtered homotopy equivalence* $\phi : (M,F) \to (N,F)$ *with* $(N,F) \in C_{\mathcal{D}(2)fil}(S)$ *such that* $a_\tau j^* H^n \operatorname{Gr}^p_F(M,F) = 0$ *for all* $n,p \in \mathbb{Z}$,

– $C_{\mathcal{D}(2)fil,Z,h}(S) := C_{\mathcal{D}(2)fil,Z}(S) \cap C_{\mathcal{D}(2)fil,h}(S) \subset C_{\mathcal{D}(2)fil}(S)$ *the full subcategory consising of* (M,F) *such that* $a_\tau H^n(M,F)$ *are filtered holonomic for all* $n \in \mathbb{Z}$ *and such that there exists* $r \in \mathbb{N}$ *and an* r*-filtered homotopy equivalence* $\phi : (M,F) \to (N,F)$ *with* $(N,F) \in C_{\mathcal{D}(2)fil}(S)$ *such that* $a_\tau j^* H^n \operatorname{Gr}^p_F(M,F) = 0$ *for all* $n,p \in \mathbb{Z}$,

– $C_{\mathcal{D}(2)fil,Z,c}(S) := C_{\mathcal{D}(2)fil,Z}(S) \cap C_{\mathcal{D}(2)fil,c}(S) \subset C_{\mathcal{D}(2)fil}(S)$ *the full subcategory consising of* (M,F) *such that* $a_\tau H^n(M,F)$ *are filtered coherent for all* $n \in \mathbb{Z}$ *and such that there exists* $r \in \mathbb{N}$ *and an* r*-filtered homotopy equivalence* $\phi : (M,F) \to (N,F)$ *with* $(N,F) \in C_{\mathcal{D}(2)fil}(S)$ *such that* $a_\tau j^* H^n \operatorname{Gr}^p_F(M,F) = 0$ *for all* $n,p \in \mathbb{Z}$.

(iii) We have then the full subcategories

– $C_{\mathcal{D}(1,0)fil,Z}(S) = C_{\mathcal{D}(1,0)fil}(S) \cap C_{\mathcal{D}2fil,Z}(S) \subset C_{\mathcal{D}2fil}(S)$,

– $C_{\mathcal{D}(1,0)fil,Z,rh}(S) = C_{\mathcal{D}(1,0)fil}(S) \cap C_{\mathcal{D}2fil,Z,rh}(S) \subset C_{\mathcal{D}2fil}(S)$.

– $C_{\mathcal{D}(1,0)fil,Z,h}(S) = C_{\mathcal{D}(1,0)fil}(S) \cap C_{\mathcal{D}2fil,Z,h}(S) \subset C_{\mathcal{D}2fil}(S)$.

Similarly :

Definition 55. *Let* $S \in \operatorname{AnSm}(\mathbb{C})$. *Let* $Z \subset S$ *a closed subset and denote by* $j : S\backslash Z \hookrightarrow S$ *the open embedding.*

(i) We denote by

– $C_{\mathcal{D}^\infty(2)fil,Z}(S) \subset C_{\mathcal{D}^\infty(2)fil}(S)$. *the full subcategory consisting of* $(M,F) \in C_{\mathcal{D}^\infty}(S)$ *such that* j^*M *is acyclic*

– $C_{\mathcal{D}^\infty(2)fil,Z,h}(S) := C_{\mathcal{D}^\infty(2)fil,Z}(S) \cap C_{\mathcal{D}^\infty(2)fil,h}(S) \subset C_{\mathcal{D}^\infty(2)fil}(S)$ *the full subcategory consising of* (M,F) *such that* $a_\tau H^n(M)$ *are holonomic and such that there exist* $r \in \mathbb{Z}$ *and an* r*-filtered homotopy equivalence* $\phi : (M,F) \to (N,F)$ *with* $(N,F) \in C_{\mathcal{D}^\infty(2)fil}(S)$ *such that* $a_\tau j^* H^n \operatorname{Gr}^p_F(M,F) = 0$.

(ii) We have then the full subcategories

– $C_{\mathcal{D}^\infty(1,0)fil,Z}(S) = C_{\mathcal{D}^\infty(1,0)fil}(S) \cap C_{\mathcal{D}^\infty 2fil,Z}(S) \subset C_{\mathcal{D}^\infty 2fil}(S)$,

– $C_{\mathcal{D}^\infty(1,0)fil,Z,h}(S) := C_{\mathcal{D}^\infty(1,0)fil}(S) \cap C_{\mathcal{D}^\infty 2fil,Z,h}(S) \subset C_{\mathcal{D}^\infty(2)fil}(S)$.

Let $S \in \operatorname{SmVar}(\mathbb{C})$ or $S \in \operatorname{AnSm}(\mathbb{C})$. We recall that a morphism $m : (M,F) \to (N,F)$ with $(M,F),(N,F) \in C_{\mathcal{D}fil}(S)$ is said to be an r-filtered quasi-isomorphism if there exist an r-filtered homotopy

$$(h,m,m') : (M,F)[1] \to (N,F)$$

such that $m' : (M,F) \to (N,F)$ is a filtered quasi-isomorphism (see section 2.1).

Definition 56. *(i) Let* $f : X \to S$ *a morphism with* $X,S \in \operatorname{SmVar}(\mathbb{C})$, *or with* $X,S \in \operatorname{AnSm}(\mathbb{C})$, *we have, for* $r = 1,\ldots\infty$, *resp.* $r = (1,\ldots\infty)^2$, *the categories*

$$D_{\mathcal{D},f^*\mathcal{D}(2)fil,r}(S) := \operatorname{Ho}_{Frtop} C_{\mathcal{D},f^*\mathcal{D}(2)fil}(S)\ ,\ D_{\mathcal{D}^{op},f^*\mathcal{D}(2)fil,r}(S) := \operatorname{Ho}_{Frtop} C_{\mathcal{D}^{op},f^*\mathcal{D}(2)fil}(S),$$

the localizations with respect to r*-filtered Zariski, resp. usu, local equivalence (see section 2.1).*

(ii) *Let $S \in \mathrm{SmVar}(\mathbb{C})$ or $S \in \mathrm{AnSm}(\mathbb{C})$. We denote by*

$$D_{\mathcal{D}(2)fil,\infty,rh}(S) \subset D_{\mathcal{D}(2)fil,\infty,h}(S) \subset D_{\mathcal{D}(2)fil,\infty}(S),$$

the full subcategories consisting of the image of $C_{\mathcal{D}(2)fil,h}(S)$, resp. $C_{\mathcal{D}(2)fil,rh}(S)$, by the localization functor

$$D(top) : C_{\mathcal{D}(2)fil}(S) \to D_{\mathcal{D}(2)fil,\infty}(S)$$

that is consisting of $(M,F) \in C_{\mathcal{D}fil}(S)$ such that $a_\tau H^n(M,F)$ are filtered holonomic, resp. filtered regular holonomic for all $n \in \mathbb{Z}$,

(iii) *Let $S \in \mathrm{SmVar}(\mathbb{C})$ or $S \in \mathrm{AnSm}(\mathbb{C})$. We denote by*

$$D_{\mathcal{D}(1,0)fil,\infty,rh}(S) \subset D_{\mathcal{D}(1,0)fil,\infty,h}(S) \subset D_{\mathcal{D}(2)fil,\infty}(S),$$

the full subcategories consisting of the image of $C_{\mathcal{D}(1,0)fil,h}(S)$, resp. $C_{\mathcal{D}(1,0)fil,rh}(S)$, by the localization functor

$$D(top) : C_{\mathcal{D}(2)fil}(S) \to D_{\mathcal{D}(2)fil,\infty}(S)$$

that is consisting of $(M,F,W) \in C_{\mathcal{D}2fil}(S)$ such that $a_\tau H^n(M,F)$ are filtered holonomic, resp. filtered regular holonomic, and $W^pM^n \subset M^n$ are D_S submodules for all $n \in \mathbb{Z}$.

Let $S \in \mathrm{SmVar}(\mathbb{C})$ or $S \in \mathrm{AnSm}(\mathbb{C})$. By definition (see section 2), we have sequences of functors

$$C_{\mathcal{D}(2)fil}(\mathcal{S}) \to K_{\mathcal{D}(2)fil}(S) \to D_{\mathcal{D}(2)fil}(\mathcal{S}) \to D_{\mathcal{D}(2)fil,2}(S) \to \cdots \to D_{\mathcal{D}(2)fil,\infty}(S).$$

and commutative diagrams of functors

$$\begin{tikzcd} K_{\mathcal{D}(2)fil}(S) \arrow[r] \arrow[d] & D_{fil}(S) \arrow[d] \\ K_{\mathcal{D}(2)fil,2}(\mathcal{S}) \arrow[r] & D_{\mathcal{D}(2)fil,2}(S) \end{tikzcd} \quad , \quad \begin{tikzcd} K_{\mathcal{D}(2)fil,r}(S) \arrow[r] \arrow[d] & D_{\mathcal{D}(2)fil,r}(S) \arrow[d] \\ K_{\mathcal{D}(2)fil,r+1}(S) \arrow[r] & D_{\mathcal{D}(2)fil,r+1}(S) \end{tikzcd}$$

where $K_{\mathcal{D}(2)fil}(S) := K(\mathrm{PSh}_{\mathcal{D}(2)fil}(S))$ and $K_{\mathcal{D}(2)fil,r}(S) := K_r(\mathrm{PSh}_{\mathcal{D}(2)fil}(S))$. are the categories where the morphisms are r-filtered homotopy classes of morphisms. Then, for $r = 1$, $K_{\mathcal{D}(2)fil}(S)$ and $D_{\mathcal{D}(2)fil}(S)$ are in the canonical way triangulated categories. However, for $r > 1$, the categories $K_{\mathcal{D}(2)fil,r}(S)$ and $D_{\mathcal{D}(2)fil,r}(S)$ together with the canonical triangles does NOT satisfy the 2 of 3 axiom of triangulated categories.

Similarly,

Definition 57. (i) *Let $f : X \to S$ a morphism with $X, S \in \mathrm{AnSm}(\mathbb{C})$, we have, for $r = 1, \ldots \infty$, resp. $r = (1, \ldots \infty)^2$, the categories*

$$D_{\mathcal{D}^\infty,f^*\mathcal{D}^\infty(2)fil,r}(S) := \mathrm{Ho}_{Frtop} C_{\mathcal{D}^\infty,f^*\mathcal{D}^\infty(2)fil}(S) \ , \ D_{\mathcal{D}^{\infty,op},f^*\mathcal{D}^\infty(2)fil,r}(S) := \mathrm{Ho}_{Frtop} C_{\mathcal{D}^{\infty,op},f^*\mathcal{D}^\infty(2)fil}(S),$$

the localizations with respect to r-filtered usu local equivalence (see section 2.1).

(ii) *Let $S \in \mathrm{AnSm}(\mathbb{C})$. We denote by*

$$D_{\mathcal{D}^\infty(2)fil,\infty,h}(S) \subset D_{\mathcal{D}^\infty(2)fil,\infty}(S),\ D_{\mathcal{D}^\infty(1,0)fil,\infty,h}(S) \subset D_{\mathcal{D}^\infty 2fil,\infty}(S)$$

the full subcategories consisting of the image of $C_{\mathcal{D}^\infty(2)fil,h}(S)$, resp. $C_{\mathcal{D}^\infty(1,0)fil,h}(S)$, by the localization functor

$$D(top) : C_{\mathcal{D}^\infty(2)fil}(S) \to D_{\mathcal{D}^\infty(2)fil,\infty}(S).$$

We begin this subsection by recalling the following well known facts

Proposition 45. *Let $S \in \mathrm{SmVar}(\mathbb{C})$ or $S \in \mathrm{AnSm}(\mathbb{C})$.*

(i) The sheaf of differential operators D_S is a locally free sheaf of O_S module. Hence, a coherent D_S module $M \in \mathrm{Coh}_{\mathcal{D}}(S)$ is a quasi-coherent sheaf of O_S modules.

(ii) A coherent sheaf $M \in \mathrm{Coh}_{O_S}(S)$ of O_S module admits a D_S module structure if and only if it is locally free (of finite rank by coherency) and admits an integrable connexion. In particular if $i : Z \hookrightarrow S$ is a closed embedding for the Zariski topology, then i_*O_Z does NOT admit a D_S module structure since it is a coherent but not locally free O_S module.

Proof. Standard. □

In order to prove a version of the first GAGA theorem for coherent D modules, we will need to following. We start by a definition (cf. [16] definition 1.4.2) :

Definition 58. *An $X \in \mathrm{SmVar}(\mathbb{C})$ is said to be D-affine if the following two condition hold:*

(i) The global section functor $\Gamma(X, \cdot) : \mathcal{Q}Coh_{\mathcal{D}}(X) \to Mod(\Gamma(X, D_X))$ is exact.

(ii) If $\Gamma(X, M) = 0$ for $M \in \mathcal{Q}Coh_{\mathcal{D}}(X)$, then $M = 0$.

Proposition 46. *If $X \in \mathrm{SmVar}(\mathbb{C})$ is D-affine, then :*

(i) Any $M \in \mathcal{Q}Coh_{\mathcal{D}}(X)$ is generated by its global sections.

(ii) The functor $\Gamma(X, \cdot) : \mathcal{Q}Coh_{\mathcal{D}}(X) \to Mod(\Gamma(X, D_X))$ is an equivalence of category whose inverse is $L \in Mod(\Gamma(X, D_X)) \mapsto D_X \otimes_{\Gamma(X,D_X)} L \in \mathcal{Q}Coh_{\mathcal{D}}(X)$.

(iii) We have $\Gamma(X, \cdot)(\mathcal{C}oh_{\mathcal{D}}(X)) = Mod(\Gamma(X, D_X))_f$, that is the global sections of a coherent D_X module is a finite module over the differential operators on X.

Proof. See [16]. □

The following proposition is from Kashiwara.

Proposition 47. *Let $S \in \mathrm{AnSm}(\mathbb{C})$.*

(i) For $K \in C_c(S)$ a complex of presheaves with constructible cohomology sheaves, we have $\mathcal{H}om(L(K), E(O_S)) \in C_{\mathcal{D}^\infty,h}(S)$.

(ii) The functor $J_S : C_{\mathcal{D}(2)fil}(S) \to C_{\mathcal{D}^\infty(2)fil}(S)$ satisfy $J_S(C_{\mathcal{D}(2)fil,rh}(S)) \subset C_{\mathcal{D}^\infty(2)fil,h}(S)$, derive trivially, and induce an equivalence of category

$$J_S : D_{\mathcal{D}(2)fil,\infty,rh}(S) \to D_{\mathcal{D}^\infty(2)fil,\infty,h}(S).$$

whose inverse satify, for $(M, F) \in \mathcal{H}ol_{\mathcal{D}^\infty(2)fil}(S)$ a (filtered) holonomic D_S^∞ module, that $J_S^{-1}(M, F) = (M_{reg}, F) \subset (M, F)$ is the D_S sub-module of M which is the regular part.

(iii) We have $J_S(C_{\mathcal{D}(1,0)fil,rh}(S)) \subset C_{\mathcal{D}^\infty(1,0)fil,h}(S)$ and $J_S(D_{\mathcal{D}(1,0)fil,\infty,rh}(S)) = D_{\mathcal{D}^\infty(1,0)fil,\infty,h}(S)$.

Proof. Follows from [18]. □

Let $S \in \mathrm{SmVar}(\mathbb{C})$ or $S \in \mathrm{AnSm}(\mathbb{C})$, and let $i : Z \hookrightarrow S$ a closed embedding and denote by $j : S\backslash Z \hookrightarrow S$ the open complementary. For $M \in \mathrm{PSh}_{\mathcal{D}}(S)$, we denote $\mathcal{I}_Z M \subset M$ the (left) D_S submodule given by, for $S^o \subset S$ an open subset, $\mathcal{I}_Z M(S^o) \subset M(S^o)$ is the (left) $D_S(S^o)$ submodule

$$\mathcal{I}_Z M(S^o) =< \{fm, f \in \mathcal{I}_Z(S^o), m \in M(S^o)\} >\subset M(S^o)$$

generated by the elements of the form fm. We denote by $b_Z(M) : \mathcal{I}_Z M \to M$ the inclusion map and $c_Z(M) : M \to M/\mathcal{I}_Z M$ the quotient map of (left) D_S modules. For $M \in \mathrm{PSh}_{\mathcal{D}}(S)$, we denote $M\mathcal{I}_Z \subset M$ the right D_S submodule given by, for $S^o \subset S$ an open subset, $\mathcal{I}_Z M(S^o) \subset M(S^o)$ is the right $D_S(S^o)$ submodule

$$\mathcal{I}_Z M(S^o) =< \{mf, f \in \mathcal{I}_Z(S^o), m \in M(S^o)\} >\subset M(S^o)$$

generated by the elements of the form mf. We denote by $b_Z(M) : \mathcal{I}_Z M \to M$ the inclusion map and $c_Z(M) : M \to M/\mathcal{I}_Z M$ the quotient map of right D_S modules.

4.2.1 Functorialities

Let $f : X \to S$ be a morphism with $X, S \in \mathrm{SmVar}(\mathbb{C})$, or let $f : X \to S$ be a morphism with $X, S \in \mathrm{AnSm}(\mathbb{C})$. Then, we recall from section 4.1, the transfers modules

- $(D_{X\to S}, F^{ord}) := f^{*mod}(D_S, F^{ord}) := f^*(D_S, F^{ord}) \otimes_{f^*O_S} (O_X, F_b)$ which is a left D_X module and a left and right f^*D_S module
- $(D_{X\leftarrow S}, F^{ord}) := (K_X, F_b) \otimes_{O_X} (D_{X\to S}, F^{ord}) \otimes_{f^*O_S} f^*(K_S, F_b)$. which is a right D_X module and a left and right f^*D_S module.

Let $f : X \to S$ be a morphism with $X, S \in \mathrm{AnSm}(\mathbb{C})$. Then, the transfers modules of infite order are

- $(D^\infty_{X\to S}, F^{ord}) := f^{*mod}(D^\infty_S, F^{ord}) := f^*(D^\infty_S, F^{ord}) \otimes_{f^*O_S} (O_X, F_b)$ which is a left D^∞_X module and a left and right $f^*D^\infty_S$ module
- $(D^\infty_{X\leftarrow S}, F^{ord}) := (K_X, F_b) \otimes_{O_X} (D^\infty_{X\to S}, F^{ord}) \otimes_{f^*O_S} f^*(K_S, F_b)$. which is a right D^∞_X module and a left and right $f^*D^\infty_S$ module.

We have the following :

Lemma 4. *Let $f_1 : X \to Y$, $f_2 : Y \to S$ be two morphism with $X, S, Y \in \mathrm{SmVar}(\mathbb{C})$, or let $f_1 : X \to Y$, $f_2 : Y \to S$ be two morphism with $X, S, Y \in \mathrm{AnSm}(\mathbb{C})$.*

(i) We have $(D_{X\to S}, F^{ord}) = f_1^*(D_{Y\to S}, F^{ord}) \otimes_{f_1^*D_Y} (D_{X\to Y}, F^{ord})$ in $C_{\mathcal{D},(f_2\circ f_1)^*\mathcal{D}fil}(X)$ and

$$(D_{X\to S}, F^{ord}) = f_1^*(D_{Y\to S}, F^{ord}) \otimes_{f_1^*D_Y} (D_{X\to Y}, F^{ord}) = f_1^*(D_{Y\to S}, F^{ord}) \otimes^L_{f_1^*D_Y} (D_{X\to Y}, F^{ord}).$$

in $D_{\mathcal{D},(f_2\circ f_1)^*\mathcal{D}fil,r}(X)$.

(ii) We have $(D_{X\leftarrow S}, F^{ord}) = f_1^*(D_{Y\to S}, F^{ord}) \otimes_{f_1^*D_Y} (D_{X\leftarrow Y}, F^{ord})$ in $C_{\mathcal{D}^{op},(f_2\circ f_1)^*\mathcal{D}fil}(X)$ and

$$(D_{X\leftarrow S}, F^{ord}) = f_1^*(D_{Y\to S}, F^{ord}) \otimes_{f_1^*D_Y} (D_{X\leftarrow Y}, F^{ord}) = f_1^*(D_{Y\leftarrow S}, F^{ord}) \otimes^L_{f_1^*D_Y} (D_{X\leftarrow Y}, F^{ord}),$$

in $D_{\mathcal{D}^{op},(f_2\circ f_1)^*\mathcal{D}fil,r}(X)$.

Proof. Follows immediately from definition. The first assertions of (i) and (ii) are particular cases of lemma 2. See [16] for example. □

In the analytical case we also have

Lemma 5. *Let $f_1 : X \to Y$, $f_2 : Y \to S$ be two morphism with $X, S, Y \in \mathrm{AnSm}(\mathbb{C})$.*

(i) We have $(D^\infty_{X\to S}, F^{ord}) = f_1^*(D^\infty_{Y\to S}, F^{ord}) \otimes_{f_1^*D^\infty_Y} (D^\infty_{X\to Y}, F^{ord})$ in $C_{\mathcal{D}^\infty,(f_2\circ f_1)^*\mathcal{D}^\infty fil}(X)$ and

$$(D^\infty_{X\to S}, F^{ord}) = f_1^*(D^\infty_{Y\to S}, F^{ord}) \otimes_{f_1^*D^\infty_Y} (D_{X\to Y}, F^{ord}) = f_1^*(D^\infty_{Y\to S}, F^{ord}) \otimes^L_{f_1^*D^\infty_Y} (D^\infty_{X\to Y}, F^{ord}).$$

in $D_{\mathcal{D}^\infty,(f_2\circ f_1)^*\mathcal{D}^\infty fil,r}(X)$.

(ii) We have $(D^\infty_{X\leftarrow S}, F^{ord}) = f_1^*(D^\infty_{Y\to S}, F^{ord}) \otimes_{f_1^*D^\infty_Y} (D^\infty_{X\leftarrow Y}, F^{ord})$ in $C_{\mathcal{D}^{\infty,op},(f_2\circ f_1)^*\mathcal{D}^\infty fil}(X)$ and

$$(D^\infty_{X\leftarrow S}, F^{ord}) = f_1^*(D^\infty_{Y\to S}, F^{ord}) \otimes_{f_1^*D^\infty_Y} (D^\infty_{X\leftarrow Y}, F^{ord}) = f_1^*(D^\infty_{Y\leftarrow S}, F^{ord}) \otimes^L_{f_1^*D^\infty_Y} (D^\infty_{X\leftarrow Y}, F^{ord}),$$

in $D_{\mathcal{D}^{\infty,op},(f_2\circ f_1)^*\mathcal{D}^\infty fil,r}(X)$.

Proof. Similar to the proof of lemma 4 □

For closed embeddings, we have :

Proposition 48. *(i) Let $i : Z \hookrightarrow S$ be a closed embedding with $Z, S \in \mathrm{SmVar}(\mathbb{C})$. Then, $D_{Z\to S} = i^*D_S/D_S\mathcal{I}_Z$ and it is a locally free (left) D_Z module. Similarly, $D_{Z\leftarrow S} = i^*D_S/\mathcal{I}_Z D_S$ and it is a locally free right D_Z module.*

(ii) Let $i : Z \to S$ be a closed embedding with $Z, S \in \mathrm{AnSm}(\mathbb{C})$. Then, $D_{Z\to S} = i^*D_S/D_S\mathcal{I}_Z$ and it is a locally free (left) D_Z module. Similarly, $D_{Z\leftarrow S} = i^*D_S/\mathcal{I}_Z D_S$ and it is a locally free right D_Z module.

(iii) Let $i : Z \to S$ be a closed embedding with $Z, S \in \mathrm{AnSm}(\mathbb{C})$. Then, $D^\infty_{Z\to S} = i^*D^\infty_S/D^\infty_S\mathcal{I}_Z$ and it is a locally free (left) D^∞_Z module. Similarly, $D^\infty_{Z\leftarrow S} = i^*D^\infty_S/\mathcal{I}_Z D^\infty_S$ and it is a locally free right D^∞_Z module.

Proof. (i): See [16].
(ii):See [25].
(iii):Similar to (ii). □

We now enumerate some functorialities we will use, all of them are particular case of the functoriality given in subsection 2.3 for any ringed spaces :

- Let $f : X \to S$ be a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$, or let $f : X \to S$ be a morphism with $X, S \in \mathrm{AnSp}(\mathbb{C})$. Then, the inverse image functor

$$f^{*mod} : \mathrm{PSh}_{O_S}(S) \to \mathrm{PSh}_{O_X}(X), \quad M \mapsto f^{*mod}M := O_X \otimes_{f^*O_S} f^*M$$

  is a Quillen adjonction which induces in the derived category the functor

$$Lf^{*mod} : D_{O_S}(S) \to D_{O_X}(X), \quad M \mapsto Lf^{*mod}M := O_X \otimes^L_{f^*O_S} f^*M = O_X \otimes_{f^*O_S} f^*L_O M,$$

  The adjonction $(f^{*mod}, f_*) : \mathrm{PSh}_{O_S}(S) \leftrightarrows \mathrm{PSh}_{O_X}(X)$ is a Quillen adjonction, the adjonction map are the maps

  - for $M \in C_{O_S}(S)$, $\mathrm{ad}(f^{*mod}, f_*)(M) : M \xrightarrow{\mathrm{ad}(f^*,f_*)(M)} f_*f^*M \xrightarrow{f_*m} f_*(f^*M \otimes_{f^*O_S} O_X) = f_*f^{*mod}M$ where $m(m) = m \otimes 1$,
  - for $M \in C_{O_X}(X)$, $\mathrm{ad}(f^{*mod}, f_*)(M) : f^{*mod}f_*M = f^*f_*M \otimes_{f^*O_S} O_X \xrightarrow{\mathrm{ad}(f^*,f_*)(M)\otimes_{f^*O_S}O_X} M \otimes_{f^*O_S} O_X \xrightarrow{n} M$, where $n(m \otimes h) = h.m$ is the multiplication map.

- Let $S \in \mathrm{SmVar}(\mathbb{C})$ or $S \in \mathrm{AnSm}(\mathbb{C})$.
  - For $M \in C_{\mathcal{D}}(S)$, we have the canonical projective resolution $q : L_D(M) \to M$ of complexes of D_S modules.
  - For $M \in C_{\mathcal{D}}(S)$, there exist a unique strucure of D_S module on the flasque presheaves $E^i(M)$ such that $E(M) \in C_{\mathcal{D}}(S)$ (i.e. is a complex of D_S modules) and that the map $k : M \to E(M)$ is a morphism of complexes of D_S modules.

  Let $S \in \mathrm{AnSm}(\mathbb{C})$.
  - For $M \in C_{\mathcal{D}^\infty}(S)$, we have the canonical projective resolution $q : L_{D^\infty}(M) \to M$ of complexes of D^∞_S modules.
  - For $M \in C_{\mathcal{D}^\infty}(S)$, there exist a unique strucure of D^∞_S module on the flasque presheaves $E^i(M)$ such that $E(M) \in C_{\mathcal{D}^\infty}(S)$ (i.e. is a complex of D^∞_S modules) and that the map $k : M \to E(M)$ is a morphism of complexes of D^∞_S modules.

- Let $S \in \mathrm{SmVar}(\mathbb{C})$ or let $S \in \mathrm{AnSm}(\mathbb{C})$. For $M \in C_{\mathcal{D}^{(op)}}(S)$, $N \in C(S)$, we will consider the induced D module structure (right D_S module in the case one is a left D_S module and the other one is a right one) on the presheaf $M \otimes N := M \otimes_{\mathbb{Z}_S} N$ (see section 2). We get the bifunctor

$$C(S) \times C_{\mathcal{D}}(S) \to C_{\mathcal{D}}(S), (M, N) \mapsto M \otimes N$$

For $S \in \mathrm{AnSm}(\mathbb{C})$, we also have the bifunctor $C(S) \times C_{\mathcal{D}^\infty}(S) \to C_{\mathcal{D}^\infty}(S)$, $(M, N) \mapsto M \otimes N$.

- Let $S \in \mathrm{SmVar}(\mathbb{C})$ or let $S \in \mathrm{AnSm}(\mathbb{C})$. For $M, N \in C_{\mathcal{D}^{(op)}}(S)$, $M \otimes_{O_S} N$ (see section 2), has a canonical structure of D_S modules (right D_S module in the case one is a left D_S module and the other one is a right one) given by (in the left case) for $S^o \subset S$ an open subset,

$$m \otimes n \in \Gamma(S^o, M \otimes_{O_S} N), \gamma \in \Gamma(S^o, D_S),\ \gamma.(m \otimes n) := (\gamma.m) \otimes n - m \otimes \gamma.n$$

This gives the bifunctor

$$C_{\mathcal{D}^{(op)}}(S)^2 \to C_{\mathcal{D}^{(op)}}(S), (M, N) \mapsto M \otimes_{O_S} N$$

More generally, let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$ or with $X, S \in \mathrm{AnSp}(\mathbb{C})$. Assume S smooth. For $M, N \in C_{f^*\mathcal{D}^{(op)}}(X)$, $M \otimes_{f^*O_S} N$ (see section 2), has a canonical structure of f^*D_S modules (right f^*D_S module in the case one is a left f^*D_S module and the other one is a right one) given by (in the left case) for $X^o \subset X$ an open subset,

$$m \otimes n \in \Gamma(X^o, M \otimes_{f^*O_S} N), \gamma \in \Gamma(X^o, f^*D_S),\ \gamma.(m \otimes n) := (\gamma.m) \otimes n - m \otimes \gamma.n$$

This gives the bifunctor

$$C_{f^*\mathcal{D}^{(op)}}(X)^2 \to C_{f^*\mathcal{D}^{(op)}}(X), (M, N) \mapsto M \otimes_{f^*O_S} N$$

For $f : X \to S$ a morphism with $X, S \in \mathrm{AnSp}(\mathbb{C})$ and S smooth, we also have the bifunctor $C_{f^*\mathcal{D}^{\infty,(op)}}(X)^2 \to C_{f^*\mathcal{D}^{\infty,(op)}}(X), (M, N) \mapsto M \otimes_{f^*O_S} N$.

- Let $S \in \mathrm{SmVar}(\mathbb{C})$ or let $S \in \mathrm{AnSm}(\mathbb{C})$. For $M \in C_{\mathcal{D}^{op}}(S)$ and $N \in C_{\mathcal{D}}(S)$, we have $M \otimes_{D_S} N \in C(S)$ (see section 2). This gives the bifunctor

$$C_{\mathcal{D}^{op}}(S) \times C_{\mathcal{D}}(S) \to C(S), (M, N) \mapsto M \otimes_{D_S} N$$

For $S \in \mathrm{AnSm}(\mathbb{C})$, we also have the bifunctor $C_{\mathcal{D}^{\infty,op}}(S) \times C_{\mathcal{D}^\infty}(S) \to C(S), (M, N) \mapsto M \otimes_{D_S^\infty} N$.

- Let $S \in \mathrm{SmVar}(\mathbb{C})$ or let $S \in \mathrm{AnSm}(\mathbb{C})$. The internal hom bifunctor

$$\mathcal{H}om(\cdot, \cdot) := \mathcal{H}om_{\mathbb{Z}_S}(\cdot, \cdot) : C(S)^2 \to C(S)$$

induces a bifunctor

$$\mathcal{H}om(\cdot, \cdot) := \mathcal{H}om_{\mathbb{Z}_S}(\cdot, \cdot) : C(S) \times C_{\mathcal{D}}(S) \to C_{\mathcal{D}}(S)$$

such that, for $F \in C(S)$ and $G \in C_{\mathcal{D}}(S)$, the D_S structure on $\mathcal{H}om^\bullet(F, G)$ is given by

$$\gamma \in \Gamma(S^o, D_S) \longmapsto (\phi \in \mathrm{Hom}^p(F^\bullet_{|S^o}, G_{|S^o}) \mapsto (\gamma \cdot \phi : \alpha \in F^\bullet(S^o) \mapsto \gamma \cdot \phi^p(S^o)(\alpha))$$

where $\phi^p(S^o)(\alpha) \in \Gamma(S^o, G)$. For $S \in \mathrm{AnSm}(\mathbb{C})$, it also induce the bifunctor

$$\mathcal{H}om(\cdot, \cdot) := \mathcal{H}om_{\mathbb{Z}_S}(\cdot, \cdot) : C(S) \times C_{\mathcal{D}^\infty}(S) \to C_{\mathcal{D}^\infty}(S)$$

- Let $S \in \mathrm{SmVar}(\mathbb{C})$ or let $S \in \mathrm{AnSm}(\mathbb{C})$. For $M, N \in C_{\mathcal{D}}(S)$, $\mathcal{H}om_{O_S}(M, N)$, has a canonical structure of D_S modules given by for $S^o \subset S$ an open subset and $\phi \in \Gamma(S^o, \mathcal{H}om(M, O_S))$, $\gamma \in \Gamma(S^o, D_S)$, $(\gamma.\phi)(m) := \gamma.(\phi(m)) - \phi(\gamma.m)$ This gives the bifunctor

$$\mathrm{Hom}^\bullet_{O_S}(-, -) : C_{\mathcal{D}}(S)^2 \to C_{\mathcal{D}}(S)^{op}, (M, N) \mapsto \mathcal{H}om^\bullet_{O_S}(M, N)$$

In particular, for $M \in C_{\mathcal{D}}(S)$, we get the dual

$$\mathbb{D}^O_S(M) := \mathcal{H}om^{\bullet}_{O_S}(M, O_S) \in C_{\mathcal{D}}(S)$$

with respect to O_S, together with the canonical map $d(M) : M \to \mathbb{D}^{O,2}_S(M)$. Let $f : X \to S$ a morphism with $X, S \in \mathrm{SmVar}(\mathbb{C})$ or with $X, S \in \mathrm{AnSm}(\mathbb{C})$. We have, for $M \in C_{\mathcal{D}}(S)$, the canonical transformation map

$$T(f, D^o)(M) : f^{*mod}\mathbb{D}^O_S M = (f^*\mathcal{H}om_{O_S}(M, O_S)) \otimes_{f^*O_S} O_X$$
$$\xrightarrow{T^{mod}(f,hom)(M,O_S)} \mathcal{H}om_{O_X}(f^*M \otimes_{f^*O_S} O_X, O_X) =: \mathbb{D}^O_X(f^{*mod}M).$$

- Let $S \in \mathrm{SmVar}(\mathbb{C})$ or let $S \in \mathrm{AnSm}(\mathbb{C})$. We have the bifunctors
  - $\mathrm{Hom}^{\bullet}_{D_S}(-,-) : C_{\mathcal{D}}(S)^2 \to C(S)$, $(M, N) \mapsto \mathcal{H}om^{\bullet}_{D_S}(M, N)$, and if N is a bimodule (i.e. has a right D_S module structure whose opposite coincide with the left one), $\mathcal{H}om_{D_S}(M, N) \in C_{\mathcal{D}^{op}}(S)$ given by for $S^o \subset S$ an open subset and $\phi \in \Gamma(S^o, \mathcal{H}om(M, N))$, $\gamma \in \Gamma(S^o, D_S)$, $(\phi.\gamma)(m) := (\phi(m)).\gamma$
  - $\mathrm{Hom}_{D_S}(-,-) : C_{\mathcal{D}^{op}}(S)^2 \to C(S)$, $(M, N) \mapsto \mathcal{H}om_{D_S}(M, N)$ and if N is a bimodule, $\mathcal{H}om_{D_S}(M, N) \in C_{\mathcal{D}}(S)$

  For $M \in C_{\mathcal{D}}(S)$, we get in particular the dual with respect $\mathbb{D}_S$,

  $$\mathbb{D}_S M := \mathcal{H}om_{D_S}(M, D_S) \in C_{\mathcal{D}}(S) \; ; \; \mathbb{D}^K_S M := \mathcal{H}om_{D_S}(M, D_S) \otimes_{O_S} \mathbb{D}^O_S w(K_S)[d_S] \in C_{\mathcal{D}}(S)$$

  and we have canonical map $d : M \to \mathbb{D}^2_S M$. This functor induces in the derived category, for $M \in D_{\mathcal{D}}(S)$,

  $$L\mathbb{D}_S M := R\mathcal{H}om_{D_S}(L_D M, D_S) \otimes_{O_S} \mathbb{D}^O_S w(K_S)[d_S] = \mathbb{D}^K_S L_D M \in D_{\mathcal{D}}(S).$$

  where $\mathbb{D}^O_S w(S) : \mathbb{D}^O_S w(K_S) \to \mathbb{D}^O_S K_S = K_S^{-1}$ is the dual of the Koczul resolution of the canonical bundle (proposition 68), and the canonical map $d : M \to L\mathbb{D}^2_S M$. For $S \in \mathrm{AnSm}(\mathbb{C})$, we also have the bifunctors
  - $\mathrm{Hom}^{\bullet}_{D^{\infty}_S}(-,-) : C_{\mathcal{D}^{\infty}}(S)^2 \to C(S)$, $(M, N) \mapsto \mathcal{H}om^{\bullet}_{D^{\infty}_S}(M, N)$, and if N is a bimodule, $\mathcal{H}om_{D^{\infty}_S}(M, N) \in C_{\mathcal{D}^{\infty}}(S)$,
  - $\mathrm{Hom}_{D_S}(-,-) : C_{\mathcal{D}^{\infty,op}}(S)^2 \to C(S)$, $(M, N) \mapsto \mathcal{H}om_{D^{\infty}_S}(M, N)$ and if N is a bimodule, $\mathcal{H}om_{D^{\infty}_S}(M, N) \in C_{\mathcal{D}^{\infty,op}}(S)$

  For $M \in C_{\mathcal{D}}(S)$, we get in particular the dual with respect $\mathbb{D}^{\infty}_S$,

  $$\mathbb{D}^{\infty}_S M := \mathcal{H}om_{D^{\infty}_S}(M, D^{\infty}_S) \in C_{\mathcal{D}^{\infty}}(S) \; , \; \mathbb{D}^{\infty,K}_S M := \mathcal{H}om_{D^{\infty}_S}(M, D^{\infty}_S) \otimes_{O_S} \mathbb{D}^O_S w(K_S)[d_S] \in C_{\mathcal{D}^{\infty}}(S)$$

  and we have canonical maps $d : M \to \mathbb{D}^{\infty,2}_S M$, $d : M \to \mathbb{D}^{\infty,K,2}_S M$. This functor induces in the derived category, for $M \in D_{\mathcal{D}^{\infty}}(S)$,

  $$L\mathbb{D}^{\infty}_S M := R\mathcal{H}om_{D^{\infty}_S}(M, D^{\infty}_S) \otimes_{O_S} \mathbb{D}^O_S w(K_S)[d_S] = \mathbb{D}^{\infty,K}_S L_{D^{\infty}} M \in D_{\mathcal{D}^{\infty}}(S).$$

  and the canonical map $d : M \to L\mathbb{D}^{\infty,2}_S M$.

- Let $f : X \to S$ a morphism with $X, S \in \mathrm{SmVar}(\mathbb{C})$ or with $X, S \in \mathrm{AnSm}(\mathbb{C})$. For $N \in C_{\mathcal{D},f^*\mathcal{D}}(X)$ and $M \in C_{\mathcal{D}}(X)$, $N \otimes_{D_X} M$ has the canonical f^*D_S module structure given by, for $X^o \subset X$ an open subset,

  $$\gamma \in \Gamma(X^o, f^*D_S), m \in \Gamma(X^o, M), n \in \Gamma(X^o, N), \; \gamma.(n \otimes m) = (\gamma.n) \otimes m.$$

  This gives the functor

  $$C_{\mathcal{D},f^*\mathcal{D}}(X) \times C_{\mathcal{D}}(X) \to C_{f^*\mathcal{D}}(X), \; (M, N) \mapsto M \otimes_{D_X} N$$

- Let $f : X \to S$ a morphism with $X, S \in \mathrm{AnSm}(\mathbb{C})$. For $N \in C_{\mathcal{D}^\infty, f^*\mathcal{D}^\infty}(X)$ and $M \in C_{\mathcal{D}^\infty}(X)$, $N \otimes_{D_X^\infty} M$ has the canonical $f^*D_S^\infty$ module structure given by, for $X^o \subset X$ an open subset,

$$\gamma \in \Gamma(X^o, f^*D_S), m \in \Gamma(X^o, M), n \in \Gamma(X^o, N),\ \gamma.(n \otimes m) = (\gamma.n) \otimes m.$$

  This gives the functor

$$C_{\mathcal{D}^\infty, f^*\mathcal{D}^\infty}(X) \times C_{\mathcal{D}^\infty}(X) \to C_{f^*\mathcal{D}^\infty}(X),\ (M, N) \mapsto M \otimes_{D_X^\infty} N$$

- Let $f : X \to S$ be a morphism with $X, S \in \mathrm{SmVar}(\mathbb{C})$, or let $f : X \to S$ be a morphism with $X, S \in \mathrm{AnSm}(\mathbb{C})$. Then, for $M \in C_{\mathcal{D}}(S)$, $O_X \otimes_{f^*O_S} f^*M$ has a canonical D_X module structure given by given by, for $X^o \subset X$ an open subset,

$$m \otimes n \in \Gamma(X^o, O_X \otimes_{f^*O_S} f^*M), \gamma \in \Gamma(X^o, D_X),\ \gamma.(m \otimes n) := (\gamma.m) \otimes n - m \otimes df(\gamma).n.$$

  This gives the inverse image functor

$$f^{*mod} : \mathrm{PSh}_{\mathcal{D}}(S) \to \mathrm{PSh}_{\mathcal{D}}(X),\ \ M \mapsto f^{*mod}M := O_X \otimes_{f^*O_S} f^*M = D_{X\to S} \otimes_{f^*D_S} f^*M$$

  which induces in the derived category the functor

$$Lf^{*mod} : D_{\mathcal{D}}(S) \to D_{\mathcal{D}}(X),\ \ M \mapsto Lf^{*mod}M := O_X \otimes^L_{f^*O_S} f^*M = O_X \otimes_{f^*O_S} f^*L_D M,$$

  We will also consider the shifted inverse image functor

$$Lf^{*mod[-]} := Lf^{*mod}[d_S - d_X] : D_{\mathcal{D}}(S) \to D_{\mathcal{D}}(X).$$

- Let $f : X \to S$ be a morphism with $X, S \in \mathrm{AnSm}(\mathbb{C})$. Then, for $M \in C_{\mathcal{D}^\infty}(S)$, $O_X \otimes_{f^*O_S} f^*M$ has a canonical D_X^∞ module structure induced by the finite order case. This gives the inverse image functor

$$f^{*mod} : \mathrm{PSh}_{\mathcal{D}^\infty}(S) \to \mathrm{PSh}_{\mathcal{D}^\infty}(X),\ \ M \mapsto f^{*mod}M := O_X \otimes_{f^*O_S} f^*M = D_{X\to S} \otimes_{f^*D_S^\infty} f^*M$$

  which induces in the derived category the functor

$$Lf^{*mod} : D_{\mathcal{D}^\infty}(S) \to D_{\mathcal{D}^\infty}(X),\ \ M \mapsto Lf^{*mod}M := O_X \otimes^L_{f^*O_S} f^*M = O_X \otimes_{f^*O_S} f^*L_{D^\infty} M,$$

  We will also consider the shifted inverse image functor

$$Lf^{*mod[-]} := Lf^{*mod}[d_S - d_X] : D_{\mathcal{D}^\infty}(S) \to D_{\mathcal{D}^\infty}(X).$$

- Let $f : X \to S$ be a morphism with $X, S \in \mathrm{SmVar}(\mathbb{C})$, or let $f : X \to S$ be a morphism with $X, S \in \mathrm{AnSm}(\mathbb{C})$. For $M \in C_{\mathcal{D}}(X)$, $D_{X\leftarrow S} \otimes_{D_X} M$ has the canonical f^*D_S module structure given above. Then, the direct image functor

$$f^0_{*mod} : \mathrm{PSh}_{\mathcal{D}}(X) \to \mathrm{PSh}_{\mathcal{D}}(S),\ \ M \mapsto f_{*mod}M := f_*(D_{X\leftarrow S} \otimes_{D_X} M)$$

  induces in the derived category the functor

$$\int_f = Rf_{*mod} : D_{\mathcal{D}}(X) \to D_{\mathcal{D}}(S),\ \ M \mapsto \int_f M = Rf_*(D_{X\leftarrow S} \otimes^L_{D_X} M).$$

- Let $f : X \to S$ be a morphism with $X, S \in \mathrm{AnSm}(\mathbb{C})$. For $M \in C_{\mathcal{D}^\infty}(X)$, $D_{X\leftarrow S} \otimes_{D_X} M$ has the canonical f^*D_S module structure given above. Then, the direct image functor

$$f^{00}_{*mod} : \mathrm{PSh}_{\mathcal{D}^\infty}(X) \to \mathrm{PSh}_{\mathcal{D}^\infty}(S),\ \ M \mapsto f_{*mod}M := f_*(D^\infty_{X\leftarrow S} \otimes_{D_X^\infty} M)$$

  induces in the derived category the functor

$$\int_f = Rf_{*mod} : D_{\mathcal{D}^\infty}(X) \to D_{\mathcal{D}^\infty}(S),\ \ M \mapsto \int_f M = Rf_*(D^\infty_{X\leftarrow S} \otimes^L_{D_X^\infty} M).$$

- Let $f : X \to S$ be a morphism with $X, S \in \mathrm{AnSm}(\mathbb{C})$. The direct image functor with compact support
$$f^{00}_{!mod} : \mathrm{PSh}_{\mathcal{D}}(X) \to \mathrm{PSh}_{\mathcal{D}}(S), \; M \mapsto f_{!mod}M := f_!(D_{S\leftarrow X} \otimes_{D_X} M)$$
induces in the derived category the functor
$$\int_{f!} = Rf_{!mod} : D_{\mathcal{D}}(X) \to D_{\mathcal{D}}(S), \; M \mapsto \int_f M = Rf_!(D_{X\leftarrow S} \otimes^L_{D_X} M).$$

- Let $f : X \to S$ be a morphism with $X, S \in \mathrm{AnSm}(\mathbb{C})$. The direct image functor with compact support
$$f^{00}_{!mod} : \mathrm{PSh}_{\mathcal{D}^\infty}(X) \to \mathrm{PSh}_{\mathcal{D}^\infty}(S), \; M \mapsto f_{!mod}M := f_!(D^\infty_{S\leftarrow X} \otimes_{D^\infty_X} M)$$
induces in the derived category the functor
$$\int_{f!} = Rf_{!mod} : D_{\mathcal{D}^\infty}(X) \to D_{\mathcal{D}^\infty}(S), \; M \mapsto \int_f M = Rf_!(D^\infty_{X\leftarrow S} \otimes^L_{D^\infty_X} M).$$

- Let $S \in \mathrm{SmVar}(\mathbb{C})$. The analytical functor of a D_S modules has a canonical structure of $D_{S^{an}}$ module:
$$(-)^{an} : C_{\mathcal{D}}(S) \to C_{\mathcal{D}}(S^{an}), \; M \mapsto M^{an} := \mathrm{an}^{*mod}_S M := M \otimes_{\mathrm{an}^*_S O_S} O_{S^{an}}$$
which induces in the derived category
$$(-)^{an} : D_{\mathcal{D}}(S) \to D_{\mathcal{D}}(S^{an}), \; M \mapsto M^{an} := \mathrm{an}^{*mod}_S M)$$
since an^{*mod}_S derive trivially.

The functorialities given above induce :

- Let $f : X \to S$ be a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$, or let $f : X \to S$ be a morphism with $X, S \in \mathrm{AnSp}(\mathbb{C})$. The adjonction map induces
  - for $(M, F) \in C_{O_S fil}(S)$, the map in $D_{O_S fil}(S)$
$$\mathrm{ad}(Lf^{*mod}, Rf_*)(M, F) : (M, F) \xrightarrow{k\circ \mathrm{ad}(f^*, f_*)(M,F)} f_*E(f^*(M, F)) = Rf_*f^*(M, F)$$
$$\xrightarrow{f_*m} Rf_*(f^*(M, F) \otimes^L_{f^*O_S} O_X) = Rf_*f^*(M, F),$$
where $m(m) = m \otimes 1$,
  - for $(M, F) \in C_{O_X fil}(X)$, the map in $D_{O_X fil}(X)$
$$\mathrm{ad}(Lf^{*mod}, Rf_*)(M, F) : Lf^{*mod}Rf_*(M, F) = f^*f_*E(M, F) \otimes^L_{f^*O_S} O_X$$
$$\xrightarrow{\mathrm{ad}(f^*, f_*)(E(M,F))\otimes^L_{f^*O_S} O_X} (M, F) \otimes^L_{f^*O_S} O_X \xrightarrow{n} (M, F),$$
where $n(m \otimes h) = h.m$ is the multiplication map.

- For a commutative diagram in $\mathrm{Var}(\mathbb{C})$ or in $\mathrm{AnSp}(\mathbb{C})$:
$$D = \begin{array}{ccc} Y & \xrightarrow{g_2} & X \\ \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} \\ T & \xrightarrow{g_1} & S \end{array} ,$$

we have, for $(M,F)\in C_{O_X fil}(X)$, the canonical map in $D_{O_T fil}(T)$

$$T^{mod}(D)(M,F) : Lg_1^{*mod}f_{1*}(M,F) \xrightarrow{\mathrm{ad}(Lf_2^{*mod},Rf_{2*})(Lg_1^{*mod}f_{1*}E(M,F))} Rf_{2*}Lf_2^{*mod}Lg_1^{*mod}Rf_{1*}(M,F) = Rf_{2*}Lg_2^{*mod}Lf_1^{*mod}Rf_{1*}(M,F) \xrightarrow{\mathrm{ad}(Lf_1^{*mod},Rf_1)(M,F)} Rf_{2*}Lg_2^{*mod}(M,F)$$

the canonical transformation map given by the adjonction maps.

- Let $S\in\mathrm{SmVar}(\mathbb{C})$ or $S\in\mathrm{AnSm}(\mathbb{C})$. For $(M,F)\in C_{fil}(S)$ and $(N,F)\in C_{fil}(S)$, recall that (see section 2)

  $$F^p((M,F)\otimes(N,F)) := \mathrm{Im}(\oplus_q F^qM\otimes F^{p-q}N\to M\otimes N)$$

  This gives the functor

  $$(\cdot,\cdot) : C_{fil}(S)\times C_{\mathcal{D}fil}(S)\to C_{\mathcal{D}fil}(S)\ ,\ ((M,F),(N,F))\mapsto (M,F)\otimes(N,F).$$

  It induces in the derived categories by taking r-projective resolutions the bifunctors, for $r = 1,\ldots,\infty$,

  $$(\cdot,\cdot) : D_{\mathcal{D}fil,r}(S)\times D_{fil,r}(S)\to D_{\mathcal{D}fil,r}(S)\ ,\ ((M,F),(N,F))\mapsto (M,F)\otimes^L(N,F) = L_D(M,F)\otimes(N,F).$$

  For $S\in\mathrm{AnSm}(\mathbb{C})$, it gives the bifunctor

  $$(\cdot,\cdot) : C_{fil}(S)\times C_{\mathcal{D}^\infty fil}(S)\to C_{\mathcal{D}^\infty fil}(S)\ ,\ ((M,F),(N,F))\mapsto (M,F)\otimes(N,F),$$

  and its derived functor.

- Let $S\in\mathrm{SmVar}(\mathbb{C})$ or $S\in\mathrm{AnSm}(\mathbb{C})$ and $O'_S\in\mathrm{PSh}(S,\mathrm{cRing})$ a sheaf of commutative ring. For $(M,F)\in C_{O'_S fil}(S)$ and $(N,F)\in C_{O'_S fil}(S)$, recall that (see section 2)

  $$F^p((M,F)\otimes_{O'_S}(N,F)) := \mathrm{Im}(\oplus_q F^qM\otimes_{O'_S}F^{p-q}N\to M\otimes_{O'_S}N)$$

  It induces in the derived categories by taking r-projective resolutions the bifunctors, for $r = 1,\ldots,\infty$,

  $$(\cdot,\cdot) : D_{\mathcal{D}fil,r}(S)^2\to D_{\mathcal{D}fil,r}(S)\ ,\ ((M,F),(N,F))\mapsto (M,F)\otimes^L_{O_S}(N,F).$$

  More generally, let $f : X\to S$ a morphism with $X,S\in\mathrm{Var}(\mathbb{C})$ or with $X,S\in\mathrm{AnSp}(\mathbb{C})$. Assume S smooth. We have the bifunctors

  $$(\cdot,\cdot) : D_{f^*\mathcal{D}fil,r}(X)^2\to D_{f^*\mathcal{D}fil,r}(X)\ ,\ ((M,F),(N,F))\mapsto (M,F)\otimes^L_{f^*O_S}(N,F) = (M,F)\otimes_{f^*O_S}L_{f^*D}(N,F).$$

- Let $S\in\mathrm{SmVar}(\mathbb{C})$ or let $S\in\mathrm{AnSm}(\mathbb{C})$. The hom functor induces the bifunctor

  $$\mathrm{Hom}(-,-) : C_{\mathcal{D}fil}(S)\times C_{fil}(S)\to C_{\mathcal{D}(1,0)fil}(S), ((M,W),(N,F))\mapsto \mathcal{H}om((M,W),(N,F)).$$

  For $S\in\mathrm{AnSm}(\mathbb{C})$, the hom functor also induces the bifunctor

  $$\mathrm{Hom}(-,-) : C_{\mathcal{D}^\infty fil}(S)\times C_{fil}(S)\to C_{\mathcal{D}^\infty(1,0)fil}(S), ((M,W),(N,F))\mapsto \mathcal{H}om((M,W),(N,F)).$$

  Note that the filtration given by W satisfy that the W^p are D_S submodule which is stronger than Griffitz transversality.

- Let $S\in\mathrm{SmVar}(\mathbb{C})$ or let $S\in\mathrm{AnSm}(\mathbb{C})$. The hom functor induces the bifunctor

  $$\mathrm{Hom}_{O_S}(-,-) : C_{\mathcal{D}fil}(S)^2\to C_{\mathcal{D}2fil}(S), ((M,W),(N,F))\mapsto \mathcal{H}om_{O_S}((M,W),(N,F)).$$

- Let $S\in\mathrm{SmVar}(\mathbb{C})$ or let $S\in\mathrm{AnSm}(\mathbb{C})$. The hom functor induces the bifunctors
  - $\mathrm{Hom}_{D_S}(-,-):C_{\mathcal{D}fil}(S)^2\to C_{2fil}(S)$, $((M,W),(N,F))\mapsto\mathcal{H}om_{D_S}((M,W),(N,F))$,
  - $\mathrm{Hom}_{D_S}(-,-):C_{\mathcal{D}^{op}fil}(S)^2\to C_{2fil}(S)$, $((M,W),(N,F))\mapsto\mathcal{H}om_{D_S}((M,W),(N,F))$.

  We get the filtered dual

  $\mathbb{D}^K_S(\cdot):C_{\mathcal{D}(2)fil}(S)\to C_{\mathcal{D}(2)fil}(S)^{op},\ (M,F)\mapsto\mathbb{D}^K_S(M,F):=\mathcal{H}om_{D_S}((M,F),D_S)\otimes_{O_S}\mathbb{D}^O_S w(K_S)[d_S]$

  together with the canonical map $d(M,F):(M,F)\to\mathbb{D}^{2,K}_S(M,F)$. Of course $\mathbb{D}^K_S(\cdot)(C_{\mathcal{D}(1,0)fil}(S))\subset C_{\mathcal{D}(1,0)fil}(S)$. It induces in the derived categories $D_{\mathcal{D}fil,r}(S)$, for $r=1,\ldots,\infty$, the functors

  $$L\mathbb{D}_S(\cdot):D_{\mathcal{D}(2)fil,r}(S)\to D_{\mathcal{D}(2)fil,r}(S)^{op},(M,F)\mapsto L\mathbb{D}_S(M,F):=\mathbb{D}^K_S L_D(M,F).$$

  together with the canonical map $d(M,F):L_D(M,F)\to\mathbb{D}^{2,K}_S L_D(M,F)$.

- Let $S\in\mathrm{AnSm}(\mathbb{C})$. The hom functor also induces the bifunctors
  - $\mathrm{Hom}_{D^\infty_S}(-,-):C_{\mathcal{D}^\infty fil}(S)^2\to C_{2fil}(S)$, $((M,W),(N,F))\mapsto\mathcal{H}om_{D^\infty_S}((M,W),(N,F))$,
  - $\mathrm{Hom}_{D_S}(-,-):C_{\mathcal{D}^{\infty,op}fil}(S)^2\to C_{2fil}(S)$, $((M,W),(N,F))\mapsto\mathcal{H}om_{D^\infty_S}((M,W),(N,F))$.

  We get the filtered dual

  $\mathbb{D}^{\infty,K}_S(\cdot):C_{\mathcal{D}^\infty(2)fil}(S)\to C_{\mathcal{D}^\infty(2)fil}(S)^{op},\ (M,F)\mapsto\mathbb{D}^{\infty,K}_S(M,F):=\mathcal{H}om_{D^\infty_S}((M,F),D^\infty_S)\otimes_{O_S}\mathbb{D}^O_S w(K_S)[d_S]$

  together with the canonical map $d(M,F):(M,F)\to\mathbb{D}^{\infty,2}_S(M,F)$. Of course $\mathbb{D}^{\infty,K}_S(\cdot)(C_{\mathcal{D}^\infty(1,0)fil}(S))\subset C_{\mathcal{D}^\infty(1,0)fil}(S)$. It induces in the derived categories $D_{\mathcal{D}fil,r}(S)$, for $r=1,\ldots,\infty$, the functors

  $$L\mathbb{D}^\infty_S(\cdot):D_{\mathcal{D}^\infty(2)fil,r}(S)\to D_{\mathcal{D}^\infty(2)fil,r}(S)^{op},(M,F)\mapsto L\mathbb{D}^\infty_S(M,F)=\mathbb{D}^{\infty,K}_S L_{D^\infty}(M,F).$$

  together with the canonical map $d(M,F):(M,F)\to L\mathbb{D}^{\infty,2}_S(M,F)$.

- Let $f:X\to S$ be a morphism with $X,S\in\mathrm{SmVar}(\mathbb{C})$, or let $f:X\to S$ be a morphism with $X,S\in\mathrm{AnSm}(\mathbb{C})$. Then, the inverse image functor

  $$f^{*mod}:C_{\mathcal{D}(2)fil}(S)\to C_{\mathcal{D}(2)fil}(X),$$
  $$(M,F)\mapsto f^{*mod}(M,F):=(O_X,F_b)\otimes_{f^*O_S}f^*(M,F)=(D_{X\to S},F^{ord})\otimes_{f^*D_S}f^*(M,F),$$

  induces in the derived categories the functors, for $r=1,\ldots,\infty$ (resp. $r\in(1,\ldots\infty)^2$),

  $$Lf^{*mod}:D_{\mathcal{D}(2)fil,r}(S)\to D_{\mathcal{D}(2)fil,r}(X),$$
  $$(M,F)\mapsto Lf^{*mod}M:=(O_X,F_b)\otimes^L_{f^*O_S}f^*(M,F)=(O_X,F_b)\otimes_{f^*O_S}f^*L_D(M,F).$$

  Of course $f^{*mod}(C_{\mathcal{D}(1,0)fil}(S))\subset C_{\mathcal{D}(1,0)fil}(X)$. Note that
  - If the M is a complex of locally free O_S modules, then $Lf^{*mod}(M,F)=f^{*mod}(M,F)$ in $D_{\mathcal{D}(2)fil,\infty}(S)$.
  - If the $\mathrm{Gr}^p_F M$ are complexes of locally free O_S modules, then $Lf^{*mod}(M,F)=f^{*mod}(M,F)$ in $D_{\mathcal{D}(2)fil}(S)$.

  We will consider also the shifted inverse image functors

  $$Lf^{*mod[-]}:=Lf^{*mod}[d_S-d_X]:D_{\mathcal{D}(2)fil,r}(S)\to D_{\mathcal{D}(2)fil,r}(X).$$

- Let $f : X \to S$ be a morphism with $X, S \in \mathrm{AnSm}(\mathbb{C})$. Then, the inverse image functor

$$f^{*mod} : C_{\mathcal{D}^\infty(2)fil}(S) \to C_{\mathcal{D}^\infty(2)fil}(X), (M,F) \mapsto f^{*mod}(M,F) := (O_X, F_b) \otimes_{f^*O_S} f^*(M,F),$$

induces in the derived categories the functors, for $r = 1, \ldots, \infty$ (resp. $r \in (1, \ldots \infty)^2$),

$$Lf^{*mod} : D_{\mathcal{D}^\infty(2)fil,r}(S) \to D_{\mathcal{D}^\infty(2)fil,r}(X),$$
$$(M,F) \mapsto Lf^{*mod}M := (O_X, F_b) \otimes^L_{f^*O_S} f^*(M,F) = (O_X, F_b) \otimes_{f^*O_S} f^* L_{D^\infty}(M,F).$$

Of course $f^{*mod}(C_{\mathcal{D}^\infty(1,0)fil}(S)) \subset C_{\mathcal{D}^\infty(1,0)fil}(X)$. Note that We will consider also the shifted inverse image functors

$$Lf^{*mod[-]} := Lf^{*mod}[d_S - d_X] : D_{\mathcal{D}^\infty(2)fil,r}(S) \to D_{\mathcal{D}^\infty(2)fil,r}(X).$$

- Let $f : X \to S$ be a morphism with $X, S \in \mathrm{SmVar}(\mathbb{C})$, or let $f : X \to S$ be a morphism with $X, S \in \mathrm{AnSm}(\mathbb{C})$. Then,the direct image functor

$$f^{00}_{*mod} : (\mathrm{PSh}_{\mathcal{D}}(X), F) \to (\mathrm{PSh}_{\mathcal{D}}(S), F), \ \ (M,F) \mapsto f_{*mod}(M,F) := f_*((D_{S\leftarrow X}, F^{ord}) \otimes_{D_X} (M,F))$$

induces in the derived categories by taking r-injective resolutions the functors, for $r = 1, \ldots, \infty$,

$$\int_f = Rf_{*mod} : D_{\mathcal{D}(2)fil,r}(X) \to D_{\mathcal{D}(2)fil,r}(S), (M,F) \mapsto \int_f (M,F) = Rf_*((D_{S\leftarrow X}, F^{ord}) \otimes^L_{D_X} (M,F)).$$

Let $f_1 : X \to Y$ and $f_2 : Y \to S$ two morphism with $X, Y, S \in \mathrm{SmVar}(\mathbb{C})$ or with $X, Y, S \in \mathrm{AnSm}(\mathbb{C})$. We have, for $(M,F) \in C_{\mathcal{D}fil}(X)$, the canonical transformation map in $D_{\mathcal{D}(2)fil,r}(S)$

$$\begin{aligned} T(\int_{f_2} \circ \int_{f_1}, \int_{f_2\circ f_1})(M,F) : \\ \int_{f_2}\int_{f_1}(M,F) := Rf_{2*}((D_{Y\leftarrow S}, F^{ord}) \otimes^L_{D_Y} Rf_{1*}((D_{X\leftarrow Y}, F^{ord}) \otimes^L_{D_X} (M,F))) \\ \xrightarrow{T(f_1,\otimes)(-,-)} Rf_{2*}Rf_{1*}(f_1^*(D_{Y\leftarrow S}, F^{ord}) \otimes^L_{D_Y} ((D_{X\leftarrow Y}, F^{ord}) \otimes^L_{D_X} (M,F))) \\ \xrightarrow{\sim} Rf_{2*}Rf_{1*}((f_1^*(D_{Y\leftarrow S}, F^{ord}) \otimes^L_{D_Y} (D_{X\leftarrow Y}, F^{ord})) \otimes^L_{D_X} (M,F)) \\ \xrightarrow{\sim} Rf_{2*}Rf_{1*}((D_{X\leftarrow S}, F^{ord}) \otimes^L_{D_X} (M,F)) := \int_{f_2\circ f_1}(M,F) \end{aligned}$$

- Let $f : X \to S$ be a morphism with $X, S \in \mathrm{AnSm}(\mathbb{C})$. Then,the direct image functor

$$f^{00}_{*mod} : (\mathrm{PSh}_{\mathcal{D}^\infty}(X), F) \to (\mathrm{PSh}_{\mathcal{D}^\infty}(S), F), \ \ (M,F) \mapsto f_{*mod}(M,F) := f_*((D^\infty_{S\leftarrow X}, F^{ord}) \otimes_{D^\infty_X} (M,F))$$

induces in the derived categories by taking r-injective resolutions the functors, for $r = 1, \ldots, \infty$,

$$\int_f = Rf_{*mod} : D_{\mathcal{D}^\infty(2)fil,r}(X) \to D_{\mathcal{D}^\infty(2)fil,r}(S), (M,F) \mapsto \int_f (M,F) = Rf_*((D^\infty_{S\leftarrow X}, F^{ord}) \otimes^L_{D^\infty_X} (M,F)).$$

We have, similarly, for $(M,F) \in C_{\mathcal{D}^\infty fil}(X)$, the canonical transformation map in $D_{\mathcal{D}^\infty(2)fil,r}(S)$

$$T(\int_{f_2} \circ \int_{f_1}, \int_{f_2\circ f_1})(M,F) : \int_{f_2}\int_{f_1}(M,F) \to \int_{f_2\circ f_1}(M,F)$$

- Let $f : X \to S$ be a morphism with $X, S \in \mathrm{AnSm}(\mathbb{C})$. Then,the direct image functor with compact support

$$f^{00}_{!mod} : (\mathrm{PSh}_{\mathcal{D}}(X), F) \to (\mathrm{PSh}_{\mathcal{D}}(S), F), \ \ (M,F) \mapsto f^{00}_{!mod}(M,F) := f_!((D_{S\leftarrow X}, F^{ord}) \otimes_{D_X} (M,F))$$

induces in the derived categories by taking r-injective resolutions the functors, for $r = 1, \ldots, \infty$,

$$\int_{f!} = Rf_{!mod} : D_{\mathcal{D}fil,r}(X) \to D_{\mathcal{D}fil,r}(S), \ \ (M, F) \mapsto \int_f (M, F) = Rf_!((D_{S\leftarrow X}, F^{ord}) \otimes^L_{D_X} (M, F)).$$

We have, similarly, for $(M, F) \in C_{\mathcal{D}fil}(X)$, the canonical transformation map in $D_{\mathcal{D}(2)fil,r}(S)$

$$T(\int_{f_2!} \circ \int_{f_1!}, \int_{(f_2 \circ f_1})!)(M, F) : \int_{f_2!} \int_{f_1!} (M, F) \to \int_{(f_2\circ f_1)!} (M, F)$$

- Let $f : X \to S$ be a morphism with $X, S \in \mathrm{AnSm}(\mathbb{C})$. Then,the direct image functor with compact support

  $$f^{00}_{!mod} : (\mathrm{PSh}_{\mathcal{D}^\infty}(X), F) \to (\mathrm{PSh}_{\mathcal{D}^\infty}(S), F), \ \ (M, F) \mapsto f^{00}_{!mod}(M, F) := f_!((D^\infty_{S\leftarrow X}, F^{ord}) \otimes_{D^\infty_X} (M, F))$$

  induces in the derived categories by taking r-injective resolutions the functors, for $r = 1, \ldots, \infty$,

  $$\int_{f!} = Rf_{!mod} : D_{\mathcal{D}^\infty fil,r}(X) \to D_{\mathcal{D}^\infty fil,r}(S), (M, F) \mapsto \int_f (M, F) = Rf_!((D^\infty_{S\leftarrow X}, F^{ord}) \otimes^L_{D^\infty_X} (M, F)).$$

  We have, similarly, for $(M, F) \in C_{\mathcal{D}^\infty fil}(X)$, the canonical transformation map in $D_{\mathcal{D}^\infty(2)fil,r}(S)$

  $$T(\int_{f_2!} \circ \int_{f_1!}, \int_{(f_2 \circ f_1})!)(M, F) : \int_{f_2!} \int_{f_1!} (M, F) \to \int_{(f_2\circ f_1)!} (M, F)$$

- Let $S \in \mathrm{SmVar}(\mathbb{C})$. The analytical functor for filtered D_S-modules is

  $$(\cdot)^{an} : C_{\mathcal{D}(2)fil}(S) \to C_{\mathcal{D}(2)fil}(S^{an}), \ (M, F) \mapsto (M, F)^{an} := \mathrm{an}^*_S(M, F) \otimes_{\mathrm{an}^*_S O_S} (O_{S^{an}}, F_b).$$

  It induces in the derived categories the functors, for $r = 1, \ldots, \infty$,

  $$(\cdot)^{an} : D_{\mathcal{D}(2)fil,r}(S) \to D_{\mathcal{D}(2)fil,r}(S^{an}), \ (M, F) \mapsto (M, F)^{an} := \mathrm{an}^*_S(M, F) \otimes^L_{\mathrm{an}^*_S O_S} (O_{S^{an}}, F_b).$$

- Let $f : X \to S$ be a morphism with $X, S \in \mathrm{SmVar}(\mathbb{C})$, or let $f : X \to S$ be a morphism with $X, S \in \mathrm{AnSm}(\mathbb{C})$. Then the functor

  $$f^{\hat{*}mod} : C_{\mathcal{D}2fil}(S) \to C_{\mathcal{D}2fil}(X), (M, F) \mapsto f^{\hat{*}mod}(M, F) := \mathbb{D}^K_X L_D f^{*mod} L_D \mathbb{D}^K_S(M, F)$$

  induces in the derived categories the exceptional inverse image functors, for $r = 1, \ldots, \infty$ (resp. $r \in (1, \ldots \infty)^2$),

  $$Lf^{\hat{*}mod} : D_{\mathcal{D}(2)fil,r}(S) \to D_{\mathcal{D}(2)fil,r}(X),$$
  $$(M, F) \mapsto Lf^{\hat{*}mod}(M, F) := L\mathbb{D}_X Lf^{*mod} L\mathbb{D}_S(M, F) := f^{\hat{*}mod} L_D(M, F).$$

  Of course $f^{\hat{*}mod}(C_{\mathcal{D}(1,0)fil}(S)) \subset C_{\mathcal{D}(1,0)fil}(X)$. We will also consider the shifted exceptional inverse image functors

  $$Lf^{\hat{*}mod[-]} := Lf^{\hat{*}mod}[d_S - d_X] : D_{\mathcal{D}(2)fil,r}(S) \to D_{\mathcal{D}(2)fil,r}(X).$$

- Let $f : X \to S$ be a morphism with $X, S \in \mathrm{AnSm}(\mathbb{C})$. Then the functor

  $$f^{\hat{*}mod} : C_{\mathcal{D}^\infty(2)fil}(S) \to C_{\mathcal{D}^\infty(2)fil}(X), (M, F) \mapsto f^{\hat{*}mod}(M, F) := \mathbb{D}^{K,\infty}_X L_D f^{*mod} L_D \mathbb{D}^{K,\infty}_S(M, F)$$

induces in the derived categories the exceptional inverse image functors, for $r = 1, \ldots, \infty$ (resp. $r \in (1, \ldots \infty)^2$),

$$Lf^{\hat{*}mod} : D_{\mathcal{D}^\infty(2)fil,r}(S) \to D_{\mathcal{D}^\infty(2)fil,r}(X),$$
$$(M,F) \mapsto Lf^{\hat{*}mod}(M,F) := L\mathbb{D}^\infty_X Lf^{*mod} L\mathbb{D}^\infty_S(M,F) := f^{\hat{*}mod}(M,F).$$

Of course $f^{\hat{*}mod}(C_{\mathcal{D}^\infty(1,0)fil}(S)) \subset C_{\mathcal{D}^\infty(1,0)fil}(X)$. We will also consider the shifted exceptional inverse image functors

$$Lf^{\hat{*}mod[-]} := Lf^{\hat{*}mod}[d_S - d_X] : D_{\mathcal{D}^\infty(2)fil,r}(S) \to D_{\mathcal{D}^\infty(2)fil,r}(X).$$

- Let $S_1, S_2 \in \mathrm{SmVar}(\mathbb{C})$ or $S_1, S_2 \in \mathrm{AnSm}(\mathbb{C})$. Consider $p : S_1 \times S_2 \to S_1$ the projection. Since p is a projection, we have a canonical embedding $p^*D_{S_1} \hookrightarrow D_{S_1 \times S_2}$. For $(M,F) \in C_{\mathcal{D}(2)fil}(S_1 \times S_2)$, (M,F) has a canonical $p^*D_{S_1}$ module structure. Moreover, with this structure, for $(M_1,F) \in C_{\mathcal{D}(2)fil}(S_1)$
$$\mathrm{ad}(p^{*mod}, p)(M_1,F) : (M_1,F) \to p_*p^{*mod}(M_1,F)$$
is a map of complexes of D_{S_1} modules, and for $(M_{12},F) \in C_{\mathcal{D}(2)fil}(S_1 \times S_2))$
$$\mathrm{ad}(p^{*mod}, p)(M_{12},F) : p^{*mod}p_*(M_{12},F) \to (M_{12},F)$$
is a map of complexes of $D_{S_1 \times S_2}$ modules. Indeed, for the first adjonction map, we note that $p^{*mod}(M_1,F)$ has a structure of $p^*D_{S_1}$ module, hence $p_*p^{*mod}(M_1,F)$ has a structure of $p_*p^*D_{S_1}$ module, hence a structure of D_{S_1} module using the adjonction map $\mathrm{ad}(p^*, p_*)(D_{S_1}) : D_{S_1} \to p_*p^*D_{S_1}$. For the second adjonction map, we note that (M_{12},F) has a structure of $p^*D_{S_1}$ module, hence $p_*(M_{12},F)$ has a structure of $p_*p^*D_{S_1}$, hence a structure of D_{S_1} module using the adjonction map $\mathrm{ad}(p^*, p_*)(D_{S_1}) : D_{S_1} \to p_*p^*D_{S_1}$.

- Let $S_1, S_2 \in \mathrm{AnSm}(\mathbb{C})$. Consider $p : S_1 \times S_2 \to S_1$ the projection. Since p is a projection, we have a canonical embedding $p^*D^\infty_{S_1} \hookrightarrow D^\infty_{S_1 \times S_2}$. For $(M,F) \in C_{\mathcal{D}^\infty(2)fil}(S_1 \times S_2)$, (M,F) has a canonical $p^*D^\infty_{S_1}$ module structure. Moreover, with this structure, for $(M_1,F) \in C_{\mathcal{D}^\infty(2)fil}(S_1)$
$$\mathrm{ad}(p^{*mod}, p)(M_1,F) : (M_1,F) \to p_*p^{*mod}(M_1,F)$$
is a map of complexes of $D^\infty_{S_1}$ modules, and for $(M_{12},F) \in C_{\mathcal{D}^\infty(2)fil}(S_1 \times S_2)$
$$\mathrm{ad}(p^{*mod}, p)(M_{12},F) : p^{*mod}p_*(M_{12},F) \to (M_{12},F)$$
is a map of complexes of $D^\infty_{S_1 \times S_2}$ modules, similarly to the finite order case.

We following proposition concern the commutativity of the inverse images functors and the commutativity of the direct image functors.

Proposition 49. *(i) Let $f_1 : X \to Y$ and $f_2 : Y \to S$ two morphism with $X, Y, S \in \mathrm{SmVar}(\mathbb{C})$.*

- *Let $(M,F) \in C_{\mathcal{D}(2)fil,r}(S)$. Then $(f_2 \circ f_1)^{*mod}(M,F) = f_1^{*mod} f_2^{*mod}(M,F)$.*
- *Let $(M,F) \in D_{\mathcal{D}(2)fil,r}(S)$. Then $L(f_2 \circ f_1)^{*mod}(M,F) = Lf_1^{*mod}(Lf_2^{*mod}(M,F))$.*

(ii) Let $f_1 : X \to Y$ and $f_2 : Y \to S$ two morphism with $X, Y, S \in \mathrm{SmVar}(\mathbb{C})$. Let $M \in D_{\mathcal{D}}(X)$. Then,
$$T(\int_{f_2} \circ \int_{f_1}, \int_{f_2 \circ f_1})(M) : \int_{f_2} \int_{f_1}(M) \xrightarrow{\sim} \int_{f_2 \circ f_1}(M)$$
is an isomorphism in $D_{\mathcal{D}}(S)$ (i.e. if we forget filtration).

(iii) Let $i_0 : Z_2 \hookrightarrow Z_1$ and $i_1 : Z_1 \hookrightarrow S$ two closed embedding, with $Z_2, Z_1, S \in \mathrm{SmVar}(\mathbb{C})$. Let $(M,F) \in C_{\mathcal{D}(2)fil}(Z_2)$. Then, $(i_1 \circ i_0)_{*mod}(M,F) = i_{1*mod}(i_{0*mod}(M,F))$ in $C_{\mathcal{D}(2)fil}(S)$.

Proof. (i): Obvious : we have

- $f_1^{*mod} f_2^{*mod}(M,F) = f_1^*(f_2^*(M,F) \otimes_{f_2^*O_S} O_Y) \otimes_{f_1^*O_Y} O_X = f_1^* f_2^*(M,F) \otimes_{f_1^* f_2^* O_S} f_1^* O_Y \otimes_{f_1^* O_Y} O_X = (f_2 \circ f_1)^{*mod}(M,F)$
- $Lf_1^{*mod} Lf_2^{*mod}(M,F) = f_1^*(f_2^*(M,F) \otimes^L_{f_2^*O_S} O_Y) \otimes^L_{f_1^*O_Y} O_X = f_1^* f_2^*(M,F) \otimes^L_{f_1^* f_2^* O_S} f_1^* O_Y \otimes^L_{f_1^* O_Y} O_X = L(f_2 \circ f_1)^{*mod}(M,F)$

.
(ii): See [16] : we have by lemma 4

$$\int_{f_2 \circ f_1} M := Rf_{2*} Rf_{1*}(D_{X \leftarrow S} \otimes^L_{D_X} M)$$
$$\xrightarrow{=} Rf_{2*} Rf_{1*}((f_1^* D_{Y \leftarrow S} \otimes^L_{f_1^* D_Y} D_{X \leftarrow Y}) \otimes^L_{D_X} M) \xrightarrow{Rf_{2*} T(f_1, \otimes)(D_{Y \leftarrow S}, D_{X \leftarrow Y} \otimes^L_{D_X} M)^{-1}}$$
$$Rf_{2*}(D_{Y \leftarrow S} \otimes^L_{D_Y} Rf_{1*}(D_{X \leftarrow Y} \otimes^L_{D_X} M)) =: \int_{f_2} \int f_1 M$$

where, $D_{Y \leftarrow S}$ being a quasi-coherent D_Y module, we used the fact that for $N \in C_{f_1^* \mathcal{D}}(X)$ and $N' \in C_{\mathcal{D}}(Y)$

$$T(f_1, \otimes)(N', N) : N' \otimes^L_{D_Y} Rf_{1*} N \to Rf_{1*}(f_1^* N' \otimes^L_{f_1^* D_Y} N)$$

is an isomorphism if N' is quasi-coherent, which follows from the fact that f_{1*} commutes with arbitrary (possibly infinite) direct sums (see [16]).
(iii): Denote $i_2 = i_1 \circ i_0 : Z_2 \hookrightarrow S$. We have

$$i_{2*mod}(M,F) = i_{2*}((M,F) \otimes_{D_{Z_2}} (D_{Z_2 \leftarrow S}, F^{ord})) \xrightarrow{=}$$
$$i_{1*} i_{0*}((M,F) \otimes_{D_{Z_2}} (D_{Z_2 \leftarrow Z_1}, F^{ord}) \otimes_{i_0^* D_{Z_1}} i_0^*(D_{Z_1 \leftarrow S}, F^{ord})) \xrightarrow{i_{1*} T(i_0, \otimes)(-)^{-1}} i_{1*mod} i_{0*mod}((M,F))$$

using proposition 4 and proposition 10. □

Remark 8. *Let $f_1 : X \to Y$ and $f_2 : Y \to S$ two morphism with $X, Y, S \in \mathrm{SmVar}(\mathbb{C})$. Then, for $(M,F) \in D_{\mathcal{D}(2)fil,\infty}(X)$, $\int_{f_2} \int_{f_1} (M,F)$ is NOT isomorphic to $\int_{f_2 \circ f_1} (M,F)$ in general, the filtrations on the isomorphic cohomology sheaves may be different.*

Proposition 50. *Let $f : X \to S$ a morphism with $X, S \in \mathrm{SmVar}(\mathbb{C})$. Then,*

(i) For $(M,F) \in C_{\mathcal{D}(2)fil,h}(S)$, we have $Lf^{*mod}(M,F) \in D_{\mathcal{D}(2)fil,\infty,h}(X)$.

(ii) For $M \in C_{\mathcal{D},h}(X)$, we have $\int_f M \in D_{\mathcal{D},h}(S)$.

(iii) If f is proper, for $(M,F) \in C_{\mathcal{D}(2)fil,h}(X)$, we have $\int_f (M,F) \in D_{\mathcal{D}(2)fil,\infty,h}(S)$.

Proof. See [16] for the non filtered case. The filtered case follows immediately from the non filtered case and the fact the pullback of a good filtration is a good filtration (since the pullback of a coherent O_S module is coherent) and the direct image of a good filtration by a proper morphism is a good filtration (since the pushforward of a coherent O_X module by a proper morphism is coherent). □

The following easy proposition says that the analytical functor commutes we the pullback of D modules and the tensor product. Again it is well known in the non filtered case. Note that for $S \in \mathrm{SmVar}(\mathbb{C})$, $D_S^{an} = D_{S^{an}}$.

Proposition 51. *(i) Let $f : T \to S$ a morphism with $T, S \in \mathrm{SmVar}(\mathbb{C})$.*

– *Let* $(M,F) \in C_{\mathcal{D}(2)fil,r}(S)$. *Then* $(f^{*mod}(M,F))^{an} = f^{*mod}(M,F)^{an}$.

– *Let* $(M,F) \in D_{\mathcal{D}fil,r}(S)$, *for* $r = 1, \ldots \infty$. *Then,* $(Lf^{*mod}(M,F))^{an} = Lf^{*mod}(M,F)^{an}$.

(ii) Let $S \in \mathrm{SmVar}(\mathbb{C})$

– *Let* $(M,F),(N,F) \in C_{\mathcal{D}fil}(S)$. *Then,* $((M,F)\otimes_{O_S}(N,F))^{an} = (M,F)^{an}\otimes_{O_{S^{an}}}(N,F)^{an}$.

– *Let* $(M,F),(N,F) \in D_{\mathcal{D}fil,r}(S)$, *for* $r = 1, \ldots \infty$. *Then,* $((M,F)\otimes^L_{O_S}(N,F))^{an} = (M,F)^{an}\otimes^L_{O_{S^{an}}} (N,F)^{an}$.

Proof. (i): For $(M,F) \in C_{\mathcal{D},fil}(S)$, we have, since $f^*\,\mathrm{an}_S^* = \mathrm{an}_X^*\,f^{an*}$,

$$\begin{aligned}(f^{*mod}(M,F))^{an} &= \mathrm{an}(X)^*(f^*(M,F)\otimes_{f^*O_S}O_X)\otimes_{\mathrm{an}(X)^*O_X}O_{X^{an}} \\ &= f^{an*}\,\mathrm{an}_S^*(M,F)\otimes_{f^{an*}O_{S^{an}}}\otimes O_{X^{an}} =: f^{an*mod}(M^{an},F)\end{aligned}$$

For $(M,F) \in D_{\mathcal{D},fil,r}(S)$, we take $(M,F) \in C_{\mathcal{D},fil}(S)$ an r-projective f^*O_S module such that $D_{top,r}(M,F) = (M,F)$ so that

$$(Lf^{*mod}(M,F))^{an} = (f^{*mod}(M,F))^{an} = f^{an*mod}(M^{an},F) = Lf^{an*mod}(M^{an},F)$$

(ii): For $(M,F),(N,F) \in C_{\mathcal{D},fil}(S)$, we have

$$\begin{aligned}((M,F)\otimes_{O_S}(N,F))^{an}: &= \mathrm{an}_S^*((M,F)\otimes_{O_S}(N,F))\otimes_{\mathrm{an}_S^*O_S}O_{S^{an}} \\ &= \mathrm{an}_S^*(M,F)\otimes_{\mathrm{an}_S^*O_S}\mathrm{an}_S^*(N,F)\otimes_{\mathrm{an}_S^*O_S}O_{S^{an}} \\ &= \mathrm{an}_S^*(M,F)\otimes_{\mathrm{an}_S^*O_S}\otimes O_{S^{an}}\otimes_{O_{S^{an}}}\mathrm{an}_S^*(N,F)\otimes_{\mathrm{an}_S^*O_S}O_{S^{an}} \\ &= :(M^{an},F)\otimes_{O_{S^{an}}}(N^{an},F)\end{aligned}$$

It implies the isomorphism in the derived category by taking an r-projective resolution of (M,F) (e.g $(L_D(M),F) = L_D(M,F)$). □

Proposition 52. *(i) Let* $f_1 : X \to Y$ *and* $f_2 : Y \to S$ *two morphism with* $X,Y,S \in \mathrm{AnSm}(\mathbb{C})$.

– *Let* $(M,F) \in C_{\mathcal{D}(2)fil}(S)$ *or let* $(M,F) \in C_{\mathcal{D}^\infty(2)fil}(S)$. *Then* $(f_2\circ f_1)^{*mod}(M,F) = f_1^{*mod}f_2^{*mod}(M,F)$.

– *Let* $(M,F) \in D_{\mathcal{D}(2)fil,r}(S)$ *or let* $(M,F) \in D_{\mathcal{D}^\infty(2)fil,r}(S)$. *Then* $L(f_2\circ f_1)^{*mod}(M,F) = Lf_1^{*mod}(Lf_2^{*mod}(M,F))$.

(ii) Let $f_1 : X \to Y$ *and* $f_2 : Y \to S$ *two morphisms with* $X,Y,S \in \mathrm{AnSm}(\mathbb{C})$. *Let* $M \in D_{\mathcal{D}}(X)$. *If* f_1 *is proper, we have* $\int_{f_2\circ f_1} M = \int_{f_2}(\int_{f_1} M)$.

(ii)' Let $f_1 : X \to Y$ *and* $f_2 : Y \to S$ *two morphisms with* $X,Y,S \in \mathrm{AnSm}(\mathbb{C})$. *Let* $M \in D_{\mathcal{D}^\infty}(X)$. *If* f_1 *is proper, we have* $\int_{f_2\circ f_1} M = \int_{f_2}(\int_{f_1} M)$.

(iii) Let $f_1 : X \to Y$ *and* $f_2 : Y \to S$ *two morphisms with* $X,Y,S \in \mathrm{AnSm}(\mathbb{C})$. *Let* $(M,F) \in D_{\mathcal{D}}(X)$. *We have* $\int_{(f_2\circ f_1)!} M = \int_{f_2!}(\int_{f_1!} M)$.

(iii)' Let $f_1 : X \to Y$ *and* $f_2 : Y \to S$ *two morphisms with* $X,Y,S \in \mathrm{AnSm}(\mathbb{C})$. *Let* $M \in D_{\mathcal{D}^\infty}(X)$. *We have* $\int_{(f_2\circ f_1)!} M = \int_{f_2!}(\int_{f_1!} M)$.

(iv) Let $i_0 : Z_2 \hookrightarrow Z_1$ *and* $i_1 : Z_1 \hookrightarrow S$ *two closed embedding, with* $Z_2, Z_1, S \in \mathrm{AnSm}(\mathbb{C})$. *Let* $(M,F) \in C_{\mathcal{D}(2)fil}(Z_2)$. *Then,* $(i_1\circ i_0)_{*mod}(M,F) = i_{1*mod}(i_{0*mod}(M,F))$ *in* $C_{\mathcal{D}(2)fil}(S)$.

(iv)' Let $i_0 : Z_2 \hookrightarrow Z_1$ *and* $i_1 : Z_1 \hookrightarrow S$ *two closed embedding, with* $Z_2, Z_1, S \in \mathrm{AnSm}(\mathbb{C})$. *Let* $(M,F) \in C_{\mathcal{D}^\infty(2)fil}(Z_2)$. *Then,* $(i_1\circ i_0)_{*mod}(M,F) = i_{1*mod}(i_{0*mod}(M,F))$ *in* $C_{\mathcal{D}^\infty(2)fil}(S)$.

Proof. (i): Similar to the proof of proposition 49(i).
(ii): Similar to the proof of proposition 49(ii) : we use lemma 4 and the fact that for $N \in C_{f_1^*\mathcal{D}}(X)$ and $N' \in C_{\mathcal{D}}(Y)$, the canonical morphism

$$T(f_1,\otimes)(N',N) : N' \otimes^L_{D_Y} Rf_{1*}N \to Rf_{1*}(f_1^*N' \otimes^L_{f_1^*D_Y} N)$$

is an isomorphism if f_1 is proper (in this case $f_{1!} = f_{1*}$).
(ii)': Similar to the proof of proposition 49(ii) : we use lemma 5 and the fact that for $N \in C_{f_1^*\mathcal{D}^\infty}(X)$ and $N' \in C_{\mathcal{D}^\infty}(Y)$, the canonical morphism

$$T(f_1,\otimes)(N',N) : N' \otimes^L_{D_Y^\infty} Rf_{1*}N \to Rf_{1*}(f_1^*N' \otimes^L_{f_1^*D_Y^\infty} N)$$

is an isomorphism if f_1 is proper (in this case $f_{1!} = f_{1*}$).
(iii): Similar to the proof of proposition 49(ii) : we use lemma 4 and the fact that for $N \in C_{f_1^*\mathcal{D}}(X)$ and $N' \in C_{\mathcal{D}}(Y)$, the canonical morphism

$$T(f_1!,\otimes)(N',N) : N' \otimes^L_{D_Y} Rf_{1!}N \to Rf_{1!}(f_1^*N' \otimes^L_{f_1^*D_Y} N)$$

is an isomorphism.
(iii)': Similar to the proof of proposition 49(ii) : we use lemma 5 and the fact that for $N \in C_{f_1^*\mathcal{D}^\infty}(X)$ and $N' \in C_{\mathcal{D}^\infty}(Y)$, the canonical morphism

$$T(f_1!,\otimes)(N',N) : N' \otimes^L_{D_Y^\infty} Rf_{1!}N \to Rf_{1!}(f_1^*N' \otimes^L_{f_1^*D_Y^\infty} N)$$

is an isomorphism
(iv): Similar to the proof of proposition 49(iii) :we have

$$i_{2*mod}(M,F) = i_{2*}((M,F) \otimes_{D_{Z_2}} (D_{Z_2\leftarrow S}), F^{ord}) \xrightarrow{=}$$
$$i_{1*}i_{0*}((M,F) \otimes_{D_{Z_2}} (D_{Z_2\leftarrow Z_1}, F^{ord}) \otimes_{i_0^*D_{Z_1}} i_0^*(D_{Z_1\leftarrow S}), F^{ord}) \xrightarrow{i_{1*}T(i_0,\otimes)(-)^{-1}} i_{1*mod}i_{0*mod}((M,F))$$

using lemma 4 and proposition 10.
(iv)':Similar to (iv): we have

$$i_{2*mod}(M,F) = i_{2*}((M,F) \otimes_{D_{Z_2}^\infty} (D^\infty_{Z_2\leftarrow S}, F^{ord})) \xrightarrow{=}$$
$$i_{1*}i_{0*}((M,F) \otimes_{D_{Z_2}^\infty} (D^\infty_{Z_2\leftarrow Z_1}, F^{ord}) \otimes_{i_0^*D_{Z_1}^\infty} i_0^*(D^\infty_{Z_1\leftarrow S}, F^{ord})) \xrightarrow{i_{1*}T(i_0,\otimes)(-)^{-1}} i_{1*mod}i_{0*mod}((M,F))$$

using lemma 5 and proposition 10.

□

Proposition 53. *(i) Let $f : X \to S$ a morphism with $X, S \in \mathrm{AnSm}(\mathbb{C})$. For $(M,F) \in C_{\mathcal{D}(2)fil,h}(S)$, we have $Lf^{*mod}(M,F) \in D_{\mathcal{D}(2)fil,\infty,h}(X)$. For $(M,F) \in C_{\mathcal{D}^\infty(2)fil,h}(S)$, we have $Lf^{*mod}(M,F) \in D_{\mathcal{D}^\infty(2)fil,\infty,h}(X)$.*

(ii) Let $f : X \to S$ a proper morphism with $X, S \in \mathrm{AnSm}(\mathbb{C})$. Then, for $(M,F) \in C_{\mathcal{D}(2)fil,h}(X)$, we have $\int_f(M,F) \in D_{\mathcal{D}(2)fil,\infty,h}(S)$.

(iii) Let $f : X \to S$ a morphism with $X, S \in \mathrm{AnSm}(\mathbb{C})$. Then, for $(M,F) \in C_{\mathcal{D}^\infty(2)fil,h}(X)$, we have $\int_f(M,F) \in D_{\mathcal{D}^\infty(2)fil,\infty,h}(S)$.

Proof. (i)and (ii):Follows imediately from the non filtered case since we look at the complex in the derived category with respect to ∞-usu local equivalence. It says that the pullback and the proper pushforward of an holonomic D module is still holonomic. See [16] for the non filtered case.

(iii):In the case the morphism is proper, it follows from the finite order case (ii). In the case of an open embedding, it follows from proposition 47(i) : we have for $j : S^o \hookrightarrow S$ an open embedding,

$$j_*E(O_{S^o}) = j_*\mathcal{H}om(\mathbb{Z}_{S^o}, E(O_{S^o})) = \mathcal{H}om(j_!\mathbb{Z}_{S^o}, E(O_S)) \in C_{\mathcal{D}^\infty,h}(S).$$

and on the other hand

$$T(j,\otimes)(-,-) = T^{mod}(j,\otimes)(-,-) : \int_j (M,F) = j_*E(M,F) = j_*E(j^*O_S \otimes_{O_{S^o}} (M,F)) \xrightarrow{\sim} j_*E(O_{S^o}) \otimes_{O_S} (M,F)$$

is an isomorphism by proposition 9. ◻

For $X, Y \in \mathrm{SmVar}(\mathbb{C})$ or $X, Y \in \mathrm{AnSm}(\mathbb{C})$, we denote by

- $C_{O_X}(X) \times C_{O_Y}(Y) \to C_{O_{X\times Y}}(X \times Y), (M, N) \mapsto M \cdot N := O_{X\times Y} \otimes_{p_X^* O_X \otimes p_Y^* O_Y} p_X^* M \otimes p_Y^* N$,
- $C_{\mathcal{D}}(X) \times C_{\mathcal{D}}(Y) \to C_{\mathcal{D}}(X \times Y), (M, N) \mapsto M \cdot N := O_{X\times Y} \otimes_{p_X^* O_X \otimes p_Y^* O_Y} p_X^* M \otimes p_Y^* N$

the natural functors which induces functors in the filtered cases and the derived categories, $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ the projections.

We have then the following easy proposition :

Proposition 54. *For $X \in \mathrm{SmVar}(\mathbb{C})$ or $X \in \mathrm{AnSm}(\mathbb{C})$, we have for $(M, F), (N, F) \in C_{O_X,fil}(X)$ or $(M, F), (N, F) \in C_{\mathcal{D},fil}(X)$,*

$$(M, F) \otimes_{O_X} (N, F) = \Delta_X^{*mod}(M, F) \cdot (N, F)$$

Proof. Standard. ◻

Definition 59. *Let $f : X \to S$ a morphism with $X, S \in \mathrm{AnSm}(\mathbb{C})$. We have the canonical map in $C_{f^*\mathcal{D},\mathcal{D}^\infty}(X)$ modules :*

$$T(f,\infty) : (D_{X\to S}, F^{ord}) \otimes_{D_X} (D_X^\infty, F^{ord}) \to (D_{X\to S}^\infty, F^{ord}),\ (h_X \otimes P_S) \otimes P_X \mapsto (P_X.h_X \otimes P_S + h_X \otimes df(P_X)P_S$$

where $h_X \in \Gamma(X^o, O_X)$, $P_S \in \Gamma(X^o, f^*D_S)$ and $P_X \in \Gamma(X^o, D_X^\infty)$. This gives, for $(M, F) \in C_{\mathcal{D}(2)fil}(S)$, the following transformation map in $C_{\mathcal{D}^\infty(2)fil}(X)$

$$T(f,\infty)(M,F) : J_X(f^{*mod}(M,F)) := f^*(M,F) \otimes_{f^*D_S} (D_{X\to S}, F^{ord} \otimes_{D_X} (D_X^\infty, F^{ord}) \xrightarrow{I\otimes T(f,\infty)} f^*(M,F) \otimes_{f^*D_S} (D_{X\to S}^\infty, F^{ord}) = f^*(M,F) \otimes_{f^*D_S} f^*D_S^\infty \otimes_{f^*D_S^\infty} (D_{X\to S}^\infty, F^{ord}) =: f^{*mod} J_S(M,F)$$

where we recall that $J_S(M, F) = (M, F) \otimes_{D_S} (D_S^\infty, F^{ord})$.

We now look at some properies of the dual functor for D modules : For complex of D module with coherent cohomology we have the following:

Proposition 55. *(i) Let $S \in \mathrm{SmVar}(\mathbb{C})$. For $M \in C_{\mathcal{D},c}(S)$, the canonical map $d(M) : M \to \mathbb{D}_S^2 L_D M$ is an equivalence Zariski local.*

(ii) Let $S \in \mathrm{AnSm}(\mathbb{C})$. For $M \in C_{\mathcal{D}}(S)$, the canonical map $d(M) : M \to \mathbb{D}_S^2 L_D(M)$ is an equivalence usu local.

iii) Let $S \in \mathrm{AnSm}(\mathbb{C})$. For $(M, F) \in C_{\mathcal{D}^\infty}(S)$, the canonical map $d(M) : M \to \mathbb{D}_S^2 L_{D^\infty}(M)$ is an equivalence usu local.

Proof. Standard :follows from the definition of coherent sheaves. See [16] for exemple. ◻

Let $S_1, S_2 \in \mathrm{SmVar}(\mathbb{C})$ or $S_1, S_2 \in \mathrm{AnSm}(\mathbb{C})$ and $p : S_{12} := S_1 \times S_2 \to S_1$ the projection. In this case we have a canonical embedding $D_{S_1} \hookrightarrow p_* D_{S_{12}}$. This gives, for $(M, F) \in C_{\mathcal{D}fil}(S_1 \times S_2)$, the following transformation map in $C_{\mathcal{D}fil}(S_1)$

$$\begin{aligned} T_*(p, D)(M, F) : p_* \mathbb{D}^K_{S_{12}}(M, F) &:= p_* \mathcal{H}om_{D_{S_{12}}}((M, F), D_{S_{12}}) \otimes_{O_{S_{12}}} \mathbb{D}^O_{S_{12}} w(K_{S_{12}})[d_{S_{12}}] \\ &\xrightarrow{T_*(p,hom)(-,-)} \mathcal{H}om_{p_* D_{S_{12}}}(p_*(M, F), p_* D_{S_{12}}) \otimes_{p_* O_{S_{12}}} \mathbb{D}^O_{p_* S_{12}} w(p_* K_{S_{12}})[d_{S_{12}}] \\ &\xrightarrow{\sim} \mathcal{H}om_{D_{S_1}}(p_*(M, F), D_{S_1}) \otimes_{O_{S_1}} \mathbb{D}^O_{S_1} w(K_{S_1})[d_{S_1}] =: \mathbb{D}^K_{S_1} p_*(M, F) \end{aligned}$$

We have the canonical map

$$p(D) : p^{*mod} D_{S_1} = p^* D_{S_1} \otimes_{p^* O_{S_1}} O_{S_{12}} \to D_{S_{12}}, \ \gamma \otimes f \mapsto f.\gamma$$

induced by the embedding $p^* D_{S_1} \hookrightarrow D_{S_{12}}$. This gives, for $(M, F) \in C_{\mathcal{D}fil}(S_1)$, the following transformation map in $C_{\mathcal{D}fil}(S_1 \times S_2)$

$$\begin{aligned} T(p, D)(M, F) : p^{*mod} \mathbb{D}^K_{S_1}(M, F) &:= p^* \mathcal{H}om_{D_{S_1}}((M, F), D_{S_1}) \otimes_{p^* O_{S_1}} p^{*mod} \mathbb{D}^O_{S_1} w(K_{S_1})[d_{S_1}] \\ &\xrightarrow{T(p,hom)(-,-) \otimes I} \mathcal{H}om_{p^* D_{S_1}}(p^*(M, F), p^* D_{S_1}) \otimes_{p^* O_{S_1}} p^{*mod} \mathbb{D}^O_{S_1} w K_{S_1})[d_{S_1}] \\ &\xrightarrow{(\phi \mapsto \phi \otimes I_{O_{S_{12}}}) \otimes I} \mathcal{H}om_{D_{S_{12}}}(p^{*mod}(M, F), p^{*mod} D_{S_1}) \otimes_{p^* O_{S_1}} p^{*mod} \mathbb{D}^O_{S_1} w(K_{S_1})[d_{S_1}] \\ &\xrightarrow{I \otimes K^{-1}(S_1/S_{12})} \mathcal{H}om_{D_{S_{12}}}(p^{*mod}(M, F), p^{*mod} D_{S_1}) \otimes_{p^* O_{S_1}} \mathbb{D}^O_{S_{12}} w(K_{S_{12}})[d_{S_{12}}] \\ &\xrightarrow{q(p^* O_{S_1}/O_{S_{12}})} \mathcal{H}om_{D_{S_{12}}}(p^{*mod}(M, F), p^{*mod} D_{S_1}) \otimes_{O_{S_{12}}} \mathbb{D}^O_{S_{12}} w(K_{S_{12}})[d_{S_{12}}] \\ &\xrightarrow{\mathcal{H}om(p^{*mod}(M,F), p(D))} \mathcal{H}om_{D_{S_{12}}}(p^{*mod}(M, F), D_{S_{12}}) \otimes_{O_{S_{12}}} \mathbb{D}^O_{S_{12}} w(K_{S_{12}})[d_{S_{12}}] =: \mathbb{D}^K_{S_{12}}(p^{*mod}(M, F)) \end{aligned}$$

whre $K^{-1}(S_1/S_{12})$ is given by the wedge product with a generator of $\wedge^{d_{S_2}} T_{S_{12}/S_1} \xrightarrow{\sim} K^{-1}_{S_2}$.

In the case $S_1, S_2 \in \mathrm{AnSm}(\mathbb{C})$, we also have the embedding $p^* D^\infty_{S_1} \hookrightarrow D^\infty_{S_{12}}$. This gives in the same way, for $(M, F) \in C_{\mathcal{D}fil}(S_1 \times S_2)$, the following transformation map in $C_{\mathcal{D}fil}(S_1)$

$$T_*(p, D^\infty)(M, F) : p_* \mathbb{D}^{\infty,K}_{S_{12}}(M, F) \to \mathbb{D}^{\infty,K}_{S_1} p_*(M, F).$$

The map

$$p(D^\infty) : p^{*mod} D_{S_1} = p^* D^\infty_{S_1} \otimes_{p^* O_{S_1}} O_{S_{12}} \to D^\infty_{S_{12}}, \ \gamma \otimes f \mapsto f.\gamma$$

induced by the embedding $p^* D^\infty_{S_1} \hookrightarrow D^\infty_{S_{12}}$, gives in the same way, for $(M, F) \in C_{\mathcal{D}^\infty fil}(S_1)$, the transformation map in $C_{\mathcal{D}^\infty fil}(S_1 \times S_2)$

$$\begin{aligned} T(p, D^\infty)(M, F) : p^{*mod} \mathbb{D}^{\infty,K}_{S_1}(M, F) := p^{*mod}(\mathbb{D}^\infty_{S_1}(M, F) \otimes_{O_{S_1}} \mathbb{D}^O_{S_1} w(K_{S_1})[d_{S_1}]) \to \\ \mathcal{H}om_{D^\infty_{S_{12}}}(p^{*mod}(M, F), D^\infty_{S_{12}}) \otimes_{O_{S_{12}}} \mathbb{D}^O_{S_{12}} w(K_{S_{12}}[d_{S_{12}}] =: \mathbb{D}^{\infty,K}_{S_{12}}(p^{*mod}(M, F)), \end{aligned}$$

given in the same way then $T(p, D)(M, F)$.

Proposition 56. *(i) Let $g : T \to S$ a morphism with $T, S \in \mathrm{SmVar}(\mathbb{C})$. We have, for $M \in D_{\mathcal{D}}(S)$ canonical maps*

- $T'(g, D)(M) : L\mathbb{D}_S Lg^{*mod} M \to Lg^{*mod} L\mathbb{D}_S M$
- $T'(g, D)(M) : Lg^{*mod} L\mathbb{D}_S M \to L\mathbb{D}_S Lg^{*mod} M$

Moreover, in the case where g is non caracteristic with respect to M (e.g if g is smooth), these maps are isomorphism.

(ii) *Let $S_1,S_2\in\mathrm{SmVar}(\mathbb{C})$, $p:S_1\times S_2\to S_1$ the projection. For $M\in D_{\mathcal{D}}(S_1)$, we have $T(p,D)(L_D(M))=T'(p,D)(M)$ in $D_{\mathcal{D}fil}(S_1\times S_2)$ (c.f.(i)).*

Proof. (i):See [16] for the first map. The second one follows from the first by proposition 55(i) and (iii). (ii):See the proof of (i) in [16] $\square$

We have the followings :

Proposition 57. *Let $f_1:X\to Y$ and $f_2:Y\to S$ two morphism with $X,Y,S\in\mathrm{SmVar}(\mathbb{C})$. Let $M\in C_{\mathcal{D},h}(S)$. Then, we have $L(f_2\circ f_1)^{\hat{*}mod}M=Lf_1^{\hat{*}mod}(Lf_2^{\hat{*}mod}M)$ in $D_{\mathcal{D},h}(X)$.*

Proof. Follows from proposition 49 (i), proposition 50 and proposition 55. $\square$

Proposition 58. *Let $f_1:X\to Y$ and $f_2:Y\to S$ two morphism with $X,Y,S\in\mathrm{AnSm}(\mathbb{C})$.*

(i) *Let $(M)\in C_{\mathcal{D},h}(S)$. Then, we have $L(f_2\circ f_1)^{\hat{*}mod}(M)=Lf_1^{\hat{*}mod}(Lf_2^{\hat{*}mod}(M))$ in $D_{\mathcal{D},h}(X)$.*

(ii) *Let $M\in C_{\mathcal{D}^\infty,h}(S)$. Then, we have $L(f_2\circ f_1)^{\hat{*}mod}M=Lf_1^{\hat{*}mod}(Lf_2^{\hat{*}mod}M)$ in $D_{\mathcal{D}^\infty,h}(X)$.*

Proof. Follows from proposition 52 (i), proposition 53 and proposition 55. $\square$

In the analytic case, we have the following transformation map which we will use in subection 5.3:

Definition 60. *Let $S\in\mathrm{AnSm}(\mathbb{C})$. We have for $(M,F)\in C_{\mathcal{D}fil}(S)$ the canonical transformation map in $C_{\mathcal{D}^\infty fil}(S)$:*

$$\begin{aligned}&T(D,\infty)(M,F):\\ J_S(\mathbb{D}_S^K(M,F)):=\mathcal{H}om_{D_S}((M,F),D_S)\otimes_{D_S}(D_S^\infty,F^{ord})\otimes_{O_S}\mathbb{D}_S^O w(K_S)[d_S]&\xrightarrow{ev_{D_S}(\hom,\otimes)(-,-,-)\otimes I}\\ \mathcal{H}om_{D_S}(L_D(M,F),D_S^\infty)\otimes_{O_S}\mathbb{D}_S^O w(K_S)[d_S]&\xrightarrow{I(D_S^\infty/D_S)((M,F),D_S^\infty)\otimes I}\\ \mathcal{H}om_{D_S^\infty}((M,F)\otimes_{D_S}(D_S^\infty,F^{ord}),D_S^\infty)\otimes_{O_S}\mathbb{D}_S^O w(K_S)[d_S]&=:\mathbb{D}_S^{\infty,K}J_S(M,F).\end{aligned}$$

which is an isomorphism.

4.2.2 The (relative) De Rahm of a (filtered) complex of a D-module and the filtered De Rham direct image

Recall that for $f:X\to S$ a morphism with $X,S\in\mathrm{Var}(\mathbb{C})$ or with $X,S\in\mathrm{AnSp}(\mathbb{C})$,

$$DR(X/S):=\Omega^\bullet_{X/S}\in C_{f^*O_S}(X)$$

denotes (see section 2) the relative De Rham complex of the morphism of ringed spaces $f:(X,O_X)\to(S,O_S)$, with $\Omega^p_{X/S}:=\wedge^p\Omega_{X/S}\in\mathrm{PSh}_{O_X}(X)$ and $\Omega_{X/S}:=\mathrm{coker}(f^*\Omega_S\to\Omega_X)\in\mathrm{PSh}_{O_X}(X)$. Recall that $\Omega^\bullet_{X/S}\in C_{f^*O_S}(S)$ is a complex of f^*O_S modules, but is NOT a complex of O_X module since the differential is a derivation hence NOT O_X linear. Recall that (see section 4.1), for $(M,F)\in C_{D(O_X)fil}(X)$, we have the relative (filtered) De rham complex of (M,F)

$$DR(X/S)(M,F):=(\Omega^\bullet_{X/S},F_b)\otimes_{O_X}(M,F)\in C_{f^*O_Sfil}(X),$$

and that if $\phi:(M_1,F)\to(M_2,F)$ a morphism with $(M_1,F),(M_2,F)\in C_{D(O_X)fil}(X)$,

$$(I\otimes\phi):DR(X/S)(M_1,F):=(\Omega^\bullet_{X/S},F)\otimes_{O_X}(M_1,F)\to DR(X/S)(M_2,F):=(\Omega^\bullet_{X/S},F)\otimes_{O_X}(M_2,F)$$

is by definition a morphism of complexes, that is a morphism in $C_{f^*O_Sfil}(X)$. For $(N,F)\in C_{D(O_X)^{op}fil}(X)$, we have the relative (filtered) Spencer complex of (N,F)

$$SP(X/S)(N,F):=(T^\bullet_{X/S},F_b)\otimes_{O_X}(N,F)\in C_{f^*O_Sfil}(X),$$

and that if $\phi:(N_1,F)\to(N_2,F)$ a morphism with $(N_1,F),(N_2,F)\in C_{D(O_X)^{op}fil}(X)$,

$$(I\otimes\phi):SP(X/S)(N_1,F):=(T^\bullet_{X/S},F)\otimes_{O_X}(N_1,F)\to SP(X/S)(N_2,F):=(T^\bullet_{X/S},F)\otimes_{O_X}(N_2,F)$$

is by definition a morphism of complexes, that is a morphism in $C_{f^*O_Sfil}(X)$.

Proposition 59. *Let $f : X \to S$ a smooth morphism with $X, S \in \mathrm{Var}(\mathbb{C})$ or with $X, S \in \mathrm{AnSp}(\mathbb{C})$, denote $d = d_X - d_S$. The inner product gives, for $(M, F) \in C_{D(O_X)fil}(X)$, an isomorphism in $C_{f^*O_S fil}(X)$ and termwise O_X linear*

$$T(DR, SP)(M, F) : T^{\bullet}_{X/S} \otimes_{O_X} (M, F) \otimes_{O_X} K_{X/S} \xrightarrow{\sim} \Omega^{d-\bullet}_{X/S} \otimes_{O_X} (M, F), \partial \otimes m \otimes \kappa \mapsto \iota(\partial)\kappa \otimes m$$

Proof. Standard. □

For a commutative diagram in $\mathrm{Var}(\mathbb{C})$ or in $\mathrm{AnSp}(\mathbb{C})$:

$$D = \begin{array}{ccc} X & \xrightarrow{f} & S \\ \uparrow{\scriptstyle g'} & & \uparrow{\scriptstyle g} \\ X' & \xrightarrow{f'} & T \end{array}$$

we have (see section 2) the relative differential map of g' given by the pullback of differential forms:

$$\Omega_{(X'/X)/(T/S)} : g^{'*}\Omega_{X/S} \to \Omega_{X'/T}, \text{ given by for } X^{'o} \subset X',\ X^o \supset g'(X^{'o})(\text{i.e. } g^{'-1}(X^o) \supset X^{'o}),$$
$$\omega \in \Gamma(X^o, \Omega^p_{X/S}) \mapsto \Omega_{(X'/X)/(T/S)}(X^{'o})(\omega) := [g^{'*}\omega] \in \Gamma(X^{'o}, \Omega^p_{X'/T}).$$

Moreover, by definition-proposition 16 (section 4.1), for $(M, F) \in C_{D(O_X)fil}(X)$ the map

$$\Omega_{(X'/X)/(T/S)}(M, F) : g'^*(\Omega^{\bullet}_{X/S} \otimes_{O_X} (M, F)) \to \Omega^{\bullet}_{X'/T} \otimes_{O_{X'}} g^{'*mod}(M, F)$$

given in degree (p, i) by, for $X^{'o} \subset X'$ an open subset and $X^o \subset X$ an open subset such that $g^{'-1}(X^o) \supset X^{'o}$ (i.e. $X^o \supset g'(X^{'o})$), $\omega \in \Gamma(X^o, \Omega^p_{X/S})$ and $m \in \Gamma(X^o, M^i)$,

$$\Omega_{(X'/X)/(T/S)}(M, F)(\omega \otimes m) = g^{'*}\omega \otimes (m \otimes 1)$$

is a map of complexes, that is a map in $C_{f^*O_S fil}(X')$. This give, for $(M, F) \in C_{D(O_X)fil}(X)$, the following transformation map in $C_{O_T fil}(T)$

$$T^O_{\omega}(D)(M, F) : g^{*mod}L_O(f_*E(\Omega^{\bullet}_{X/S} \otimes_{O_X} (M, F))) \xrightarrow{T(g, L_O)(-)}$$
$$(g^* f_* E(\Omega^{\bullet}_{X/S} \otimes_{O_X} (M, F))) \otimes_{g^*O_S} O_T \xrightarrow{T(g', E)(-) \circ T(D)(E(\Omega^{\bullet}_{X/S} \otimes_{O_X} M))}$$
$$(f'_*E(g'^*(\Omega^{\bullet}_{X/S} \otimes_{O_X} (M, F)))) \otimes_{g^*O_S} O_T \xrightarrow{m \circ E(\Omega_{(Y/X)/(T/S)}(M))} f'_*E(\Omega^{\bullet}_{X'/T} \otimes_{O_{X'}} g^{'*mod}(M, F)),$$

with $m'(m) = m \otimes 1$. Under the canonical isomorphism $\Omega^{\bullet}_{X/S} \xrightarrow{\sim} \Omega^{\bullet}_{X/S} \otimes_{O_X} O_X$ given by $\omega \mapsto \omega \otimes 1$, we have (see remark 7)

$$T^O_{\omega}(D)(O_X) = T^O_{\omega}(D) : g^{*mod}L_O(f_*E(\Omega^{\bullet}_{X/S})) \to f'_*E(\Omega^{\bullet}_{X'/T}).$$

Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$. Again by definition-proposition 16 (section 4.1), for $(M, F) \in C_{D(O_X)fil}(X)$ the map

$$\Omega_{(X^{an}/X)/(S^{an}/S)}(M, F) : an^*_X(\Omega^{\bullet}_{X/S} \otimes_{O_X} (M, F)) \to \Omega^{\bullet}_{X^{an}/S^{an}} \otimes_{O_{X^{an}}} M^{an}$$

given in degree (p, i) by, for $X^o \subset X$ and $X^o \supset X^{oo}$ an open subsets of X for the usual, resp. Zariski topology, $\omega \in \Gamma(X^o, \Omega^p_{X/S})$ and $m \in \Gamma(X^o, M^i)$,

$$\Omega_{(X^{an}/X)/(S^{an}/S)}(M, F)(\omega \otimes m = \omega \otimes (m \otimes 1)$$

is a map of complexes, that is a map in $C_{f^*O_{S^{an}}fil}(X^{an})$. This gives, for $(M,F)\in C_{D(O_X)fil}(X)$, we have the following transformation map in $C_{O_{S^{an}}fil}(S^{an})$

$$\begin{aligned}T^O_\omega(an,f)(M,F):(f_*E(\Omega^\bullet_{X/S}\otimes_{O_X}(M,F)))^{an}&:=\mathrm{an}_S^*(f_*E(\Omega^\bullet_{X/S}\otimes_{O_X}(M,F)))\otimes_{\mathrm{an}_S^*O_S}O_{S^{an}}\\&\xrightarrow{T(an(X),E)(-)\circ T(an,f)(E(\Omega^\bullet_{X/S}\otimes_{O_X}M))}(f_*E(\mathrm{an}_X^*(\Omega^\bullet_{X/S}\otimes_{O_X}(M,F))))\otimes_{\mathrm{an}_S^*O_S}O_{S^{an}}\\&\xrightarrow{m\circ E(\Omega_{(X^{an}/X)/(S^{an}/S)}(M,F))}f_*E(\Omega^\bullet_{X/S}\otimes_{O_X^{an}}(M^{an},F))\end{aligned}$$

with $m(n\otimes s)=s.n$.Under the canonical isomorphism $\Omega^\bullet_{X/S}\xrightarrow{\sim}\Omega^\bullet_{X/S}\otimes_{O_X}O_X$ given by $\omega\mapsto\omega\otimes 1$, we have (see remark 7)

$$T^O_\omega(an,f)(O_X)=T^O_\omega(an,f):(f_*E(\Omega^\bullet_{X/S}))^{an}\to f_*E(\Omega^\bullet_{X^{an}/S^{an}})$$

Let $f:X\to S$ a morphism with $X,S\in\mathrm{Var}(\mathbb{C})$ or with $X,S\in\mathrm{AnSp}(\mathbb{C})$. In the case where X is smooth, for $(M,F)=(M^\bullet,F)\in C_{\mathcal{D}fil}(X)$, the differential of the relative De Rham complex of (M,F)

$$DR(X/S)(M,F):=(\Omega^\bullet_{X/S},F)\otimes_{O_X}(M,F)=\mathrm{Tot}((\Omega^\bullet_{X/S},F)\otimes_{O_X}(M^\bullet,F))\in C_{f^*O_Sfil}(X)$$

are given by

- $d_{p,p+1}:\Omega^p_{X/S}\otimes_{O_X}M^i\to\Omega^{p+1}_{X/S}\otimes_{O_X}M^i$, with for $X^o\subset X$ an open affine subset with $(x_1,\ldots,x_n)$ local coordinate (since X is smooth, T_X is locally free), $m\in\Gamma(X^o,M^i)$ and $\omega\in\Gamma(X^o,\Omega^p_{X/S})$,

$$d_{p,p+1}(\omega\otimes m):=(d\omega)\otimes m+\sum_{i=1}^n(dx_i\wedge\omega)\otimes(\partial_i)m$$

- $d_{i,i+1}:\Omega^p_{X/S}\otimes_{O_X}M^i\to\Omega^p_{X/S}\otimes_{O_X}M^{i+1}$, with for $X^o\subset X$ an open subset, $m\in\Gamma(X^o,M^i)$ and $\omega\in\Gamma(X^o,\Omega^p_{X/S})$, $d_{i,i+1}(\omega\otimes m):=(\omega\otimes dm)$.

For D_X only, the differential of its De Rahm complex $(\Omega^\bullet_{X/S},F)\otimes_{O_X}D_X$ are right linear, so that

$$(\Omega^\bullet_{X/S},F_b)\otimes_{O_X}(D_X,F^{ord})\in C_{\mathcal{D}^{op},f^*O_Sfil}(X)$$

In the particular case of a projection $p:Y\times S\to S$ with $Y,S\in\mathrm{SmVar}(\mathbb{C})$ or with $Y,S\in\mathrm{AnSm}(\mathbb{C})$ we have :

Proposition 60. *Let $Y,S\in\mathrm{SmVar}(\mathbb{C})$ or $Y,S\in\mathrm{AnSm}(\mathbb{C})$. Let $p:Y\times S\to S$ the projection. For $(M,F)\in C_{\mathcal{D}fil}(Y\times S)$,*

$$DR(Y\times S/S)(M,F):=(\Omega^\bullet_{Y\times S/S},F_b)\otimes_{O_{Y\times S}}(M,F)\in C_{p^*O_Sfil}(Y\times S)$$

is a naturally a complex of filtered p^*D_S modules, that is

$$DR(Y\times S/S)(M,F):=(\Omega^\bullet_{Y\times S/S},F_b)\otimes_{O_{Y\times S}}(M,F)\in C_{p^*\mathcal{D}fil}(Y\times S),$$

where the p^*D_S module structure on $\Omega^p_{Y\times S/S}\otimes_{O_{Y\times S}}M^n$ is given by for $(Y\times S)^o\subset Y\times S$ an open subset,

$$(\gamma\in\Gamma((Y\times S)^o,T_{Y\times S}),\hat\omega\otimes m\in\Gamma((Y\times S)^o,\Omega^p_{Y\times S/S}\otimes_{O_{Y\times S}}M^n))\mapsto\gamma.(\hat\omega\otimes m):=(\hat\omega\otimes(\gamma.m).$$

Moreover, if $\phi:(M_1,F)\to(M_2,F)$ a morphism with $(M_1,F),(M_2,F)\in C_{\mathcal{D}fil}(Y\times S)$,

$$DR(Y\times S/S)(\phi):=(I\otimes\phi):(\Omega^\bullet_{Y\times S/S},F_b)\otimes_{O_{Y\times S}}(M_1,F)\to(\Omega^\bullet_{Y\times S/S},F_b)\otimes_{O_{Y\times S}}(M_2,F)$$

is a morphism in $C_{p^*\mathcal{D}fil}(Y\times S)$.

Proof. Standard. $\square$

In the analytic case, we also have

Proposition 61. *Let $Y, S \in \mathrm{AnSm}(\mathbb{C})$. Let $p : Y \times S \to S$ the projection. For $(M, F) \in C_{\mathcal{D}^\infty fil}(Y \times S)$,*

$$DR(Y \times S/S)(M, F) := (\Omega^\bullet_{Y\times S/S}, F_b) \otimes_{O_{Y\times S}} (M, F) \in C_{p^*O_S fil}(Y \times S)$$

is naturally a complex of filtered $p^*D_S^\infty$ modules, that is

$$DR(Y \times S/S)(M, F) := (\Omega^\bullet_{Y\times S/S}, F_b) \otimes_{O_{Y\times S}} (M, F) \in C_{p^*\mathcal{D}^\infty fil}(Y \times S),$$

where the $p^*D_S^\infty$ module structure on $\Omega^p_{Y\times S/S} \otimes_{O_{Y\times S}} M^n$ is given by for $(Y \times S)^o \subset Y \times S$ an open subset,

$$(\gamma \in \Gamma((Y \times S)^o, T_{Y\times S}), \hat{\omega} \otimes m \in \Gamma((Y \times S)^o, \Omega^p_{Y\times S/S} \otimes_{O_{Y\times S}} M^n)) \mapsto \gamma.(\hat{\omega} \otimes m) := (\hat{\omega} \otimes (\gamma.m).$$

Moreover, if $\phi : (M_1, F) \to (M_2, F)$ a morphism with $(M_1, F), (M_2, F) \in C_{\mathcal{D}^\infty fil}(Y \times S)$,

$$DR(Y \times S/S)(\phi) := (I \otimes \phi) : (\Omega^\bullet_{Y\times S/S}, F_b) \otimes_{O_{Y\times S}} (M_1, F) \to (\Omega^\bullet_{Y\times S/S}, F_b) \otimes_{O_{Y\times S}} (M_2, F)$$

is a morphism in $C_{p^*\mathcal{D}^\infty fil}(Y \times S)$.

Proof. Standard : follows from the finite order case (proposition 60). $\square$

We state on the one hand the commutativity of the tensor product with respect to D_S and with respect to O_S, for $S \in \mathrm{SmVar}(\mathbb{C})$ or $S \in \mathrm{AnSm}(\mathbb{C})$ in the filtered case, and on the other hand the commutativity between the tensor product with respect to D_S by D_S and the De Rahm complex :

Proposition 62.

(i) Let $f : X \to S$ a morphism with $X, S \in \mathrm{SmVar}(\mathbb{C})$ or with $X, S \in \mathrm{AnSm}(\mathbb{C})$. For $(M', F) \in C_{\mathcal{D}^{op} fil, f^*\mathcal{D}}(X)$ and $(M, F), (N, F) \in C_{\mathcal{D}fil}(X)$. we have canonical isomorphisms of filtered f^*D_S modules, i.e. isomorphisms in $C_{f^*\mathcal{D}}(X)$,

$$\begin{aligned}(M', F) \otimes_{O_X} (N, F) \otimes_{D_X} (M, F) &= (M', F) \otimes_{D_X} ((M, F) \otimes_{O_X} (N, F)) \\ &= ((M', F) \otimes_{O_X} (M, F)) \otimes_{D_X} (N, F)\end{aligned}$$

(ii) Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$ or with $X, S \in \mathrm{AnSp}(\mathbb{C})$. For $(M, F) \in C_{D(O_X)fil}(X)$, we have a canonical isomorphisms of filtered f^*O_S modules, i.e. isomorphisms in $C_{f^*O_S fil}(X)$,

$$(\Omega^\bullet_{X/S}, F_b) \otimes_{O_X} (M, F) = ((\Omega^\bullet_{X/S}, F_b) \otimes_{O_X} (D(O_X), F_b)) \otimes_{D(O_X)} (M, F)$$

(iii) Let $p : Y \times S \to S$ a projection with $Y, S \in \mathrm{SmVar}(\mathbb{C})$ or with $X, S \in \mathrm{AnSm}(\mathbb{C})$. For $(M, F) \in C_{\mathcal{D}fil}(Y \times S)$, the isomorphisms of filtered p^*O_S modules of (ii)

$$(\Omega^\bullet_{Y\times S/S}, F_b) \otimes_{O_{Y\times S}} (M, F) = ((\Omega^\bullet_{Y\times S/S}, F_b) \otimes_{O_{Y\times S}} (D_{Y\times S}, F_b) \otimes_{D_{Y\times S}} (M, F)$$

are isomorphisms of filtered p^*D_S modules, that is isomorphism in $C_{p^*\mathcal{D}fil}(Y \times S)$.

Proof. (i) and (ii) are particular case of proposition 42.
(iii): follows immediately by definition of the p^*D_S module structure. $\square$

We now look at the functorialities of the relative De Rham complex of a smooth morphisms of smooth complex algebraic varieties :

Proposition 63. *Consider a commutative diagram in* SmVar($\mathbb{C}$) *or in* AnSm($\mathbb{C}$) *:*

$$D = \begin{tikzcd} Y \times S \arrow[r, "p"] & S \\ Y' \times T \arrow[u, "g''=(g''_0 \times g)"] \arrow[r, "p'"] & T \arrow[u, "g"'] \end{tikzcd}$$

with p *and* p' *the projections. For* $(M,F) \in C_{\mathcal{D}fil}(Y \times S)$ *the map in* $C_{g''^*p^*O_S fil}(Y' \times T)$

$$\Omega_{(Y' \times T/Y \times S)/(T/S)}(M,F) : g''^*((\Omega^\bullet_{Y \times S/S}, F_b) \otimes_{O_{Y \times S}} (M,F)) \to (\Omega^\bullet_{Y' \times T/T}, F_b) \otimes_{O_{Y' \times T}} g''^{*mod}(M,F)$$

given in definition-proposition 16 is a map in $C_{g''^*p^*\mathcal{D}fil}(Y' \times T)$*. Hence, for* $(M,F) \in C_{\mathcal{D}fil}(Y \times S)$*, the map in* $C_{O_T fil}(T)$ *(with* L_D *instead of* L_O*)*

$$T^O_\omega(D)(M) : g^{*mod} L_D(p_* E((\Omega^\bullet_{Y \times S/S}, F_b) \otimes_{O_{Y \times S}} (M,F))) \to p'_* E((\Omega^\bullet_{Y' \times T/T}, F_b) \otimes_{O_{Y' \times T}} g''^{*mod}(M,F)),$$

is a map in $C_{\mathcal{D}fil}(T)$.

Proof. Follows imediately by definition. □

In the analytic case, we also have

Proposition 64. *Consider a commutative diagram in* AnSm($\mathbb{C}$) *:*

$$D = \begin{tikzcd} Y \times S \arrow[r, "p"] & S \\ Y' \times T \arrow[u, "g''=(g''_0 \times g)"] \arrow[r, "p'"] & T \arrow[u, "g"'] \end{tikzcd}$$

with p *and* p' *the projections. For* $(M,F) \in C_{\mathcal{D}^\infty fil}(Y \times S)$ *the map in* $C_{g''^*p^*O_S fil}(Y' \times T)$

$$\Omega_{(Y' \times T/Y \times S)/(T/S)}(M,F) : g''^*((\Omega^\bullet_{Y \times S/S}, F_b) \otimes_{O_{Y \times S}} (M,F)) \to (\Omega^\bullet_{Y' \times T/T}, F_b) \otimes_{O_{Y' \times T}} g''^{*mod}(M,F)$$

is a map in $C_{g''^*p^*\mathcal{D}^\infty fil}(Y' \times T)$*. Hence, for* $(M,F) \in C_{\mathcal{D}^\infty fil}(Y \times S)$*, the map in* $C_{O_T fil}(T)$ *(with* L_D *instead of* L_O*)*

$$T^O_\omega(D)(M) : g^{*mod} L_D(p_* E((\Omega^\bullet_{Y \times S/S}, F_b) \otimes_{O_{Y \times S}} (M,F))) \to p'_* E((\Omega^\bullet_{Y' \times T/T}, F_b) \otimes_{O_{Y' \times T}} g''^{*mod}(M,F)),$$

is a map in $C_{\mathcal{D}^\infty fil}(T)$.

Proof. Follows immediately by definition. □

Similarly, we have

Proposition 65. *Let* $p : Y \times S \to S$ *a projection with* $Y, S \in$ SmVar($\mathbb{C}$)*. For* $(M,F) \in C_{\mathcal{D}fil}(Y \times S)$ *the map in* $C_{p^*O_{S^{an}}}(Y^{an} \times S^{an})$

$$\Omega_{(Y^{an} \times S^{an}/Y \times S)/(S^{an}/S)}(M,F) : \mathrm{an}(Y \times S)^*((\Omega^\bullet_{Y \times S/S}, F_b) \otimes_{O_{Y \times S}} (M,F)) \to (\Omega^\bullet_{Y^{an} \times S^{an}/S^{an}}, F_b) \otimes_{O^{an}_{Y \times S}} (M^{an}, F)$$

is a map in $C_{p^{\mathcal{D}} fil}(Y^{an} \times S^{an})$*. For* $(M,F) \in C_{\mathcal{D}fil}(Y \times S)$*, the map in* $C_{O_{S^{an}} fil}(S^{an})$

$$T^O_\omega(an, h)(M,F) : (p_* E((\Omega^\bullet_{Y \times S/S}, F_b) \otimes_{O_{Y \times S}} (M,F)))^{an} \to p_* E((\Omega^\bullet_{Y \times S/S}, F_b) \otimes_{O^{an}_{Y \times S}} (M,F)^{an})$$

is a map in $C_{\mathcal{D}fil}(S^{an})$.

Proof. Similar to the proof of proposition 63. □

Proposition 66. *Let $p : Y \times S \to S$ a projection with $Y, S \in \mathrm{SmVar}(\mathbb{C})$ or with $Y, S \in \mathrm{AnSm}(\mathbb{C})$.*

(i) If $\phi : (M, F) \to (N, F)$ is an r-filtered Zariski, resp. usu, local equivalence with $(M_1, F), (M_2, F) \in C_{\mathcal{D}fil}(Y \times S)$, then

$$DR(Y \times S/S)(\phi) : (\Omega^\bullet_{Y\times S/S}, F_b) \otimes_{O_{Y\times S}} (M, F) \to \Omega^\bullet_{Y\times S/S} \otimes_{O_{Y\times S}} (N, F)$$

is an r-filtered equivalence Zariski, resp. usu, local in $C_{p^*\mathcal{D}fil}(Y \times S)$.

(ii) Consider a commutative diagram in $\mathrm{SmVar}(\mathbb{C})$ or in $\mathrm{AnSm}(\mathbb{C})$

$$\begin{tikzcd} D = Y \times S \arrow[r, "p"] & S \\ V \arrow[u, "l"] \arrow[r, "k"] & S \arrow[u, "="'] \end{tikzcd} .$$

with p the projection. For $(N, F) \in C_{\mathcal{D},l^*\mathcal{D}fil}(V)$, the map in $C_{p^*O_S}(Y \times S)$ (see definition 52)

$$\begin{aligned} k \circ T^O_\omega(l, \otimes)(E(N, F)) : (\Omega^\bullet_{Y\times S/S}, F_b) \otimes_{O_{Y\times S}} l_*E(N, F) &\to l_*((\Omega^\bullet_{V/S}, F_b) \otimes_{O_V} E(N, F)) \\ &\to l_*E((\Omega^\bullet_{V/S}, F_b) \otimes_{O_V} E(N, F)) \end{aligned}$$

is a filtered equivalence Zariski, resp. usu, local in $C_{p^*\mathcal{D}fil}(Y \times S)$.

Proof. (i):Follows from proposition 60that it is a morphism of p^*D_S module. The fact that it is an equivalence Zariski, resp usu, local is a particular case of proposition 43(i).
(ii):Follows from proposition 60 and the first part of proposition 63 that it is a morphism of h^*D_S module. The fact that it is an equivalence Zariski, resp usu, local is a particular case of proposition 43(ii). □

In the analytical case, we also have

Proposition 67. *Let $p : Y \times S \to S$ a projection with $Y, S \in \mathrm{SmVar}(\mathbb{C})$ or with $Y, S \in \mathrm{AnSm}(\mathbb{C})$.*

(i) If $\phi : (M, F) \to (N, F)$ is an r-filtered usu local equivalence with $(M_1, F), (M_2, F) \in C_{\mathcal{D}^\infty fil}(Y \times S)$, then

$$DR(Y \times S/S)(\phi) : (\Omega^\bullet_{Y\times S/S}, F_b) \otimes_{O_{Y\times S}} (M, F) \to \Omega^\bullet_{Y\times S/S} \otimes_{O_{Y\times S}} (N, F)$$

is an r-filtered equivalence usu local in $C_{p^*\mathcal{D}^\infty fil}(Y \times S)$.

(ii) Consider a commutative diagram in $\mathrm{AnSm}(\mathbb{C})$

$$\begin{tikzcd} D = Y \times S \arrow[r, "p"] & S \\ V \arrow[u, "l"] \arrow[r, "k"] & S \arrow[u, "="'] \end{tikzcd} .$$

with p the projection. For $(N, F) \in C_{\mathcal{D}^\infty,l^*\mathcal{D}^\infty fil}(V)$, the map in $C_{p^*O_S}(Y \times S)$ (see definition 52)

$$\begin{aligned} k \circ T^O_\omega(l, \otimes)(E(N, F)) : (\Omega^\bullet_{Y\times S/S}, F_b) \otimes_{O_{Y\times S}} l_*E(N, F) &\to l_*((\Omega^\bullet_{V/S}, F_b) \otimes_{O_V} E(N, F)) \\ &\to l_*E((\Omega^\bullet_{V/S}, F_b) \otimes_{O_V} E(N, F)) \end{aligned}$$

is a filtered equivalence usu local in $C_{p^*\mathcal{D}^\infty fil}(Y \times S)$.

Proof. Follows from the finite order case : proposition 66. □

Dually of the De Rahm complex of a D_S module M, we have the Spencer complex of M. In the particular case of D_S, we have the following:

Proposition 68. *Let $S \in \mathrm{SmVar}(\mathbb{C})$ or $S \in \mathrm{AnSm}(\mathbb{C})$.*

- *We have the filtered resolutions of K_S by the following complex of locally free right D_S modules: $\omega(S) : \omega(K_S) := (\Omega^\bullet_S, F_b)[d_S] \otimes_{O_S} (D_S, F_b) \to (K_S, F_b)$ and $\omega(S) : \omega(K_S, F^{ord}) := (\Omega^\bullet_S, F_b)[d_S] \otimes_{O_S} (D_S, F^{ord}) \to (K_S, F^{ord})$*
- *Dually, we have the filtered resolution of O_S by the following complex of locally free (left) D_S modules: $\omega^\vee(S) : \omega(O_S) := (\wedge^\bullet T_S, F_b)[d_S] \otimes_{O_S} (D_S, F_b) \to (O_S, F_b)$ and $\omega^\vee(S) : \omega(O_S, F^{ord}) := (\wedge^\bullet T_S, F_b)[d_S] \otimes_{O_S} (D_S, F^{ord}) \to (O_S, F^{ord})$.*

Let $S_1, S_2 \in \mathrm{SmVar}(\mathbb{C})$ or $S_1, S_2 \in \mathrm{AnSm}(\mathbb{C})$. Consider the projection $p = p_1 : S_1 \times S_2 \to S_1$.

- *We have the filtered resolution of $D_{S_1 \times S_2 \to S_1}$ by the following complexes of (left) ($p^* D_{S_1}$ and right $D_{S_1 \times S_2}$) modules :*

$$\omega(S_1 \times S_2/S_1) : (\Omega^\bullet_{S_1 \times S_2/S_1}[d_{S_2}], F_b) \otimes_{O_{S_1 \times S_2}} (D_{S_1 \times S_2}, F^{ord}) \to (D_{S_1 \times S_2 \leftarrow S_1}, F^{ord}).$$

- *Dually, we have the filtered resolution of $D_{S_1 \times S_2 \to S_1}$ by the following complexes of (left) ($p^* D_{S_1}, D_{S_1 \times S_2}$) modules :*

$$\omega^\vee(S_1 \times S_2/S_1) : (\wedge^\bullet T_{S_1 \times S_2/S_1}[d_{S_2}], F_b) \otimes_{O_{S_1 \times S_2}} (D_{S_1 \times S_2}, F^{ord}) \to (D_{S_1 \times S_2 \to S_1}, F^{ord}),$$

Proof. See [16]. □

In the analytical case, we also have

Proposition 69. *Let $S \in \mathrm{AnSm}(\mathbb{C})$.*

- *We have the filtered resolutions of K_S by the following complex of locally free right D_S modules: $\omega(S) : \omega(K_S) := (\Omega^\bullet_S, F_b)[d_S] \otimes_{O_S} (D^\infty_S, F^{ord}) \to (K_S, F_b)$.*
- *Dually, we have the filtered resolution of O_S by the following complex of locally free (left) D_S modules: $\omega^\vee(S) : \omega(O_S) := (\wedge^\bullet T_S, F_b)[d_S] \otimes_{O_S} (D^\infty_S, F^{ord}) \to (O_S, F_b)$.*

Let $S_1, S_2 \in \mathrm{AnSm}(\mathbb{C})$. Consider the projection $p = p_1 : S_1 \times S_2 \to S_1$.

- *We have the filtered resolution of $D^\infty_{S_1 \times S_2 \to S_1}$ by the following complexes of (left) ($p^* D^\infty_{S_1}$ and right $D^\infty_{S_1 \times S_2}$) modules :*

$$\omega(S_1 \times S_2/S_1) : (\Omega^\bullet_{S_1 \times S_2/S_1}[d_{S_2}], F_b) \otimes_{O_{S_1 \times S_2}} (D^\infty_{S_1 \times S_2}, F^{ord}) \to (D^\infty_{S_1 \times S_2 \leftarrow S_1}, F^{ord}).$$

- *Dually, we have the filtered resolution of $D^\infty_{S_1 \times S_2 \to S_1}$ by the following complexes of (left) ($p^* D^\infty_{S_1}, D^\infty_{S_1 \times S_2}$) modules :*

$$\omega^\vee(S_1 \times S_2/S_1) : (\wedge^\bullet T_{S_1 \times S_2/S_1}[d_{S_2}], F_b) \otimes_{O_{S_1 \times S_2}} (D^\infty_{S_1 \times S_2}, F^{ord}) \to (D^\infty_{S_1 \times S_2 \to S_1}, F^{ord}),$$

Proof. Similar to the finite order case : the first map on the right is a surjection and the kernel are obtained by tensoring D^∞_S with the kernel of the kozcul resolution of K_S (note that D^∞_S is a locally free hence flat O_S module). □

Motivated by these resolutions, we make the following definition

Definition 61. *(i) Let $i : Z \hookrightarrow S$ be a closed embedding, with $Z, S \in \mathrm{SmVar}(\mathbb{C})$ or with $Z, S \in \mathrm{AnSm}(\mathbb{C})$. Then, for $(M, F) \in C_{\mathcal{D}fil}(Z)$, we set*

$$i_{*mod}(M, F) := i^0_{*mod}(M, F) := i_*((M, F) \otimes_{D_Z} (D_{Z \leftarrow S}, F^{ord})) \in C_{\mathcal{D}fil}(S)$$

(ii) *Let $S_1, S_2 \in \mathrm{SmVar}(\mathbb{C})$ or $S_1, S_2 \in \mathrm{AnSm}(\mathbb{C})$ and $p : S_1 \times S_2 \to S_1$ be the projection. Then, for $(M,F) \in C_{\mathcal{D}fil}(S_1 \times S_2)$, we set*

- $p^0_{*mod}(M,F) := p_*(DR(S_1 \times S_2/S_1)(M,F)) := p_*((\Omega^{\bullet}_{S_1\times S_2/S_1}, F_b) \otimes_{O_{S_1\times S_2}} (M,F))[d_{S_2}] \in C_{\mathcal{D}fil}(S_1)$,
- $p_{*mod}(M,F) := p_*E(DR(S_1 \times S_2/S_1)(M,F)) := p_*E((\Omega^{\bullet}_{S_1\times S_2/S_1}, F_b) \otimes_{O_{S_1\times S_2}} (M,F))[d_{S_2}] \in C_{\mathcal{D}fil}(S_1)$.

(iii) *Let $f : X \to S$ be a morphism, with $X, S \in \mathrm{SmVar}(\mathbb{C})$ or $X, S \in \mathrm{AnSm}(\mathbb{C})$. Consider the factorization $f : X \xrightarrow{i} X \times S \xrightarrow{p_S} S$, where i is the graph embedding and $p_S : X \times S \to S$ is the projection. Then, for $(M,F) \in C_{\mathcal{D}fil}(X)$ we set*

- $f^{FDR}_{*mod}(M,F) := p_{S*mod}i_{*mod}(M,F) \in C_{\mathcal{D}fil}(S)$,
- $\int^{FDR}_f (M,F) := f^{FDR}_{*mod}(M,F) := p_{S*mod}i_{*mod}(M,F) \in D_{\mathcal{D}fil,\infty}(S)$.

By proposition 70 below, we have $\int^{FDR}_f M = \int_f M \in D_{\mathcal{D}}(X)$.

(iii) *Let $f : X \to S$ be a morphism, with $X, S \in \mathrm{SmVar}(\mathbb{C})$ or $X, S \in \mathrm{AnSm}(\mathbb{C})$. Consider the factorization $f : X \xrightarrow{i} X \times S \xrightarrow{p_S} S$, where i is the graph embedding and $p_S : X \times S \to S$ is the projection. Then, for $(M,F) \in C_{\mathcal{D}fil}(X)$ we set*

- $f^{FDR}_{!mod}(M,F) := \mathbb{D}^K_S L_D f^{FDR}_{*mod} \mathbb{D}^K_S L_D(M,F) := \mathbb{D}^K_S L_D p_{S*mod} i_{*mod} \mathbb{D}^K_{X\times S} L_D(M,F) \in C_{\mathcal{D}fil}(S)$,
- $\int^{FDR}_{f!}(M,F) := f^{FDR}_{!mod}(M,F) := \mathbb{D}^K_S L_D p_{S*mod} i_{*mod} \mathbb{D}^K_{X\times S} L_D(M,F) \in D_{\mathcal{D}fil,\infty}(S)$.

In the analytical case we also consider :

Definition 62. *(i) Let $i : Z \hookrightarrow S$ be a closed embedding with $Z, S \in \mathrm{AnSm}(\mathbb{C})$. Then, for $(M,F) \in C_{\mathcal{D}^\infty fil}(Z)$, we set*

$$i_{*mod}(M,F) := i^0_{*mod}(M,F) := i_*((M,F) \otimes_{D^\infty_Z} (D^\infty_{Z\leftarrow S}, F^{ord})) \in C_{\mathcal{D}^\infty fil}(S)$$

(ii) *Let $S_1, S_2 \in \mathrm{AnSm}(\mathbb{C})$ and $p : S_1 \times S_2 \to S_1$ be the projection. For $(M,F) \in C_{\mathcal{D}^\infty fil}(S_1 \times S_2)$, we consider*

- $p^0_{*mod}(M,F) := p_*(DR(S_1 \times S_2/S_1)(M,F)) := p_*((\Omega^{\bullet}_{S_1\times S_2/S_1}, F_b) \otimes_{O_{S_1\times S_2}} (M,F))[d_{S_2}] \in C_{\mathcal{D}^\infty fil}(S_1)$,
- $p_{*mod}(M,F) := p_*E(DR(S_1 \times S_2/S_1)(M,F)) := p_*E((\Omega^{\bullet}_{S_1\times S_2/S_1}, F_b) \otimes_{O_{S_1\times S_2}} (M,F))[d_{S_2}] \in C_{\mathcal{D}^\infty fil}(S_1)$.

(iii) *Let $S_1, S_2 \in \mathrm{AnSm}(\mathbb{C})$ and $p : S_1 \times S_2 \to S_1$ be the projection. For $(M,F) \in C_{\mathcal{D}fil}(S_1 \times S_2)$ or $(M,F) \in C_{\mathcal{D}^\infty fil}(S_1 \times S_2)$, we set*

- $p^0_{!mod}(M,F) := p_!(DR(S_1 \times S_2/S_1)(M,F)) := p_!((\Omega^{\bullet}_{S_1\times S_2/S_1}, F_b) \otimes_{O_{S_1\times S_2}} (M,F))[d_{S_2}] \in C_{\mathcal{D}fil}(S_1)$,
- $p_{!mod}(M,F) := p_!E(DR(S_1 \times S_2/S_1)(M,F)) := p_!E((\Omega^{\bullet}_{S_1\times S_2/S_1}, F_b) \otimes_{O_{S_1\times S_2}} (M,F))[d_{S_2}] \in C_{\mathcal{D}fil}(S_1)$.

(iv) *Let $f : X \to S$ be a morphism, with $X, S \in \mathrm{AnSm}(\mathbb{C})$. Consider the factorization $f : X \xrightarrow{i} X \times S \xrightarrow{p_S} S$, where i is the graph embedding and $p_S : X \times S \to S$ is the projection. Then, for $(M,F) \in C_{\mathcal{D}^\infty fil}(X)$ we set*

- $f^{FDR}_{*mod}(M,F) := p_{S*mod}i_{*mod}(M,F) \in C_{\mathcal{D}^\infty fil}(S)$,

- $\int_f^{FDR}(M,F) := f_{*mod}^{FDR}(M,F) := p_{S*mod}i_{*mod}(M,F) \in D_{\mathcal{D}^\infty fil,\infty}(S)$,
- $f_{!mod}^{FDR}(M,F) := p_{S!mod}i_{*mod}(M,F) \in C_{\mathcal{D}^\infty fil}(S)$,
- $\int_{f!}^{FDR}(M,F) := f_{!*mod}^{FDR}(M,F) := p_{S!mod}i_{*mod}(M,F) \in D_{\mathcal{D}^\infty fil,\infty}(S)$.

By proposition 71 below, we have $\int_{f!}^{FDR} M = \int_{f!} M \in D_{\mathcal{D}^\infty}(X)$ *and* $\int_f^{FDR} M = \int_f M \in D_{\mathcal{D}^\infty}(X)$.

(v) Let $f : X \to S$ *be a morphism, with* $X, S \in \mathrm{AnSm}(\mathbb{C})$. *Consider the factorization* $f : X \xrightarrow{i} X \times S \xrightarrow{p_S} S$, *where* i *is the graph embedding and* $p_S : X \times S \to S$ *is the projection. Then, for* $(M,F) \in C_{\mathcal{D}fil}(X)$ *we set*

- $f_{!mod}^{FDR}(M,F) := p_{S!mod}i_{*mod}(M,F) \in C_{\mathcal{D}fil}(S)$,
- $\int_{f!}^{FDR}(M,F) := f_{!*mod}^{FDR}(M,F) := p_{S!mod}i_{*mod}(M,F) \in D_{\mathcal{D}fil,\infty}(S)$.

By proposition 71 below, we have $\int_{f!}^{FDR} M = \int_{f!} M \in D_{\mathcal{D}}(X)$.

Proposition 70. *(i) Let* $i : Z \hookrightarrow S$ *a closed embedding with* $S, Z \in \mathrm{SmVar}(\mathbb{C})$ *or with* $S, Z \in \mathrm{AnSm}(\mathbb{C})$. *Then for* $(M,F) \in C_{\mathcal{D}fil}(Z)$, *we have*

$$\int_i (M,F) := Ri_*((M,F) \otimes^L_{D_Z} (D_{Z\leftarrow S}, F^{ord}) = i_*((M,F) \otimes_{D_Z} (D_{Z\leftarrow S}, F^{ord})) = i_{*mod}(M,F).$$

(ii) Let $S_1, S_2 \in \mathrm{SmVar}(\mathbb{C})$ *or* $S_1, S_2 \in \mathrm{AnSm}(\mathbb{C})$ *and* $p : S_{12} := S_1 \times S_2 \to S_1$ *be the projection. Then, for* $(M,F) \in C_{\mathcal{D}fil}(S_1 \times S_2)$ *we have*

$$\begin{aligned}\int_p (M,F) : &= Rp_*((M,F) \otimes^L_{D_{S_1\times S_2}} (D_{S_1\times S_2 \leftarrow S_1}, F^{ord})) \\ &= p_* E((\Omega^\bullet_{S_1\times S_2/S_1}, F_b) \otimes_{O_{S_1\times S_2}} (D_{S_1\times S_2}, F^{ord}) \otimes_{D_{S_1\times S_2}} (M,F))[d_{S_2}] \\ &= p_* E((\Omega^\bullet_{S_1\times S_2/S_1}, F_b) \otimes_{O_{S_1\times S_2}} (M,F))[d_{S_2}] =: p_{*mod}(M,F).\end{aligned}$$

where the second equality follows from Griffitz transversality (the canonical isomorphism map respect by definition the filtration).

(iii) Let $f : X \to S$ *be a morphism with* $X, S \in \mathrm{SmVar}(\mathbb{C})$ *or with* $X, S \in \mathrm{AnSm}(\mathbb{C})$. *Then for* $M \in C_{\mathcal{D}}(X)$, *we have* $\int_f^{FDR} M = \int_f M$.

Proof. (i):Follows from the fact that $D_{Z\leftarrow S}$ is a locally free D_Z module and that i_* is an exact functor. (ii): Since $\Omega^\bullet_{S_{12}/S_1}[d_{S_2}], F_b) \otimes_{O_{S_{12}}} D_{S_{12}}$ is a complex of locally free $D_{S_1\times S_2}$ modules, we have in $D_{fil}(S_1 \times S_2)$, using proposition 68,

$$(D_{S_1\times S_2\leftarrow S_1}, F^{ord}) \otimes^L_{D_{S_1\times S_2}} (M,F) = (\Omega^\bullet_{S_{12}/S_1}[d_{S_2}], F_b) \otimes_{O_{S_{12}}} (D_{S_{12}}, F^{ord}) \otimes_{D_{S_{12}}} (M,F).$$

(iii): Follows from (i) and (ii) by proposition 49 (ii) in the algebraic case and by proposition 52(ii) in the analytic case since a closed embedding is proper.

□

In the analytical case, we also have :

Proposition 71. *(i) Let* $i : Z \hookrightarrow S$ *a closed embedding with* $S, Z \in \mathrm{AnSm}(\mathbb{C})$. *Then for* $(M,F) \in C_{\mathcal{D}^\infty fil}(Z)$, *we have* $\int_i (M,F) = i_{*mod}(M,F)$.

(ii) Let $S_1, S_2 \in \mathrm{AnSm}(\mathbb{C})$ and $p : S_{12} := S_1 \times S_2 \to S_1$ be the projection. Then, for $(M,F) \in C_{\mathcal{D}^\infty fil}(S_1 \times S_2)$ we have

$$\begin{aligned}\int_p (M,F): &= Rp_*((M,F) \otimes^L_{D^\infty_{S_1\times S_2}} (D^\infty_{S_1\times S_2\leftarrow S_1}, F^{ord})) \\ &= p_*E((\Omega^\bullet_{S_1\times S_2/S_1}, F_b) \otimes_{O_{S_1\times S_2}} (D_{S_1\times S_2}, F^{ord}) \otimes_{D_{S_1\times S_2}} (M,F)[d_{S_2}]) \\ &= p_*E((\Omega^\bullet_{S_1\times S_2/S_1}, F_b) \otimes_{O_{S_1\times S_2}} (M,F)[d_{S_2}]) =: p_{*mod}(M,F).\end{aligned}$$

(ii)' Let $S_1, S_2 \in \mathrm{AnSm}(\mathbb{C})$ and $p : S_{12} := S_1 \times S_2 \to S_1$ be the projection. Then, for $(M,F) \in C_{\mathcal{D}fil}(S_1 \times S_2)$ or $(M,F) \in C_{\mathcal{D}fil}(S_1 \times S_2)$, we have

$$\begin{aligned}\int_{p!} (M,F): &= Rp_!((M,F) \otimes^L_{D_{S_1\times S_2}} (D_{S_1\times S_2\leftarrow S_1}, F^{ord}) \\ &= p_!E((\Omega^\bullet_{S_1\times S_2/S_1}, F_b) \otimes_{O_{S_1\times S_2}} (D_{S_1\times S_2}, F^{ord}) \otimes_{D_{S_1\times S_2}} (M,F)[d_{S_2}]) \\ &= p_!E((\Omega^\bullet_{S_1\times S_2/S_1}, F_b) \otimes_{O_{S_1\times S_2}} (M,F)[d_{S_2}]) =: p_{!mod}(M,F).\end{aligned}$$

(iii) Let $f : X \to S$ be a morphism with $X, S \in \mathrm{AnSm}(\mathbb{C})$. For $M \in C_{\mathcal{D}^\infty}(X)$, we have $\int_f^{FDR} M = \int_f M$ and $\int_{f!}^{FDR} M = \int_{f!} M$. For $M \in C_{\mathcal{D}}(X)$, we have $\int_{f!}^{FDR} M = \int_{f!} M$.

Proof. (i):Follows from the fact that $D^\infty_{Z\leftarrow S}$ is a locally free D^∞_Z module and that i_* is an exact functor.
(ii): Similar to the proof of proposition 70(ii):follows from proposition 69.
(ii)': Similar to the proof of proposition 70(ii):follows from proposition 69.
(iii):The first assertion follows from (i), (ii) and (ii)' by proposition 52. The second one follows from proposition 70(i) and (ii)' and by proposition 52.

□

Proposition 72. *Let $f_1 : X \to Y$ and $f_2 : Y \to S$ two morphism with $X, Y, S \in \mathrm{SmVar}(\mathbb{C})$.*

(i) Let $(M,F) \in C_{\mathcal{D}(2)fil}(X)$. Then $\int_{f_2\circ f_1}^{FDR}(M,F) = \int_{f_1}^{FDR}\int_{f_2}^{FDR}(M,F) \in D_{\mathcal{D}(2)fil,\infty}(S)$.

(ii) Let $(M,F) \in C_{\mathcal{D}(2)fil,h}(X)$. Then $\int_{(f_2\circ f_1)!}^{FDR}(M,F) = \int_{f_1!}^{FDR}\int_{f_2!}^{FDR}(M,F) \in D_{\mathcal{D}(2)fil,\infty}(S)$.

Proof. See [21]. □

Proposition 73. *Let $f_1 : X \to Y$ and $f_2 : Y \to S$ two morphism with $X, Y, S \in \mathrm{AnSm}(\mathbb{C})$.*

(i) Let $(M,F) \in C_{\mathcal{D}^\infty(2)fil}(X)$. Then $\int_{(f_2\circ f_1)!}^{FDR}(M,F) = \int_{f_1!}^{FDR}\int_{f_2!}^{FDR}(M,F)$.

(ii) Let $(M,F) \in C_{\mathcal{D}^\infty(2)fil,h}(X)$. Then $\int_{f_2\circ f_1}^{FDR}(M,F) = \int_{f_1}^{FDR}\int_{f_2}^{FDR}(M,F)$.

Proof. Similar to proposition 72. □

Definition 63. *(i) Let $f : X \to S$ be a morphism, with $X, S \in \mathrm{SmVar}(\mathbb{C})$ or $X, S \in \mathrm{AnSm}(\mathbb{C})$. Consider the graph factorization $f : X \xrightarrow{l} X \times S \xrightarrow{p} S$, with l the graph embedding and p the projection. We have the transformation map given by, for $(M,F) \in C_{\mathcal{D}fil}(X)$,*

$$T(\int_f^{FDR}, \int_f)(M,F) : \int_f^{FDR}(M,F) := \int_p\int_l (M,F) \xrightarrow{T(\int_p\circ\int_l, \int_{p\circ l})(M,F)} \int_f (M,F)$$

(ii) Let $j : S^o \hookrightarrow S$ an open embedding with $S \in \mathrm{Var}(\mathbb{C})$. Consider the graph factorization $j : S^o \xrightarrow{l} S^o \times S \xrightarrow{p} S$, with l the graph embedding and p the projection. We have, for $(M,F) \in C_{\mathcal{D}fil}(S^o)$, the canonical map in $C_{\mathcal{D}fil}(S)$,

$$T(j^{FDR}_{*mod}, j_*)(M,F) : j^{FDR}_{*mod}(M,F) := p_*E((\Omega^\bullet_{S^o\times S/S}, F_b) \otimes_{O_{S^o\times S}} l_{*mod}(M,F)) \xrightarrow{k\circ\omega(S^o\times S/S)}$$
$$p_*E((D_{S^o\times S\leftarrow S}, F^{ord}) \otimes_{D_{S^o\times S}} l_*(D_{S^o\leftarrow S^o\times S} \otimes_{D_{S^o}} E(M,F)) \xrightarrow{T(l,\otimes)(-,-)} j_*E(M,F)$$

We have, for $(M,F) \in C_{\mathcal{D}fil}(S)$, the canonical map in $C_{\mathcal{D}fil}(S)$,

$$\mathrm{ad}(j^*, j^{FDR}_{*mod})(M,F) : (M,F) \xrightarrow{\mathrm{ad}(p^{*mod},p_*)(M,F)} p_*E((\Omega^\bullet_{S^o\times S/S}, F_b) \otimes_{O_{S^o\times S}} p^{*mod}(M,F))$$

4.2.3 The support section functors for D modules and the graph inverse image

Let $S \in \mathrm{SmVar}(\mathbb{C})$ or $S \in \mathrm{AnSm}(\mathbb{C})$. Let $i : Z \hookrightarrow S$ a closed embedding and denote $j : S\backslash Z \hookrightarrow S$ the complementary open embedding. More generally, let $h : Y \to S$ a morphism with $Y, S \in \mathrm{Var}(\mathbb{C})$ or $Y, S \in \mathrm{AnSp}(\mathbb{C})$, S smooth, and let $i : X \hookrightarrow Y$ a closed embedding and denote by $j : Y\backslash X \hookrightarrow Y$ the open complementary. We then get from section 2 the following functors :

- We get the functor
$$\Gamma_Z : C_{\mathcal{D}(2)fil}(S) \to C_{\mathcal{D}(2)fil}(S),$$
$$(M,F) \mapsto \Gamma_Z(M,F) := \mathrm{Cone}(\mathrm{ad}(j^*, j_*)((M,F)) : (M,F) \to j_*j^*(M,F))[-1],$$
together we the canonical map $\gamma_Z(M,F) : \Gamma_Z(M,F) \to (M,F)$, and more generally the functor
$$\Gamma_X : C_{h^*\mathcal{D}(2)fil}(Y) \to C_{h^*\mathcal{D}(2)fil}(Y),$$
$$(M,F) \mapsto \Gamma_X(M,F) := \mathrm{Cone}(\mathrm{ad}(j^*, j_*)((M,F)) : (M,F) \to j_*j^*(M,F))[-1],$$
together we the canonical map $\gamma_X(M,F) : \Gamma_X(M,F) \to (M,F)$.
- We get the functor
$$\Gamma^\vee_Z : C_{\mathcal{D}(2)fil}(S) \to C_{\mathcal{D}(2)fil}(S),$$
$$(M,F) \mapsto \Gamma^\vee_Z(M,F) := \mathrm{Cone}(\mathrm{ad}(j_!, j^*)((M,F)) : j_!j^*(M,F) \to (M,F)),$$
together we the canonical map $\gamma^\vee_Z(M,F) : (M,F) \to \Gamma^\vee_Z(M,F)$, and more generally the functor
$$\Gamma^\vee_X : C_{h^*\mathcal{D}(2)fil}(Y) \to C_{h^*\mathcal{D}(2)fil}(Y),$$
$$(M,F) \mapsto \Gamma^\vee_X(M,F) := \mathrm{Cone}(\mathrm{ad}(j_!, j^*)((M,F)) : j_!j^*(M,F) \to (M,F)),$$
together we the canonical map $\gamma^\vee_X(M,F) : (M,F) \to \Gamma^\vee_X(M,F)$.
- We get the functor
$$\Gamma^{\vee,h}_Z : C_{\mathcal{D}(2)fil}(S) \to C_{\mathcal{D}(2)fil}(S),\ (M,F) \mapsto \Gamma^{\vee,h}_Z(M,F) := \mathbb{D}^K_S L_D \Gamma_Z E(\mathbb{D}^K_S(M,F)),$$
together with the factorization
$$\gamma^{\vee,h}_Z(L_D(M,F)) : L_D(M,F) \xrightarrow{\gamma^\vee_Z(L_D(M,F))} \Gamma^\vee_Z L_D(M,F) \xrightarrow{k\circ\mathbb{D}^K I(j_!,j^*)(-)\circ d(-)} \Gamma^{\vee,h}_Z L_D(M,F),$$
and more generally the functor
$$\Gamma^{\vee,h}_X : C_{h^*\mathcal{D}(2)fil}(Y) \to C_{h^*\mathcal{D}(2)fil}(Y),\ (M,F) \mapsto \Gamma^{\vee,h}_X(M,F) := \mathbb{D}^{h^*D,K}_Y L_{h^*D} \Gamma_X E(\mathbb{D}^{h^*D,K}_Y(M,F)),$$
together with the factorization
$$\gamma^{\vee,h}_X(L_{h^*D}(M,F)) : L_{h^*D}(M,F) \xrightarrow{\gamma^\vee_X(L_{h^*D}(M,F))} \Gamma^\vee_X L_{h^*D}(M,F)$$
$$\xrightarrow{k\circ\mathbb{D}^{h^*D,K} I(j_!,j^*)(-)\circ d(-)} \Gamma^{\vee,h}_X L_{h^*D}(M,F).$$

- We get the functor

$$\Gamma^{\vee,O}_Z : C_{\mathcal{D}(2)fil}(S) \to C_{\mathcal{D}(2)fil}(S),$$
$$(M,F) \mapsto \Gamma^{\vee,O}_Z(M,F) := \mathrm{Cone}(b_Z((M,F)) : \mathcal{I}_Z(M,F) \to (M,F)),$$

together we the factorization

$$\gamma^{\vee,O}_Z(M,F) : (M,F) \xrightarrow{\gamma^{\vee}_Z(M,F)} \Gamma^{\vee}_Z(M,F) \xrightarrow{b_{S/Z}(M,F)} \Gamma^{\vee,O}_Z(M,F).$$

Since $M \mapsto M/\mathcal{I}_Z M$ is a right exact functor, $M \mapsto \Gamma^{\vee,O}_Z M$ send Zariski, resp. usu, local equivalence between projective complexes of presheaves to Zariski, resp. usu local equivalence, and thus induces in the derived category

$$L\Gamma^{\vee,O}_Z : D_{\mathcal{D}fil,\infty}(S) \to D_{\mathcal{D}fil,\infty}(S),$$
$$(M,F) \mapsto \Gamma^{\vee,O}_Z L_D(M,F) := \mathrm{Cone}(b_Z(L_D(M,F)) : \mathcal{I}_Z L_D(M,F) \to L_D(M,F)).$$

- We get the functor

$$\Gamma^{O}_Z : C_{\mathcal{D}(2)fil}(S) \to C_{\mathcal{D}(2)fil}(S),$$
$$(M,F) \mapsto \Gamma^{O}_Z(M,F) := \mathrm{Cone}(b'_Z((M,F)) : (M,F) \to (M,F) \otimes_{O_S} \mathbb{D}^K_S(\mathcal{I}_Z D_S)),$$

together we the factorization

$$\gamma^{O}_Z(M,F) : (M,F)\Gamma^{O}_Z \xrightarrow{b'_{S/Z}(M,F)} \Gamma_Z(M,F) \xrightarrow{\gamma_Z(M,F)} (M,F).$$

- We have, for $(M,F) \in C_{\mathcal{D}fil}(S)$, a canonical isomorphism

$$I(D,\gamma^O)(M,F) : \mathbb{D}^K_S \Gamma^{\vee,O}_Z(M,F) \xrightarrow{\sim} \Gamma^{O}_Z \mathbb{D}^K_S(M,F)$$

which gives the transformation map in $C_{\mathcal{D}fil}(S)$

$$T(D,\gamma^O)(M,F) : \Gamma^{\vee,O}_Z \mathbb{D}^K_S(M,F) \to \mathbb{D}^K_S \Gamma^{O}_Z(M,F)$$

Let $S \in \mathrm{AnSm}(\mathbb{C})$. Let $i : Z \hookrightarrow S$ a closed embedding and denote $j : S\backslash Z \hookrightarrow S$ the complementary open embedding. More generally, let $h : Y \to S$ a morphism with $Y, S \in \mathrm{AnSp}(\mathbb{C})$, S smooth, and let $i : X \hookrightarrow Y$ a closed embedding and denote by $j : Y\backslash X \hookrightarrow Y$ the open complementary.

- We get the functor

$$\Gamma_Z : C_{\mathcal{D}^\infty(2)fil}(S) \to C_{\mathcal{D}^\infty(2)fil}(S),$$
$$(M,F) \mapsto \Gamma_Z(M,F) := \mathrm{Cone}(\mathrm{ad}(j^*, j_*)((M,F)) : (M,F) \to j_* j^*(M,F))[-1],$$

together we the canonical map $\gamma_Z(M,F) : \Gamma_Z(M,F) \to (M,F)$, and more generally the functor

$$\Gamma_X : C_{h^*\mathcal{D}^\infty(2)fil}(Y) \to C_{h^*\mathcal{D}^\infty(2)fil}(Y),$$
$$(M,F) \mapsto \Gamma_X(M,F) := \mathrm{Cone}(\mathrm{ad}(j^*, j_*)((M,F)) : (M,F) \to j_* j^*(M,F))[-1],$$

together we the canonical map $\gamma_X(M,F) : \Gamma_X(M,F) \to (M,F)$.

- We get the functor

$$\Gamma^{\vee}_Z : C_{\mathcal{D}^\infty(2)fil}(S) \to C_{\mathcal{D}^\infty(2)fil}(S),$$
$$(M,F) \mapsto \Gamma^{\vee}_Z(M,F) := \mathrm{Cone}(\mathrm{ad}(j_!, j^*)((M,F)) : j_! j^*(M,F) \to (M,F)),$$

together we the canonical map $\gamma^{\vee}_Z(M,F) : (M,F) \to \Gamma^{\vee}_Z(M,F)$, and more generally the functor

$$\Gamma^{\vee}_X : C_{h^*\mathcal{D}^\infty(2)fil}(Y) \to C_{h^*\mathcal{D}^\infty(2)fil}(Y),$$
$$(M,F) \mapsto \Gamma^{\vee}_X(M,F) := \mathrm{Cone}(\mathrm{ad}(j_!, j^*)((M,F)) : j_! j^*(M,F) \to (M,F)),$$

together we the canonical map $\gamma^{\vee}_X(M,F) : (M,F) \to \Gamma^{\vee}_X(M,F)$.

- We get the functor

$\Gamma_Z^{\vee,h} : C_{\mathcal{D}^\infty(2)fil}(S) \to C_{\mathcal{D}^\infty(2)fil}(S),\ (M,F) \mapsto \Gamma_Z^{\vee,h}(M,F) := \mathbb{D}_S^{\infty,K} L_{D^\infty}\Gamma_Z E(\mathbb{D}_S^{\infty,K}(M,F)),$

together with the factorization

$\gamma_Z^{\vee,h}(L_{D^\infty}(M,F)) : L_{D^\infty}(M,F) \xrightarrow{\gamma_Z^\vee(L_{D^\infty}(M,F))} \Gamma_Z^\vee L_{D^\infty}(M,F) \xrightarrow{k\circ\mathbb{D}^\infty I(j_!,j^*)(-)\circ d(-)} \Gamma_Z^{\vee,h} L_{D^\infty}(M,F),$

and more generally the functor

$\Gamma_X^{\vee,h} : C_{h^*\mathcal{D}^\infty(2)fil}(Y) \to C_{h^*\mathcal{D}^\infty(2)fil}(Y),\ (M,F) \mapsto \Gamma_X^{\vee,h}(M,F) := \mathbb{D}_Y^{h^*\infty,K} L_{h^*D^\infty}\Gamma_X E(\mathbb{D}_Y^{h^*\infty,K}(M,F)),$

together with the factorization

$$\gamma_X^{\vee,h}(L_{h^*D^\infty}(M,F)) : L_{h^*D^\infty}(M,F) \xrightarrow{\gamma_X^\vee(L_{h^*D^\infty}(M,F))} \Gamma_X^\vee L_{h^*D^\infty}(M,F) \xrightarrow{k\circ\mathbb{D}^{h^*D^\infty,K} I(j_!,j^*)(-)\circ d(-)} \Gamma_X^{\vee,h} L_{h^*D^\infty}(M,F).$$

- We get the functor

$$\Gamma_Z^{\vee,O} : C_{\mathcal{D}^\infty(2)fil}(S) \to C_{\mathcal{D}^\infty(2)fil}(S),$$
$$(M,F) \mapsto \Gamma_Z^{\vee,O}(M,F) := \mathrm{Cone}(b_Z((M,F)) : \mathcal{I}_Z(M,F) \to (M,F)),$$

together we the factorization

$$\gamma_Z^{\vee,O}(M,F) : (M,F) \xrightarrow{\gamma_Z^\vee(M,F)} \Gamma_Z^\vee(M,F) \xrightarrow{b_{S/Z}(M,F)} \Gamma_Z^{\vee,O}(M,F).$$

- We get the functor

$$\Gamma_Z^O : C_{\mathcal{D}^\infty(2)fil}(S) \to C_{\mathcal{D}^\infty(2)fil}(S),$$
$$(M,F) \mapsto \Gamma_Z^O(M,F) := \mathrm{Cone}(b'_Z((M,F)) : (M,F) \to (M,F) \otimes_{O_S} \mathbb{D}_S^K(\mathcal{I}_Z D_S)),$$

together we the factorization

$$\gamma_Z^O(M,F) : (M,F)\Gamma_Z^O \xrightarrow{b'_{S/Z}(M,F)} \Gamma_Z(M,F) \xrightarrow{\gamma_Z(M,F)} (M,F).$$

- We have, for $(M,F) \in C_{\mathcal{D}^\infty fil}(S)$, a canonical isomorphism

$$I(D,\gamma^O)(M,F) : \mathbb{D}_S^{K,\infty}\Gamma_Z^{\vee,O}(M,F) \xrightarrow{\sim} \Gamma_Z^O\mathbb{D}_S^{K,\infty}(M,F)$$

which gives the transformation map in $C_{\mathcal{D}^\infty fil}(S)$

$$T(D,\gamma^O)(M,F) : \Gamma_Z^{\vee,O}\mathbb{D}_S^{\infty,K}(M,F) \to \mathbb{D}_S^{\infty,K}\Gamma_Z^O(M,F)$$

In the analytic case, we have

Definition 64. *Let $S \in \mathrm{AnSm}(\mathbb{C})$. For $(M,F) \in C_{\mathcal{D}fil}(S)$, we have the map in $C_{\mathcal{D}^\infty fil}(S)$*

$$T(\infty,\gamma)(M,F) := (I, T(j,\otimes)(-,-)) :$$
$$J_S(\Gamma_Z(M,F)) := \Gamma_Z(M,F) \otimes_{D_S} (D_S^\infty, F^{ord}) \to \Gamma_Z((M,F) \otimes_{D_S} (D_S^\infty, F^{ord})) =: \Gamma_Z J_S(M,F)$$

Let $i : Z \hookrightarrow S$ a closed embedding, with $Z,S \in \mathrm{SmVar}(\mathbb{C})$ or $Z,S \in \mathrm{AnSm}(\mathbb{C})$. We have the functor

$$i^\sharp : C_{\mathcal{D}fil}(S) \to C_{\mathcal{D}fil}(Z), (M,F) \mapsto i^\sharp(M,F) := \mathcal{H}om_{i^*D_S}((D_{S\leftarrow Z}, F^{ord}), i^*(M,F))$$

where the (left) D_Z module structure on $i^\sharp M$ comes from the right module structure on $D_{S\leftarrow Z}$, resp. O_Z. We denote by

- for $(M,F)\in C_{\mathcal{D}fil}(S)$, the canonical map in $C_{\mathcal{D}fil}(S)$

$$\begin{aligned}\mathrm{ad}(i_{*mod},i^\sharp)(M,F) : i_{*mod}i^\sharp(M,F) := i_*(\mathcal{H}om_{i^*D_S}((D_{S\leftarrow Z},F^{ord}),i^*(M,F))\otimes_{D_Z}(D_{S\leftarrow Z},F^{ord})) \\ \to (M,F), \phi\otimes P\mapsto \phi(P)\end{aligned}$$

- for $(N,F)\in C_{\mathcal{D}fil}(Z)$, the canonical map in $C_{\mathcal{D}fil}(Z)$

$$\begin{aligned}\mathrm{ad}(i_{*mod},i^\sharp)(N,F) : (N,F)\to i^\sharp i_{*mod}(N,F) := \mathcal{H}om_{i^*D_S}(D_{S\leftarrow Z},i^*i_*((N,F)\otimes_{D_Z}(D_{S\leftarrow Z},F^{ord}))) \\ n\mapsto (P\mapsto n\otimes P)\end{aligned}$$

The functor $i^\sharp$ induces in the derived category the functor :

$$Ri^\sharp : D_{\mathcal{D}(2)fil,r}(S)\to D_{\mathcal{D}(2)fil,r}(Z),$$
$$(M,F)\mapsto Ri^\sharp(M,F) := R\mathcal{H}om_{i^*D_S}((D_{Z\leftarrow S},F^{ord}),i^*(M,F)) = \mathcal{H}om_{i^*D_S}((D_{Z\leftarrow S},F^{ord}),E(i^*(M,F))).$$

Proposition 74. *Let $i:Z\hookrightarrow S$ a closed embedding, with $Z,S\in\mathrm{SmVar}(\mathbb{C})$ or $Z,S\in\mathrm{AnSm}(\mathbb{C})$. The functor $i_{*mod}:C_{\mathcal{D}}(Z)\to C_{\mathcal{D}}(S)$ admit a right adjoint which is the functor $i^\sharp:C_{\mathcal{D}}(S)\to C_{\mathcal{D}}(Z)$ and*

$$\mathrm{ad}(i_{*mod},i^\sharp)(N) : N\to i^\sharp i_{*mod}N \ \text{ and } \ \mathrm{ad}(i_{*mod},i^\sharp)(M) : i_{*mod}i^\sharp M\to M$$

are the adjonction maps.

Proof. See [16] for the algebraic case. The analytic case is completely analogue. □

One of the main results in D modules is Kashiwara equivalence :

Theorem 20. *(i) Let $i:Z\hookrightarrow S$ a closed embedding with $Z,S\in\mathrm{SmVar}(\mathbb{C})$.*

- *The functor $i_{*mod}:\mathcal{Q}Coh_{\mathcal{D}}(Z)\to\mathcal{Q}Coh_{\mathcal{D}}(S)$ is an equivalence of category whose inverse is $i^\sharp := a_\tau i^\sharp : \mathcal{Q}Coh_{\mathcal{D}}(S)\to\mathcal{Q}Coh_{\mathcal{D}}(Z)$. That is, for $M\in QCoh_{\mathcal{D}}(S)$ and $N\in QCoh_{\mathcal{D}}(Z)$, the adjonction maps*

$$\mathrm{ad}(i_{*mod},i^\sharp)(M) : i_{*mod}i^\sharp M\xrightarrow{\sim}M\ ,\ \mathrm{ad}(i_{*mod},i^\sharp)(N) : i^\sharp i_{*mod}N\xrightarrow{\sim}N$$

  are isomorphisms.
- *The functor $\int_i = i_{*mod} : D_{\mathcal{D}}(Z)\to D_{\mathcal{D}}(S)$ is an equivalence of category whose inverse is $Ri^\sharp : D_{\mathcal{D}}(S)\to D_{\mathcal{D}}(Z)$. That is, for $M\in D_{\mathcal{D}}(S)$ and $N\in D_{\mathcal{D}}(Z)$, the adjonction maps*

$$\mathrm{ad}(\int_i,Ri^\sharp)(M) : \int_i Ri^\sharp M\xrightarrow{\sim}M\ ,\ \mathrm{ad}(\int_i,Ri^\sharp)(N) : Ri^\sharp\int_i N\xrightarrow{\sim}N$$

  are isomorphisms.

(ii) Let $i:Z\hookrightarrow S$ a closed embedding with $Z,S\in\mathrm{AnSm}(\mathbb{C})$.

- *The functor $i_{*mod}:\mathcal{C}oh_{\mathcal{D}}(Z)\to\mathcal{C}oh_{\mathcal{D}}(S)$ is an equivalence of category whose inverse is $i^\sharp := a_\tau i^\sharp : \mathcal{C}oh_{\mathcal{D}}(S)\to\mathcal{C}oh_{\mathcal{D}}(Z)$. That is, for $M\in Coh_{\mathcal{D}}(S)$ and $N\in Coh_{\mathcal{D}}(Z)$, the adjonction maps*

$$\mathrm{ad}(i_{*mod},i^\sharp)(M) : i_{*mod}i^\sharp M\xrightarrow{\sim}M\ ,\ \mathrm{ad}(i_{*mod},i^\sharp)(N) : i^\sharp i_{*mod}N\xrightarrow{\sim}N$$

  are isomorphisms.
- *The functor $\int_i = i_{*mod} : D_{\mathcal{D},c}(Z)\to D_{\mathcal{D},c}(S)$ is an equivalence of category whose inverse is $Ri^\sharp : D_{\mathcal{D},c}(S)\to D_{\mathcal{D},c}(Z)$. That is, for $M\in D_{\mathcal{D},c}(S)$ and $N\in D_{\mathcal{D},c}(Z)$, the adjonction maps*

$$\mathrm{ad}(\int_i,Ri^\sharp)(M) : \int_i Ri^\sharp M\xrightarrow{\sim}M\ ,\ \mathrm{ad}(\int_i,Ri^\sharp)(N) : Ri^\sharp\int_i N\xrightarrow{\sim}N$$

  are isomorphisms.

Proof. (i):Standard. Note that the second point follows from the first.
(ii):Standard. Note that the second point follows from the first. □

We have a canonical embedding of rings $D_Z \hookrightarrow D_{Z\to S} := i^*D_S \otimes_{i^*O_S} O_Z$. We denote by $C_{i^*\mathcal{D},Z}(Z)$ the category whose objects are complexes of presheaves M of i^*D_S modules on Z such that the cohomology presheaves H^nM have an induced structure of D_Z modules. We denote by

$$q_K : K_{O_S}(i_*O_Z) \to i_*O_Z$$

the Kozcul complex which is a resolution of the O_S module i_*O_Z of lenght $c = \mathrm{codim}(Z,S)$ by locally free sheaves of finite rank. The fact that it is a locally free resolution of finite rank comes from the fact that Z is a locally complete intersection in S since both Z and S are smooth. We denote again

$$q_K = i^*q_K : K_{i^*O_S}(O_Z) := i^*K_{O_S}(i_*O_Z) \to i^*i_*O_Z = O_Z$$

We denote by $K^{\vee}_{i^*O_S}(O_Z) := \mathcal{H}om_{i^*O_S}(K_{i^*O_S}(O_Z), i^*O_S)$ its dual, so that we have a canonical map

$$q^{\vee}_K : K^{\vee}_{i^*O_S}(O_Z) \to O_Z[-c].$$

Let $M \in C_{\mathcal{D}}(S)$. The i^*D_S module structure on $\mathcal{H}om_{i^*O_S}(K_{i^*O_S}(O_Z), i^*M)$ and $K_{i^*O_S}(O_Z) \otimes_{i^*O_S} i^*M$ induce a canonical D_Z module structure on the cohomology groups $H^n\mathcal{H}om_{i^*O_S}(K_{i^*O_S}(O_Z), i^*M)$ and $H^n(K_{i^*O_S}(O_Z) \otimes_{i^*O_S} i^*M)$ for all $n \in \mathbb{Z}$.

The projection formula for ringed spaces (proposition 9) implies the following lemma :

Lemma 6. *Let $i : Z \hookrightarrow S$ a closed embedding with $Z, S \in \mathrm{Var}(\mathbb{C})$ or with $Z, S \in \mathrm{AnSp}(\mathbb{C})$. Denote by $j : U := S\backslash Z \hookrightarrow Z$ the open complementary embedding. Then, if i is a locally complete intersection embedding (e.g. if Z, S are smooth), we have for $M \in C_{O_U}(U)$, $Li^{*mod}Rj_*M = 0$.*

Proof. We have

$$\begin{aligned} i_*Li^{*mod}Rj_*M &:= i_*(i^*L_O(j_*E(M)) \otimes_{i^*O_S} O_Z) \xrightarrow{T(i,\otimes)(L_O(j_*E(M)),O_Z)^{-1}} L_O(j_*E(M)) \otimes_{O_S} i_*O_Z \\ &\xrightarrow{q\circ(i_*q_K)^{-1}} (j_*E(M)) \otimes_{O_S} i_*K_{i^*O_S}(O_Z) \xrightarrow{T(j,\otimes)(E(M),K_{O_S}(i_*O_Z))} j_*(E(M) \otimes_{O_U} j^*K_{O_S}(i_*O_Z)), \end{aligned}$$

$T(i,\otimes)(L_O(j_*E(M)), O_Z)$ being an equivalence Zariski, resp. usu, local by proposition 10 and follows from the fact that $j^*K_{O_S}(i_*O_Z)$ is acyclic. But

$$T(j,\otimes)(E(M), K_{O_S}(i_*O_Z)) : (j_*E(M)) \otimes_{O_S} K_{O_S}(i_*O_Z) \to j_*(E(M) \otimes_{O_U} j^*i_*K_{i^*O_S}(O_Z))$$

is an equivalence Zariski, resp. usu, local by proposition 9 since $K_{O_S}(i_*O_Z)$ is a finite complex of locally free O_S modules of finite rank. □

We deduce from theorem 20(i) and lemma 6 the localization for D-modules for a closed embedding of smooth algebraic varieties:

Theorem 21. *Let $i : Z \hookrightarrow S$ a closed embedding with $Z, S \in \mathrm{SmVar}(\mathbb{C})$. Denote by $c = \mathrm{codim}(Z, S)$. Then, for $M \in C_{\mathcal{D}}(S)$, we have by Kashiwara equivalence the following map in $C_{\mathcal{D}}(S)$:*

$$\begin{aligned} \mathcal{K}_{Z/S}(M) : \Gamma_Z E(M) &\xrightarrow{\mathrm{ad}(i_{*mod},i^{\sharp})(-)^{-1}} i_{*mod}i^{\sharp}\Gamma_Z E(M) \\ \xrightarrow{\gamma_Z(-)} i_{*mod}i^{\sharp}(E(M)) &\xrightarrow{\mathcal{H}om(q_K,E(i^*M))\circ\mathcal{H}om(O_Z,T(i,E)(M)} i_{*mod}K^{\vee}_{i^*O_S}(O_Z) \otimes_{i^*O_S} M \end{aligned}$$

which is an equivalence Zariski local. It gives the isomorphism in $D_{\mathcal{D}}(S)$

$$\mathcal{K}_{Z/S}(M) : R\Gamma_Z M \to i_{*mod}K^{\vee}_{i^*O_S}(O_Z) = i_{*mod}Li^{*mod}M[c]$$

Proof. Follows from theorem 20(i) and lemma 6 : see [16] for example. □

Definition 65. *Let $f : X \to S$ be a morphism, with $X, S \in \mathrm{SmVar}(\mathbb{C})$ or $X, S \in \mathrm{AnSm}(\mathbb{C})$. Consider the factorization $f : X \xrightarrow{i} X \times S \xrightarrow{p_S} S$, where i is the graph embedding and $p_S : X \times S \to S$ is the projection.*

(i) Then, for $(M, F) \in C_{\mathcal{D}(2)fil}(S)$ we set

$$f^{*mod[-],\Gamma}(M, F) := \Gamma_X E(p_S^{*mod[-]}(M, F)) \in C_{\mathcal{D}(2)fil,\infty}(X \times S),$$

It induces in the derived category

$$Rf^{*mod[-],\Gamma}(M, F) := f^{*mod[-],\Gamma}(M, F) := \Gamma_X E(p_S^{*mod[-]}(M, F)) \in D_{\mathcal{D}(2)fil,\infty}(X \times S),$$

By definition-proposition 21, we have in the algebraic case $Li^{*mod} f^{*mod,\Gamma} M = Lf^{*mod} M \in D_{\mathcal{D}}(X)$.

(ii) Then, for $(M, F) \in C_{\mathcal{D}(2)fil}(S)$ we set

$$Lf^{\hat{*}mod[-],\Gamma}(M, F) := \Gamma_X^{\vee,h} L_D p_S^{*mod[-]}(M, F) := \mathbb{D}_S^K L_D \Gamma_X E(\mathbb{D}_S^K L_D p_S^{*mod[-]}(M, F)) \in D_{\mathcal{D}(2)fil,\infty}(X \times S).$$

In the analytical case we also have

Definition 66. *Let $f : X \to S$ be a morphism, with $X, S \in \mathrm{AnSm}(\mathbb{C})$. Consider the factorization $f : X \xrightarrow{i} X \times S \xrightarrow{p_S} S$, where i is the graph embedding and $p_S : X \times S \to S$ is the projection.*

(i) Then, for $(M, F) \in C_{\mathcal{D}^\infty(2)fil}(S)$ we set

$$f^{*mod[-],\Gamma}(M, F) := \Gamma_X E(p_S^{*mod[-]}(M, F)) \in C_{\mathcal{D}^\infty(2)fil,\infty}(X \times S),$$

It induces in the derived category

$$Rf^{*mod[-],\Gamma}(M, F) := f^{*mod[-],\Gamma}(M, F) := \Gamma_X E(p_S^{*mod[-]}(M, F)) \in D_{\mathcal{D}^\infty(2)fil,\infty}(X \times S),$$

(ii) Then, for $(M, F) \in C_{\mathcal{D}^\infty(2)fil}(S)$ we set

$$Lf^{\hat{*}mod[-],\Gamma}(M, F) := \Gamma_X^{\vee,h} L_D p_S^{*mod[-]}(M, F) := \mathbb{D}_S^K L_D \Gamma_X E(\mathbb{D}_S^K L_D p_S^{*mod[-]}(M, F)) \in D_{\mathcal{D}^\infty(2)fil,\infty}(X \times S).$$

4.2.4 The 2 functors and transformations maps for D modules on the smooth complex algebraic varieties and the complex analytic manifolds

By the definitions and the propositions 49, 50, 72, for the algebraic case, and the propositions 52, 53, 73, for the analytic case,

- we have the 2 functors on $\mathrm{SmVar}(\mathbb{C})$:
  - $C_{\mathcal{D}(2)fil}(\cdot) : \mathrm{SmVar}(\mathbb{C}) \to C_{\mathcal{D}(2)fil}(\cdot),\ S \mapsto C_{\mathcal{D}(2)fil}(S)$, $(f : T \to S) \mapsto f^{*mod}$, $(f : T \to S) \mapsto f^{*mod[-]}$,
  - $D_{\mathcal{D}(2)fil,r}(\cdot) : \mathrm{SmVar}(\mathbb{C}) \to D_{\mathcal{D}(2)fil,r}(\cdot),\ S \mapsto D_{\mathcal{D}(2)fil,r}(S)$, $(f : T \to S) \mapsto Lf^{*mod}$, $(f : T \to S) \mapsto Lf^{*mod[-]}$,
  - $D_{\mathcal{D}(2)fil,\infty}(\cdot) : \mathrm{SmVar}(\mathbb{C}) \to D_{\mathcal{D}(2)fil,\infty}(\cdot),\ S \mapsto D_{\mathcal{D}(2)fil,\infty}(S)$, $(f : T \to S) \mapsto \int_f^{FDR}$,
- we have the 2 functors on $\mathrm{AnSm}(\mathbb{C})$:
  - $C_{\mathcal{D}(2)fil}(\cdot) : \mathrm{AnSm}(\mathbb{C}) \to C_{\mathcal{D}(2)fil}(\cdot),\ S \mapsto C_{\mathcal{D}(2)fil}(S)$, $(f : T \to S) \mapsto f^{*mod}$, $(f : T \to S) \mapsto f^{*mod[-]}$,

- $D_{\mathcal{D}(2)fil,r}(\cdot) : \mathrm{AnSm}(\mathbb{C}) \to D_{\mathcal{D}(2)fil,r}(\cdot),\ S \mapsto D_{\mathcal{D}(2)fil,r}(S)$, $(f : T \to S) \mapsto Lf^{*mod}$, $(f : T \to S) \mapsto Lf^{*mod[-]}$,
- $D_{\mathcal{D}(2)fil,\infty}(\cdot) : \mathrm{AnSm}(\mathbb{C}) \to D_{\mathcal{D}(2)fil,\infty}(\cdot),\ S \mapsto D_{\mathcal{D}(2)fil,\infty}(S)$, $(f : T \to S) \mapsto \int_{f!}^{FDR}$,

- we have also the 2 functors on $\mathrm{AnSm}(\mathbb{C})$:
  - $C_{\mathcal{D}^\infty(2)fil}(\cdot) : \mathrm{AnSm}(\mathbb{C}) \to C_{\mathcal{D}^\infty(2)fil}(\cdot),\ S \mapsto C_{\mathcal{D}^\infty(2)fil}(S)$, $(f : T \to S) \mapsto f^{*mod}$, $(f : T \to S) \mapsto f^{*mod[-]}$,
  - $D_{\mathcal{D}^\infty(2)fil,r}(\cdot) : \mathrm{AnSm}(\mathbb{C}) \to D_{\mathcal{D}^\infty(2)fil,r}(\cdot),\ S \mapsto D_{\mathcal{D}^\infty(2)fil,r}(S)$, $(f : T \to S) \mapsto Lf^{*mod}$, $(f : T \to S) \mapsto Lf^{*mod[-]}$,
  - $D_{\mathcal{D}^\infty(2)fil,r}(\cdot) : \mathrm{AnSm}(\mathbb{C}) \to D_{\mathcal{D}^\infty(2)fil,r}(\cdot),\ S \mapsto D_{\mathcal{D}^\infty(2)fil,r}(S)$, $(f : T \to S) \mapsto \int_{f!}^{FDR}$,

inducing the following commutative diagrams of functors :

$$\begin{tikzcd} \mathrm{SmVar}(\mathbb{C}) \arrow[r, "f\mapsto f^{*mod}"] \arrow[d, "\mathrm{An}"] & C_{\mathcal{D}(2)fil}(\cdot) \arrow[d, "an"] \arrow[dr] & \\ \mathrm{AnSm}(\mathbb{C}) \arrow[r, "f\mapsto f^{*mod}"] & C_{\mathcal{D}(2)fil}(\cdot) \arrow[r, "J."] & C_{\mathcal{D}^\infty(2)fil}(\cdot) \end{tikzcd}, \quad \begin{tikzcd} \mathrm{SmVar}(\mathbb{C}) \arrow[r, "f\mapsto Lf^{*mod}"] \arrow[d, "\mathrm{An}"] & D_{\mathcal{D}(2)fil,r}(\cdot) \arrow[d, "an"] \arrow[dr] & \\ \mathrm{AnSm}(\mathbb{C}) \arrow[r, "f\mapsto Lf^{*mod}"] & D_{\mathcal{D}(2)fil,r}(\cdot) \arrow[r, "J."] & D_{\mathcal{D}^\infty(2)fil,r}(\cdot) \end{tikzcd},$$

where, for $S \in \mathrm{AnSm}(\mathbb{C})$,

- $D_{\mathcal{D}(2)fil,\infty,rh}(S) \subset D_{\mathcal{D}(2)fil,\infty,h}(S)$ is the full subcategory consisting of filtered complexes of D_S module whose cohomology sheaves are regular holonomic,
- $J : C_{\mathcal{D}(2)fil}(S) \to C_{\mathcal{D}^\infty(2)fil}(S)$ is the functor $(M, F) \mapsto J(M, F) := (M, F) \otimes_{D_S} D_S^\infty$, which derive trivially.

We first look at the pullback map and the transformation map of De Rahm complexes (see definition 16 and definition-proposition 17) together with the support section functor :

Proposition 75. *Consider a commutative diagram and a factorization*

$$D_0 = \begin{tikzcd} X \arrow[r, "f"] & S \\ X' \arrow[u, "g'"] \arrow[r, "f'"] & T \arrow[u, "g"] \end{tikzcd} \quad D_0 = \begin{tikzcd} f : X \arrow[r, "i"] & Y \times S \arrow[r, "p"] & S \\ f' : X' \arrow[u, "g'"] \arrow[r, "i'"] & Y \times T \arrow[u, "g''=I\times g"] \arrow[r, "p'"] & T \arrow[u, "g"] \end{tikzcd}$$

with $X, X', Y, S, T \in \mathrm{Var}(\mathbb{C})$ or $X, X', Y, S, T \in \mathrm{AnSp}(\mathbb{C})$, i, i' being closed embeddings, and p, p' the projections. Denote by D the right square of D_0. We have a factorization $i' : X' \xrightarrow{i'_1} X_T = X \times_{Y\times S} Y \times T \xrightarrow{i'_0} Y \times T$, where i'_0, i'_1 are closed embedding. Assume S, T, Y, Y' are smooth.

(i) For $(M, F) \in C_{\mathcal{D}fil}(Y \times S)$, the canonical map in $C_{p'^*O_T fil}(Y \times T)$ (c.f. definition-proposition 17),

$$E(\Omega_{((Y'\times T)/(X\times S))/(T/S)}(M, F)) \circ T(g'', E)(-) \circ T(g'', \gamma)(-) :$$
$$g''^{*}\Gamma_X E((\Omega^\bullet_{Y\times S}, F_b) \otimes_{O_{Y\times S}} (M, F)) \to \Gamma_{X_T} E((\Omega^\bullet_{Y\times T/T}, F_b) \otimes_{O_{Y\times T}} g''^{*mod}(M, F))$$

is a map in $C_{p'^*\mathcal{D}fil}(Y \times T)$.

(ii) For $(M, F) \in C_{\mathcal{D}fil}(Y \times S)$, the canonical map in $C_{O_T fil}(T)$ (c.f. definition-proposition 17 with L_D instead of L_O)

$$T^O_\omega(D)(M, F)^\gamma : g^{*mod} L_D p_* \Gamma_X E(\Omega^\bullet_{Y\times S} \otimes_{O_{Y\times S}} (M, F)) \to p'_* \Gamma_{X_T} E(\Omega^\bullet_{Y\times T/T} \otimes_{O_{Y\times T}} g''^{*mod}(M, F))$$

is a map in $C_{\mathcal{D}fil}(T)$.

(iii) *For $(N,F)\in C_{\mathcal{D}fil}(Y\times T)$, the canonical map in $C_{p'^*O_Tfil}(Y\times T)$*

$$T(X'/X_T,\gamma)(-):\Gamma_{X'}E((\Omega^{\bullet}_{Y\times T/T},F_b)\otimes_{O_{Y\times T}}(N,F))\to\Gamma_{X_T}E((\Omega^{\bullet}_{Y\times T/T},F_b)\otimes_{O_{Y\times T}}(N,F))$$

is a map in $C_{p'^*\mathcal{D}0fil}(Y\times T)$.

(iv) *For $M=O_Y$, we have $T^O_\omega(D)(O_{Y\times S})^\gamma=T^O_\omega(D)^\gamma$ as complexes of D_T modules and $T^O_\omega(X_T/Y\times T)(O_{Y\times T})^\gamma=T^O_\omega(X_T/Y\times T)^\gamma$. as complexes of $p^{'*}D_T$ modules.*

Proof. Follows by definition from proposition 63. □

In the analytical case, we also have

Proposition 76. *Consider a commutative diagram and a factorization*

$$D_0=\begin{tikzcd} X \arrow[r, "f"] & S \\ X' \arrow[u, "g'"] \arrow[r, "f'"] & T \arrow[u, "g"] \end{tikzcd}\quad D_0=\begin{tikzcd} f:X \arrow[r, "i"] & Y\times S \arrow[r, "p"] & S \\ f':X' \arrow[u, "g'"] \arrow[r, "i'"] & Y\times T \arrow[u, "g''=I\times g"] \arrow[r, "p'"] & T \arrow[u, "g"] \end{tikzcd}$$

with $X,X',Y,S,T\in\mathrm{AnSp}(\mathbb{C})$, i, i' being closed embeddings, and p, p' the projections. Denote by D the right square of D_0. We have a factorization $i':X'\xrightarrow{i'_1}X_T=X\times_{Y\times S}Y\times T\xrightarrow{i'_0}Y\times T$, where i'_0,i'_1 are closed embedding. Assume S,T,Y,Y' are smooth.

(i) *For $(M,F)\in C_{\mathcal{D}^\infty fil}(Y\times S)$, the canonical map in $C_{p'^*O_Tfil}(Y\times T)$ (c.f. definition-proposition 17),*

$$\begin{gathered}E(\Omega_{((Y'\times T)/(X\times S))/(T/S)}(M,F))\circ T(g'',E)(-)\circ T(g'',\gamma)(-):\\ g^{''*}\Gamma_XE((\Omega^{\bullet}_{Y\times S},F_b)\otimes_{O_{Y\times S}}(M,F))\to\Gamma_{X_T}E((\Omega^{\bullet}_{Y\times T/T},F_b)\otimes_{O_{Y\times T}}g^{''*mod}(M,F))\end{gathered}$$

is a map in $C_{p'^*\mathcal{D}^\infty fil}(Y\times T)$.

(ii) *For $(M,F)\in C_{\mathcal{D}^\infty fil}(Y\times S)$, the canonical map in $C_{O_Tfil}(T)$ (c.f. definition-proposition 17 with L_{D^∞} instead of L_O)*

$$T^O_\omega(D)(M,F)^\gamma:g^{*mod}L_{D^\infty}p_*\Gamma_XE((\Omega^{\bullet}_{Y\times S},F_b)\otimes_{O_{Y\times S}}(M,F))\to p'_*\Gamma_{X_T}E((\Omega^{\bullet}_{Y\times T/T},F_b)\otimes_{O_{Y\times T}}g^{''*mod}(M,F))$$

is a map in $C_{\mathcal{D}^\infty fil}(T)$.

(iii) *For $(N,F)\in C_{\mathcal{D}^\infty fil}(Y\times T)$, the canonical map in $C_{p'^*O_Tfil}(Y\times T)$*

$$T(X'/X_T,\gamma)(-):\Gamma_{X'}E(\Omega^{\bullet}_{Y\times T/T}\otimes_{O_{Y\times T}}(N,F))\to\Gamma_{X_T}E(\Omega^{\bullet}_{Y\times T/T}\otimes_{O_{Y\times T}}(N,F))$$

is a map in $C_{p'^*\mathcal{D}^\infty fil}(Y\times T)$.

(iv) *For $M=O_Y$, we have $T^O_\omega(D)(O_{Y\times S})^\gamma=T^O_\omega(D)^\gamma$ as complexes of D^∞_T modules and $T^O_\omega(X_T/Y\times T)(O_{Y\times T})^\gamma=T^O_\omega(X_T/Y\times T)^\gamma$. as complexes of $p^{'*}D^\infty_T$ modules.*

Proof. Follows from proposition 75. □

Similarly, we have :

Proposition 77. *Let $p:Y\times S\to S$ a projection and $i:X\hookrightarrow Y\times S$ a closed embedding with $S,Y\in\mathrm{SmVar}(\mathbb{C})$.*

(i) *For $(M,F)\in C_{\mathcal{D}fil}(Y\times S)$ the canonical map in $C_{p^*O_Sfil}(Y^{an}\times S^{an})$ (see definition-proposition 17)*

$$\begin{gathered}E(\Omega_{(Y^{an}\times S^{an}/Y\times S)/(S^{an}/S)}(M,F))\circ T(an,\gamma)(-):\\(\Gamma_X E((\Omega^{\bullet}_{Y\times S/S},F_b)\otimes_{O_Y}(M,F)))^{an}\to\Gamma_{X^{an}}E((\Omega^{\bullet}_{Y\times S/S},F_b)\otimes_{O_{Y^{an}\times S^{an}}}(M,F)^{an})\end{gathered}$$

is a map in $C_{h^*\mathcal{D}fil}(Y^{an}\times S)$.

(ii) *For $(M,F)\in C_{\mathcal{D}fil}(Y\times S)$ the canonical map in $C_{O_Sfil}(S^{an})$ (see definition-proposition 17)*

$$T^O_\omega(an,p)(M,F)^\gamma:(p_*\Gamma_X E((\Omega^{\bullet}_{Y\times S/S},F_b)\otimes_{O_Y}(M,F)))^{an}\to p_*\Gamma_{X^{an}}E((\Omega^{\bullet}_{Y\times S/S},F_b)\otimes_{O_{Y^{an}}}(M,F)^{an})$$

is a map in $C_{\mathcal{D}fil}(S^{an})$.

(iii) *For $M=O_Y$, we have $T^O_\omega(an,h)(O_Y)^\gamma=T^O_\omega(an,h)^\gamma$ as complexes of D_S modules*

Proof. Follows by definition from proposition 65 □

Let $p:Y\times S\to S$ a projection with $Y,S\in\mathrm{SmVar}(\mathbb{C})$ or with $Y,S\in\mathrm{AnSm}(\mathbb{C})$. Let $j:V\hookrightarrow Y\times S$ an open embedding. Consider (see proposition 66), for $(M,F)\in C_{\mathcal{D}fil}(Y\times S)$, the canonical transformation map in $C_{p^*\mathcal{D}fil}(Y\times S)$

$$\begin{gathered}k\circ T^O_w(j,\otimes)(E(M,F)):(\Omega^{\bullet}_{Y\times S/S},F_b)\otimes_{O_Y}j_*j^*E(M,F)\\\xrightarrow{DR(Y\times S/S)(\mathrm{ad}(j^*,j_*)(-))}\\j_*j^*((\Omega^{\bullet}_{Y\times S/S},F_b)\otimes_{O_Y}j_*j^*E(M,F))=j_*j^*(\Omega^{\bullet}_{Y\times S/S},F_b)\otimes_{O_Y}j^*j_*j^*E(M,F)\\\xrightarrow{k\circ DR(Y\times S/S)(\mathrm{ad}(j^*,j_*)(E(M,F)))}\\j_*E(j^*(\Omega^{\bullet}_{Y\times S/S},F_b)\otimes_{O_{Y\times S}}j^*E(M,F))=j_*E(j^*((\Omega^{\bullet}_{Y\times S/S},F_b)\otimes_{O_{Y\times S}}E(M,F)))\end{gathered}$$

We have then :

Proposition 78. *Let $p:Y\times S\to S$ a projection with $Y,S\in\mathrm{SmVar}(\mathbb{C})$ or with $Y,S\in\mathrm{AnSm}(\mathbb{C})$. Let $i:X\hookrightarrow Y\times S$ a closed embedding. Then, for $(M,F)\in C_{\mathcal{D}fil}(Y\times S)$*

(i) *the canonical map in $C_{p^*\mathcal{D}fil}(Y)$ (definition 53)*

$$\begin{gathered}T^O_w(\gamma,\otimes)(M,F):=(I,k\circ T^O_w(j,\otimes)(E(M,F))):\\(\Omega^{\bullet}_{Y\times S/S},F_b)\otimes_{O_{Y\times S}}\Gamma_X E(M,F)\to\Gamma_X E((\Omega^{\bullet}_{Y\times S/S},F_b)\otimes_{O_{Y\times S}}E(M,F)),\end{gathered}$$

is a (1-)filtered Zariski, resp usu, local equivalence.

(ii) *the map of point (i) gives the following canonical isomorphism in $D_{p^*\mathcal{D}fil}(Y)$*

$$\begin{gathered}T^O_w(\gamma,\otimes)(M,F):(\Omega^{\bullet}_{Y\times S/S},F_b)\otimes_{O_{Y\times S}}\Gamma_X E(M,F)\xrightarrow{T^O_w(\gamma,\otimes)(M,F)}\\\Gamma_X E((\Omega^{\bullet}_{Y\times S/S},F_b)\otimes_{O_{Y\times S}}E(M,F))\xrightarrow{DR(Y\times S/S)(k)^{-1}}\Gamma_X E((\Omega^{\bullet}_{Y\times S/S},F_b)\otimes_{O_{Y\times S}}(M,F)).\end{gathered}$$

Proof. (i): By proposition 66

- $T^O_w(j,\otimes)(M,F):(\Omega^{\bullet}_{Y\times S/S},F_b)\otimes_{O_{Y\times S}}j_*j^*E(M,F)\to j_*E(j^*((\Omega^{\bullet}_{Y\times S/S},F_b)\otimes_{O_{Y\times S}}E(M,F)))$ is a filtered Zariski, resp usu, local equivalence in $C_{p^*\mathcal{D}fil}(Y\times S)$ and
- $DR(Y\times S/S)(k):(\Omega^{\bullet}_{Y\times S/S},F_b)\otimes_{O_{Y\times S}}(M,F)\to(\Omega^{\bullet}_{Y\times S/S},F_b)\otimes_{O_{Y\times S}}E(M,F)$ is a filtered Zariski, resp usu, local equivalence in $C_{p^*\mathcal{D}fil}(Y\times S)$.

(ii): Follows from (i). $\square$

In the analytic case, we also have

Proposition 79. *Let $p : Y \times S \to S$ a projection with $Y, S \in \mathrm{AnSm}(\mathbb{C})$. Let $i : X \hookrightarrow Y$ a closed embedding. Then, for $(M,F) \in C_{\mathcal{D}^\infty fil}(Y \times S)$*

(i) the canonical map in $C_{p^*\mathcal{D}fil}(Y)$

$$T^O_w(\gamma,\otimes)(M,F) := (I, T^O_w(j,\otimes)(E(M,F))) :$$
$$(\Omega^\bullet_{Y\times S/S}, F_b) \otimes_{O_{Y\times S}} \Gamma_X E(M,F) \to \Gamma_X E((\Omega^\bullet_{Y\times S/S}, F_b) \otimes_{O_{Y\times S}} E(M,F)).$$

is a map in $C_{p^*\mathcal{D}^\infty fil}(Y \times S)$. Proposition 78 says that it is a filtered equivalence usu local,

(ii) the map of point (i) gives the following canonical isomorphism in $D_{p^*\mathcal{D}^\infty,fil}(Y \times S)$

$$T^O_w(\gamma,\otimes)(M,F) : (\Omega^\bullet_{Y\times S/S}, F_b) \otimes_{O_{Y\times S}} \Gamma_X E(M,F) \xrightarrow{T^O_w(\gamma,\otimes)(M,F)}$$
$$\Gamma_X E((\Omega^\bullet_{Y\times S/S}, F_b) \otimes_{O_{Y\times S}} E(M,F)) \xrightarrow{DR(Y\times S/S)(k)^{-1}} \Gamma_X E((\Omega^\bullet_{Y\times S/S}, F_b) \otimes_{O_{Y\times S}} (M,F)).$$

Proof. (i): By proposition 67

- $T^O_w(j,\otimes)(M) : \Omega^\bullet_{Y\times S/S} \otimes_{O_{Y\times S}} j_*j^*E(M) \to j_*E(j^*(\Omega^\bullet_{Y\times S/S} \otimes_{O_{Y\times S}} E(M)))$ is an equivalence usu local in $C_{p^*\mathcal{D}^\infty}(Y \times S)$ and
- $DR(Y \times S/S)(k) : \Omega^\bullet_{Y\times S/S} \otimes_{O_{Y\times S}} M \to \Omega^\bullet_{Y\times S/S} \otimes_{O_{Y\times S}} E(M)$ is an equivalence usu local in $C_{p^*\mathcal{D}^\infty}(Y \times S)$.

(ii): Follows from (i). $\square$

In the projection case, we consider the following canonical maps : Let $S_1, S_2 \in \mathrm{SmVar}(\mathbb{C})$ or let $S_1, S_2 \in \mathrm{AnSm}(\mathbb{C})$. Denote by $p = p_1 : S_{12} = S_1 \times S_2 \to S_1$ and $p_2 : S_{12} = S_1 \times S_2 \to S_1$ the projection. We consider

- $p(M_1,F) : (M_1,F) \to p_{*mod}p^{*mod[-]}(M_1,F)$ in $C_{\mathcal{D}(2)fil}(S_1)$, for $(M_1,F) \in C_{\mathcal{D}fil(2)}(S_1)$, which is the composite

$$p(M_1,F) : (M_1,F) \xrightarrow{\mathrm{ad}(p^*,p_*)(M_1)} p_*p^*(M_1,F) \xrightarrow{m_1} p_*((\Omega^\bullet_{S_{12}/S_1}, F_b) \otimes_{p^*O_{S_1}} p^*(M_1,F))$$
$$\xrightarrow{=} p_*((\Omega^\bullet_{S_{12}/S_1}, F_b) \otimes_{O_{S_{12}}} p^{*mod}(M_1,F) = p_{*mod}p^{*mod[-]}(M_1,F)$$

  where $m_1 : p^*M_1 \to p^*M_1 \otimes_{p^*O_{S_1}} \Omega^\bullet_{S_{12}/S_1}$ is given by $m_1(m) = m \otimes 1$,

- $p(M_{12},F) : p^{*mod[-]}p_{*mod}(M_{12},F) \to (M_{12},F)$ in $C_{\mathcal{D}fil}(S_1 \times S_2)$, for $(M_{12},F) \in C_{\mathcal{D}}(S_1 \times S_2)$, which is the composite

$$p(M_{12},F) : p^{*mod[-]}p_{*mod}(M_{12},F) = p^*p_*((M_{12,F}) \otimes_{O_{S_{12}}} (\Omega^\bullet_{S_{12}/S_1}, F_b)) \otimes_{p^*O_{S_1}} O_{S_{12}}$$
$$\xrightarrow{\mathrm{ad}(p^*,p_*)(-)\otimes_{p^*O_{S_1}} I} (M_{12},F) \otimes_{O_{S_{12}}} \Omega^\bullet_{S_{12}/S_1} \otimes_{p^*O_{S_1}} O_{S_{12}} = (M_{12},F) \otimes_{p^*O_{S_1}} \Omega^\bullet_{S_{12}/S_1} \xrightarrow{m_{12}} (M_{12},F)$$

  where $m_{12} : M_{12} \otimes_{p^*O_{S_1}} \Omega^\bullet_{S_{12}/S_1} \to M_{12}$ is the multiplication map:

  - $m_{12}(M_{12} \otimes_{p^*O_{S_1}} \Omega^p_{S_{12}/S_1}) = 0$ for $p \neq 0$ and
  - $m_{12} : M_{12} \otimes_{p^*O_{S_1}} \Omega^0_{S_{12}/S_1} = M_{12} \otimes_{p^*O_{S_1}} O_{S_{12}} \to M_{12}$ is given by $m_{12}(m \otimes f) = fm$

We have then $p(p^{*mod[-]}(M_1,F)) \circ p^{*mod[-]}p(M_1,F) = I_{p^{*mod[-]}(M_1,F)}$. It gives the following maps

- $p_!(M_{12}) : (M_{12},F) \to p^{*mod[-]}\int_{p!}(M_{12,F})$ in $D_{\mathcal{D}(2)fil}(S_1\times S_2)$, for $(M_{12},F)\in C_{\mathcal{D}fil,h}(S_1\times S_2)$, given by

$$p_!(M_{12}) : (M_{12},F) \xrightarrow{d(M_{12},F)} L\mathbb{D}_S^2(M_{12},F) \xrightarrow{L\mathbb{D}_S(p(-)\circ q)} \mathbb{D}_S^K L_D p^{*mod[-]}p_{*mod}E(\mathbb{D}_S^K L_D(M_{12},F))$$
$$\xrightarrow{T(p,D)(-)^{-1}} p^{*mod}\mathbb{D}_S^K L_D p_{*mod}E(\mathbb{D}_S^K L_D(M_{12},F)) = p^{*mod[-]}\int_{p!}(M_{12},F)$$

- $p_!(M_1,F) : \int_{p!}p^{*mod[-]}(M_1,F)\to(M_1,F)$ in $D_{\mathcal{D}fil}(S_1)$, for $M_1\in C_{\mathcal{D}fil,h}(S_1)$, given by

$$p_!(M_1,F) : \int_{p!}p^{*mod[-]}(M_1,F) = \mathbb{D}_S^K L_D p_{*mod}E(\mathbb{D}_S^K p^{*mod[-]}L_D(M_1,F)) \xrightarrow{(\mathbb{D}_S^K k)\circ T(p,D)(-)^{-1}}$$
$$\mathbb{D}_S^K p_{*mod}p^{*mod[-]}\mathbb{D}_S^K L_D(M_1,F) \xrightarrow{\mathbb{D}_S^K p(\mathbb{D}_S^K L_D(M_1,F))} \mathbb{D}_S^{K,2}L_D(M_1,F) \xrightarrow{d(M_1,F)^{-1}} (M_1,F)$$

so that $p^{*mod[-]}(p_!(M_1,F))\circ p_!(p^{*mod[-]}(M_1,F)) = I_{p^{*mod[-]}(M_1,F)}$.

Definition 67. *(i) Consider a commutative diagram in* SmVar$(\mathbb{C})$ *which is cartesian, together with its factorization*

$$D=(f,q)= \begin{tikzcd} X\times T \arrow[r, "f'"] \arrow[d, "g'"'] & T\times S \arrow[d, "q"] \\ X \arrow[r, "f"] & S \end{tikzcd} \quad D=(f,q)= \begin{tikzcd} f'' : X\times T \arrow[r, "i''"] \arrow[d, "q'"] & X\times T\times S \arrow[r, "p''"] \arrow[d, "q''=I\times q"] & T\times S \arrow[d, "q"] \\ f : X \arrow[r, "i"] & X\times S \arrow[r, "p"] & S \end{tikzcd},$$

where the squares are cartesian, $f=p\circ i$ being the graph factorization and q, q' the projections. We have, for $(M,F)\in C_{\mathcal{D}(2)fil,\infty}(X)$, the following transformation map in $C_{\mathcal{D}(2)fil,\infty}(T\times S)$:

$$T^{\mathcal{D}mod}(f,q)(M,F) : q^{*mod}p_*E((\Omega_{X\times S/S},F_b)\otimes_{O_{X\times S}} i_{*mod}(M,F))$$
$$\xrightarrow{T^O_\omega(q,p)(i_{*mod}(M,F))} p''_*E((\Omega_{X\times T\times S/T\times S},F_b)\otimes_{O_{X\times T\times S}} q^{''*mod}i_{*mod}(M,F))$$
$$\xrightarrow{p_*E(T^{\mathcal{D}mod}(i,q'')(M,F)\otimes I)} p''_*E((\Omega_{X\times T\times S/T\times S},F_b)\otimes_{O_{X\times T\times S}} i''_{*mod}q^{'*mod}(M,F))$$

where

$$T^{\mathcal{D}mod}(i,q'')(M,F) : q^{''*mod}i_{*mod}(M,F) := q^{''*mod}i_*((M,F)\otimes_{D_X} i^{*mod}(D_{X\times S},F^{ord}))$$
$$\xrightarrow{T^{mod}(q'',i)(-)} i_*q^{'*mod}((M,F)\otimes_{D_X} i^{*mod}(D_{X\times S},F^{ord}))$$
$$\xrightarrow{=} i_*(q^{'*mod}(M,F)\otimes_{q'^*D_X} q^{'*}i^{*mod}(D_{X\times S},F^{ord})) \xrightarrow{=}$$
$$i_*(q^{'*mod}(M,F)\otimes_{D_{X\times T}} i^{'*mod}(D_{X\times S\times T},F^{ord})) =: i'_{*mod}q^{'*mod}(M,F)$$

(ii) Consider a commutative diagram in SmVar$(\mathbb{C})$ *which is cartesian, together with its factorization*

$$D=(f,g)= \begin{tikzcd} X_T \arrow[r, "f'"] \arrow[d, "g'"'] & T \arrow[d, "g"] \\ X \arrow[r, "f"] & S \end{tikzcd} \quad D=(f,g)= \begin{tikzcd} f' : X_T \arrow[r, "i'"] \arrow[d, "l'"] & X\times T \arrow[r, "p'"] \arrow[d, "l''=I\times l"] & T \arrow[d, "l"] \\ f'' : X\times T \arrow[r, "i''"] \arrow[d, "q'"] & X\times T\times S \arrow[r, "p''"] \arrow[d, "q''=I\times q"] & T\times S \arrow[d, "q"] \\ f : X \arrow[r, "i"] & X\times S \arrow[r, "p"] & S \end{tikzcd},$$

where the squares are cartesian, $f = p \circ i$, $g = q \circ l$ being the graph factorizations. We have, for $(M, F) \in D_{\mathcal{D}(2)fil,\infty}(X)$, the following transformation map in $D_{\mathcal{D}(2)fil,\infty}(T \times S)$:

$$T^{\mathcal{D}mod}(f,g)((M,F)) :$$

$$Rg^{*mod,\Gamma}(M,F)\int_f^{FDR}(M,F) := \Gamma_T E(q^{*mod}p_*E((\Omega_{X\times S/S},F_b)\otimes_{O_{X\times S}} i_{*mod}(M,F)))$$

$$\xrightarrow{\Gamma_T E(T^{\mathcal{D}mod}(f,q)(M,F))} \Gamma_T E(p''_*E((\Omega_{X\times T\times S/T\times S},F_b)\otimes_{O_{X\times T\times S}} i''_{*mod}q^{'*mod}(M,F)))$$

$$\xrightarrow{=} p''_*\Gamma_{X\times T}E((\Omega_{X\times T\times S/T\times S},F_b)\otimes_{O_{X\times T\times S}} i''_{*mod}q^{'*mod}(M,F))$$

$$\xrightarrow{T^O_w(\gamma,\otimes)(-)} p''_*E((\Omega_{X\times T\times S/T\times S},F_b)\otimes_{O_{X\times T\times S}} \Gamma_{X\times T}E(i''_{*mod}q^{'*mod}(M,F)))$$

$$\xrightarrow{=} p''_*E((\Omega_{X\times T\times S/T\times S},F_b)\otimes_{O_{X\times T\times S}} (i''_{*mod}q^{'*mod}\Gamma_{X_T}E(M,F))) =: \int_{f'}^{FDR} Rg^{'*mod,\Gamma}(M,F)$$

(ii)' We have, for $M \in D_{\mathcal{D}}(X)$, the following transformation map in $D_{\mathcal{D}}(T)$:

$$T^{\mathcal{D}mod}(f,g)(M) :$$

$$g^{*mod}(M,F)\int_f(M) = l^{*mod}q^{*mod}\int_f M \xrightarrow{q'(M)} l^{*mod}q^{*mod}\int_f q'_{*mod}q^{'*mod}M$$

$$\xrightarrow{=} l^{*mod}q^{*mod}q_{*mod}\int_{f''} q^{'*mod}M \xrightarrow{q(-)} l^{*mod}\int_{f''} q^{'*mod}M$$

$$\xrightarrow{l^{*mod}\mathrm{ad}(l^{'\sharp},l'_{*mod})(-)^{-1}} l^{*mod}\int_{f''} l'_{*mod}l^{'\sharp}q^{'*mod}M \xrightarrow{=} l^{\sharp}l_{*mod}\int_{f'} l^{'*mod}q^{*mod}M$$

$$\xrightarrow{\mathrm{ad}(l^{\sharp},l_{*mod})(-)} \int_{f'} l^{'*mod}q^{*mod}M =: \int_{f'} g^{'*mod}(M)$$

where $l^{*mod}\mathrm{ad}(l^{'\sharp},l'_{*mod})(-)$ is an isomorphism by lemma 6.

In the analytic case, we have :

Definition 68. *Consider a commutative diagram in $\mathrm{AnSm}(\mathbb{C})$ which is cartesian together with a factorization*

$$D=(f,g)= \begin{tikzcd} X_T \arrow[r,"f'"] \arrow[d,"g'"'] & T \arrow[d,"g"] \\ X \arrow[r,"f"] & S \end{tikzcd} \quad D=(f,g)= \begin{tikzcd} f' : X_T \arrow[r,"i'"] \arrow[d,"g'"] & Y\times T \arrow[r,"p'"] \arrow[d,"g''=I\times g"] & T \arrow[d,"g"] \\ f : X \arrow[r,"i"] & Y\times S \arrow[r,"p"] & S \end{tikzcd},$$

where $Y \in \mathrm{AnSm}(\mathbb{C})$, i, i' are closed embeddings and p, p' the projections.

(i) We have, for $(M,F) \in D_{\mathcal{D}(2)fil,\infty,h}(X)$, the following transformation map in $D_{\mathcal{D}(2)fil,\infty}(T \times S)$

$$T^{\mathcal{D}mod}(f,g)((M,F)) : Rg^{*mod,\Gamma}\int_f^{FDR}(M,F) \to \int_{f'}^{FDR} Rg^{'*mod,\Gamma}(M,F)$$

define in the same way as in definition 67

(ii) For $(M,F) \in D_{\mathcal{D}^\infty(2)fil,\infty}(X)$, the following transformation map in $D_{\mathcal{D}^\infty(2)fil,\infty}(T \times S)$

$$T^{\mathcal{D}mod}(f,g)((M,F)) : Rg^{*mod,\Gamma}\int_f^{FDR}(M,F) \to \int_{f'}^{FDR} Rg^{'*mod,\Gamma}(M,F)$$

is defined in the same way as in (ii) : see definition 67.

In the algebraic case, we have the following proposition:

Proposition 80. *Consider a cartesian square in* SmVar($\mathbb{C}$)

$$D = \begin{tikzcd} X_T \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ T \arrow[r, "g"] & S \end{tikzcd}$$

(i) For $(M,F) \in D_{\mathcal{D}(2)fil,\infty,c}(X)$,

$$T^{\mathcal{D}mod}(f,g)((M,F)) : Rg^{*mod,\Gamma}\int_f^{FDR}(M,F) \xrightarrow{\sim} \int_{f'}^{FDR} Rg'^{*mod,\Gamma}(M,F)$$

is an isomorphism in $D_{\mathcal{D}(2)fil,\infty}(T)$.

(ii) For $M \in D_{\mathcal{D},c}(X)$,

$$T^{\mathcal{D}mod}(f,g)(M) : g^{*mod}\int_f M \xrightarrow{\sim} \int_{f'} g'^{*mod} M$$

is an isomorphism in $D_{\mathcal{D}}(T)$.

Proof. Follows from the projection case and the closed embedding case. □

In the analytic case, we have similarly:

Proposition 81. *Consider a cartesian square in* AnSm($\mathbb{C}$)

$$D = \begin{tikzcd} X_T \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ T \arrow[r, "g"] & S \end{tikzcd}$$

(i) Assume that f, hence f' is proper. For $(M,F) \in D_{\mathcal{D}(2)fil,\infty,h}(X)$,

$$T^{\mathcal{D}mod}(f,g)((M,F)) : Rg^{*mod,\Gamma}\int_f^{FDR}(M,F) \xrightarrow{\sim} \int_{f'}^{FDR} Rg'^{*mod,\Gamma}(M,F)$$

is an isomorphism in $D_{\mathcal{D}(2)fil,\infty}(T)$.

(ii) For $(M,F) \in D_{\mathcal{D}^\infty(2)fil,\infty,h}(X)$,

$$T^{\mathcal{D}mod}(f,g)((M,F)) : Rg^{*mod,\Gamma}\int_f^{FDR}(M,F) \xrightarrow{\sim} \int_{f'}^{FDR} Rg'^{*mod,\Gamma}(M,F)$$

is an isomorphism in $D_{\mathcal{D}^\infty(2)fil,\infty}(T)$.

Proof. (i):Similar to the proof of proposition 80.
(ii):Similar to the proof of proposition 80. □

Definition 69. *Let $f : X \to S$ a morphism with $X, S \in$* SmVar($\mathbb{C}$).

(i) We have, for $(M,F) \in C_{\mathcal{D}fil}(S)$ and $(N,F) \in C_{\mathcal{D}fil}(X)$, we have the map in $C_{\mathcal{D}fil}(S)$

$$\begin{aligned} &T^{\mathcal{D}mod,0}(\otimes,f)((M,F),(N,F)) : \\ &(M,F)\otimes_{O_S} f^0_{*mod}(N,F) := (M,F)\otimes_{O_S} f_*((N,F)\otimes_{D_X}(D_{X\leftarrow S},F^{ord})) \\ &\xrightarrow{T(\otimes,f)(-,-)} f_*(f^*(M,F)\otimes_{f^*O_S}(N,F)\otimes_{D_X}(D_{X\leftarrow S},F^{ord})) \xrightarrow{=} \\ &f_*(f^{*mod}(M,F)\otimes_{O_X}(N,F)\otimes_{D_X}(D_{X\leftarrow S},F^{ord})) =: f^0_{*mod}(f^{*mod}(M,F)\otimes_{O_X}(N,F)) \end{aligned}$$

(ii) *Consider the cartesian square*

$$\begin{tikzcd} D = X \arrow[r, "i"] \arrow[d, "f"] & X \times S \arrow[d, "f\times I_S"] \\ S \arrow[r, "\Delta_S"] & S \times S \end{tikzcd},$$

where $i_f = (f \times I_S) \circ \Delta_X : X \hookrightarrow X \times S$ is the graph embedding. Then, for $(M,F) \in C_{\mathcal{D}(2)fil}(S)$ and $(N,F) \in C_{\mathcal{D}fil}(X)$, we have the map in $D_{\mathcal{D}(2)fil,r}(S)$

$$T^{\mathcal{D}mod}(\otimes, f)((M,F),(N,F)) : \int_f^{FDR}((N,F) \otimes_{O_X} f^{*mod}_{FDR}(M,F)) = \int_f^{FDR} i^{*mod}_{f,FDR}(p_X^* N \otimes p_S^* M)$$
$$\xrightarrow{T^{\mathcal{D}mod}(\Delta_S, f\times I_S)(-)} \Delta^{*mod}_{S,FDR} \int_{(f\times I_S)}^{FDR}(p_X^* N \otimes p_S^* M) = (\int_f (N,F)) \otimes^L_{O_S} (M,F).$$

Clearly if $i : Z \hookrightarrow S$ is a closed embedding with $Z, S \in \mathrm{SmVar}(\mathbb{C})$ or with $Z, S \in \mathrm{AnSm}(\mathbb{C})$, then $T^{\mathcal{D},0}(\otimes, i)(M,N) = T^{\mathcal{D}}(\otimes, i)(M,N)$ in $D_{\mathcal{D}(2)fil,\infty}(S)$.

We have then the following :

Proposition 82. (i) *Let $i : Z \hookrightarrow S$ is a closed embedding with $Z, S \in \mathrm{SmVar}(\mathbb{C})$, then for $(M,F) \in C_{\mathcal{D}fil}(S)$ and $(N,F) \in C_{\mathcal{D}fil}(Z)$*

$$T^{\mathcal{D},0}(\otimes, i)((M,F),(N,F)) : (M,F) \otimes_{O_S} i_{*mod}(N,F) \xrightarrow{\sim} i_{*mod}(i^{*mod}(M,F) \otimes_{O_Z} (N,F))$$

is an isomorphism in $C_{\mathcal{D}fil}(S)$.

(ii) *Let $f : X \to S$ a morphism with $X, S \in \mathrm{SmVar}(\mathbb{C})$. Then, for $(M,F) \in C_{\mathcal{D}(2)fil}(X)$ and $(N,F) \in C_{\mathcal{D}(2)fil}(S)$,*

$$T^{\mathcal{D}mod}(\otimes, f)((M,F),(N,F)) : \int_f^{FDR}((M,F) \otimes^L_{O_X} f^{*mod}_{FDR}(N,F)) \xrightarrow{\sim} (\int_f^{FDR}(M,F)) \otimes^L_{O_Y} (N,F)$$

is an isomorphism in $D_{\mathcal{D}(2)fil,\infty}(S)$.

Proof. (i): Follows from proposition 10.
(ii):Follows from proposition 80(i). □

Let $f : X \to S$ a morphism with $X, S \in \mathrm{SmVar}(\mathbb{C})$. Consider the graph embedding $f : X \xrightarrow{i} X \times S \xrightarrow{p} S$, with $X, Y, S \in \mathrm{SmVar}(\mathbb{C})$. We have, for $(M,F) \in C_{\mathcal{D}fil}(X)$, the canonical isomorphism in $C_{\mathcal{D}(2)fil}(S^{an})$

$$\mathrm{an}_X^{*mod} i^{*mod} L_D(p^{*mod}(M,F) \otimes_{O_{X\times S}} (O_{X\times S}, \mathcal{V}_X)) \xrightarrow{=}$$
$$i^{*mod} L_D p^{*mod}((M,F)^{an} \otimes_{O_{X^{an}\times S^{an}}} (O_{X^{an}\times S^{an}}, \mathcal{V}_{X^{an}}))$$

We then define and study the transformation map between the direct image functor and the analytical functor for D-modules :

Definition 70. *Let $f : X \to S$ a morphism with $X, S \in \mathrm{SmVar}(\mathbb{C})$.*

(i) *We have for $(M,F) \in C_{\mathcal{D}(2)fil}(X)$ the canonical map in $C_{\mathcal{D}(2)fil}(S^{an})$*

$$T^{\mathcal{D}mod0}(an, f)(M,F) : \mathrm{an}_S^{*mod}(f_* E((D_{X\leftarrow S}, F^{ord}) \otimes_{D_X} L_D(M,F))) \xrightarrow{T^{mod}(an,f)(-)}$$
$$f_*(E((D_{X\leftarrow S}, F^{ord}) \otimes_{D_X} L_D(M,F)))^{an} \xrightarrow{=} f_* E(D_{X^{an}\leftarrow S^{an}} \otimes_{D_{X^{an}}} L_D(M^{an}, F))$$

(ii) *Consider the graph embedding* $f : X \xrightarrow{i} X \times S \xrightarrow{p} S$, *with* $X, Y, S \in \mathrm{SmVar}(\mathbb{C})$. *We have, for* $(M, F) \in C_{\mathcal{D}fil}(X)$, *the canonical map in* $C_{\mathcal{D}(2)fil}(S^{an})$

$$T^{\mathcal{D}mod}(an, f)(M, F) : \mathrm{an}_S^{*mod}(p_*E((\Omega^\bullet_{Y\times S/S}, F_b) \otimes_{O_{Y\times S}} i_{*mod}(M, F)))$$
$$\xrightarrow{T^O_\omega(an,p)(i_{*mod}(M,F))} p_*E((\Omega^\bullet_{Y\times S/S}, F_b) \otimes_{O_{Y^{an}\times S^{an}}} (i_{*mod}(M, F))^{an})$$
$$\xrightarrow{p_{*mod}T^{\mathcal{D}mod0}(an,i)((M,F))} p_*E((\Omega^\bullet_{Y\times S/S}, F_b) \otimes_{O_{Y^{an}\times S^{an}}} i_{*mod}((M, F)^{an})).$$

In order to prove that this map gives an isomorphism in the derived category in the non filtered case if f is proper and M coherent, we will need the following (c.f.[16]):

Theorem 22. *A product* $X \times S$ *of a smooth projective variety* X *and a smooth affine variety* S *is D-affine.*

Proof. See [16] theorem 1.6.5. □

A main result is that we have the following version of the first GAGA theorem for coherent D-modules :

Theorem 23. *Let* $f : X \to S$ *a morphism with* $X, S \in \mathrm{SmVar}(\mathbb{C})$. *Let* $M \in D_{\mathcal{D}(2)fil,c}(X)$, *for* $r = 1, \ldots \infty$. *If* f *is proper,*

$$T^{\mathcal{D}mod}(an, f)(M, F) : (\int_f M)^{an} \xrightarrow{\sim} \int_{f^{an}} (M^{an})$$

is an isomorphism.

Proof. We may assume that f is projective, so that we have a factorization $f : X \xrightarrow{i} \mathbb{P}^N \times S \xrightarrow{p} S$ where i is a closed embedding and p the projection. The question being local on S, we may assume that S is affine. Since $\mathbb{P}^N \times S$ is D-affine by theorem 22, we have by proposition 46(iii) a complex $F \in C_{\mathcal{D}}(\mathbb{P}^N \times S)$ such that $i_{*mod}M = F \simeq F \in D_{\mathcal{D},r}(\mathbb{P}^N \times S)$ and each F^n is a direct summand of a free $D_{\mathbb{P}^N\times S}$ module of finite rank. The theorem now follows from the fact that $\int_p D_{\mathbb{P}^N\times S} \simeq D_S[-N]$ and the fact that $(D_S)^{an} = D_{S^{an}}$. □

We also have

Definition 71. (i) *Let* $f : X \to S$ *a morphism with* $X, S \in \mathrm{SmVar}(\mathbb{C})$ *or with* $X, S \in \mathrm{AnSm}(\mathbb{C})$. *We have, for* $M, N \in C_{\mathcal{D}fil}(X)$, *the canonical transformation map in* $D_{\mathcal{D}fil,\infty}(S)$

$$T^{0,\mathcal{D}}(f, hom)((M, F), (N, F)) : Rf_*R\mathcal{H}om_{f^*D_S}((M, F), (N, F)) \to$$
$$Rf_*R\mathcal{H}om_{D_X}((M, F) \otimes_{D_X} L_D(D_{X\leftarrow S}, F^{ord}), (N, F) \otimes_{D_X} L_D(D_{X\leftarrow S}, F^{ord})) \xrightarrow{T^0(f,hom)(E(-),E(-))}$$
$$R\mathcal{H}om_{D_X}(Rf_*((M, F) \otimes_{D_X} L_D(D_{X\leftarrow S}, F^{ord})), Rf_*((N, F) \otimes_{D_X} L_D(D_{X\leftarrow S}, F^{ord}))) =$$
$$R\mathcal{H}om_{D_X}(\int_f (M, F), \int_f (N, F))$$

(ii) *Let* $f : X \to S$ *a morphism with* $X, S \in \mathrm{AnSm}(\mathbb{C})$. *We have, for* $(M, F), (N, F) \in C_{\mathcal{D}fil}(X)$, *the canonical transformation map in* $D_{\mathcal{D}fil,\infty}(S)$

$$T^{0,\mathcal{D}}(f_!, hom)((M, F), (N, F)) : Rf_*\mathcal{H}om_{f^*D_S}((M, F), (N, F)) \to$$
$$Rf_*\mathcal{H}om_{D_X}((M, F) \otimes_{D_X} L_D(D_{X\leftarrow S}, F^{ord}), (N, F) \otimes_{D_X} L_D(D_{X\leftarrow S}, F^{ord})) \xrightarrow{T^0(f_!,hom)(E(-),E(-))}$$
$$R\mathcal{H}om_{D_X}(Rf_!((M, F) \otimes_{D_X} L_D(D_{X\leftarrow S}, F^{ord})), Rf_!((N, F) \otimes_{D_X} L_D(D_{X\leftarrow S}, F^{ord}))) =$$
$$R\mathcal{H}om_{D_X}(\int_{f!} (M, F), \int_{f!} (N, F))$$

Definition 72. *Let $f : X \to S$ a morphism with $X, S \in \mathrm{SmVar}(\mathbb{C})$ or with $X, S \in \mathrm{AnSm}(\mathbb{C})$. We have, for $(M, F), (N, F) \in C_{\mathcal{D}fil}(S)$, the canonical transformation map in $C_{\mathcal{D}fil}(X)$*

$$\begin{aligned}T^{\mathcal{D}}(f, hom)((M, F), (N, F)) : f^*\mathcal{H}om_{D_S}((M, F), (N, F)) \\ \xrightarrow{T(f,hom)((M,F),(N,F))} \mathcal{H}om_{f^*D_S}(f^*(M, F), (f^*(N, F)) \\ \to \mathcal{H}om_{D_X}(f^*(M, F) \otimes_{f^*D_S} L_{f^*D}(D_{X\to S}, F^{ord}), f^*(N, F) \otimes_{f^*D_S} L_{f^*D}(D_{X\to S}, F^{ord})) \\ \xrightarrow{=} \mathcal{H}om_{D_X}(f^{*mod}(M, F), f^{*mod}(N, F))\end{aligned}$$

which is the one given by Kashiwara (see [19]).

In the algebraic case, we have, in the non filtered case, the six functor formalism for holonomic D-modules :

Theorem 24. *Let $f : X \to S$ a morphism with $X, S \in \mathrm{SmVar}(\mathbb{C})$.*

(i) We have, for $M \in D_{\mathcal{D},h}(X)$ and $N \in D_{\mathcal{D},h}(S)$ a canonical isomorphism in $D_{\mathcal{D}}(S)$

$$I^{\mathcal{D}mod}(Lf^{\hat{*}mod[-]}, \int_f)(M, N) : Rf_*R\mathcal{H}om_{D_X}(Lf^{\hat{*}mod[-]}N, M) \xrightarrow{\sim} R\mathcal{H}om_{D_S}(N, \int_f M).$$

(ii) We have, for $M \in D_{\mathcal{D},h}(X)$ and $N \in D_{\mathcal{D},h}(S)$ a canonical isomorphism in $D_{\mathcal{D}}(X)$

$$I^{\mathcal{D}mod}(\int_{f!}, Lf^{*mod[-]})(M, N) : R\mathcal{H}om_{D_X}(\int_{f!} M, N) \xrightarrow{\sim} Rf_*R\mathcal{H}om_{D_S}(M, Lf^{*mod[-]}N).$$

Proof. Follows from the projection case and the closed embedding case. □

Corollary 2. *Let $f : X \to S$ a morphism with $X, S \in \mathrm{SmVar}(\mathbb{C})$. Then,*

- *$(Lf^{\hat{*}mod[-]}, \int_f) : D_{\mathcal{D},h}(S) \to D_{\mathcal{D},h}(X)$ is a pair of adjoint functors.*
- *$(\int_{f!}, Lf^{*mod[-]}) : D_{\mathcal{D},h}(S) \to D_{\mathcal{D},h}(X)$ is a pair of adjoint functors.*

Proof. Follows immediately from theorem 24 by taking global sections. □

Consider a commutative diagram in $\mathrm{SmVar}(\mathbb{C})$,

$$D = \begin{array}{ccc} X' & \xrightarrow{f'} & T \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & S \end{array} .$$

We have, for $M \in C_{\mathcal{D},h}(X)$, the following transformation maps

$$\begin{aligned}T_1^{\mathcal{D}mod}(D)(M) : Lg^{\hat{*}mod[-]} \int_f M \xrightarrow{\mathrm{ad}(Lf'^{\hat{*}mod[-]}, \int_{f'})(-)} \int_{f'} Lf'^{\hat{*}mod[-]} Lg^{\hat{*}mod[-]} \int_f M \xrightarrow{=} \\ \int_{f'} Lg'^{\hat{*}mod[-]}_{FDR} Lf^{\hat{*}mod[-]} \int_f M \xrightarrow{\mathrm{ad}(Lf^{\hat{*}mod[-]}, \int_f)(M)} \int_{f'} Lg'^{\hat{*}mod[-]}_{FDR} M\end{aligned}$$

and

$$\begin{aligned}T_2^{\mathcal{D}mod}(D)(M, F) : \int_{f'!} Lg'^{*mod[-]} M \xrightarrow{\mathrm{ad}(\int_{f!}, Lf^{*mod[-]})(-)} \int_{f'!} Lg'^{*mod[-]} Lf^{*mod[-]} \int_{f!} M \xrightarrow{=} \\ \int_{f'!} Lf'^{*mod[-]} Lg^{*mod[-]} M \int_{f!} \xrightarrow{\mathrm{ad}(\int_{f'!}, Lf'^{*mod[-]})(-)} Lg^{*mod[-]} \int_{f!}^{FDR} M\end{aligned}$$

Proposition 83. *Consider a cartesian square in* SmVar($\mathbb{C}$)

$$\begin{tikzcd} D = & X_T \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ & T \arrow[r, "g"] & S \end{tikzcd}$$

Assume that f (and hence f') is proper. Then, for $(M,F) \in D_{\mathcal{D}(2)fil,\infty,h}(X)$,

- $T_1^{\mathcal{D}mod}(f,g)(M) : Lg^{\hat{*}mod[-]} \int_f M \xrightarrow{\sim} \int_{f'} Lg'^{\hat{*}mod[-]} M$ *and*
- $T_2^{\mathcal{D}mod}(f,g)(M) : \int_{f'!} Lg'^{*mod[-]} M \xrightarrow{\sim} Lg^{*mod[-]} \int_{f!} M$

are isomorphisms in $D_{\mathcal{D}}(T)$.

Proof. Follows from proposition 80 and the fact that the map $T_1^{\mathcal{D}mod}(f,g)(M)$ is given by the composite

$$T_1^{\mathcal{D}mod}(f,g)(M)[d_T - d_S] : Lg^{\hat{*}mod} \int_f (M,F) = L\mathbb{D}_T Lg^{*mod} L\mathbb{D}_S \int_f M \xrightarrow{T(f_*,f_!)(-)}$$

$$L\mathbb{D}_T Lg^{*mod} \int_f L\mathbb{D}_X M \xrightarrow{(L\mathbb{D}_T T^{\mathcal{D}mod}(f,g)(\mathbb{D}_X M))^{-1}} \mathbb{D}_T \int_{f'} Lg'^{*mod} \mathbb{D}_X M$$

$$\xrightarrow{T(f'_!,f'_*)(-)} \int_{f'} L\mathbb{D}_{X_T} Lg'^{\hat{*}mod} \mathbb{D}_X(M,F) = \int_{f'} Lg'^{\hat{*}mod} M$$

and the map $T_2^{\mathcal{D}mod}(f,g)(M,F)$ is given by the composite

$$T_2^{\mathcal{D}mod}(f,g)(M)[d_T - d_S] : \int_{f'!} Lg'^{*mod} M = L\mathbb{D}_T \int_{f'} \mathbb{D}_{X_T} Lg'^{*mod} M \xrightarrow{d(-)\circ T(f_*,f_!)(-)}$$

$$\int_{f'} Lg'^{*mod} M \xrightarrow{T^{\mathcal{D}mod}(f,g)(M)^{-1}} L\mathbb{D}_T Lg^{*mod} \int_f L\mathbb{D}_X M = Lg^{*mod}(M,F) \int_{f!} M$$

□

4.3 The D modules on singular algebraic varieties and singular complex analytic spaces

In this subsection by defining the category of complexes of filtered D-modules in the singular case and there functorialities.

4.3.1 Definition

In all this subsection, we fix the notations:

- For $S \in \mathrm{Var}(\mathbb{C})$, we denote by $S = \cup_i S_i$ an open cover such that there exits closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. We have then closed embeddings $i_I : S_I := \cap_{i\in I} S_i \hookrightarrow \tilde{S}_I := \Pi_{i\in I}\tilde{S}_I$. Then for $I \subset J$, we denote by $j_{IJ} : S_J \hookrightarrow S_I$ the open embedding and $p_{IJ} : \tilde{S}_J \to \tilde{S}_I$ the projection, so that $p_{IJ} \circ i_J = i_I \circ j_{IJ}$. This gives the diagram of algebraic varieties $(\tilde{S}_I) \in \mathrm{Fun}(\mathcal{P}(\mathbb{N}), \mathrm{Var}(\mathbb{C}))$ which gives the diagram of sites $(\tilde{S}_I) := \mathrm{Ouv}(\tilde{S}_I) \in \mathrm{Fun}(\mathcal{P}(\mathbb{N}), \mathrm{Cat})$. For $I \subset J$, we denote by $m : \tilde{S}_I \backslash (S_I \backslash S_J) \hookrightarrow \tilde{S}_I$ the open embedding.
- For $S \in \mathrm{AnSp}(\mathbb{C})$ we denote by $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{AnSm}(\mathbb{C})$. We have then closed embeddings $i_I : S_I = \cap_{i\in I} S_i \hookrightarrow \tilde{S}_I = \Pi_{i\in I}\tilde{S}_I$. Then for $I \subset J$, we denote by $j_{IJ} : S_J \hookrightarrow S_I$ the open embedding and $p_{IJ} : \tilde{S}_J \to \tilde{S}_I$ the projection, so that $p_{IJ} \circ i_J = i_I \circ j_{IJ}$. This gives the diagram of analytic spaces $(\tilde{S}_I) \in \mathrm{Fun}(\mathcal{P}(\mathbb{N}), \mathrm{AnSp}(\mathbb{C}))$ which gives the diagram of sites $(\tilde{S}_I) := \mathrm{Ouv}(\tilde{S}_I) \in \mathrm{Fun}(\mathcal{P}(\mathbb{N}), \mathrm{Cat})$. For $I \subset J$, we denote by $m : \tilde{S}_I \backslash (S_I \backslash S_J) \hookrightarrow \tilde{S}_I$ the open embedding.

The first definition is from [27] remark 2.1.20, where we give a shifted version to have compatibility with perverse sheaves.

Definition 73. *Let $S \in \mathrm{Var}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$; or let $S \in \mathrm{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exit closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{AnSm}(\mathbb{C})$. Then, $\mathrm{PSh}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)) \subset \mathrm{PSh}_{\mathcal{D}(2)fil}((\tilde{S}_I))$ is the full subcategory*

- *whose objects are $(M, F) = ((M_I, F)_{I \subset [1,\cdots l]}, s_{IJ})$, with*
  - *$(M_I, F) \in \mathrm{PSh}_{\mathcal{D}(2)fil}(\tilde{S}_I)$ such that $\mathcal{I}_{S_I} M_I = 0$, in particular $(M_I, F) \in \mathrm{PSh}_{\mathcal{D}(2)fil,S_I}(\tilde{S}_I)$*
  - *$s_{IJ} : m^*(M_I, F) \xrightarrow{\sim} m^* p_{IJ*}(M_J, F)[d_{\tilde{S}_I} - d_{\tilde{S}_J}]$ for $I \subset J$, are isomorphisms, $p_{IJ} : \tilde{S}_J \to \tilde{S}_I$ being the projection, satisfying for $I \subset J \subset K$, $p_{IJ*} s_{JK} \circ s_{IJ} = s_{IK}$;*
- *the morphisms $m : (M, F) \to (N, F)$ between $(M, F) = ((M_I, F)_{I \subset [1,\cdots l]}, s_{IJ})$ and $(N, F) = ((N_I, F)_{I \subset [1,\cdots l]}, r_{IJ})$ are by definition a family of morphisms of complexes,*

  $$m = (m_I : (M_I, F) \to (N_I, F))_{I \subset [1,\cdots l]}$$

  such that $r_{IJ} \circ m_J = p_{IJ*} m_J \circ s_{IJ}$ in $C_{\mathcal{D},S_J}(\tilde{S}_J)$.

We denote by

$$\mathrm{PSh}_{\mathcal{D}(2)fil,rh}(S/(\tilde{S}_I)) \subset \mathrm{PSh}_{\mathcal{D}(2)fil,h}(S/(\tilde{S}_I)) \subset \mathrm{PSh}_{\mathcal{D}(2)fil,c}(S/(\tilde{S}_I)) \subset \mathrm{PSh}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$$

the full subcategory consisting of $((M_I, F), s_{IJ})$ such that (M_I, F) is filtered coherent, resp. filtered holonomic, resp. filtered regular holonomic, i.e. M_I are coherent, resp. holonomic,resp. filtered regular holonomic, sheaves of $D_{\tilde{S}_I}$ modules and F is a good filtration. We have the full subcategories

$$\mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \subset \mathrm{PSh}_{\mathcal{D}2fil,rh}(S/(\tilde{S}_I)),\ \mathrm{PSh}_{\mathcal{D}(1,0)fil,h}(S/(\tilde{S}_I)) \subset \mathrm{PSh}_{\mathcal{D}2fil,h}(S/(\tilde{S}_I)),$$
$$\mathrm{PSh}_{\mathcal{D}(1,0)fil,h}(S/(\tilde{S}_I)) \subset \mathrm{PSh}_{\mathcal{D}2fil,h}(S/(\tilde{S}_I)),$$

consisting of $((M_I, F, W), s_{IJ})$ such that $W^p M_I$ are $D_{\tilde{S}_I}$ submodules.

A morphism $m = (m_I) : ((M_I), s_{IJ}) \to ((N_I), r_{IJ})$ in $C(\mathrm{PSh}_{\mathcal{D}}(S/(\tilde{S}_I)))$ is a Zariski, resp. usu, local equivalence if all the m_I are Zariski, resp. usu, local equivalences. A morphism $m = (m_I) : ((M_I, F), s_{IJ} \to ((N_I, F), r_{IJ}))$ in $C(\mathrm{PSh}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)))$ is an r-filtered Zariski, resp. usu, local equivalence if all the m_I are r-filtered Zariski, resp. usu, local equivalence.

Let $S \in \mathrm{Var}(\mathbb{C})$ or $S \in \mathrm{AnSp}(\mathbb{C})$.

- If $S \in \mathrm{Var}(\mathbb{C})$, let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$, and let $S = \cup_{i'=1}^{l'} S_{i'}$ an other open cover such that there exist closed embeddings $i_{i'} : S_{i'} \hookrightarrow \tilde{S}_{i'}$ with $\tilde{S}_{i'} \in \mathrm{SmVar}(\mathbb{C})$.
- If $S \in \mathrm{AnSp}(\mathbb{C})$, let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{AnSm}(\mathbb{C})$, and let $S = \cup_{i'=1}^{l'} S_{i'}$ an other open cover such that there exist closed embeddings $i_{i'} : S_{i'} \hookrightarrow \tilde{S}_{i'}$ with $\tilde{S}_{i'} \in \mathrm{AnSm}(\mathbb{C})$.

Denote $L = [1, \ldots, l]$, $L' = [1, \ldots, l']$ and $L'' := [1, \ldots, l] \sqcup [1, \ldots, l']$. We have then the refined open cover $S = \cup_{k \in L} S_k$ and we denote for $I \sqcup I' \subset L''$, $S_{I \sqcup I'} := \cap_{k \in I \sqcup I'} S_k$ and $\tilde{S}_{I \sqcup I'} := \Pi_{k \in I \sqcup I'} \tilde{S}_k$, so that we have a closed embedding $i_{I \sqcup I'} : S_{I \sqcup I'} \hookrightarrow \tilde{S}_{I \sqcup I'}$. For $I \sqcup I' \subset J \sqcup J'$, denote by $p_{I \sqcup I', J \sqcup J'} : \tilde{S}_{J \sqcup J'} \to \tilde{S}_{I \sqcup I'}$ the projection. We then have a natural transfer map

$$T_S^{L/L'} : \mathrm{PSh}_{\mathcal{D}fil}(S/(S_I)) \to \mathrm{PSh}_{\mathcal{D}fil}(S/(S_{I'})),$$
$$((M_I, F), s_{IJ}) \mapsto (\operatorname{ho}\lim_{I \in L} p_{I'(I \sqcup I')*}(p_{I(I \sqcup I')}^{*mod}(M_I, F))/\mathcal{I}_{S_{I \sqcup I'}}, s_{I'J'}),$$

with, in the homotopy limit, the natural transition morphisms

$$p_{I'(I\sqcup I')*}\operatorname{ad}(p_{IJ}^{*mod},p_{IJ*})(p_{I(I\sqcup I')}^{*mod[-]}(M_I,F)):$$
$$p_{I'(J\sqcup I')*}(p_{J(J\sqcup I')}^{*mod[-]}(M_J,F))/\mathcal{I}_{S_{J\sqcup I'}}\to p_{I'(I\sqcup I')*}(p_{I(I\sqcup I')}^{*mod[-]}(M_I,F))/\mathcal{I}_{S_{I\sqcup I'}}$$

for $J\subset I$, and

$$s_{I'J'}:\operatorname{holim}_{I\in L}m^*p_{I'(I\sqcup I')*}(p_{I(I\sqcup I')}^{*mod[-]}(M_I,F)/\mathcal{I}_{S_{I\sqcup I'}})\to$$
$$\operatorname{holim}_{I\in L}p_{I'J'*}(p_{I'J'}^{*mod[-]}m^*p_{I'(I\sqcup I')*}p_{I(I\sqcup I')}^{*mod[-]}((M_I,F)/\mathcal{I}_{S_{I\sqcup I'}}))/\mathcal{I}_{S_{J'}}$$
$$\to\operatorname{holim}_{I\in L}p_{I'J'*}p_{J'(I\sqcup J')*}(p_{I(I\sqcup J')}^{*mod[-]}(M_I,F)\mathcal{I}_{S_{I\sqcup I'}})$$

Definition-Proposition 18. *Let $S\in\mathrm{Var}(\mathbb{C})$ and let $S=\cup_iS_i$ an open cover such that there exist closed embeddings $i_iS_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$; or let $S\in\mathrm{AnSp}(\mathbb{C})$ and let $S=\cup_iS_i$ an open cover such that there exist closed embeddings $i_iS_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{AnSm}(\mathbb{C})$. Then $\mathrm{PSh}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$ does not depend on the open covering of S and the closed embeddings and we set*

$$\mathrm{PSh}_{\mathcal{D}(2)fil}(S):=\mathrm{PSh}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$$

We denote by $C^0_{\mathcal{D}(2)fil}(S):=C(\mathrm{PSh}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)))$ and by $D^0_{\mathcal{D}(2)fil,r}(S):=\mathrm{Ho}_{Frtop}(C^0_{\mathcal{D}(2)fil}(S))$ its localization with respect to r-filtered Zariski, resp. usu, local equivalences.

Proof. It is obvious that $T_S^{L/L'}:\mathrm{PSh}_{\mathcal{D}fil}(S/(S_I))\to\mathrm{PSh}_{\mathcal{D}}(S/(S_{I'}))$ is an equivalence of category with inverse $T_S^{L'/L}:\mathrm{PSh}_{\mathcal{D}fil}(S/(S_{I'}))\to\mathrm{PSh}_{\mathcal{D}}(S/(S_I))$. ☐

We now give the definition of our category :

Definition 74. *Let $S\in\mathrm{Var}(\mathbb{C})$ and let $S=\cup_iS_i$ an open cover such that there exist closed embeddings $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$; or let $S\in\mathrm{AnSp}(\mathbb{C})$ and let $S=\cup_iS_i$ an open cover such that there exist closed embeddings $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{AnSm}(\mathbb{C})$. Then, $C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))\subset C_{\mathcal{D}(2)fil}((\tilde{S}_I))$ is the full subcategory*

- *whose objects are $(M,F)=((M_I,F)_{I\subset[1,\cdots l]},u_{IJ})$, with*
  - *$(M_I,F)\in C_{\mathcal{D}(2)fil,S_I}(\tilde{S}_I)$ (see definition 54),*
  - *$u_{IJ}:m^*(M_I,F)\to m^*p_{IJ*}(M_J,F)[d_{\tilde{S}_I}-d_{\tilde{S}_J}]$ for $J\subset I$, are morphisms, $p_{IJ}:\tilde{S}_J\to\tilde{S}_I$ being the projection, satisfying for $I\subset J\subset K$, $p_{IJ}*u_{JK}\circ u_{IJ}=u_{IK}$ in $C_{\mathcal{D}fil}(\tilde{S}_I)$;*
- *the morphisms $m:((M_I,F),u_{IJ})\to((N_I,F),v_{IJ})$ between $(M,F)=((M_I,F)_{I\subset[1,\cdots l]},u_{IJ})$ and $(N,F)=((N_I,F)_{I\subset[1,\cdots l]},v_{IJ})$ being a family of morphisms of complexes,*

  $$m=(m_I:(M_I,F)\to(N_I,F))_{I\subset[1,\cdots l]}$$

  such that $v_{IJ}\circ m_I=p_{IJ*}m_J\circ u_{IJ}$ in $C_{\mathcal{D}fil}(\tilde{S}_I)$.

We denote by $C^{\sim}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))\subset C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$ the full subcategory consisting of objects $((M_I,F),u_{IJ})$ such that the u_{IJ} are ∞-filtered Zariski, resp. usu, local equivalences.

Let $S\in\mathrm{Var}(\mathbb{C})$ and let $S=\cup_iS_i$ an open cover such that there exist closed embeddings $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$; or let $S\in\mathrm{AnSp}(\mathbb{C})$ and let $S=\cup_iS_i$ an open cover such that there exist closed embeddings $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{AnSm}(\mathbb{C})$. Then, We denote by

$$C_{\mathcal{D}(2)fil,rh}(S/(\tilde{S}_I))\subset C_{\mathcal{D}(2)fil,h}(S/(\tilde{S}_I))\subset C_{\mathcal{D}(2)fil,c}(S/(\tilde{S}_I))\subset C^{\sim}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$$

the full subcategories consisting of those $((M_I,F),u_{IJ})\in C^{\sim}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$ such that $(M_I,F)\in C_{\mathcal{D}(2)fil,S_I,c}(\tilde{S}_I)$, that is such that $a_\tau H^n(M_I,F)$ are filtered coherent for all $n\in\mathbb{Z}$ and all $I\subset[1,\cdots l]$ (i.e. $a_\tau H^n(M_I)$ are coherent sheaves of $D_{\tilde{S}_I}$ modules and F induces a good filtration on $a_\tau H^n(M_I)$), resp. such that $(M_I,F)\in C_{\mathcal{D}(2)fil,S_I,h}(\tilde{S}_I)$, that is such that $a_\tau H^n(M_I,F)$ are filtered holonomic for all $n\in\mathbb{Z}$ and all $I\subset[1,\cdots l]$ (i.e. $a_\tau H^n(M_I)$ are holonomic sheaves of $D_{\tilde{S}_I}$ modules and F induces a good filtration on $a_\tau H^n(M_I)$), resp. such that $(M_I,F)\in C_{\mathcal{D}(2)fil,S_I,rh}(\tilde{S}_I)$, that is such that $a_\tau H^n(M_I,F)$ are filtered regular holonomic for all $n\in\mathbb{Z}$ and all $I\subset[1,\cdots l]$ (i.e. $a_\tau H^n(M_I)$ are regular holonomic sheaves of $D_{\tilde{S}_I}$ modules and F induces a good filtration on $a_\tau H^n(M_I)$).

We denote by

$$C_{\mathcal{D}(1,0)fil,h}(S/(\tilde{S}_I))\subset C_{\mathcal{D}2fil,h}(S/(\tilde{S}_I)),\ C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\subset C_{\mathcal{D}2fil,rh}(S/(\tilde{S}_I)),$$
$$C^{\sim}_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))\subset C^{\sim}_{\mathcal{D}2fil}(S/(\tilde{S}_I)),$$

the full subcategories consisting of those $((M_I,F,W),u_{IJ})\in C^{\sim}_{\mathcal{D}2fil}(S/(\tilde{S}_I))$ such that W^pM_I are $D_{\tilde{S}_I}$ submodules (resp. and $a_\tau H^n(M_I,F)$ are filtered holonomic).

A morphism $m=(m_I):((M_I),u_{IJ})\to((N_I),v_{IJ})$ in $C_{\mathcal{D}}(S/(\tilde{S}_I))$ is a Zariski, resp. usu, local equivalence if all the m_I are Zariski, resp. usu, local equivalence. A morphism $m=(m_I):((M_I,F),u_{IJ}\to((N_I,F),v_{IJ}))$ in $C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$ is an r-filtered Zariski, resp. usu, local equivalence if all the m_I are r-filtered Zariski, resp. usu, local equivalence.

In the analytic case, we also define in the same way :

Definition 75. *Let $S\in\mathrm{AnSp}(\mathbb{C})$ and let $S=\cup_iS_i$ an open cover such that there exist closed embeddings $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{AnSm}(\mathbb{C})$. Then, $C_{\mathcal{D}^\infty(2)fil}(S/(\tilde{S}_I))\subset C_{\mathcal{D}^\infty(2)fil}((\tilde{S}_I))$ is the full subcategory*

- *whose objects are $(M,F)=((M_I,F)_{I\subset[1,\cdots l]},u_{IJ})$, with*
  - *$(M_I,F)\in C_{\mathcal{D}^\infty fil,S_I}(\tilde{S}_I)$ (see definition 55),*
  - *$u_{IJ}:m^*(M_I,F)\to m^*p_{IJ*}(M_I,F)[d_{\tilde{S}_I}-d_{\tilde{S}_J}]$, for $J\subset I$, are morphisms, $p_{IJ}:\tilde{S}_J\to\tilde{S}_I$ being the projection, satisfying for $I\subset J\subset K$, $p_{IJ*}u_{JK}\circ u_{IJ}=u_{IK}$ in $C_{\mathcal{D}^\infty fil}(\tilde{S}_I)$;*
- *the morphisms $m:((M_I,F),u_{IJ})\to((N_I,F),v_{IJ})$ between $(M,F)=((M_I,F)_{I\subset[1,\cdots l]},u_{IJ})$ and $(N,F)=((N_I,F)_{I\subset[1,\cdots l]},v_{IJ})$ being a family of morphisms of complexes,*

  $$m=(m_I:(M_I,F)\to(N_I,F))_{I\subset[1,\cdots l]}$$

  such that $v_{IJ}\circ m_I=p_{IJ*}m_J\circ u_{IJ}$ in $C_{\mathcal{D}^\infty fil}(\tilde{S}_I)$.

We denote by $C^{\sim}_{\mathcal{D}^\infty(2)fil}(S/(\tilde{S}_I))\subset C_{\mathcal{D}^\infty(2)fil}(S/(\tilde{S}_I))$ the full subcategory consisting of objects $((M_I,F),u_{IJ})$ such that the u_{IJ} are ∞-filtered usu local equivalence.

Let $S\in\mathrm{AnSp}(\mathbb{C})$ and let $S=\cup_iS_i$ an open cover such that there exist closed embeddings $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{AnSm}(\mathbb{C})$. We denote by

$$C_{\mathcal{D}^\infty(2)fil,h}(S/(\tilde{S}_I))\subset C_{\mathcal{D}^\infty(2)fil,c}(S/(\tilde{S}_I))\subset C^{\sim}_{\mathcal{D}^\infty(2)fil}(S/(\tilde{S}_I))$$

the full subcategories consisting of $((M_I,F),u_{IJ})\in C^{\sim}_{\mathcal{D}^\infty(2)fil}(S/(\tilde{S}_I))$ such that $(M_I,F)\in C_{\mathcal{D}^\infty(2)fil,S_I,c}(\tilde{S}_I)$, that is such that $a_\tau H^n(M_I,F)$ are filtered coherent for all $n\in\mathbb{Z}$ and all $I\subset[1,\cdots l]$, resp. such that $(M_I,F)\in C_{\mathcal{D}^\infty(2)fil,S_I,h}(\tilde{S}_I)$, that is such that $a_\tau H^n(M_I,F)$ are filtered holonomic for all $n\in\mathbb{Z}$ and all $I\subset[1,\cdots l]$. We denote by

$$C_{\mathcal{D}^\infty(1,0)fil}(S/(\tilde{S}_I))\subset C^{\sim}_{\mathcal{D}^\infty 2fil}(S/(\tilde{S}_I)),\ C_{\mathcal{D}^\infty(1,0)fil,h}(S/(\tilde{S}_I))\subset C^{\sim}_{\mathcal{D}^\infty 2fil,h}(S/(\tilde{S}_I))$$

the full subcategories consisting of those $((M_I,F,W),u_{IJ})\in C^{\sim}_{\mathcal{D}2fil}(S/(\tilde{S}_I))$ such that W^pM_I are $D_{\tilde{S}_I}$ submodules (resp. and $a_\tau H^n(M_I,F)$ filtered holonomic).

A morphism $m = (m_I) : ((M_I), u_{IJ}) \to ((N_I), v_{IJ})$ in $C_{\mathcal{D}^\infty}(S/(\tilde{S}_I))$ is said to an usu local equivalence if all the m_I are usu local equivalences. A morphism $m = (m_I) : ((M_I, F), u_{IJ} \to ((N_I, F), v_{IJ}))$ in $C_{\mathcal{D}^\infty(2)fil}(S/(\tilde{S}_I))$ is said to an r-filtered usu local equivalence if all the m_I are r-filtered usu local equivalences.

Definition 76. *Let $S \in \mathrm{Var}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$; or let $S \in \mathrm{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{AnSm}(\mathbb{C})$. We denote by*

$$D_{\mathcal{D}(2)fil,\infty}(S/(\tilde{S}_I)) := \mathrm{Ho}_{F\infty,top}(C^{\sim}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)))$$

the localizations with respect to ∞-filtered Zariski, resp. usu, local equivalences. We have

$$D_{\mathcal{D}(1,0)fil,\infty,h}(S/(\tilde{S}_I)) \subset D_{\mathcal{D}2fil,\infty,h}(S/(\tilde{S}_I)) \subset D_{\mathcal{D}2fil,\infty}(S/(\tilde{S}_I))$$

the full subcategories which are the image of $C_{\mathcal{D}2fil,h}(S/(\tilde{S}_I))$, resp. of $C_{\mathcal{D}(1,0)fil,h}(S/(\tilde{S}_I))$, by the localization functor $D(top) : C^{\sim}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)) \to D_{\mathcal{D}(2)fil,\infty}(S/(\tilde{S}_I))$.

In the analytic case, we also have

Definition 77. *Let $S \in \mathrm{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{AnSm}(\mathbb{C})$. We denote by*

$$D_{\mathcal{D}^\infty(2)fil,\infty}(S/(\tilde{S}_I)) := \mathrm{Ho}_{Frtop}(C^{\sim}_{\mathcal{D}^\infty(2)fil}(S/(\tilde{S}_I)))$$

the localizations with respect to usu local equivalence. We have then

$$D_{\mathcal{D}^\infty(1,0)fil,\infty,h}(S/(\tilde{S}_I)) \subset D_{\mathcal{D}^\infty 2fil,\infty,h}(S/(\tilde{S}_I)) \subset D_{\mathcal{D}^\infty 2fil,\infty}(S/(\tilde{S}_I))$$

the full subcategories wich are the image of $C_{\mathcal{D}^\infty 2fil,h}(S/(\tilde{S}_I))$, resp. $C_{\mathcal{D}^\infty(1,0)fil,h}(S/(\tilde{S}_I))$, by the localization functor $D(usu) : C^{\sim}_{\mathcal{D}^\infty(2)fil}(S/(\tilde{S}_I)) \to D_{\mathcal{D}^\infty(2)fil,\infty}(S/(\tilde{S}_I))$.

Definition 78. *Let $S \in \mathrm{Var}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. Or let $S \in \mathrm{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{AnSm}(\mathbb{C})$.*

(i) We denote by

$$C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))^0 \subset C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$$

the full subcategory consisting of $((M_I, F), u_{IJ}) \in C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$ such that

$$H^n((M_I, F), u_{IJ}) = (H^n(M_I, F), H^n u_{IJ}) \in \mathrm{PSh}^0_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$$

that is such that the $H^n u_{IJ}$ are isomorphism. We denote by $D_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))^0 := D(top)(C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))^0)$ its image by the localization functor.

(ii) We have the full embedding functor

$$\iota^0_{S/(\tilde{S}_I)} : C^0_{\mathcal{D}(2)fil}(S) := C^0_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)) \hookrightarrow C^{\sim}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)),$$
$$((M_I, F), s_{IJ}) \mapsto ((M_I, F), s_{IJ})$$

By definition, $\iota^0_{S/(\tilde{S}_I)}(C^0_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))) \subset C^{\sim}_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))$. This full embedding induces in the derived category the functor

$$\iota^0_{S/(\tilde{S}_I)} : D^0_{\mathcal{D}(2)fil,\infty}(S) := D^0_{\mathcal{D}(2)fil,\infty}(S/(\tilde{S}_I)) \to D_{\mathcal{D}(2)fil,\infty}(S/(\tilde{S}_I)),$$
$$((M_I, F), s_{IJ}) \mapsto ((M_I, F), s_{IJ}).$$

Proposition 84. *Let $S \in \mathrm{Var}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. Or let $S \in \mathrm{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{AnSm}(\mathbb{C})$. Then,*

$$\iota^0_{S/(\tilde{S}_I)} : D^0_{\mathcal{D}(2)fil,\infty}(S) \to D_{\mathcal{D}(2)fil,\infty}(S/(\tilde{S}_I))$$

is a full embedding whose image is $D_{\mathcal{D}(2)fil,\infty}(S/(\tilde{S}_I))^0$, that is consists of $((M_I, F), s_{IJ}) \in C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$ such that

$$H^n((M_I, F), s_{IJ}) := (H^n(M_I, F), H^n(s_{IJ})) \in \mathrm{PSh}^0_{\mathcal{D}}(S/(\tilde{S}_I)).$$

and

$$\iota^0_S := \iota^0_{S/(\tilde{S}_I)} : D^0_{\mathcal{D}(2)fil,\infty}(S) \xrightarrow{\sim} D_{\mathcal{D}(2)fil,\infty}(S/(\tilde{S}_I))^0$$

the induced equivalence of categories.

Proof. Standard. □

We finish this subsection by the statement a result of kashiwara in the singular case.

Definition 79. *Let $S \in \mathrm{AnSp}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{AnSm}(\mathbb{C})$. We will consider the functor*

$$J_S : C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)) \to C_{\mathcal{D}^\infty(2)fil}(S/(\tilde{S}_I)),$$
$$((M_I, F), u_{IJ}) \mapsto J_S((M_I, F), u_{IJ}) := (J_{\tilde{S}_I}(M_I, F), J(u_{IJ})) := ((M_I \otimes_{D_S} D^\infty_S, F), J(u_{IJ}))$$

with, denoting for short $d_{IJ} := d_{\tilde{S}_J} - d_{\tilde{S}_I}$,

$$J(u_{IJ}) : J(M_I, F) \xrightarrow{J(u_{IJ})} J(p_{IJ*}(M_J, F)[d_{IJ}]) \xrightarrow{T_*(p_{IJ}, J)(-)} p_{IJ*}J(M_J, F)[d_{IJ}].$$

Of course $J_S(C_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))) \subset C_{\mathcal{D}^\infty(1,0)fil}(S/(\tilde{S}_I))$.

Proposition 85. *Let $S \in \mathrm{AnSp}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{AnSm}(\mathbb{C})$. Then the functor*

$$J_S : C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)) \to C_{\mathcal{D}^\infty(2)fil}(S/(\tilde{S}_I)),$$

satisfy $J_S : C^\sim_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)) \subset C^\sim_{\mathcal{D}^\infty(2)fil}(S/(\tilde{S}_I))$ and induces an equivalence of category

$$J_S : D_{\mathcal{D}(2)fil,\infty,rh}(S/(\tilde{S}_I)) \to D_{\mathcal{D}^\infty(2)fil,\infty,h}(S/(\tilde{S}_I)).$$

and $J_S(D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I)) \subset D_{\mathcal{D}^{\infty,h}(1,0)fil,\infty}(S/(\tilde{S}_I))$.

Proof. Follows immediately from the smooth case (proposition 47). □

4.3.2 Duality in the singular case

The definition of Saito's category comes with a dual functor :

Definition 80. *Let $S \in \mathrm{Var}(\mathbb{C})$ and let $S = \cup S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$; or let $S \in \mathrm{AnSp}(\mathbb{C})$ and $S = \cup S_i$ an open cover such that there exist closed embedding $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{AnSm}(\mathbb{C})$. We have the dual functor :*

$$\mathbb{D}^K_S : C^0_{\mathcal{D}fil}(S/(\tilde{S}_I)) \to C^0_{\mathcal{D}fil}(S/(\tilde{S}_I)),\ ((M_I, F), s_{IJ}) \mapsto (\mathbb{D}^K_{\tilde{S}_I}(M_I, F), s^d_{IJ}),$$

with, denoting for short $d_{IJ} := d_{\tilde{S}_J} - d_{\tilde{S}_I}$,

$$u^q_{IJ} : \mathbb{D}^K_{\tilde{S}_I}(M_I, F) \xrightarrow{\mathbb{D}^K(s^{-1}_{IJ})} \mathbb{D}^K_{\tilde{S}_I} p_{IJ*}(M_J, F)[d_{IJ}] \xrightarrow{T_*(p_{IJ},D)(-)} p_{IJ*}\mathbb{D}^K_{\tilde{S}_J}(M_J, F)[d_{IJ}]$$

It induces in the derived category the functor

$$L\mathbb{D}^K_S : D^0_{\mathcal{D}fil}(S/(\tilde{S}_I)) \to D^0_{\mathcal{D}fil}(S/(\tilde{S}_I)),\ ((M_I, F), s_{IJ}) \mapsto \mathbb{D}^K_S Q((M_I, F), s_{IJ}),$$

with $q : Q((M_I, F), s_{IJ}) \to ((M_I, F), s_{IJ})$ *a projective resolution.*

In the analytic case we also define

Definition 81. *Let* $S \in \mathrm{AnSp}(\mathbb{C})$ *and* $S = \cup S_i$ *an open cover such that there exist closed embedding* $i_i : S_i \hookrightarrow \tilde{S}_i$ *with* $\tilde{S}_i \in \mathrm{AnSm}(\mathbb{C})$. *We have the dual functor :*

$$\mathbb{D}^{K,\infty}_S : C_{\mathcal{D}^\infty fil}(S/(\tilde{S}_I)) \to C_{\mathcal{D}^\infty fil}(S/(\tilde{S}_I)),\ ((M_I, F), u_{IJ}) \mapsto (\mathbb{D}^{K,\infty}_{\tilde{S}_I}(M_I, F), u^d_{IJ}),$$

with u^d_{IJ} *defined similarly as in definition 80. It induces in the derived category the functor*

$$L\mathbb{D}^{K,\infty}_S : D_{\mathcal{D}^\infty fil}(S/(\tilde{S}_I)) \to D_{\mathcal{D}^\infty fil}(S/(\tilde{S}_I)),\ ((M_I, F), u_{IJ}) \mapsto (\mathbb{D}^{K,\infty}_S Q((M_I, F), u^{q,d}_{IJ}),$$

with $q : Q((M_I, F), s_{IJ}) \to ((M_I, F), s_{IJ})$ *a projective resolution.*

4.3.3 Inverse image in the singular case

We give in this subsection the inverse image functors between our categories.

Let $n : S^o \hookrightarrow S$ be an open embedding with $S \in \mathrm{Var}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$; or let $n : S^o \hookrightarrow S$ be an open embedding with $S \in \mathrm{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{AnSm}(\mathbb{C})$. Denote $S^o_I := n^{-1}(S_I) = S_I \cap S^o$ and $n_I := n_{|S^o_I} : S^o_I \hookrightarrow S^o$ the open embeddings. Consider open embeddings $\tilde{n}_I : \tilde{S}^o_I \hookrightarrow \tilde{S}_I$ such that $\tilde{S}^o_I \cap S_I = S^o_I$, that is which are lift of n_I. We have the functor

$$\begin{aligned} n^* : C_{\mathcal{D}fil}(S/(\tilde{S}_I)) &\to C_{\mathcal{D}fil}(S^o/(\tilde{S}^o_I)), \\ (M, F) = ((M_I, F), u_{IJ}) &\mapsto n^*(M, F) := (\tilde{n}_I)^*(M, F) := (\tilde{n}^*_I(M_I, F), n^* u_{IJ}) \end{aligned}$$

which derive trivially.

Let $f : X \to S$ be a morphism, with $X, S \in \mathrm{Var}(\mathbb{C})$, such that there exist a factorization $f ; X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection, and consider $S = \cup^l_{i=1} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$, with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$; or let $f : X \to S$ be a morphism, with $X, S \in \mathrm{AnSp}(\mathbb{C})$, such that there exist a factorization $f ; X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{AnSm}(\mathbb{C})$, l a closed embedding and p_S the projection and consider $S = \cup^l_{i=1} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$, with $\tilde{S}_i \in \mathrm{AnSm}(\mathbb{C})$. Then, $X = \cup^l_{i=1} X_i$ with $X_i := f^{-1}(S_i)$. Denote by $p_{IJ} : \tilde{S}_J \to \tilde{S}_I$ and $p'_{IJ} : Y \times \tilde{S}_J \to Y \times \tilde{S}_I$ the projections and by

$$E_{IJ} = \begin{tikzcd} \tilde{S}_J \backslash S_J \arrow[r, "m_J"] \arrow[d, "p_{IJ}"] & \tilde{S}_J \arrow[d, "p_{IJ}"] \\ \tilde{S}_I \backslash (S_I \backslash S^o_J) \arrow[r, "m = m_{IJ}"] & \tilde{S}_I \end{tikzcd},\ E'_{IJ} = \begin{tikzcd} Y \times \tilde{S}_J \backslash X_J \arrow[r, "m'_J"] \arrow[d, "p'_{IJ}"] & Y \times \tilde{S}_J \arrow[d, "p'_{IJ}"] \\ Y \times \tilde{S}_I \backslash (X_I \backslash X^o_J) \arrow[r, "m' = m'_{IJ}"] & Y \times \tilde{S}_I \end{tikzcd},\ E_{fIJ} \begin{tikzcd} \tilde{X}_J \arrow[r, "\tilde{f}_J"] \arrow[d, "p'_{IJ}"] & \tilde{S}_J \arrow[d, "p_{IJ}"] \\ Y \times \tilde{S}_I \arrow[r, "\tilde{f}_I"] & \tilde{S}_I \end{tikzcd}$$

the commutative diagrams. The (graph) inverse image functors is :

$$\begin{aligned} f^{*mod[-],\Gamma} : C_{\mathcal{D}fil}(S/(\tilde{S}_I)) &\to C_{\mathcal{D}fil}(X/(Y \times \tilde{S}_I)), \\ (M, F) = ((M_I, F), u_{IJ}) &\mapsto f^{*mod[-],\Gamma}(M, F) := (\Gamma_{X_I} E(p^{*mod[-]}_{\tilde{S}_I}(M_I, F)), \tilde{f}^{*mod[-]}_J u_{IJ}) \end{aligned}$$

with, denoting for short $d_{IJ} := d_{\tilde{S}_I} - d_{\tilde{S}_J}$,

$$\tilde{f}_J^{*mod[-]}u_{IJ} : \Gamma_{X_I}E(p_{\tilde{S}_I}^{*mod[-]}(M_I,F)) \xrightarrow{\Gamma_{X_I}E(p_{\tilde{S}_I}^{*mod[-]})(u_{IJ})} \Gamma_{X_I}E(p_{\tilde{S}_I}^{*mod[-]}p_{IJ*}(M_J,F)[d_{IJ}])$$
$$\xrightarrow{\Gamma_{X_I}E(T(p_{IJ}^{*mod},p_{\tilde{S}_I})(-)^{-1})[d_Y+d_{IJ}]} \Gamma_{X_I}E(p'_{IJ*}p_{\tilde{S}_J}^{*mod}(M_J,F)[d_Y+d_{IJ}])$$
$$\xrightarrow{=} p'_{IJ*}\Gamma_{X_J}E(p_{\tilde{S}_J}^{*mod[-]}(M_J,F))[d_{IJ}].$$

It induces in the derived categories the functor

$$Rf^{*mod[-],\Gamma} : D_{\mathcal{D}(2)fil,\infty}(S/(\tilde{S}_I)) \to D_{\mathcal{D}(2)fil,\infty}(X/(Y\times\tilde{S}_I)),$$
$$(M,F) = ((M_I,F),u_{IJ}) \mapsto f^{*mod[-],\Gamma}(M,F) := (\Gamma_{X_I}E(p_{\tilde{S}_I}^{*mod[-]}(M_I,F)), \tilde{f}_J^{*mod[-]}u_{IJ}).$$

It gives by duality the functor

$$Lf^{\hat{*}mod[-],\Gamma} : D_{\mathcal{D}(2)fil,\infty}(S/(\tilde{S}_I))^0 \to D_{\mathcal{D}(2)fil,\infty}(X/(Y\times\tilde{S}_I))^0,$$
$$(M,F) = ((M_I,F),u_{IJ}) \mapsto Lf^{\hat{*}mod[-],\Gamma}(M,F) := L\mathbb{D}_S^K Rf^{*mod[-],\Gamma} L\mathbb{D}_S^K \iota_S^{0,-1}(M,F).$$

where $\iota_S^0 : D^0_{\mathcal{D}(2)fil,\infty}(S/(\tilde{S}_I)) \xrightarrow{\sim} D_{\mathcal{D}(2)fil,\infty}(S/(\tilde{S}_I))^0$ is the isomorphism of definition 78.

Let $f : X \to S$ be a morphism, with $X, S \in \mathrm{AnSp}(\mathbb{C})$, such that there exist a factorization $f; X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{AnSm}(\mathbb{C})$, l a closed embedding and p_S the projection and consider $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$, with $\tilde{S}_i \in \mathrm{AnSm}(\mathbb{C})$. Then, $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. We have also the functors,

$$f^{*mod[-],\Gamma} : C_{\mathcal{D}^\infty fil}(S/(\tilde{S}_I)) \to C_{\mathcal{D}^\infty fil}(X/(Y\times\tilde{S}_I)),$$
$$(M,F) = ((M_I,F),u_{IJ}) \mapsto f^{*mod[-],\Gamma}(M,F) := (\Gamma_{X_I}(p_{\tilde{S}_I}^{*mod[-]}(M_I,F), \tilde{f}_J^{*mod[-]}u_{IJ}))$$

with, denoting for short $d_{IJ} := d_{\tilde{S}_J} - d_{\tilde{S}_I}$,

$$\tilde{f}_J^{*mod[-]}u_{IJ} : \Gamma_{X_I}E(p_{\tilde{S}_I}^{*mod[-]}(M_I,F)) \xrightarrow{\Gamma_{X_I}E(p_{\tilde{S}_I}^{*mod[-]})(u_{IJ})} \Gamma_{X_I}E(p_{\tilde{S}_I}^{*mod[-]}p_{IJ*}(M_J,F)[d_{IJ}])$$
$$\xrightarrow{\Gamma_{X_I}E(T(p_{IJ}^{*mod},p_{\tilde{S}_I})(-)^{-1})[d_Y+d_{IJ}]} \Gamma_{X_I}E(p'_{IJ*}p_{\tilde{S}_J}^{*mod}(M_J,F)[d_Y+d_{IJ}])$$
$$\xrightarrow{=} p'_{IJ*}\Gamma_{X_J}E(p_{\tilde{S}_J}^{*mod[-]}(M_J,F))[d_{IJ}].$$

It induces in the derived categories, the functor

$$Rf^{*mod[-],\Gamma} : D_{\mathcal{D}^\infty(2)fil,\infty}(S/(\tilde{S}_I)) \to D_{\mathcal{D}^\infty(2)fil,\infty}(X/(Y\times\tilde{S}_I)),$$
$$(M,F) = ((M_I,F),u_{IJ}) \mapsto Rf^{*mod[-],\Gamma}(M,F) := (\Gamma_{X_I}E(p_{\tilde{S}_I}^{*mod[-]}(M_I,F)), \tilde{f}_J^{*mod[-]}u_{IJ}).$$

It gives by duality the functor

$$Lf^{\hat{*}mod[-],\Gamma} : D_{\mathcal{D}^\infty(2)fil,\infty}(S/(\tilde{S}_I))^0 \to D_{\mathcal{D}^\infty(2)fil,\infty}(X/(Y\times\tilde{S}_I))^0,$$
$$(M,F) = ((M_I,F),u_{IJ}) \mapsto Lf^{\hat{*}mod[-],\Gamma}(M,F) := L\mathbb{D}^{K,\infty} Rf^{*mod[-],\Gamma} L\mathbb{D}^{K,\infty} \iota_S^{0,-1}(M,F).$$

where $\iota_S^0 : D^0_{\mathcal{D}^\infty(2)fil,\infty}(S/(\tilde{S}_I)) \xrightarrow{\sim} D_{\mathcal{D}^\infty(2)fil,\infty}(S/(\tilde{S}_I))^0$ is the isomorphism of definition 78.

The following proposition are then easy :

Proposition 86. *Let $f_1 : X \to Y$ and $f_2 : Y \to S$ two morphism with $X, Y, S \in \mathrm{Var}(\mathbb{C})$. Assume there exist factorizations $f_1 : X \xrightarrow{l_1} Y' \times Y \xrightarrow{p_Y} Y$ and $f_2 : Y \xrightarrow{l_2} Y'' \times S \xrightarrow{p_S} S$ with $Y', Y'' \in \mathrm{SmVar}(\mathbb{C})$, l_1, l_2 closed embeddings and p_S, p_Y the projections. We have then the factorization*

$$f_2 \circ f_1 : X \xrightarrow{(l_2\circ I_{Y'})\circ l_1} Y' \times Y'' \times S \xrightarrow{p_S} S.$$

We have, for $(M,F) \in C_{\mathcal{D}(2)fil}^{\sim}(S/(\tilde{S}_I))$, $R(f_2 \circ f_1)^{*mod[-],\Gamma}(M,F) = Rf_2^{*mod[-],\Gamma} \circ Rf_1^{*mod[-],\Gamma}(M,F)$.

Proof. Follows from the the fact that for $(M,F)=((M_I,F),u_{IJ})\in C_{\tilde{\mathcal{D}(2)fil}}(S/(\tilde{S}_I))$,

$$(\Gamma_{X_I}E(\tilde{f}_{1I}^{*mod[-]}\Gamma_{Y_I}E(\tilde{f}_{2I}^{*mod[-]}(M_I,F))),\tilde{f}_{1J}^{*mod[-]}(\tilde{f}_{2J}^{*mod[-]}u_{IJ}))\xrightarrow{=}$$
$$(\Gamma_{X_I}E((\tilde{f}_{1I}\circ\tilde{f}_{2I})^{*mod[-]}(M_I,F)),(\tilde{f}_{1J}\circ\tilde{f}_{2J})^{*mod[-]}u_{IJ}^q)$$

by proposition 49(i) and the fact that $X_I\subset\tilde{f}_{1I}^{-1}(Y_I)$. □

Proposition 87. *Let $f_1:X\to Y$ and $f_2:Y\to S$ two morphism with $X,Y,S\in\mathrm{Var}(\mathbb{C})$. Assume there exist factorizations $f_1:X\xrightarrow{l_1}Y'\times Y\xrightarrow{p_Y}Y$ and $f_2:Y\xrightarrow{l_2}Y''\times S\xrightarrow{p_S}S$ with $Y',Y''\in\mathrm{SmVar}(\mathbb{C})$, l_1,l_2 closed embeddings and p_S,p_Y the projections. We have then the factorization*

$$f_2\circ f_1:X\xrightarrow{(l_2\circ I_{Y'})\circ l_1}Y'\times Y''\times S\xrightarrow{p_S}S.$$

We have, for $(M,F)\in C_{\tilde{\mathcal{D}(2)fil}}(S/(\tilde{S}_I))$ or $(M,F)\in C_{\tilde{\mathcal{D}^\infty(2)fil}}(S/(\tilde{S}_I))$, $R(f_2\circ f_1)^{*mod[-],\Gamma}(M,F)=Rf_2^{*mod[-],\Gamma}\circ Rf_1^{*mod[-],\Gamma}(M,F)$.

Proof. Similar to the proof of proposition 86. □

4.3.4 Direct image functor in the singular case

We define the direct image functors between our category.

Let $f:X\to S$ be a morphism with $X,S\in\mathrm{Var}(\mathbb{C})$, and assume there exist a factorization $f:X\xrightarrow{l}Y\times S\xrightarrow{p_S}S$ with $Y\in\mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S a the projection ; or let $f:X\to S$ be a morphism with $X,S\in\mathrm{AnSp}(\mathbb{C})$, and assume there exist a factorization $f:X\xrightarrow{l}Y\times S\xrightarrow{p_S}S$ with $Y\in\mathrm{AnSm}(\mathbb{C})$, l a closed embedding and p_S a the projection. Let $S=\cup_{i=1}^lS_i$ an open cover such that there exist closed embeddings $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$; resp. let $S=\cup_{i=1}^lS_i$ an open cover such that there exist closed embeddings $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{AnSm}(\mathbb{C})$. Then $X=\cup_{i=1}^lX_i$ with $X_i:=f^{-1}(S_i)$. Denote, for $I\subset[1,\cdots l]$, $S_I=\cap_{i\in I}S_i$ and $X_I=\cap_{i\in I}X_i$. For $I\subset[1,\cdots l]$, denote by $\tilde{S}_I=\Pi_{i\in I}\tilde{S}_i$, We then have, for $I\subset[1,\cdots l]$, closed embeddings $i_I:S_I\hookrightarrow\tilde{S}_I$ and the following commutative diagrams which are cartesian (we take $Y=\mathbb{P}^{N,o}$ in the algebraic case)

$$\begin{tikzcd} f_I=f_{|X_I}:X_I \arrow[r,"l_I"] \arrow[dr] & Y\times S_I \arrow[r,"p_{S_I}"] \arrow[d,"i'_I"] & S_I \arrow[d,"i_I"] \\ & Y\times\tilde{S}_I \arrow[r,"p_{\tilde{S}_I}"] & \tilde{S}_I \end{tikzcd}\quad,\quad \begin{tikzcd} Y\times\tilde{S}_J \arrow[r,"p_{\tilde{S}_J}"] \arrow[d,"p'_{IJ}"'] & \tilde{S}_J \arrow[d,"p_{IJ}"] \\ Y\times\tilde{S}_I \arrow[r,"p_{\tilde{S}_I}"] & \tilde{S}_I \end{tikzcd}$$

with $l_I:l_{|X_I}$, $i'_I=I\times i_I$, p_{S_I} and $p_{\tilde{S}_I}$ are the projections and $p'_{IJ}=I\times p_{IJ}$. Then $\tilde{f}_I:=p_{\tilde{S}_I}:Y\times\tilde{S}_I\to\tilde{S}_I$ is a lift of $f_I=f_{|X_I}$. We define the direct image functor on our category by

$$f_{*mod}^{FDR}:C_{\mathcal{D}(2)fil}(X/(Y\times\tilde{S}_I))\to C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)),$$
$$((M_I,F),u_{IJ})\mapsto(\tilde{f}_{I*mod}^{FDR}(M_I,F),f^k(u_{IJ})):=(p_{\tilde{S}_I*}E((\Omega^\bullet_{Y\times\tilde{S}_I/\tilde{S}_I},F_b)\otimes_{O_{Y\times\tilde{S}_I}}(M_I,F)[d_Y]),f^k(u_{IJ}))$$

with, denoting for short $d_{IJ}:=d_{\tilde{S}_J}-d_{\tilde{S}_I}$,

$$f^k(u_{IJ})[d_Y]:p_{\tilde{S}_I*}E((\Omega^\bullet_{Y\times\tilde{S}_I/\tilde{S}_I},F_b)\otimes_{O_{Y\times\tilde{S}_I}}(M_I,F))$$
$$\xrightarrow{p_{\tilde{S}_I*}E(DR(Y\times\tilde{S}_I/\tilde{S}_I)(u_{IJ}))}p_{\tilde{S}_I*}E((\Omega^\bullet_{Y\times\tilde{S}_I/\tilde{S}_I},F_b)\otimes_{O_{Y\times\tilde{S}_I}}p'_{IJ*}(M_J,F)[d_{IJ}])$$
$$\xrightarrow{T_w^O(p_{IJ},\otimes)(M_I,F)}p_{\tilde{S}_I*}E(p_{IJ*}(\Omega^\bullet_{Y\times\tilde{S}_J/\tilde{S}_J},F_b)\otimes_{O_{Y\times\tilde{S}_J}}(M_J,F)[d_{IJ}])$$
$$\xrightarrow{=}p_{\tilde{S}_J*}E((\Omega^\bullet_{Y\times\tilde{S}_J/\tilde{S}_J},F_b)\otimes_{O_{Y\times\tilde{S}_J}}(M_J,F))[d_{IJ}].$$

It induces in the derived categories the functor

$$\int_f^{FDR}: D_{\mathcal{D}(2)fil,\infty}(X)\to D_{\mathcal{D}(2)fil,\infty}(S),\ ((M_I,F),u_{IJ})\mapsto(\tilde{f}^{FDR}_{I*mod}(M_I,F),f^k(u_{IJ}))$$

Let $f:X\to S$ be a morphism with $X,S\in\mathrm{AnSp}(\mathbb{C})$, and assume there exist a factorization $f:X\xrightarrow{l}Y\times S\xrightarrow{p_S}S$ with $Y\in\mathrm{AnSm}(\mathbb{C})$, l a closed embedding and p_S a the projection. Let $S=\cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i=\in\mathrm{AnSm}(\mathbb{C})$. Then $X=\cup_{i=1}^l X_i$ with $X_i:=f^{-1}(S_i)$. We also have the functors

$$f^{FDR}_{*mod}:C_{\mathcal{D}^\infty(2)fil}(X/(Y\times\tilde{S}_I))\to C_{\mathcal{D}^\infty(2)fil}(S/(\tilde{S}_I)),$$
$$((M_I,F),u_{IJ})\mapsto(\tilde{f}^{FDR}_{I*mod}(M_I,F),f^k(u_{IJ})):=(p_{\tilde{S}_I*}((\Omega^\bullet_{Y\times\tilde{S}_I/\tilde{S}_I},F_b)\otimes_{O_{Y\times\tilde{S}_I}}(M_I,F)[d_Y]),f^k(u_{IJ}))$$

where $f^k(u_{IJ})[d_Y]$ is given as above,

$$\int_f^{FDR}:D_{\mathcal{D}^\infty(2)fil,\infty}(X)\to D_{\mathcal{D}^\infty(2)fil,\infty}(S),$$
$$((M_I,F),u_{IJ})\mapsto(\tilde{f}^{FDR}_{I*mod}(M_I,F),f^k(u_{IJ})):=(p_{\tilde{S}_I*}E((\Omega^\bullet_{Y\times\tilde{S}_I/\tilde{S}_I},F_b)\otimes_{O_{Y\times\tilde{S}_I}}(M_I,F)[d_Y]),f^k(u_{IJ}))$$

where $f^k(u_{IJ})[d_Y]$ is given as above.

In the algebraic case, we have the followings:

Proposition 88. *Let $f_1:X\to Y$ and $f_2:Y\to S$ two morphism with $X,Y,S\in\mathrm{QPVar}(\mathbb{C})$ quasi-projective. Then there exist factorizations $f_1:X\xrightarrow{l_1}Y'\times Y\xrightarrow{p_Y}Y$ and $f_2:Y\xrightarrow{l_2}Y''\times S\xrightarrow{p_S}S$ with $Y'=\mathbb{P}^{N,o}\subset\mathbb{P}^N$,$Y''=\mathbb{P}^{N',o}\subset\mathbb{P}^{N'}$ open subsets, l_1,l_2 closed embeddings and p_S,p_Y the projections. We have then the factorization $f_2\circ f_1:X\xrightarrow{(l_2\circ I_{Y'})\circ l_1}Y'\times Y''\times S\xrightarrow{p_S}S$. Let $i:S\hookrightarrow\tilde{S}$ a closed embedding with $\tilde{S}=\mathbb{P}^{n,o}\subset\mathbb{P}^n$ an open subset.*

- *(i) Let $(M,F)\in C_{\mathcal{D}(2)fil}(X/(Y'\times Y''\times\tilde{S}))$. Then, we have $\int^{FDR}_{f_2\circ f_1}(M,F)=\int^{FDR}_{f_2}(\int^{FDR}_{f_1}(M,F))$ in $D_{\mathcal{D}(2)fil,\infty}(S/(\tilde{S}_I))$.*
- *(ii) Let $(M,F)\in C_{\mathcal{D}(2)fil,h}(X/(Y'\times Y''\times\tilde{S}))$. Then, we have $\int^{FDR}_{(f_2\circ f_1)!}(M,F)=\int^{FDR}_{f_2!}(\int^{FDR}_{f_1!}(M,F))$ in $D_{\mathcal{D}(2)fil,\infty,h}(S/(\tilde{S}_I))$.*

Proof. (i):By the smooth case : proposition 72, we have en isomorphism

$$\int_{f_2}^{FDR}\int_{f_1}^{FDR}(M,F):=\int_{p_{\tilde{S}}}^{FDR}\int_{p_{Y'\times\tilde{S}}}^{FDR}(M,F)\xrightarrow{\sim}\int_{p_{\tilde{S}}}^{FDR}(M,F):=\int_{(f_2\circ f_1)}^{FDR}(M,F).$$

(ii):Follows from (i). □

In the analytic case, we have the followings:

Proposition 89. *Let $f_1:X\to Y$ and $f_2:Y\to S$ two morphism with $X,Y,S\in\mathrm{AnSp}(\mathbb{C})$ quasi-projective. Then there exist factorizations $f_1:X\xrightarrow{l_1}Y'\times Y\xrightarrow{p_Y}Y$ and $f_2:Y\xrightarrow{l_2}Y''\times S\xrightarrow{p_S}S$ with $Y'=\mathbb{P}^{N,o}\subset\mathbb{P}^N$,$Y''=\mathbb{P}^{N',o}\subset\mathbb{P}^{N'}$ open subsets, l_1,l_2 closed embeddings and p_S,p_Y the projections. We have then the factorization $f_2\circ f_1:X\xrightarrow{(l_2\circ I_{Y'})\circ l_1}Y'\times Y''\times S\xrightarrow{p_S}S$. Let $i:S\hookrightarrow\tilde{S}$ a closed embedding with $\tilde{S}=\mathbb{P}^{n,o}\subset\mathbb{P}^n$ an open subset.*

- *(i) Let $(M,F)\in C_{\mathcal{D}^\infty(2)fil,h}(X/(Y'\times Y''\times\tilde{S}))$. Then, we have $\int^{FDR}_{f_2\circ f_1}(M,F)=\int^{FDR}_{f_2}(\int^{FDR}_{f_1}(M,F))$ in $D_{\mathcal{D}^\infty(2)fil,\infty}(S/(\tilde{S}_I))$.*

(ii) *Let* $(M,F)\in C_{\mathcal{D}^\infty(2)fil,h}(X/(Y'\times Y''\times\tilde{S}))$. *Then, we have* $\int^{FDR}_{(f_2\circ f_1)!}(M,F)=\int^{FDR}_{f_2!}(\int^{FDR}_{f_1!}(M,F))$ *in* $D_{\mathcal{D}^\infty(2)fil,\infty}(S/(\tilde{S}_I))$.

Proof. (i): By the smooth case : proposition 73, we have en isomorphism

$$\int^{FDR}_{f_2}\int^{FDR}_{f_1}(M,F):=\int^{FDR}_{p_{\tilde{S}}}\int^{FDR}_{p_{Y'\times\tilde{S}}}(M,F)\xrightarrow{\sim}\int^{FDR}_{p_{\tilde{S}}}(M,F):=\int^{FDR}_{(f_2\circ f_1)}(M,F).$$

(ii):Follows from (i). □

4.3.5 Tensor product in the singular case

Let $S\in\mathrm{Var}(\mathbb{C})$ and let $S=\cup S_i$ an open cover such that there exist closed embeddings $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$; or let $S\in\mathrm{AnSp}(\mathbb{C})$ and $S=\cup S_i$ an open cover such that there exist closed embeddings $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{AnSm}(\mathbb{C})$. We have the tensor product functors

$$(-)\otimes^{[-]}_{O_S}(-):C^2_{\mathcal{D}fil}(S/(\tilde{S}_I))\to C_{\mathcal{D}fil}(S/(\tilde{S}_I)),$$
$$(((M_I,F),u_{IJ}),((N_I,F),v_{IJ}))\mapsto((M_I,F)\otimes_{O_{\tilde{S}_I}}(N_I,F)[d_{\tilde{S}_I}],u_{IJ}\otimes v_{IJ}),$$

with, denoting for short $d_{IJ}:=d_{\tilde{S}_J}-d_{\tilde{S}_I}$ and $d_I:=d_{\tilde{S}_I}$,

$$\begin{aligned}u_{IJ}\otimes v_{IJ}:(M_I,F)\otimes_{O_{\tilde{S}_I}}(N_I,F)[d_I]&\xrightarrow{T(p_{IJ}^{*mod},p_{IJ})(-)[d_I]}p_{IJ*}p_{IJ}^{*mod}((M_I,F)\otimes_{O_{\tilde{S}_I}}(N_I,F))[d_I]\\&\xrightarrow{=}p_{IJ*}(p_{IJ}^{*mod}(M_I,F)\otimes_{O_{\tilde{S}_J}}p_{IJ}^{*mod}(N_I,F))[d_I]\\&\xrightarrow{I(p_{IJ}^{*mod},p_{IJ})(-,-)(u_{IJ})\otimes I(p_{IJ}^{*mod},p_{IJ})(-,-)(v_{IJ})[d_I]}p_{IJ*}((M_J,F)\otimes_{O_{\tilde{S}_J}}(N_J,F))[d_J+d_{IJ}].\end{aligned}$$

Let $S\in\mathrm{AnSp}(\mathbb{C})$ and $S=\cup S_i$ an open cover such that there exist closed embeddings $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{AnSm}(\mathbb{C})$. We have the tensor product functors

$$(-)\otimes^{[-]}_{O_S}(-):C^2_{\mathcal{D}^\infty fil}(S/(\tilde{S}_I))\to C_{\mathcal{D}^\infty fil}(S/(\tilde{S}_I)),$$
$$(((M_I,F),u_{IJ}),((N_I,F),v_{IJ}))\mapsto((M_I,F)\otimes_{O_{\tilde{S}_I}}(N_I,F),u_{IJ}\otimes v_{IJ}),$$

with $u_{IJ}\otimes v_{IJ}$ as above.

Proposition 90. *Let* $S\in\mathrm{Var}(\mathbb{C})$. *Denote* $\Delta_S:S\hookrightarrow S\times S$ *the diagonal embedding. Let* $S=\cup S_i$ *an open cover such that there exist closed embeddings* $i_i:S_i\hookrightarrow\tilde{S}_i$ *closed embedding with* $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$ *; or let* $S\in\mathrm{AnSp}(\mathbb{C})$ *and* $S=\cup S_i$ *an open cover such that there exist closed embedding* $i_i:S_i\hookrightarrow\tilde{S}_i$ *with* $\tilde{S}_i\in\mathrm{AnSm}(\mathbb{C})$. *We have, for* $((M_I,F),u_{IJ}),((N_I,F),v_{IJ})\in C_{\mathcal{D}fil}(S/(\tilde{S}_I))$,

$$((M_I,F),u_{IJ})\otimes^{[-]}_{O_{S_I}}((N_I,F),v_{IJ})=\Delta_S^{*mod}(((M_I,F),u_{IJ}).((N_I,F),v_{IJ}))$$

Proof. Follows from proposition 54. □

4.3.6 The 2 functors of D modules on the category of complex algebraic varieties and on the category of complex analytic spaces, and the transformation maps

Definition 82. *Consider a commutative diagram in* $\mathrm{Var}(\mathbb{C})$ *which is cartesian :*

$$D=\begin{tikzcd}X_T\arrow[r,"f'"]\arrow[d,"g'"]&T\arrow[d,"g"]\\X\arrow[r,"f"]&S\end{tikzcd}.$$

Assume there exist factorizations $f : X \xrightarrow{l_1} Y_1 \times S \xrightarrow{p_S} S$, $g : T \xrightarrow{l_2} Y_2 \times S \xrightarrow{p_S} S$, *with* $Y_1, Y_2 \in \mathrm{SmVar}(\mathbb{C})$, l_1, l_2 *closed embeddings and* p_S, p_S *the projections. Then, the above commutative diagram factors through*

$$D = \begin{tikzcd} f' : X_T \arrow[r, "l'_1"] \arrow[d, "l'_2"] & Y_1 \times T \arrow[r, "p_T"] \arrow[d, "l''_2 = I \times l_2"] & T \arrow[d, "l_2"] \\ f'' : X \times Y_2 \arrow[r, "l''_1 = I \times l"] \arrow[d, "p_X"] & Y_1 \times Y_2 \times S \arrow[r, "p_{Y_2 \times S}"] \arrow[d, "p_{Y_1 \times S}"] & Y_2 \times S \arrow[d, "p_S"] \\ f : X \arrow[r, "l"] & Y_1 \times S \arrow[r, "p_S"] & S \end{tikzcd} .$$

whose squares are cartesian. Let $S = \cup_i S_i$ *be an open cover such that there exist closed embeddings* $i_i : S_i \hookrightarrow \tilde{S}_i$ *with* $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. *Then* $X = \cup_i X_i$ *and* $T = \cup_i T_i$ *with* $X_i := f^{-1}(S_i)$ *and* $T_i := f^{-1}(S_i)$. *Moreover,* $f_i = f_{|X_i} : X_i \to S_i$ *lift to* $\tilde{f}_i := p_{\tilde{S}_i} : Y_1 \times \tilde{S}_i \to \tilde{S}_i$ *and* $g_i = g_{|T_i} : T_i \to S_i$ *lift to* $\tilde{g}_i := p_{\tilde{S}_i} : Y_2 \times \tilde{S}_i \to \tilde{S}_i$. *We then have the following commutative diagram whose squares are cartesian*

$$\begin{tikzcd} f' : X_{IT} \arrow[r, "l'"] \arrow[d, "l'_{2I}"] & Y_1 \times T_I \arrow[r, "p_T"] \arrow[d, "l''_{2I}"] & T_I \arrow[d, "l_{2I}"] \\ X_I \times Y_2 \arrow[r, "l''_1 = I \times l_1"] \arrow[d, "p_X"] & Y_1 \times Y_2 \times \tilde{S}_I \arrow[r, "p_{Y_2 \times \tilde{S}_I}"] \arrow[d, "p_{Y_1 \times \tilde{S}_I}"] & Y_2 \times \tilde{S}_I \arrow[d, "\tilde{g}_I"] \\ X_I \arrow[r, "i_I \circ l_I"] & Y_1 \times \tilde{S}_I \arrow[r, "\tilde{f}_I"] & \tilde{S}_I \end{tikzcd}$$

We then define, for $(M, F) = ((M_I, F), u_{IJ}) \in C_{\mathcal{D}(2)fil}(X/(Y_1 \times \tilde{S}_I))$, *the following canonical transformation map in* $D_{\mathcal{D}(2)fil,\infty}(T/(Y_2 \times \tilde{S}_I))$, *using proposition 78,*

$$\begin{aligned} & T^{\mathcal{D}mod}(f, g)(M, F) : \\ Rg^{*mod,\Gamma} \int_f^{FDR} (M, F) := & (\Gamma_{T_I} E(\tilde{g}_I^{*mod} p_{\tilde{S}_I *} E((\Omega^\bullet_{Y_1 \times \tilde{S}_I/\tilde{S}_I}, F_b) \otimes_{O_{Y_1 \times \tilde{S}_I}} (M_I, F))), \tilde{g}_J^{*mod} f^k(u_{IJ})) \\ & \xrightarrow{(T^O_\omega(p_{\tilde{S}_I}, \tilde{g}_I)(M_I, F))} \\ & (\Gamma_{T_I} E(p_{Y_2 \times \tilde{S}_I *} E((\Omega^\bullet_{Y_1 \times Y_2 \times \tilde{S}_I / Y_2 \times \tilde{S}_I}, F_b) \otimes_{O_{Y_1 \times Y_2 \times \tilde{S}_I}} p^{*mod}_{Y_1 \times \tilde{S}_I}(M_I, F))), f'^k(p^{*mod}_{Y_1 \times \tilde{S}_J}(u_{IJ}))) \\ & \xrightarrow{(T^O_\omega(\gamma, \otimes)(p^{*mod}_{Y_1 \times \tilde{S}_I}(M_I, F)))^{-1}} \\ & (p_{Y_2 \times \tilde{S}_I *} E((\Omega^\bullet_{Y_1 \times Y_2 \times \tilde{S}_I / Y_2 \times \tilde{S}_I}, F_b) \otimes_{O_{Y_1 \times Y_2 \times \tilde{S}_I}} \Gamma_{Y_1 \times T_I} E(p^{*mod}_{Y_1 \times \tilde{S}_I}((M_I, F)))), f'^k(\tilde{g}_J''^{*mod}(u^q_{IJ}))) \\ & =: \int_{f'}^{FDR} Rg'^{*mod,\Gamma}(M, F). \end{aligned}$$

In the analytic case, we have

Definition 83. *Consider a commutative diagram in* $\mathrm{AnSp}(\mathbb{C})$ *which is cartesian :*

$$D = (f, g) = \begin{tikzcd} X_T \arrow[r, "f'"] \arrow[d, "g'"] & T \arrow[d, "g"] \\ X \arrow[r, "f"] & S \end{tikzcd} .$$

Assume there exist factorizations $f : X \xrightarrow{l_1} Y_1 \times S \xrightarrow{p_S} S$, $g : T \xrightarrow{l_2} Y_2 \times S \xrightarrow{p_S} S$, *with* $Y_1, Y_2 \in \mathrm{AnSm}(\mathbb{C})$, l_1, l_2 *closed embeddings and* p_S, p_S *the projections.*

(i) *We have, for* $(M,F)\in D_{\mathcal{D}(2)fil,\infty,h}(X/(Y_1\times\tilde{S}_I))$, *the following transformation map in* $D_{\mathcal{D}(2)fil,\infty}(T/(Y_2\times\tilde{S}_I))$

$$T^{\mathcal{D}mod}(f,g)((M,F)):Rg^{*mod,\Gamma}\int_f^{FDR}(M,F)\to\int_{f'}^{FDR}Rg^{'*mod,\Gamma}(M,F)$$

define in the same way as in definition 82

(ii) *For* $(M,F)\in D_{\mathcal{D}^\infty(2)fil,\infty}(X/(Y_1\times\tilde{S}_I))$, *the following transformation map in* $D_{\mathcal{D}^\infty(2)fil,\infty}(T/(Y_2\times\tilde{S}_I))$

$$T^{\mathcal{D}mod}(f,g)((M,F)):Rg^{*mod,\Gamma}\int_f^{FDR}(M,F)\to\int_{f'}^{FDR}Rg^{'*mod,\Gamma}(M,F)$$

is defined in the same way as in (ii) : see definition 82.

In the algebraic case, we have the following :

Proposition 91. *Consider a commutative diagram in* $\mathrm{Var}(\mathbb{C})$

$$D=(f,g)=\begin{tikzcd} X_T \arrow[r, "f'"] \arrow[d, "g'"] & T \arrow[d, "g"] \\ X \arrow[r, "f"] & S \end{tikzcd}.$$

which is cartesian. Assume there exist factorizations $f:X\xrightarrow{l_1}Y_1\times S\xrightarrow{p_S}S$, $g:T\xrightarrow{l_2}Y_2\times S\xrightarrow{p_S}S$, *with* $Y_1,Y_2\in\mathrm{SmVar}(\mathbb{C})$, l_1,l_2 *closed embeddings and* p_S, p_S *the projections. For* $(M,F)=((M_I,F),u_{IJ})\in C_{\mathcal{D}(2)fil,c}(X/(Y\times\tilde{S}_I))$,

$$T^{\mathcal{D}mod}(f,g):Rg^{*mod,\Gamma}\int_f^{FDR}(M,F)\to\int_{f'}^{FDR}Rg^{'*mod,\Gamma}(M,F)$$

is an isomorphism in $D_{\mathcal{D}(2)fil,\infty}(T/(Y_2\times\tilde{S}_I))$.

Proof. Similar to the proof of proposition 80: the maps

$$T^O_\omega(p_{\tilde{S}_I},\tilde{g}_I)(M_I,F):\tilde{g}_I^{*mod}p_{\tilde{S}_I*}E((\Omega^\bullet_{Y\times\tilde{S}_I/\tilde{S}_I},F_b)\otimes_{O_{Y\times\tilde{S}_I}}(M_I,F))\to \\ p_{\tilde{T}_I*}E((\Omega_{Y\times\tilde{T}_I/\tilde{T}_I},F_b)\otimes_{O_{Y\times\tilde{T}_I}}\tilde{g}_I^{''*mod}(M_I,F))$$

are ∞-filtered Zariski local equivalences since $\tilde{g}_I:Y_2\times\tilde{S}_I\to\tilde{S}_I$ are projections. □

Proposition 92. *Consider a commutative diagram in* $\mathrm{AnSp}(\mathbb{C})$

$$D=(f,g)=\begin{tikzcd} X_T \arrow[r, "f'"] \arrow[d, "g'"] & T \arrow[d, "g"] \\ X \arrow[r, "f"] & S \end{tikzcd}.$$

which is cartesian. Assume that f *(hence* f'*) is proper and that there exist factorizations* $f:X\xrightarrow{l_1}Y_1\times S\xrightarrow{p_S}S$, $g:T\xrightarrow{l_2}Y_2\times S\xrightarrow{p_S}S$, *with* $Y_1,Y_2\in\mathrm{AnSm}(\mathbb{C})$, l_1,l_2 *closed embeddings and* p_S, p_S *the projections.*

(i) *For* $(M,F)=((M_I,F),u_{IJ})\in C_{\mathcal{D}(2)fil,h}(X/(Y_1\times\tilde{S}_I))$

$$T^{\mathcal{D}mod}(f,g):Rg^{*mod,\Gamma}\int_f^{FDR}(M,F)\to\int_{f'}^{FDR}Rg'^{*mod,\Gamma}(M,F)$$

is an isomorphism in $D_{\mathcal{D}(2)fil,\infty}(T/Y_2\times\tilde{S}_I)$.

(ii) *For* $(M,F)=((M_I,F),u_{IJ})\in C_{\mathcal{D}^\infty(2)fil,h}(X/(Y_1\times\tilde{S}_I))$

$$T^{\mathcal{D}mod}(f,g):Rg^{*mod,\Gamma}\int_f^{FDR}(M,F)\to\int_{f'}^{FDR}Rg'^{*mod,\Gamma}(M,F)$$

is an isomorphism in $D_{\mathcal{D}^\infty(2)fil,\infty}(T/(Y_2\times\tilde{S}_I))$.

Proof. (i):Similar to the proof of proposition 91.
(ii):Similar to the proof of proposition 91. □

Definition 84. *Let* $f:X\to S$ *be a morphism, with* $X,S\in\mathrm{Var}(\mathbb{C})$*, such that there exist a factorization* $f;X\xrightarrow{l}Y\times S\xrightarrow{p_S}S$ *with* $Y\in\mathrm{SmVar}(\mathbb{C})$*,* l *a closed embedding and* p_S *the projection, and consider* $S=\cup_{i=1}^l S_i$ *an open cover such that there exist closed embeddings* $i_i:S_i\hookrightarrow\tilde{S}_i$*, with* $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$ *; Then,* $X=\cup_{i=1}^l X_i$ *with* $X_i:=f^{-1}(S_i)$*. We have, for* $(M,F)=((M_I,F),u_{IJ})\in C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$*, the canonical transformation map in* $D_{\mathcal{D}(2)fil}(T^{an}/(\tilde{T}_I^{an}))$

$$\begin{aligned}&T^{mod}(an,\gamma_T)(M,F):\\ f^{*mod[-],\Gamma}(M,F))^{an}&:=((\Gamma_{T_I}E(p_{\tilde{S}_I}^{*mod[-]}(M_I,F)))^{an},(f^{*mod[-]}u_{IJ})^{an})\\ &\xrightarrow{(T^{mod}(an,\gamma_{T_I})(-))}(\Gamma_{T_I^{an}}E((p_{\tilde{S}_I}^{*mod[-]}(M_I,F))^{an}),f^{*mod[-]}u_{IJ}^{an})\\ &\xrightarrow{=}(\Gamma_{T_I^{an}}E(p_{\tilde{S}_I}^{*mod[-]}(M_I^{an},F)),f^{*mod[-]}u_{IJ}^{an})=:f^{*mod[-],\Gamma}((M,F)^{an})\end{aligned}$$

where the equality is obvious (see proposition 51).

Definition 85. *Let* $f:X\to S$ *a morphism with* $X,S\in\mathrm{Var}(\mathbb{C})$*. Assume there exist a factorization* $f:X\xrightarrow{l}Y\times S\xrightarrow{p_S}S$ *with* $Y\in\mathrm{SmVar}(\mathbb{C})$*,* l *a closed embedding and* p_S *the projection. Let* $S=\cup_{i=1}^l S_i$ *be an open cover such that there exist closed embeddings* $i_i:S_i\hookrightarrow\tilde{S}_i$ *closed embeddings with* $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$*. We have, for* $(M,F)=((M_I,F),u_{IJ})\in C_{\mathcal{D}fil}(X/Y\times\tilde{S}_I)$*, the following transformation map in* $D_{\mathcal{D}fil}(X^{an}/(Y\times\tilde{S}_I)^{an})$

$$\begin{aligned}&T^{\mathcal{D}mod}(an,f)(M,F):(\int_f^{FDR}(M,F))^{an}=(p_{\tilde{S}_I*}E((\Omega^\bullet_{Y\times\tilde{S}_I/\tilde{S}_I},F_b)\otimes_{O_{Y\times\tilde{S}_I}}L_D(M_I,F)))^{an},(f^k(u_{IJ}^q))^an)\\ &\xrightarrow{(T^O_\omega(p_{\tilde{S}_I},an)(M_I,F))}(p_{\tilde{T}_I*}E((\Omega^\bullet_{Y\times\tilde{T}_I/\tilde{T}_I},F_b)\otimes_{O_{(Y\times\tilde{T}_I)^{an}}}L_D(M_I,F)^{an}),f'^k((u_{IJ}^q)^{an}))=:\int_{f^{an}}^{FDR}(M,F)^{an}\end{aligned}$$

Theorem 25. *Let* $f:X\to S$ *a morphism with* $X,S\in\mathrm{Var}(\mathbb{C})$*. Assume there exist a factorization* $f:X\xrightarrow{l}Y\times S\xrightarrow{p_S}S$ *with* $Y\in\mathrm{SmVar}(\mathbb{C})$*,* l *a closed embedding and* p_S *the projection. Let* $S=\cup_{i=1}^l S_i$ *be an open cover such that there exist closed embeddings* $i_i:S_i\hookrightarrow\tilde{S}_i$ *closed embeddings with* $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$*. Let* $M\in D_{\mathcal{D}fil,c}(X/Y\times\tilde{S}_I)$*. If* f *is proper,*

$$T^{\mathcal{D}}(an,f)(M):(\int_f M)^{an}\xrightarrow{\sim}\int_{f^{an}}(M)^{an}$$

is an isomorphism.

Proof. By theorem 23, $T^O_\omega(p_{\tilde{S}_I}, an)(M_I)$ are usu local equivalences. $\square$

In the analytic case, we have the following canonical transformation maps

Definition 86. *Let $f : X \to S$ be a morphism, with $X, S \in \mathrm{AnSp}(\mathbb{C})$, such that there exist a factorization $f; X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{AnSm}(\mathbb{C})$, l a closed embedding and p_S the projection, and consider $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$, with $\tilde{S}_i \in \mathrm{AnSm}(\mathbb{C})$; Then, $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. We have, for $(M, F) = ((M_I, F), u_{IJ}) \in C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$, the canonical transformation map in $D_{\mathcal{D}^\infty fil}(T/(\tilde{T}_I))$ obtained by the canonical maps given in definition 59 and definition 64 :*

$$T(f,\infty)(M,F) : J_T(f^{*mod[-],\Gamma}(M,F)) := (J_{\tilde{T}_I}(\Gamma_{T_I}E(p^{*mod[-]}_{\tilde{S}_I}(M_I,F))), J(f^{*mod[-]}u_{IJ}))$$
$$\xrightarrow{(T(\infty,\gamma_{T_I})(-))} (\Gamma_{T_I}E(J_{\tilde{T}_I}(p^{*mod[-]}_{\tilde{S}_I}(M_I,F))), N_{IJ})$$
$$\xrightarrow{(T(p_{\tilde{S}_I},\infty)(-))} (\Gamma_{T_I}E(p^{*mod[-]}_{\tilde{S}_I}J_{\tilde{S}_I}(M_I,F)), f^{*mod[-]}J(u_{IJ})) =: f^{*mod[-],\Gamma}(J_S(M,F))$$

4.4 The category of complexes of quasi-coherent sheaves whose cohomology sheaves has a structure of D-modules

4.4.1 Definition on a smooth complex algebraic variety or smooth complex analytic space and the functorialities

Let $X \in \mathrm{SmVar}(\mathbb{C})$ or let $X \in \mathrm{AnSm}(\mathbb{C})$. Recall that (see definition 49 section 4.1) $C_{O_X fil,\mathcal{D}}(X)$ is the category

- whose objects $(M, F) \in C_{O_X fil,\mathcal{D}}(X)$ are filtered complexes of presheaves of O_X modules $(M, F) \in C_{O_X fil}(X)$ whose cohomology presheaves $H^n(M, F) \in \mathrm{PSh}_{O_X fil}(X)$ are emdowed with a structure of filtered D_X modules for all $n \in \mathbb{Z}$.
- whose set of morphisms $\mathrm{Hom}_{C_{O_X fil,\mathcal{D}}(X)}((M, F), (N, F)) \subset \mathrm{Hom}_{C_{O_X fil}(X)}((M, F), (N, F))$ between $(M, F), (N, F) \in C_{O_X fil,\mathcal{D}}(X)$ are the morphisms of filtered complexes of O_X modules $m : (M, F) \to (N, F)$ such that $H^n m : H^n(M, F) \to H^n(N, F)$ is D_X linear, i.e. is a morphism of (filtered) D_X modules, for all $n \in \mathbb{Z}$.

More generally, let $h : X \to S$ a morphism with $X, S \in \mathrm{SmVar}(\mathbb{C})$ or with $X, S \in \mathrm{AnSm}(\mathbb{C})$. Then, $C_{h^*O_S fil,h^*\mathcal{D}}(X)$ the category

- whose objects $(M, F) \in C_{h^*O_S fil,h^*\mathcal{D}}(X)$ are filtered complexes of presheaves of h^*O_S modules $(M, F) \in C_{h^*O_S fil}(X)$ whose cohomology presheaves $H^n(M, F) \in \mathrm{PSh}_{h^*O_S fil}(X)$ are emdowed with a structure of filtered h^*D_S modules for all $n \in \mathbb{Z}$.
- whose set of morphisms $\mathrm{Hom}_{C_{h^*O_S fil,h^*\mathcal{D}}(X)}((M, F), (N, F)) \subset \mathrm{Hom}_{C_{h^*O_S fil}(X)}((M, F), (N, F))$ between $(M, F), (N, F) \in C_{h^*O_S fil,h^*\mathcal{D}}(X)$ are the morphisms of filtered complexes of h^*D_S modules $m : (M, F) \to (N, F)$ such that $H^n m : H^n(M, F) \to H^n(N, F)$ is h^*D_S linear, i.e. is a morphism of (filtered) h^*D_S modules, for all $n \in \mathbb{Z}$.

Definition 87. *Let $S \in \mathrm{SmVar}(\mathbb{C})$ or $S \in \mathrm{AnSm}(\mathbb{C})$. Let $Z \subset S$ a closed subset. Denote by $j : S\backslash Z \hookrightarrow S$ the open complementary embedding.*

(i) We denote by $C_{O_S,\mathcal{D},Z}(S) \subset C_{O_S,\mathcal{D}}(S)$ the full subcategory consisting of $M \in C_{O_S,\mathcal{D}}(S)$ such that such that $j^*H^nM = 0$ for all $n \in \mathbb{Z}$.

(ii) We denote by $C_{O_S fil,\mathcal{D},Z}(S) \subset C_{O_S fil,\mathcal{D}}(S)$ the full subcategory consisting of $(M, F) \in C_{O_S fil,\mathcal{D}}(S)$ such that there exist $r \in \mathbb{N}$ such that $j^*E_r^{p,q}(M, F) = 0$ for all $p, q \in \mathbb{Z}$, note that by definition r does NOT depend on p and q.

We look at functoriality

- Let $S \in \mathrm{SmVar}(\mathbb{C})$ or $S \in \mathrm{AnSm}(\mathbb{C})$. Let $(M,F) \in C_{O_S fil,\mathcal{D}}(S)$. Then, the canonical morphism $q : L_O(M,F) \to (M,F)$ in $C_{O_S fil}(S)$ being a quasi-isomorphism of O_S modules, we get in a unique way $L_O(M,F) \in C_{O_S fil,\mathcal{D}}(S)$ such that $q : L_O(M,F) \to (M,F)$ is a morphism in $C_{O_S fil,\mathcal{D}}(S)$

- Let $f : X \to S$ be a morphism with $X, S \in \mathrm{SmVar}(\mathbb{C})$, or let $f : X \to S$ be a morphism with $X, S \in \mathrm{AnSm}(\mathbb{C})$. Let $(M,F) \in C_{O_S fil,\mathcal{D}}(S)$. Then, $f^{*mod}H^n(M,F) := (O_X, F_b) \otimes_{f^*O_S} f^*H^n(M,F)$ is canonical a filtered D_X module (see section 4.1 or 4.2). Consider the canonical surjective map $q(f) : H^n f^{*mod}(M,F) \to f^{*mod}H^n(M,F)$. Then, $q(f)$ is an isomorphism if f is smooth. Let $h : U \to S$ be a smooth morphism with $U, S \in \mathrm{SmVar}(\mathbb{C})$, or let $h : U \to S$ be a smooth morphism with $U, S \in \mathrm{AnSm}(\mathbb{C})$. We get the functor

$$h^{*mod} : C_{O_S fil,\mathcal{D}}(S) \to C_{O_U fil,\mathcal{D}}(U), (M,F) \mapsto h^{*mod}(M,F),$$

- Let $S \in \mathrm{SmVar}(\mathbb{C})$ or $S \in \mathrm{AnSm}(\mathbb{C})$, and let $i : Z \hookrightarrow S$ a closed embedding and denote by $j : S\backslash Z \hookrightarrow S$ the open complementary. For $M \in C_{O_S,\mathcal{D}}(S)$, the cohomology presheaves of

$$\Gamma_Z M := \mathrm{Cone}(\mathrm{ad}(j^*, j_*)(M) : M \to j_* j^* M)[-1]$$

has a canonical D_S-module structure (as j^*H^nM is a j^*D_S module, $H^n j_* j^* M = j_* j^* H^n M$ has an induced structure of D_S module), and $\gamma_Z(M) : \Gamma_Z M \to M$ is a map in $C_{O_S,\mathcal{D}}(S)$. For $Z_2 \subset Z$ a closed subset and $M \in C_{O_S,\mathcal{D}}(S)$, $T(Z_2/Z, \gamma)(M) : \Gamma_{Z_2} M \to \Gamma_Z M$ is a map in $C_{O_S,\mathcal{D}}(S)$. We get the functor

$$\begin{aligned}\Gamma_Z : C_{O_S fil,\mathcal{D}}(S) \to C_{O_S fil,\mathcal{D}}(S),\\ (M,F) \mapsto \Gamma_Z(M,F) := \mathrm{Cone}(\mathrm{ad}(j^*, j_*)((M,F)) : (M,F) \to j_* j^*(M,F))[-1],\end{aligned}$$

together we the canonical map $\gamma_Z(M,F) : \Gamma_Z(M,F) \to (M,F)$

More generally, let $h : Y \to S$ a morphism with $Y, S \in \mathrm{Var}(\mathbb{C})$ or $Y, S \in \mathrm{AnSp}(\mathbb{C})$, S smooth, and let $i : X \hookrightarrow Y$ a closed embedding and denote by $j : Y\backslash X \hookrightarrow Y$ the open complementary. For $M \in C_{h^*O_S,h^*\mathcal{D}}(Y)$,

$$\Gamma_X M := \mathrm{Cone}(\mathrm{ad}(j^*, j_*)(M) : M \to j_* j^* M)[-1]$$

has a canonical h^*D_S-module structure, (as j^*H^nM is a $j^*h^*D_S$ module, $H^n j_* j^* M = j_* j^* H^n M$ has an induced structure of $j^*h^*D_S$ module), and $\gamma_X(M) : \Gamma_X M \to M$ is a map in $C_{h^*O_S,h^*\mathcal{D}}(Y)$. For $X_2 \subset X$ a closed subset and $M \in C_{h^*O_S,h^*\mathcal{D}}(Y)$, $T(Z_2/Z, \gamma)(M) : \Gamma_{X_2} M \to \Gamma_X M$ is a map in $C_{h^*O_S,h^*\mathcal{D}}(Y)$. We get the functor

$$\begin{aligned}\Gamma_X : C_{h^*O_S fil,h^*\mathcal{D}}(Y) \to C_{h^*O_S fil,h^*\mathcal{D}}(Y),\\ (M,F) \mapsto \Gamma_X(M,F) := \mathrm{Cone}(\mathrm{ad}(j^*, j_*)((M,F)) : (M,F) \to j_* j^*(M,F))[-1],\end{aligned}$$

together we the canonical map $\gamma_X(M,F) : \Gamma_X(M,F) \to (M,F)$

- Let $f : X \to S$ be a morphism with $X, S \in \mathrm{SmVar}(\mathbb{C})$, or let $f : X \to S$ be a morphism with $X, S \in \mathrm{AnSm}(\mathbb{C})$. Consider the factorization $f : X \xrightarrow{l} X \times S \xrightarrow{p} S$, where l is the graph embedding and p the projection. We get from the two preceding points the functor

$$f^{*mod,\Gamma} : C_{O_S fil,\mathcal{D}}(S) \to C_{O_X fil,\mathcal{D}}(X \times S), (M,F) \mapsto f^{*mod,\Gamma}(M,F) := \Gamma_X p^{*mod}(M,F),$$

and

$$f^{*mod[-],\Gamma} : C_{O_S fil,\mathcal{D}}(S) \to C_{O_X fil,\mathcal{D}}(X \times S), (M,F) \mapsto f^{*mod[-],\Gamma}(M,F) := \Gamma_X E(p^{*mod}(M,F))[-d_X],$$

which induces in the derived categories the functor

$$Rf^{*mod[-],\Gamma}:D_{O_Sfil,\mathcal{D}}(S)\to D_{O_Xfil,\mathcal{D}}(X\times S),(M,F)\mapsto Rf^{*mod[-],\Gamma}(M,F):=\Gamma_XE(p^{*mod[-]}(M,F)).$$

For $(M,F)\in C_{O_Sfil,\mathcal{D}}(S)$ or $(M,F)\in C_{O_Sfil}(S)$, the canonical map in $C_{O_Xfil}(X\times S)$

$$\mathrm{ad}(i^{*mod},i_*)(-):L_O\Gamma_XE(p^{*mod}(M,F))\to i_*i^{*mod}L_O\Gamma_XE(p^{*mod}(M,F))$$

gives in the derived category, the canonical map in $D_{O_Xfil,\infty}(X\times S)$

$$I(f^{*mod,\Gamma})(M,F):Rf^{*mod,\Gamma}(M,F)=L_O\Gamma_XE(p^{*mod}(M,F))\xrightarrow{\mathrm{ad}(i^{*mod},i_*)(-)}$$
$$i_*i^{*mod}L_O\Gamma_XE(p^{*mod}(M,F))\xrightarrow{\sim}i_*i^{*mod}L_O(p^{*mod}(M,F))=Lf^{*mod}(M,F)$$

where the isomorphism is given by lemma 6.

- Let $S\in\mathrm{SmVar}(\mathbb{C})$. We have the analytical functor :

  $$(-)^{an}:C_{O_Sfil,\mathcal{D}}(S)\to C_{O_Sfil,\mathcal{D}}(S^{an}),\ (M,F)\mapsto(M,F)^{an}:=\mathrm{an}_S^{*mod}(M,F):=(M,F)\otimes_{\mathrm{an}_S^*O_S}O_{S^{an}}$$

  which induces in the derived category

  $$(-)^{an}:D_{O_Sfil,\mathcal{D}}(S)\to D_{O_Sfil,\mathcal{D}}(S^{an}),\ ((M,F)\mapsto(M,F)^{an}:=\mathrm{an}_S^{*mod}(M,F))$$

  since an_S^{*mod} is an exact functor.

We have, for $f:T\to S$ with $T,S\in\mathrm{SmVar}(\mathbb{C})$ or with $T,S\in\mathrm{AnSm}(\mathbb{C})$, the commutative diagrams of functors

$$\begin{tikzcd}
C_{\mathcal{D}fil}(S) \arrow[r, "o_S"] \arrow[d, "f^{*mod[-],\Gamma}"] & C_{Ofil,\mathcal{D}}(S) \arrow[d, "f^{*mod[-],\Gamma}"] \\
C_{\mathcal{D}fil}(T) \arrow[r, "o_T"] & C_{Ofil,\mathcal{D}}(T)
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
D_{\mathcal{D}fil,r}(S) \arrow[r, "o_S"] \arrow[d, "Rf^{*mod[-],\Gamma}"] & D_{Ofil,\mathcal{D},r}(S) \arrow[d, "Rf^{*mod[-],\Gamma}"] \\
D_{\mathcal{D}fil,r}(T) \arrow[r, "o_T"] & D_{Ofil,\mathcal{D},r}(T)
\end{tikzcd}$$

where o_S and o_T are the forgetfull functors.

4.4.2 Definition on a singular complex algebraic variety or singular complex analytic space and the functorialities

Definition 88. *Let $S\in\mathrm{Var}(\mathbb{C})$ and let $S=\cup_iS_i$ an open cover such that there exist closed embeddings $i_i:S_i\hookrightarrow\tilde S_i$ with $\tilde S_i\in\mathrm{SmVar}(\mathbb{C})$; or let $S\in\mathrm{AnSp}(\mathbb{C})$ and let $S=\cup_iS_i$ an open cover such that there exist closed embeddings $i_i:S_i\hookrightarrow\tilde S_i$ with $\tilde S_i\in\mathrm{AnSm}(\mathbb{C})$. Then, $C_{Ofil,\mathcal{D}}(S/(\tilde S_I))$ is the category*

- *whose objects are $(M,F)=((M_I,F)_{I\subset[1,\cdots l]},u_{IJ})$, with*
  - *$(M_I,F)\in C_{O_{\tilde S_I}fil\mathcal{D},S_I}(\tilde S_I)$,*
  - *$u_{IJ}:m^*(M_I,F)\to m^*p_{IJ*}(M_J,F)[d_{\tilde S_J}-d_{\tilde S_I}]$ for $J\subset I$, are morphisms, $p_{IJ}:\tilde S_J\to\tilde S_I$ being the projection, satisfying for $I\subset J\subset K$, $p_{IJ*}u_{JK}\circ u_{IJ}=u_{IK}$ in $C_{O_{\tilde S_I}fil,\mathcal{D}}(\tilde S_I)$;*
- *whose morphisms $m:((M_I,F),u_{IJ})\to((N_I,F),v_{IJ})$ between $(M,F)=((M_I,F)_{I\subset[1,\cdots l]},u_{IJ})$ and $(N,F)=((N_I,F)_{I\subset[1,\cdots l]},v_{IJ})$ are a family of morphisms of complexes,*

  $$m=(m_I:(M_I,F)\to(N_I,F))_{I\subset[1,\cdots l]}$$

  such that $v_{IJ}\circ m_I=p_{IJ*}m_J\circ u_{IJ}$ in $C_{O_{\tilde S_I}fil,\mathcal{D}}(\tilde S_I)$.

We denote by $C^{\sim}_{Ofil,\mathcal{D}}(S/(\tilde{S}_I)) \subset C_{Ofil,\mathcal{D}}(S/(\tilde{S}_I))$ the full subcategory consisting of objects $((M_I,F),u_{IJ})$ such that the u_{IJ} are ∞-filtered Zariski, resp. usu, local equivalences,and

$$D_{Ofil,\mathcal{D},\infty}(S/(\tilde{S}_I)) := Ho_{\top,\infty}C^{\sim}_{Ofil,\mathcal{D}}(S/(\tilde{S}_I))$$

the derived category.

Let $f : X \to S$ be a morphism, with $X, S \in \mathrm{Var}(\mathbb{C})$, such that there exist a factorization $f; X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection, and consider $S = \cup_{i=1}^{l} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$, with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$; or let $f : X \to S$ be a morphism, with $X, S \in \mathrm{AnSp}(\mathbb{C})$, such that there exist a factorization $f; X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{AnSm}(\mathbb{C})$, l a closed embedding and p_S the projection and consider $S = \cup_{i=1}^{l} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$, with $\tilde{S}_i \in \mathrm{AnSm}(\mathbb{C})$. Then, $X = \cup_{i=1}^{l} X_i$ with $X_i := f^{-1}(S_i)$. Denote by $p_{IJ} : \tilde{S}_J \to \tilde{S}_I$ and $p'_{IJ} : Y \times \tilde{S}_J \to Y \times \tilde{S}_I$ the projections and by

$$E_{IJ} = \begin{tikzcd} \tilde{S}_J\backslash S_J \arrow[r, "m_J"] \arrow[d, "p_{IJ}"] & \tilde{S}_J \arrow[d, "p_{IJ}"] \\ \tilde{S}_I\backslash(S_I\backslash S_J) \arrow[r, "m=m_{IJ}"] & \tilde{S}_I \end{tikzcd}, \quad E'_{IJ} = \begin{tikzcd} Y \times \tilde{S}_J\backslash X_J \arrow[r, "m'_J"] \arrow[d, "p'_{IJ}"] & Y \times \tilde{S}_J \arrow[d, "p'_{IJ}"] \\ Y \times \tilde{S}_I\backslash(X_I\backslash X_J) \arrow[r, "m'=m'_{IJ}"] & Y \times \tilde{S}_I \end{tikzcd}, \quad E_{fIJ} \begin{tikzcd} \tilde{X}_J \arrow[r, "\tilde{f}_J"] \arrow[d, "p'_{IJ}"] & \tilde{S}_J \arrow[d, "p_{IJ}"] \\ Y \times \tilde{S}_I \arrow[r, "\tilde{f}_I"] & \tilde{S}_I \end{tikzcd}$$

the commutative diagrams. We then have the filtered De Rham the inverse image functor :

$$\begin{aligned} f^{*mod[-],\Gamma} : C_{Ofil,\mathcal{D}}(S/(\tilde{S}_I)) \to C_{Ofil,\mathcal{D}}(X/(Y \times \tilde{S}_I)), \quad (M,F) = ((M_I,F),u_{IJ}) \mapsto \\ f^{*mod[-],\Gamma}(M,F) := (\Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]}(M_I,F))), \tilde{f}_J^{*mod[-]} u_{IJ}) \end{aligned}$$

with, denoting for short $d_{IJ} := d_{\tilde{S}_J} - d_{\tilde{S}_I}$

$$\begin{aligned} \tilde{f}_J^{*mod[-]} u_{IJ} : \Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]}(M_I,F)) &\xrightarrow{\Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]})(u_{IJ})} \Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]} p_{IJ*}(M_J,F)[d_{IJ}]) \\ &\xrightarrow{\Gamma_{X_I} E(T(p_{IJ}^{*mod}, p_{\tilde{S}_I})(-)^{-1})[d_Y + d_{IJ}]} \Gamma_{X_I} E(p'_{IJ*} p_{\tilde{S}_J}^{*mod}(M_J,F)[d_Y + d_{IJ}]) \\ &\xrightarrow{=} p'_{IJ*} \Gamma_{X_J} E(p_{\tilde{S}_J}^{*mod[-]})(M_J,F)[d_{IJ}]. \end{aligned}$$

It induces in the derived categories, the functor

$$\begin{aligned} Rf^{*mod[-],\Gamma} : D_{Ofil,\mathcal{D},\infty}(S/(\tilde{S}_I) \to D_{Ofil,\mathcal{D},\infty}(X/(Y \times \tilde{S}_I)), \\ (M,F) = ((M_I,F),u_{IJ}) \mapsto \\ Rf^{*mod[-],\Gamma} := f^{*mod[-],\Gamma}(M,F) := (\Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]}(M_I,F)), \tilde{f}_J^{*mod[-]} u_{IJ}). \end{aligned}$$

By definition, for $f : T \to S$ with $T, S \in \mathrm{QPVar}(\mathbb{C})$ or with $T, S \in \mathrm{AnSp}(\mathbb{C})^{QP}$, after considering a factorization $f : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection, the commutative diagrams of functors

$$\begin{tikzcd} C_{\mathcal{D}fil}(S/(\tilde{S}_I)) \arrow[r, "o_S"] \arrow[d, "f^{*mod[-],\Gamma}"] & C_{Ofil,\mathcal{D}}(S/(\tilde{S}_I)) \arrow[d, "f^{*mod[-],\Gamma}"] \\ C_{\mathcal{D}fil}(T/(Y \times \tilde{S}_I)) \arrow[r, "o_T"] & C_{Ofil,\mathcal{D}}(T/(Y \times \tilde{S}_I)) \end{tikzcd}, \quad \begin{tikzcd} D_{\mathcal{D}fil,r}(S/(\tilde{S}_I)) \arrow[r, "o_S"] \arrow[d, "Rf^{*mod[-],\Gamma}"] & D_{Ofil,\mathcal{D},\infty}(S/(\tilde{S}_I)) \arrow[d, "Rf^{*mod[-],\Gamma}"] \\ D_{\mathcal{D}fil,\infty}(T/(Y \times \tilde{S}_I)) \arrow[r, "o_T"] & D_{Ofil,\mathcal{D},r}(T/(Y \times \tilde{S}_I)) \end{tikzcd}$$

where o_S and o_T are the forgetful functors.

Let $f : X \to S$ be a morphism, with $X, S \in \mathrm{Var}(\mathbb{C})$, such that there exist a factorization $f ; X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection, and consider $S = \cup_{i=1}^{l} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$, with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$; Then, $X = \cup_{i=1}^{l} X_i$ with $X_i := f^{-1}(S_i)$. We have, for $(M, F) = ((M_I, F), u_{IJ}) \in C_{Ofil,\mathcal{D}}(S/(\tilde{S}_I))^{\vee}$, the canonical transformation map in $D_{Ofil,\mathcal{D}}(T^{an}/(\tilde{T}_I^{an}))^{\vee}$

$$\begin{aligned}
&T^{mod}(an, \gamma_T)(M, F) : \\
f^{*mod[-],\Gamma}(M, F))^{an} &:= ((\Gamma_{T_I} E(p_{\tilde{S}_I}^{*mod[-]}(M_I, F)))^{an}, (f^{*mod}[-]u_{IJ})^{an}) \\
&\xrightarrow{(T^{mod}(an, \gamma_{T_I})(-))} (\Gamma_{T_I^{an}} E((p_{\tilde{S}_I}^{*mod[-]}(M_I, F))^{an}), f^{*mod[-]} u_{IJ}^{an}) \\
&\xrightarrow{=} (\Gamma_{T_I^{an}} E(p_{\tilde{S}_I}^{*mod[-]}(M_I^{an}, F)), f^{*mod[-]} u_{IJ}^{an}) =: f^{*mod[-],\Gamma}((M, F)^{an})
\end{aligned}$$

where the equality is obvious.

5 The category of mixed Hodge modules on complex algebraic varieties and complex analytic spaces and the functorialities

For $S \in \mathrm{Top}$ a topological space endowed with a stratification $S = \sqcup_{k=1}^{d} S_k$ by locally closed subsets S_k together with the perversity $p(S_k)$, we denote by $\mathcal{P}(S, W) \subset D_{fil}(S)$ the category of filtered perverse sheaves of abelian groups. For a locally compact (hence Hausdorf) topological space, we denote by $D_c(S) \subset D(S)$ the full subcategory of complexes of presheaves whose cohomology sheaves are constructible.

5.1 The De Rahm functor for D modules on a complex analytic space

Let $S \in \mathrm{AnSm}(\mathbb{C})$. Recall we have the dual functor

$$\mathbb{D}_S : C(S) \to C(S), \ K \mapsto \mathbb{D}_S(K) := \mathcal{H}om(K, E(\mathbb{Z}_S))$$

which induces the functor

$$L\mathbb{D}_S : D(S) \to D(S), \ K \mapsto L\mathbb{D}_S(K) := \mathbb{D}_S(LK) := \mathcal{H}om(LK, E_{et}(\mathbb{Z}_S)).$$

Let $S \in \mathrm{AnSm}(\mathbb{C})$.

- The functor
$$M \in \mathrm{PSh}_{\mathcal{D}}(S) \mapsto DR(S)(M) := \Omega_S^{\bullet} \otimes_{O_S} M \in C_{\mathbb{C}_S}(S)$$
which sends a D_S module to its De Rham complex (see section 4) induces, after shifting by d_S in order to send holonomic module (degree zero) to perverse sheaves, in the derived category the functor
$$\begin{aligned}
&DR(S)^{[-]} : D_{\mathcal{D}}(S) \to D_{\mathbb{C}_S}(S), M \mapsto \\
&DR(S)^{[-]}(M) := DR(S)(M)[d_S] := \Omega_S^{\bullet} \otimes_{O_S} M[d_S] \simeq K_S \otimes_{D_S}^{L} M \simeq \mathcal{H}om_{D_S}(\mathbb{D}_S L_D M, E(O_S))[d_S]
\end{aligned}$$
and, by functoriality, the functor
$$\begin{aligned}
&DR(S)^{[-]} : D_{\mathcal{D}0fil,\infty}(S) \to D_{\mathbb{C}_S fil,\infty}(S), \\
&(M, W) \mapsto DR(S)^{[-]}(M, W) := (\Omega_S^{\bullet}, F_b) \otimes_{O_S} (M, W)[d_S] = K_S \otimes_{D_S}^{L} (M, W)
\end{aligned}$$

- On the other hand, we have the functor
$$C_{\mathbb{C}_S}(S) \to C_{\mathcal{D}^{\infty}}(S), \ K \mapsto \mathcal{H}om_{\mathbb{C}_S}(L_{\mathbb{C}}\mathbb{D}_S(LK), E(O_S))[-d_S]$$

together with, for $K \in C_{\mathbb{C}_S}(S)$, the canonical map

$$\begin{aligned}
s(K) : K \to DR(S)^{[-]}(J_S^{-1}\mathcal{H}om_{\mathbb{C}_S}(L_{\mathbb{C}}(\mathbb{D}_S LK), E(O_S))[-d_S]) \\
\xrightarrow{=} \mathcal{H}om_{D_S}(\mathbb{D}_S^K L_D J_S^{-1}\mathcal{H}om_{\mathbb{C}_S}(L_{\mathbb{C}}(\mathbb{D}_S LK), E(O_S)), E(O_S)), \\
c \in \Gamma(S^o, L(K)) \longmapsto s(K)(c) = (\phi \in \Gamma(S^{oo}, L_D\mathcal{H}om(L_{\mathbb{C}}(K), E(O_S))) \mapsto \phi(c))
\end{aligned}$$

where $S^{oo} \subset S^o \subset S$ are open subsets.

The main result is Riemann-Hilbert equivalence :

Theorem 26. *Let $S \in \mathrm{AnSm}(\mathbb{C})$.*

(i) The functor $J_S : D_{\mathcal{D},rh}(S) \to D_{\mathcal{D}^\infty,h}(S)$ is an equivalence of category. Moreover, for $K \in C(S)$, we have $\mathcal{H}om(L(K), E(O_S)) \in C_{\mathcal{D}^\infty,h}(S)$.

(ii) The restriction of the De Rahm functor to the full subcategory $D_{\mathcal{D},rh}(S) \subset D_{\mathcal{D}}(S)$ is an equivalence of category

$$DR(S)^{[-]} : D_{\mathcal{D},rh}(S) \xrightarrow{\sim} D_{\mathbb{C}_S,c}(S)$$

whose inverse is the functor

$$K \in C_{\mathbb{C}_S,c}(S) \mapsto J^{-1}\mathcal{H}om_{\mathbb{C}_S}(\mathbb{D}_S L(K), E(O_S))[-d_S],$$

the map $s(K) : K \xrightarrow{\sim} DR(S)^{[-]}(J^{-1}\mathcal{H}om_{\mathbb{C}_S}(L_{\mathbb{C}}\mathbb{D}_S L(K), E(O_S)))$ being an isomorphism.

(iii) The De Rahm functor $DR(S)^{[-]}$ sends regular holonomic modules to perverse sheaves.

Proof. See [18]. □

Let $S \in \mathrm{AnSp}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$.

- The De Rham functor is in this case

$$\begin{aligned}
DR(S)^{[-]} : D_{\mathcal{D}0fil,\infty}(S) \to D_{\mathbb{C}_S fil,\infty}(S), M = ((M_I, W), u_{IJ}) \mapsto \\
DR(S)^{[-]}(M, W) := (DR(\tilde{S}_I)^{[-]}(M_I, W), DR^{[-]}(u_{IJ})) := (\Omega^{\bullet}_{\tilde{S}_I} \otimes_{O_{\tilde{S}_I}} (M_I, W), DR^{[-]}(u_{IJ}))
\end{aligned}$$

  with, denoting for short $d_I = d_{\tilde{S}_I}$

$$\begin{aligned}
DR^{[-]}(u_{IJ}) : \Omega^{\bullet}_{\tilde{S}_I} \otimes_{O_{\tilde{S}_I}} (M_I, W)[d_I] \xrightarrow{\mathrm{ad}(p_{IJ}, p_{IJ*})(-))} p_{IJ*}p_{IJ}^*\Omega^{\bullet}_{\tilde{S}_I} \otimes_{O_{\tilde{S}_I}} (M_I, W)[d_I] \\
\xrightarrow{p_{IJ*}\Omega_{\tilde{S}_J/\tilde{S}_I}[d_I]} p_{IJ*}\Omega^{\bullet}_{\tilde{S}_J} \otimes_{O_{\tilde{S}_J}} p_{IJ}^{*mod}(M_I, W)[d_I] \\
\xrightarrow{p_{IJ*}I(p_{IJ}^{*mod}, p_{IJ})(-,-)(u_{IJ})[d_I]} p_{IJ*}\Omega^{\bullet}_{\tilde{S}_J} \otimes_{O_{\tilde{S}_J}} (M_J, W)[d_J + d_{IJ}]
\end{aligned}$$

- Considering the diagrams

$$D_{IJ} = \begin{array}{ccc} \tilde{S}_J & \xrightarrow{p_{IJ}} & \tilde{S}_I \\ {\scriptstyle i_J}\uparrow & & \uparrow{\scriptstyle i_I} \\ S_J & \xrightarrow{j_{IJ}} & S_I \end{array}$$

  we get the functor

$$\begin{aligned}
C_{\mathbb{C}_S fil}(S) \xrightarrow{T(S/(S_I))} C_{\mathbb{C}_S fil}(S/(S_I)) \to C_{\mathcal{D}0fil}(S/(S_I)), \\
(K, W) \mapsto (\mathcal{H}om_{\mathbb{C}_{\tilde{S}_I}}(L_{\mathbb{C}}\mathbb{D}_{\tilde{S}_I}(Li_{I*}j_I^*(K, W)), E(O_{\tilde{S}_I}))[-d_{\tilde{S}_I}], u_{IJ}(K, W))
\end{aligned}$$

where

$$u_{IJ}(K,W):\mathcal{H}om_{\mathbb{C}_{\tilde{S}_I}}(L_{\mathbb{C}}\mathbb{D}_{\tilde{S}_I}L(i_{I*}j_I^*(K,W)),E(O_{\tilde{S}_I}))[-d_{\tilde{S}_I}]$$
$$\xrightarrow{\mathrm{ad}(p_{IJ}^{*mod},p_{IJ})(-)} p_{IJ*}p_{IJ}^{*mod[-]}\mathcal{H}om_{\mathbb{C}_{\tilde{S}_I}}(L_{\mathbb{C}}\mathbb{D}_{\tilde{S}_I}L(i_{I*}j_I^*(K,W)),E(O_{\tilde{S}_I}))[-d_{\tilde{S}_J}]$$
$$\xrightarrow{\mathcal{H}om(-,Eo(p_{IJ}))\circ T(p_{IJ},\mathrm{hom})(-,-)} p_{IJ*}\mathcal{H}om_{\mathbb{C}_{\tilde{S}_I}}(p_{IJ}^*L_{\mathbb{C}}\mathbb{D}_{\tilde{S}_I}L(i_{I*}j_I^*(K,W)),E(O_{\tilde{S}_J}))[-d_{\tilde{S}_J}]$$
$$\xrightarrow{\mathcal{H}om(T(p_{IJ},\mathbb{D})(-)^{-1},-))} p_{IJ*}\mathcal{H}om_{\mathbb{C}_{\tilde{S}_I}}(L_{\mathbb{C}}\mathbb{D}_{\tilde{S}_J}p_{IJ}^*L(i_{I*}j_I^*(K,W)),E(O_{\tilde{S}_J}))[-d_{\tilde{S}_J}]$$
$$\xrightarrow{\mathcal{H}om(L_{\mathbb{C}}\mathbb{D}_{\tilde{S}_J}T^q(D_{IJ})(j_I^*(K,W)),E(O_{\tilde{S}_J}))} p_{IJ*}\mathcal{H}om_{\mathbb{C}_{\tilde{S}_J}}(L_{\mathbb{C}}\mathbb{D}_{\tilde{S}_J}L(i_{J*}j_J^*(K,W)),E(O_{\tilde{S}_J}))[-d_{\tilde{S}_J}].$$

Moreover, for $(K,W)\in C_{fil}(S)$, we have

$$(\mathcal{H}om_{\mathbb{C}_{\tilde{S}_I}}(L_{\mathbb{C}}\mathbb{D}_{\tilde{S}_I}L(i_{I*}j_I^*(K,W)),E(O_{\tilde{S}_I}))[-d_{\tilde{S}_I}],u_{IJ}(K,W))\in C_{\mathcal{D}^\infty 0fil,h}(S)^0$$

and a canonical map in $D_{fil}(S)=D_{fil}(S/(\tilde{S}_I))$

$$s(K):T(S/(S_I))(K,W):=(L(i_{I*}j_I^*(K,W)),I)\to$$
$$DR(S)^{[-]}(J_S^{-1}\mathcal{H}om_{\mathbb{C}_{\tilde{S}_I}}(L_{\mathbb{C}}\mathbb{D}_{\tilde{S}_I}L(i_{I*}j_I^*(K,W)),E(O_{\tilde{S}_I}))[-d_{\tilde{S}_I}],u_{IJ}(K,W))$$

Corollary 3. *Let $S\in\mathrm{AnSp}(\mathbb{C})$. Let $S=\cup_{i=1}^l S_i$ an open cover such that there exists closed embeddings $i_i:S_i\hookrightarrow\tilde{S}_i$. The restriction of the De Rahm functor to the full subcategory $D^0_{\mathcal{D},rh}(S)\subset D^0_{\mathcal{D}}(S)$ is an equivalence of category*

$$DR(S)^{[-]}:D^0_{\mathcal{D},rh}(S)\xrightarrow{\sim}D_{\mathbb{C}_S,c}(S)$$

whose inverse is the functor

$$K\mapsto J_S^{-1}(\mathcal{H}om_{\mathbb{C}_{\tilde{S}_I}}(L_{\mathbb{C}}\mathbb{D}_{\tilde{S}_I}L(i_{I*}j_I^*K),E(O_{\tilde{S}_I}))[-d_{\tilde{S}_I}],u_{IJ}(K))$$

the map

$$s(K):T(S/(S_I))(K,W):=(L(i_{I*}j_I^*(K,W)),I)\to$$
$$DR(S)^{[-]}(J_S^{-1}\mathcal{H}om_{\mathbb{C}_{\tilde{S}_I}}(L_{\mathbb{C}}\mathbb{D}_{\tilde{S}_I}L(i_{I*}j_I^*(K,W)),E(O_{\tilde{S}_I}))[-d_{\tilde{S}_I}],u_{IJ}(K,W))$$

being an isomorphism.

Proof. Follows from theorem 26(ii), see [27]. □

Proposition 93. *(i) Let $S\in\mathrm{AnSm}(\mathbb{C})$.Then, for $M\in C_{\mathcal{D},c}(S)$, there is a canonical isomorphism*

$$T(D,DR)(M):\mathbb{D}_S^{\mathbb{C}}DR(S)^{[-]}(M)\xrightarrow{\sim}DR(S)^{[-]}(\mathbb{D}_S^K L_D M)$$

(ii) Let $S\in\mathrm{AnSp}(\mathbb{C})$. Let $S=\cup_{i=1}^l S_i$ an open cover such that there exists closed embeddings $i_i:S_i\hookrightarrow\tilde{S}_i$. Then, for $M=(M_I,u_{IJ})\in C^0_{\mathcal{D},c}(S/(\tilde{S}_I))$, there is a canonical isomorphism

$$T(D,DR)(M):\mathbb{D}_S^{\mathbb{C}}DR(S)^{[-]}(M)\xrightarrow{\sim}DR(S)^{[-]}(L\mathbb{D}_S^K M)$$

Proof. (i):See [16].
(ii):Follows from (i), see [27]. □

We have the following transformation maps :

- Let $g : T \to S$ a morphism with $T, S \in \mathrm{AnSm}(\mathbb{C})$. We have, for $(M,W) \in C_{\mathcal{D}0fil}(S)$, the canonical transformation map in $D_{fil}(T)$:

$$T(g,DR)(M,W) : g^*DR(S)^{[-]}(M,W) := g^*(\Omega^\bullet_S \otimes_{O_S} L_D(M,W))[d_S]$$
$$\xrightarrow{\Omega_{T/S}(L_D(M,W))[d_S]} \Omega^\bullet_T \otimes_{O_T} g^{*mod} L_D(M,W)[d_S]$$
$$\xrightarrow{=} \Omega^\bullet_T \otimes_{O_T} g^{*mod[-]} L_D(M,W)[d_S] =: DR(T)^{(-)}(Lg^{*mod[-]}(M,W))$$

Note that this transformation map is NOT an isomorphism in general. It is an isomorphism if g is a smooth morphism. If g is a closed embedding, it is an isomorphism for M non caracteristic with respect to g.

- Let $j : S^o \hookrightarrow S$ an open embedding with $S \in \mathrm{AnSm}(\mathbb{C})$. We have, for $(M,W) \in C_{\mathcal{D}0fil}(S^o)$, the canonical transformation map in $D_{fil}(S)$:

$$T_*(j,DR)(M,W) : DR(S)^{[-]}(j_*(M,W)) := \Omega^\bullet_S \otimes_{O_S} j_*(M,W)[d_S]$$
$$\xrightarrow{T^O_w(j,\otimes)(L_D(M,W))[d_S]} j_*(\Omega^\bullet_{S^o} \otimes_{O_{S^o}} L_D(M,W))[d_S] =: j_*DR(S)^{[-]}(M,W)$$

which is an isomorphism (see proposition 78).

- Let $g : T \to S$ a morphism with $T, S \in \mathrm{AnSp}(\mathbb{C})$. Assume there exist a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{AnSm}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_i S_i$ an open covers such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{AnSm}(\mathbb{C})$. We have, for $M = (M_I, u_{IJ}) \in C_{\mathcal{D}}(S/(\tilde{S}_I))$, the canonical transformation map in $D_{fil}(T/(Y \times \tilde{S}_I))$

$$T^!(g,DR)(M) : T(T/(Y \times \tilde{S}_I))(g^! DR(S)^{[-]}(M,W))$$
$$\xrightarrow{:=} (\Gamma_{T_I} E(\tilde{g}^*_I(\Omega^\bullet_{\tilde{S}_I} \otimes_{O_{\tilde{S}_I}} L_D(M_I,W))), \tilde{g}^*_I DR(u_{IJ}))$$
$$\xrightarrow{(\Omega_{(Y\times\tilde{S}_I/\tilde{S}_I)}(L_D(M_I,W)))} (\Gamma_{T_I} E(\Omega^\bullet_{Y\times\tilde{S}_I} \otimes_{O_{Y\times\tilde{S}_I}} \tilde{g}^{*mod}_I(M_I,W)), DR(\tilde{g}^{*mod}_I u_{IJ}))$$
$$\xrightarrow{(T^O_w(\gamma,\otimes)(\tilde{g}^{*mod}_I L_D(M,W)))} (\Omega^\bullet_{Y\times\tilde{S}_I} \otimes_{O_{Y\times\tilde{S}_I}} \Gamma_{T_I} E(\tilde{g}^{*mod}_I(M_I,W)), DR(\tilde{g}^{*mod}_I u_{IJ}))$$
$$\xrightarrow{=:} DR(T)^{[-]}(Rg^{*mod[-],\Gamma}(M,W))$$

which is an isomorphism.

- Let $f : T \to S$ a morphism with $T, S \in \mathrm{Var}(\mathbb{C})$. Assume there exist a factorization $f : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_i S_i$ an open covers such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. We have, for $M = (M_I, u_{IJ}) \in C_{\mathcal{D}}(S/(\tilde{S}_I))$, the canonical map in $D_{fil}(T/(Y \times \tilde{S}_I))$

$$DR(T)^{[-]}(T^{mod}(an,\gamma_T)(M)) :$$
$$DR(T)^{[-]}((Rf^{*mod[-],\Gamma}M)^{an}) := DR(T)^{[-]}(((\Gamma_{T_I} E(\tilde{f}^{*mod[-]}_I(M_I)))^{an}, (f^{*mod[-]}u^q_{IJ})^{an}))$$
$$\to DR(T)^{[-]}(Rf^{*mod[-],\Gamma}(M^{an})) := DR(T)^{[-]}((\Gamma_{T^{an}_I} E(\tilde{f}^{*mod[-]}_I(M^{an}_I)), f^{*mod[-]}u^{q,an}_{IJ})).$$

Proposition 94. *Let $f : T \to S$ a morphism with $T, S \in \mathrm{Var}(\mathbb{C})$. Assume there exist a factorization $f : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_i S_i$ an open covers such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. Then, for $M = (M_I, u_{IJ}) \in C_{\mathcal{D},rh}(S/(\tilde{S}_I))$, the map in $D_{fil}(T/(Y \times \tilde{S}_I))$*

$$DR(T)^{[-]}(T^{mod}(an,\gamma_T)(M)) :$$
$$DR(T)^{[-]}((Rf^{*mod[-],\Gamma}M)^{an}) := DR(T)^{[-]}(((\Gamma_{T_I} E(\tilde{f}^{*mod[-]}_I(M_I)))^{an}, (f^{*mod[-]}u^q_{IJ})^{an}))$$
$$\to DR(T)^{[-]}(Rf^{*mod[-],\Gamma}(M^{an})) := DR(T)^{[-]}((\Gamma_{T^{an}_I} E(\tilde{f}^{*mod[-]}_I(M^{an}_I)), f^{*mod[-]}u^{q,an}_{IJ}))$$

given above is an isomorphism.

Proof. See [16]. □

In the algebraic case, we have, by proposition 94, for complexes of D-modules whose cohomology sheaves are regular holonomic the following canonical isomorphisms:

Definition 89. *(i) Let $f : T \to S$ a morphism with $T, S \in \mathrm{Var}(\mathbb{C})$. Assume there exist a factorization $f : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have, for $M = (M_I, u_{IJ}) \in C_{\mathcal{D},rh}(S/(\tilde{S}_I))^0$, the canonical map*

$$T^!(f, DR)(M) : f^! DR(S)^{[-]}(M^{an}) \xrightarrow{T^!(f,DR)(M^{an})} DR(T)^{[-]}(Rf^{*mod[-],\Gamma}(M^{an}))$$
$$\xrightarrow{DR(T)^{[-]}(T^{mod}(an,\gamma_T)(M))} DR(T)^{[-]}((Rf^{*mod[-],\Gamma}M)^{an}) =: DR(T)^{[-]}((Rf^{*mod[-],\Gamma}M)^{an}).$$

which is an isomorphism by proposition 94.

(ii) Let $f : T \to S$ a morphism with $T, S \in \mathrm{Var}(\mathbb{C})$. Assume there exist a factorization $f : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have, for $M = (M_I, u_{IJ}) \in C_{\mathcal{D},rh}(S/(\tilde{S}_I))^0$, the canonical transformation map

$$T(f, DR)(M) : DR(T)^{[-]}((Lf^{\hat{*}mod[-],\Gamma}M)^{an}) := DR(T)^{[-]}((L\mathbb{D}_T^K Rf^{*mod[-],\Gamma} L\mathbb{D}_S^K M)^{an})$$
$$\xrightarrow{T(D,DR)(-)} L\mathbb{D}_T^K DR(T)^{[-]}((Rf^{*mod[-],\Gamma} L\mathbb{D}_S^K M)^{an})) \xrightarrow{L\mathbb{D}_T^K T^!(f,DR)(-)} L\mathbb{D}_T^K f^! DR(S)^{[-]}(L\mathbb{D}_S^K M^{an})$$
$$\xrightarrow{L\mathbb{D}_T^K f^! T(D,DR)(-)^{-1}} L\mathbb{D}_T^K f^! L\mathbb{D}_S^K DR(S)^{[-]}(M^{an}) = f^* DR(S)^{[-]}(M^{an})$$

which is an isomorphism by (i) and proposition 93.

(iii) Let $f : T \to S$ a morphism with $T, S \in \mathrm{QPVar}(\mathbb{C})$. Consider a factorization $f : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y = \mathbb{P}^{N,o} \subset \mathbb{P}^N$ an open subset, l a closed embedding, and p_S the projection. We have, for $M \in C_{\mathcal{D},rh}(T/Y \times \tilde{S})^0$, the canonical transformation map

$$T_*(f, DR)(M) : DR(S)^{[-]}((\int_f M)^{an}) \xrightarrow{\mathrm{ad}(f^*, Rf_*)(-)} Rf_* f^* DR(S)^{[-]}((\int_f M)^{an}) \xrightarrow{Rf_* T(f,DR)((\int_f M))}$$
$$Rf_* DR(T)^{[-]}((Lf^{\hat{*}mod[-],\Gamma} \int_f M)^{an}) \xrightarrow{Rf_* DR(T)^{[-]}((\mathrm{ad}(Lf^{\hat{*}mod[-],\Gamma}, \int_f)(M))^{an})} Rf_* DR(T)^{[-]}(M^{an})$$

which is an isomorphism by GAGA in the proper case and by the open embedding case (c.f. proposition 94).

(iv) Let $f : T \to S$ a morphism with $T, S \in \mathrm{QPVar}(\mathbb{C})$. Consider a factorization $f : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y = \mathbb{P}^{N,o} \subset \mathbb{P}^N$ an open subset, l a closed embedding, and p_S the projection. We have, for $M \in C_{\mathcal{D},rh}(T)$, the canonical transformation map

$$T_!(f, DR)(M) : Rf_! DR(T)^{[-]}(M^{an})$$
$$\xrightarrow{Rf_! DR(T)^{[-]}(\mathrm{ad}(\int_{f!} Rf^{*mod[-],\Gamma})(M)^{an})} Rf_! DR(T)^{[-]}((Rf^{*mod[-],\Gamma} \int_{f!} (M))^{an})$$
$$\xrightarrow{T^!(f,DR)(\int_{f!} M)} Rf_! f^! DR(S)^{[-]}(\int_{f!} (M)) \xrightarrow{\mathrm{ad}(Rf_!, f^!)(-)} DR(S)^{[-]}(\int_{f!} M)$$

which is an isomorphism by (iii) and proposition 93.

5.2 The filtered Hodge direct image, the filtered Hodge inverse image, and the hodge support section functors for mixed hodge modules

We consider in the algebraic and analytic case the following categories :

- Let $S \in \mathrm{AnSm}(\mathbb{C})$. The category $C_{\mathcal{D}(1,0)fil,rh}(S) \times_I C_{fil}(S)$ is the category
  - whose set of objects is the set of triples $\{((M,F,W),(K,W),\alpha)\}$ with

    $$(M,F,W) \in C_{\mathcal{D}(1,0)fil,rh}(S),\ (K,W) \in C_{fil}(S),\ \alpha : (K,W) \otimes \mathbb{C}_S \to DR(S)^{[-]}((M,W))$$

    where $DR(S)^{[-]}$ is the De Rahm functor and α is an isomorphism in $D_{fil}(S)$,
  - and whose set of morphisms are

    $$\phi = (\phi_D, \phi_C) : ((M_1,F,W),(K_1,W),\alpha_1) \to ((M_2,F,W),(K_2,W),\alpha_2)$$

    where $\phi_D : (M_1,F,W) \to (M_2,F,W)$ and $\phi_C : (K_1,W) \to (K_2,W)$ are morphisms such that $\alpha_2 \circ DR(S)^{[-]}(\phi_D) = \phi_C \circ \alpha_1$ in $D_{fil}(S)$.

  We have then the full embedding

  $$\mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S) \times_I P_{fil}(S) \hookrightarrow C_{\mathcal{D}(1,0)fil,rh}(S) \times_I D_{fil}(S)$$

- Let $S \in \mathrm{AnSp}(\mathbb{C})$. Let $S = \cup_{i \in I} S_i$ an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_I \in \mathrm{AnSm}(\mathbb{C})$. The category $C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I C_{fil}(S)$ is the category
  - whose set of objects is the set of triples $\{(((M_I,F,W),u_{IJ}),(K,W),\alpha)\}$ with

    $$((M_I,F,W),u_{IJ}) \in C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)),\ (K,W) \in C_{fil}(S),$$
    $$\alpha : T(S/(\tilde{S}_I))(K,W) \otimes \mathbb{C}_S \to DR(S)^{[-]}((M_I,W),u_{IJ})$$

    where $DR(S)^{[-]}$ is the De Rahm functor and α is an isomorphism in $D_{fil}(S)$,
  - and whose set of morphisms are

    $$\phi = (\phi_D, \phi_C) : (((M_{1I},F,W),u_{IJ}),(K_1,W),\alpha_1) \to (((M_{2I},F,W),u_{IJ}),(K_2,W),\alpha_2)$$

    where $\phi_D : (M_1,F,W) \to (M_2,F,W)$ and $\phi_C : (K_1,W) \to (K_2,W)$ are morphisms such that $\alpha_2 \circ DR(S)^{[-]}(\phi_D) = \phi_C \circ \alpha_1$ in $D_{fil}(S)$.

  We have then full embeddings

  $$\mathrm{PSh}^0_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I P_{fil}(S) \hookrightarrow C^0_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S)$$
  $$\xrightarrow{\iota^0_{S/\tilde{S}_I}} C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))^0 \times_I D_{fil}(S) \hookrightarrow C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S)$$

- Let $S \in \mathrm{SmVar}(\mathbb{C})$. The category $C_{\mathcal{D}(1,0)fil,rh}(S) \times_I C_{fil}(S^{an})$ is the category
  - whose set of objects is the set of triples $\{((M,F,W),(K,W),\alpha)\}$ with

    $$(M,F,W) \in C_{\mathcal{D}(1,0)fil,rh}(S),\ (K,W) \in C_{fil}(S^{an}),\ \alpha : (K,W) \otimes \mathbb{C}_{S^{an}} \to DR(S)^{[-]}((M,W)^{an})$$

    where $DR(S)^{[-]}$ is the De Rahm functor and α is an isomorphism in $D_{fil}(S^{an})$,
  - and whose set of morphisms are

    $$\phi = (\phi_D, \phi_C) : ((M_1,F,W),(K_1,W),\alpha_1) \to ((M_2,F,W),(K_2,W),\alpha_2)$$

    where $\phi_D : (M_1,F,W) \to (M_2,F,W)$ and $\phi_C : (K_1,W) \to (K_2,W)$ are morphisms such that $\alpha_2 \circ DR(S)^{[-]}(\phi_D) = \phi_C \circ \alpha_1$ in $D_{fil}(S^{an})$.

We have then the full embedding

$$\mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S) \times_I P_{fil}(S^{an}) \hookrightarrow C_{\mathcal{D}(1,0)fil,rh}(S) \times_I D_{fil}(S^{an})$$

- Let $S \in \mathrm{Var}(\mathbb{C})$. Let $S = \cup_{i \in I} S_i$ an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_I \in \mathrm{SmVar}(\mathbb{C})$. The category $C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I C_{fil}(S^{an})$ is the category
  - whose set of objects is the set of triples $\{(((M_I, F, W), u_{IJ}), (K, W), \alpha)\}$ with

    $$((M_I, F, W), u_{IJ}) \in C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)),\ (K, W) \in C_{fil}(S^{an}),$$
    $$\alpha : T(S/(\tilde{S}_I))(K, W) \otimes \mathbb{C}_S \to DR(S)^{[-]}(((M_I, W), u_{IJ})^{an})$$

    where $DR(S)^{[-]}$ is the De Rahm functor and α is an isomorphism in $D_{fil}(S^{an})$,
  - and whose set of morphisms are

    $$\phi = (\phi_D, \phi_C) : (((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1) \to (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2)$$

    where $\phi_D : ((M_1, F, W), u_{IJ}) \to ((M_2, F, W), u_{IJ})$ and $\phi_C : (K_1, W) \to (K_2, W)$ are morphisms such that $\alpha_2 \circ DR(S)^{[-]}(\phi_D^{an}) = \phi_C \circ \alpha_1$ in $D_{fil}(S^{an})$.

  We have then full embeddings

  $$\mathrm{PSh}^0_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I P_{fil}(S^{an}) \hookrightarrow C^0_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$$
  $$\xrightarrow{\iota^0_{S/\tilde{S}_I}} C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))^0 \times_I D_{fil}(S^{an}) \hookrightarrow C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$$

For holonomic D-modules on a smooth variety $S \in \mathrm{SmVar}(\mathbb{C})$, there exist for a closed embedding $Z \subset S$ with Z smooth, a V_Z-filtration (see definition 48) satisfying further hypothesis so that it is unique:

Definition 90. *Let $S \in \mathrm{SmVar}(\mathbb{C})$ or $S \in \mathrm{AnSm}(\mathbb{C})$.*

(i) Let $D = V(s) \subset S$ be a smooth (Cartier) divisor, where $s \in \Gamma(S, L)$ is a section of the line bundle $L = L_D$ associated to D. Let $M \in \mathrm{PSh}_{\mathcal{D}}(S)$. A V_D-filtration V for M (see definition 48) is called a Kashiwara-Malgrange V_D-filtration for M if

- *$V_k M$ are coherent O_S modules for all $k \in \mathbb{Z}$, that is V is a good filtration,*
- *$sV_k M = V_{k-1} M$ for $k << 0$,*
- *all eigenvalues of $s\partial_s : \mathrm{Gr}_{V,k} := V_k M / V_{k-1} M \to \mathrm{Gr}_{V,k} M := V_k M / V_{k-1} M$ have real part between $k-1$ and k.*

Almost by definition, a Kashiwara-Malgrange V_D-filtration for M if it exists is unique (see [28]) so that we denote it by $(M, V_D) \in \mathrm{PSh}_{O_S fil}(S)$ and (M, V_D) is strict. In particular if $m : (M_1, F) \to (M_2, F)$ a morphism with $(M_1, F), (M_2, F) \in \mathrm{PSh}_{\mathcal{D}(2)fil}(S)$ such that M_1 and M_2 admit the Kashiwara-Malgrange filtration for $D \subset S$, we have $m(V_{D,q} F^p M_1) \subset V_{D,q} F^p M_2$, that is we get $m : (M_1, F, V_D) \to (M_2, F, V_D)$ a filtered morphism, and if $0 \to M' \to M \to M'' \to 0$ is an exact sequence, $0 \to (M', V_D) \to (M, V_D) \to (M'', V_D) \to 0$ is an exact sequence (strictness).

(ii) More generally, let $Z = V(s_1, \ldots, s_r) = D_1 \cap \cdots \cap D_r \subset S$ be a smooth Zariski closed subset, where $s_i \in \Gamma(S, L_i)$ is a section of the line bundle $L = L_{D_i}$ associated to D_i. Let $M \in \mathrm{PSh}_{\mathcal{D}}(S)$. A V_Z-filtration V for M (see definition 48) is called a Kashiwara-Malgrange V_Z-filtration for M if

- *$V_k M$ are coherent O_S modules for all $k \in \mathbb{Z}$,*
- *$\sum_{i=1}^r s_i V_k M = V_{k-1} M$ for $k << 0$,*

– *all eigenvalues of* $\sum_{i=1}^r s_i\partial_{s_i} : \mathrm{Gr}_{V,k} M := V_kM/V_{k-1}M \to \mathrm{Gr}_k^V M := V_kM/V_{k-1}M$ *have real part between* $k-1$ *and* k.

Almost by definition, a Kashiwara-Malgrange V_Z*-filtration for* M *if it exists is unique (see [28]) so that we denote it by* $(M, V_Z) \in \mathrm{PSh}_{O_S fil}(S)$ *and* (M, V_Z) *is strict. In particular if* $m : (M_1, F) \to (M_2, F)$ *a morphism with* $(M_1, F), (M_2, F) \in \mathrm{PSh}_{\mathcal{D}(2)fil}(S)$ *such that* M_1 *and* M_2 *admit the Kashiwara-Malgrange filtration for* $D \subset S$*, we have* $m(V_{Z,q}F^pM_1) \subset V_{Z,q}F^pM_2$*, that is we get* $m : (M_1, F, V_Z) \to (M_2, F, V_Z)$ *a filtered morphism, and if* $0 \to M' \to M \to M'' \to 0$ *is an exact sequence,* $0 \to (M', V_Z) \to (M, V_Z) \to (M'', V_Z) \to 0$ *is an exact sequence (strictness).*

Proposition 95. *(i) Let* $S \in \mathrm{AnSm}(\mathbb{C})$.

– *Let* $D = V(s) \subset S$ *a smooth (Cartier) divisor, where* $s \in \Gamma(S, L)$ *is a section of the line bundle* $L = L_D$ *associated to* D*. If* $M \in \mathrm{PSh}_{\mathcal{D},rh}(S)$*, the Kashiwara-Malgrange* V_D*-filtration for* M *(see definition 90) exist so that we denote it by* $(M, V_D) \in \mathrm{PSh}_{O_S fil}(S)$.
– *More generally, let* $Z = V(s_1, \ldots, s_r) = D_1 \cap \cdots \cap D_r \subset S$ *be a smooth Zariski closed subset, where* $s_i \in \Gamma(S, L_i)$ *is a section of the line bundle* $L = L_{D_i}$ *associated to* D_i*. If* $M \in \mathrm{PSh}_{\mathcal{D}(2)rh}(S)$*, the Kashiwara-Malgrange* V_Z*-filtration for* M *(see definition 90) exist so that we denote it by* $(M, V_Z) \in \mathrm{PSh}_{O_S fil}(S)$.

(ii) Let $S \in \mathrm{SmVar}(\mathbb{C})$.

– *Let* $D = V(s) \subset S$ *a smooth (Cartier) divisor, where* $s \in \Gamma(S, L)$ *is a section of the line bundle* $L = L_D$ *associated to* D*. If* $M \in \mathrm{PSh}_{\mathcal{D},rh}(S)$*, the Kashiwara-Malgrange* V_D*-filtration for* M *(see definition 90) exist so that we denote it by* $(M, V_D) \in \mathrm{PSh}_{O_S fil}(S)$.
– *More generally, let* $Z = V(s_1, \ldots, s_r) = D_1 \cap \cdots \cap D_r \subset S$ *be a smooth Zariski closed subset, where* $s_i \in \Gamma(S, L_i)$ *is a section of the line bundle* $L = L_{D_i}$ *associated to* D_i*. If* $M \in \mathrm{PSh}_{\mathcal{D}(2)rh}(S)$*, the Kashiwara-Malgrange* V_Z*-filtration for* M *(see definition 90) exist so that we denote it by* $(M, V_Z) \in \mathrm{PSh}_{O_S fil}(S)$.

Proof. (i):Follows from the work of Kashiwara. Note that the second point is a particular case of the first by induction. (ii): Take a compactification $\bar{S} \in \mathrm{PSmVar}(\mathbb{C})$ of S and denote by $\bar{D} \subset \bar{S}$ the closure of D. Using the closed embedding $i : \bar{S} \hookrightarrow L_{\bar{D}}$ given by the zero section, we may assume that $\bar{D}$ is smooth. Denote by $j : \bar{S}\backslash\bar{D} \hookrightarrow \bar{S}$ the open complementary. Then, $j_*M \in \mathrm{PSh}_{\mathcal{D},rh}(\bar{S})$ is regular holonomic. The result then follows by (i) and GAGA for j_*M and we get $(j_*M, V_D) \in \mathrm{PSh}_{O_{\bar{S}} fil}(\bar{S})$ and $(M, V_D) = (j^*j_*M, j^*V_D) \in \mathrm{PSh}\, O_S fil(S)$. We can also prove the algebraic case directly using the theory of meromorphic connexions since a simple holonomic D_S-module with support $Z \subset S$ is an integrable connexion on $Z^o = Z \cap S^o$, $S^o \subset S$ being an open subset. □

We have from Kashiwara or Malgrange the following which relates the graded piece of the Kashiwara-Magrange V-filtration V_D of a D_S module $M \in \mathrm{PSh}_{\mathcal{D},rh}(S)$ along a smooth divisor D with the nearby and vanishing cycle functors of $DR(S)(M)$ with respect to D :

Theorem 27. *Let* $S \in \mathrm{AnSm}(\mathbb{C})$*. Let* $D = V(s) \subset S$ *be a smooth (Cartier) divisor, where* $s \in \Gamma(S, L)$ *is a section of the line bundle* $L = L_D$ *associated to* D*. Denote by* $j : S^o := S\backslash D \hookrightarrow S$ *the open complementary embedding and by* $k : \tilde{S}^o \xrightarrow{k} S^o \xrightarrow{j} S$ *with* k *the universal covering of* S^o *For* $M \in \mathrm{PSh}_{\mathcal{D},rh}(S)$ *a regular holonomic* D_S *module, consider* $(M, V_D) \in \mathrm{PSh}_{O_S fil}(S)$ *it together with its* V_D *filtration. Then,*

- *there is canonical isomorphism*

$$T(V, DR)(M) : DR(S)(\mathrm{Gr}_{V_D,0} M) \xrightarrow{\sim} \psi_D(DR(S)(M)) := Rk_*k^*DR(S)(M)$$

- *there is canonical isomorphism*

$$T(V, DR)(M) : DR(S)(\mathrm{Gr}_{V_D,-1} M) \xrightarrow{\sim}$$
$$\phi_D(DR(S)(M)) := \mathrm{Cone}(DR(S)(M) \xrightarrow{\mathrm{ad}(k^*,Rk_*)(-)} \psi_D DR(S)(M))[-1]$$

- $T(V,DR)(M) : DR(S)(\partial_s) \simeq can$, *with* $can : \psi_D DR(S)(M) \to \phi_D(DR(S))$ *the structural embedding of complexes of the cone,*

- $T(V,DR)(M) : DR(S)(T) \simeq s\partial_s$, *with* $T : \psi_D DR(S)(M) \to \psi_D(DR(S))$ *the monodromy morphism.*

- $T(V,DR)(M) : DR(S)(s) \simeq var$ *with* $var : \phi_D DR(S)(M) \to \psi_D(DR(S))$.

Proof. See [28]. □

The main tool is the nearby and vanishing cycle functors for Cartier divisors. We need for the definition of Hodge modules on a smooth complex algebraic variety S to extend the V-filtration associated to a smooth Cartier divisor $D \subset S$ of regular holonomic D_S module M such that the monodromy morphism $T : \psi_D(DR(S)(M)) \to \psi_D(DR(S))$ is quasi-unipotent by a rational V-filtration (i.e. indexed by rational numbers).

Definition 91. *Let* $S \in \mathrm{SmVar}(\mathbb{C})$ *or* $S \in \mathrm{AnSm}(\mathbb{C})$*. Let* $D = V(s) \subset S$ *a (Cartier) divisor, where* $s \in \Gamma(S, L)$ *is a section of the line bundle* $L = L_D$ *associated to* D*. We then have the zero section embedding* $i : S \hookrightarrow L$*. We denote* $L_0 = i(S)$ *and* $j : L^o := L\backslash L_0 \hookrightarrow L$ *the open complementary subset. We denote* $\mathrm{PSh}_{\mathcal{D}(2)fil,rh}(S)^{sp_D 0} \subset \mathrm{PSh}_{\mathcal{D}(2)fil,rh}(S)$ *the full subcategory consiting of objects such that the monodromy operator* $T : \psi_D(DR(S)(M^{(an)})) \to \psi_D(DR(S)(M^{(an)}))$ *is quasi-unipotent.*

(i) Let $(M, F) \in \mathrm{PSh}_{\mathcal{D}(2)fil,rh}(S)^{sp_D 0}$ *By proposition 95, we have the Kashiwara-Malgrange* V_S*-filtration on* $i_{*mod}M$*. We refine it to all rational numbers as follows : for* $\alpha = k - 1 + r/q \in \mathbb{Q}$*,* $k, q, r \in \mathbb{Z}$*,* $q \leq 0$*,* $0 \leq r \leq q - 1$*, we set*

$$V_{S,\alpha}M := q_{V,k}^{-1}(\oplus_{k-1<\beta\leq\alpha} \mathrm{Gr}_{k,\beta}^{V_S} M \subset V_{S,k}M$$

with $\mathrm{Gr}_{k,\beta}^{V_S} M := \ker(\partial_s s - \beta I) \subset \mathrm{Gr}_k^{V_S} M$ *and* $q_{V,k} : V_{S,k}M \to \mathrm{Gr}_k^{V_S} M$ *is the projection. We set similarly*

$$V_{S,<\alpha}M := q_{V,k}^{-1}(\oplus_{k-1<\beta<\alpha} \mathrm{Gr}_{k,\beta}^{V_S} M \subset V_{S,k}M$$

The Hodge filtration induced on $\mathrm{Gr}_\alpha^V M$ *is*

$$F^p \mathrm{Gr}_\alpha^{V_S} M := (F^p M \cap V_{S,\alpha}M)/(F^p M \cap V_{S,<\alpha}M)$$

(ii) we have using (i) the nearby cycle functors

$$\psi_D : \mathrm{PSh}_{\mathcal{D}fil,rh}(S)^{sp_D 0} \to \mathrm{PSh}_{\mathcal{D}fil,rh}(D/(S)),\ (M, F) \mapsto \psi_D(M, F) := \oplus_{-1\leq\alpha<0} \mathrm{Gr}_{V_S,\alpha}\, i_{*mod}(M, F)$$

and

$$\psi_{D1} : \mathrm{PSh}_{\mathcal{D}(2)fil,rh}(S)^{sp_D 0} \to \mathrm{PSh}_{\mathcal{D}fil,rh}(D/(S)),\ (M, F) \mapsto \psi_{D1}(M, F) := \mathrm{Gr}_{V_S,-1}\, i_{*mod}(M, F)$$

and the vanishing cycle functor

$$\phi_{D1} : \mathrm{PSh}_{\mathcal{D}(2)fil,rh}(S)^{sp_D 0} \to \mathrm{PSh}_{\mathcal{D}(2)fil,rh}(D/(S)),\ (M, F) \mapsto \phi_{D1}(M, F) := \mathrm{Gr}_{V_S,0}\, i_{*mod}(M, F).$$

(iii) This induces, by theorem 27, the nearby cycle functors

$$\psi_D : \mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S)^{sp_D 0} \times_I P_{fil}(S^{(an)}) \to \mathrm{PSh}_{\mathcal{D}(1,0)fil,rh,D}(S) \times_I P_{fil,D}(S^{an}),$$
$$((M, F, W), (K, W), \alpha) \mapsto \psi_D((M, F, W), (K, W), \alpha) := (\psi_D(M, F, W), \psi_D(K, W), \psi_D(\alpha))$$

and

$$\psi_{D1} : \mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S)^{sp_D 0} \times_I P_{fil}(S^{(an)}) \to \mathrm{PSh}_{\mathcal{D}(1,0)fil,rh,D}(S) \times_I P_{fil,D}(S^{an}),$$
$$((M, F, W), (K, W), \alpha) \mapsto \psi_{D1}((M, F, W), (K, W), \alpha) := (\psi_{D1}(M, F, W), \psi_{D1}(K, W), \psi_{D1}(\alpha))$$

and the vanishing cycle functor

$$\phi_{D1} : \mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S)^{sp_D 0} \times_I D_{fil}(S^{(an)}) \to \mathrm{PSh}_{\mathcal{D}(1,0)fil,rh,D}(S) \times_I P_{fil,D}(S^{an}),$$
$$((M,F,W),(K,W),\alpha) \mapsto \phi_{D1}((M,F,W),(K,W),\alpha) := (\phi_{D1}(M,F,W),\phi_D(K,W),\phi_D(\alpha))$$

We have the category of mixed Hodge modules over a complex algebraic variety or a complex analytic space S defined by, for S smooth, by induction on dimension of S, and for S singular using embeddings into smooth complex algebraic varieties, resp. smooth complex analytic spaces:

Definition 92. *[27]*

(i) Let $S \in \mathrm{SmVar}(\mathbb{C})$ or $S \in \mathrm{AnSm}(\mathbb{C})$. Denote $\mathrm{PSh}_{\mathcal{D}fil,rh}(S)^{sp,ssd} \subset \mathrm{PSh}_{\mathcal{D}fil,rh}(S)$ the full subcategory consisting of objects (M,F)

- *such that for all Cartier divisor $D = V(s) \subset S$, $s \in \Gamma(S,L)$, denoting $i : S \hookrightarrow L$ the closed embedding the monodromy morphism $T : \psi_D(DR(S)(M^{(an)})) \to \psi_D(DR(S)(M^{(an)}))$ is quasi-unipotent, $sF^pV_{S,\alpha}i_{*mod}M = F^pV_{S,\alpha-1}i_{*mod}M$ for $\alpha < 0$, $\partial_s F^p \mathrm{Gr}^{V_S}_\alpha i_{*mod}M = \mathrm{Gr}^{V_S}_{\alpha+1} i_{*mod}M$ for $\alpha > -1$, the filtration induced by F on $\mathrm{Gr}^{V_S}_\alpha i_{*mod}M$ is good,*
- *which admits a decomposition with D_S module with strict support on closed irreducible subvarieties.*

The category of Hodge modules over S of weight w is the full subcategory

$$\iota_S : HM(S,w) = \oplus_{d\in\mathbb{N}} HM_{\leq d}(S,w) \hookrightarrow \mathrm{PSh}_{\mathcal{D}fil,rh}(S)^{sp,ssd} \times_I P(S^{(an)}), \hookrightarrow \mathrm{PSh}_{\mathcal{D}fil,rh}(S) \times_I P(S^{(an)})$$

given inductively by, d being the dimension of the support of the D_S modules,

- *for $i_0 : s_0 \hookrightarrow S$ a closed point, $i_{0*}\iota_{\mathrm{pt}} : HM_{s_0}(S,w) = HS \hookrightarrow \mathrm{PSh}_{\mathcal{D}fil,rh,s_0}(S) \times_I P_{s_0}(S^{(an)})$ consist of Hodge structures of weight w, this gives $HM_0(S,w)$*
- *for $Z \subset S$ an irreducible closed subvariety of dimension d, $((M,F),K,\alpha) \in \mathrm{PSh}_{\mathcal{D}fil,rh}(S) \times_I P(S^{(an)})$ belongs to $HM_Z(S,w)$ if and only if M has strict support Z (i.e. $\mathrm{supp}(M) = Z$ and for all non trivial subobject N or quotient of M $\mathrm{supp}(N) = Z$), and for all proper maps $f : S^o \to \mathbb{A}^1$ such that $f_{|Z\cap S^o} \neq 0$, $j : S^o \hookrightarrow S$ being an open subset,*

  $$\mathrm{Gr}^{W(N)}_k \psi_{f^{-1}(0)}(j^*(M,F), j^*K, j^*\alpha) \in HM_{\leq d-1}(S^o, w-1+k) \hookrightarrow \mathrm{PSh}_{\mathcal{D}fil,rh,f^{-1}(0)}(S^o) \times_I P_{f^{-1}(0)}(S^{o(an)})$$

  for all $k \in \mathbb{Z}$, see definition 91, $W(N)$ being the weight filtration associated to the monodromy morphism $T : \psi_{f^{-1}(0)}(DR(S)(M^{(an)})) \to \psi_{f^{-1}(0)}(DR(S)(M^{(an)}))$, we then set $HM_{\leq d}(S,w) := \oplus_{Z\subset S,\dim(Z)=d} HM_Z(S,w)$.

(ii) Let $S \in \mathrm{SmVar}(\mathbb{C})$ or $S \in \mathrm{AnSm}(\mathbb{C})$. The category of mixed Hodge modules over S is the full subcategory

$$\iota_S : MHM(S) \hookrightarrow MHW(S) \hookrightarrow \mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S) \times_I P_{fil}(S^{(an)}),$$

where the full subcategory $MHW(S)$ consists of objects $((M,F,W),(K,W),\alpha) \in \mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S) \times_I P_{fil}(S^{(an)})$ satisfy

$$(\mathrm{Gr}^W_i((M,F,W), \mathrm{Gr}^W_i(K,W), \mathrm{Gr}^W_i \alpha) \in HM(S).$$

and the objects of $MHM(S)$ satisfy in addition an admissibility condition (in particular the three filtration F, W, V_Z are compatible). As usual, for $Z \subset S$ a closed subset and $j : S\backslash Z \hookrightarrow S$ the open complementary subset, we denote $MHM_Z(S) \subset MHM(S)$ the full subcategory consisting of $((M,F,W),(K,W),\alpha) \in MHM(S)$ such that

$$j^*((M,F,W),(K,W),\alpha) := (j^*(M,F,W), j^*(K,W), j^*\alpha) = 0.$$

(iii) Let $S\in\mathrm{Var}(\mathbb{C})$ or $S\in\mathrm{AnSp}(\mathbb{C})$ non smooth. Take an open cover $S=\cup_i S_i$ so that there are closed embedding $S_I\hookrightarrow\tilde{S}_I$, with $S_I\in\mathrm{SmVar}(\mathbb{C})$, resp $S_I\in\mathrm{AnSm}(\mathbb{C})$. The category of mixed Hodge modules over S is the full subcategory

$$\iota_S:MHM(S)\hookrightarrow MHW(S)\hookrightarrow\mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I P_{fil}(S^{(an)})$$

consisting of objects

$$(((M_I,F,W),u_{IJ}),(K,W),\alpha)\in\mathrm{PSh}^0_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I P_{fil}(S^{(an)})$$

such that $((M_I,F,W),T(S/(\tilde{S}_I))(K,W),\alpha)\in(MHM_{S_I}(\tilde{S}_I))$ (see (ii)). The category $MHM(S)$ does NOT depend on the open cover an the closed embedding by proposition 97.

(iv) Let $S\in\mathrm{Var}(\mathbb{C})$ or $S\in\mathrm{AnSp}(\mathbb{C})$. We get from (iii) $D(MHM(S)):=\mathrm{Ho}_{top}C(MHM(S))$. By induction, using the result for mixed hodge structure and the strictness of the Kashiwara-Malgrange V-filtration for morphism of D-module, the morphism of $MHM(S)$ are strict for F and W (see [27]).

- Let $S\in\mathrm{SmVar}(\mathbb{C})$. We consider the canonical functor

$$\pi_S:C(MHW(S))\xrightarrow{\iota_S}C_{\mathcal{D}(1,0)fil}(S)\times_I C_{fil}(S^{an})\xrightarrow{p_S}C_{\mathcal{D}(1,0)fil}(S),$$
$$((M,F,W),(K,W),\alpha)\mapsto(M,F,W)$$

  where p_S is the projection functor. Then $\pi_S(MHW(S))\subset\mathrm{PSh}_{\mathcal{D}(1,0)fil}(S)$ is the subcategory consisting of $(M,F,W)\in\mathrm{PSh}_{\mathcal{D}(1,0)fil}(S)$ such that $((M,F,W),(K,W),\alpha)\in MHW(S)$ is a W filtered Hodge module for some $(K,W)\in C_{fil}(S)$. It induces in the derived category the functor

$$\pi_S:D(MHW(S))\xrightarrow{\iota_S}D_{\mathcal{D}(1,0)fil,\infty}(S)\times_I D_{fil}(S^{an})\xrightarrow{p_S}D_{\mathcal{D}(1,0)fil,\infty}(S),$$
$$((M,F,W),(K,W),\alpha)\mapsto(M,F,W)$$

  after localization with respect to ∞-filtered Zariski and usu local equivalence.

- Let $S\in\mathrm{Var}(\mathbb{C})$ non smooth. Take an open cover $S=\cup_i S_i$ such that there are closed embedding $S_I\hookrightarrow\tilde{S}_I$, with $S_I\in\mathrm{SmVar}(\mathbb{C})$. We consider the canonical functor

$$\pi_S:C(MHW(S))\hookrightarrow C_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))\times_I C_{fil}(S^{an})\xrightarrow{p_S}C_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)),$$
$$(((M_I,F,W),u_{IJ}),(K,W),\alpha)\mapsto((M_I,F,W),u_{IJ})$$

  where p_S is the projection functor. Then $\pi_S(MHW(S))\subset\mathrm{PSh}_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))$ is the subcategory consisting of $((M,F,W),u_{IJ})\in\mathrm{PSh}_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))$ such that $(((M_I,F,W),u_{IJ}),(K,W),\alpha)\in MHW(S)$ is a W filtered Hodge module for some $(K,W)\in C_{fil}(S)$. It induces in the derived category the functor

$$\pi_S:D(MHW(S))\xrightarrow{\iota_S}D_{\mathcal{D}(1,0)fil,\infty}(S/(\tilde{S}_I))\times_I D_{fil}(S^{an})\xrightarrow{p_S}D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)),$$
$$(((M_I,F,W),u_{IJ},(K,W),\alpha)\mapsto((M,F,W),u_{IJ})$$

  after localization with respect to ∞-filtered Zariski and usu local equivalence.

We have from [27] the following proposition which shows us how to construct inductively mixed Hodge modules, as we do for perverse sheaves :

Proposition 96. *(i) Let $S\in\mathrm{AnSm}(\mathbb{C})$. Let $D=V(s)\subset S$ a (Cartier) divisor, where $s\in\Gamma(S,L)$ is a section of the line bundle $L=L_D$ associated to D. We then have the zero section embedding*

$i : S \hookrightarrow L$. We denote $L_0 = i(S)$ and $j : L^o := L\backslash L_0 \hookrightarrow L$ the open complementary subset. We denote by $MHW(S\backslash D)^{ex} \times_J MHW(D)$ the category whose set of objects consists of

$$\{(\mathcal{M},\mathcal{N},a,b),\mathcal{M}\in MHW(S\backslash D)^{ex},\mathcal{N}\in MHW(D),a:\psi_{D1}\mathcal{M}\to N,b:N\to\psi_{D1}M\}$$

where $MHW(S\backslash D)^{ex} \subset MHW(S\backslash D)$ is the full subcategory of extendable objects. The functor (see definition 91)

$$(j^*,\phi_{D1},c,v):MHW(S)\to MHW(S\backslash D)^{ex}\times_J MHW(D),$$
$$((M,F,W),(K,W),\alpha)\mapsto((j^*(M,F,W),j^*(K,W),j^*\alpha),\phi_{D1}((M,F,W),(K,W),\alpha),can,var)$$

is an equivalence of category.

(ii) *Let $S \in \mathrm{SmVar}(\mathbb{C})$. Let $D = V(s) \subset S$ a (Cartier) divisor, where $s \in \Gamma(S,L)$ is a section of the line bundle $L = L_D$ associated to D. We then have the zero section embedding $i : S \hookrightarrow L$. We denote $L_0 = i(S)$ and $j : L^o := L\backslash L_0 \hookrightarrow L$ the open complementary subset. We denote by $MHW(S\backslash D) \times_J MHW(D)$ the category whose set of objects consists of*

$$\{(\mathcal{M},\mathcal{N},a,b),\mathcal{M}\in MHW(S\backslash D),\mathcal{N}\in MHW(D),a:\psi_{D1}\mathcal{M}\to N,b:N\to\psi_{D1}M\}$$

The functor (see definition 91)

$$(j^*,\phi_{D1},c,v):MHW(S)\to MHW(S\backslash D)\times_J MHW(D),$$
$$((M,F,W),(K,W),\alpha)\mapsto((j^*(M,F,W),j^*(K,W),j^*\alpha),\phi_{D1}((M,F,W),(K,W),\alpha),can,var)$$

is an equivalence of category.

Proof. See [27]. □

Let $S \in \mathrm{Var}(\mathbb{C})$ or $S \in \mathrm{AnSp}(\mathbb{C})$.

- If $S \in \mathrm{Var}(\mathbb{C})$, let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$, and let $S = \cup_{i'=1}^{l'} S_{i'}$ an other open cover such that there exist closed embeddings $i_{i'} : S_{i'} \hookrightarrow \tilde{S}_{i'}$ with $\tilde{S}_{i'} \in \mathrm{SmVar}(\mathbb{C})$.
- If $S \in \mathrm{AnSp}(\mathbb{C})$, let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{AnSm}(\mathbb{C})$, and let $S = \cup_{i'=1}^{l'} S_{i'}$ an other open cover such that there exist closed embeddings $i_{i'} : S_{i'} \hookrightarrow \tilde{S}_{i'}$ with $\tilde{S}_{i'} \in \mathrm{AnSm}(\mathbb{C})$.

Denote $L = [1,\ldots,l]$, $L' = [1,\ldots,l']$ and $L'' := [1,\ldots,l] \sqcup [1,\ldots,l']$. We have then the refined open cover $S = \cup_{k\in L} S_k$ and we denote for $I \sqcup I' \subset L''$, $S_{I\sqcup I'} := \cap_{k\in I\sqcup I'} S_k$ and $\tilde{S}_{I\sqcup I'} := \Pi_{k\in I\sqcup I'}\tilde{S}_k$, so that we have a closed embedding $i_{I\sqcup I'} : S_{I\sqcup I'} \hookrightarrow \tilde{S}_{I\sqcup I'}$. Consider $\pi_S^L(MHM(S)) \subset \mathrm{PSh}_{\mathcal{D}fil}(S/(S_I))$ and $\pi_S^{L'}(MHM(S)) \subset \mathrm{PSh}_{\mathcal{D}fil}(S/(S_{I'}))$. For $I \sqcup I' \subset J \sqcup J'$, denote by $p_{I\sqcup I',J\sqcup J'} : \tilde{S}_{J\sqcup J'} \to \tilde{S}_{I\sqcup I'}$ the projection. We then have a natural transfer map

$$T_S^{L/L'}:\pi_S^L(MHM(S))\to\pi_S^{L'}(MHM(S)),$$
$$((M_I,F,W),s_{IJ})\mapsto(\mathrm{ho}\lim_{I\in L}p_{I'(I\sqcup I')*}\mathrm{Gr}_{V_{I\sqcup I'}}p^{*mod}_{I(I\sqcup I')}(M_I,F)),s_{I'J'}),$$

with, in the homotopy limit, the natural transition morphisms

$$p_{I'(I\sqcup I')*}\mathrm{ad}(p^{*mod}_{IJ},p_{IJ*})(p^{*mod[-]}_{I(I\sqcup I')}(M_I,F)):$$
$$p_{I'(J\sqcup I')*}(\mathrm{Gr}_{V_{J\sqcup I'}}p^{*mod[-]}_{J(J\sqcup I')}(M_J,F))\to p_{I'(I\sqcup I')*}(\mathrm{Gr}_{V_{I\sqcup I'}}p^{*mod[-]}_{I(I\sqcup I')}(M_I,F))$$

for $J \subset I$, and

$$s_{I'J'}:\mathrm{holim}_{I\in L}m^*p_{I'(I\sqcup I')*}(\mathrm{Gr}_{V_{I\sqcup I'}}p^{*mod[-]}_{I(I\sqcup I')}(M_I,F)\to$$
$$\mathrm{holim}_{I\in L}p_{I'J'*}\mathrm{Gr}_{V_{J'}}(p^{*mod[-]}_{I'J'}m^*p_{I'(I\sqcup I')*}\mathrm{Gr}_{V_{I\sqcup I'}}p^{*mod[-]}_{I(I\sqcup I')}((M_I,F)))$$
$$\to\mathrm{holim}_{I\in L}p_{I'J'*}p_{J'(I\sqcup J')*}\mathrm{Gr}_{V_{I\sqcup I'}}p^{*mod[-]}_{I(I\sqcup J')}(M_I,F)$$

Proposition 97. *(i) Let $S \in \mathrm{Var}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. Then $\pi_S(MHM(S) \subset \mathrm{PSh}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$ does not depend on the open covering of S and the closed embeddings. More precisely, let $S = \cup_{i'=1}^{l'} S_{i'}$ an other open cover such that there exist closed embeddings $i_{i'} : S_{i'} \hookrightarrow \tilde{S}_{i'}$ with $\tilde{S}_{i'} \in \mathrm{SmVar}(\mathbb{C})$. Then,*

$$T_S^{L/L'} : \pi_S^L(MHM(S)) \to \pi_S^{L'}(MHM(S)),$$

is an equivalence of category with inverse is $T_S^{L'/L} : \pi_S^{L'}(MHM(S)) \to \pi_S^L(MHM(S))$.

(ii) Let $S \in \mathrm{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{AnSm}(\mathbb{C})$. Then $\pi_S(MHM(S) \subset \mathrm{PSh}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$ does not depend on the open covering of S and the closed embeddings. More precisely, let $S = \cup_{i'=1}^{l'} S_{i'}$ an other open cover such that there exist closed embeddings $i_{i'} : S_{i'} \hookrightarrow \tilde{S}_{i'}$ with $\tilde{S}_{i'} \in \mathrm{AnSm}(\mathbb{C})$. Then,

$$T_S^{L/L'} : \pi_S^L(MHM(S)) \to \pi_S^{L'}(MHM(S)),$$

is an equivalence of category with inverse is $T_S^{L'/L} : \pi_S^{L'}(MHM(S)) \to \pi_S^L(MHM(S))$.

Proof. Follows from proposition 96(see [27]). □

The main results of Saito, which implies in the algebraic case the six functor formalism on $DMHM(-)$ are the followings

Theorem 28. *Let $S \in \mathrm{Var}(\mathbb{C})$. The category of mixed Hodge modules is the full subcategory*

$$\iota_S : MHM(S) \hookrightarrow MHW(S) \hookrightarrow \mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S) \times_I P_{fil}(S^{an})$$

consisting of objects

$$((M, F, W), (K, W), \alpha) = (((M_I, F, W), u_{IJ}), (K, W), \alpha) \in \mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S) \times_I P_{fil}(S^{an})$$

such that $((M, F, W)^{an}, (K, W), \alpha) := (((M_I^{an}, F, W), u_{IJ}), (K, W), \alpha) \in MHM(S^{an})$.

Proof. Follows from GAGA and the extendableness in the algebraic case (proposition 96). □

Definition 93. *Let $S \in \mathrm{SmVar}(\mathbb{C})$ or $S \in \mathrm{AnSm}(\mathbb{C})$. We denote by $VMHS(S) \subset \mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S) \times_I P_{fil}(S^{an})$ the full subcategory consisting of variation of mixed Hodge structure, whose objects consist of*

$$(((L_S, W) \otimes O_S, F), (L_S, W), \alpha) \subset \mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S) \times_I P_{fil}(S^{an})$$

with

- *$L_S \in \mathrm{PSh}(S^{an})$ a local system,*
- *the D_S module structure on $(L_S, W) \otimes O_S$ is given by the flat connection associated to the local system L_S,*
- *$F^p(W^q L_S \otimes O_S) \subset (W^q L_S \otimes O_S)$ are locally free O_S subbundle satisfying Griffitz transversality for the D_S module structure (i.e. for the flat connection).*
- *$\alpha : (L_S, W) \to DR(S)^{[-]}((L_S, W) \otimes O_S)$ is the isomorphism given by theorem 26.*

Theorem 29. *Let $S \in \mathrm{SmVar}(\mathbb{C})$ or $S \in \mathrm{AnSm}(\mathbb{C})$.*

(i) A variation of mixed Hodge structure $(((L_S, W) \otimes O_S, F), (L_S, W), \alpha) \in VMHS(S)$ (see definition 93) is a mixed module. That is $VMHS(S) \subset MHM(S)$.

(ii) *For $((M,F,W),(K,W),\alpha)\in MHM(S)$ a mixed Hodge module with support* $\operatorname{supp} M = Z$*, there exist an open subset $j : S^o \hookrightarrow S$, such that $j^*((M,F,W),(K,W),\alpha) := (j^*(M,F,W), j^*(K,W), j^*\alpha) \in VMHS(Z\cap S^o)$. That is a mixed Hodge module is generically a variation of mixed Hodge structure.*

Proof. See [27]. □

Theorem 30. *(i) Let $f : X \to S$ a projective morphism with $X, S \in \mathrm{AnSp}(\mathbb{C})$, where projective means that there exist a factorization $f : X \xrightarrow{l} \mathbb{P}^N \times S \xrightarrow{p_S} S$ with l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^s S_i$ an open cover such that there exits closed embeddings $i_I : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{AnSm}(\mathbb{C})$. For $I \subset [1,\ldots,s]$, recall that we denote $S_I := \cap_{i\in I} S_i$ and $X_I := f^{-1}(S_I)$. We have then the following commutative diagram*

$$\begin{tikzcd} X_I \arrow[r, "i_I\circ l_I"] & \mathbb{P}^N\times\tilde{S}_I \arrow[r, "p_{\tilde{S}_I}"] & \tilde{S}_I \\ X_J \arrow[u, "j'_{IJ}"] \arrow[r, "i_J\circ l_J"] & \mathbb{P}^N\times\tilde{S}_J \arrow[u, "p'_{IJ}"] \arrow[r, "p_{\tilde{S}_J}"] & \tilde{S}_J \arrow[u, "p_{IJ}"'] \end{tikzcd}$$

whose right square is cartesian (see section 5).Then, for

$$((M,F,W),(K,W),\alpha) = (((M_I,F,W),u_{IJ}),(K,W),\alpha) \in MHM(X),$$

where $((M_I,F,W),u_{IJ}) \in C_{\mathcal{D}2fil}(X_I/(\mathbb{P}^N\times\tilde{S}_I))$, $(K,W)\in C_{fil}(X)$, we have for all $n\in\mathbb{Z}$,

$$(H^n\int_f^{FDR}((M_I,F,W),u_{IJ}), R^nf_*(K,W), H^nf_*(\alpha)) \in MHM(S)$$

and for all $p\in\mathbb{Z}$, the differentials of $\mathrm{Gr}_F^p\int_f^{FDR}((M_I,F,W),u_{IJ})$ are strict for the the Hodge filtration F.

(ii) *Let $f : X \to S$ a projective morphism with $X, S \in \mathrm{Var}(\mathbb{C})$, where projective means that there exist a factorization $f : X \xrightarrow{l} \mathbb{P}^N \times S \xrightarrow{p_S} S$ with l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^s S_i$ an open cover such that there exits closed embeddings $i_I : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. For $I \subset [1,\ldots,s]$, recall that we denote $S_I := \cap_{i\in I} S_i$ and $X_I := f^{-1}(S_I)$. We have then the following commutative diagram*

$$\begin{tikzcd} X_I \arrow[r, "i_I\circ l_I"] & \mathbb{P}^N\times\tilde{S}_I \arrow[r, "p_{\tilde{S}_I}"] & \tilde{S}_I \\ X_J \arrow[u, "j'_{IJ}"] \arrow[r, "i_J\circ l_J"] & \mathbb{P}^N\times\tilde{S}_J \arrow[u, "p'_{IJ}"] \arrow[r, "p_{\tilde{S}_J}"] & \tilde{S}_J \arrow[u, "p_{IJ}"'] \end{tikzcd}$$

whose right square is cartesian (see section 5). Then, for

$$((M,F,W),(K,W),\alpha) = (((M_I,F,W),u_{IJ}),(K,W),\alpha) \in D(MHM(X)),$$

where $((M_I,F,W),u_{IJ}) \in C_{\mathcal{D}2fil}(X_I/(\mathbb{P}^N\times\tilde{S}_I))$, $(K,W)\in C_{fil}(X^{an})$, we have

$$H^n(\int_f^{FDR}((M_I,F,W),u_{IJ}), Rf_*(K,W), f_*(\alpha)) \in MHM(S)$$

for all $n\in\mathbb{Z}$, and for all $p\in\mathbb{Z}$, the differentials of $\mathrm{Gr}_F^p\int_f^{FDR}((M_I,F,W),u_{IJ})$ are strict for the the Hodge filtration F.

Proof. (i):See [27].
(ii): By (i) $(H^n\int_f((M,F,W)^{an}), R^nf_*(K,W), H^nf_*(\alpha)) \in MHM(S^{an})$ for all $n\in\mathbb{Z}$. On the other hand, $T^{\mathcal{D}}(an,f)(M,F,W) : (\int_f(M,F,W))^{an} \xrightarrow{\sim} \int_f((M,F,W)^{an})$ is an isomorphism since f is proper by theorem GAGA for mixed hodge modules : see [27]. □

Theorem 31. *(i) Let $S\in\mathrm{AnSp}(\mathbb{C})$. Let $Y\in\mathrm{AnSm}(\mathbb{C})$ and $p_S : Y\times S\to S$ the projection. Let $S=\cup_{i=1}^sS_i$ an open cover such that there exits closed embeddings $i_I : S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{AnSm}(\mathbb{C})$. For $I\subset[1,\ldots,s]$, recall that we denote $S_I:=\cap_{i\in I}S_i$. We have then the following commutative diagram*

$$\begin{tikzcd} Y\times\tilde{S}_I \arrow[r, "p_{\tilde{S}_I}"] & \tilde{S}_I \\ Y\times\tilde{S}_J \arrow[u, "p'_{IJ}"] \arrow[r, "p_{\tilde{S}_J}"] & \tilde{S}_J \arrow[u, "p_{IJ}"'] \end{tikzcd}$$

which is cartesian (see section 5). Then, for

$$((M,F,W),(K,W),\alpha) = (((M_I,F,W),u_{IJ}),(K,W),\alpha)\in MHM(S),$$

where $((M_I,F,W),u_{IJ})\in C_{\mathcal{D}2fil}(S_I/(\tilde{S}_I))$, $(K,W)\in C_{fil}(S)$,

- $(p_S^{*mod[-]}(M,F,W), p_S^*(K,W), p_S^*(\alpha)) := ((p_{\tilde{S}_I}^{*mod[-]}(M_I,F,W), p_{\tilde{S}_J}^{*mod[-]}u_{IJ}), p_S^*(K,W), p_S^*(\alpha)) \in MHM(S)$
- $(p_S^{\hat{*}mod[-]}(M,F,W), p_S^*(K,W), p_S^*(\alpha)) := ((p_{\tilde{S}_I}^{\hat{*}mod[-]}(M_I,F,W), p_{\tilde{S}_J}^{*mod[-]}u_{IJ}), p_S^*(K,W), p_S^*(\alpha)) \in MHM(S)$

(ii) Let $S\in\mathrm{Var}(\mathbb{C})$. Let $Y\in\mathrm{SmVar}(\mathbb{C})$ and $p_S : Y\times S\to S$ the projection. Let $S=\cup_{i=1}^sS_i$ an open cover such that there exits closed embeddings $i_I : S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$. For $I\subset[1,\ldots,s]$, recall that we denote $S_I:=\cap_{i\in I}S_i$. We have then the following commutative diagram

$$\begin{tikzcd} Y\times\tilde{S}_I \arrow[r, "p_{\tilde{S}_I}"] & \tilde{S}_I \\ Y\times\tilde{S}_J \arrow[u, "p'_{IJ}"] \arrow[r, "p_{\tilde{S}_J}"] & \tilde{S}_J \arrow[u, "p_{IJ}"'] \end{tikzcd}$$

which is cartesian (see section 5).Then, for

$$((M,F,W),(K,W),\alpha) = (((M_I,F,W),u_{IJ}),(K,W),\alpha)\in D(MHM(S))$$

where $((M_I,F,W),u_{IJ})\in C_{\mathcal{D}2fil}(S_I/(\tilde{S}_I))$, $(K,W)\in C_{fil}(S^{an})$, we have

- $(p_S^{*mod[-]}, p_S^!)((M,F,W),(K,W),\alpha) := ((p_{\tilde{S}_I}^{*mod[-]}(M_I,F,W), p_{\tilde{S}_J}^{*mod[-]}u_{IJ}), p_S^*(K,W), p_S^*(\alpha)) \in D(MHM(S))$
- $(p_S^{\hat{*}mod[-]}, p_S^*)((M,F,W),(K,W),\alpha) := ((p_{\tilde{S}_I}^{\hat{*}mod[-]}(M_I,F,W), p_{\tilde{S}_J}^{*mod[-]}u_{IJ}), p_S^*(K,W), p_S^*(\alpha)) \in D(MHM(S))$.

Proof. (i):See [27].
(ii):Follows immediately from (i) since $(p_{\tilde{S}_I}^{*mod[-]}(M_I,F,W))^{an} = p_{\tilde{S}_I}^{*mod[-]}((M_I,F,W)^{an})$. □

We have, by the results of Saito, the following key definition.

Definition 94. *(i) Let $S \in \mathrm{SmVar}(\mathbb{C})$ or $S \in \mathrm{AnSm}(\mathbb{C})$. Let $D = V(s) \subset S$ a divisor with $s \in \Gamma(S, L)$ and L a line bundle (S being smooth, D is Cartier). Denote by $j : S^o := S\backslash D \hookrightarrow S$ the open complementary embedding. Let $(M, F, W) \in \pi_{S^o}(MHW(S^o))$. Consider the V_S-filtration on $i_{*mod}M$ (see proposition 95). If (M, F, W) is extendable (which is always the case in the algebraic case), then, by proposition 96,*

- *there exist*

$$\begin{aligned} j_*^{Hdg}(M, F, W) : &= (j^*, \phi_{D1}, c, v)^{-1}((M, F, W), \psi_{D1}(M, F, W)(-1)) \\ &= (j_*M, F, W) \in \pi_S(MHM(S)), \end{aligned}$$

with $F^p j_*M = \sum_{k\in\mathbb{N}} \partial_s^k F^{p+k} V_{S,0} j_*M,$

unique such that $j^* j_*^{Hdg}(M, F, W) = (M, F, W)$ *and* $DR(S)(j_*^{Hdg}(M, F, W)) = j_*DR(S^o)(M, W)$,

- *there exist*

$$\begin{aligned} j_!^{Hdg}(M, F, W) : &= (j^*, \phi_{D1}, c, v)^{-1}((M, F, W), \psi_{D1}(M, F, W)) \\ &= \mathbb{D}_S^{Hdg} j_*^{Hdg} \mathbb{D}_S^{Hdg}(M, F, W) \in \pi_S(MHM(S)) \end{aligned}$$

unique such that $j^* j_!^{Hdg}(M, F, W) = (M, F, W)$ *and* $DR(S)(j_!^{Hdg}(M, F, W)) = j_!DR(S^o)(M, W)$.

Moreover for $(M', F, W) \in \pi_S(MHM(S))$, by proposition 96

- *there is a canonical map* $\mathrm{ad}(j^*, j_*^{Hdg})(M', F, W) : (M', F, W) \to j_*^{Hdg} j^*(M', F, W)$ *in* $\pi_S(MHM(S))$,
- *there is a canonical map* $\mathrm{ad}(j_!^{Hdg}, j^*)(M', F, W) : j_!^{Hdg} j^*(M', F, W) \to (M', F, W)$ *in* $\pi_S(MHM(S))$.

(ii) Let $S \in \mathrm{SmVar}(\mathbb{C})$. Let $Z = V(\mathcal{I}) \subset S$ an arbitrary closed subset, $\mathcal{I} \subset O_S$ being an ideal subsheaf. Taking generators $\mathcal{I} = (s_1, \ldots, s_r)$, we get $Z = V(s_1, \ldots, s_r) = \cap_{i=1}^r Z_i \subset S$ with $Z_i = V(s_i) \subset S$, $s_i \in \Gamma(S, \mathcal{L}_i)$ and L_i a line bundle. Note that Z is an arbitrary closed subset, $d_Z \geq d_X - r$ needing not be a complete intersection. Denote by $j : S^o := S\backslash Z \hookrightarrow S$, $j_I : S^{o,I} := \cap_{i\in I}(S\backslash Z_i) = S\backslash(\cup_{i\in I} Z_i) \xrightarrow{j_I^o} S^o \xrightarrow{j} S$ the open complementary embeddings, where $I \subset \{1, \cdots, r\}$. For $(M, F, W) \in \pi_{S^o}(C(MHM(S^o)))$, we define by (i)

- *the (bi)-filtered complex of D_S-modules*

$$j_*^{Hdg}(M, F, W) := \varinjlim_{\{(Z_i)_{i\in[1,\ldots r]}, Z_i\subset S, \cap Z_i = Z\}, Z_i'\subset Z_i} \mathrm{Tot}_{cardI=\bullet}(j_{I*}^{Hdg} j_I^{o*}(M, F, W)) \in \pi_S(C(MHM(S))),$$

where the horizontal differential are given by, if $I \subset J$, $d_{IJ} := \mathrm{ad}(j_{IJ}^*, j_{IJ*}^{Hdg})(j_I^{o*}(M, F, W))$, $j_{IJ} : S^{oJ} \hookrightarrow S^{oI}$ being the open embedding, and $d_{IJ} = 0$ if $I \notin J$,

- *the (bi)-filtered complex of D_S-modules*

$$\begin{aligned} j_!^{Hdg}(M, F, W) : &= \varprojlim_{\{(Z_i)_{i\in[1,\ldots r]}, Z_i\subset S, \cap Z_i = Z\}, Z_i'\subset Z_i} \mathrm{Tot}_{cardI=-\bullet}(j_{I!}^{Hdg} j_I^{o*}(M, F, W)) \\ &= \mathbb{D}_S^{Hdg} j_*^{Hdg} \mathbb{D}_S^{Hdg}(M, F, W) \in \pi_S(C(MHM(S))), \end{aligned}$$

where the horizontal differential are given by, if $I \subset J$, $d_{IJ} := \mathrm{ad}(j_{IJ!}^{Hdg}, j_{IJ}^*)(j_I^{o*}(M, F, W))$, $j_{IJ} : S^{oJ} \hookrightarrow S^{oI}$ being the open embedding, and $d_{IJ} = 0$ if $I \notin J$.

By definition, we have for $(M, F, W) \in \pi_{S^o}(C(MHM(S^o)))$, $j^* j_*^{Hdg}(M, F, W) = (M, F, W)$ and $j^* j_!^{Hdg}(M, F, W) = (M, F, W)$. For $(M', F, W) \in \pi_S(C(MHM(S)))$, there is, by construction,

– *a canonical map* $\mathrm{ad}(j^*, j_*^{Hdg})(M', F, W) : (M', F, W) \to j_*^{Hdg} j^*(M', F, W)$,
– *a canonical map* $\mathrm{ad}(j_!^{Hdg}, j^*)(M', F, W) : j_!^{Hdg} j^*(M', F, W) \to (M', F, W)$.

For $(M, F, W) \in \pi_{S^o}(C(MHM(S^o)))$,

– *we have the canonical map in* $C_{\mathcal{D}(1,0)fil}(S)$

$$T(j_*^{Hdg}, j_*)(M, F, W) := k \circ \mathrm{ad}(j^*, j_*)(j_*^{Hdg}(M, F, W)) : j_*^{Hdg}(M, F, W) \to j_* E(M, F, W),$$

– *we have the canonical map in* $C_{\mathcal{D}(1,0)fil}(S)$

$$T(j_!, j_!^{Hdg})(M, F, W) := \mathbb{D}_S^K L_D(k \circ \mathrm{ad}(j^*, j_*)(-)) :$$
$$j_!(M, F, W) := \mathbb{D}_S^K L_D j_* E(\mathbb{D}_S^K(M, F, W)) \to \mathbb{D}_S^K L_D j_*^{Hdg} \mathbb{D}_S^K(M, F, W) = j_!^{Hdg}(M, F, W)$$

the canonical maps.

Remark 9. *Let* $j : S^o \hookrightarrow S$ *an open embedding, with* $S \in \mathrm{SmVar}(\mathbb{C})$*. Then, for* $((M, F, W), (K, W), \alpha) \in MHM(S^o)$,

- *the map* $T(j_!, j_!^{Hdg})(M, W) : j_!(M, W) \to j_!^{Hdg}(M, W)$ *in* $C_{\mathcal{D}0fil}(S)$ *is a filtered quasi-isomorphism (apply the functor* $DR^{[-]}(S^o)$ *and use theorem 26 and theorem 89).*
- *the map* $T(j_*^{Hdg}, j_*)(M, W) : j_*^{Hdg}(M, W) \to j_* E(M, W)$ *in* $C_{\mathcal{D}0fil}(S)$ *is a filtered quasi-isomorphism (apply the functor* $DR^{[-]}(S^o)$ *and use theorem 26 and theorem 89).*

Hence, for $((M, F, W), (K, W), \alpha) \in MHM(S^o)$,

- *we get, for all* $p, n \in \mathbb{N}$*, monomorphisms*
$$F^p H^n T(j_!, j_!^{Hdg})(M, F, W) : F^p H^n j_!(M, F, W) \hookrightarrow F^p H^n j_!^{Hdg}(M, F, W)$$
in $\mathrm{PSh}_{O_S}(S)$*, but* $F^p H^n j_!(M, F, W) \neq F^p H^n j_!^{Hdg}(M, F, W)$ *(it leads to different F-filtrations), since* $F^p H^n j_!(M, F) \subset H^n j_! M$ *are sub* D_S *module while the F-filtration on* $H^n j_!^{Hdg}(M, F)$ *is given by Kashiwara-Malgrange* V*-filtrations, hence satisfy a non trivial Griffith transversality property, thus* $H^n j_!(M, F)$ *and* $H^n j_!^{Hdg}(M, F)$ *are isomorphic as* D_S*-modules but NOT isomorphic as filtered* D_S*-modules.*
- *we get, for all* $p, n \in \mathbb{N}$*, monomorphisms*
$$T(j_*^{Hdg}, j_*)(M, F, W) : F^p H^n j_*^{Hdg}(M, F, W) \hookrightarrow F^p H^n j_* E(M, F, W)$$
in $\mathrm{PSh}_{O_S}(S)$*, but* $F^p H^n j_{*Hdg}(M, F, W) \neq F^p H^n j_* E(M, F, W)$ *(it leads to different F-filtrations), since* $F^p H^n j_* E(M, F) \subset H^n j_* E(M)$ *are sub* D_S *module while the F-filtration on* $H^n j_*^{Hdg}(M, F)$ *is given by Kashiwara-Malgrange* V*-filtrations, hence satisfy a non trivial Griffith transversality property, thus* $H^n j_* E(M, F)$ *and* $H^n j_*^{Hdg}(M, F)$ *are isomorphic as* D_S*-modules but NOT isomorphic as filtered* D_S*-modules.*

Definition 95. *Let* $f : X \to S$ *a morphism with* $X, S \in \mathrm{SmVar}(\mathbb{C})$*. Consider a compactification* $f : X \xrightarrow{j} \bar{X} \xrightarrow{\bar{f}} S$ *of* f*, in particular* j *is an open embedding and* $\bar{f}$ *is proper.*

(i) For $(M, F, W) \in \pi_X(C(MHM(X)))$*, we define, using definition 94,*
$$\int_f^{Hdg}(M, F, W) := \int_{\bar{f}}^{FDR} j_*^{Hdg}(M, F, W) \in D_{\mathcal{D}(1,0)fil,\infty}(S)$$
It does not depends on the choice of the compactification by the unicity of proposition 96. By theorem 30, for $(M, F, W) \in \pi_X(C(MHM(X)))$*,* $H^i \int_f^{Hdg}(M, F, W) \in \pi_S(C(MHM(S)))$ *for all* $i \in \mathbb{Z}$*. Note that* $H^i \int_f^{Hdg}(M, F, W) = 0$ *for all* $i < 0$ *if* $(M, F, W) \in \pi_X(MHM(X))$*. We then set*

- *for* $(M,F,W) \in \pi_X(MHM(X))$, $f_*^{Hdg}(M,F,W) := H^0 \int_f^{Hdg}(M,F,W) \in \pi_S(MHM(S))$,
- $Rf_*^{Hdg}(M,F,W) := f_*^{Hdg} I(M,F,W) \in \pi_S(D(MHM(S)))$ *where* $k : (M,F,W) \to I(M,F,W)$ *is the image by* π_S *of an injective resolution in* $MHM(S)$.

(ii) For $(M,F,W) \in\in \pi_X(C(MHM(X)))$, *we define, using definition 94,*

$$\int_{f!}^{Hdg}(M,F,W) := \int_{\bar{f}}^{FDR} j_!^{Hdg}(M,F,W) \in D_{\mathcal{D}(1,0)fil,\infty}(S)$$

It does not depends on the choice of the compactification by the unicity of proposition 96. By theorem 30, for $(M,F,W) \in\in \pi_X(C(MHM(X)))$, $H^i \int_{f!}^{Hdg}(M,F,W) \in\in \pi_S(C(MHM(S)))$ *for all* $i \in \mathbb{Z}$. *Note that* $H^i \int_{f!}^{Hdg}(M,F,W) = 0$ *for all* $i < 0$ *if* $(M,F,W) \in \pi_X(MHM(X))$. *We then set*

- *for* $(M,F,W) \in \pi_X(MHM(X))$, $f_!^{Hdg}(M,F,W) := H^0 \int_{f!}^{Hdg}(M,F,W) \in\in \pi_S(MHM(S))$,
- $Rf_!^{Hdg}(M,F,W) := f_!^{Hdg} I(M,F,W) \in \pi_S(D(MHM(S)))$ *where* $k : (M,F,W) \to I(M,F,W)$ *is the image by* π_S *of an injective resolution in* $MHM(S)$.

Proposition 98. *Let* $f_1 : X \to Y$ *and* $f_2 : Y \to S$ *two morphism with* $X, Y, S \in \mathrm{SmVar}(\mathbb{C})$.

(i) Let $(M,F,W) \in \pi_X(C(MHM(X)))$. *Then,*

$$R(f_2 \circ f_1)_*^{Hdg}(M,F) = Rf_{2*}^{Hdg} Rf_{1*}^{Hdg}(M,F) \in \pi_S(D(MHM(S))).$$

(ii) Let $(M,F,W) \in \pi_S(C(MHM(S)))$. *Then,*

$$R(f_2 \circ f_1)_!^{Hdg}(M,F) = Rf_{2!}^{Hdg} Rf_{1!}^{Hdg}(M,F) \in \pi_S(D(MHM(S)))$$

Proof. (i):Follows from the unicity of of the functor j_*^{Hdg} by proposition 96.
(ii):Follows from the unicity of the functor $j_!^{Hdg}$ by proposition 96. □

We make the following key definition

Definition 96. *Let* $S \in \mathrm{SmVar}(\mathbb{C})$. *Let* $Z \subset S$ *a closed subset. Denote by* $j : S \backslash Z \hookrightarrow S$ *the complementary open embedding.*

(i) We define using definition 94, the filtered Hodge support section functor

$$\Gamma_Z^{Hdg} : \pi_S(C(MHM(S)) \to \pi_S(C(MHM(S)),$$
$$(M,F,W) \mapsto \Gamma_Z^{Hdg}(M,F,W) := \mathrm{Cone}(\mathrm{ad}(j^*, j_*^{Hdg})(M,F) : (M,F) \to j_*^{Hdg} j^*(M,F))[-1],$$

together we the canonical map $\gamma_Z^{Hdg}(M,F,W) : \Gamma_Z^{Hdg}(M,F,W) \to (M,F,W)$. *We then have the canonical map in* $C_{\mathcal{D}(2)fil}(S)$

$$T(\Gamma_Z^{Hdg}, \Gamma_Z)(M,F,W) := (I, T(j_*^{Hdg}, j_*)(M,F,W)) : \Gamma_Z^{Hdg}(M,F,W) \to \Gamma_Z E(M,F,W)$$

unique up to homotopy such that $\gamma_Z^{Hdg}(M,F,W) = \gamma_Z(E(M,F,W)) \circ T(\Gamma_Z^{Hdg}, \Gamma_Z)(M,F,W)$.

(i)' Since $j_*^{Hdg} : \pi_{S^o}(C(MHM(S^o)) \to \pi_S(C(MHM(S))$ *is an exact functor,* Γ_Z^{Hdg} *induces the functor*

$$\Gamma_Z^{Hdg} : \pi_S(D(MHM(S)) \to \pi_S(D(MHM(S)),\ (M,F,W) \mapsto \Gamma_Z^{Hdg}(M,F,W)$$

(ii) We define using definition 94, the dual filtered Hodge support section functor

$$\Gamma^{\vee,Hdg}_Z : \pi_S(C(MHM(S)) \to \pi_S(C(MHM(S)),$$
$$(M,F,W) \mapsto \Gamma^{\vee,Hdg}_Z(M,F,W) := \mathrm{Cone}(\mathrm{ad}(j^{Hdg}_!, j^*)(M,F,W) : j^{Hdg}_!, j^*(M,F,W) \to (M,F,W)),$$

together we the canonical map $\gamma^{\vee,Hdg}_Z(M,F,W) : (M,F,W) \to \Gamma^{\vee,Hdg}_Z(M,F)$. We then have the canonical map in $C_{\mathcal{D}(2)fil}(S)$

$$T(\Gamma^{\vee,h}_Z, \Gamma^{\vee,Hdg}_Z)(M,F,W) := (I, T(j_!, j^{Hdg}_!)(M,F,W)) : \Gamma^{\vee,h}_Z(M,F,W) \to \Gamma^{\vee,Hdg}_Z(M,F,W)$$

unique up to homotopy such that $\gamma^{\vee,Hdg}_Z(M,F) = T(\Gamma^{\vee,h}_Z, \Gamma^{\vee,Hdg}_Z)(M,F,W) \circ \gamma^{\vee,h}_Z(M,F,W)$.

(ii)' Since $j^{Hdg}_! : \pi_{S^o}(C(MHM(S^o)) \to \pi_S(C(MHM(S))$ is an exact functor, $\Gamma^{Hdg,\vee}_Z$ induces the functor

$$\Gamma^{\vee,Hdg}_Z : \pi_S(D(MHM(S)) \to \pi_S(D(MHM(S)),\ (M,F,W) \mapsto \Gamma^{\vee,Hdg}_Z(M,F,W)$$

We now give the definition of the filtered Hodge inverse image functor :

Definition 97. *(i) Let $i : Z \hookrightarrow S$ be a closed embedding, with $Z, S \in \mathrm{SmVar}(\mathbb{C})$. Then, for $(M,F,W) \in \pi_S(C(MHM(S))$, we set*

$$i^{*mod}_{Hdg}(M,F,W) := i^*\mathcal{S}^{-1}_Z\Gamma^{Hdg}_Z(M,F,W) \in \pi_Z(D(MHM(Z))$$

and

$$i^{\hat{*}mod}_{Hdg}(M,F,W) := i^*\mathcal{S}^{-1}_Z\Gamma^{\vee,Hdg}_Z(M,F,W) \in \pi_Z(D(MHM(Z))$$

using the fact that $\mathcal{S}_Z : \pi_Z(D(MHM(Z)) \to \pi_S(D(MHM_Z(S))$ is an equivalence of category since $\mathcal{S}_Z : D(MHM_Z(S)) \to D(MHM_Z(S))$ is an equivalence of category by [27].

(ii) Let $f : X \to S$ be a morphism, with $X, S \in \mathrm{SmVar}(\mathbb{C})$. Consider the factorization $f : X \xrightarrow{i} X \times S \xrightarrow{p_S} S$, where i is the graph embedding and $p_S : X \times S \to S$ is the projection.

- *For $(M,F,W) \in \pi_S(C(MHM(S))$ we set*

$$f^{*mod}_{Hdg}(M,F,W) := i^{*mod}_{Hdg}p^{*mod[-]}_S(M,F,W)(d_X)[2d_X] \in \pi_X(D(MHM(X)),$$

- *For $(M,F,W) \in \pi_S(C(MHM(S))$ we set*

$$f^{\hat{*}mod}_{Hdg}(M,F,W) := i^{\hat{*}mod}_{Hdg}p^{*mod[-]}_S(M,F,W)\pi_X(D(MHM(X)),$$

If $j : S^o \hookrightarrow S$ is a closed embedding, we have (see [27]), for $(M,F,W) \in \pi_S(C(MHM(S)))$,

$$j^{*mod}_{Hdg}(M,F,W) = j^{\hat{*}mod}_{Hdg}(M,F,W) = j^*(M,F,W) \in \pi_{S^o}(D(MHM(S^o)))$$

(iii) Let $f : X \to S$ be a morphism, with $X, S \in \mathrm{SmVar}(\mathbb{C})$ or $X, S \in \mathrm{AnSm}(\mathbb{C})$. Consider the factorization $f : X \xrightarrow{i} X \times S \xrightarrow{p_S} S$, where i is the graph embedding and $p_S : X \times S \to S$ is the projection.

- *For $(M,F,W) \in \pi_S(C(MHM(S))$ we set*

$$f^{*mod}_{Hdg}(M,F,W) := \Gamma^{Hdg}_X p^{*mod[-]}_S(M,F,W)(d_X)[2d_X] \in \pi_{X\times S}(C(MHM(X \times S))),$$

We have for $(M,F,W) \in \pi_S(C(MHM(S))$, the canonical map in $C_{\mathcal{D}(1,0)fil}(X \times S)$

$$T(f^{*mod}_{Hdg}, f^{*mod,\Gamma})(M,F,W) : f^{*mod,\Gamma}_{Hdg}(M,F,W) := \Gamma^{Hdg}_X p^{*mod[-]}_S(M,F,W)$$
$$\xrightarrow{T(\Gamma^{Hdg}_X, \Gamma_X)(-)} \Gamma_X E(p^{*mod[-]}_S(M,F,W)) =: f^{*mod[-],\Gamma}(M,F,W)$$

– *For* $(M,F,W)\in\pi_S(C(MHM(S)))$ *we set*

$$f^{\hat{*}mod}_{Hdg}(M,F,W):=\Gamma^{\vee,Hdg}_X p_S^{*mod[-]}(M,F,W)\in\pi_{X\times S}(C(MHM(X\times S))),$$

We have for $(M,F,W)\in\pi_S(C(MHM(S))$*, the canonical map in* $C_{\mathcal{D}(1,0)fil}(X\times S)$

$$T(f^{\hat{*}mod,\Gamma},f^{\hat{*}mod}_{Hdg})(M,F,W):f^{\hat{*}mod,\Gamma}(M,F,W):=\Gamma^{\vee,h}_X p_S^{*mod[-]}(M,F,W)$$
$$\xrightarrow{T(\Gamma^{\vee,h}_X,\Gamma^{\vee,Hdg}_X)(-)}\Gamma^{\vee,Hdg}_X p_S^{*mod[-]}(M,F,W)=:f^{\hat{*}mod}_{Hdg}(M,F,W)$$

Proposition 99. *Let* $f_1:X\to Y$ *and* $f_2:Y\to S$ *two morphism with* $X,Y,S\in\mathrm{SmVar}(\mathbb{C})$*.*

(i) Let $(M,F,W)\in\pi_S(C(MHM(S)))$*. Then,*

$$(f_2\circ f_1)^{*mod}_{Hdg}(M,F)=f^{*mod}_{1Hdg}f^{*mod}_{2Hdg}(M,F)\in\pi_X(D(MHM(X))).$$

(ii) Let $(M,F,W)\in\pi_S(C(MHM(S)))$*. Then,*

$$(f_2\circ f_1)^{\hat{*}mod}_{Hdg}(M,F)=f^{\hat{*}mod}_{1Hdg}f^{\hat{*}mod}_{2Hdg}(M,F)\in\pi_X(D(MHM(X)))$$

Proof. (i):Follows from the unicity of the functor j_*^{Hdg}.
(ii):Follows from the unicity of the functor $j_!^{Hdg}$. □

Definition-Proposition 19. *(i) Let* $g:S'\to S$ *a morphism with* $S',S\in\mathrm{SmVar}(\mathbb{C})$ *and* $i:Z\hookrightarrow S$ *a closed subset. Then, for* $(M,F,W)\in\pi_S(C(MHM(S)))$*, there is a canonical map in* $\pi_S(C(MHM_{S'}(S'\times S)))$

$$T^{Hdg}(g,\gamma)(M,F,W):g^{*mod,\Gamma}_{Hdg}\Gamma^{Hdg}_Z(M,F,W)\to\Gamma^{Hdg}_{Z\times_S S'}g^{*mod,\Gamma}_{Hdg}(M,F,W)$$

unique up to homotopy such that

$$\gamma^{Hdg}_{Z\times_S S'}(g^{*mod,\Gamma}_{Hdg}(M,F,W))\circ T^{Hdg}(g,\gamma)(M,F,W)=g^{*mod,\Gamma}_{Hdg}\gamma^{Hdg}_Z(M,F,W).$$

(i)' Let $g:S'\to S$ *a morphism with* $S',S\in\mathrm{SmVar}(\mathbb{C})$ *and* $i:Z\hookrightarrow S$ *a closed subset. Then, for* $(M,F,W)\in\pi_S(C(MHM(S)))$*, there is a canonical isomorphism in* $\pi_S(C(MHM_{S'}(S'\times S)))$

$$T^{Hdg}(g,\gamma^\vee)(M,F,W):\Gamma^{Hdg}_{Z\times_S S'}g^{\hat{*}mod,\Gamma}_{Hdg}(M,F,W)\xrightarrow{\sim}g^{\hat{*}mod,\Gamma}_{Hdg}\Gamma^{Hdg}_Z(M,F,W)$$

unique up to homotopy such that

$$\gamma^{\vee,Hdg}_{Z\times_S S'}(g^{\hat{*}mod,\Gamma}_{Hdg}(M,F,W))\circ g^{\hat{*}mod,\Gamma}_{Hdg}\gamma^{\vee,Hdg}_Z(M,F,W)=T^{Hdg}(g,\gamma)(M,F,W).$$

(ii) Let $S\in\mathrm{SmVar}(\mathbb{C})$ *and* $i_1:Z_1\hookrightarrow S$*,* $i_2:Z_2\hookrightarrow Z_1$ *be closed embeddings. Then, for* $(M,F,W)\in\pi_S(C(MHM(S)))$*,*

– *there is a canonical map* $T(Z_2/Z_1,\gamma^{Hdg})(M,F,W):\Gamma^{Hdg}_{Z_2}(M,F,W)\to\Gamma^{Hdg}_{Z_1}(M,F,W)$ *in* $\pi_S(C(MHM(S)))$ *unique up to homotopy such that*

$$\gamma^{Hdg}_{Z_1}(G,F)\circ T(Z_2/Z_1,\gamma^{Hdg})(G,F)=\gamma^{Hdg}_{Z_2}(G,F)$$

together with a distinguish triangle in $K(\pi_S(MHM(S)))$

$$\Gamma^{Hdg}_{Z_2}(M,F,W)\xrightarrow{T(Z_2/Z_1,\gamma^{Hdg})(M,F,W)}\Gamma^{Hdg}_{Z_1}(M,F,W)$$
$$\xrightarrow{\mathrm{ad}(j_2^*,j_{2*}^{Hdg})(\Gamma^{Hdg}_{Z_1}(G,F))}\Gamma^{Hdg}_{Z_1/\backslash Z_2}(G,F)\to\Gamma^{Hdg}_{Z_2}(G,F)[1]$$

– *there is a canonical map* $T(Z_2/Z_1,\gamma^{\vee,Hdg})(M,F,W):\Gamma^{\vee,Hdg}_{Z_1}(M,F,W)\to\Gamma^{\vee,Hdg}_{Z_2}(M,F,W)$ *in* $\pi_S(C(MHM(S)))$ *unique up to homotopy such that*

$$\gamma^{\vee,Hdg}_{Z_2}(M,F,W)=T(Z_2/Z_1,\gamma^{\vee,Hdg})(M,F,W)\circ\gamma^{\vee,Hdg}_{Z_1}(M,F,W).$$

together with a distinguish triangle in $K(\pi_S((MHM(S))))$

$$\Gamma^{\vee,Hdg}_{Z_1\backslash Z_2}(M,F,W)\xrightarrow{\mathrm{ad}(j_{2!}^{Hdg},j_2^*)(M,F,W)}\Gamma^{\vee,Hdg}_{Z_1}(M,F,W)$$
$$\xrightarrow{T(Z_2/Z_1,\gamma^{\vee,Hdg})(M,F,W))}\Gamma^{\vee,Hdg}_{Z_2}(M,F,W)\to\Gamma^{\vee,Hdg}_{Z_2\backslash Z_1}(M,F,W)[1]$$

Proof. Follows from the projection case and the closed embedding case using the adjonction maps. □

We have by proposition 99 and proposition 98 the 2 functors on SmVar($\mathbb{C}$) :

- $\pi(D(MHM(-))):\mathrm{SmVar}(\mathbb{C})\to\pi(D(MHM(-)))$, $S\mapsto\pi_S(D(MHM(S)))$, $(f:T\to S)\mapsto Rf_*^{Hdg}$,
- $\pi(D(MHM(-))):\mathrm{SmVar}(\mathbb{C})\to\pi(D(MHM(-)))$, $S\mapsto\pi_S(D(MHM(S)))$, $(f:T\to S)\mapsto Rf_!^{Hdg}$,
- $\pi(D(MHM(-))):\mathrm{SmVar}(\mathbb{C})\to\pi(D(MHM(-)))$, $S\mapsto\pi_S(D(MHM(S)))$, $(f:T\to S)\mapsto f^{*mod}_{Hdg}$,
- $\pi(D(MHM(-))):\mathrm{SmVar}(\mathbb{C})\to\pi(D(MHM(-)))$, $S\mapsto\pi_S(D(MHM(S)))$, $(f:T\to S)\mapsto f^{\hat{*}mod}_{Hdg}$,

The definitions 96 and 97 immediately extends to the non smooth case :

Definition 98. *Let* $S\in\mathrm{Var}(\mathbb{C})$. *Let* $Z\subset S$ *a closed subset. Let* $S=\cup_i S_i$ *an open cover such that there exist closed embeddings* $i_i:S_i\hookrightarrow\tilde{S}_i$ *with* $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$. *Denote* $Z_I:=Z\cap S_I$. *Denote by* $j:S\backslash Z\hookrightarrow S$ *and* $\tilde{j}_I:\tilde{S}_I\backslash Z_I\hookrightarrow\tilde{S}_I$ *the complementary open embeddings.*

(i) *We define using definition 94, the filtered Hodge support section functor*

$$\Gamma^{Hdg}_Z:\pi(C(MHM(S)))\to\pi(C(MHM(S))),\ ((M_I,F,W),u_{IJ})\mapsto\Gamma^{Hdg}_Z((M_I,F,W),u_{IJ}):=$$
$$\mathrm{Cone}(\mathrm{ad}(j^*,j_*^{Hdg})((M_I,F,W),u_{IJ}):((M_I,F,W),u_{IJ})\to(\tilde{j}^{Hdg}_{I*}\tilde{j}_I^*(M_I,F,W)),\tilde{j}_J(u_{IJ}))[-1],$$

together with the canonical map $\gamma^{Hdg}_Z((M_I,F,W),u_{IJ}):\Gamma^{Hdg}_Z((M_I,F,W),u_{IJ})\to((M_I,F,W),u_{IJ})$. *We then have the canonical map in* $C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)$

$$T(\Gamma^{Hdg}_Z,\Gamma_Z)((M_I,F,W),u_{IJ}):=(I,T(j_*^{Hdg},j_*)(M,F,W)):$$
$$\Gamma^{Hdg}_Z((M_I,F,W),u_{IJ})\to(\Gamma_Z E(M_I,F,W),\Gamma(u_{IJ}))$$

unique up to homotopy such that

$$\gamma^{Hdg}_Z((M_I,F,W),u_{IJ})=(\gamma_{Z_I}(E(M_I,F,W)))\circ T(\Gamma^{Hdg}_Z,\Gamma_Z)((M_I,F,W),u_{IJ}).$$

(i)' *Since* $\tilde{j}^{Hdg}_{I*}:\pi_{\tilde{S}_I}(C(MHM(\tilde{S}_I\backslash S_I)))\to\pi_{\tilde{S}_I}(C(MHM(\tilde{S}_I)))$ *are exact functors,* Γ^{Hdg}_Z *induces the functor*

$$\Gamma^{Hdg}_Z:\pi_S(D(MHM(S))\to\pi_S(D(MHM(S)),((M_I,F,W),u_{IJ})\mapsto\Gamma^{Hdg}_Z((M_I,F,W),u_{IJ})$$

(ii) We define using definition 94, the dual filtered Hodge support section functor

$$\Gamma^{\vee,Hdg}_Z : \pi(C(MHM(S))) \to \pi(C(MHM(S))),\ ((M_I,F,W),u_{IJ}) \mapsto \Gamma^{\vee,Hdg}_Z((M_I,F,W),u_{IJ}) := \mathrm{Cone}(\mathrm{ad}(j^{Hdg}_!,j^*)((M_I,F,W),u_{IJ}) : \tilde{j}^{Hdg}_{I!},\tilde{j}^*_I((M_I,F,W),u_{IJ}) \to ((M_I,F,W),u_{IJ})),$$

together we the canonical map $\gamma^{\vee,Hdg}_Z((M_I,F,W),u_{IJ}) : ((M_I,F,W),u_{IJ}) \to \Gamma^{\vee,Hdg}_Z((M_I,F,W),u_{IJ})$. *We then have the canonical map in* $C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$

$$T(\Gamma^{\vee,h}_Z,\Gamma^{\vee,Hdg}_Z)((M_I,F,W),u_{IJ}) := (I,T(j_!,j^{Hdg}_!)((M_I,F,W),u_{IJ})) : (\Gamma^{\vee,h}_Z(M_I,F,W),\Gamma^{\vee,h}_Z(u_{IJ})) \to \Gamma^{\vee,Hdg}_Z((M_I,F,W),u_{IJ})$$

unique up to homotopy such that

$$\gamma^{\vee,Hdg}_Z((M_I,F,W),u_{IJ}) = T(\Gamma^{\vee,h}_Z,\Gamma^{\vee,Hdg}_Z)((M_I,F,W),u_{IJ}) \circ (\gamma^{\vee,h}_{Z_I}(M_I,F,W)).$$

(ii)' Since $\tilde{j}^{Hdg}_{I!} : \pi_{\tilde{S}_I}(C(MHM(\tilde{S}_I\backslash S_I))) \to \pi_{\tilde{S}_I}(C(MHM(\tilde{S}_I)))$ *are exact functors,* $\Gamma^{Hdg,\vee}_Z$ *induces the functor*

$$\Gamma^{\vee,Hdg}_Z : \pi_S(D(MHM(S)) \to \pi_S(D(MHM(S)),\ ((M_I,F,W),u_{IJ}) \mapsto \Gamma^{\vee,Hdg}_Z((M_I,F,W),u_{IJ})$$

Definition 99. *Let* $f : X \to S$ *a morphism with* $X,S \in Var(\mathbb{C})$. *Assume there exist a factorization* $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ *with* $Y \in \mathrm{SmVar}(\mathbb{C})$, l *a closed embedding and* p_S *the projection. Let* $S = \cup_{i\in I}$ *an open cover such that there exist closed embeddings* $i : S_i \hookrightarrow \tilde{S}_i$ *with* $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. *Denote* $X_I := f^{-1}(S_I)$. *We have then* $X = \cup_{i\in I} X_i$ *and the commutative diagrams*

$$\begin{tikzcd} f : X_I \arrow[r, "l_I"] \arrow[dr] & Y \times S_I \arrow[r, "p_{S_I}"] \arrow[d, "i'_I := (I\times i_I)"] & S_I \arrow[d, "i_I"] \\ & Y \times \tilde{S}_I \arrow[r, "p_{\tilde{S}_I} =: \tilde{f}_I"] & \tilde{S}_I \end{tikzcd}$$

(i) For $((M_I,F,W),u_{IJ}) \in \pi_S(C(MHM(S))$ *we set (see definition 98 for* l*)*

$$f^{*mod}_{Hdg}((M_I,F,W),u_{IJ}) := \Gamma^{Hdg}_X(p^{*mod[-]}_{\tilde{S}_I}(M_I,F,W),u_{IJ})(d_Y)[2d_Y] \in \pi_X(C(MHM(X))),$$

We have for $((M_I,F,W),u_{IJ}) \in \pi_S(C(MHM(S))$, *the canonical map in* $C_{\mathcal{D}(1,0)fil}(X/(Y\times\tilde{S}_I))$

$$T(f^{*mod}_{Hdg},f^{*mod,\Gamma})((M_I,F,W),u_{IJ}) : f^{*mod}_{Hdg}((M_I,F,W),u_{IJ}) := \Gamma^{Hdg}_X(p^{*mod[-]}_{\tilde{S}_I}(M_I,F,W),p^{*mod[-]}_{\tilde{S}_I}u_{IJ}) \xrightarrow{T(\Gamma^{Hdg}_X,\Gamma_X)(-)} (\Gamma_X E(p^{*mod[-]}_{\tilde{S}_I}(M_I,F,W)),\tilde{f}^{*mod[-]}_I u_{IJ})) =: f^{*mod[-],\Gamma}(M,F,W)$$

(ii) For $((M_I,F,W),u_{IJ}) \in \pi_S(C(MHM(S)))$ *we set (see definition 98 for* l*)*

$$f^{\hat{*}mod}_{Hdg}(M,F,W) := \Gamma^{\vee,Hdg}_X(p^{*mod[-]}_{\tilde{S}_I}(M_I,F,W),p^{*mod[-]}_{\tilde{S}_I}u_{IJ}) \in \pi_X(C(MHM(X)),$$

We have for $(M,F,W) \in \pi_S(C(MHM(S))$, *the canonical map in* $C_{\mathcal{D}(1,0)fil}(X/(Y\times\tilde{S}_I))$

$$T(f^{\hat{*}mod,\Gamma},f^{\hat{*}mod[-]}_{Hdg})((M_I,F,W),u_{IJ}) : f^{\hat{*}mod[-],\Gamma}(M,F,W) := \mathbb{D}^K_S f^{*mod[-],\Gamma}\mathbb{D}^K_X((M_I,F,W),u_{IJ}) \xrightarrow{\mathbb{D}^K_S T(\Gamma^{Hdg}_X,\Gamma_X)(-)} \Gamma^{\vee,Hdg}_X(p^{*mod[-]}_S(M_I,F,W),p^{*mod[-]}_{\tilde{S}_I}u_{IJ}) =: f^{\hat{*}mod[-]}_{Hdg}(M,F,W)$$

From the D-module case on algebraic varieties and the constructible sheaves case on CW complexes, we get :

Definition 100. *Let $f : X \to S$ a morphism with $X, S \in \mathrm{QPVar}(\mathbb{C})$. Then, since X is quasi-projective, there exist a factorization $f : X \xrightarrow{l} \mathbb{P}^{N,o} \times S \xrightarrow{p_S} S$ with $n_0 : \mathbb{P}^{N,o} \hookrightarrow \mathbb{P}^N$ an open subset, l a closed embedding and p_S the projection. Since S is quasi-projective, there exist a closed embedding $i : S \hookrightarrow \tilde{S}$ with $\tilde{S} \in \mathrm{SmVar}(\mathbb{C})$. We have then the commutative diagram*

$$\begin{tikzcd}
f : X \arrow[r, "l"] \arrow[dr] & \mathbb{P}^{N,o} \times S \arrow[r, "p_S"] \arrow[d, "i' := (I \times i)"] & S \arrow[d, "i"] \\
& \mathbb{P}^{N,o} \times \tilde{S} \arrow[r, "p_{\tilde{S}}"] \arrow[d, "n := (n_0 \times I)"] & \tilde{S} \arrow[d, "="] \\
& \mathbb{P}^N \times \tilde{S} \arrow[r, "\bar{p}_{\tilde{S}}"] & \tilde{S}
\end{tikzcd}$$

(i) For $((M, F, W), (K, W), \alpha) \in D(MHM(X))$, where $(M, F, W) \in C_{\mathcal{D}(1,0)fil}(X/\mathbb{P}^{N,o} \times \tilde{S})$ and $(K, W) \in C_{fil}(X^{an})$, we define, using theorem 30(ii) for $\bar{p}_S$ and definition 94 for n,

$$\begin{aligned} f_{*Hdg}((M, F, W), (K, W), \alpha) : &= (Rf_*^{Hdg}(M, F, W), Rf_*(K, W), f_*(\alpha)) \\ : &= (R\bar{p}_{\tilde{S}*}^{Hdg} n_*^{Hdg}(M, F, W), Rf_*(K, W), f_*(\alpha)) \in D(MHM(S)) \end{aligned}$$

with

$$f_*(\alpha) : Rf_*(K, W) \xrightarrow{Rf_*\alpha} Rf_* DR(T)^{[-]}((M, W)^{an})$$
$$\xrightarrow{T_*(f, DR)(M, W)} DR(S)^{[-]}((\int_f (M, W))^{an}) = DR(S)^{[-]}((Rf_*^{Hdg}(M, W))^{an})$$

see definition 89 and remark 9.

(ii) For $((M, F, W), (K, W), \alpha) \in D(MHM(X))$, where $(M, F, W) \in C_{\mathcal{D}(1,0)fil}(X/\mathbb{P}^{N,o} \times \tilde{S})$ and $(K, W) \in C_{fil}(X^{an})$, we define, using theorem 30(ii) for $\bar{p}_S$ and definition 94 for n,

$$\begin{aligned} f_{!Hdg}((M, F, W), (K, W), \alpha) : &= (Rf_!^{Hdg}(M, F, W), Rf_!(K, W), f_!(\alpha)) \\ : &= (R\bar{p}_{\tilde{S}*}^{Hdg} n_!^{Hdg}(M, F, W)), Rf_!(K, W), f_!(\alpha)) \in D(MHM(S)) \end{aligned}$$

with

$$f_!(\alpha) : Rf_!(K, W) \xrightarrow{Rf_!\alpha} Rf_! DR(T)((M, W)^{an})$$
$$\xrightarrow{T_!(f, DR)(M, W)} DR(S)^{[-]}((\int_{f!} (M, W))^{an}) = DR(S)^{[-]}((Rf_!^{Hdg}(M, W))^{an})$$

see definition 89 and remark 9.

(iii) For $((M, F, W), (K, W), \alpha) \in D(MHM(S))$, where $(M, F, W) \in C_{\mathcal{D}(1,0)fil}(S/(\tilde{S}))$, $(K, W) \in C_{fil}(S^{an})$, we define, using definition 99 (see theorem 31(ii) for p_S and definition 96 for $i \circ l$),

$$\begin{aligned} f^{*Hdg}((M, F, W), (K, W), \alpha) : &= (f_{Hdg}^{\hat{*}mod}(M, F, W), f^*(K, W), f^*\alpha) \\ : &= (\Gamma_X^{\vee, Hdg} p_{\tilde{S}}^{\hat{*}mod[-]}(M, F, W), \Gamma_X^{\vee} p_S^*(K, W), f^*(\alpha)) \in D(MHM(X)) \end{aligned}$$

with

$$f^*(\alpha) : f^*(K, W) \xrightarrow{f^*\alpha} f^* DR(S)((M, W)^{an})$$
$$\xrightarrow{T(f, DR)((M, W))} DR(T)^{[-]}((Lf^{\hat{*}mod[-], \Gamma}(M, W))^{an}) = DR(T)^{[-]}((f_{Hdg}^{\hat{*}mod}(M, W))^{an})$$

see definition 89 and remark 9. For $j : S^o \hookrightarrow S$ an open embedding and $((M,F,W),(K,W),\alpha) \in D(MHM(S))$, we have (see [27])

$$j^{*Hdg}((M,F,W),(K,W),\alpha) = (j^*(M,F,W), j^*(K,W), j^*\alpha) \in D(MHM(S^o)).$$

(iv) For $((M,F,W),(K,W),\alpha) \in D(MHM(S))$, where $(M,F,W) \in C_{\mathcal{D}(1,0)fil}(S/(\tilde{S}))$, $(K,W) \in C_{fil}(S^{an})$, we define, using definition 99 (see theorem 31(ii) for p_S and definition 96 for $i \circ l$),

$$\begin{aligned} f^{!Hdg}((M,F,W),(K,W),\alpha) :&= (f^{*mod}_{Hdg}(M,F,W), f^!(K,W), f^!\alpha) \\ :&= (\Gamma^{Hdg}_X p^{*mod[-]}_{\tilde{S}_I}(M,F,W)(d_X)[2d_X], R\Gamma_X p_S^*(K,W), f^!(\alpha)) \in D(MHM(X)) \end{aligned}$$

with

$$f^!(\alpha) : f^!(K,W) \xrightarrow{f^!\alpha} f^! DR(S)((M,W)^{an})$$
$$\xrightarrow{T^!(f,DR)((M,W))^{-1}} DR(T)^{[-]}((Rf^{*mod[-],\Gamma}(M,W))^{an}) = DR(T)^{[-]}((f^{*mod}_{Hdg}(M,W))^{an})$$

see definition 89 and remark 9. For $j : S^o \hookrightarrow S$ an open embedding and $((M,F,W),(K,W),\alpha) \in D(MHM(S))$, we have (see [27])

$$j^{!Hdg}((M,F,W),(K,W),\alpha) = (j^*(M,F,W), j^*(K,W), j^*\alpha) \in D(MHM(S^o)).$$

Using the unicity of proposition 96, we see that these definitions does NOT depends on the choice of the factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ of f. Moreover, using the unicity of proposition 96 and proposition 49, we see that they are 2 functors on the category of quasi-projective complex algebraic varieties $(\mathrm{Var}(\mathbb{C}))^{QP}$.

- By definition, we have

$$\iota_S^{-1}(\int^{FDR}_{\bar{p}_{\tilde{S}}} n^{Hdg}_*(M,F,W), Rf_*(K,W), f_*(\alpha)) = Rf_{*Hdg}((M,F,W),(K,W),\alpha) \in D(MHM(S)).$$

  and for $j : S^o \hookrightarrow S$ an open embedding and $((M,F,W),(K,W),\alpha) \in D(MHM(S^o))$,

$$j_{*Hdg}((M,F,W),(K,W),\alpha) = (j^{Hdg}_*(M,F,W), Rj_*(K,W), j_*\alpha) \in D(MHM(S)).$$

- By definition, we have

$$\iota_S^{-1}(\int^{FDR}_{\bar{p}_{\tilde{S}}} n^{Hdg}_!(M,F,W), Rf_!(K,W), f_!(\alpha)) = Rf_{Hdg!}((M,F,W),(K,W),\alpha) \in D(MHM(S)).$$

  and for $j : S^o \hookrightarrow S$ an open embedding and $((M,F,W),(K,W),\alpha) \in D(MHM(S^o))$,

$$j_{!Hdg}((M,F,W),(K,W),\alpha) = (j^{Hdg}_!(M,F,W), j_!(K,W), j_!\alpha) \in D(MHM(S)).$$

We have then the following

Theorem 32. *Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$, X quasi-projective. Then,*

(i) $(f^{*Hdg}, f_{*Hdg}) : D(MHM(S)) \to D(MHM(X))$ is a pair of adjoint functors,

(ii) $(f^{*Hdg}, f_{*Hdg}) : D(MHM(S)) \to D(MHM(X))$ is a pair of adjoint functors.

Proof. For the projection case see section 4. For the open embedding see definition 94. $\square$

Definition 100 gives by proposition 99 and proposition 98 the following 2 functors :

- We have the following 2 functor on the category of complex algebraic varieties

$$D(MHW(\cdot)) : \mathrm{Var}(\mathbb{C}) \to \mathrm{TriCat},\ S \mapsto D(MHW(S)),$$
$$(f : T \to S) \longmapsto (f^{*Hdg} : ((M,F,W),(K,W),\alpha) \mapsto$$
$$f^{*Hdg}((M,F,W),(K,W),\alpha) := (f_{Hdg}^{\hat{*}mod}(M,F,W), f^*(K,W), f^*(\alpha))).$$

- We have the following 2 functor on the category of complex quasi-projective algebraic varieties

$$D(MHW(\cdot)) : \mathrm{QPVar}(\mathbb{C}) \to \mathrm{TriCat},\ S \mapsto D(MHW(S)),$$
$$(f : T \to S) \longmapsto (f_{*Hdg} : ((M,F,W),(K,W),\alpha) \mapsto$$
$$f_{*Hdg}((M,F,W),(K,W),\alpha) := (Rf_*^{Hdg}(M,F,W), Rf_*(K,W), f_*(\alpha))).$$

- We have the following 2 functor on the category of complex quasi-projective algebraic varieties

$$D(MHW(\cdot)) : \mathrm{QPVar}(\mathbb{C}) \to \mathrm{TriCat},\ S \mapsto D(MHW(S)),$$
$$(f : T \to S) \longmapsto (f_{!Hdg} : ((M,F,W),(K,W),\alpha) \mapsto$$
$$f_{!Hdg}((M,F,W),(K,W),\alpha) := (Rf_!^{Hdg}(M,F,W), Rf_!(K,W), f_!(\alpha))).$$

- We have the following 2 functor on the category of complex algebraic varieties

$$D(MHW(\cdot)) : \mathrm{Var}(\mathbb{C}) \to \mathrm{TriCat},\ S \mapsto D(MHW(S)),$$
$$(f : T \to S) \longmapsto (f^{!Hdg} : ((M,F,W),(K,W),\alpha) \mapsto$$
$$f^{!Hdg}((M,F,W),(K,W),\alpha) := (f_{Hdg}^{*mod}(M,F,W), f^!(K,W), f^!(\alpha))).$$

For a commutative diagram in $\mathrm{Var}(\mathbb{C})$

$$D = \begin{tikzcd} X \arrow[r, "f"] & S \\ X' \arrow[u, "g'"] \arrow[r, "f'"] & T \arrow[u, "g"'] \end{tikzcd}$$

with S, T, X', X quasi-projective, we have, for $((M,F,W),(K,W),\alpha) \in D(MHM(X))$ using theorem 32, the following transformations maps

$$T_1^{Hdg}(D)((M,F,W),(K,W),\alpha) :$$
$$g^{*Hdg} f_{*Hdg}((M,F,W),(K,W),\alpha) \xrightarrow{\mathrm{ad}(f'^{*Hdg}, f_{*Hdg})(-)} f'_{*Hdg} f'^{*Hdg} g^{*Hdg} f_{*Hdg}((M,F,W),(K,W),\alpha)$$
$$\xrightarrow{=} f'_{*Hdg} g'^{*Hdg} f^{*Hdg} f_{*Hdg}((M,F,W),(K,W),\alpha) \xrightarrow{\mathrm{ad}(f^{*Hdg}, f_{*Hdg})(-)} f'_{*Hdg} g'^{*Hdg}((M,F,W),(K,W),\alpha)$$

and

$$T_2^{Hdg}(D)((M,F,W),(K,W),\alpha) :$$
$$f'_{!Hdg} g'^{!Hdg}((M,F,W),(K,W),\alpha) \xrightarrow{\mathrm{ad}(f^{!Hdg}, f_{!Hdg})(-)} f'_{!Hdg} g'^{!Hdg} f^{!Hdg} f_{!Hdg}((M,F,W),(K,W),\alpha)$$
$$\xrightarrow{=} f'_{!Hdg} f'^{!Hdg} g^{!Hdg} f_{!Hdg}((M,F,W),(K,W),\alpha) \xrightarrow{\mathrm{ad}(f'^{*Hdg}, f'_{*Hdg})(-)} g^{!Hdg} f_{!Hdg}((M,F,W),(K,W),\alpha)$$

One consequence of the unicity of proposition 96 is the following :

Proposition 100. *For a commutative diagram in* $\mathrm{Var}(\mathbb{C})$

$$\begin{tikzcd} D = & X \arrow[r, "f"] & S \\ & X_T \arrow[u, "g'"] \arrow[r, "f'"] & T \arrow[u, "g"] \end{tikzcd}$$

which is cartesian, with S, T, X', X *quasi-projective and* f *(hence* f' *proper), and* $((M,F,W),(K,W),\alpha) \in D(MHM(X))$

$$T_1^{Hdg}(f,g) : ((M,F,W),(K,W),\alpha) : \\ g^{*Hdg} f_{*Hdg}((M,F,W),(K,W),\alpha) \xrightarrow{\sim} f'_{*Hdg} g'^{*Hdg}((M,F,W),(K,W),\alpha)$$

is an isomorphism.

Proof. See [27]. □

Proposition 101. *(i) Let* $S \in \mathrm{AnSp}(\mathbb{C})$*. Take an open cover* $S = \cup_{i=1}^{l} S_i$ *such that there exists closed embeddings* $i_i : S_i \hookrightarrow \tilde{S}_i$ *with* $\tilde{S}_i \in \mathrm{AnSm}(\mathbb{C})$*. Then for*

$$(((M_I,F,W),u_{IJ}),(K,W),\alpha),(((N_I,F,W),v_{IJ}),(K',W),\alpha') \in MHM(S)$$

we have

$$(((M_I,F,W),u_{IJ}) \otimes_{O_S} ((N_I,F,W),v_{IJ}),(K,W)\otimes(K',W),\alpha\otimes\alpha') \in MHM(S)$$

(ii) Let $S \in \mathrm{Var}(\mathbb{C})$*. Take an open cover* $S = \cup_{i=1}^{l} S_i$ *such that there exists closed embeddings* $i_i : S_i \hookrightarrow \tilde{S}_i$ *with* $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$*. Then for*

$$(((M_I,F,W),u_{IJ}),(K,W),\alpha),(((N_I,F,W),v_{IJ}),(K',W),\alpha') \in MHM(S)$$

we have

$$(((M_I,F,W),u_{IJ}) \otimes_{O_S} ((N_I,F,W),v_{IJ}),(K,W)\otimes(K',W),\alpha\otimes\alpha') \in MHM(S)$$

Proof. See [27]. □

- Let $S \in \mathrm{AnSp}(\mathbb{C})$. Take an open cover $S = \cup_{i=1}^{l} S_i$ such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{AnSm}(\mathbb{C})$. By proposition 101(i), the functor

$$((-)\otimes_{O_S}^{[-]}(-),(-)\otimes(-)) : (C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I C_{fil}(S))^2 \to C_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \times_I C(S), \\ ((((M_I,F,W),u_{IJ}),(K,W),\alpha),(((N_I,F,W),v_{IJ}),(K',W),\alpha')) \mapsto \\ (((M_I,F,W),u_{IJ}) \otimes_{O_S} ((N_I,F,W),v_{IJ}),(K,W)\otimes(K',W),\alpha\otimes\alpha')$$

restricts to a functor $((-)\otimes_{O_S}(-),(-)\otimes(-)) : C(MHM(S)^2 \to C(MHM(S))$.

- Let $S \in \mathrm{Var}(\mathbb{C})$. Take an open cover $S = \cup_{i=1}^{l} S_i$ such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. By proposition 101(ii), the functor

$$(-)\otimes(-) : (C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I C_{fil}(S^{an}))^2 \to C_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \times_I C(S^{an}), \\ ((((M_I,F,W),u_{IJ}),(K,W),\alpha),(((N_I,F,W),v_{IJ}),(K',W),\alpha')) \mapsto \\ (((M_I,F,W),u_{IJ}) \otimes_{O_S} ((N_I,F,W),v_{IJ}),(K,W)\otimes(K',W),\alpha\otimes\alpha')$$

restricts to a functor $((-)\otimes_{O_S}(-),(-)\otimes(-)) : C(MHM(S)^2 \to C(MHM(S))$.

For $X \in \mathrm{SmVar}(\mathbb{C})$, we have, by definition

$$\mathbb{Z}_X^{Hdg} := a_X^{*Hdg}\mathbb{Z}_{\mathrm{pt}}^{Hdg} := ((O_X, F_b)[d_X], \mathbb{Z}_X, \alpha(X)) \in D(MHM(X)),$$

with $\alpha(X) : \mathbb{C}_X \hookrightarrow (0 \to O_X \to \Omega_X \to \cdots K_X)$. If $X \in \mathrm{SmVar}(\mathbb{C})$,

$$\mathbb{Z}_X^{Hdg} := a_X^{*Hdg}\mathbb{Z}_{\mathrm{pt}}^{Hdg} := ((O_X, F_b)[d_X], \mathbb{Z}_{X^{an}}, \alpha(X^{an})) \in D(MHM(X)).$$

Let $X \in \mathrm{Var}(\mathbb{C})$ non smooth. Take an open cover $X = \cup_{i=1}^l X_i$ such that there exists closed embeddings $i_i : X_i \hookrightarrow \tilde{X}_i$ with $\tilde{X}_i \in \mathrm{SmVar}(\mathbb{C})$. Then, by definition

$$\mathbb{Z}_X^{Hdg} := a_X^*\mathbb{Z}_{\mathrm{pt}}^{Hdg} := ((\Gamma_{X_I}^{\vee,Hdg}(O_{\tilde{X}_I}, F_b)[d_{\tilde{X}_I}], o_{\tilde{S}_J/\tilde{S}_I}), (\mathbb{Z}_{X^{an}}, W), \alpha(X/\tilde{X}_I)) \in D(MHM(X)).$$

with

$$\alpha(X/\tilde{X}_I) : (\Gamma_{X_I}^{\vee}\alpha(\tilde{X}_I)) : T(X/(\tilde{X}_I))(\mathbb{Z}_{X^{an}}) := (i_{I*}\mathbb{Z}_{X_I^{an}}, I) \to DR(X)^{[-]}((\Gamma_{X_I}^{\vee,Hdg}(O_{\tilde{X}_I})[d_{\tilde{X}_I}]), o_{\tilde{S}_J/\tilde{S}_I})$$

We have the following proposition

Proposition 102. *Let $Y \in \mathrm{PSmVar}(\mathbb{C})$ and $i : Z \hookrightarrow S$ a closed embedding with Z smooth. Denote by $j : U := S\backslash Z \hookrightarrow Y$ the complementary open subset.*

(i) We have

$$a_{UHdg!}\mathbb{Z}_U^{Hdg} := a_{UHdg!}((O_U, F_b), \mathbb{Z}_{U^{an}}, \alpha(U)) \xrightarrow{=} (\int_{a_Y}^{FDR} j_!^{Hdg}(O_U, F_b), (Ra_{U!}\mathbb{Z}_{U^{an}}, W), a_{U*}\alpha(U))$$
$$\xrightarrow{\sim} (\int_{a_Y}^{FDR} \mathrm{Cone}(\mathbb{D}_S^K \,\mathrm{ad}(i_{*mod}, i^\sharp)(-) : (O_Y, F_b) \to i_{*mod}(O_Z, F_b)), (Ra_{U!}\mathbb{Z}_{U^{an}}, W), a_{U!}\alpha(U))$$
$$\xrightarrow{\sim} (\mathrm{Cone}(E(\Omega_{D/Y})(D) : \Gamma(Y, E(\Omega_Y^\bullet, F_b)) \to \Gamma(Z, E(\Omega_Z^\bullet, F_b)), W), (Ra_{U!}\mathbb{Z}_{U^{an}}, W), a_{U!}\alpha(U))$$

(ii) We have

$$a_{UHdg*}\mathbb{Z}_U^{Hdg} := a_{UHdg*}((O_U, F), \mathbb{Z}_{U^{an}}, \alpha(U)) \xrightarrow{=} (\int_{a_Y}^{FDR} j_*^{Hdg}(O_U, F, W), (Ra_{U*}\mathbb{Z}_{U^{an}}, W), a_{U!}\alpha(U))$$
$$\xrightarrow{\sim} (\int_{a_Y}^{FDR} \mathrm{Cone}(\mathrm{ad}(i_{*mod}, i^\sharp)(-) : i_{*mod}(O_Z, F_b)[c] \to (O_Y, F_b)), (Ra_{U*}\mathbb{Z}_{U^{an}}, W), a_{U*}\alpha(U))$$
$$\xrightarrow{\sim} (\mathrm{Cone}(i_{Z*} : \Gamma(Z, E(\Omega_Z^\bullet, F_b))(-c)[-2c] \to \Gamma(Y, E(\Omega_Y^\bullet, F_b), (Ra_{U*}\mathbb{Z}_{U^{an}}, W), a_{U*}\alpha(U))$$

Proof. See [27]. □

In the case where $D = \cup D_i \subset Y$ is a normal crossing divisor, proposition 102 gives

$$a_{HdgU*}\mathbb{Z}_U^{Hdg} \xrightarrow{\sim} (\Gamma(Y, E(\Omega_Y^\bullet(\log D), F, W)), (Ra_{U*}\mathbb{Z}_{U^{an}}, W), a_{U*}\alpha(U))$$

and

$$a_{HdgU!}\mathbb{Z}_U^{Hdg} \quad := \quad (\Gamma(Y, E(\Omega_Y^\bullet(\mathrm{nul}\, D), F, W)), (Ra_{U!}\mathbb{Z}_{U^{an}}, W), a_{U!}\alpha(U))$$

- Let $S \in \mathrm{SmVar}(\mathbb{C})$. The category $D_{\mathcal{D}(1,0)fil,rh}(S) \times_I D_{fil}(S^{an})$ is then the category
  - whose set of objects is the set of triples $\{((M, F, W), (K, W), \alpha)\}$

    $(M, F, W) \in D_{\mathcal{D}(1,0)fil,rh}(S)$, $(K, W) \in D_{fil}(S^{an})$, $\alpha : (K, W) \otimes \mathbb{C}_S \to DR(S)^{[-]}((M, W)^{an})$

    where $DR(S)^{[-]}$ is the De Rahm functor and α is an isomorphism in $D_{fil}(S^{an})$,

– and whose set of morphisms are

$$\phi = (\phi_D, \phi_C) : ((M_1, F, W), (K_1, W), \alpha_1) \to ((M_2, F, W), (K_2, W), \alpha_2)$$

where $\phi_D : (M_1, F, W) \to (M_2, F, W)$ and $\phi_C : (K_1, W) \to (K_2, W)$ are morphisms such that $\alpha_2 \circ DR(S)^{[-]}(\phi_D^{an}) = \phi_C \circ \alpha_1$ in $D_{fil}(S^{an})$.

together with the localization functor

$$(D(zar), D(usu)) : C_{\mathcal{D}(1,0)fil,rh}(S) \times_I C_{fil}(S^{an}) \to D_{\mathcal{D}(1,0)fil,rh}(S) \times_I D_{fil}(S^{an})$$

The category $D_{\mathcal{D}(1,0)fil,\infty,rh}(S)^{gm} \times_I D_{fil}(S^{an})$ is then the category

– whose set of objects is the set of triples $\{((M, F, W), (K, W), \alpha)\}$

$(M, F, W) \in D_{\mathcal{D}(1,0)fil,\infty,rh}(S)$, $(K, W) \in D_{fil}(S^{an})$, $\alpha : (K, W) \otimes \mathbb{C}_S \to DR(S)^{[-]}((M, W)^{an})$

where $DR(S)^{[-]}$ is the De Rahm functor and α is an isomorphism in $D_{fil}(S^{an})$,

– and whose set of morphisms are

$$\phi = (\phi_D, \phi_C) : ((M_1, F, W), (K_1, W), \alpha_1) \to ((M_2, F, W), (K_2, W), \alpha_2)$$

where $\phi_D : (M_1, F, W) \to (M_2, F, W)$ and $\phi_C : (K_1, W) \to (K_2, W)$ are morphisms such that $\alpha_2 \circ DR(S)^{[-]}(\phi_D^{an}) = \phi_C \circ \alpha_1$ in $D_{fil}(S^{an})$.

together with the localization functor

$$(D(zar), D(usu)) : C_{\mathcal{D}(1,0)fil,rh}(S) \times_I C_{fil}(S^{an}) \to \\ D_{\mathcal{D}(1,0)fil,rh}(S) \times_I D_{fil}(S^{an}) \to D_{\mathcal{D}(1,0)fil,\infty,rh}(S) \times_I D_{fil}(S^{an})$$

- Let $S \in \mathrm{Var}(\mathbb{C})$. Let $S = \cup_{i \in I} S_i$ an open cover such that there exists closed embedding $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. The category $D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$ is then the category

  – whose set of objects is the set of triples $\{(((M_I, F, W), u_{IJ}), (K, W), \alpha)\}$ with

  $$((M_I, F, W), u_{IJ}) \in D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)), (K, W) \in D_{fil}(S^{an}), \\ \alpha : (K, W) \otimes \mathbb{C}_{S^an} \to DR(S)^{[-]}(((M_I, W), u_{IJ})^{an})$$

  where $DR(S)$ is the De Rahm functor and α is an isomorphism in $D_{fil}(S)$,

  – and whose set of morphisms are

  $$\phi = (\phi_D, \phi_C) : (((M_1, F, W), u_{IJ}), (K_1, W), \alpha_1) \to (((M_2, F, W), u_{IJ}), (K_2, W), \alpha_2)$$

  where $\phi_D : ((M_1, F, W), u_{IJ}) \to ((M_2, F, W), u_{IJ})$ and $\phi_C : (K_1, W) \to (K_2, W)$ are morphisms such that $\alpha_2 \circ DR(S)^{[-]}(\phi_D^{an}) = \phi_C \circ \alpha_1$.

  together with the localization functor

  $$(D(zar), D(usu)) : C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I C_{fil}(S^{an}) \to D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$$

  The category $D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$ is then the category

  – whose set of objects is the set of triples $\{(((M_I, F, W), u_{IJ}), (K, W), \alpha)\}$ with

  $$((M_I, F, W), u_{IJ}) \in D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I)), (K, W) \in D_{fil}(S^{an}), \\ \alpha : (K, W) \otimes \mathbb{C}_{S^an} \to DR(S)^{[-]}(((M_I, W), u_{IJ})^{an})$$

  where $DR(S)$ is the De Rahm functor and α is an isomorphism in $D_{fil}(S)$,

– and whose set of morphisms are

$$\phi=(\phi_D,\phi_C):(((M_1,F,W),u_{IJ}),(K_1,W),\alpha_1)\to(((M_2,F,W),u_{IJ}),(K_2,W),\alpha_2)$$

where $\phi_D:((M_1,F,W),u_{IJ})\to((M_2,F,W),u_{IJ})$ and $\phi_C:(K_1,W)\to(K_2,W)$ are morphisms such that $\alpha_2\circ DR(S)^{[-]}(\phi_D^{an})=\phi_C\circ\alpha_1$.

together with the localization functor

$$(D(zar),D(usu)):C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I C_{fil}(S^{an})\to$$
$$D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I D_{fil}(S^{an})\to D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I))\times_I D_{fil}(S^{an})$$

We now state and prove the following key theorem :

Theorem 33. *(i) Let $S\in\mathrm{Var}(\mathbb{C})$. Let $S=\cup_{i\in I}S_i$ an open cover such that there exists closed embedding $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$. Then the full embedding*

$$\iota_S:MHM(S)\hookrightarrow\mathrm{PSh}^0_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I P_{fil}(S^{an})\hookrightarrow C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I C_{fil}(S^{an})$$

induces a full embedding

$$\iota_S:D(MHM(S))\hookrightarrow D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I D_{fil}(S^{an})$$

whose image consists of $(((M_I,F,W),u_{IJ}),(K,W),\alpha)\in D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I D_{fil}(S^{an})$ such that

$$((H^n(M_I,F,W),H^n(u_{IJ})),H^n(K,W),H^n\alpha)\in MHM(S)$$

for all $n\in\mathbb{Z}$ and such that for all $p\in\mathbb{Z}$, the differentials of $\mathrm{Gr}^p_W(M_I,F)$ are strict for the filtrations F.

(ii) Let $S\in\mathrm{Var}(\mathbb{C})$. Let $S=\cup_{i\in I}S_i$ an open cover such that there exists closed embedding $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$. Then the full embedding

$$\iota_S:MHM(S)\hookrightarrow\mathrm{PSh}^0_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I P_{fil}(S^{an})\hookrightarrow C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I C_{fil}(S^{an})$$

induces a full embedding

$$\iota_S:D(MHM(S))\hookrightarrow D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I))\times_I D_{fil}(S^{an})$$

whose image consists of $(((M_I,F,W),u_{IJ}),(K,W),\alpha)\in D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I))\times_I D_{fil}(S^{an})$ such that

$$((H^n(M_I,F,W),H^n(u_{IJ})),H^n(K,W),H^n\alpha)\in MHM(S)$$

for all $n\in\mathbb{Z}$ and such that there exist $r\in\mathbb{Z}$ and an r-filtered homotopy equivalence $((M_I,F,W),u_{IJ})\to((M'_I,F,W),u_{IJ})$ such that for all $p\in\mathbb{Z}$ the differentials of $\mathrm{Gr}^p_W(M'_I,F)$ are strict for the filtrations F.

Proof. (i): We first show that ι_S is fully faithfull, that is for all $\mathcal{M}=(((M_I,F,W),u_{IJ}),(K,W),\alpha),\mathcal{M}'=(((M'_I,F,W),u_{IJ}),(K',W),\alpha')\in MHM(S)$ and all $n\in\mathbb{Z}$,

$$\iota_S:\mathrm{Ext}^n_{D(MHM(S))}(\mathcal{M},\mathcal{M}'):=\mathrm{Hom}_{D(MHM(S))}(\mathcal{M},\mathcal{M}'[n])$$
$$\to\mathrm{Ext}^n_{\mathcal{D}(S)}(\mathcal{M},\mathcal{M}'):=\mathrm{Hom}_{\mathcal{D}(S):=D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I D_{fil}(S^{an})}(\mathcal{M},\mathcal{M}'[n])$$

For this it is enough to assume S smooth. We then proceed by induction on $max(\dim\mathrm{supp}(M),\dim\mathrm{supp}(M'))$.

- For $\mathrm{supp}(M)=\mathrm{supp}(M')=\{s\}$, it is the theorem for mixed hodge complexes or absolute Hodge complexes, see [9]. If $\mathrm{supp}(M)=\{s\}$ and $\mathrm{supp}(M')=\{s'\}$, then by the localization exact sequence

$$\mathrm{Ext}^n_{D(MHM(S))}(\mathcal{M},\mathcal{M}')=0=\mathrm{Ext}^n_{\mathcal{D}(S)}(\mathcal{M},\mathcal{M}')$$

- Denote $\mathrm{supp}(M)=Z\subset S$ and $\mathrm{supp}(M')=Z'\subset S$. There exist an open subset $S^o\subset S$ such that $Z^o:=Z\cap S^o$ and $Z'^o:=Z'\cap S^o$ are smooth, and $\mathcal{M}_{|Z^o}:=((i^*\mathrm{Gr}_{V_{Z^o},0}M_{|S^o},F,W),i^*j^*(K,W),\alpha^*(i))\in MHM(Z^o)$ and $\mathcal{M}'_{|Z'^o}:=((i'^*\mathrm{Gr}_{V_{Z'^o},0}M'_{|S^o},F,W),i'^*j^*K,\alpha^*(i'))\in MHM(Z'^o)$ are variation of mixed Hodge structure, where $j:S^o\hookrightarrow S$ is the open embedding, and $i:Z^o\hookrightarrow S^o$, $i:Z^o\hookrightarrow S^o$ the closed embeddings. Considering the connected components of Z^o and Z'^o, we way assume that Z^o and Z'^o are connected. Shrinking S^o if necessary, we may assume that either $Z^o=Z'^o$ or $Z^o\cap Z'^o=\emptyset$, We denote $D=S\backslash S^o$. Shrinking S^o if necessary, we may assume that D is a divisor and denote by $l:S\hookrightarrow L_D$ the zero section embedding.
  - If $Z^o=Z'^o$, denote $i:Z^o\hookrightarrow S^o$ the closed embedding. We have then the following commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^n_{D(MHM(S^o))}(\mathcal{M}_{|S^o},\mathcal{M}'_{|S^o}) \arrow[r, "\iota_{S^o}"] \arrow[d, shift left, "(i_{*mod},i_*,\alpha_*(i))"] & \mathrm{Ext}^n_{\mathcal{D}(S^o)}(\mathcal{M}_{|S^o},\mathcal{M}'_{|S^o}) \arrow[d, shift left, "(i^*\mathrm{Gr}_{V_{Z^o},0},i^*,\alpha^*(i))"] \\
\mathrm{Ext}^n_{D(MHM(Z^o))}(\mathcal{M}_{|Z^o},\mathcal{M}'_{|Z^o}) \arrow[r, "\iota_{Z^o}"] \arrow[u, shift left, "(i^*\mathrm{Gr}_{V_{Z^o},0},i^*,\alpha^*(i))"] & \mathrm{Ext}^n_{\mathcal{D}(Z^o)}(\mathcal{M}_{|Z^o},\mathcal{M}'_{|Z^o}) \arrow[u, shift left, "(i_{*mod},i_*,\alpha_*(i))"]
\end{tikzcd}$$

    Now we prove that ι_{Z^o} is an isomorphism similarly to the proof the the generic case of 33. On the other hand the left and right colummn are isomorphisms. Hence ι_{S^o} is an isomorphism by the diagram.
  - If $Z^o\cap Z'^o=\emptyset$, we consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^n_{D(MHM(S^o))}(\mathcal{M}_{|S^o},\mathcal{M}'_{|S^o}) \arrow[r, "\iota_{S^o}"] \arrow[d, shift left, "(i_{*mod},i_*,\alpha_*(i))"] & \mathrm{Ext}^n_{\mathcal{D}(S^o)}(\mathcal{M}_{|S^o},\mathcal{M}'_{|S^o}) \arrow[d, shift left, "(i^*\mathrm{Gr}_{V_{Z^o},0},i^*,\alpha^*(i))"] \\
\mathrm{Ext}^n_{D(MHM(Z^o))}(\mathcal{M}_{|Z^o},0)=0 \arrow[r, "\iota_{Z^o}"] \arrow[u, shift left, "(i^*\mathrm{Gr}_{V_{Z^o},0},i^*,\alpha^*(i))"] & \mathrm{Ext}^n_{\mathcal{D}(Z^o)}(\mathcal{M}_{|Z^o},0)=0 \arrow[u, shift left, "(i_{*mod},i_*,\alpha_*(i))"]
\end{tikzcd}$$

    where the left and right column are isomorphism by strictness of the V_{Z^o} filtration (use a bi-filtered injective resolution with respect to F and V_{Z^o} for the right column).

- We consider now the following commutative diagram in $C(\mathbb{Z})$ where we denote for short $H:=D(MHM(S))$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}^\bullet_H(\Gamma^{\vee,Hdg}_D\mathcal{M},\Gamma^{Hdg}_D\mathcal{M}') \arrow[r, "\mathrm{Hom}(-,\gamma^{Hdg}_D(\mathcal{M}'))"] \arrow[d, "\iota_S"] & \mathrm{Hom}^\bullet_H(\Gamma^{\vee,Hdg}_D\mathcal{M},\mathcal{M}') \arrow[r, "\mathrm{Hom}(-,\mathrm{ad}(j^*,j_{*Hdg})(\mathcal{M}'))"] \arrow[d, "\iota_S"] & \mathrm{Hom}^\bullet_H(\Gamma^{\vee,Hdg}_D\mathcal{M},j_{*Hdg}j^*\mathcal{M}') \arrow[r] \arrow[d, "\iota_S"] & 0 \\
0 \arrow[r] & \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(\Gamma^{\vee,Hdg}_D\mathcal{M},\Gamma^{Hdg}_D\mathcal{M}') \arrow[r, "\mathrm{Hom}(-,\gamma^{Hdg}_D(\mathcal{M}'))"] & \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(\Gamma^{\vee,Hdg}_D\mathcal{M},\mathcal{M}') \arrow[r, "\mathrm{Hom}(-,\mathrm{ad}(j^*,j_{*Hdg})(\mathcal{M}'))"] & \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(\Gamma^{\vee,Hdg}_D\mathcal{M},j_{*Hdg}j^*\mathcal{M}') \arrow[r] & 0
\end{tikzcd}$$

  whose lines are exact sequence. We have on the one hand,

$$\mathrm{Hom}^\bullet_{D(MHM(S))}(\Gamma^{\vee,Hdg}_D\mathcal{M},j_{*Hdg}j^*\mathcal{M}')=0=\mathrm{Hom}^\bullet_{\mathcal{D}(S)}(\Gamma^{\vee,Hdg}_D\mathcal{M},j_{*Hdg}j^*\mathcal{M}')$$

  On the other hand by induction hypothesis

$$\iota_S:\mathrm{Hom}^\bullet_{D(MHM(S))}(\Gamma^{\vee,Hdg}_D\mathcal{M},\Gamma^{Hdg}_D\mathcal{M}')\to\mathrm{Hom}^\bullet_{\mathcal{D}(S)}(\Gamma^{\vee,Hdg}_D\mathcal{M},\Gamma^{Hdg}_D\mathcal{M}')$$

is a quasi-isomorphism. Hence, by the diagram

$$\iota_S : \mathrm{Hom}^\bullet_{D(MHM(S))}(\Gamma^{\vee,Hdg}_D\mathcal{M},\mathcal{M}') \to \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(\Gamma^{\vee,Hdg}_D\mathcal{M},\mathcal{M}')$$

is a quasi-isomorphism.

- We consider now the following commutative diagram in $C(\mathbb{Z})$ where we denote for short $H := D(MHM(S))$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}^\bullet_H(\Gamma^{\vee,Hdg}_D\mathcal{M},\mathcal{M}') \arrow[r, "\mathrm{Hom}(\gamma^{\vee,Hdg}_D(\mathcal{M}),-)"] \arrow[d, "\iota_S"] & \mathrm{Hom}^\bullet_H(\mathcal{M},\mathcal{M}') \arrow[r, "\mathrm{Hom}(\mathrm{ad}(j_{!Hdg},j^*)(\mathcal{M}'),-)"] \arrow[d, "\iota_S"] & \mathrm{Hom}^\bullet_H(j_{!Hdg}j^*\mathcal{M},\mathcal{M}') \arrow[r] \arrow[d, "\iota_S"] & 0 \\
0 \arrow[r] & \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(\Gamma^{\vee,Hdg}_D\mathcal{M},\mathcal{M}') \arrow[r, "\mathrm{Hom}(\gamma^{\vee,Hdg}_D(\mathcal{M}),-)"] & \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(\mathcal{M},\mathcal{M}') \arrow[r, "\mathrm{Hom}(\mathrm{ad}(j_{!Hdg},j^*)(\mathcal{M}),-)"] & \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(j_{!Hdg}j^*\mathcal{M},\mathcal{M}') \arrow[r] & 0
\end{tikzcd}$$

whose lines are exact sequence. On the one hand, the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}^\bullet_{D(MHM(S))}(j_{!Hdg}j^*\mathcal{M},\mathcal{M}') \arrow[r, "j^*"] \arrow[d, "\iota_S"] & \mathrm{Hom}^\bullet_{D(MHM(S^o))}(j^*\mathcal{M},j^*\mathcal{M}') \arrow[d, "\iota_{S^o}"] \\
\mathrm{Hom}^\bullet_{\mathcal{D}(S)}(j_{!Hdg}j^*\mathcal{M},\mathcal{M}') \arrow[r, "j^*"] & \mathrm{Hom}^\bullet_{\mathcal{D}(S^o)}(j^*\mathcal{M},j^*\mathcal{M}')
\end{tikzcd}$$

together with the fact that the horizontal arrows j^* are quasi-isomorphism by the functoriality given the uniqueness of the V_S filtration for the embedding $l : S \hookrightarrow L_D$, (use a bi-filtered injective resolution with respect to F and V_S for the lower arrow) and the fact that ι_{S^o} is a quasi-isomorphism by the first two point, show that

$$\iota_S : \mathrm{Hom}^\bullet_{D(MHM(S))}(j_{!Hdg}j^*\mathcal{M},\mathcal{M}') \to \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(j_{!Hdg}j^*\mathcal{M},\mathcal{M}')$$

is a quasi-isomorphism. On the other hand, by the third point

$$\iota_S : \mathrm{Hom}^\bullet_{D(MHM(S))}(\Gamma^{\vee,Hdg}_D\mathcal{M},\mathcal{M}') \to \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(\Gamma^{\vee,Hdg}_D\mathcal{M},\mathcal{M}')$$

is a quasi-isomorphism. Hence, by the diagram

$$\iota_S : \mathrm{Hom}^\bullet_{D(MHM(S))}(\Gamma^{\vee,Hdg}_D\mathcal{M},\mathcal{M}') \to \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(\Gamma^{\vee,Hdg}_D\mathcal{M},\mathcal{M}')$$

is a quasi-isomorphism.

This shows the fully faithfulness. We now prove the essential surjectivity : let $(((M_I,F,W),u_{IJ}),(K,W),\alpha) \in C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I C_{fil}(S^{an})$ such that the cohomology are mixed hodge modules and such that the differential are strict. We proceed by induction on $card\{n \in \mathbb{Z}\}$, s.t.$H^n(M_I,F,W) \neq 0$ by taking for the cohomological troncation

$$\tau^{\leq n}(((M_I,F,W),u_{IJ}),(K,W),\alpha) := ((\tau^{\leq n}(M_I,F,W),\tau^{\leq n}u_{IJ}),\tau^{\leq n}(K,W),\tau^{\leq n}\alpha)$$

and using the fact that the differential are strict for the filtration F and the fully faithfullness.
(ii):Follows from (i) and the strictness of mixed Hodge modules. $\square$

6 The algebraic and analytic filtered De Rham realizations for Voevodsky relative motives

6.1 The algebraic filtered De Rham realization functor

6.1.1 The algebraic Gauss-Manin filtered De Rham realization functor and its transformation map with pullbacks

Consider, for $S \in \mathrm{Var}(\mathbb{C})$, the following composition of morphism in RCat (see section 2)

$$\tilde{e}(S) : (\mathrm{Var}(\mathbb{C})/S, O_{\mathrm{Var}(\mathbb{C})/S}) \xrightarrow{\rho_S} (\mathrm{Var}(\mathbb{C})^{sm}/S, O_{\mathrm{Var}(\mathbb{C})^{sm}/S}) \xrightarrow{e(S)} (S, O_S)$$

with, for $X/S = (X, h) \in \mathrm{Var}(\mathbb{C})/S$,

- $O_{\mathrm{Var}(\mathbb{C})/S}(X/S) := O_X(X)$,
- $(\tilde{e}(S)^* O_S(X/S) \to O_{\mathrm{Var}(\mathbb{C})/S}(X/S)) := (h^* O_S \to O_X)$.

and $O_{\mathrm{Var}(\mathbb{C})^{sm}/S} := \rho_{S*} O_{\mathrm{Var}(\mathbb{C})/S}$, that is, for $U/S = (U, h) \in \mathrm{Var}(\mathbb{C})^{sm}/S$, $O_{\mathrm{Var}(\mathbb{C})^{sm}/S}(U/S) := O_{\mathrm{Var}(\mathbb{C})/S}(U/S) := O_U(U)$

Definition 101. *(i) For $S \in \mathrm{Var}(\mathbb{C})$, we consider the complexes of presheaves*

$$\Omega^\bullet_{/S} := \mathrm{coker}(\Omega_{O_{\mathrm{Var}(\mathbb{C})/S}/\tilde{e}(S)^* O_S} : \Omega^\bullet_{\tilde{e}(S)^* O_S} \to \Omega^\bullet_{O_{\mathrm{Var}(\mathbb{C})/S}}) \in C_{O_S}(\mathrm{Var}(\mathbb{C})/S)$$

which is by definition given by

- *for X/S a morphism $\Omega^\bullet_{/S}(X/S) = \Omega^\bullet_{X/S}(X)$*
- *for $g : X'/S \to X/S$ a morphism,*

$$\begin{aligned}\Omega^\bullet_{/S}(g) := \Omega_{(X'/X)/(S/S)}(X') : \Omega^\bullet_{X/S}(X) &\to g^* \Omega_{X/S}(X') \to \Omega^\bullet_{X'/S}(X') \\ \omega &\mapsto \Omega_{(X'/X)/(S/S)}(X')(\omega) := g^*(\omega) : (\alpha \in \wedge^k T_{X'}(X') \mapsto \omega(dg(\alpha)))\end{aligned}$$

(ii) For $S \in \mathrm{Var}(\mathbb{C})$, we consider the complexes of presheaves

$$\Omega^\bullet_{/S} := \rho_{S*} \tilde{\Omega}^\bullet_{/S} = \mathrm{coker}(\Omega_{O_{\mathrm{Var}(\mathbb{C})^{sm}/S}/e(S)^* O_S} : \Omega^\bullet_{e(S)^* O_S} \to \Omega^\bullet_{O_{\mathrm{Var}(\mathbb{C})^{sm}/S}}) \in C_{O_S}(\mathrm{Var}(\mathbb{C})^{sm}/S)$$

which is by definition given by

- *for U/S a smooth morphism $\Omega^\bullet_{/S}(U/S) = \Omega^\bullet_{U/S}(U)$*
- *for $g : U'/S \to U/S$ a morphism,*

$$\begin{aligned}\Omega^\bullet_{/S}(g) := \Omega_{(U'/U)/(S/S)}(U') : \Omega^\bullet_{U/S}(U) &\to g^* \Omega_{U/S}(U') \to \Omega^\bullet_{U'/S}(U') \\ \omega &\mapsto \Omega_{(U'/U)/(S/S)}(U')(\omega) := g^*(\omega) : (\alpha \in \wedge^k T_{U'}(U') \mapsto \omega(dg(\alpha)))\end{aligned}$$

Remark 10. *For $S \in \mathrm{Var}(\mathbb{C})$, $\Omega^\bullet_{/S} \in C(\mathrm{Var}(\mathbb{C})/S)$ is by definition a natural extension of $\Omega^\bullet_{/S} \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$. However $\Omega^\bullet_{/S} \in C(\mathrm{Var}(\mathbb{C})/S)$ does NOT satisfy cdh descent.*

For a smooth morphism $h : U \to S$ with $S, U \in \mathrm{SmVar}(\mathbb{C})$, the cohomology presheaves $H^n \Omega^\bullet_{U/S}$ of the relative De Rham complex

$$DR(U/S) := \Omega^\bullet_{U/S} := \mathrm{coker}(h^* \Omega_S \to \Omega_U) \in C_{h^* O_S}(U)$$

for all $n \in \mathbb{Z}$, have a canonical structure of a complex of $h^* D_S$ modules given by the Gauss Manin connexion : for $S^o \subset S$ an open subset, $U^o = h^{-1}(S^o)$, $\gamma \in \Gamma(S^o, T_S)$ a vector field and $\hat{\omega} \in \Omega^p_{U/S}(U^o)^c$ a closed form, the action is given by

$$\gamma \cdot [\hat{\omega}] = [\widehat{\iota(\tilde{\gamma}) \partial \omega}],$$

$\omega \in \Omega^p_U(U^o)$ being a representative of $\hat{\omega}$ and $\tilde{\gamma} \in \Gamma(U^o, T_U)$ a relevement of γ (h is a smooth morphism), so that

$$DR(U/S) := \Omega^\bullet_{U/S} := \mathrm{coker}(h^*\Omega_S \to \Omega_U) \in C_{h^*O_S, h^*\mathcal{D}}(U)$$

with this h^*D_S structure. Hence we get $h_*\Omega^\bullet_{U/S} \in C_{O_S,\mathcal{D}}(S)$ considering this structure. Since h is a smooth morphism, $\Omega^p_{U/S}$ are locally free O_U modules.

The point (ii) of the definition 112 above gives the object in $\mathrm{DA}(S)$ which will, for S smooth, represent the algebraic Gauss-Manin De Rham realisation. It is the class of an explicit complex of presheaves on $\mathrm{Var}(\mathbb{C})^{sm}/S$.

Proposition 103. *Let $S \in \mathrm{Var}(\mathbb{C})$.*

(i) For $U/S = (U, h) \in \mathrm{Var}(\mathbb{C})^{sm}/S$, we have $e(U)_*h^*\Omega^\bullet_{/S} = \Omega^\bullet_{U/S}$.

(ii) The complex of presheaves $\Omega^\bullet_{/S} \in C_{O_S}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is $\mathbb{A}^1$ homotopic, in particular $\mathbb{A}^1$ invariant. Note that however, for $p > 0$, the complexes of presheaves $\Omega^{\bullet \geq p}$ are NOT $\mathbb{A}^1$ local. On the other hand, $(\Omega^\bullet_{/S}, F_b)$ admits transferts (recall that means $\mathrm{Tr}(S)_* \mathrm{Tr}(S)^*\Omega^p_{/S} = \Omega^p_{/S}$).

(iii) If S is smooth, we get $(\Omega^\bullet_{/S}, F_b) \in C_{O_S fil, D_S}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ with the structure given by the Gauss Manin connexion. Note that however the D_S structure on the cohomology groups given by Gauss Main connexion does NOT comes from a structure of D_S module structure on the filtered complex of O_S module. The D_S structure on the cohomology groups satisfy a non trivial Griffitz transversality (in the non projection cases), whereas the filtration on the complex is the trivial one.

Proof. (i): Let $h' : V \to U$ a smooth morphism with $V \in \mathrm{Var}(\mathbb{C})$. We have then

$$h^*\Omega^p_{/S}(V \xrightarrow{h'} U) = \Omega^p_{/S}(V \xrightarrow{h'} U \xrightarrow{h} S).$$

Hence, if $h' : V \hookrightarrow U$ is in particular an open embedding, $h^*\Omega^p_{/S}(V \xrightarrow{h'} U) = \Omega^p_{U/S}(V)$. This proves the equality.
(ii): We prove that $E_{et}(\Omega^\bullet_{/S}, F_b) \in C_{O_S fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is 2-filtered $\mathbb{A}^1_S$ invariant. We follow [20]. Consider the map in $C(\mathrm{Var}(\mathbb{C})^{sm}/S)$

$$\phi := \mathrm{ad}(p_a^*, p_{a*})(-) : \Omega^\bullet_{/S} \to p_{a*}p_a^*\Omega^\bullet_{/S}$$

which is given, for $U/S \in \mathrm{Var}(\mathbb{C})^{sm}/S$ by

$$\mathrm{ad}(p_a^*, p_{a*})(-)(U/S) = \Omega_{(U\times\mathbb{A}^1/U)/(S/S)}(U \times \mathbb{A}^1) : \Omega^\bullet_{U/S}(U) \to \Omega^\bullet_{U\times\mathbb{A}^1/S}(U \times \mathbb{A}^1),\ \omega \mapsto p^*\omega$$

where $p : U \times \mathbb{A}^1 \to U$ is the projection. On the other hand consider the map in $C(\mathrm{Var}(\mathbb{C})^{sm}/S)$

$$\psi := I_0^* : p_{a*}p_a^*\Omega^\bullet_{/S} \to \Omega^\bullet_{/S}$$

given, for $U/S \in \mathrm{Var}(\mathbb{C})^{sm}/S$ by

$$I_0^*(U/S) : \Omega^\bullet_{U\times\mathbb{A}^1/S}(U \times \mathbb{A}^1) \to \Omega^\bullet_{U/S}(U),\ \omega \mapsto i_0^*\omega$$

where $i_0 : U \hookrightarrow: U \times \mathbb{A}^1$ is closed embedding given by $i_0(x) := (x, 0)$. Then,

- we have $\phi \circ \psi = I$
- considering the map in $\mathrm{PSh}(\mathbb{N} \times \mathrm{Var}(\mathbb{C})^{sm}/S)$

$$H : p_{a*}p_a^*\Omega^\bullet_{/S}[1] \to p_{a*}p_a^*\Omega^\bullet_{/S}$$

given for $U/S \in \mathrm{Var}(\mathbb{C})^{sm}/S$ by

$$H(U/S)\Omega^p_{U\times\mathbb{A}^1/S}(U\times\mathbb{A}^1)\to\Omega^{p-1}_{U\times\mathbb{A}^1/S}(U\times\mathbb{A}^1),$$

$$H(U/S)(p^*\omega\wedge q^*(f(s)ds))=(\int_0^t f(s)ds)p^*\omega,\ H(U/S)(p^*\omega\wedge q^*f)=0,$$

note that $g(t)=\int_0^t f(s)ds$ is algebraic since $f\in O_{\mathbb{A}^1}(\mathbb{A}^1)$ is a polynomial, we have $\psi\circ\phi-I=\partial H+H\partial$.

This shows that

$$\mathrm{ad}(p_a^*,p_{a*})(-):\Omega^\bullet_{/S}\to p_{a*}p_a^*\Omega^\bullet_{/S}$$

is an homotopy equivalence whose inverse is I_0^*. Hence,

$$\mathrm{ad}(p_a^*,p_{a*})(-):(\Omega^\bullet_{/S},F_b)\to p_{a*}p_a^*(\Omega^\bullet_{/S},F_b)$$

is a 2-filtered homotopy equivalence whose inverse is

$$I_0^*:p_{a*}p_a^*(\Omega^\bullet_{/S},F_b)\to(\Omega^\bullet_{/S},F_b).$$

(iii):For $h:U\to S$ a smooth morphism with $U,S\in\mathrm{SmVar}(\mathbb{C})$, recall that the $h^*D_S(U)=D_S(h(U))$ structure on $H^p\Omega^\bullet_{/S}(U/S):=H^p\Omega^\bullet_{U/S}(U)$ is given by, for $\hat\omega\in\Omega^p_{U/S}(U)^c$, $\gamma\cdot[\hat\omega]=[\widehat{\iota(\tilde\gamma)\partial\omega}]$, $\omega\in\Omega^p_U(U^o)$ being a representative of $\hat\omega$ and $\tilde\gamma\in\Gamma(U^o,T_U)$ a relevement of γ (h is a smooth morphism). Now, if $g:V/S\to U/S$ is a morphism, where $h':V\to S$ is a smooth morphism with $V\in\mathrm{SmVar}(\mathbb{C})$, we have

$$g^*(\gamma\cdot\hat\omega)=g^*\widehat{(\iota(\tilde\gamma)\partial\omega)}=\widehat{\iota(\tilde\gamma)\partial g^*}\omega=\gamma\cdot(g^*\hat\omega)$$

that is $H^p\Omega^\bullet_{/S}(g):H^p\Omega^\bullet(U/S)\to H^p\Omega^\bullet(V/S)$ is a map of $D_S(h(U))$ modules. $\square$

We have the following canonical transformation map given by the pullback of (relative) differential forms:

Let $g:T\to S$ a morphism with $T,S\in\mathrm{Var}(\mathbb{C})$. Consider the following commutative diagram in RCat :

$$\begin{tikzcd}
D(g,e):\ (\mathrm{Var}(\mathbb{C})^{sm}/T,O_{\mathrm{Var}(\mathbb{C})^{sm}/T}) \arrow[r,"P(g)"] \arrow[d,"e(T)"] & (\mathrm{Var}(\mathbb{C})^{sm}/S,O_{\mathrm{Var}(\mathbb{C})^{sm}/S}) \arrow[d,"e(S)"] \\
(T,O_T) \arrow[r,"P(g)"] & (S,O_S)
\end{tikzcd}$$

It gives (see section 2) the canonical morphism in $C_{g^*O_Sfil}(\mathrm{Var}(\mathbb{C})^{sm}/T)$

$$\Omega_{/(T/S)}:=\Omega_{(O_{\mathrm{Var}(\mathbb{C})^{sm}/T}/g^*O_{\mathrm{Var}(\mathbb{C})^{sm}/S})/(O_T/g^*O_S)}:$$
$$g^*(\Omega^\bullet_{/S},F_b)=\Omega^\bullet_{g^*O_{\mathrm{Var}(\mathbb{C})^{sm}/S}/g^*e(S)^*O_S}\to(\Omega^\bullet_{/T},F_b)=\Omega^\bullet_{O_{\mathrm{Var}(\mathbb{C})^{sm}/T}/e(T)^*O_T}$$

which is by definition given by the pullback on differential forms : for $(V/T)=(V,h)\in\mathrm{Var}(\mathbb{C})^{sm}/T$,

$$\Omega_{/(T/S)}(V/T):g^*(\Omega^\bullet_{/S})(V/T):=\lim_{(h':U\to S\mathrm{sm},g':V\to U,h,g)}\Omega^\bullet_{U/S}(U)\xrightarrow{\Omega_{(V/U)/(T/S)}(V/T)}\Omega^\bullet_{V/T}(V)=:\Omega^\bullet_{/T}(V/T)$$
$$\hat\omega\mapsto\Omega_{(V/U)/(T/S)}(V/T)(\omega):=\hat{g'^*\omega}.$$

If S and T are smooth, $\Omega_{/(T/S)}:g^*(\Omega^\bullet_{/S},F_b)\to(\Omega^\bullet_{/T},F_b)$ is a map in $C_{g^*O_Sfil,g^*D_S}(\mathrm{Var}(\mathbb{C})^{sm}/T)$ It induces the canonical morphisms in $C_{g^*O_Sfil,g^*D_S}(\mathrm{Var}(\mathbb{C})^{sm}/T)$:

$$E\Omega_{/(T/S)}:g^*E_{et}(\Omega^\bullet_{/S},F_b)\xrightarrow{T(g,E_{et})(\Omega^\bullet_{/S},F_b)}E_{et}(g^*(\Omega^\bullet_{/S},F_b))\xrightarrow{E_{et}(\Omega_{/(T/S)})}E_{et}(\Omega^\bullet_{/T},F_b).$$

and

$$E\Omega_{/(T/S)}:g^*E_{zar}(\Omega^\bullet_{/S},F_b)\xrightarrow{T(g,E_{zar})(\Omega^\bullet_{/S},F_b)}E_{zar}(g^*(\Omega^\bullet_{/S},F_b))\xrightarrow{E_{zar}(\Omega_{/(T/S)})}E_{zar}(\Omega^\bullet_{/T},F_b).$$

Definition 102. *(i) Let $g : T \to S$ a morphism with $T, S \in \mathrm{Var}(\mathbb{C})$. We have, for $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$, the canonical transformation in $C_{O_T fil}(T)$:*

$$\begin{aligned}
T^O(g,\Omega_{/\cdot})(F) : g^{*mod} L_O e(S)_* \mathcal{H}om^\bullet(F, E_{et}(\Omega^\bullet_{/S}, F_b)) \\
\xrightarrow{:=} (g^* L_O e(S)_* \mathcal{H}om^\bullet(F, E_{et}(\Omega^\bullet_{/S}, F_b))) \otimes_{g^* O_S} O_T \\
\xrightarrow{T(e,g)(-)\circ T(g,L_O)(-)} L_O(e(T)_* g^* \mathcal{H}om^\bullet(F, E_{et}(\Omega^\bullet_{/S}, F)) \otimes_{g^* O_S} O_T) \\
\xrightarrow{T(g,hom)(F,E_{et}(\Omega^\bullet_{/S}))\otimes I} L_O(e(T)_* \mathcal{H}om^\bullet(g^*F, g^*E_{et}(\Omega^\bullet_{/S}, F_b)) \otimes_{g^* O_S} O_T) \\
\xrightarrow{ev(hom,\otimes)(-,-,-)} L_O e(T)_* \mathcal{H}om^\bullet(g^*F, g^*E_{et}(\Omega^\bullet_{/S}, F_b) \otimes_{g^* e(S)^* O_S} e(T)^* O_T) \\
\xrightarrow{\mathcal{H}om^\bullet(g^*F, E\Omega_{/(T/S)}\otimes I)} L_O e(T)_* \mathcal{H}om^\bullet(g^*F, E_{et}(\Omega^\bullet_{/T}, F_b) \otimes_{g^* e(S)^* O_S} e(T)^* O_T) \\
\xrightarrow{m} L_O e(T)_* \mathcal{H}om^\bullet(g^*F, E_{et}(\Omega^\bullet_{/T}, F_b)
\end{aligned}$$

where $m(\alpha \otimes h) := h.\alpha$ is the multiplication map.

(ii) Let $g : T \to S$ a morphism with $T, S \in \mathrm{Var}(\mathbb{C})$, S smooth. Assume there is a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have, for $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$, the canonical transformation in $C_{O_T fil}(Y \times S)$:

$$\begin{aligned}
T(g,\Omega_{/\cdot})(F) : g^{*mod,\Gamma} e(S)_* \mathcal{H}om^\bullet(F, E_{et}(\Omega^\bullet_{/S}, F_b)) \\
\xrightarrow{:=} \Gamma_T E_{zar}(p_S^{*mod} e(S)_* \mathcal{H}om^\bullet(F, E_{et}(\Omega^\bullet_{/S}, F_b))) \\
\xrightarrow{T^O(p_S,\Omega_{/\cdot})(F)} \Gamma_T E_{zar}(e(T \times S)_* \mathcal{H}om^\bullet(p_S^*F, E_{et}(\Omega^\bullet_{/Y\times S}, F_b))) \\
\xrightarrow{=} e(T \times S)_* \Gamma_T(\mathcal{H}om^\bullet(p_S^*F, E_{et}(\Omega^\bullet_{/Y\times S}, F_b))) \\
\xrightarrow{I(\gamma,\mathrm{hom})(-,-)} e(T \times S)_* \mathcal{H}om^\bullet(\Gamma_T^\vee p_S^*F, E_{et}(\Omega^\bullet_{/Y\times S}, F_b)).
\end{aligned}$$

For $Q \in Proj\,\mathrm{PSh}(\mathrm{Var}(\mathbb{C})^{sm}/S)$,

$$T(g,\Omega_{/\cdot})(Q) : g^{*mod,\Gamma} e(S)_* \mathcal{H}om^\bullet(Q, E_{et}(\Omega^\bullet_{/S}, F_b)) \to e(T \times S)_* \mathcal{H}om^\bullet(\Gamma_T^\vee p_S^*Q, E_{et}(\Omega^\bullet_{/Y\times S}, F_b))$$

is a map in $C_{O_T fil,\mathcal{D}}(Y \times S)$.

The following easy lemma describe these transformation map on representable presheaves :

Lemma 7. *Let $g : T \to S$ a morphism with $T, S \in \mathrm{Var}(\mathbb{C})$ and $h : U \to S$ is a smooth morphism with $U \in \mathrm{Var}(\mathbb{C})$. Consider a commutative diagram whose square are cartesian :*

$$\begin{tikzcd}
g : T \arrow[r, "l"] & S \times Y \arrow[r, "p_S"] & S \\
g' : U_T \arrow[u, "h'"] \arrow[r, "l'"] & U \times Y \arrow[u, "h'':=h\times I"] \arrow[r, "p_U"] & U \arrow[u, "h"]
\end{tikzcd}$$

with l, l' the graph embeddings and p_S, p_U the projections. Then $g^*\mathbb{Z}(U/S) = \mathbb{Z}(U_T/T)$ and *(i) we have the following commutative diagram in $C_{O_T fil}(T)$ (see definition 1 and definition 102(i)) :*

$$\begin{tikzcd}
g^{*mod}L_O e(S)_*\mathcal{H}om^\bullet(\mathbb{Z}(U/S),E_{et}(\Omega^\bullet_{/S},F_b)) \arrow[r, "T(g{,}\Omega_{/.})(\mathbb{Z}(U/S))"] & e(T)_*\mathcal{H}om^\bullet(\mathbb{Z}(U_T/T),E_{et}(\Omega^\bullet_{/T},F_b)) \\
g^{*mod}L_O e(S)_*\mathcal{H}om^\bullet(\mathbb{Z}(U/S),E_{zar}(\Omega^\bullet_{/S},F_b)) \arrow[r, "T(g{,}\Omega_{/.})(\mathbb{Z}(U/S))"] \arrow[u, "k"] \arrow[d, "="'] & e(T)_*\mathcal{H}om^\bullet(\mathbb{Z}(U_T/T),E_{zar}(\Omega^\bullet_{/T},F_b)) \arrow[u, "k"'] \arrow[d, "="] \\
g^{*mod}L_O h_*E_{zar}(\Omega^\bullet_{U/S},F_b) \arrow[r, "T^{mod}_\omega(g{,}h)"] & h'_*E_{zar}(\Omega^\bullet_{U_T/T},F_b)
\end{tikzcd}$$

(ii) if $Y,S\in\mathrm{SmVar}(\mathbb{C})$, we have the following commutative diagram in $C_{O_T fil,\mathcal{D}}(Y\times S)$ (see definition 1 and definition 102(ii)) :

$$\begin{tikzcd}
g^{*mod,\Gamma}e(S)_*\mathcal{H}om^\bullet(\mathbb{Z}(U/S),E_{et}(\Omega^\bullet_{/S},F_b)) \arrow[r, "T(g{,}\Omega_{/.})(\mathbb{Z}(U/S))"] & e(Y\times S)_*\mathcal{H}om^\bullet(\Gamma^\vee_{U_T}\mathbb{Z}(U\times Y/S\times Y),E_{et}(\Omega^\bullet_{/Y\times S} \\
g^{*mod,\Gamma}L_O e(S)_*\mathcal{H}om^\bullet(\mathbb{Z}(U/S),E_{zar}(\Omega^\bullet_{/S},F_b)) \arrow[r, "T(g{,}\Omega_{/.})(\mathbb{Z}(U/S))"] \arrow[u, "k"] \arrow[d, "="'] & e(Y\times S)_*\mathcal{H}om^\bullet(\Gamma^\vee_{U_T}\mathbb{Z}(U\times Y/S\times Y),E_{zar}(\Omega^\bullet_{/Y\times} \arrow[u, "k"'] \arrow[d, "="] \\
g^{*mod,\Gamma}h_*E_{zar}(\Omega^\bullet_{U/S},F_b) \arrow[r, "T^O_\omega(p_S{,}h)(-)"] & h''_*\Gamma_{U_T}E_{zar}(\Omega^\bullet_{U\times Y/S\times Y},F_b)
\end{tikzcd}$$

where $j:T\backslash T\times S\hookrightarrow T\times S$ is the open complementary embedding,

with

$$k:E_{zar}(h^*\Omega^\bullet_{/S},F_b)\to E_{et}(E_{zar}(h^*\Omega^\bullet_{/S},F_b))=E_{et}(h^*\Omega^\bullet_{/S},F_b).$$

which is a filtered Zariski local equivalence.

Proof. The commutative diagram follows from Yoneda lemma and proposition 103(i). On the other hand, $k:E_{zar}(\Omega^\bullet_{/S},F_b)\to E_{et}(\Omega^\bullet_{/S},F_b)$ is a (1-)filtered Zariski local equivalence by theorem 10 and proposition 103(ii) □

In the projection case, we have the following :

Proposition 104. *Let $p:S_{12}\to S_1$ is a smooth morphism with $S_1,S_{12}\in\mathrm{AnSp}(\mathbb{C})$. Then if $Q\in C(\mathrm{Var}(\mathbb{C})^{sm}/S_1)$ is projective,*

$$T(p,\Omega_{/.})(Q):p^{*mod}e(S_1)_*\mathcal{H}om^\bullet(Q,E_{et}(\Omega^\bullet_{/S_1},F_b))\to e(S_{12})_*\mathcal{H}om^\bullet(p^*Q,E_{et}(\Omega^\bullet_{/S_{12}},F_b))$$

is an isomorphism.

Proof. Follows from lemma 7 and base change by smooth morphisms of quasi-coherent sheaves. □

Let $S\in\mathrm{Var}(\mathbb{C})$ and $h:U\to S$ a morphism with $U\in\mathrm{Var}(\mathbb{C})$. We then have the canonical map given by the wedge product

$$w_{U/S}:\Omega^\bullet_{U/S}\otimes_{O_S}\Omega^\bullet_{U/S}\to\Omega^\bullet_{U/S};\alpha\otimes\beta\mapsto\alpha\wedge\beta.$$

Let $S\in\mathrm{Var}(\mathbb{C})$ and $h_1:U_1\to S$, $h_2:U_2\to S$ two morphisms with $U_1,U_2\in\mathrm{Var}(\mathbb{C})$. Denote $h_{12}:U_{12}:=U_1\times_S U_2\to S$ and $p_{112}:U_1\times_S U_2\to U_1$, $p_{212}:U_1\times_S U_2\to U_2$ the projections. We then have the canonical map given by the wedge product

$$w_{(U_1,U_2)/S}:p^*_{112}\Omega^\bullet_{U_1/S}\otimes_{O_S}p^*_{212}\Omega^\bullet_{U_2/S}\to\Omega^\bullet_{U_{12}/S};\alpha\otimes\beta\mapsto p^*_{112}\alpha\wedge p^*_{212}\beta$$

which gives the map

$$\begin{aligned}
&Ew_{(U_1,U_2)/S}:h_{1*}E_{zar}(\Omega^{\bullet}_{U_1/S})\otimes_{O_S}h_{2*}E_{zar}(\Omega^{\bullet}_{U_2/S})\\
&\xrightarrow{\mathrm{ad}(p_{112}^*,p_{112*})(-)\otimes\mathrm{ad}(p_{212}^*,p_{212*})(-)}(h_{1*}p_{112*}p_{112}^*E_{zar}(\Omega^{\bullet}_{U_1/S}))\otimes_{O_S}(h_{2*}p_{212*}p_{212}^*E_{zar}(\Omega^{\bullet}_{U_2/S}))\\
&\xrightarrow{=}h_{12*}(p_{112}^*E_{zar}(\Omega^{\bullet}_{U_1/S})\otimes_{h_{12}^*O_S}p_{212}^*E_{zar}(\Omega^{\bullet}_{U_2/S})\\
&\xrightarrow{T(\otimes,E)(-)\circ(T(p_{112},E)(-)\otimes T(p_{212},E)(-))}h_{12*}E_{zar}(p_{112}^*\Omega^{\bullet}_{U_1/S}\otimes_{O_S}p_{212}^*\Omega^{\bullet}_{U_2/S})
\end{aligned}$$

Let $S\in\mathrm{Var}(\mathbb{C})$. We have the canonical map in $C_{O_Sfil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$

$$w_S:(\Omega^{\bullet}_{/S},F_b)\otimes_{O_S}(\Omega^{\bullet}_{/S},F_b)\to(\Omega^{\bullet}_{/S},F_b)$$

given by for $h:U\to S\in\mathrm{Var}(\mathbb{C})^{sm}/S$

$$w_S(U/S):(\Omega^{\bullet}_{U/S},F_b)\otimes_{h^*O_S}(\Omega^{\bullet}_{U/S},F_b)(U)\xrightarrow{w_{U/S}(U)}(\Omega^{\bullet}_{U/S},F_b)(U)$$

It gives the map

$$Ew_S:E_{et}(\Omega^{\bullet}_{/S},F_b)\otimes_{O_S}E_{et}(\Omega^{\bullet}_{/S},F_b)\xrightarrow{=}E_{et}((\Omega^{\bullet}_{/S},F_b)\otimes_{O_S}(\Omega^{\bullet}_{/S},F_b))\xrightarrow{E_{et}(w_S)}E_{et}(\Omega^{\bullet}_{/S},F_b)$$

If $S\in\mathrm{SmVar}(\mathbb{C})$,

$$w_S:(\Omega^{\bullet}_{/S},F_b)\otimes_{O_S}(\Omega^{\bullet}_{/S},F_b)\to(\Omega^{\bullet}_{/S},F_b)$$

is a map in $C_{O_Sfil,D_S}(\mathrm{Var}(\mathbb{C})^{sm}/S)$.

Definition 103. *Let $S\in\mathrm{Var}(\mathbb{C})$. We have, for $F,G\in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$, the canonical transformation in $C_{O_Sfil}(S)$:*

$$\begin{aligned}
&T(\otimes,\Omega)(F,G):e(S)_*\mathcal{H}om(F,E_{et}(\Omega^{\bullet}_{/S},F_b))\otimes_{O_S}e(S)_*\mathcal{H}om(G,E_{et}(\Omega^{\bullet}_{/S},F_b))\\
&\xrightarrow{=}e(S)_*(\mathcal{H}om(F,E_{et}(\Omega^{\bullet}_{/S},F_b))\otimes_{O_S}\mathcal{H}om(G,E_{et}(\Omega^{\bullet}_{/S},F_b)))\\
&\xrightarrow{e(S)_*T(\mathcal{H}om,\otimes)(-)}e(S)_*\mathcal{H}om(F\otimes G,E_{et}(\Omega^{\bullet}_{/S},F_b)\otimes_{O_S}E_{et}(\Omega^{\bullet}_{/S},F_b))\\
&\xrightarrow{\mathcal{H}om(F\otimes G,Ew_S)}e(S)_*\mathcal{H}om(F\otimes G,E_{et}(\Omega^{\bullet}_{/S},F_b))
\end{aligned}$$

If $S\in\mathrm{SmVar}(\mathbb{C})$, $T(\otimes,\Omega)(F,G)$ is a map in $C_{O_Sfil,\mathcal{D}}(S)$.

Lemma 8. *Let $S\in\mathrm{Var}(\mathbb{C})$ and $h_1:U_1\to S$, $h_2:U_2\to S$ two smooth morphisms with $U_1,U_2\in\mathrm{Var}(\mathbb{C})$. Denote $h_{12}:U_{12}:=U_1\times_S U_2\to S$ and $p_{112}:U_1\times_S U_2\to U_1$, $p_{212}:U_1\times_S U_2\to U_2$ the projections. We then have the following commutative diagram*

$$\begin{tikzcd}
e(S)_*\mathcal{H}om(F,E_{et}(\Omega^{\bullet}_{/S},F_b))\otimes_{O_S}e(S)_*\mathcal{H}om(G,E_{et}(\Omega^{\bullet}_{/S},F_b)) \arrow[r, "T(\otimes,\Omega)(F,G)"] & e(S)_*\mathcal{H}om(F\otimes G,E_{et}(\Omega^{\bullet}_{/S},F)) \\
h_{1*}E_{zar}(\Omega^{\bullet}_{U_1/S},F_b)\otimes_{O_S}h_{2*}E_{zar}(\Omega^{\bullet}_{U_2/S},F_b) \arrow[u, "k"] \arrow[r, "Ew_{(U_1,U_2)/S}"] & h_{12*}E_{zar}(\Omega^{\bullet}_{U_{12}/S},F_b) \arrow[u, "k"']
\end{tikzcd}$$

with

$$k:E_{zar}(\Omega^{\bullet}_{/S},F_b)\to E_{et}(E_{zar}(\Omega^{\bullet}_{/S},F_b))=E_{et}(\Omega^{\bullet}_{/S},F_b).$$

which is a filtered Zariski local equivalence.

Proof. Follows from Yoneda lemma. □

Let $S\in\mathrm{Var}(\mathbb{C})$ and $S=\cup_{i=1}^{l}S_i$ an open cover such that there exist closed embeddings $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$. For $I\subset[1,\cdots l]$, denote by $S_I:=\cap_{i\in I}S_i$ and $j_I:S_I\hookrightarrow S$ the open embedding. We then have closed embeddings $i_I:S_I\hookrightarrow\tilde{S}_I:=\Pi_{i\in I}\tilde{S}_i$. Consider, for $I\subset J$, the following commutative diagram

$$D_{IJ}=\begin{tikzcd} S_I \arrow[r, "i_I"] & \tilde{S}_I \\ S_J \arrow[u, "j_{IJ}"] \arrow[r, "i_J"] & \tilde{S}_J \arrow[u, "p_{IJ}"'] \end{tikzcd}$$

and $j_{IJ}:S_J\hookrightarrow S_I$ is the open embedding so that $j_I\circ j_{IJ}=j_J$. Considering the factorization of the diagram D_{IJ} by the fiber product :

$$D_{IJ}=\begin{tikzcd} \tilde{S}_J=\tilde{S}_I\times\tilde{S}_{J\backslash I} \arrow[rr, "p_{IJ}"] & & \tilde{S}_I \\ & S_I\times\tilde{S}_{J\backslash I} \arrow[ul, "i_I\times I"'] \arrow[dr, "p_{IJ}^0"] & \\ S_J \arrow[uu, "i_J"] \arrow[ur, "l_J"] \arrow[rr, "j_{IJ}"] & & S_I \arrow[uu, "i_I"'] \end{tikzcd}$$

the square of this factorization being cartesian, we have for $F\in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ the canonical map in $C(\mathrm{Var}(\mathbb{C})^{sm}/\tilde{S}_J)$

$$S(D_{IJ})(F):Li_{J*}j_J^*F\xrightarrow{q}i_{J*}j_J^*F=(i_I\times I)*l_{J*}j_J^*F\xrightarrow{(i_I\times I)_*\,\mathrm{ad}(p^o_{IJ\sharp},p^{o*}_{IJ})(-)}$$
$$(i_I\times I)_*p^{o*}_{IJ}p^0_{IJ\sharp}l_{J*}j_J^*F\xrightarrow{T(p_{IJ},i_I)(-)^{-1}}p^*_{IJ}i_{I*}p^0_{IJ\sharp}l_{J*}j_J^*F=p^*_{IJ}i_{I*}j_I^*F$$

which factors through

$$S(D_{IJ})(F):Li_{J*}j_I^*F\xrightarrow{S^q(D_{IJ})(F)}p^*_{IJ}Li_{I*}j_I^*F\xrightarrow{q}p^*_{IJ}i_{I*}j_I^*F$$

Definition 104. *(i) Let $S\in\mathrm{SmVar}(\mathbb{C})$. We have the functor*

$$C(\mathrm{Var}(\mathbb{C})^{sm}/S)^{op}\to C_{Ofil,\mathcal{D}}(S),\quad F\mapsto e(S)_*\mathcal{H}om^\bullet(L(i_{I*}j_I^*F),E_{et}(\Omega^\bullet_{/S},F_b))[-d_S].$$

(ii) Let $S\in\mathrm{Var}(\mathbb{C})$ and $S=\cup_{i=1}^{l}S_i$ an open cover such that there exist closed embeddings $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$. For $I\subset[1,\cdots l]$, denote by $S_I:=\cap_{i\in I}S_i$ and $j_I:S_I\hookrightarrow S$ the open embedding. We then have closed embeddings $i_I:S_I\hookrightarrow\tilde{S}_I:=\Pi_{i\in I}\tilde{S}_i$. We have the functor

$$C(\mathrm{Var}(\mathbb{C})^{sm}/S)^{op}\to C_{Ofil,\mathcal{D}}(S/(\tilde{S}_I)),\quad F\mapsto(e(\tilde{S}_I)_*\mathcal{H}om^\bullet(L(i_{I*}j_I^*F),E_{et}(\Omega^\bullet_{/\tilde{S}_I},F_b))[-d_{\tilde{S}_I}],u^q_{IJ}(F))$$

where

$$u^q_{IJ}(F)[d_{\tilde{S}_J}]:e(\tilde{S}_I)_*\mathcal{H}om^\bullet(L(i_{I*}j_I^*F),E_{et}(\Omega^\bullet_{/\tilde{S}_I},F_b))$$
$$\xrightarrow{\mathrm{ad}(p_{IJ}^{*mod},p_{IJ*})(-)}p_{IJ*}p_{IJ}^{*mod}e(\tilde{S}_I)_*\mathcal{H}om^\bullet(L(i_{I*}j_I^*F),E_{et}(\Omega^\bullet_{/\tilde{S}_I},F_b))$$
$$\xrightarrow{p_{IJ*}T(p_{IJ},\Omega.)(L(i_{I*}j_I^*F))}p_{IJ*}e(\tilde{S}_J)_*\mathcal{H}om^\bullet(p_{IJ}^*L(i_{I*}j_I^*F),E_{et}(\Omega^\bullet_{/\tilde{S}_J},F_b))$$
$$\xrightarrow{p_{IJ*}e(\tilde{S}_J)_*\mathcal{H}om(S^q(D_{IJ})(F),E_{et}(\Omega^{\bullet,\Gamma}_{/\tilde{S}_J},F_b))}p_{IJ*}e(\tilde{S}_J)_*\mathcal{H}om^\bullet(L(i_{J*}j_J^*F),E_{et}(\Omega^\bullet_{/\tilde{S}_J},F_b)).$$

For $I\subset J\subset K$, we have obviously $p_{IJ*}u_{JK}(F)\circ u_{IJ}(F)=u_{IK}(F)$.

We will prove in corollary 4 below that $u_{IJ}(F)$ are ∞-filtered Zariski local equivalence.

We then have the following key proposition

Proposition 105. *Let $S \in \mathrm{Var}(\mathbb{C})$. Let $m : Q_1 \to Q_2$ be an equivalence $(\mathbb{A}^1, et)$ local in $C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ with Q_1, Q_2 complexes of projective presheaves. Then,*

$$e(S)_*\mathcal{H}om(m, E_{et}(\Omega^\bullet_{/S}, F_b)) : e(S)_*\mathcal{H}om^\bullet(Q_2, E_{et}(\Omega^\bullet_{/S}, F_b)) \to e(S)_*\mathcal{H}om^\bullet(Q_1, E_{et}(\Omega^\bullet_{/S}, F_b))$$

is an 2-filtered quasi-isomorphism. It is thus an isomorphism in $D_{O_S fil,\mathcal{D},\infty}(S)$ if S is smooth.

Proof. By definition of an $(\mathbb{A}^1, et)$ local equivalence (see proposition 17), there exist

$$\{X_{1,\alpha}/S, \alpha \in \Lambda_1\}, \ldots, \{X_{s,\alpha}/S, \alpha \in \Lambda_s\} \subset \mathrm{Var}(\mathbb{C})^{(sm)}/S$$

such that we have in $\mathrm{Ho}_{et}(C(\mathrm{Var}(\mathbb{C})^{(sm)}/S))$

$$\begin{aligned}\mathrm{Cone}(m) \xrightarrow{\sim} \mathrm{Cone}(&\oplus_{\alpha\in\Lambda_1} \mathrm{Cone}(\mathbb{Z}(X_{1,\alpha} \times \mathbb{A}^1/S) \to \mathbb{Z}(X_{1,\alpha}/S)) \\ &\to \cdots \to \oplus_{\alpha\in\Lambda_s} \mathrm{Cone}(\mathbb{Z}(X_{s,\alpha} \times \mathbb{A}^1/S) \to \mathbb{Z}(X_{s,\alpha}/S)))\end{aligned}$$

This gives in $D_{fil}(\mathbb{Z}) := \mathrm{Ho}_{fil}(\mathbb{Z})$,

$$\begin{aligned}\mathrm{Cone}(\mathrm{Hom}(m, E_{et}(\Omega^\bullet_{/S}, F_b))) \xrightarrow{\sim} \mathrm{Cone}(&\oplus_{\alpha\in\Lambda_1} \mathrm{Cone}(E_{et}(\Omega^\bullet_{/S}, F_b)(X_{1,\alpha}/S) \to E_{et}(\Omega^\bullet_{/S}, F_b)(X_{1,\alpha} \times \mathbb{A}^1/S)) \\ &\to \cdots \to \oplus_{\alpha\in\Lambda_s} \mathrm{Cone}(E_{et}(\Omega^\bullet_{/S}, F_b)(X_{s,\alpha}/S) \to E_{et}(\Omega^\bullet_{/S}, F_b)(X_{s,\alpha} \times \mathbb{A}^1/S)))\end{aligned}$$

Since $\Omega^\bullet_{/S} \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is $\mathbb{A}^1$ homotopic, for all $1 \leq i \leq s$ and all $\alpha \in \Lambda_i$,

$$\mathrm{Cone}(E_{et}(\Omega^\bullet_{/S})(X_{i,\alpha}/S) \to E_{et}(\Omega^\bullet_{/S})(X_{i,\alpha} \times \mathbb{A}^1/S)) \to 0$$

are homotopy equivalence. Hence $\mathrm{Cone}(\mathrm{Hom}(m, E_{et}(G, F)) \to 0$ is a 2-filtered quasi-isomorphism. $\square$

Definition 105. *(i) We define, using definition 104, by proposition 105, the filtered algebraic Gauss-Manin realization functor defined as*

$$\mathcal{F}_S^{GM} : \mathrm{DA}_c(S)^{op} \to D_{O_S fil,\mathcal{D},\infty}(S), \;\; M \mapsto \mathcal{F}_S^{GM}(M) := e(S)_*\mathcal{H}om^\bullet(L(F), E_{et}(\Omega^\bullet_{/S}, F_b))[-d_S]$$

where $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$,

(ii) Let $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. For $I \subset [1, \cdots l]$, denote by $S_I = \cap_{i\in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \Pi_{i\in I}\tilde{S}_i$. We define, using definition 104 and corollary 4,by proposition 105 the filtered algebraic Gauss-Manin realization functor defined as

$$\begin{aligned}\mathcal{F}_S^{GM} : \mathrm{DA}_c(S)^{op} \to D_{Ofil,\mathcal{D},\infty}(S/(\tilde{S}_I)),\; M \mapsto \\ \mathcal{F}_S^{GM}(M) := ((e(\tilde{S}_I)_*\mathcal{H}om^\bullet(L(i_{I*}j_I^*F), E_{et}(\Omega^\bullet_{/\tilde{S}_I}), F_b))[-d_{\tilde{S}_I}], u^q_{IJ}(F))\end{aligned}$$

where $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$.

Proposition 106. *For $S \in \mathrm{Var}(\mathbb{C})$, the functor $\mathcal{F}_S^{GM}$ is well defined.*

Proof. Let $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. Denote, for $I \subset [1, \cdots, l]$, $S_I = \cap_{i\in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \Pi_{i\in I}\tilde{S}_i$. Let $M \in \mathrm{DA}(S)$. Let $F, F' \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}_1, et)(F) = D(\mathbb{A}_1, et)(F')$. Then there exist by definition a sequence of morphisms in $C(\mathrm{Var}(\mathbb{C})^{sm}/S)$:

$$F = F_1 \xrightarrow{s_1} F_2 \xleftarrow{s_2} F_3 \xrightarrow{s_3} F_4 \to \cdots \xrightarrow{s_l} F' = F_s$$

where, for $1 \leq k \leq s$, and s_k are $(\mathbb{A}^1, et)$ local equivalence. But if $s : F_1 \to F_2$ is an equivalence $(\mathbb{A}^1, et)$ local,

$$L(i_{I*}j_I^*s) : L(i_{I*}j_I^*F_1) \to L(i_{I*}j_I^*F_2)$$

is an equivalence $(\mathbb{A}^1, et)$ local, hence

$$\begin{aligned}\mathcal{H}om(L(i_{I*}j_I^*s), E_{et}(\Omega^{\bullet}_{/\tilde{S}_I}, F_b)) : (e(\tilde{S}_I)_*\mathcal{H}om(L(i_{I*}j_I^*F_2), E_{et}(\Omega^{\bullet}_{/\tilde{S}_I}, F_b)), u_{IJ}(F_2)) \\ \to (e(\tilde{S}_I)_*\mathcal{H}om(L(i_{I*}j_I^*F_1), E_{et}(\Omega^{\bullet}_{/\tilde{S}_I}, F_b)), u_{IJ}(F_1))\end{aligned}$$

is an ∞-filtered quasi-isomorphism by proposition 105. □

Let $f : X \to S$ a morphism with $S, X \in \mathrm{Var}(\mathbb{C})$. Assume that there is a factorization

$$f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$$

of f, with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^{l} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. We have $X = \cup_{i=1}^{l} X_i$ with $X_i := f^{-1}(S_i)$. Denote, for $I \subset [1, \cdots l]$, $S_I = \cap_{i \in I} S_i$ and $X_I = \cap_{i \in I} X_i$. For $I \subset [1, \cdots l]$, denote by $\tilde{S}_I = \Pi_{i \in I}\tilde{S}_i$, We then have, for $I \subset [1, \cdots l]$, closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I$ and the following commutative diagrams which are cartesian

$$\begin{tikzcd} f_I = f_{|X_I} : X_I \arrow[r, "l_I"] \arrow[dr] & Y \times S_I \arrow[r, "p_{S_I}"] \arrow[d, "i'_I"] & S_I \arrow[d, "i_I"] \\ & Y \times \tilde{S}_I \arrow[r, "p_{\tilde{S}_I}"] & \tilde{S}_I \end{tikzcd} \quad , \quad \begin{tikzcd} Y \times \tilde{S}_J \arrow[r, "p_{\tilde{S}_J}"] \arrow[d, "p'_{IJ}"'] & \tilde{S}_J \arrow[d, "p_{IJ}"] \\ Y \times \tilde{S}_I \arrow[r, "p_{\tilde{S}_I}"] & \tilde{S}_I \end{tikzcd}$$

with $l_I : l_{|X_I}$, $i'_I = I \times i_I$, p_{S_I} and $p_{\tilde{S}_I}$ are the projections and $p'_{IJ} = I \times p_{IJ}$, and we recall that we denote by $j_I : \tilde{S}_I \backslash S_I \hookrightarrow \tilde{S}_I$ and $j'_I : Y \times \tilde{S}_I \backslash X_I \hookrightarrow Y \times S_I$ the open complementary embeddings. We then have the commutative diagrams

$$D_{IJ} = \begin{tikzcd} S_J \arrow[r, "i_J"] \arrow[d, "j_{IJ}"] & \tilde{S}_J \arrow[d, "p_{IJ}"] \\ S_I \arrow[r, "i_I"] & \tilde{S}_I \end{tikzcd} \quad , \; D'_{IJ} = \begin{tikzcd} X_J \arrow[r, "i'_J \circ l_J"] \arrow[d, "j'_{IJ}"] & Y \times \tilde{S}_J \arrow[d, "p'_{IJ}"] \\ X_I \arrow[r, "i'_I \circ l_I"] & Y \times \tilde{S}_I \end{tikzcd} .$$

and the factorization of D'_{IJ} by the fiber product:

$$D'_{IJ} = \begin{tikzcd} X_J \arrow[r, "i'_I \circ l_I"] \arrow[d, "j'_{IJ}"] & Y \times \tilde{S}_J \arrow[d, "p'_{IJ}"] \\ X_I \arrow[r, "i'_I \circ l_I"] & Y \times \tilde{S}_I \end{tikzcd} , \quad D'_{IJ} = \begin{tikzcd} X_J \arrow[rr, "i'_I \circ l_I"] \arrow[dr, "\hat{l}_J"] \arrow[dd, "j'_{IJ}"] & & Y \times \tilde{S}_J \arrow[dd, "p'_{IJ}"] \\ & X_I \times_{Y \times \tilde{S}_I} Y \times \tilde{S}_J = X_I \times \tilde{S}_{J \backslash I} \arrow[ur, "\hat{l}_I"] \arrow[dl, "\hat{p}_{IJ}"] & \\ X_I \arrow[rr, "i'_I \circ l_I"] & & Y \times \tilde{S}_I \end{tikzcd} \tag{49}$$

where $j'_{IJ} : X_J \hookrightarrow X_I$ is the open embedding. Consider

$$F(X/S) := p_{S,\sharp}\Gamma_X^{\vee}\mathbb{Z}(Y \times S/Y \times S) \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$$

so that $D(\mathbb{A}^1, et)(F(X/S)) = M(X/S)$. Then, by definition,

$$\mathcal{F}_S^{GM}(M^{BM}(X/S)) := (e(\tilde{S}_I)_*\mathcal{H}om(L(i_{I*}j_I^*F(X/S)), E_{et}(\Omega_{/\tilde{S}_I}, F_b))[-d_{\tilde{S}_I}], u_{IJ}^q(F(X/S)))$$

On the other hand, let

$$Q(X_I/\tilde{S}_I) := p_{\tilde{S}_I,\sharp}\Gamma^{\vee}_{X_I}\mathbb{Z}(Y\times\tilde{S}_I/Y\times\tilde{S}_I) \in C(\mathrm{Var}(\mathbb{C})^{sm}/\tilde{S}_I),$$

see definition 10. We have then for $I\subset[1,l]$ the following map in $C(\mathrm{Var}(\mathbb{C})^{sm}/\tilde{S}_J)$:

$$N_I(X/S) : Q(X_I/\tilde{S}_I) = p_{\tilde{S}_I\sharp}\Gamma^{\vee}_{X_I}\mathbb{Z}(Y\times\tilde{S}_I/Y\times\tilde{S}_I) \xrightarrow{p_{\tilde{S}_I\sharp}\mathrm{ad}(i_I'^*,i'_{I*})(-)} \tag{50}$$

$$p_{\tilde{S}_I\sharp}i'_{I*}i_I'^*\Gamma^{\vee}_{X_I}\mathbb{Z}(Y\times\tilde{S}_I/Y\times\tilde{S}_I)[d_Y] \xrightarrow{p_{\tilde{S}\sharp}(T(i'_I,\gamma^{\vee})(-))^{-1}} p_{\tilde{S}_I\sharp}i'_{I*}\Gamma^{\vee}_{X_I}\mathbb{Z}(Y\times S_I/Y\times S_I)[d_Y] \tag{51}$$

$$\xrightarrow{\hat{T}_{\sharp}(p_{S_I},i_I)(-)} i_{I*}p_{S_I\sharp}\Gamma^{\vee}_{X_I}\mathbb{Z}(Y\times S_I/Y\times S_I)[d_Y] = i_{I*}j_I^*F(X/S) \tag{52}$$

We have then for $I\subset J$ the following commutative diagram in $C(\mathrm{Var}(\mathbb{C})^{sm}/\tilde{S}_J)$:

$$\begin{tikzcd} p_{IJ}^*p_{\tilde{S}_I\sharp}\Gamma^{\vee}_{X_I}\mathbb{Z}(Y\times\tilde{S}_I/Y\times\tilde{S}_I) \arrow[r, "p_{IJ}^*N_I(X/S)"] & p_{IJ}^*(i_{I*}j_I^*F(X/S)) \\ p_{\tilde{S}_J\sharp}\Gamma^{\vee}_{X_J}\mathbb{Z}(Y\times\tilde{S}_J/Y\times\tilde{S}_J) \arrow[r, "N_J(X/S)"] \arrow[u, "H_{IJ}"] & i_{J*}j_J^*F(X/S) \arrow[u, "S(D_{IJ})(F(X/S))"'] \end{tikzcd} \tag{53}$$

with

$$H_{IJ} : p_{\tilde{S}_J\sharp}\Gamma^{\vee}_{X_J}\mathbb{Z}(Y\times\tilde{S}_J/Y\times\tilde{S}_J)$$

$$\xrightarrow{=} p_{\tilde{S}_J\sharp}p_{X\sharp}\Gamma^{\vee}_{X_J}p_{IJ}'^*\mathbb{Z}(Y\times\tilde{S}_I/Y\times\tilde{S}_I) \xrightarrow{\mathrm{Cone}(\mathrm{ad}(p'_{IJ\sharp},p_{IJ}'^*)(-),I)} p_{\tilde{S}_J\sharp}\Gamma^{\vee}_{X_I\times\tilde{S}_{J\setminus I}}p_{IJ}'^*\mathbb{Z}(Y\times\tilde{S}_I/Y\times\tilde{S}_I)$$

$$\xrightarrow{T(p_{IJ},\gamma^{\vee})(-)} p_{\tilde{S}_J\sharp}p_{IJ}^*\Gamma^{\vee}_{X_I}\mathbb{Z}(Y\times\tilde{S}_I/Y\times\tilde{S}_I) \xrightarrow{T_{\sharp}(p_{IJ},p_{\tilde{S}_I})(-)} p_{IJ}^*p_{\tilde{S}_I\sharp}\Gamma^{\vee}_{X_I}\mathbb{Z}(Y\times\tilde{S}_I/Y\times\tilde{S}_I).$$

This say that the maps $N_I(X/S)$ induces a map in $C(\mathrm{Var}(\mathbb{C})^{sm}/(S/\tilde{S}_I))$

$$(N_I(X/S)) : (Q(X_I/\tilde{S}_I),H_{IJ}) \to (i_{I*}j_I^*F(X/S), S(D_{IJ})(F(X/S))).$$

We denote by $v^q_{IJ}(F(X/S))$ the composite

$$v^q_{IJ}(F(X/S))[d_{\tilde{S}_J}] : e(\tilde{S}_I)_*\mathcal{H}om(Q(X_I/\tilde{S}_I),E_{et}(\Omega^{\bullet}_{/\tilde{S}_I},F_b))$$

$$\xrightarrow{\mathrm{ad}(p_{IJ}^{*mod},p_{IJ})(-)} p_{IJ*}p_{IJ}^{*mod}e(\tilde{S}_I)_*\mathcal{H}om(Q(X_I/\tilde{S}_I),E_{et}(\Omega^{\bullet}_{/\tilde{S}_I},F_b))$$

$$\xrightarrow{p_{IJ*}T(p_{IJ},\Omega.)(Q(X_I/\tilde{S}_I))} p_{IJ*}e(\tilde{S}_J)_*\mathcal{H}om(p_{IJ}^*Q(X_I/\tilde{S}_I),E_{et}(\Omega^{\bullet}_{/\tilde{S}_J},F_b))$$

$$\xrightarrow{\mathcal{H}om(H_{IJ},E_{et}(\Omega^{\bullet}_{/\tilde{S}_J},F_b))} p_{IJ*}e(\tilde{S}_J)_*\mathcal{H}om(Q(X_J/\tilde{S}_J),E_{et}(\Omega^{\bullet}_{/\tilde{S}_J},F_b)).$$

On the other hand, we have the following map in $C_{Ofil,\mathcal{D},S_J}(\tilde{S}_J)$

$$w_{IJ}(X/S)[d_{\tilde{S}_J}] : p_{\tilde{S}_I*}\Gamma_{X_I}E_{zar}(\Omega^{\bullet}_{Y\times\tilde{S}_I/\tilde{S}_I},F_b) \xrightarrow{\mathrm{ad}(p_{IJ}^{*mod},p_{IJ})(-)} p_{IJ*}p_{IJ}^{*mod}p_{\tilde{S}_I*}\Gamma_{X_I}E_{zar}(\Omega^{\bullet}_{Y\times\tilde{S}_I/\tilde{S}_I},F_b)$$

$$\xrightarrow{T^O_w(p_{IJ},p_{\tilde{S}_I})^{\gamma}} p_{IJ*}p_{\tilde{S}_J*}\Gamma_{X_I\times\tilde{S}_{J\setminus I}}E_{zar}(\Omega^{\bullet}_{Y\times\tilde{S}_I/\tilde{S}_I},F_b) \xrightarrow{\mathrm{Cone}(I,\mathrm{ad}(p_{IJ}'^*,p_{IJ*})(-))} p_{\tilde{S}_J*}\Gamma_{X_J}E_{zar}(\Omega^{\bullet}_{Y\times\tilde{S}_J/\tilde{S}_J},F_b).$$

Lemma 9. *(i) The map in* $C(\mathrm{Var}(\mathbb{C})^{sm}/(S/\tilde{S}_I))$

$$(N_I(X/S)) : (Q(X_I/\tilde{S}_I),H_{IJ}) \to (L(i_{I*}j_I^*F(X/S)), S^q(D_{IJ})(F(X/S))).$$

is an equivalence $(\mathbb{A}^1,et)$ *local.*

(ii) The maps $(N_I(X/S))$ induces an ∞-filtered quasi-isomorphism in $C_{Ofil,\mathcal{D}}(S/(\tilde{S}_I))$

$$
\begin{aligned}
&(\mathcal{H}om(N_I(X/S), E_{et}(\Omega^\bullet_{/\tilde{S}_I}, F_b))) : \\
&(e(\tilde{S}_I)_* \mathcal{H}om(L(i_{I*}j_I^* F(X/S)), E_{et}(\Omega^\bullet_{/\tilde{S}_I}, F_b))[-d_{\tilde{S}_I}], u^q_{IJ}(F(X/S))) \to \\
&(e(\tilde{S}_I)_* \mathcal{H}om(Q(X_I/\tilde{S}_I), E_{et}(\Omega^\bullet_{/\tilde{S}_I}, F_b))[-d_{\tilde{S}_I}], v^q_{IJ}(F(X/S)))
\end{aligned}
$$

(iii) The maps $(I(\gamma, \hom)(-,-))$ and $(k : E_{zar}(p^*_{\tilde{S}_I}\Omega^\bullet_{/\tilde{S}_I}, F_b) \to E_{et}(p^*_{\tilde{S}_I}\Omega^\bullet_{/\tilde{S}_I}, F_b))$ induce an (1-)filtered Zariski local equivalence in $C_{Ofil,\mathcal{D}}(S/(\tilde{S}_I))$

$$
\begin{aligned}
(k \circ I(\gamma, \hom)(-,-)) : &(p_{\tilde{S}_I*}\Gamma_{X_I} E_{zar}(\Omega^\bullet_{Y\times\tilde{S}_I/\tilde{S}_I}, F_b)[-d_{\tilde{S}_I}], w_{IJ}(X/S)) \\
&\to (e(\tilde{S}_I)_* \mathcal{H}om(Q(X_I/\tilde{S}_I), E_{et}(\Omega^\bullet_{/\tilde{S}_I}, F_b))[-d_{\tilde{S}_I}], v^q_{IJ}(F(X/S)))
\end{aligned}
$$

Proof. (i): Follows from theorem 16.
(ii): These maps induce a morphism in $C_{Ofil,\mathcal{D}}(S/(\tilde{S}_I))$ by construction. The fact that it is an ∞-filtered quasi-isomorphism follows from (i) and proposition 105.
(iii):These maps induce a morphism in $C_{Ofil,\mathcal{D}}(S/(\tilde{S}_I))$ by construction. □

Proposition 107. *Let $f : X \to S$ a morphism with $S, X \in \mathrm{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. Then $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. Denote, for $I \subset [1, \cdots l]$, $S_I = \cap_{i\in I} S_i$ and $X_I = \cap_{i\in I} X_i$. Assume there exist a factorization*

$$f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$$

of f with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We then have, for $I \subset [1, \cdots l]$, the following commutative diagrams which are cartesian

$$
\begin{tikzcd}
f_I = f_{|X_I} : X_I \arrow[r, "l_I"] \arrow[dr] & Y \times S_I \arrow[r, "p_{S_I}"] \arrow[d, "i'_I"] & S_I \arrow[d, "i_I"] \\
& Y \times \tilde{S}_I \arrow[r, "p_{\tilde{S}_I}"] & \tilde{S}_I
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
Y \times \tilde{S}_J \arrow[r, "p_{\tilde{S}_J}"] \arrow[d, "p'_{IJ}"'] & \tilde{S}_J \arrow[d, "p_{IJ}"] \\
Y \times \tilde{S}_I \arrow[r, "p_{\tilde{S}_I}"] & \tilde{S}_I
\end{tikzcd}
$$

Let $F(X/S) := p_{S,\sharp}\Gamma^\vee_X \mathbb{Z}(Y\times S/Y\times S)$. The transformations maps $(N_I(X/S) : Q(X_I/\tilde{S}_I) \to i_{I*}j_I^* F(X/S))$ and $(k \circ I(\gamma, \hom)(-,-))$, for $I \subset [1, \cdots, l]$, induce an isomorphism in $D_{Ofil,\mathcal{D},\infty}(S/(\tilde{S}_I))$

$$
\begin{aligned}
&I^{GM}(X/S) : \\
&\mathcal{F}^{GM}_S(M(X/S)) := (e(\tilde{S}_I)_* \mathcal{H}om(L(i_{I*}j_I^* F(X/S)), E_{et}(\Omega^\bullet_{/\tilde{S}_I}, F_b))[-d_{\tilde{S}_I}], u^q_{IJ}(F(X/S))) \\
&\xrightarrow{(e(\tilde{S}_I)_*\mathcal{H}om(LN_I(X/S), E_{et}(\Omega^\bullet_{/\tilde{S}_I}, F_b)))} (e(\tilde{S}_I)_* \mathcal{H}om(Q(X_I/\tilde{S}_I), E_{et}(\Omega^\bullet_{/\tilde{S}_I}, F_b))[-d_{\tilde{S}_I}], v^q_{IJ}(F(X/S))) \\
&\xrightarrow{(k\circ I(\gamma,\hom)(-,-))^{-1}} (p_{\tilde{S}_I*}\Gamma_{X_I} E_{zar}(\Omega^\bullet_{Y\times\tilde{S}_I/\tilde{S}_I}, F_b)[-d_{\tilde{S}_I}], w_{IJ}(X/S)).
\end{aligned}
$$

Proof. Follows from lemma 9. □

Corollary 4. *Let $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. For $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $D(\mathbb{A}^1, et)(F) \in \mathrm{DA}_c(S)$, $u^q_{IJ}(F)$ are ∞-filtered Zariski local equivalence.*

Proof. Let $f : X \to S$ a morphism with $X \in \mathrm{Var}(\mathbb{C})$ such that there exist a factorization, $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Then, by lemma 9(ii) and (iii), $u^q_{IJ}(F(X/S))$ are Zariski local equivalences since $w_{IJ}(X/S)$ are isomorphisms. □

We now define the functorialities of $\mathcal{F}_S^{GM}$ with respect to S which makes $\mathcal{F}_{GM}^-$ a morphism of 2-functor.

Definition 106. *Let $g : T \to S$ a morphism with $T, S \in \mathrm{SmVar}(\mathbb{C})$. Consider the factorization $g : T \xrightarrow{l} T \times S \xrightarrow{p_S} S$ where l is the graph embedding and p_S the projection. Let $M \in \mathrm{DA}_c(S)$ and $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1_S, et)(F)$. Then, $D(\mathbb{A}^1_T, et)(g^*F) = g^*M$.*

(i) We have then the canonical transformation in $D_{Ofil,\mathcal{D},\infty}(T \times S)$ (see definition 102) :

$$T(g, \mathcal{F}^{GM})(M) : Rg^{*mod[-],\Gamma}\mathcal{F}_S^{GM}(M) := g^{*mod,\Gamma}e(S)_*\mathcal{H}om^\bullet(LF, E_{et}(\Omega^\bullet_{/S}, F_b)))[-d_T] \xrightarrow{T(g,\Omega_{/\cdot})(LF)}$$
$$e(T \times S)_*\mathcal{H}om^\bullet(\Gamma^\vee_T p_S^* LF, E_{et}(\Omega^\bullet_{/T\times S}, F_b))[-d_T] =: \mathcal{F}^{GM}_{T\times S}(l_* g^*(M, W)).$$

(ii) We have then the canonical transformation in $D_{Ofil,\infty}(T)$(see definition 102) :

$$T^O(g, \mathcal{F}^{GM})(M, W) : Lg^{*mod[-]}\mathcal{F}_S^{GM}(M) := g^{*mod}e(S)_*\mathcal{H}om^\bullet(LF, E_{et}(\Omega^\bullet_{/S}, F_b)))[-d_T] \xrightarrow{T^O(g,\Omega_{/\cdot})(LF)}$$
$$e(T)_*\mathcal{H}om^\bullet(g^*LF, E_{et}(\Omega^\bullet_{/T}, F_b))[-d_T] =: \mathcal{F}_T^{GM}(g^*M).$$

We give now the definition in the non smooth case Let $g : T \to S$ a morphism with $T, S \in \mathrm{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$ Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$, Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \to \tilde{S}_I$ is a lift of $g_I := g_{|T_I} : T_I \to S_I$. We recall the commutative diagram :

$$\begin{tikzcd} E_{IJg} = (Y \times \tilde{S}_I)\backslash T_I \arrow[r, "m'_I"] \arrow[d, "p_{\tilde{S}_I}"] & Y \times \tilde{S}_J \arrow[d, "\tilde{g}_I"] \\ \tilde{S}_I\backslash S_I \arrow[r, "m_I"] & \tilde{S}_I \end{tikzcd}, \quad \begin{tikzcd} E_{IJ} = \tilde{S}_J\backslash S_J \arrow[r, "m_J"] \arrow[d, "p_{IJ}"] & \tilde{S}_J \arrow[d, "p_{IJ}"] \\ \tilde{S}_I\backslash(S_I\backslash S_J) \arrow[r, "m=m_{IJ}"] & \tilde{S}_I \end{tikzcd} \quad \begin{tikzcd} E'_{IJ} = (Y \times \tilde{S}_J)\backslash T_J \arrow[r, "m'_J"] \arrow[d, "p'_{IJ}"] & Y \times \tilde{S}_J \arrow[d, "p'_{IJ}"] \\ (Y \times \tilde{S}_I)\backslash(T_I\backslash T_J) \arrow[r, "m'=m'_{IJ}"] & Y \times \tilde{S}_I \end{tikzcd}$$

For $I \subset J$, denote by $p_{IJ} : \tilde{S}_J \to \tilde{S}_I$ and $p'_{IJ} := I_Y \times p_{IJ} : Y \times \tilde{S}_J \to Y \times \tilde{S}_I$ the projections, so that $\tilde{g}_I \circ p'_{IJ} = p_{IJ} \circ \tilde{g}_J$. Consider, for $I \subset J \subset [1, \ldots, l]$, resp. for each $I \subset [1, \ldots, l]$, the following commutative diagrams in $\mathrm{Var}(\mathbb{C})$

$$\begin{tikzcd} D_{IJ} = S_I \arrow[r, "i_I"] & \tilde{S}_I \\ S_J \arrow[u, "j_{IJ}"] \arrow[r, "i_J"] & \tilde{S}_J \arrow[u, "p_{IJ}"] \end{tikzcd}, \quad \begin{tikzcd} D'_{IJ} = T_I \arrow[r, "i'_I"] & Y \times \tilde{S}_I \\ T_J \arrow[u, "j'_{IJ}"] \arrow[r, "i'_J"] & Y \times \tilde{S}_J \arrow[u, "p'_{IJ}"] \end{tikzcd} \quad \begin{tikzcd} D_{gI} = S_I \arrow[r, "i_I"] & \tilde{S}_I \\ T_I \arrow[u, "g_I"] \arrow[r, "i'_I"] & Y \times \tilde{S}_I \arrow[u, "\tilde{g}_I"] \end{tikzcd},$$

and $j_{IJ} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. Let $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$. Recall (see section 2) that since $j_I'^* i'_{I*} j_I'^* g^*F = 0$, the morphism $T(D_{gI})(j_I^*F) : \tilde{g}_I^* i_{I*} j_I^* F \to i'_{I*} j_I'^* g^*F$ factors trough

$$T(D_{gI})(j_I^*F) : \tilde{g}_I^* i_{I*} j_I^* F \xrightarrow{\gamma^\vee_{X_I}(-)} \Gamma^\vee_{X_I}\tilde{g}_I^* i_{I*} j_I^* F \xrightarrow{T^\gamma(D_{gI})(j_I^*F)} i'_{I*} j_I'^* g^*F$$

We then have, for each $I \subset [1, \ldots, l]$, the morphism

$$T^{q,\gamma}(D_{gI})(j_I^*F) : \Gamma^\vee_{T_I}\tilde{g}_I^* L(i_{I*}j_I^*F) \xrightarrow{T(\tilde{g}_I, L)(-)}$$
$$\Gamma^\vee_{T_I} L(\tilde{g}_I^* i_{I*} j_I^* F) = L(\Gamma^\vee_{T_I}\tilde{g}_I^* i_{I*} j_I^* F) \xrightarrow{L(T^\gamma(D_{gI})(j_I^*F))} L(i'_{I*} j_I'^* g^*F)$$

and the following diagram in $C(\mathrm{Var}(\mathbb{C})^{sm}/Y\times\tilde{S}_I)$ commutes

$$\begin{tikzcd}
\Gamma^{\vee}_{T_I}\tilde{g}_I^*L(i_{I*}j_I^*F) \arrow[r, "\Gamma^{\vee}_{T_I}\tilde{g}_I^*q_I(F)"] \arrow[d, "T^{q,\gamma}(D_{gI})(j_I^*F)"'] & \Gamma^{\vee}_{T_I}\tilde{g}_I^*i_{I*}j_I^*F \arrow[d, "T^{\gamma}(D_{gI})(j_I^*F)"] \\
L(i'_{I*}j_I^{'*}g^*F) \arrow[r, "q_I(g^*F)"] & i'_{I*}g_I^*j_I^*F = i'_{I*}j_I^{'*}g^*F
\end{tikzcd}$$

We have the following commutative diagram in $C(\mathrm{Var}(\mathbb{C})^{sm}/Y\times\tilde{S}_J)$

$$\begin{tikzcd}
p_{IJ}^{'*}\tilde{g}_I^*i_{I*}j_I^*F = \tilde{g}_J^*p_{IJ}^*i_{I*}j_I^*F \arrow[r, "p_{IJ}^{'*}T(D_{gI})(j_I^*F)"] & p_{IJ}^{'*}i'_{I*}g_I^*j_I^*F = p_{IJ}^{'*}i'_{I*}j_I^{'*}g^*F \\
\tilde{g}_J^*i_{J*}j_{IJ}^*j_I^*F = \tilde{g}_J^*i_{J*}j_J^*F \arrow[u, "\tilde{g}_J^*S(D_{IJ})(F)"] \arrow[r, "T(D_{gJ})(j_J^*F)"] & i'_{J*}g_J^*j_J^*F = i'_{J*}j_{IJ}^{'*}j_I^{'*}g^*F = i'_{J*}j_J^{'*}g^*F \arrow[u, "S(D'_{IJ})(g^*F)"']
\end{tikzcd} \tag{54}$$

This gives, after taking the functor L, the following commutative diagram in $C(\mathrm{Var}(\mathbb{C})^{sm}/Y\times\tilde{S}_J)$

$$\begin{tikzcd}
\Gamma^{\vee}_{T_J}p_{IJ}^{'*}\Gamma^{\vee}_{T_I}\tilde{g}_I^*L(i_{I*}j_I^*F) = \Gamma^{\vee}_{T_J}\tilde{g}_J^*p_{IJ}^*L(i_{I*}j_I^*F) \arrow[r, "p_{IJ}^{'*}T^{q,\gamma}(D_{gI})(j_I^*F)"] & \Gamma^{\vee}_{T_J}p_{IJ}^{'*}L(i'_{I*}j_I^{'*}g^*F) \\
\Gamma^{\vee}_{T_J}\tilde{g}_J^*L(i_{J*}j_J^*g^*F) \arrow[u, "\tilde{g}_J^*S^q(D_{IJ})(F)"] \arrow[r, "T^{q,\gamma}(D_{gJ})(j_J^*F)"] & L(i'_{J*}j_J^{'*}g^*F) \arrow[u, "S^q(D'_{IJ})(g^*F)"']
\end{tikzcd} \tag{55}$$

The fact that the diagrams (55) commutes says that the maps $T^{q,\gamma}(D_{gI})(j_I^*F)$ define a morphism in $C(\mathrm{Var}(\mathbb{C})^{sm}/(T/(Y\times\tilde{S}_I)))$

$$(T^{q,\gamma}(D_{gI})(j_I^*F)) : (\Gamma^{\vee}_{T_I}\tilde{g}_I^*L(i_{I*}j_I^*F), \tilde{g}_J^*S^q(D_{IJ})(F)) \to (L(i'_{I*}j_I^{'*}g^*F), S^q(D'_{IJ})(g^*F))$$

We denote by $\tilde{g}_J^*u_{IJ}^q(F)_1$ the composite

$$\begin{aligned}
\tilde{g}_J^*u_{IJ}^q(F)_1[d_Y+d_{\tilde{S}_I}] &: e(Y\times\tilde{S}_I)_*\Gamma_{T_I}\mathcal{H}om(\tilde{g}_I^*L(i_{I*}j_I^*F), E_{et}(\Omega^{\bullet}_{/Y\times\tilde{S}_I}, F_b)) \\
&\xrightarrow{\mathrm{ad}(p'_{IJ},p'_{IJ*})(-)} p'_{IJ*}p_{IJ}^{'*mod}e(Y\times\tilde{S}_I)_*\Gamma_{T_I}\mathcal{H}om(\tilde{g}_I^*L(i_{I*}j_I^*F), E_{et}(\Omega^{\bullet}_{/Y\times\tilde{S}_I}, F_b)) \\
&\xrightarrow{T^{mod}(p'_{IJ},\gamma)(-)} p'_{IJ*}e(Y\times\tilde{S}_J)_*\Gamma_{T_I\times\tilde{S}_{J\setminus I}}p_{IJ}^{'*mod}\mathcal{H}om(\tilde{g}_J^*L(i_{J*}j_J^*F), E_{et}(\Omega^{\bullet}_{/Y\times\tilde{S}_J}, F_b)) \\
&\xrightarrow{\mathrm{Cone}(\mathrm{ad}(p'_{IJ\sharp},p_{IJ}^{'*})(-),I)} p'_{IJ*}e(Y\times\tilde{S}_J)_*\Gamma_{T_J}p_{IJ}^{'*mod}\mathcal{H}om(\tilde{g}_J^*L(i_{J*}j_J^*F), E_{et}(\Omega^{\bullet}_{/Y\times\tilde{S}_J}, F_b)) \\
&\xrightarrow{T(p'_{IJ},\Omega_{/\cdot})(-)} p'_{IJ*}e(Y\times\tilde{S}_J)_*\Gamma_{T_J}\mathcal{H}om(\tilde{g}_J^*p_{IJ}^*L(i_{I*}j_I^*F), E_{et}(\Omega^{\bullet}_{/Y\times\tilde{S}_J}, F_b)) \\
&\xrightarrow{\mathcal{H}om(\tilde{g}_J^*(S^q(D_{IJ})(F)), E_{et}(\Omega^{\bullet}_{/Y\times\tilde{S}_J}, F_b))} p'_{IJ*}e(Y\times\tilde{S}_J)_*\Gamma_{T_J}\mathcal{H}om(\tilde{g}_J^*L(i_{J*}j_J^*F), E_{et}(\Omega^{\bullet}_{/Y\times\tilde{S}_J}, F_b))
\end{aligned}$$

We denote by $\tilde{g}_J^* u_{IJ}^q(F)_2$ the composite

$$\tilde{g}_J^* u_{IJ}^q(F)_2[d_Y + d_{\tilde{S}_I}] : e(\tilde{T}_I)_* \mathcal{H}om(\Gamma_{T_I}^\vee \tilde{g}_I^* L(i_{I*} j_I^* F), E_{et}(\Omega^\bullet_{/Y\times\tilde{S}_I}, F_b))$$
$$\xrightarrow{\mathrm{ad}(p_{IJ}^{'*mod}, p'_{IJ*})(-)} p'_{IJ*} p_{IJ}^{'*mod} e(\tilde{T}_I)_* \mathcal{H}om(\Gamma_{T_I}^\vee \tilde{g}_I^* L(i_{I*} j_I^* F), E_{et}(\Omega^\bullet_{/Y\times\tilde{S}_I}, F_b))$$
$$\xrightarrow{T(p'_{IJ}, \Omega_{/.})(-)} p'_{IJ*} e(\tilde{T}_I)_* \mathcal{H}om(p_{IJ}^{'*} \Gamma_{T_I}^\vee \tilde{g}_I^* L(i_{I*} j_I^* F), E_{et}(\Omega^\bullet_{/Y\times\tilde{S}_J}, F_b))$$
$$\xrightarrow{\mathcal{H}om(T(p'_{IJ}, \gamma^\vee)(-), E_{et}(\Omega^\bullet_{/Y\times\tilde{S}_J}, F_b))} p'_{IJ*} e(\tilde{T}_I)_* \mathcal{H}om(\Gamma^\vee_{T_I\times\tilde{S}_{J\setminus I}} p_{IJ}^{'*} \tilde{g}_I^* L(i_{I*} j_I^* F), E_{et}(\Omega^\bullet_{/Y\times\tilde{S}_J}, F_b))$$
$$\xrightarrow{\mathrm{Cone}(\mathrm{ad}(p'_{IJ\sharp}, p_{IJ}^{'*})(-), I)} p'_{IJ*} e(\tilde{T}_I)_* \mathcal{H}om(\Gamma_{T_J}^\vee p_{IJ}^{'*} \tilde{g}_I^* L(i_{I*} j_I^* F), E_{et}(\Omega^\bullet_{/Y\times\tilde{S}_J}, F_b))$$
$$\xrightarrow{\mathcal{H}om(\Gamma_{T_J}^\vee \tilde{g}_J^*(S^q(D_{IJ})(F)), E_{et}(\Omega^\bullet_{/Y\times\tilde{S}_J}, F_b))} p'_{IJ*} e(Y\times\tilde{S}_J)_* \mathcal{H}om(\Gamma_{T_J}^\vee \tilde{g}_J^* L(i_{J*} j_J^* F), E_{et}(\Omega^\bullet_{/Y\times\tilde{S}_J}, F_b))$$

We then have then the following lemma :

Lemma 10. *(i) The morphism in $C(\mathrm{Var}(\mathbb{C})^{sm}/(T/(Y\times\tilde{S}_I)))$*

$$(T^{q,\gamma}(D_{gI})(j_I^* F)) : (\Gamma_{T_I}^\vee L \tilde{g}_I^* i_{I*} j_I^* F, \tilde{g}_J^* S^q(D_{IJ})(F)) \to (i'_{I*} j_I^{'*} g^* F, S^q(D'_{IJ})(F)(j_I^{'*} g^* F))$$

is an equivalence $(\mathbb{A}^1, et)$ local.

(ii) Denote for short $d_{YI} = -d_Y - d_{\tilde{S}_I}$. The maps $\mathcal{H}om((T^{q,\gamma}(D_{gI})(j_I^* F)), E_{et}(\Omega^\bullet_{/Y\times\tilde{S}_I}), F_b))$ induce an ∞-filtered quasi-isomorphism in $C_{Ofil,\mathcal{D}}(T/(Y\times\tilde{S}_I))$

$$(\mathcal{H}om(T^{q,\gamma}(D_{gI})(j_I^* F), E_{et}(\Omega^\bullet_{/Y\times\tilde{S}_I}, F_b))) :$$
$$(e(Y\times\tilde{T}_I)_* \mathcal{H}om(L(i'_{I*} j_I^{'*} g^* F), E_{et}(\Omega^\bullet_{/Y\times\tilde{S}_I}, F_b))[d_{YI}], u_{IJ}^q(g^* F)) \to$$
$$(e(Y\times\tilde{T}_I)_* \mathcal{H}om((\Gamma_{T_I}^\vee L \tilde{g}_I^* i_{I*} j_I^* F), E_{et}(\Omega^\bullet_{/Y\times\tilde{S}_I}, F_b))[d_{YI}], \tilde{g}_J^* u_{IJ}^q(F)_2)$$

(iii) The maps $T(\tilde{g}_I, \Omega_\cdot)(L(i_{I*} j_I^* F))$ (see definition 102) induce a morphism in $C_{Ofil,\mathcal{D}}(T/(Y\times\tilde{S}_I))$

$$(T(\tilde{g}_I, \Omega_{/.})(L(i_{I*} j_I^* F))) :$$
$$(\Gamma_{T_I} E_{zar}(\tilde{g}_I^{*mod[-]} e(\tilde{S}_I)_* \mathcal{H}om^\bullet(L(i_{I*} j_I^* F), E_{et}(\Omega^\bullet_{/\tilde{S}_I}, F_b)))[d_{YI}], \tilde{g}_J^{*mod} u_{IJ}^q(F)) \to$$
$$(\Gamma_{T_I}(e(Y\times\tilde{S}_I)_* \mathcal{H}om(\tilde{g}_I^* L(i_{I*} j_I^* F), E_{et}(\Omega^\bullet_{/Y\times\tilde{S}_I}, F_b)))[d_{YI}], \tilde{g}_J^* u_{IJ}^q(F)_1).$$

Proof. (i): Follows from theorem 16.
(ii): These maps induce a morphism in $C_{Ofil,\mathcal{D}}(T/(Y\times\tilde{S}_I))$ by construction. The fact that this morphism is an ∞-filtered equivalence Zariski local follows from (i) and proposition 105.
(iii): These maps induce a morphism in $C_{Ofil,\mathcal{D}}(T/(Y\times\tilde{S}_I))$ by construction.

□

Definition 107. *Let $g : T \to S$ a morphism with $T, S \in \mathrm{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$ Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$, Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \to \tilde{S}_I$ is a lift of $g_I := g_{|T_I} : T_I \to S_I$. Denote for short $d_{YI} := d_Y + d_{\tilde{S}_I}$. Let $M \in \mathrm{DA}_c(S)$ and $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1_S, et)(F)$. Then, $D(\mathbb{A}^1_T, et)(g^* F) = g^* M$. We* *have, by lemma 10, the canonical transformation in* $D_{Ofil,\mathcal{D},\infty}(T/(Y\times\tilde{S}_I))$

$$\begin{aligned}
&T(g,\mathcal{F}^{GM})(M): Rg^{*mod[-],\Gamma}\mathcal{F}_S^{GM}(M) := \\
&(\Gamma_{T_I}E_{zar}(\tilde{g}_I^{*mod}e(\tilde{S}_I)_*\mathcal{H}om^\bullet(L(i_{I*}j_I^*F),E_{et}(\Omega^\bullet_{/\tilde{S}_I},F_b)))[-d_Y-d_{\tilde{S}_I}],\tilde{g}_J^{*mod}u^q_{IJ}(F)) \\
&\xrightarrow{(\Gamma_{T_I}E(T(\tilde{g}_I,\Omega_{/\cdot})(L(i_{I*}j_I^*(F,W)))))} \\
&(\Gamma_{T_I}e(Y\times\tilde{S}_I)_*\mathcal{H}om^\bullet(\tilde{g}_I^*L(i_{I*}j_I^*F),E_{et}(\Omega^\bullet_{/Y\times\tilde{S}_I},F_b))[-d_Y-d_{\tilde{S}_I}],\tilde{g}_J^*u^q_{IJ}(F)_1) \\
&\xrightarrow{(I(\gamma,\hom(-,-)))} \\
&(e(Y\times\tilde{S}_I)_*\mathcal{H}om^\bullet(\Gamma^\vee_{T_I}\tilde{g}_I^*L(i_{I*}j_I^*F),E_{et}(\Omega^\bullet_{/Y\times\tilde{S}_I},F_b))[-d_Y-d_{\tilde{S}_I}],\tilde{g}_J^*u^q_{IJ}(F)_2) \\
&\xrightarrow{(e(Y\times\tilde{S}_I)_*\mathcal{H}om(T^{q,\gamma}(D_{gI})(j_I^*F),E_{et}(\Omega^\bullet_{/Y\times\tilde{S}_I},F_b)))^{-1}} \\
&(e(Y\times\tilde{S}_I)_*\mathcal{H}om^\bullet(L(i'_{I*}j_I^{'*}g^*F),E_{et}(\Omega^\bullet_{/Y\times\tilde{S}_I},F_b))[-d_Y-d_{\tilde{S}_I}],u^q_{IJ}(g^*F)) =: \mathcal{F}_T^{GM}(g^*M).
\end{aligned}$$

Proposition 108. *(i) Let $g:T\to S$ a morphism with $T,S\in\mathrm{Var}(\mathbb{C})$. Assume we have a factorization $g:T\xrightarrow{l}Y_2\times S\xrightarrow{p_S}S$ with $Y_2\in\mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S=\cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$ Then, $T=\cup_{i=1}^l T_i$ with $T_i:=g^{-1}(S_i)$ and we have closed embeddings $i'_i:=i_i\circ l:T_i\hookrightarrow Y_2\times\tilde{S}_i$, Moreover $\tilde{g}_I:=p_{\tilde{S}_I}:Y\times\tilde{S}_I\to\tilde{S}_I$ is a lift of $g_I:=g_{|T_I}:T_I\to S_I$. Let $f:X\to S$ a morphism with $X\in\mathrm{Var}(\mathbb{C})$. Assume that there is a factorization $f:X\xrightarrow{l}Y_1\times S\xrightarrow{p_S}S$, with $Y_1\in\mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have then the following commutative diagram whose squares are cartesians*

$$\begin{tikzcd}
f': X_T \arrow[r] \arrow[d] & Y_1\times T \arrow[r] \arrow[d] & T \arrow[d] \\
f''=f\times I: Y_2\times X \arrow[r] \arrow[d] & Y_1\times Y_2\times S \arrow[r] \arrow[d] & Y_2\times S \arrow[d] \\
f: X \arrow[r] & Y_1\times S \arrow[r] & S
\end{tikzcd}$$

Consider $F(X/S):=p_{S,\sharp}\Gamma^\vee_X\mathbb{Z}(Y_1\times S/Y_1\times S)$ and the isomorphism in $C(\mathrm{Var}(\mathbb{C})^{sm}/S)$

$$T(f,g,F(X/S)): g^*F(X/S):=g^*p_{S,\sharp}\Gamma^\vee_X\mathbb{Z}(Y_1\times S/Y_1\times S)\xrightarrow{\sim} p_{T,\sharp}\Gamma^\vee_{X_T}\mathbb{Z}(Y_1\times T/Y_1\times T)=:F(X_T/T).$$

which gives in $\mathrm{DA}(S)$ the isomorphism $T(f,g,F(X/S)):g^*M(X/S)\xrightarrow{\sim}M(X_T/T)$. Then, the following diagram in $D_{Ofil,\mathcal{D},\infty}(T/(Y_2\times\tilde{S}_I))$ commutes

$$\begin{tikzcd}
Rg^{*mod,\Gamma}\mathcal{F}_S^{GM}(M(X/S)) \arrow[rr, "T(g,\mathcal{F}^{GM})(M(X/S))"] \arrow[d, "I^{GM}(X/S)"] & & \mathcal{F}_T^{GM}(M(X_T/T)) \arrow[d, "I^{GM}(X_T/T)"] \\
\begin{array}{c} g^{*mod[-],\Gamma}(p_{\tilde{S}_I*}\Gamma_{X_I}E_{zar}(\Omega^\bullet_{Y_1\times\tilde{S}_I/\tilde{S}_I},F_b)[-d_{\tilde{S}_I}], \\ w_{IJ}(X/S)) \end{array} \arrow[rr, "(T(\tilde{g}_I\times I,\gamma)(-)\circ T^O_w(\tilde{g}_I,p_{\tilde{S}_I}))"] & & \begin{array}{c} (p_{Y_2\times\tilde{S}_I*}\Gamma_{X_{T_I}}E_{zar}(\Omega^\bullet_{Y_2\times Y_1\times\tilde{S}_I/Y_2\times\tilde{S}_I},F_b)[-d_{Y_2} - \\ w_{IJ}(X_T/T)) \end{array}
\end{tikzcd}$$

(ii) Let $g:T\to S$ a morphism with $T,S\in\mathrm{SmVar}(\mathbb{C})$. Let $f:X\to S$ a morphism with $X\in\mathrm{Var}(\mathbb{C})$. Assume that there is a factorization $f:X\xrightarrow{l}Y\times S\xrightarrow{p_S}S$, with $Y\in\mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Consider $F(X/S):=p_{S,\sharp}\Gamma^\vee_X\mathbb{Z}(Y\times S/Y\times S)$ and the isomorphism

in $C(\mathrm{Var}(\mathbb{C})^{sm}/S)$

$$T(f,g,F(X/S)) : g^*F(X/S) := g^*p_{S,\sharp}\Gamma_X^\vee\mathbb{Z}(Y\times S/Y\times S) \xrightarrow{\sim} p_{T,\sharp}\Gamma_{X_T}^\vee\mathbb{Z}(Y\times T/Y\times T) =: F(X_T/T).$$

which gives in $\mathrm{DA}(S)$ *the isomorphism* $T(f,g,F(X/S)) : g^*M(X/S) \xrightarrow{\sim} M(X_T/T)$. *Then, the following diagram in* $D_{Ofil,\infty}(T)$ *commutes*

$$\begin{tikzcd}
Lg^{*mod[-]}\mathcal{F}_S^{GM}(M(X/S)) \arrow[r, "T^O(g,\mathcal{F}^{GM})(M(X/S))"] \arrow[d, "I^{GM}(X/S)"] & \mathcal{F}_T^{GM}(M(X_T/T)) \arrow[d, "I^{GM}(X_T/T)"] \\
g^{*mod}L_O(p_{S*}\Gamma_X E_{zar}(\Omega^\bullet_{Y\times S/S},F_b)[-d_T] \arrow[r, "(T(g\times I{,}\gamma)(-)\circ T_w^O(g{,}p_S))"] \arrow[d, "T_w(\otimes{,}\gamma)(O_{Y\times S})"] & p_{Y\times T*}\Gamma_{X_T}E_{zar}(\Omega^\bullet_{Y\times T/T},F_b)[-d_T] \arrow[d, "T_w(\otimes{,}\gamma)(O_{Y\times T})"] \\
Lg^{*mod}\int_{p_S}^{FDR}\Gamma_X E(O_{Y\times S},F_b)[-d_Y-d_T] \arrow[r, "T^{\mathcal{D}mod}(g{,}f)(\Gamma_X E(O_{Y\times S}{,}F_b))"] & \int_{p_T}^{FDR}\Gamma_{X_T}E(O_{Y\times T},F_b)[-d_Y-d_T].
\end{tikzcd}$$

Proof. Follows immediately from definition. □

We have the following theorem:

Theorem 34. *(i) Let* $g : T\to S$ *is a morphism with* $T,S\in\mathrm{Var}(\mathbb{C})$. *Assume there exist a factorization* $g : T \xrightarrow{l} Y\times S \xrightarrow{p_S}$ *with* $Y\in\mathrm{SmVar}(\mathbb{C})$, l *a closed embedding and* p_S *the projection. Let* $S=\cup_{i=1}^l S_i$ *be an open cover such that there exists closed embeddings* $i_i : S_i \hookrightarrow \tilde{S}_i$ *with* $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$. *Then, for* $M\in\mathrm{DA}_c(S)$

$$T(g,\mathcal{F}^{GM})(M) : Rg^{*mod[-],\Gamma}\mathcal{F}_S^{GM}(M) \xrightarrow{\sim} \mathcal{F}_T^{GM}(g^*M)$$

is an isomorphism in $D_{O_T fil,\mathcal{D},\infty}(T/(Y\times\tilde{S}_I))$.

(ii) Let $g : T\to S$ *is a morphism with* $T,S\in\mathrm{SmVar}(\mathbb{C})$. *Then, for* $M\in\mathrm{DA}_c(S)$

$$T^O(g,\mathcal{F}^{GM})(M) : Lg^{*mod[-]}\mathcal{F}_S^{GM}(M) \xrightarrow{\sim} \mathcal{F}_T^{GM}(g^*M)$$

is an isomorphism in $D_{O_T}(T)$.

(iii) A base change theorem for algebraic De Rham cohomology : Let $g : T\to S$ *is a morphism with* $T,S\in\mathrm{SmVar}(\mathbb{C})$. *Let* $h : U\to S$ *a smooth morphism with* $U\in\mathrm{Var}(\mathbb{C})$. *Then the map (see definition 1)*

$$T_w^O(g,h) : Lg^{*mod}Rh_*(\Omega^\bullet_{U/S},F_b) \xrightarrow{\sim} Rh'_*(\Omega^\bullet_{U_T/T},F_b)$$

is an isomorphism in $D_{O_T}(T)$.

Proof. (i):Follows from proposition 104.
(ii): Follows from proposition 108(ii) and the base change for algebraic D modules (proposition 80).
(iii):Follows from (ii) and lemma 7. □

We finish this subsection by some remarks on the absolute case and on a particular case of the relative case:

Proposition 109. *(i) Let* $X\in\mathrm{PSmVar}(\mathbb{C})$ *and* $D=\cup D_i\subset X$ *a normal crossing divisor. Consider the open embedding* $j : U := X\backslash D\hookrightarrow X$. *Then,*

- *The map in* $D_{fil,\infty}(\mathbb{C})$

$$\begin{aligned}\mathrm{Hom}(L\mathbb{D}(\mathbb{Z}(U)),k)^{-1}\circ\mathrm{Hom}((0,\mathrm{ad}(j^*,j_*)(\mathbb{Z}(X/X))),E_{et}(\Omega^\bullet,F_b)):\\ \mathcal{F}^{GM}(\mathbb{D}(\mathbb{Z}(U))):=\mathrm{Hom}(L\mathbb{D}(\mathbb{Z}(U)),E_{et}(\Omega^\bullet,F_b))\\ \xrightarrow{\sim}\mathrm{Hom}(\mathrm{Cone}(\mathbb{Z}(D)\to\mathbb{Z}(X)),E_{zar}(\Omega^\bullet,F_b))=\Gamma(X,E_{zar}(\Omega^\bullet_X(\mathrm{nul}\,D),F_b)).\end{aligned}$$

is an isomorphism, where we recall $\mathbb{D}(\mathbb{Z}(U):=a_{X*}j_*E_{et}(\mathbb{Z}(U/U))=a_{U*}E_{et}(\mathbb{Z}(U/U))$,

- $\mathcal{F}^{GM}(\mathbb{Z}(U))=\Gamma(U,E_{zar}(\Omega^\bullet_U,F_b))\in D_{fil,\infty}(\mathbb{C})$ *is NOT isomorphic to* $\Gamma(X,E_{zar}(\Omega^\bullet_X(\log D),F_b))$ *in* $D_{fil,\infty}(\mathbb{C})$ *in general. For exemple* U *is affine, then* $H^n(U,\Omega^p_U)=0$ *for all* $p\in\mathbb{N}$, $p\neq 0$, *so that the* $E^{p,q}_\infty(\Gamma(U,E_{zar}(\Omega^\bullet_U,F_b)))$ *are NOT isomorphic to* $E^{p,q}_\infty(\Gamma(X,E_{zar}(\Omega^\bullet_X(\log D),F_b)))$ *in this case. In particular, the map,*

$$j^*:=\mathrm{ad}(j^*,j_*)(-):H^n\Gamma(X,E_{zar}(\Omega^\bullet_X(\log D)))\xrightarrow{\sim}H^n\Gamma(U,E_{zar}(\Omega^\bullet_U))$$

which is an isomorphism in $D(\mathbb{C})$ *(i.e. if we forgot filtrations), gives embeddings*

$$j^*:=\mathrm{ad}(j^*,j_*)(-):F^pH^n(U,\mathbb{C}):=F^pH^n\Gamma(X,E_{zar}(\Omega^\bullet_X(\log D),F_b))\hookrightarrow F^pH^n\Gamma(U,E_{zar}(\Omega^\bullet_U,F_b))$$

which are NOT an isomorphism in general for $n,p\in\mathbb{Z}$. *Note that, since* $a_U:U\to\{\mathrm{pt}\}$ *is not proper,*

$$[\Delta_U]:\mathbb{Z}(U)\to a_{U*}E_{et}(\mathbb{Z}(U/U))[2d_U]$$

is NOT an equivalence $(\mathbb{A}^1,et)$ *local.*

- *Let* $Z\subset X$ *a smooth subvariety and denote* $U:=X\backslash Z$ *the open complementary. Denote* $M_Z(X):=\mathrm{Cone}(M(U)\to M(X))\in\mathrm{DA}(\mathbb{C})$. *The map in* $D_{fil,\infty}(\mathbb{C})$

$$\begin{aligned}\mathrm{Hom}(G(X,Z),E_{et}(\Omega^\bullet,F_b))^{-1}\circ\mathrm{Hom}(a_{X\sharp}\Gamma^\vee_Z\mathbb{Z}(X/X)),k)^{-1}:\\ \mathcal{F}^{GM}(M_Z(X)):=\mathrm{Hom}(a_{X\sharp}\Gamma^\vee_Z\mathbb{Z}(X/X)),E_{et}(\Omega^\bullet,F_b))\xrightarrow{\sim}\\ \Gamma(X,\Gamma_ZE_{zar}(\Omega^\bullet_X,F_b))=\Gamma_Z(X,E_{zar}(\Omega^\bullet_X,F_b))\\ \xrightarrow{\sim}\mathcal{F}^{GM}(M(Z)(c)[2c])=\Gamma(Z,E_{zar}(\Omega^\bullet_Z,F_b))(-c)[-2c]\end{aligned}$$

is an isomorphism, where $c=\mathrm{codim}(Z,X)$ *and* $G(X,Z):a_{X\sharp}\Gamma^\vee_Z\mathbb{Z}(X/X)\to\mathbb{Z}(Z)(c)[2c]$ *is the Gynsin morphism.*

- *Let* $D\subset X$ *a smooth divisor and denote* $U:=X\backslash Z$ *the open complementary Note that the canonical distinguish triangle in* $\mathrm{DA}(\mathbb{C})$

$$M(U)\xrightarrow{\mathrm{ad}(j_\sharp,j^*)(\mathbb{Z}(X/X))}M(X)\xrightarrow{\gamma^\vee_Z(\mathbb{Z}(X/X))}M_D(X)\to M(U)[1]$$

give a canonical triangle in $D_{fil,\infty}(\mathbb{C})$

$$\mathcal{F}^{GM}(M_D(X))\xrightarrow{\mathcal{F}^{GM}(\gamma^\vee_Z(\mathbb{Z}(X/X)))}\mathcal{F}^{GM}(M(X))\xrightarrow{\mathcal{F}^{GM}(\mathrm{ad}(j_\sharp,j^*)(\mathbb{Z}(X/X))}\mathcal{F}^{GM}(M(U))\to\mathcal{F}^{GM}(M_D(X))[1],$$

which is NOT the image of a distinguish triangle in $\pi(D(MHM(\mathbb{C})))$, *as* $\mathcal{F}^{GM}(M(U))\notin\pi(D(MHM(\mathbb{C})))$ *since the morphism*

$$\mathrm{ad}(j^*,j_*):H^n(X,E_{zar}(\Omega^\bullet_X(\log D),F_b)\to H^n(U,E_{zar}(\Omega^\bullet_U,F_b))$$

is not strict. Note that if $U:=S\backslash D$ *is affine, then by the exact sequence in* $C(\mathbb{Z})$

$$0\to\Gamma_Z(X,E_{zar}(\Omega^p_X))\to\Gamma(X,E_{zar}(\Omega^p_X))\to\Gamma(U,E_{zar}(\Omega^p_U))\to 0$$

we have $H^q\Gamma_Z(X,E_{zar}(\Omega^p_X))=H^q(\Gamma(X,E_{zar}(\Omega^p_X)))$. *In particular, the map,*

$$j^*:=\mathrm{ad}(j^*,j_*)(-):\Gamma(X,E_{zar}(\Omega^\bullet_X(\log D),F_b))\to\Gamma(U,E_{zar}(\Omega^\bullet_U,F_b))$$

and hence the map

$$j^* := \mathrm{ad}(j^*, j_*)(-) : \mathrm{Cone}(\Gamma(X, \Omega^\bullet_X, F_b) \to \Gamma(X, E_{zar}(\Omega^\bullet_X(\log D), F_b)) \to \mathrm{Cone}(\Gamma(X, \Omega^\bullet_X, F_b) \to \Gamma(U, E_{zar}(\Omega^\bullet_U, F_b))) =: \Gamma(X, \Gamma_Z E_{zar}(\Omega^\bullet_X, F_b))$$

are quasi-isomorphisms (i.e. if we forgot filtrations), but the first one is NOT an ∞-filtered quasi-isomorphism whereas the second one is an ∞-filtered quasi-isomorphism (recall that for $r > 1$ the r-filtered quasi-isomorphisms does NOT satisfy the 2 of 3 property for morphism of canonical triangles : see section 2.1).

(ii) More generally, let $f : X \to S$ a smooth projective morphism with $S, X \in \mathrm{SmVar}(\mathbb{C})$. Let $D = \cup D_i \subset X$ a normal crossing divisor such that $f_{|D_I} := f \circ i_I : D_I \to S$ are SMOOTH morphisms (note that it is a very special case), with $i_I : D_I \hookrightarrow X$ the closed embeddings. Consider the open embedding $j : U := X \backslash D \hookrightarrow X$ and $h := f \circ j : U \to S$.

- *The map in $D_{\mathcal{D}fil,\infty}(S)$*

$$\mathrm{Hom}(L\mathbb{D}(\mathbb{Z}(U)), k)^{-1} \circ \mathrm{Hom}(\mathrm{ad}(j^*, j_*)(\mathbb{Z}(X/X)), E_{et}(\Omega^\bullet_{/S}, F_b)) : \mathcal{F}^{GM}_S(\mathbb{D}(\mathbb{Z}(U/S))) := \mathrm{Hom}(L\mathbb{D}(\mathbb{Z}(U/S)), E_{et}(\Omega^\bullet_{/S}, F_b)) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{Cone}(\mathbb{Z}(D) \to \mathbb{Z}(X)), E_{zar}(\Omega^\bullet_{/S}, F_b)) = f_* E_{zar}(\Omega^\bullet_{X/S}(\mathrm{nul}\, D), F_b).$$

  is an isomorphism, where we recall $\mathbb{D}(\mathbb{Z}(U) := f_* j_* E_{et}(\mathbb{Z}(U/U)) = h_* E_{et}(\mathbb{Z}(U/U))$,

- *$\mathcal{F}^{GM}_S(\mathbb{Z}(U/S)) = h_* E_{zar}\Omega^\bullet_{U/S}, F_b) \in D_{\mathcal{D}fil,\infty}(S)$ is NOT isomorphic to $f_* E_{zar}(\Omega^\bullet_{X/S}(\log D), F_b)$ in $D_{\mathcal{D}fil,\infty}(S)$ in general. In particular, the map,*

$$j^* := \mathrm{ad}(j^*, j_*)(-) : H^n f_* E_{zar}(\Omega^\bullet_{X/S}(\log D)) \xrightarrow{\sim} H^n h_* E_{zar}(\Omega^\bullet_{U/S})$$

  which is an isomorphism in $D_{\mathcal{D}}(S)$ (i.e. if we forgot filtrations), gives embeddings

$$j^* := \mathrm{ad}(j^*, j_*)(-) : F^p H^n h_* \mathbb{C}_U := F^p H^n f_* E_{zar}(\Omega^\bullet_{X/S}(\log D), F_b) \hookrightarrow F^p H^n h_* E_{zar}(\Omega^\bullet_{U/S}, F_b)$$

  which are NOT an isomorphism in general for $n, p \in \mathbb{Z}$. Note that, since $a_U : U \to \{\mathrm{pt}\}$ is not proper,

$$[\Delta_U] : \mathbb{Z}(U/S) \to h_* E_{et}(\mathbb{Z}(U/U))[2d_U]$$

  is NOT an equivalence $(\mathbb{A}^1, et)$ local.

- *Let $Z \subset X$ a subvariety and denote $U := X \backslash Z$ the open complementary. Denote $M_Z(X/S) :=$ $\mathrm{Cone}(M(U/S) \to M(X/S)) \in \mathrm{DA}(S)$. If $f_{|Z} := f \circ i_Z : Z \to S$ is a SMOOTH morphism, the map in $D_{\mathcal{D}fil,\infty}(S)$*

$$\mathrm{Hom}(G(X, Z), E_{et}(\Omega^\bullet, F_b)) \circ \mathrm{Hom}(\Gamma^\vee_Z \mathbb{Z}(X/X)), k)^{-1} : \mathcal{F}^{GM}_S(M_Z(X/S)) := \mathrm{Hom}(f_\sharp \Gamma^\vee_Z \mathbb{Z}(X/X)), E_{et}(\Omega^\bullet_{/S}, F_b)) \xrightarrow{\sim} f_* \Gamma_Z E_{zar}(\Omega^\bullet_{X/S}, F_b)) \xrightarrow{\sim} \mathcal{F}^{GM}_S(M(Z/S)(c)[2c]) = f_{Z*} E_{zar}(\Omega^\bullet_{Z/S}, F_b)(-c)[-2c]$$

  is an isomorphism, where $c = \mathrm{codim}(Z, X)$ and $G(X, Z) : f_\sharp \Gamma^\vee_Z \mathbb{Z}(X/X) \to \mathbb{Z}(Z/S)(c)[2c]$ is the Gynsin morphism.

- *Let $D \subset X$ a smooth divisor and denote $U := X \backslash Z$ the open complementary Note that the canonical distinguish triangle in $\mathrm{DA}(S)$*

$$M(U/S) \xrightarrow{\mathrm{ad}(j_\sharp, j^*)(\mathbb{Z}(X/X))} M(X/S) \xrightarrow{\gamma^\vee_Z(\mathbb{Z}(X/X))} M_D(X/S) \to M(U/S)[1]$$

  give a canonical triangle in $D_{\mathcal{D}fil,\infty}(S)$

$$\mathcal{F}^{GM}_S(M_D(X/S)) \xrightarrow{\mathcal{F}^{GM}(\gamma^\vee_Z(\mathbb{Z}(X/X)))} \mathcal{F}^{GM}_S(M(X/S)) \xrightarrow{\mathcal{F}^{GM}(\mathrm{ad}(j_\sharp, j^*)(\mathbb{Z}(X/X))} \mathcal{F}^{GM}_S(M(U/S)) \to \mathcal{F}^{GM}_S(M_D(X/S))[1],$$

  which is NOT the image of a distinguish triangle in $D(MHM(S))$.

Proof. (i):For simplicity, we may assume that $i : D \hookrightarrow X$ is a smooth divisor. Then, by theorem 16, the map

$$(0, \mathrm{ad}(j_\sharp, j_*)(\mathbb{Z}(X/X)) : \mathbb{Z}(D) \to \mathbb{Z}(X)) \to \mathbb{D}(\mathbb{Z}(U/U))$$

is an equivalence $(\mathbb{A}^1, et)$ local in $C(\mathrm{SmVar}(\mathbb{C}))$. The result then follows from proposition 103. By theorem 16, we have an equivalence $(\mathbb{A}^1, et)$ local in $C(\mathrm{SmVar}(\mathbb{C}))$

$$G(X,Z) : a_{X\sharp}\Gamma^\vee\mathbb{Z}(X/X) \to \mathbb{Z}(Z)(c)[2c]$$

The result then follows from proposition 103.
(ii):For simplicity, we may assume that $i : D \hookrightarrow X$ is a smooth divisor. Then, by theorem 16, the map

$$(0, \mathrm{ad}(j_\sharp, j_*)(\mathbb{Z}(X/X)) : \mathbb{Z}(D/S) \to \mathbb{Z}(X/S)) \to \mathbb{D}(\mathbb{Z}(U/U))$$

is an equivalence $(\mathbb{A}^1, et)$ local in $C(\mathrm{Var}(\mathbb{C})^{sm}/S)$. The result then follows from proposition 103. By theorem 16, we have an equivalence $(\mathbb{A}^1, et)$ local in $C(\mathrm{SmVar}(\mathbb{C}))$

$$G(X,Z) : f_\sharp\Gamma^\vee\mathbb{Z}(X/X) \to \mathbb{Z}(Z/S)(c)[2c]$$

The result then follows from proposition 103. □

Definition 108. *Let $S \in \mathrm{SmVar}(\mathbb{C})$. We have, for $M, N \in \mathrm{DA}(S)$ and $F, G \in C(\mathrm{Var}(\mathbb{C})^{sm}/S))$ projective such that $M = D(\mathbb{A}^1, et)(F)$ and $N = D(\mathbb{A}^1, et)(G)$, the following transformation map in $D_{Ofil,\mathcal{D}}(S)$*

$$\begin{aligned}
T(\mathcal{F}_S^{GM}, \otimes)(M,N) : \mathcal{F}_S^{GM}(M) \otimes_{O_S}^{L[-]} \mathcal{F}_S^{GM}(N) :=&\\
(e(S)_*\mathcal{H}om(F, E_{et}(\Omega^\bullet_{/S}, F_b))) \otimes_{O_S} (e(S)_*\mathcal{H}om(G, E_{et}(\Omega^\bullet_{/S}, F_b)))[-d_S]&\\
\xrightarrow{T(\otimes,\Omega_{/S})(F,G)} e(S)_*\mathcal{H}om(F \otimes G, E_{et}(\Omega^\bullet_{/S}, F_b))[-d_S]&\\
\xrightarrow{=} e(S)_*\mathcal{H}om(F \otimes G, E_{et}(\Omega^\bullet_{/S}, F_b))[-d_S] =: \mathcal{F}_S^{GM}(M \otimes N)&
\end{aligned}$$

We now give the definition in the non smooth case :

Definition 109. *Let $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open affine covering and denote, for $I \subset [1, \cdots l]$, $S_I = \cap_{i\in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. We have, for $M, N \in \mathrm{DA}(S)$ and $F, G \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1, et)(F)$ and $N = D(\mathbb{A}^1, et)(G)$, the following transformation map in $D_{Ofil,\mathcal{D}}(S/(\tilde{S}_I))$*

$$\begin{aligned}
T(\mathcal{F}_S^{GM}, \otimes)(M,N) : \mathcal{F}_S^{GM}(M) \otimes_{O_S}^{L[-]} \mathcal{F}_S^{GM}(N) :=&\\
(e(\tilde{S}_I)_*\mathcal{H}om(L(i_{I*}j_I^*F), E_{et}(\Omega^\bullet_{/\tilde{S}_I}, F_b))[-d_{\tilde{S}_I}], u_{IJ}(F)) \otimes_{O_S}^{[-]}&\\
(e(\tilde{S}_I)_*\mathcal{H}om(L(i_{I*}j_I^*G), E_{et}(\Omega^\bullet_{/\tilde{S}_I}, F_b))[-d_{\tilde{S}_I}], u_{IJ}(G))&\\
\xrightarrow{=} ((e(\tilde{S}_I)_*\mathcal{H}om(L(i_{I*}j_I^*F), E_{et}(\Omega^\bullet_{/\tilde{S}_I}, F_b)) \otimes_{O_{\tilde{S}_I}}&\\
e(\tilde{S}_I)_*\mathcal{H}om(L(i_{I*}j_I^*G), E_{et}(\Omega^\bullet_{/\tilde{S}_I}, F_b)))[-d_{\tilde{S}_I}], u_{IJ}(F) \otimes u_{IJ}(G))&\\
\xrightarrow{(T(\otimes,\Omega_{/\tilde{S}_I})(L(i_{I*}j_I^*F), L(i_{I*}j_I^*G)))}&\\
(e(\tilde{S}_I)_*\mathcal{H}om(L(i_{I*}j_I^*F) \otimes L(i_{I*}j_I^*G), E_{et}(\Omega^\bullet_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], v_{IJ}(F \otimes G))&\\
\xrightarrow{=} (e(\tilde{S}_I)_*\mathcal{H}om(L(i_{I*}j_I^*(F \otimes G), E_{et}(\Omega^\bullet_{/\tilde{S}_I}, F_b)))[-d_{\tilde{S}_I}], u_{IJ}(F \otimes G)) =: \mathcal{F}_S^{GM}(M \otimes N)&
\end{aligned}$$

Proposition 110. *Let $f_1 : X_1 \to S$, $f_2 : X_2 \to S$ two morphism with $X_1, X_2, S \in \mathrm{Var}(\mathbb{C})$. Assume that there exist factorizations $f_1 : X_1 \xrightarrow{l_1} Y_1 \times S \xrightarrow{p_S} S$, $f_2 : X_2 \xrightarrow{l_2} Y_2 \times S \xrightarrow{p_S} S$ with $Y_1, Y_2 \in \mathrm{SmVar}(\mathbb{C})$, l_1, l_2 closed embeddings and p_S the projections. We have then the factorization*

$$f_1 \times f_2 : X_{12} := X_1 \times_S X_2 \xrightarrow{l_1 \times l_2} Y_1 \times Y_2 \times S \xrightarrow{p_S} S$$

Let $S=\cup_{i=1}^{l}S_i$ an open affine covering and denote, for $I\subset[1,\cdots l]$, $S_I=\cap_{i\in I}S_i$ and $j_I:S_I\hookrightarrow S$ the open embedding. Let $i_i:S_i\hookrightarrow\tilde{S}_i$ closed embeddings, with $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$. We have, for $M,N\in\mathrm{DA}(S)$ and $F,G\in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $M=D(\mathbb{A}^1,et)(F)$ and $N=D(\mathbb{A}^1,et)(G)$, the following commutative diagram in $D_{Ofil,\mathcal{D}}(S/(\tilde{S}_I))$

$$\begin{tikzcd}
\mathcal{F}_S^{GM}(M(X_1/S))\otimes^L_{O_S}\mathcal{F}_S^{GM}(M(X_2/S)) \arrow[r, "T(\mathcal{F}_S^{GM},\otimes)(M(X_1/S),M(X_2/S))"] \arrow[d, "I^{GM}(X_1/S)\otimes I^{GM}(X_2/S)"] & \begin{matrix}\mathcal{F}_S^{GM}(M(X_1/S)\otimes M(X_2/S))\\ =\mathcal{F}_S^{GM}(M(X_1\times_S X_2/S))\end{matrix} \arrow[d, "I^{GM}(X_{12}/S)"] \\
\begin{matrix}(p_{\tilde{S}_I*}\Gamma_{X_{1I}}E_{zar}(\Omega^{\bullet}_{Y_1\times\tilde{S}_I/\tilde{S}_I},F_b)[-d_{\tilde{S}_I}],w_{IJ}(X_1/S))\otimes_{O_S}\\ (p_{\tilde{S}_I*}\Gamma_{X_{2I}}E_{zar}(\Omega^{\bullet}_{Y_2\times\tilde{S}_I/\tilde{S}_I},F_b)[-d_{\tilde{S}_I}],w_{IJ}(X_2/S))\end{matrix} \arrow[r, "(Ew_{(Y_1\times\tilde{S}_I,Y_2\times\tilde{S}_I)/\tilde{S}_I})"] & \begin{matrix}(p_{\tilde{S}_I*}\Gamma_{X_{12I}}E_{zar}(\Omega^{\bullet}_{Y_1\times Y_2\times\tilde{S}_I/\tilde{S}_I},F_b)[-d_{\tilde{S}_I}],\\ w_{IJ}(X_{12}/S))\end{matrix}
\end{tikzcd}.$$

Proof. Immediate from definition. □

6.1.2 The algebraic filtered De Rham realization functor and the commutativity with the six operation

We recall (see section 2), for $f:T\to S$ a morphism with $T,S\in\mathrm{Var}(\mathbb{C})$, the commutative diagrams of sites (29) and (30)

$$\begin{tikzcd}
\mathrm{Var}(\mathbb{C})^2/T \arrow[rr, "\mu_T"] \arrow[dr, "\rho_T"] \arrow[dd, "P(f)"'] & & \mathrm{Var}(\mathbb{C})^{2,pr}/T \arrow[dr, "\rho_T"] \arrow[dd, "P(f)"] & \\
& \mathrm{Var}(\mathbb{C})^{2,sm}/T \arrow[rr, "\mu_T"] \arrow[dd, "P(f)"] & & \mathrm{Var}(\mathbb{C})^{2,smpr}/T \arrow[dd, "P(f)"] \\
\mathrm{Var}(\mathbb{C})^2/S \arrow[rr, "\mu_S"] \arrow[dr, "\rho_S"] & & \mathrm{Var}(\mathbb{C})^{2,pr}/S \arrow[dr, "\rho_S"] & \\
& \mathrm{Var}(\mathbb{C})^{2,sm}/S \arrow[rr, "\mu_S"] & & \mathrm{Var}(\mathbb{C})^{2,smpr}/S
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Var}(\mathbb{C})^{2,pr}/T \arrow[rr, "\mathrm{Gr}_T^{12}"] \arrow[dr, "\rho_T"] \arrow[dd, "P(f)"'] & & \mathrm{Var}(\mathbb{C})/T \arrow[dr, "\rho_T"] \arrow[dd, "P(f)"] & \\
& \mathrm{Var}(\mathbb{C})^{2,smpr}/T \arrow[rr, "\mathrm{Gr}_T^{12}"] \arrow[dd, "P(f)"] & & \mathrm{Var}(\mathbb{C})^{sm}/T \arrow[dd, "P(f)"] \\
\mathrm{Var}(\mathbb{C})^{2,pr}/S \arrow[rr, "\mathrm{Gr}_S^{12}"] \arrow[dr, "\rho_S"] & & \mathrm{Var}(\mathbb{C})/S \arrow[dr, "\rho_S"] & \\
& \mathrm{Var}(\mathbb{C})^{2,sm}/S \arrow[rr, "\mathrm{Gr}_S^{12}"] & & \mathrm{Var}(\mathbb{C})^{sm}/S
\end{tikzcd}.$$

Let $S\in\mathrm{Var}(\mathbb{C})$. We have for $F\in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ the canonical map in $C(\mathrm{Var}(\mathbb{C})^{sm}/S)$

$$\mathrm{Gr}(F):\mathrm{Gr}^{12}_{S*}\mu_{S*}F^{\Gamma}\to F,$$

$$\mathrm{Gr}(F)(U/S):\Gamma_U^{\vee}p^*F(U\times S/U\times S)\xrightarrow{\mathrm{ad}(l^*,l_*)(p^*F)(U\times S/U\times S)}h^*F(U/U)=F(U/S)$$

where $h:U\to S$ is a smooth morphism with $U\in\mathrm{Var}(\mathbb{C})$ and $h:U\xrightarrow{l}U\times S\xrightarrow{p}S$ is the graph factorization with l the graph embedding and p the projection.

Definition 110. *(i) For $S \in \mathrm{Var}(\mathbb{C})$, we have the filtered complexes of presheaves*

$$(\Omega^{\bullet,\Gamma}_{/S}, F_b) \in C_{O_S fil}(\mathrm{Var}(\mathbb{C})^2/S)$$

given by

- *for $((X,Z),h) = (X,Z)/S \in \mathrm{Var}(\mathbb{C})^2/S$,*

$$(\Omega^{\bullet,\Gamma}_{/S}((X,Z)/S), F_b) := \Gamma^{\vee,h}_Z L_{h^*O}(\Omega^{\bullet}_{X/S}, F_b)(X) := \mathbb{D}_{h^*O_S} L_{h^*O} \Gamma_Z E_{zar}(\mathbb{D}_{h^*O_S} L_{h^*O}(\Omega^{\bullet}_{X/S}, F_b))(X)$$

- *for $g : (X_1, Z_1)/S = ((X_1,Z_1),h_1) \to (X,Z)/S = ((X,Z),h)$ a morphism in $\mathrm{Var}(\mathbb{C})^2/S$,*

$$\begin{aligned}
\Omega^{\bullet,\Gamma}_{/S}(g) : \Gamma^{\vee,h}_Z L_{h^*O}(\Omega^{\bullet}_{X/S}, F_b))(X) &\xrightarrow{i_-} g^*\mathbb{D}_{h^*O_S} L_{h^*O} \Gamma_Z E_{zar}(\mathbb{D}_{h^*O_S} L_{h^*O}(\Omega^{\bullet}_{X/S}, F_b))(X_1) \\
&\xrightarrow{=} \mathbb{D}_{h_1^*O_S} L_{h_1^*O} g^* \Gamma_Z E_{zar}(\mathbb{D}_{h^*O_S} L_{h^*O}(\Omega^{\bullet}_{X/S}, F_b))(X_1) \\
&\xrightarrow{\mathbb{D}_{h_1^*O_S}((T(g,E)(-)\circ T(g,\gamma)(-)))^{-1}(X_1)} \mathbb{D}_{h_1^*O_S} L_{h_1^*O} \Gamma_{Z\times_X X_1} E_{zar}(g^*\mathbb{D}_{h^*O_S} L_{h^*O}(\Omega^{\bullet}_{X/S}, F_b))(X_1) \\
&\xrightarrow{=} \mathbb{D}_{h_1^*O_S} L_{h_1^*O} \Gamma_{Z\times_X X_1} E_{zar}(\mathbb{D}_{h_1^*O_S} L_{h_1^*O} g^*(\Omega^{\bullet}_{X/S}, F_b))(X_1) \\
&\xrightarrow{\mathbb{D}_{h_1^*O_S} T(Z_1/(Z\times_X X_1)\cap Z_1,\gamma)(-)(X_1)} \mathbb{D}_{h_1^*O_S} L_{h_1^*O} \Gamma_{Z_1} E_{zar}(\mathbb{D}_{h_1^*O_S} L_{h_1^*O} g^*(\Omega^{\bullet}_{X/S}, F_b))(X_1) \\
&\xrightarrow{\Gamma^{\vee,h}_{Z_1} L_{h_1^*O}(\Omega_{(X_1/X)/(S/S)})(X_1)} \mathbb{D}_{h_1^*O_S} L_{h_1^*O} \Gamma_{Z_1} E_{zar}(\mathbb{D}_{h_1^*O_S} L_{h_1^*O}(\Omega^{\bullet}_{X_1/S}, F_b))(X_1)
\end{aligned}$$

where i_- is the arrow of the inductive limit.

For $S \in \mathrm{SmVar}(\mathbb{C})$, we get

$$(\Omega^{\bullet,\Gamma}_{/S}, F_b) := \rho_{S*}(\Omega^{\bullet,\Gamma}_{/S}, F_b) \in C_{O_S fil, D_S}(\mathrm{Var}(\mathbb{C})^{2,sm}/S)$$

(ii) For $S \in \mathrm{SmVar}(\mathbb{C})$, we have the canonical map $C_{O_S fil, D_S}(\mathrm{Var}(\mathbb{C})^{sm}/S)$

$$\mathrm{Gr}^O(\Omega_{/S}) : \mathrm{Gr}^{12}_{S*} \mu_{S*}(\Omega^{\bullet,\Gamma}_{/S}, F_b) \to (\Omega^{\bullet}_{/S}, F_b)$$

given by, for $U/S = (U,h) \in \mathrm{Var}(\mathbb{C})^{sm}/S$

$$\begin{aligned}
\mathrm{Gr}^O(\Omega_{/S})(U/S) : \mathrm{Gr}^{12}_{S*} \mu_{S*}(\Omega^{\bullet,\Gamma}_{/S}, F_b)(U/S) &:= \Gamma^{\vee,h}_U L_{h^*O}(\Omega^{\bullet}_{U\times S/S}, F_b)(U\times S) \\
&\xrightarrow{\Gamma^{\vee,h}_U L_{h^*O}\,\mathrm{ad}(i_U^*, i_{U*})(-)(U\times S)} \Gamma^{\vee,h}_U L_{h^*O} i_{U*} i_U^*(\Omega^{\bullet}_{U\times S/S}, F_b)(U\times S) \\
&\xrightarrow{\Gamma^{\vee,h}_U L_{h^*O} i_{U*}\Omega_{(U/U\times S)/(S/S)}(U\times S)} \Gamma^{\vee,h}_U L_{h^*O} i_{U*}(\Omega^{\bullet}_{U/S}, F_b)(U\times S) \\
&\xrightarrow{=} (\Omega^{\bullet}_{U/S}, F_b)(U) =: (\Omega^{\bullet}_{/S}, F_b)(U/S)
\end{aligned}$$

where $h : U \xrightarrow{i_U} U\times S \xrightarrow{p_S} S$ is the graph factorization with i_U the graph embedding and p_S the projection.

We will use the following map from the property of mixed Hodge module (see section 5) together with the specialization map of a filtered D module for a closed embedding (see definition 48) :

Definition-Proposition 20. *(i) Let $l : Z \hookrightarrow S$ a closed embedding with $S, Z \in \mathrm{SmVar}(\mathbb{C})$. Consider an open embedding $j : S^o \hookrightarrow S$. We then have the cartesian square*

$$\begin{tikzcd}
S^o \arrow[r, "j"] & S \\
Z^o := Z\times_S S^o \arrow[u, "l'"] \arrow[r, "j'"] & Z \arrow[u, "l"]
\end{tikzcd}$$

where j' is the open embedding given by base change. Using proposition 95, the morphisms $Q^{p,0}_{V_Z,V_D}(O_S,F_b)$ for $D\subset S$ a closed subset of definition-proposition 15 induces a canonical morphism in $C_{l^*O_Sfil}(Z)$

$$Q(Z,j_!)(O_S,F_b):l^*Q_{V_Z,0}j_!^{Hdg}(O_{S^o},F_b)\to j_!^{'Hdg}(O_{Z^o},F_b),$$

where V_Z is the Kashiwara-Malgrange V_Z-filtration and V_D is the Kashiwara-Malgrange V_D-filtration, which commutes with the action of T_Z.

(ii) *Let $l:Z\hookrightarrow S$ and $k:Z'\hookrightarrow Z$ be closed embeddings with $S,Z,Z'\in\mathrm{SmVar}(\mathbb{C})$. Consider an open embedding $j:S^o\hookrightarrow S$. We then have the commutative diagram whose squares are cartesian.*

$$\begin{tikzcd} S^o \arrow[r, "j"] & S \\ Z^o:=Z\times_S S^o \arrow[u, "l'"] \arrow[r, "j'"] & Z \arrow[u, "l"] \\ Z'^o:=Z'\times_S S^o \arrow[u, "k'"] \arrow[r, "j''"] & Z' \arrow[u, "k"] \end{tikzcd}$$

where j' is the open embedding given by base change. Then,

$$Q(Z',j_!)(O_S,F_b)=Q(Z',j'_!)(O_Z,F_b)\circ(k^*Q_{V_{Z'},0}Q(Z,j_!)(O_S,F_b)):$$

$$k^*Q_{V_{Z'},0}l^*Q_{V_Z,0}j_!^{Hdg}(O_{S^o},F_b)\xrightarrow{k^*Q_{V_{Z'},0}Q(Z,j_!)(O_S,F_b)}k^*Q_{V_{Z'},0}j_!^{'Hdg}(O_{Z^o},F_b)$$
$$\xrightarrow{Q(Z',j'_!)(O_Z,F_b)}j_!^{''Hdg}(O_{Z'^o},F_b)$$

in $C_{k^*l^*O_Sfil}(Z')$ which commutes with the action of $T_{Z'}$.

(iii) *Consider a commutative diagram whose squares are cartesian*

$$\begin{tikzcd} S^{oo} \arrow[r, "j_2"] & S^o \arrow[r, "j_1"] & S \\ Z^{oo}:=Z\times_S S^{oo} \arrow[u, "l''"] \arrow[r, "j'_2"] & Z^o:=Z\times_S S^o \arrow[u, "l'"] \arrow[r, "j'_1"] & Z \arrow[u, "l"] \end{tikzcd}$$

where j_1, j_2, and hence j'_1,j'_2 are open embeddings. We have then the following commutative diagram

$$\begin{tikzcd} l^*Q_{V_Z,0}j_{1!}^{Hdg}(O_{S^o},F_b) \arrow[r, "\mathrm{ad}(j_{2!}^{Hdg},j_2^*)(O_{S^o},F_b)"] \arrow[d, "Q(Z,j_!)(O_S,F_b)"] & l^*Q_{V_Z,0}(j_1\circ j_2)_!^{Hdg}(O_{S^{oo}},F_b) \arrow[d, "Q(Z,(j_1\circ j_2)_!)(O_S,F_b)"] \\ j_{1!}^{'Hdg}(O_{Z^o},F_b) \arrow[r, "\mathrm{ad}(j_{2!}^{'Hdg},j_2^{'*})(O_{Z^o},F_b)"] & (j'_1\circ j'_2)_!^{Hdg}(O_{Z^{oo}},F_b) \end{tikzcd}$$

in $C_{l^*O_Sfil}(Z)$ which commutes with the action of T_Z.

Proof. (i): By definition of $j_!^{Hdg}:\pi_{S^o}(MHM(S^o))\to\pi_S(C(MHM(S)))$, we have to construct the isomorphism for each complement of a (Cartier) divisor $j=j_D:S^o=S\backslash D\hookrightarrow S$. In this case, we have the closed embedding $i:S\hookrightarrow L$ given by the zero section of the line bundle $L=L_D$ associated to D. We have then, using definition-proposition 15, the canonical morphism in $PSh_{l^*O_Sfil}(Z)$ which commutes with the action of T_Z

$$Q(Z,j_!)(O_S,F_b):l^*Q_{V_Z,0}j_!^{Hdg}(O_{S^o},F_b)\xrightarrow{T_!(l,j)(-)^{-1}}j_!^{Hdg}Q_{V_{Z^o},0}(O_{S^o},F_b)=j_!^{'Hdg}(O_{Z^o},F_b).$$

and $V_Z^p T_!(l,j)(-)^{-1} = Q^{p,0}_{V_Z,V_S}(i_{*mod}(O_S,F_b))$. Now for $j : S^o = S\backslash R \hookrightarrow S$ an arbitrary open embedding, we set

$$Q(Z,j_!)(O_S,F_b) := \varprojlim_{(D_i),R\subset D_i\subset S}(Q(Z,j_{D_J!})(j^*_{D_I}(O_S,F_b)) : l^*Q_{V_Z,0}j_!^{Hdg}(O_{S^o},F_b) \xrightarrow{\sim} j_!^{'Hdg}(O_{Z^o},F_b)$$

(ii): Follows from definition-proposition 15.
(iii): Follows from definition-proposition 15. $\square$

Using definition-proposition 19 in the projection case, and the specialization map given in definition 48 and the isomorphism of definition-proposition 20, in the closed embedding case, we have the following canonical map :

Definition 111. *Consider a commutative diagram in* $\mathrm{SmVar}(\mathbb{C})$ *whose square are cartesian*

$$\begin{tikzcd}
Z_T \arrow[rr, "i'"] \arrow[dr, "l'"] \arrow[dd, "g'"'] & & T \arrow[dr, "l"] \arrow[dd] & & T\backslash Z_T \arrow[ll, "j'"'] \arrow[dr, "l"] \arrow[dd] & \\
& T\times_g Z \arrow[rr, "I\times i"] \arrow[dl, "p_Z"] & & T\times_g S \arrow[dl, "p_S"] & & T\times S\backslash(T\times Z) \arrow[ll, "I\times j"] \arrow[dl, "p_S"] \\
Z \arrow[rr, "i"] & & S & & S\backslash Z \arrow[ll, "j"] &
\end{tikzcd}$$

where i and hence $I\times i$ and i', are closed embeddings, j, $I\times j$, j' are the complementary open embeddings and $g : T \xrightarrow{l} T\times S \xrightarrow{p_S} S$ is the graph factorization, where l is the graph embedding and p_S the projection. Then, the map in $C_{l^*O_{T\times S}fil}(T)$

$$sp_{V_T}(\Gamma^{\vee,Hdg}_{T\times Z}(O_{T\times S},F_b)) : l^*\Gamma^{\vee,Hdg}_{T\times Z}(O_{T\times S},F_b) \xrightarrow{q_{V_T,0}} l^*Q_{V_T,0}(\Gamma^{\vee,Hdg}_{T\times Z}(O_{T\times S},F_b))$$
$$\xrightarrow{Q(T,(I\times j)_!)(O_{T\times S},F_b):=T_!(l,(I\times j))(-)} \Gamma^{\vee,Hdg}_{Z_T}(O_T,F_b)$$

which commutes with the action of T_T, where the first map is given in definition 48 and the last map is studied definition-proposition 20, factors through

$$sp_{V_T}(\Gamma^{\vee,Hdg}_{T\times Z}(O_{T\times S},F_b)) : l^*\Gamma^{\vee,Hdg}_{T\times Z}(O_{T\times S},F_b) \xrightarrow{n} l^{*mod}\Gamma^{\vee,Hdg}_{T\times Z}(O_{T\times S},F_b)$$
$$\xrightarrow{\bar{sp}_{V_T}(\Gamma^{\vee,Hdg}_{T\times Z}(O_{T\times S},F_b))} \Gamma^{\vee,Hdg}_{Z_T}(O_T,F_b),$$

with for $U\subset T\times S$ an open subset, $m\in\Gamma(U,O_{T\times S})$ and $h\in\Gamma(U_T,O_T)$, $n(m) := n\otimes 1$ and $\bar{sp}_{V_T}(-)(m\otimes h) = h\cdot sp_{V_T}(m)$; see definition-proposition 19, proposition 95 and theorem 29. Then,

$$\bar{sp}_{V_T}(\Gamma^{\vee,Hdg}_{T\times Z}(O_{T\times S},F_b)) : l^{*mod}\Gamma^{\vee,Hdg}_{T\times Z}(O_{T\times S},F_b) \to \Gamma^{\vee,Hdg}_{Z_T}(O_T,F_b),$$

is a map in $C_{\mathcal{D}(1,0)fil}(T)$, i.e. is D_T linear. We then consider the canonical map in $C_{\mathcal{D}(1,0)fil}(T)$

$$a(g,Z)(O_S,F_b) : g^{*mod}\Gamma^{\vee,Hdg}_Z(O_S,F_b) = l^{*mod}p_S^{*mod}\Gamma^{\vee,Hdg}_Z(O_S,F_b) \xrightarrow{l^{*mod}T^{Hdg}(p,\gamma^\vee)(O_S,F_b)^{-1}}$$
$$l^{*mod}\Gamma^{\vee,Hdg}_{T\times Z}(O_{T\times S},F_b) \xrightarrow{\bar{sp}_{V_T}(\Gamma^{\vee,Hdg}_{T\times Z}(O_{T\times S},F_b))} \Gamma^{\vee,Hdg}_{Z_T}(O_T,F_b).$$

Lemma 11. *(i) For $g : T \to S$ and $g : T' \to T$ two morphism with $S, T, T' \in \mathrm{SmVar}(\mathbb{C})$, considering the commutative diagram whose squares are cartesian*

$$\begin{tikzcd} Z_{T'} \arrow[r, "i''"] \arrow[d, "g'"] & T' \arrow[d, "g'"] & T' \backslash Z_{T'} \arrow[l, "j''"'] \arrow[d, "g'"] \\ Z_T \arrow[r, "i'"] \arrow[d, "g"] & T \arrow[d, "g"] & T \backslash Z_T \arrow[l, "j'"'] \arrow[d, "g"] \\ Z \arrow[r, "i"] & S & S \backslash Z \arrow[l, "j"'] \end{tikzcd}$$

we have then

$$a(g \circ g', Z)(O_S, F_b) = a(g', Z_T)(O_T, F_b) \circ (g'^{*mod} a(g, Z)(O_S, F_b)) :$$
$$(g \circ g')^{*mod} \Gamma_Z^{\vee, Hdg}(O_S, F_b) = g'^{*mod} g^{*mod} \Gamma_Z^{\vee, Hdg}(O_S, F_b) \xrightarrow{g'^{*mod} a(g,Z)(O_S,F_b)} g'^{*mod} \Gamma_{Z_T}^{\vee, Hdg}(O_T, F_b)$$
$$\xrightarrow{a(g',Z_T)(O_T,F_b)} \Gamma_{Z_{T'}}^{\vee, Hdg}(O_{T'}, F_b).$$

(ii) For $g : T \to S$ a morphism with $S, T \in \mathrm{SmVar}(\mathbb{C})$, considering the commutative diagram whose squares are cartesian

$$\begin{tikzcd} Z'_T \arrow[r, "k'"] \arrow[d, "g"] & Z_T \arrow[r, "i'"] \arrow[d, "g"] & T \arrow[d, "g"] \\ Z' \arrow[r, "k"] & Z \arrow[r, "i"] & S \end{tikzcd}$$

we have then the following commutative diagram

$$\begin{tikzcd} g^{*mod} \Gamma_Z^{\vee, Hdg}(O_S, F_b) \arrow[r, "g^{*mod} T(Z'/Z, \gamma^{\vee, Hdg})(O_S, F_b)"] \arrow[d, "a(g,Z)(O_S,F_b)"] & g^{*mod} \Gamma_{Z'}^{\vee, Hdg}(O_S, F_b) \arrow[d, "a(g,Z')(O_S,F_b)"] \\ \Gamma_{Z_T}^{\vee, Hdg}(O_T, F_b) \arrow[r, "T(Z'_T/Z_T, \gamma^{\vee, Hdg})(O_T, F_b)"] & \Gamma_{Z'_T}^{\vee, Hdg}(O_T, F_b) \end{tikzcd}$$

Proof. (i):Follows from definition-proposition 20 (ii)
(ii):Follows from definition-proposition 20 (iii) □

We can now define the main object :

Definition 112. *(i) For $S \in \mathrm{SmVar}(\mathbb{C})$, we consider the filtered complexes of presheaves*

$$(\Omega^{\bullet, \Gamma, pr}_{/S}, F_{DR}) \in C_{D_S fil}(\mathrm{Var}(\mathbb{C})^{2, smpr}/S)$$

given by,

- *for $(Y \times S, Z)/S = ((Y \times S, Z), p) \in \mathrm{Var}(\mathbb{C})^{2, smpr}/S$,*

$$(\Omega^{\bullet, \Gamma, pr}_{/S}((Y \times S, Z)/S), F_{DR}) := ((\Omega^{\bullet}_{Y \times S/S}, F_b) \otimes_{O_{Y \times S}} \Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b))(Y \times S)$$

with the structure of $p^* D_S$ module given by proposition 60,

– *for* $g : (Y_1 \times S, Z_1)/S = ((Y_1 \times S, Z_1), p_1) \to (Y \times S, Z)/S = ((Y \times S, Z), p)$ *a morphism in* $\mathrm{Var}(\mathbb{C})^{2,smpr}/S$, *denoting for short* $\hat{Z} := Z \times_{Y\times S} (Y_1 \times S)$,

$$\begin{aligned}\Omega^{\bullet,\Gamma,pr}_{/S}(g) : ((\Omega^{\bullet}_{Y\times S/S}, F_b) \otimes_{O_{Y\times S}} \Gamma^{\vee,Hdg}_{Z}(O_{Y\times S}, F_b))(Y \times S) \\ \xrightarrow{i_-} g^*((\Omega^{\bullet}_{Y\times S/S}, F_b) \otimes_{O_{Y\times S}} \Gamma^{\vee,Hdg}_{Z}(O_{Y\times S}, F_b))(Y_1 \times S) \\ \xrightarrow{\Omega_{(Y_1\times S/Y\times S)/(S/S)}(\Gamma^{\vee,Hdg}_{Z}(O_{Y\times S},F_b))(Y_1\times S)} (\Omega^{\bullet}_{Y_1\times S/S}, F_b) \otimes_{O_{Y_1\times S}} g^{*mod}\Gamma^{\vee,Hdg}_{Z}(O_{Y\times S}, F_b))(Y_1 \times S) \\ \xrightarrow{DR(Y_1\times S/S)(a(g,Z)(O_{Y\times S},F_b))(Y_1\times S)} (\Omega^{\bullet}_{Y_1\times S/S}, F_b) \otimes_{O_{Y_1\times S}} \Gamma^{\vee,Hdg}_{\hat{Z}}(O_{Y_1\times S}, F_b))(Y_1 \times S) \\ \xrightarrow{DR(Y_1\times S/S)(T(Z_1/\hat{Z},\gamma^{\vee,Hdg})(O_{Y_1\times S},F_b))(Y_1\times S)} (\Omega^{\bullet}_{Y_1\times S/S}, F_b) \otimes_{O_{Y_1\times S}} \Gamma^{\vee,Hdg}_{Z_1}(O_{Y_1\times S}, F_b))(Y_1 \times S),\end{aligned}$$

where

∗ i_- *is the arrow of the inductive limit,*

∗ *we recall that*

$$\begin{aligned}\Omega_{(Y_1\times S/Y\times S)/(S/S)}(\Gamma^{\vee,Hdg}_{Z}(O_{Y\times S}, F_b)) : g^*((\Omega^{\bullet}_{Y\times S/S}, F_b) \otimes_{O_{Y\times S}} \Gamma^{\vee,Hdg}_{Z}(O_{Y\times S}, F_b)) \\ \to (\Omega^{\bullet}_{Y_1\times S/S}, F_b) \otimes_{O_{Y_1\times S}} g^{*mod}\Gamma^{\vee,Hdg}_{Z}(O_{Y\times S}, F_b))\end{aligned}$$

is the map given in definition-proposition 16, which is $p_1^*D_S$ *linear by proposition 63,*

∗ *the map*

$$a(g, Z)(O_{Y\times S}, F_b) : g^{*mod}\Gamma^{\vee,Hdg}_{Z}(O_{Y\times S}, F_b) \to \Gamma^{\vee,Hdg}_{\hat{Z}}(O_{Y_1\times S}, F_b)$$

is the map given in definition 111

∗ *the map*

$$T(Z_1/\hat{Z}, \gamma^{\vee,Hdg})(O_{Y_1\times S}, F_b) : \Gamma^{\vee,Hdg}_{\hat{Z}}(O_{Y_1\times S}, F_b) \to \Gamma^{\vee,Hdg}_{Z_1}(O_{Y_1\times S}, F_b)$$

is given in definition-proposition 19.

For $g : ((Y_1 \times S, Z_1), p_1) \to ((Y \times S, Z), p)$ *and* $g' : ((Y'_1 \times S, Z'_1), p_1) \to ((Y_1 \times S, Z_1), p)$ *two morphisms in* $\mathrm{Var}(\mathbb{C})^{2,smpr}/S$, *we have*

$$\begin{aligned}\Omega^{\bullet,\Gamma,pr}_{/S}(g \circ g') = \Omega^{\bullet,\Gamma,pr}_{/S}(g') \circ \Omega^{\bullet,\Gamma,pr}_{/S}(g) : ((\Omega^{\bullet}_{Y\times S/S}, F_b) \otimes_{O_{Y\times S}} \Gamma^{\vee,Hdg}_{Z}(O_{Y\times S}, F_b))(Y \times S) \\ \xrightarrow{\Omega^{\bullet,\Gamma,pr}_{/S}(g)} (\Omega^{\bullet}_{Y_1\times S/S}, F_b) \otimes_{O_{Y_1\times S}} \Gamma^{\vee,Hdg}_{Z_1}(O_{Y_1\times S}, F_b))(Y_1 \times S) \\ \xrightarrow{\Omega^{\bullet,\Gamma,pr}_{/S}(g')} (\Omega^{\bullet}_{Y'_1\times S/S}, F_b) \otimes_{O_{Y'_1\times S}} \Gamma^{\vee,Hdg}_{Z'_1}(O_{Y'_1\times S}, F_b))(Y'_1 \times S),\end{aligned}$$

since, denoting for short $\hat{Z} := Z \times_{Y\times S} (Y_1 \times S)$ *and* $\hat{Z}' := Z \times_{Y\times S} (Y'_1 \times S)$

– *we have by lemma 11(i)*

$$a(g \circ g', \hat{Z}')(O_{Y\times S}, F_b) = a(g', \hat{Z})(O_{Y_1\times S}, F_b) \circ g'^{*mod}a(g, Z)(O_{Y\times S}, F_b),$$

– *we have by lemma 11(ii)*

$$\begin{aligned}T(Z'_1/\hat{Z}', \gamma^{\vee,Hdg})(O_{Y'_1\times S}, F_b) \circ a(g', \hat{Z})(O_{Y_1\times S}, F_b) \\ = a(g', Z_1)(O_{Y_1\times S}, F_b) \circ g'^{*mod}T(Z_1/\hat{Z}, \gamma^{\vee,Hdg})(O_{Y_1\times S}, F_b).\end{aligned}$$

(ii) For $S\in\mathrm{SmVar}(\mathbb{C})$, we have the canonical map $C_{O_Sfil,D_S}(\mathrm{Var}(\mathbb{C})^{sm}/S)$

$$\mathrm{Gr}(\Omega_{/S}):\mathrm{Gr}^{12}_{S*}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})\to(\Omega^{\bullet}_{/S},F_b)$$

given by, for $U/S=(U,h)\in\mathrm{Var}(\mathbb{C})^{sm}/S$

$$\begin{aligned}\mathrm{Gr}(\Omega_{/S})(U/S):\mathrm{Gr}^{12}_{S*}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})(U/S):=((\Omega^{\bullet}_{U\times S/S},F_b)\otimes_{O_{U\times S}}\Gamma^{\vee,Hdg}_{U}(O_{U\times S},F_b))(U\times S)\\ \xrightarrow{\mathrm{ad}(i_U^*,i_{U*})(-)(U\times S)} i^*((\Omega^{\bullet}_{U\times S/S},F_b)\otimes_{O_{U\times S}}\Gamma^{\vee,Hdg}_{U}(O_{U\times S},F_b))(U)\\ \xrightarrow{\Omega_{(U/U\times S)/(S/S)}(-)(U)}((\Omega^{\bullet}_{U/S},F_b)\otimes_{O_U}i_U^{*mod}\Gamma^{\vee,Hdg}_{U}(O_{U\times S},F_b))(U)\\ \xrightarrow{DR(U/S)(a(i_U,U))(U)}(\Omega^{\bullet}_{U/S},F_b)(U)=:(\Omega^{\bullet}_{/S},F_b)(U/S)\end{aligned}$$

where $h:U\xrightarrow{i_U}U\times S\xrightarrow{p_S}S$ is the graph factorization with i_U the graph embedding and p_S the projection, note that $a(i_U,U)$ is an isomorphism since for $j_U:U\times S\backslash U\hookrightarrow U\times S$ the open complementary $i_U^{*mod}j_{U!}^{Hdg}(M,F,W)=0$.

Definition 113. *For $S\in\mathrm{SmVar}(\mathbb{C})$, we have the canonical map $C_{O_Sfil,D_S}(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$*

$$T(\Omega^{\Gamma}_{/S}):\mu_{S*}(\Omega^{\bullet,\Gamma}_{/S},F_b)\to(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})$$

given by, for $(Y\times S,X)/S=((Y\times S,Z),p)\in\mathrm{Var}(\mathbb{C})^{2,smpr}/S$

$$\begin{aligned}T(\Omega^{\Gamma}_{/S})((Y\times S,Z)/S):\\ (\Omega^{\bullet,\Gamma}_{/S},F_b)((Y\times S,Z)/S):=\mathbb{D}_{p^*O_S}L_{p^*O}\Gamma_ZE_{zar}(\mathbb{D}_{p^*O_S}L_{p^*O}(\Omega^{\bullet}_{Y\times S/S},F_b))(Y\times S)\\ \xrightarrow{DR(Y\times S/S)(\gamma^{\vee,Hdg}_{Z}(O_{Y\times S}))(Y\times S)}\\ \mathbb{D}_{p^*O_S}L_{p^*O}\Gamma_ZE_{zar}(\mathbb{D}_{p^*O_S}L_{p^*O}((\Omega^{\bullet}_{Y\times S/S},F_b)\otimes_{O_{Y\times S}}\Gamma^{\vee,Hdg}_{Z}(O_{Y\times S},F_b))(Y\times S)\xrightarrow{=}\\ ((\Omega^{\bullet}_{Y\times S/S},F_b)\otimes_{O_{Y\times S}}\Gamma^{\vee,Hdg}_{Z}(O_{Y\times S},F_b))(Y\times S)=:(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})((Y\times S,Z)/S).\end{aligned}$$

By definition $\mathrm{Gr}(\Omega_{/S})\circ\mathrm{Gr}_{S*}T(\Omega^{\Gamma}_{/S})=\mathrm{Gr}^{O}(\Omega_{/S})$.

Remark 11. *(i) Let $S\in\mathrm{Var}(\mathbb{C})$. We have by definition $o_{12*}(\Omega^{\bullet,\Gamma}_{/S},F_b)=(\Omega^{\bullet}_{/S},F_b)\in C_{O_Sfil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$. Moreover, if $S\in\mathrm{SmVar}(\mathbb{C})$, $o_{12*}(\Omega^{\bullet,\Gamma}_{/S},F_b)=(\Omega^{\bullet}_{/S},F_b)\in C_{O_Sfil,D_S}(\mathrm{Var}(\mathbb{C})^{sm}/S)$.*

(ii) Let $S\in\mathrm{Var}(\mathbb{C})$. Then, $(\Omega^{\bullet,\Gamma}_{/S},F_b)\in C_{O_Sfil}(\mathrm{Var}(\mathbb{C})^2/S)$ is a natural extension of

$$(\Omega^{\bullet,\Gamma}_{/S},F_b):=\rho_{S*}(\Omega^{\bullet,\Gamma}_{/S},F_b)\in C_{O_Sfil}(\mathrm{Var}(\mathbb{C})^{2,sm}/S),$$

but does NOT satisfy cdh descent.

We have the following canonical transformation map given by the pullback of (relative) differential forms:

- Let $g:T\to S$ a morphism with $T,S\in\mathrm{Var}(\mathbb{C})$. We have the canonical morphism in $C_{g^*O_Sfil,g^*D_S}(\mathrm{Var}(\mathbb{C})^{2,sm}/T)$

$$\Omega^{\Gamma}_{/(T/S)}:g^*(\Omega^{\bullet,\Gamma}_{/S},F_b)\to(\Omega^{\bullet,\Gamma}_{/T},F_b)$$

induced by the pullback of differential forms : for $((V,Z_1)/T)=((V,Z_1),h)\in\mathrm{Var}(\mathbb{C})^{2,sm}/T$,

$$\begin{aligned}\Omega^{\Gamma}_{/(T/S)}((V,Z_1)/T):\\ g^*\Omega^{\bullet,\Gamma}_{/S}((V,Z_1)/T):=\lim_{(h:(U,Z)\to S\mathrm{sm},g_1:(V,Z_1)\to(U_T,Z_T),h,g)}\Omega^{\bullet,\Gamma}_{/S}((U,Z)/S)\\ \xrightarrow{\Omega^{\bullet,\Gamma}_{/S}(g'\circ g_1)}\Omega^{\bullet,\Gamma}_{/S}((V,Z_1)/S)\xrightarrow{\Gamma^{\vee,h}_{Z_1}q(Y_1\times T)}\Omega^{\bullet,\Gamma}_{/T}((V,Z_1)/T),\end{aligned}$$

where $g':U_T:=U\times_S T\to U$ is the base change map and $q:\Omega^{\bullet}_{Y_1\times T/S}\to\Omega^{\bullet}_{Y_1\times T/T}$ is the quotient map. If $T,S\in\mathrm{SmVar}(\mathbb{C})$,

$$\Omega^{\Gamma}_{/(T/S)}:g^*(\Omega^{\bullet,\Gamma}_{/S},F_b)\to(\Omega^{\bullet,\Gamma}_{/T},F_b)$$

is a morphism in $C_{g^*O_Sfil,g^*D_S}(\mathrm{Var}(\mathbb{C})^{2,sm}/T)$ It induces the canonical morphisms in $C_{g^*O_Sfil,g^*D_S}(\mathrm{Var}(\mathbb{C})^{2,sm}/T)$:

$$E\Omega^{\Gamma}_{/(T/S)}:g^*E_{et}(\Omega^{\bullet,\Gamma}_{/S},F_b)\xrightarrow{T(g,E_{et})(\Omega^{\bullet,\Gamma}_{/S},F_b)}E_{et}(g^*(\Omega^{\bullet,\Gamma}_{/S},F_b))\xrightarrow{E_{et}(\Omega^{\Gamma}_{/(T/S)})}E_{et}(\Omega^{\bullet,\Gamma}_{/T},F_b)$$

and

$$E\Omega^{\Gamma}_{/(T/S)}:g^*E_{zar}(\Omega^{\bullet,\Gamma}_{/S},F_b)\xrightarrow{T(g,E_{zar})(\Omega^{\bullet,\Gamma}_{/S},F_b)}E_{zar}(g^*(\Omega^{\bullet,\Gamma}_{/S},F_b))\xrightarrow{E_{zar}(\Omega^{\Gamma}_{/(T/S)})}E_{zar}(\Omega^{\bullet,\Gamma}_{/T},F_b).$$

- Let $g:T\to S$ a morphism with $T,S\in\mathrm{SmVar}(\mathbb{C})$. We have the canonical morphism in $C_{g^*D_Sfil}(\mathrm{Var}(\mathbb{C})^{2,smpr}/T)$

$$\Omega^{\Gamma,pr}_{/(T/S)}:g^*(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})\to(\Omega^{\bullet,\Gamma,pr}_{/T},F_{DR})$$

induced by the pullback of differential forms : for $((Y_1\times T,Z_1)/T)=((Y_1\times T,Z_1),p)\in\mathrm{Var}(\mathbb{C})^{2,smpr}/T$,

$$\begin{aligned}&\Omega^{\Gamma,pr}_{/(T/S)}((Y_1\times T,Z_1)/T):\\ g^*\Omega^{\bullet,\Gamma,pr}_{/S}((Y_1\times T,Z_1)/T):=&\lim_{(h:(Y\times S,Z)\to S,\,g_1:(Y_1\times T,Z_1)\to(Y\times T,Z_T),h,g)}\Omega^{\bullet,\Gamma,pr}_{/S}((Y\times T,Z)/S)\\ &\xrightarrow{\Omega^{\bullet,\Gamma,pr}_{/S}(g'\circ g_1)}\Omega^{\bullet,\Gamma,pr}_{/S}((Y_1\times T,Z_1)/S)\xrightarrow{q(-)(Y_1\times T)}\Omega^{\bullet,\Gamma,pr}_{/T}((Y_1\times T,Z_1)/T),\end{aligned}$$

where $g'=(I_Y\times g):Y\times T\to Y\times S$ is the base change map and $q(M):\Omega_{Y_1\times T/S}\otimes_{O_{Y_1\times T}}(M,F)\to\Omega_{Y_1\times T/T}\otimes_{O_{Y_1\times T}}(M,F)$ is the quotient map. It induces the canonical morphisms in $C_{g^*D_Sfil}(\mathrm{Var}(\mathbb{C})^{2,smpr}/T)$:

$$E\Omega^{\Gamma,pr}_{/(T/S)}:g^*E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})\xrightarrow{T(g,E)(-)}E_{et}(g^*(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))\xrightarrow{E_{et}(\Omega^{\Gamma,pr}_{/(T/S)})}E_{et}(\Omega^{\bullet,\Gamma,pr}_{/T},F_{DR})$$

and

$$E\Omega^{\Gamma,pr}_{/(T/S)}:g^*E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})\xrightarrow{T(g,E)(-)}E_{zar}(g^*(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))\xrightarrow{E_{zar}(\Omega^{\Gamma,pr}_{/(T/S)})}E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/T},F_{DR}).$$

Definition 114. *Let $g:T\to S$ a morphism with $T,S\in\mathrm{SmVar}(\mathbb{C})$. We have, for $F\in C(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$, the canonical transformation in $C_{\mathcal{D}fil}(T)$:*

$$\begin{aligned}T(g,\Omega^{\Gamma,pr}_{/.})(F):g^{*mod}L_De(S)_*\,\mathrm{Gr}^{12}_{S*}\,\mathcal{H}om^{\bullet}(F,E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))\\ \xrightarrow{=}(g^*L_De(S)_*\mathcal{H}om^{\bullet}(F,E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})))\otimes_{g^*O_S}O_T\\ \xrightarrow{T(g,\mathrm{Gr}^{12})(-)\circ T(e,g)(-)\circ q}e(T)_*\,\mathrm{Gr}^{12}_{T*}\,g^*\mathcal{H}om^{\bullet}(F,E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))\otimes_{g^*O_S}O_T\\ \xrightarrow{(T(g,hom)(-,-)\otimes I)}e(T)_*\,\mathrm{Gr}^{12}_{T*}\,\mathcal{H}om^{\bullet}(g^*F,g^*E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))\otimes_{g^*O_S}O_T\\ \xrightarrow{ev(hom,\otimes)(-,-,-)}e(T)_*\,\mathrm{Gr}^{12}_{T*}\,\mathcal{H}om^{\bullet}(g^*F,g^*E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))\otimes_{g^*e(S)^*O_S}e(T)^*O_T\\ \xrightarrow{\mathcal{H}om^{\bullet}(g^*F,(E\Omega^{\Gamma,pr}_{/(T/S)}\otimes m))}e(T)_*\,\mathrm{Gr}^{12}_{T*}\,\mathcal{H}om^{\bullet}(g^*F,E_{et}(\Omega^{\bullet,\Gamma,pr}_{/T},F_{DR}))\end{aligned}$$

Let $S\in\mathrm{Var}(\mathbb{C})$. Recall that for and $h:U\to S$ a morphism with $U\in\mathrm{Var}(\mathbb{C})$, we have the canonical map given by the wedge product

$$w_{U/S}:\Omega^{\bullet}_{U/S}\otimes_{O_S}\Omega^{\bullet}_{U/S}\to\Omega^{\bullet}_{U/S};\alpha\otimes\beta\mapsto\alpha\wedge\beta.$$

Let $S\in\mathrm{Var}(\mathbb{C})$ and $h_1:U_1\to S$, $h_2:U_2\to S$ two morphisms with $U_1,U_2\in\mathrm{Var}(\mathbb{C})$. Denote $h_{12}:U_{12}:=U_1\times_S U_2\to S$ and $p_{112}:U_1\times_S U_2\to U_1$, $p_{212}:U_1\times_S U_2\to U_2$ the projections. Recall we have the canonical map given by the wedge product

$$w_{(U_1,U_2)/S}:p_{112}^*\Omega^{\bullet}_{U_1/S}\otimes_{O_S}p_{212}^*\Omega^{\bullet}_{U_2/S}\to\Omega^{\bullet}_{U_{12}/S};\alpha\otimes\beta\mapsto p_{112}^*\alpha\wedge p_{212}^*\beta$$

which gives the map

$$Ew_{(U_1,U_2)/S}:h_{1*}E_{zar}(\Omega^{\bullet}_{U_1/S})\otimes_{O_S}h_{2*}E_{zar}(\Omega^{\bullet}_{U_2/S})\to h_{12*}E_{zar}(p_{112}^*\Omega^{\bullet}_{U_1/S}\otimes_{O_S}p_{212}^*\Omega^{\bullet}_{U_2/S})$$

Let $S\in\mathrm{SmVar}(\mathbb{C})$.

- The complex of presheaves $(\Omega^{\bullet,\Gamma}_{/S},F_b)\in C_{O_Sfil,D_S}(\mathrm{Var}(\mathbb{C})^{2,sm}/S)$ have a monoidal structure given by the wedge product of differential forms: for $h:(U,Z)\to S\in\mathrm{Var}(\mathbb{C})^2/S$, the map

  $$DR(-)(\gamma^{\vee,h}_Z(-))\circ w_{U/S}:(\Omega^{\bullet}_{U/S},F_b)\otimes_{p^*O_S}(\Omega^{\bullet}_{U/S},F_b)\to\Gamma^{\vee,h}_Z L_{h^*O_S}(\Omega^{\bullet}_{U/S},F_b)$$

  factors trough

  $$\begin{aligned}&DR(-)(\gamma^{\vee,Hdg}_Z(-))\circ w_{U/S}:(\Omega^{\bullet}_{U/S},F_b)\otimes_{p^*O_S}(\Omega^{\bullet}_{U/S},F_b)\\&\xrightarrow{DR(-)(\gamma^{\vee,h}_Z(-))\otimes DR(-)(\gamma^{\vee,h}_Z(-))}\Gamma^{\vee,h}_Z L_{h^*O_S}(\Omega^{\bullet}_{U/S},F_b)\otimes_{p^*O_S}\Gamma^{\vee,h}_Z L_{h^*O_S}(\Omega^{\bullet}_{U/S},F_b)\\&\xrightarrow{(DR(-)(\gamma^{\vee,h}_Z(-))\circ w_{U/S})^{\gamma}}\Gamma^{\vee,h}_Z L_{h^*O_S}(\Omega^{\bullet}_{U/S},F_b)\end{aligned}$$

  unique up to homotopy, giving the map in $C_{O_Sfil,D_S}(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$:

  $$w_S:(\Omega^{\bullet,\Gamma}_{/S},F_b)\otimes_{O_S}(\Omega^{\bullet,\Gamma}_{/S},F_b)\to(\Omega^{\bullet,\Gamma}_{/S},F_b)$$

  given by for $h:(U,Z)\to S\in\mathrm{Var}(\mathbb{C})^{2,sm}/S$,

  $$\begin{aligned}&w_S((U,Z)/S):(\Gamma^{\vee,h}_Z L_{h^*O_S}(\Omega^{\bullet}_{U/S},F_b)\otimes_{p^*O_S}\Gamma^{\vee,h}_Z L_{h^*O_S}(\Omega^{\bullet}_{U/S},F_b))(U)\\&\xrightarrow{(DR(-)(\gamma^{\vee,h}_Z(-))\circ w_{U/S})^{\gamma}(U)}\Gamma^{\vee,h}_Z L_{h^*O_S}(\Omega^{\bullet}_{U/S},F_b)(U)\end{aligned}$$

  which induces the map in $C_{O_Sfil,D_S}(\mathrm{Var}(\mathbb{C})^{2,sm}/S)$

  $$Ew_S:E_{et}(\Omega^{\bullet,\Gamma}_{/S},F_b)\otimes_{O_S}E_{et}(\Omega^{\bullet,\Gamma}_{/S},F_b)\xrightarrow{=}E_{et}((\Omega^{\bullet,\Gamma}_{/S},F_b)\otimes_{O_S}(\Omega^{\bullet,\Gamma}_{/S},F_b))\xrightarrow{E_{et}(w_S)}E_{et}(\Omega^{\bullet,\Gamma}_{/S},F_b)$$

  given by the functoriality of the Godement resolution (see section 2).

- The complex of presheaves $(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})\in C_{D_Sfil}(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$ have a monoidal structure given by the wedge product of differential forms: for $p:(Y\times S,Z)\to S\in\mathrm{Var}(\mathbb{C})^{2,smpr}/S$, the map

  $$\begin{aligned}&DR(-)(\gamma^{\vee,Hdg}_Z(-))\circ w_{Y\times S/S}:(\Omega^{\bullet}_{Y\times S/S}\otimes_{O_{Y\times S}}(O_{Y\times S},F_b))\otimes_{p^*O_S}(\Omega^{\bullet}_{Y\times S/S}\otimes_{O_{Y\times S}}(O_{Y\times S},F_b))\\&\to\Omega^{\bullet}_{Y\times S/S}\otimes_{O_{Y\times S}}\Gamma^{\vee,Hdg}_Z(O_{Y\times S},F_b)\end{aligned}$$

factors trough

$$DR(-)(\gamma_Z^{\vee,Hdg}(-))\circ w_{Y\times S/S}:$$
$$(\Omega^{\bullet}_{Y\times S/S}\otimes_{O_{Y\times S}}(O_{Y\times S},F_b))\otimes_{p^*O_S}(\Omega^{\bullet}_{Y\times S/S}\otimes_{O_{Y\times S}}(O_{Y\times S},F_b))$$
$$\xrightarrow{DR(-)(\gamma_Z^{\vee,Hdg}(-))\otimes DR(-)(\gamma_Z^{\vee,Hdg}(-))}$$
$$(\Omega^{\bullet}_{Y\times S/S}\otimes_{O_{Y\times S}}\Gamma_Z^{\vee,Hdg})(O_{Y\times S},F_b)\otimes_{p^*O_S}\Omega^{\bullet}_{Y\times S/S}\otimes_{O_{Y\times S}}\Gamma_Z^{\vee,Hdg}(O_{Y\times S},F_b)$$
$$\xrightarrow{(DR(-)(\gamma_Z^{\vee,Hdg}(-))\circ w_{Y\times S/S})^{\gamma}}\Omega^{\bullet}_{Y\times S/S}\otimes_{O_{Y\times S}}\Gamma_Z^{\vee,Hdg}(O_{Y\times S},F_b)$$

unique up to homotopy, giving the map in $C_{D_Sfil}(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$:

$$w_S:(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})\otimes_{O_S}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})\to(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})$$

given by for $p:(Y\times S,Z)\to S\in\mathrm{Var}(\mathbb{C})^{2,smpr}/S$,

$$w_S((Y\times S,Z)/S):$$
$$(((\Omega^{\bullet}_{Y\times S/S}\otimes_{O_{Y\times S}}\Gamma_Z^{\vee,Hdg})(O_{Y\times S},F_b))\otimes_{p^*O_S}(\Omega^{\bullet}_{Y\times S/S}\otimes_{O_{Y\times S}}\Gamma_Z^{\vee,Hdg}(O_{Y\times S},F_b)))(Y\times S)$$
$$\xrightarrow{(DR(-)(\gamma_Z^{\vee,Hdg}(-))\circ w_{Y\times S/S})^{\gamma}(Y\times S)}(\Omega^{\bullet}_{Y\times S/S}\otimes_{O_{Y\times S}}\Gamma_Z^{\vee,Hdg}(O_{Y\times S},F_b))(Y\times S)$$

which induces the map in $C_{D_Sfil}(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$

$$Ew_S:E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})\otimes_{O_S}E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})\xrightarrow{=}$$
$$E_{et}((\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})\otimes_{O_S}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))\xrightarrow{E_{et}(w_S)}E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})$$

by the functoriality of the Godement resolution (see section 2).

Definition 115. *Let $S\in\mathrm{SmVar}(\mathbb{C})$. We have, for $F,G\in C(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$, the canonical transformation in $C_{\mathcal{D}fil}(S)$:*

$$T(\otimes,\Omega)(F,G):$$
$$e(S)_*\,\mathrm{Gr}^{12}_{S*}\,\mathcal{H}om(F,E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))\otimes_{O_S}e(S)_*\,\mathrm{Gr}^{12}_{S*}\,\mathcal{H}om(G,E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))$$
$$\xrightarrow{=}e(S)_*\,\mathrm{Gr}^{12}_{S*}(\mathcal{H}om(F,E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))\otimes_{O_S}\mathcal{H}om(G,E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})))$$
$$\xrightarrow{T(\mathcal{H}om,\otimes)(-)}e(S)_*\,\mathrm{Gr}^{12}_{S*}\,\mathcal{H}om(F\otimes G,E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})\otimes_{O_S}E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})))$$
$$\xrightarrow{=}e(S)_*\,\mathrm{Gr}^{12}_{S*}\,\mathcal{H}om(F\otimes G,(E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})\otimes_{O_S}E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})))$$
$$\xrightarrow{\mathcal{H}om(F\otimes G,Ew_S)}e(S)_*\,\mathrm{Gr}^{12}_{S*}\,\mathcal{H}om(F\otimes G,E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})).$$

Recall, see section 2 that we have the projection morphism of site $p_a:\mathrm{Var}(\mathbb{C})^{2,smpr}/S\to\mathrm{Var}(\mathbb{C})^{2,smpr}/S$ given by the functor $p_a((Y\times S,Z)/S)=(Y\times\mathbb{A}^1\times S,Z\times\mathbb{A}^1)/S$ and $p_a(g)=g\times I$.

We have the following key proposition :

Proposition 111. *Let $S\in\mathrm{Var}(\mathbb{C})$.*

(i) The complex of presheaves $\Omega^{\bullet,\Gamma}_{/S}\in C_{O_S}(\mathrm{Var}(\mathbb{C})^{2,sm}/S)$ is $\mathbb{A}^1$ homotopic and admits transferts (i.e. $\mathrm{Tr}(S)_*\,\mathrm{Tr}(S)^*\Omega^{\bullet,\Gamma}_{/S}=\Omega^{\bullet,\Gamma}_{/S}$).

(ii1) The complex of presheaves $(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})\in C_{D_Sfil}(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$ is 2-filtered $\mathbb{A}^1$ homotopic, that is

$$\mathrm{ad}(p_a^*,p_{a*})(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}):(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})\to p_{a*}p_a^*(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})$$

is a 2-filtered homotopy.

(ii2) The complex of presheaves $(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})\in C_{D_Sfil}(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$ admits transferts, i.e.

$$\mathrm{Tr}(S)_*\,\mathrm{Tr}(S)^*(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}=(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}).$$

(iii) Let $m:Q_1\to Q_2$ be an equivalence $(\mathbb{A}^1,et)$ local with $Q_1,Q_2\in C(\mathrm{Var}(\mathbb{C})^{smpr}/S)$ complexes of representable presheaves. Then,

$$e(S)_*\,\mathrm{Gr}^{12}_{S*}\,\mathcal{H}om^\bullet(m,E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})):e(S)_*\,\mathrm{Gr}^{12}_{S*}\,\mathcal{H}om^\bullet(Q_2,E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))$$
$$\to e(S)_*\,\mathrm{Gr}^{12}_{S*}\,\mathcal{H}om^\bullet(Q_1,E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))$$

is an 2-filtered quasi-isomorphism. It is thus an isomorphism in $D_{\mathcal{D}fil,\infty}(S)$.

Proof. (i):Follows from proposition 103.
(ii1):Let $(Y\times S,Z)/S=((Y\times S,Z),p)\in\mathrm{Var}(\mathbb{C})^{2,smpr}/S$ so that $p_a:(Y\times\mathbb{A}^1\times S,Z\times\mathbb{A}^1)\to(Y\times S,Z)$ is the projection, and $i_0:(Y\times S,Z)\to(Y\times\mathbb{A}^1\times S,Z\times\mathbb{A}^1)$ the closed embedding. Then,

$$a(p_a,Z):p_a^{*mod}\Gamma^{\vee,Hdg}_Z(O_{Y\times\times S},F_b))\to\Gamma^{\vee,Hdg}_{Z\times\mathbb{A}^1}(O_{Y\times\mathbb{A}^1\times S},F_b)).$$

a quasi-isomorphism in $\pi_{Y\times\mathbb{A}^1\times S}(C(MHM(Y\times\mathbb{A}^1\times S)))$. Since a morphism of mixed Hodge module is strict for the F-filtration,

$$a(p_a,Z):p_a^{*mod}\Gamma^{\vee,Hdg}_Z(O_{Y\times\times S},F_b)\to\Gamma^{\vee,Hdg}_{Z\times\mathbb{A}^1}(O_{Y\times\mathbb{A}^1\times S},F_b).$$

is a filtered quasi-isomorphism in $C_{\mathcal{D}fil}(Y\times\mathbb{A}^1\times S)$. Hence, as

$$I(p_a^*,p_{a*})(-,-)(\Omega_{(Y\times\mathbb{A}^1\times S/Y\times S)(S/S)}(\Gamma^{\vee,Hdg}_Z(O_{Y\times S},F_b))):$$
$$(\Omega^\bullet_{Y\times S/S},F_b)\otimes_{O_{Y\times S}}\Gamma^{\vee,Hdg}_Z(O_{Y\times S},F_b)$$
$$\to p_{a*}((\Omega^\bullet_{Y\times\mathbb{A}^1\times S/S},F_b)\otimes_{O_{Y\times\mathbb{A}^1\times S}}p_a^{*mod}\Gamma^{\vee,Hdg}_Z(O_{Y\times\times S},F_b))$$

is a 2-filtered homotopy equivalence whose inverse is

$$p_{a*}I(i_0^*,i_{0*})(-,-)(\Omega_{(Y\times S/Y\times\mathbb{A}^1\times S)(S/S)}(p_a^{*mod}(\Gamma^{\vee,Hdg}_Z(O_{Y\times S},F_b)))):$$
$$p_{a*}((\Omega^\bullet_{Y\times\mathbb{A}^1\times S/S},F_b)\otimes_{O_{Y\times\mathbb{A}^1\times S}}p_a^{*mod}(\Gamma^{\vee,Hdg}_Z(O_{Y\times\times S},F_b)))$$
$$\to p_{a*}i_{0*}(\Omega^\bullet_{Y\times S/S}\otimes_{O_{Y\times S}}(i_0^{*mod}p_a^{*mod}(\Gamma^{\vee,Hdg}_Z(O_{Y\times S},F_b))$$
$$\xrightarrow{=}(\Omega^\bullet_{Y\times S/S},F_b)\otimes_{O_{Y\times S}}(\Gamma^{\vee,Hdg}_Z(O_{Y\times S},F_b))$$

(see the proof of proposition 103), the map

$$\mathrm{ad}(p_a^*,p_{a*})(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}):(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})\to p_{a*}p_a^*(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})$$

is an homotopy equivalence whose inverse is

$$\mathrm{ad}(i_0^*,i_{0*})(p_{a*}p_a^*(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})):p_{a*}p_a^*(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})\to(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}).$$

(ii2):Let us shows that $\Omega^{\bullet,\Gamma,pr}_{/S}\in C_{\mathcal{D}_Sfil}(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$ admits transferts. Let $\alpha\in\mathrm{Cor}(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)((Y_1\times S,Z_1)/S,(Y_2\times S,Z_2)/S)$ irreducible. Denote by $i:\alpha\hookrightarrow Y_1\times Y_2\times S$ the closed embedding, and $p_1:Y_1\times Y_2\times S\to Y_1\times S$, $p_2:Y_1\times Y_2\times S\to Y_2\times S$ the projections. The morphism $p_1\circ i:\alpha\to Y_1\times S$ is then finite surjective and $(Z_1\times Y_2)\cap\alpha\subset Y_1\times Z_2$ (i.e. $p_2(p_1^{-1}(Z_2)\cap\alpha)\subset Z_2$). Then, the transfert map is given by

$$\begin{aligned}
\Omega^{\bullet,\Gamma,pr}_{/S}(\alpha) : ((\Omega^{\bullet}_{Y_2\times S/S},F_b)\otimes_{O_{Y_2\times S}}\Gamma^{\vee,Hdg}_{Z_2}(O_{Y_2\times S},F_b))(Y_2\times S) \\
\xrightarrow{i_-} p_2^*((\Omega^{\bullet}_{Y_2\times S/S},F_b)\otimes_{O_{Y_2\times S}}\Gamma^{\vee,Hdg}_{Z_2}(O_{Y_2\times S},F_b))(Y_1\times Y_2\times S) \\
\xrightarrow{\Omega_{(Y_1\times Y_2\times S/Y_2\times S)/(S/S)}(-)(-)} ((\Omega^{\bullet}_{Y_1\times Y_2\times S/S},F_b)\otimes_{O_{Y_1\times Y_2\times S}}\Gamma^{\vee,Hdg}_{Y_1\times Z_2}(O_{Y_1\times Y_2\times S},F_b))(Y_1\times Y_2\times S) \\
\xrightarrow{DR(-)(T((Z_1\times Y_2)\cap\alpha/Y_1\times Z_2,\gamma^{\vee,Hdg})(-)(-)} ((\Omega^{\bullet}_{Y_1\times Y_2\times S/S},F_b)\otimes_{O_{Y_1\times Y_2\times S}}\Gamma^{\vee,Hdg}_{(Z_1\times Y_2)\cap\alpha}(O_{Y_1\times Y_2\times S},F_b))(Y_1\times Y_2\times S) \\
\xrightarrow{i_-} i^*((\Omega^{\bullet}_{Y_1\times Y_2\times S/S},F_b)\otimes_{O_{Y_1\times Y_2\times S}}\Gamma^{\vee,Hdg}_{(Z_1\times Y_2)\cap\alpha}(O_{Y_1\times Y_2\times S},F_b))(\alpha) \\
\xrightarrow{\Omega_{(\alpha/Y_1\times Y_2\times S)/(S/S)}(-)(-)} ((\Omega^{\bullet}_{\alpha/S},F_b)\otimes_{O_\alpha} i^{*mod}\Gamma^{\vee,Hdg}_{(Z_1\times Y_2)\cap\alpha}(O_{Y_1\times Y_2\times S},F_b))(\alpha) \\
\xrightarrow{\Omega_{(\alpha/Y_1\times S)(S/S)}(-)(-)^{tr}} ((\Omega^{\bullet}_{Y_1\times S/S},F_b)\otimes_{O_{Y_1\times S}}\Gamma^{\vee,Hdg}_{Z_1}(O_{Y_1\times S},F_b))(Y_1\times S).
\end{aligned}$$

(iii): Denote for short $e'(S) := e(S)\circ\mathrm{Gr}^{12}_S$. By stictness of the Hodge filtration F of mixed hodge modules, it suffice to show that

$$\mathrm{Cone}(e'(S)_*\mathcal{H}om^\bullet(m,E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})))\to 0$$

there is a 2-filtered quasi-isomorphism. Indeed, if

$$(h,I,\phi):\mathrm{Cone}(e'(S)_*\mathcal{H}om^\bullet(m,E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})))[1]\to\mathrm{Cone}(e'(S)_*\mathcal{H}om^\bullet(m,E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})))$$

is a 2-filtered homotopy with $H^n\,\mathrm{Gr}^p_F\,\phi=0$ for all $p,n\in\mathbb{Z}$, then by strictness of the Hodge filtration F of mixed hodge modules,

- the canonical map

  $$a_1:\mathrm{Cone}(e'(S)_*\mathcal{H}om^\bullet(m,E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})))\to e'(S)_*\mathcal{H}om^\bullet(Q_2,E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))[1]$$

  will be a filtered homotopy equivalence with inverse a'_1
- the canonical map

  $$b_1:e'(S)_*\mathcal{H}om^\bullet(Q_1,E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))\to\mathrm{Cone}(e'(S)_*\mathcal{H}om^\bullet(m,E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})))$$

  will be a filtered homotopy equivalence with inverse b'_1

and

$$\begin{aligned}
(h\circ a'_1,e(S)_*\mathcal{H}om^\bullet(m,E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})),b'_1\circ\phi\circ a'_1): \\
e'(S)_*\mathcal{H}om^\bullet(Q_2,E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))[1]\to e'(S)_*\mathcal{H}om^\bullet(Q_1,E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))
\end{aligned}$$

will be a 2-filtered homotopy and $i_2\circ\phi\circ a'_1$ will be a filtered quasi-isomorphism.

By definition of an $(\mathbb{A}^1,et)$ local equivalence (see proposition 21), there exists

$$\{(Y_{1,\alpha}\times S,Z_{1,\alpha})/S,\alpha\in\Lambda_1\},\ldots,\{(Y_{s,\alpha}\times S,Z_{s,\alpha})/S,\alpha\in\Lambda_s\}\subset\mathrm{Var}(\mathbb{C})^{2,(sm)pr}/S$$

such that we have in $\mathrm{Ho}_{et}(C(\mathrm{Var}(\mathbb{C})^{2,(sm)}/S))$

$$\begin{aligned}
\mathrm{Cone}(m)\xrightarrow{\sim}\mathrm{Cone}(\oplus_{\alpha\in\Lambda_1}\mathrm{Cone}(\mathbb{Z}((Y_{1,\alpha}\times\mathbb{A}^1\times S,Z_{1,\alpha}\times\mathbb{A}^1)/S)\to\mathbb{Z}((Y_{1,\alpha}\times S,Z_{1,\alpha})/S)) \\
\to\cdots\to\oplus_{\alpha\in\Lambda_s}\mathrm{Cone}(\mathbb{Z}((Y_{s,\alpha}\times\mathbb{A}^1\times S,Z_{s,\alpha}\times\mathbb{A}^1)/S)\to\mathbb{Z}((Y_{s,\alpha}\times S,Z_{s,\alpha})/S)))
\end{aligned}$$

This gives in $D_{fil}(S) := \mathrm{Ho}_{zar,fil}(S)$,

$$\mathrm{Cone}(e'(S)_*\mathcal{H}om^\bullet(m, E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR}))) \xrightarrow{\sim}$$
$$\mathrm{Cone}(\oplus_{\alpha\in\Lambda_1}\mathrm{Cone}(e'(S)_*\mathcal{H}om^\bullet(\mathbb{Z}((Y_{1,\alpha}\times S, Z_{1,\alpha})/S), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR})) \to$$
$$e'(S)_*\mathcal{H}om^\bullet(\mathbb{Z}((Y_{1,\alpha}\times\mathbb{A}^1\times S, Z_{1,\alpha}\times\mathbb{A}^1)/S), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR})))$$
$$\to\cdots\to\oplus_{\alpha\in\Lambda_s}\mathrm{Cone}(e'(S)_*\mathcal{H}om^\bullet(\mathbb{Z}((Y_{s,\alpha}\times S, Z_{s,\alpha})/S), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR})) \to$$
$$e'(S)_*\mathcal{H}om^\bullet(\mathbb{Z}((Y_{s,\alpha}\times\mathbb{A}^1\times S, Z_{s,\alpha}\times\mathbb{A}^1)/S), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR}))))$$

Then by (ii1), for all $1\leq i\leq s$ and all $\alpha\in\Lambda_i$

$$\mathrm{Cone}(e'(S)_*\mathcal{H}om^\bullet(\mathbb{Z}((Y_{i,\alpha}\times S, Z_{i,\alpha})/S), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR})) \to$$
$$e'(S)_*\mathcal{H}om^\bullet(\mathbb{Z}((Y_{i,\alpha}\times\mathbb{A}^1\times S, Z_{i,\alpha}\times\mathbb{A}^1)/S), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR}))) \to 0$$

are 2-filtered homotopy equivalence. Hence $\mathrm{Cone}(e'(S)_*\mathcal{H}om^\bullet(m, E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR}))) \to 0$ is a 2-filtered quasi-isomorphism.

□

We now define the filtered De Rahm realization functor.

Definition 116. *(i) Let $S\in\mathrm{SmVar}(\mathbb{C})$. We have, using definition 112 and definition 34, the functor*

$$\mathcal{F}^{FDR}_S : C(\mathrm{Var}(\mathbb{C})^{sm}/S)\to C_{\mathcal{D}fil}(S),\ F\mapsto$$
$$\mathcal{F}^{FDR}_S(F) := e(S)_*\,\mathrm{Gr}^{12}_{S*}\,\mathcal{H}om^\bullet(L\rho_{S*}\mu_{S*}R^{CH}(\rho_S^*L(F)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR}))[-d_S]$$

(ii) Let $S\in\mathrm{Var}(\mathbb{C})$ and $S=\cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$. For $I\subset[1,\cdots l]$, denote by $S_I := \cap_{i\in I}S_i$ and $j_I : S_I\hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I\hookrightarrow\tilde{S}_I := \Pi_{i\in I}\tilde{S}_i$. Consider, for $I\subset J$, the following commutative diagram

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ \uparrow{\scriptstyle j_{IJ}} & & \uparrow{\scriptstyle p_{IJ}} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}$$

and $j_{IJ} : S_J\hookrightarrow S_I$ is the open embedding so that $j_I\circ j_{IJ} = j_J$. We have, using definition 112 and definition 34, the functor

$$\mathcal{F}^{FDR}_S : C(\mathrm{Var}(\mathbb{C})^{sm}/S)\to C_{\mathcal{D}fil}(S/(\tilde{S}_I)),\ F\mapsto$$
$$\mathcal{F}^{FDR}_S(F) := (e'(\tilde{S}_I)_*\mathcal{H}om^\bullet(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho^*_{\tilde{S}_I}L(i_{I*}j_I^*F)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u^q_{IJ}(F))$$

where we have denoted for short $e'(\tilde{S}_I)=e(\tilde{S}_I)\circ\mathrm{Gr}^{12}_{\tilde{S}_I}$, *and*

$$u^q_{IJ}(F)[d_{\tilde{S}_J}]:e'(\tilde{S}_I)_*\mathcal{H}om^\bullet(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho^*_{\tilde{S}_I}L(i_{I*}j^*_IF)),E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I},F_{DR}))$$

$$\xrightarrow{\mathrm{ad}(p^{*mod}_{IJ},p_{IJ})(-)}p_{IJ*}p^{*mod}_{IJ}e'(\tilde{S}_I)_*\mathcal{H}om^\bullet(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho^*_{\tilde{S}_I}L(i_{I*}j^*_IF)),E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I},F_{DR}))$$

$$\xrightarrow{p_{IJ*}T(p_{IJ},\Omega^{\gamma,pr}_{\cdot})(-)}p_{IJ*}e'(\tilde{S}_J)_*\mathcal{H}om^\bullet(L\rho_{\tilde{S}_J*}\mu_{\tilde{S}_J*}p^*_{IJ}R^{CH}(\rho^*_{\tilde{S}_I}L(i_{I*}j^*_IF)),E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_J},F_{DR}))$$

$$\xrightarrow{\mathcal{H}om(T(p_{IJ},R^{CH})(Li_{I*}j^*_IF)^{-1},E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_J},F_{DR}))}$$

$$p_{IJ*}e'(\tilde{S}_J)_*\mathcal{H}om^\bullet(L\rho_{\tilde{S}_J*}\mu_{\tilde{S}_J*}R^{CH}(\rho^*_{\tilde{S}_J}p^*_{IJ}L(i_{I*}j^*_IF)),E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_J},F_{DR}))$$

$$\xrightarrow{\mathcal{H}om(R^{CH}_{\tilde{S}_J}(T^q(D_{IJ})(j^*_IF)),E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_J},F_{DR}))}$$

$$p_{IJ*}e'(\tilde{S}_J)_*\mathcal{H}om^\bullet(L\rho_{\tilde{S}_J*}\mu_{\tilde{S}_J*}R^{CH}(\rho^*_{\tilde{S}_J}L(i_{J*}j^*_JF)),E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_J},F_{DR})).$$

For $I\subset J\subset K$, *we have obviously* $p_{IJ*}u_{JK}(F)\circ u_{IJ}(F)=u_{IK}(F)$. *We will prove in corollary 5 below that* $u_{IJ}(F)$ *are* ∞*-filtered Zariski local equivalence.*

We have the following key proposition :

Proposition 112. *Let* $S\in\mathrm{SmVar}(\mathbb{C})$.

(i) Let $m:Q_1\to Q_2$ *be an etale local equivalence local with* $Q_1,Q_2\in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ *complexes of projective presheaves. Then,*

$$e(S)_*\mathrm{Gr}^{12}_{S*}\mathcal{H}om^\bullet(L\rho_{S*}\mu_{S*}R^{CH}_S(m),E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))[-d_S]:$$
$$e(S)_*\mathrm{Gr}^{12}_{S*}\mathcal{H}om^\bullet(L\rho_{S*}\mu_{S*}R^{CH}(\rho^*_SQ_1),E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))[-d_S]$$
$$\to e(S)_*\mathrm{Gr}^{12}_{S*}\mathcal{H}om^\bullet(L\rho_{S*}\mu_{S*}R^{CH}(\rho^*_SQ_2),E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))[-d_S]$$

is a 2-filtered quasi-isomorphism. It is thus an isomorphism in $D_{\mathcal{D}fil,\infty}(S)$.

(ii) Let $m:Q_1\to Q_2$ *be an equivalence* $(\mathbb{A}^1,et)$ *local with* $Q_1,Q_2\in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ *complexes of representable presheaves. Then,*

$$e(S)_*\mathrm{Gr}^{12}_{S*}\mathcal{H}om^\bullet(L\rho_{S*}\mu_{S*}R^{CH}_S(m),E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))[-d_S]:$$
$$e(S)_*\mathrm{Gr}^{12}_{S*}\mathcal{H}om^\bullet(L\rho_{S*}\mu_{S*}R^{CH}(\rho^*_SQ_1),E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))[-d_S]$$
$$\to e(S)_*\mathrm{Gr}^{12}_{S*}\mathcal{H}om^\bullet(L\rho_{S*}\mu_{S*}R^{CH}(\rho^*_SQ_2),E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))[-d_S]$$

is an 2-filtered quasi-isomorphism. It is thus an isomorphism in $D_{\mathcal{D}fil,\infty}(S)$.

Proof. (i): If $m:Q_1\to Q_2$ is a quasi-isomorphism, hence an homotopy equivalence, then obviously $R^{CH}_S(m):R^{CH}(\rho^*_SQ_2)\to R^{CH}(\rho^*_SQ_1)$ is an homotopy equivalence, hence

$$e(S)_*\mathrm{Gr}^{12}_{S*}\mathcal{H}om^\bullet(L\rho_{S*}\mu_{S*}R^{CH}_S(m),E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))[-d_S]:$$
$$e(S)_*\mathrm{Gr}^{12}_{S*}\mathcal{H}om^\bullet(L\rho_{S*}\mu_{S*}R^{CH}(\rho^*_SQ_1),E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))[-d_S]$$
$$\to e(S)_*\mathrm{Gr}^{12}_{S*}\mathcal{H}om^\bullet(L\rho_{S*}\mu_{S*}R^{CH}(\rho^*_SQ_2),E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))[-d_S]$$

is an homotopy equivalence. Let $h:U\to S$ a smooth morphism with $U\in\mathrm{Var}(\mathbb{C})$. Note that U is smooth since S is smooth. Let $(r_i:U_i\to U)_{i\in L=[1,\ldots,s]}$ an etale cover of U. Consider the Chech cover

$$(U_\bullet/S):=((U_L,h_L):=(U_1\times_UU_2\times\cdots\times_UU_s,h\circ(r_1\times\ldots\times r_s))$$
$$\xrightarrow{r_{L,I}}\cdots\xrightarrow{i_{I,i}}\sqcup_{i\in L}(U_i,h_i):=\sqcup_{i\in L}(U_i,h\circ r_i))\xrightarrow{r=\sqcup_{i\in L}r_i}(U,h)$$

Take (see definition-proposition 12) a compactification of $r : (U_\bullet/S) \to U/S$

$$\bar{X}_{\bullet 0}/\bar{S} := ((\bar{X}_{L0}, \bar{f}_{L0}) \xrightarrow{\bar{r}_{L,I0}} \cdots \xrightarrow{\bar{r}_{Ii0}} \sqcup_{i\in L}(\bar{X}_i, \bar{f}_{i0})) \xrightarrow{\bar{r}_0} (\bar{X}_0, \bar{f}_0)$$

where $\bar{f}_0 : \bar{X}_0 \to \bar{S}$ is a compactification of $h : U \to S$ and $\bar{f}_{I0} : \bar{X}_{I0} \to \bar{S}$ is a compactification of $h_I : U_I := U_{i_1} \times U \times \cdots \times U_{i_l} \to S$ with equidimensional fibers. Denote $\bar{Z} := \bar{X}_0 \backslash U$ and $\bar{Z}_I := \bar{X}_{I0} \backslash U_I$. Take (see again definition-proposition 12) a strict resolution $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, Z)$ of the pair $(\bar{X}_0, \bar{Z})$ and strict resolutions $\bar{\epsilon}_I : (\bar{X}_I, \bar{D}_I) \to (\bar{X}_{I0}, \bar{Z}_I)$ of the pairs $(\bar{X}_{I0}, \bar{Z}_I)$, fitting in an other compactification of $r : (U_\bullet/S) \to U/S$

$$(\bar{X}_\bullet/S : (\bar{X}_L, \bar{f}_L) \xrightarrow{\bar{r}_{L,I}} \cdots \xrightarrow{\bar{r}_{Ii}} \sqcup_{i\in L}(\bar{X}_i, \bar{f}_i)) \xrightarrow{\bar{r}} (\bar{X}, \bar{f})$$

Denote by $j : U \hookrightarrow \bar{X}$ and $j_\bullet : U_\bullet \hookrightarrow \bar{X}_\bullet$ the open embeddings. We have the factorization

$$j_\bullet : U_\bullet \xrightarrow{j^{10}_\bullet} \bar{r}^{-1}(U) \xrightarrow{j^0_\bullet} \bar{X}^\bullet$$

where j^{10} and j^0 are open embeddings. Consider the graph embeddings $\bar{r} : \bar{X}_\bullet \xrightarrow{l_\bullet} \bar{X}_\bullet \times \bar{X} \xrightarrow{p_{I_o}} X$ Denote for short $j^s := j \times I : U \times S \hookrightarrow \bar{X} \times S$, $j^s_\bullet := j_\bullet \times I : U_\bullet \times S \hookrightarrow \bar{X}_\bullet \times S$ and $\bar{r}^s := \bar{r} \times I : \bar{X}_\bullet \times S \to \bar{X} \times S$ We have the factorization

$$j^s_\bullet : U_\bullet \times S \xrightarrow{j^{s10}_\bullet := (j^{10}_\bullet \times I)} \bar{r}^{s,-1}(U \times S) \xrightarrow{j^{s0}_\bullet := (j^0_\bullet \times I)} \bar{X}^\bullet \times S$$

We have then by definition the following commutative diagram, with for short $e = e(S) \circ \mathrm{Gr}^{12}_S$,

$$\begin{tikzcd}
e_*\mathcal{H}om^\bullet(L\rho_{S*}\mu_{S*}R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR})) \arrow[r, "\mathcal{H}om(T^{\mu,q}(p_{S\sharp},p_{S*})(-,-)"] & p_{S*}E_{et}((\Omega^\bullet_{\bar{X}\times S/S}, F_b) \otimes_{O_{X\times S}} j^{s,Hdg}_!(O_{U\times S}, F_b))(-d_X) \arrow[d, "{'p_{\bar{X}*}E_{et}(\Omega_{(\bar{X}_\bullet\times S/\bar{X}\times S)/(S/S)}(j^{s,Hdg}_!(O_{U\times S},F_b))\circ\mathrm{ad}(\bar{r}^{s*},\bar{r}^s_*)(-))}"'] \\
 & p_{S*}E_{et}((\Omega^\bullet_{\bar{X}_\bullet\times S/S}, F_b) \otimes_{O_{\bar{X}_\bullet\times S}} \bar{r}^{s*mod}j^{s,Hdg}_!(O_{U\times S}, F_b)(- \arrow[d, "{'DR(\bar{X}_\bullet\times S/S)(G(U_\bullet\times S,j^s_!)(O_{U\times S},F_b)\circ T^{\vee,Hdg}(p_{I_o}\times I,j_s)(-))}"'] \\
 & p_{S*}E_{et}((\Omega^\bullet_{X_\bullet\times S/S}, F_b) \otimes_{O_{\bar{X}_\bullet\times S}} j^{s0,Hdg}_{\bullet !}(O_{\bar{r}_s^{-1}(U\times S)}, F_b))(- \\
e_*\mathcal{H}om^\bullet(L\rho_{S*}\mu_{S*}R_{(\bar{X}_\bullet,\bar{D}_\bullet)/S}(\mathbb{Z}(U_\bullet/S)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR})) \arrow[uuu, "{\mathcal{H}om^\bullet(L\rho_{S*}\mu_{S*}R^{CH}(r),-)}"] \arrow[r, "\mathcal{H}om(T^{\mu,q}(p_{S\sharp},p_{S*})(-,-)"] & p_{S*}E_{et}((\Omega^\bullet_{\bar{X}_\bullet\times S/S}, F_b) \otimes_{O_{\bar{X}_\bullet\times S}} j^{s,Hdg}_{\bullet !}(O_{U_\bullet\times S}, F_b))(-d \arrow[u, "{DR(\bar{X}_\bullet\times S/S)(j^{s0,Hdg}_{\bullet !}\,\mathrm{ad}(j^{s10,Hdg}_{\bullet !},j^{s10*}_\bullet)(O_{\bar{r}_s^{-1}(U\times S)},F_b))}"']
\end{tikzcd}$$

where,

- the map in $C(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$

$$\begin{aligned}
T^{\mu,q}(p_{S\sharp}, p_{S*})(\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/\bar{X} \times S), \mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S)) \\
\mathrm{Cone}(\mathbb{Z}(i_\bullet \times I) : (\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/S), u_{IJ}) \to \mathbb{Z}((\bar{X} \times S, X)/S)) \to \\
L\rho_{S*}\mu_{S*}p_{S*}E_{et}\,\mathrm{Cone}(\mathbb{Z}(i_\bullet \times I) : L\rho_{S*}\mu_{S*}p_{S*}E_{et}((\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/\bar{X} \times S), u_{IJ}) \to \\
\mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S)))(d_X)[2d_X] =: L\rho_{S*}\mu_{S*}R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S))(d_X)[2d_X]
\end{aligned}$$

  given in definition 35 is an equivalence $(\mathbb{A}^1, et)$ local by proposition 36,

- the map in $C(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$

$$\begin{aligned}
T^{\mu,q}(p_{S\sharp}, p_{S*})(\mathbb{Z}((\bar{D}_{\bullet,\bullet} \times S, D_{\bullet,\bullet})/\bar{X}_\bullet \times S), \mathbb{Z}((\bar{X}_\bullet \times S, X_\bullet)/\bar{X}_\bullet \times S)) \\
\mathrm{Cone}(\mathbb{Z}(i_{\bullet,\bullet} \times I) : (\mathbb{Z}((\bar{D}_{\bullet,\bullet} \times S, D_{\bullet,\bullet})/S), u_{IJ}) \to \mathbb{Z}((\bar{X}_\bullet \times S, X_\bullet)/S)) \to \\
L\rho_{S*}\mu_{S*}p_{S*}E_{et}\,\mathrm{Cone}(\mathbb{Z}(i_{\bullet,\bullet} \times I) : L\rho_{S*}\mu_{S*}p_{S*}E_{et}((\mathbb{Z}((\bar{D}_{\bullet,\bullet} \times S, D_{\bullet,\bullet})/\bar{X}_\bullet \times S), u_{IJ}) \to \\
\mathbb{Z}((\bar{X}_\bullet \times S, X_\bullet)/\bar{X}_\bullet \times S)))(d_X)[2d_X] =: L\rho_{S*}\mu_{S*}R_{(\bar{X}_\bullet,\bar{D}_\bullet)/S}(\mathbb{Z}(U_\bullet/S))(d_X)[2d_X]
\end{aligned}$$

  given in definition 35 is an equivalence $(\mathbb{A}^1, et)$ local by proposition 36.

Now, the two horizontal arrows are 2-filtered quasi-isomorphism by proposition 111. The arrow given by the composition of the two maps of the right column is a filtered quasi-isomorphism by proposition 100. The upper arrow of the right column is a filtered quasi-isomorphism by proposition 98. Hence the arrow of the left column is a 2-filtered quasi-isomorphism which proves (i).
(ii):It is enough to consider the case of $m = \mathbb{Z}(p) : \mathbb{Z}(U \times \mathbb{A}^1/S) \to \mathbb{Z}(U/S)$ where $p : U \times \mathbb{A}^1 \to U$ is the projection. So, let $h : U \to S$ a smooth morphism with $U \in \mathrm{Var}(\mathbb{C})$ and $p : U \times \mathbb{A}^1 \to U$ the projection. Let $\bar{f}_0 : \bar{X}_0 \to \bar{S}$ is a compactification of $h : U \to S$ (see definition-proposition 12) Denote $\bar{Z} := \bar{X}_0 \backslash U$ and $\bar{Z}_I := \bar{X}_{I0} \backslash U_I$. Then $\bar{p}_0 : \bar{X}_0 \times \mathbb{P}^1 \to \bar{X}_0$ is a compactification of $p : U \times \mathbb{A}^1 \to U$. Take (see again definition-proposition 12) a strict resolution $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ of the pair $(\bar{X}_0, \bar{Z})$. Then

$$\epsilon' : (\bar{X} \times \mathbb{P}^1, (\bar{D} \times \mathbb{P}^1) \cup (\bar{X} \times \{\infty\})) \to (\bar{X}_0 \times \mathbb{P}^1, \bar{Z} \times \mathbb{P}^1) \cup (\bar{X}_0 \times \{\infty\})$$

is a strict resolution of the pair $(\bar{X}_0 \times \mathbb{P}^1, \bar{Z} \times \mathbb{P}^1) \cup (\bar{X}_0 \times \{\infty\})$. Denote by $j : U \hookrightarrow X$ the open embedding. Denote for short $j^s := j \times I : U \times S \hookrightarrow \bar{X} \times S$, $j^{sa} := j \times j_a \times I : U \times \mathbb{A}^1 \times S \hookrightarrow X \times \mathbb{P}^1 \times S$ $\bar{p}^s := \bar{p} \times I : \bar{X} \times \mathbb{P}^1 \times S \to \bar{X} \times S$. We have then by definition the following commutative diagram, with for short $e = e(S) \circ \mathrm{Gr}_S^{12}$,

$$\begin{tikzcd}
e_* \mathcal{H}om^\bullet(L\rho_{S*}\mu_{S*}R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR})) \arrow[r, "\mathcal{H}om(T^{\mu,q}(p_{S\sharp},p_{S*})(-),-)"] & p_{S*}E_{et}((\Omega^\bullet_{\bar{X}\times S/S}, F_b) \otimes_O j_!^{s,Hdg}(O_{U\times S}, F_b)) \arrow[d, "'p_{S*}E_{et}(\Omega_{(\bar{X}\times\mathbb{P}^1\times S/\bar{X}\times S)/(S/S)}(j_!^{s,Hdg}(O_{U\times S},F_b))\circ\mathrm{ad}(\bar{p}^*,\bar{p}_*)(-))"'] \\
 & p_{S*}E_{et}((\Omega^\bullet_{\bar{X}\times\mathbb{P}^1\times S/S}, F_b) \otimes_O \bar{p}^{s*mod} j_!^{s,Hdg}(O_{U\times S} \arrow[d, "'DR(\bar{X}\times\mathbb{P}^1\times S/S)(T^{\vee,Hdg}(\bar{p}^s,j^s)(-))"'] \\
e_* \mathcal{H}om^\bullet(L\rho_{S*}\mu_{S*}R_{(\bar{X}\times\mathbb{P}^1,-)/S}(\mathbb{Z}(U\times\mathbb{A}^1/S)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR})) \arrow[uu, "\mathcal{H}om^\bullet(L\rho_{S*}\mu_{S*}R^{CH}(p),-)"] \arrow[r, "\mathcal{H}om(T^{\mu,q}((p_S\circ\bar{p}_s)_\sharp,(p_S\circ\bar{p}_s)_*)(-),-)"] & p_{S*}E_{et}((\Omega^\bullet_{\bar{X}\times\mathbb{P}^1\times S/S}, F_b) \otimes_O j_!^{sa,Hdg}(O_{U\times\mathbb{A}^1\times S}, F_b))
\end{tikzcd}$$

where,

- the map in $C(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$

$$\begin{aligned} T^{\mu,q}(p_{S\sharp}, p_{S*})(\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/\bar{X} \times S), \mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S)) \\ \mathrm{Cone}(\mathbb{Z}(i_\bullet \times I) : (\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/S), u_{IJ}) \to \mathbb{Z}((\bar{X} \times S, X)/S)) \to \\ L\rho_{S*}\mu_{S*}p_{S*}E_{et}\,\mathrm{Cone}(\mathbb{Z}(i_\bullet \times I) : L\rho_{S*}\mu_{S*}p_{S*}E_{et}((\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/\bar{X} \times S), u_{IJ}) \to \\ \mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S)))(d_X)[2d_X] =: L\rho_{S*}\mu_{S*}R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S))(d_X)[2d_X] \end{aligned}$$

  given in definition 35 is an equivalence $(\mathbb{A}^1, et)$ local by proposition 36,

- the map in $C(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$

$$\begin{aligned} T^{\mu,q}((p_S \circ \bar{p}_s)_\sharp, (p_S \circ \bar{p}_s)_*)(\mathbb{Z}((\bar{D}_\bullet \times \mathbb{P}^1 \times S, D_\bullet \times \mathbb{P}^1)/\bar{X} \times \mathbb{P}^1 \times S), \mathbb{Z}((\bar{X} \times \mathbb{P}^1 \times S, X \times \mathbb{P}^1)/\bar{X} \times \mathbb{P}^1 \times S)) : \\ \mathrm{Cone}(\mathbb{Z}(i_\bullet \times I) : (\mathbb{Z}((\bar{D}_\bullet \times \mathbb{P}^1 \times S, D_\bullet \times \mathbb{P}^1)/S), u_{IJ}) \to \mathbb{Z}((\bar{X} \times \mathbb{P}^1 \times S, X \times \mathbb{P}^1)/S)) \to \\ L\rho_{S*}\mu_{S*}p_{S*}E_{et}\,\mathrm{Cone}(\mathbb{Z}(i_\bullet \times I) : L\rho_{S*}\mu_{S*}p_{S*}E_{et}((\mathbb{Z}((\bar{D}_\bullet \times \mathbb{P}^1 \times S, D_\bullet \times \mathbb{P}^1)/\bar{X} \times \mathbb{P}^1 \times S), u_{IJ}) \to \\ \mathbb{Z}((\bar{X} \times \mathbb{P}^1 \times S, X \times \mathbb{P}^1)/\bar{X} \times \mathbb{P}^1 \times S)))(d_X + 1)[2d_X + 2] \\ =: L\rho_{S*}\mu_{S*}R_{(\bar{X}\times\mathbb{P}^1,\bar{D}\times\mathbb{P}^1)/S}(\mathbb{Z}(U \times \mathbb{A}^1/S))(d_X + 1)[2d_X + 2] \end{aligned}$$

  given in definition 35 is an equivalence $(\mathbb{A}^1, et)$ local by proposition 36.

Now, by proposition 111, the two horizontal arrows of the right column are 2-filtered quasi-isomorphism. The arrow given by the composition of the two maps of the right column is a filtered quasi-isomorphism by proposition 100. Hence the arrow of the left column is a 2-filtered quasi-isomorphism which proves (ii). $\square$

Definition 117. *(i) Let $S \in \mathrm{SmVar}(\mathbb{C})$. We define using definition 116(i) and proposition 112(ii) the filtered algebraic De Rahm realization functor defined as*

$$\mathcal{F}_S^{FDR} : \mathrm{DA}_c(S) \to D_{\mathcal{D}fil,\infty}(S),\ M \mapsto$$
$$\mathcal{F}_S^{FDR}(M) := e(S)_* \operatorname{Gr}_{S*}^{12} \mathcal{H}om^\bullet(L\rho_{S*}\mu_{S*}R^{CH}(\rho_S^* L(F)), E_{et}(\Omega_{/S}^{\bullet,\Gamma,pr}, F_{DR}))[-d_S]$$

where $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$.

(i)' For the Corti-Hanamura weight structure W on $\mathrm{DA}_c(S)^-$, we define using definition 116(i) and proposition 112(ii)

$$\mathcal{F}_S^{FDR} : \mathrm{DA}_c^-(S) \to D^-_{\mathcal{D}(1,0)fil,\infty}(S),\ M \mapsto$$
$$\mathcal{F}_S^{FDR}((M,W)) := e(S)_* \operatorname{Gr}_{S*}^{12} \mathcal{H}om^\bullet(L\rho_{S*}\mu_{S*}R^{CH}(\rho_S^* L(F,W)), E_{et}(\Omega_{/S}^{\bullet,\Gamma,pr}, F_{DR}))[-d_S]$$

where $(F,W) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)((F,W))$ using corollary 1. Note that the filtration induced by W is a filtration by sub D_S module, which is a stronger property then Griffitz transversality. Of course, the filtration induced by F satisfy only Griffitz transversality in general.

(ii) Let $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. For $I \subset [1, \cdots l]$, denote by $S_I = \cap_{i\in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \Pi_{i\in I}\tilde{S}_i$. We define, using definition 116(ii), proposition 112(ii) and corollary 5, the filtered algebraic De Rahm realization functor defined as

$$\mathcal{F}_S^{FDR} : \mathrm{DA}_c(S) \to D_{\mathcal{D}fil,\infty}(S/(\tilde{S}_I)),\ M \mapsto$$
$$\mathcal{F}_S^{FDR}(M) := (e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*}j_I^*F)), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet,\Gamma,pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F))$$

where $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$, see definition 116.

(ii)' For the Corti-Hanamura weight structure W on $\mathrm{DA}_c^-(S)$, using definition 116(ii), proposition 112(ii) and corollary 5,

$$\mathcal{F}_S^{FDR} : \mathrm{DA}_c^-(S) \to D^-_{\mathcal{D}(1,0)fil,\infty}(S/(\tilde{S}_I)),\ M \mapsto \mathcal{F}_S^{FDR}((M,W)) :=$$
$$(e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*}j_I^*(F,W))), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet,\Gamma,pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F,W))$$

where $(F,W) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is such that $(M,W) = D(\mathbb{A}^1, et)(F,W)$ using corollary 1. Note that the filtration induced by W is a filtration by sub $D_{\tilde{S}_I}$-modules, which is a stronger property then Griffitz transversality. Of course, the filtration induced by F satisfy only Griffitz transversality in general.

Proposition 113. *For $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$, the functor $\mathcal{F}_S^{FDR}$ is well defined.*

Proof. Let $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. Denote, for $I \subset [1, \cdots, l]$, $S_I = \cap_{i\in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \Pi_{i\in I}\tilde{S}_i$. Let $M \in \mathrm{DA}(S)$. Let $F, F' \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}_1, et)(F) = D(\mathbb{A}_1, et)(F')$. Then there exist by definition a sequence of morphisms in $C(\mathrm{Var}(\mathbb{C})^{sm}/S)$:

$$F = F_1 \xrightarrow{s_1} F_2 \xleftarrow{s_2} F_3 \xrightarrow{s_3} F_4 \to \cdots \xrightarrow{s_l} F' = F_s$$

where, for $1 \le k \le s$, and s_k are $(\mathbb{A}^1, et)$ local equivalence. But if $s : F_1 \to F_2$ is an equivalence $(\mathbb{A}^1, et)$ local,

$$L(i_{I*}j_I^*s) : L(i_{I*}j_I^*F_1) \to L(i_{I*}j_I^*F_2)$$

is an equivalence $(\mathbb{A}^1, et)$ local, hence

$$\begin{aligned}&\mathcal{H}om(R^{CH}_{\tilde{S}_I}(L(i_{I*}j_I^*s)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR})) : \\ &(e'(\tilde{S}_I)_*\mathcal{H}om(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho^*_{\tilde{S}_I}L(i_{I*}j_I^*F_1)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR})), u^q_{IJ}(F_1)) \\ &\to (e'(\tilde{S}_I)_*\mathcal{H}om(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho^*_{\tilde{S}_I}L(i_{I*}j_I^*F_2)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR})), u^q_{IJ}(F_2))\end{aligned}$$

is an ∞-filtered quasi-isomorphism by proposition 112. $\square$

Let $f : X \to S$ a morphism with $S, X \in \mathrm{Var}(\mathbb{C})$. Assume there exists a factorization

$$f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$$

of f, with l a closed embedding, $Y \in \mathrm{SmVar}(\mathbb{C})$ and p_S the projection. Let $\bar{Y} \in \mathrm{PSmVar}(\mathbb{C})$ a smooth compactification of Y with $\bar{Y}\backslash Y = D$ a normal crossing divisor, denote $k : D \hookrightarrow \bar{Y}$ the closed embedding and $n : Y \hookrightarrow \bar{Y}$ the open embedding. Denote $\bar{X} \subset \bar{Y} \times S$ the closure of $X \subset \bar{Y} \times S$. We have then the following commutative diagram in $\mathrm{Var}(\mathbb{C})$

$$\begin{tikzcd} X \arrow[r, "l"] \arrow[d] & Y \times S \arrow[d, "(n\times I)"] \arrow[dr, "p_S"] & \\ \bar{X} \arrow[r, "l"] & \bar{Y} \times S \arrow[r, "\bar{p}_S"] & S \\ Z := \bar{X}\backslash X \arrow[r] \arrow[u] \arrow[ur, "l_Z"] & D \times S \arrow[u, "(k\times I)"] \arrow[ur] & \end{tikzcd} .$$

Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. We have $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. Denote, for $I \subset [1, \cdots l]$, $S_I = \cap_{i\in I} S_i$ and $X_I = \cap_{i\in I} X_i$. For $I \subset [1, \cdots l]$, denote by $\tilde{S}_I = \Pi_{i\in I}\tilde{S}_i$. We then have, for $I \subset [1, \cdots l]$, closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I$ and for $I \subset J$, the following commutative diagrams which are cartesian

$$\begin{tikzcd} f_I = f_{|X_I} : X_I \arrow[r, "l_I"] \arrow[dr] & Y \times S_I \arrow[r, "p_{S_I}"] \arrow[d, "i'_I"] & S_I \arrow[d, "i_I"] \\ & Y \times \tilde{S}_I \arrow[r, "p_{\tilde{S}_I}"] & \tilde{S}_I \end{tikzcd} , \quad \begin{tikzcd} Y \times \tilde{S}_J \arrow[r, "p_{\tilde{S}_J}"] \arrow[d, "p'_{IJ}"'] & \tilde{S}_J \arrow[d, "p_{IJ}"] \\ Y \times \tilde{S}_I \arrow[r, "p_{\tilde{S}_I}"] & \tilde{S}_I \end{tikzcd} ,$$

with $l_I : l_{|X_I}$, $i'_I = I \times i_I$, p_{S_I} and $p_{\tilde{S}_I}$ are the projections and $p'_{IJ} = I \times p_{IJ}$, and we recall that we denote by $j_I : \tilde{S}_I\backslash S_I \hookrightarrow \tilde{S}_I$ and $j'_I : Y \times \tilde{S}_I\backslash X_I \hookrightarrow Y \times S_I$ the open complementary embeddings. We then have the commutative diagrams

$$D_{IJ} = \begin{tikzcd} S_J \arrow[r, "i_J"] \arrow[d, "j_{IJ}"] & \tilde{S}_J \arrow[d, "p_{IJ}"] \\ S_I \arrow[r, "i_I"] & \tilde{S}_I \end{tikzcd} , \quad D'_{IJ} = \begin{tikzcd} X_J \arrow[r, "i'_J\circ l_J"] \arrow[d, "j'_{IJ}"] & Y \times \tilde{S}_J \arrow[d, "p'_{IJ}"] \\ X_I \arrow[r, "i'_I\circ l_I"] & Y \times \tilde{S}_I \end{tikzcd} .$$

and the factorization of D'_{IJ} by the fiber product:

$$D'_{IJ} = \begin{tikzcd} X_J \arrow[r, "i'_I\circ l_I"] \arrow[d, "j'_{IJ}"] & Y\times\tilde{S}_J \arrow[d, "p'_{IJ}"] \\ X_I \arrow[r, "i'_I\circ l_I"] & Y\times\tilde{S}_I \end{tikzcd}, \quad D'_{IJ} = \begin{tikzcd} X_J \arrow[rr, "i'_I\circ l_I"] \arrow[dr, "\hat{l}_J"] \arrow[dd, "j'_{IJ}"] & & Y\times\tilde{S}_J \arrow[dd, "p'_{IJ}"] \\ & X_I\times_{Y\times\tilde{S}_I} Y\times\tilde{S}_J = X_I\times\tilde{S}_{J\setminus I} \arrow[ur, "\hat{i}l_I"] \arrow[dl, "\hat{p}_{IJ}"] & \\ X_I \arrow[rr, "i'_I\circ l_I"] & & Y\times\tilde{S}_I \end{tikzcd} \tag{56}$$

where $j'_{IJ}: X_J \hookrightarrow X_I$ is the open embedding. Consider

$$F(X/S) := p_{S,\sharp}\Gamma^{\vee}_X\mathbb{Z}(Y\times S/Y\times S) \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$$

so that $D(\mathbb{A}^1, et)(F(X/S)) = M(X/S)$ since Y is smooth. Then, by definition,

$$\begin{aligned}\mathcal{F}^{FDR}_S(M(X/S)) := (e'(\tilde{S}_I)_*\mathcal{H}om(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho^*_{\tilde{S}_I}L(i_{I*}j^*_IF(X/S))), \\ E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u^q_{IJ}(F(X/S)))\end{aligned}$$

On the other hand, let

$$Q(X_I/\tilde{S}_I) := p_{\tilde{S}_I,\sharp}\Gamma^{\vee}_{X_I}\mathbb{Z}(Y\times\tilde{S}_I/Y\times\tilde{S}_I). \in C(\mathrm{Var}(\mathbb{C})^{sm}/\tilde{S}_I),$$

We have then for $I\subset[1,l]$ the map (50) in $C(\mathrm{Var}(\mathbb{C})^{sm}/\tilde{S}_J)$:

$$\begin{aligned}N_I(X/S): Q(X_I/\tilde{S}_I) = p_{\tilde{S}_I\sharp}\Gamma^{\vee}_{X_I}\mathbb{Z}(Y\times\tilde{S}_I/Y\times\tilde{S}_I) \xrightarrow{\mathrm{ad}(i'^*_I, i'_{I*})(-)} \\ p_{\tilde{S}_I\sharp}i'_{I*}i'^*_I\Gamma^{\vee}_{X_I}\mathbb{Z}(Y\times S_I/Y\times S_I) \xrightarrow{T(i'_I,\gamma^{\vee})(-)} p_{\tilde{S}_I\sharp}i'_{I*}\Gamma^{\vee}_{X_I}\mathbb{Z}(Y\times S_I/Y\times S_I) \\ \xrightarrow{\hat{T}_\sharp(p_{S_I}, i_I)(-)} i_{I*}p_{S_I\sharp}\Gamma^{\vee}_{X_I}\mathbb{Z}(Y\times S_I/Y\times S_I) = i_{I*}j^*_IF(X/S).\end{aligned}$$

We then have the commutative diagram in $C(\mathrm{Var}(\mathbb{C})^{sm}/\tilde{S}_J)$

$$\begin{tikzcd} p^*_{IJ}Q(X_I/\tilde{S}_I) \arrow[r, "p^*_{IJ}N_I(X/S)"] \arrow[d, "H_{IJ}"] & p^*_{IJ}Li_{I*}j^*_IF(X/S) \arrow[d, "T^q(D_{IJ})(j^*_IF(X/S))"] \\ Q(X_J/\tilde{S}_J) \arrow[r, "N_J(X/S)"] & Li_{J*}j^*_JF(X/S) \end{tikzcd} \tag{57}$$

with

$$\begin{aligned}H_{IJ}: p^*_{IJ}p_{\tilde{S}_I\sharp}\Gamma^{\vee}_{X_I}\mathbb{Z}(Y\times\tilde{S}_I/Y\times\tilde{S}_I) \xrightarrow{T_\sharp(p_{IJ}, p_{\tilde{S}_I})(-)^{-1}} p_{\tilde{S}_J\sharp}p'^*_{IJ}\Gamma^{\vee}_{X_I}\mathbb{Z}(Y\times\tilde{S}_I/Y\times\tilde{S}_I) \\ \xrightarrow{p_{\tilde{S}_J\sharp}T(p_{IJ},\gamma^{\vee})(-)} p_{\tilde{S}_J\sharp}\Gamma^{\vee}_{X_I\times\tilde{S}_{J\setminus I}}\mathbb{Z}(Y\times\tilde{S}_J/Y\times\tilde{S}_J) \xrightarrow{p_{\tilde{S}_J\sharp}T(X_J/X_I\times\tilde{S}_{J\setminus I},\gamma^{\vee})(-)} p_{\tilde{S}_J\sharp}\Gamma^{\vee}_{X_J}\mathbb{Z}(Y\times\tilde{S}_J/Y\times\tilde{S}_J).\end{aligned}$$

The diagram 57 say that the maps $N_I(X/S)$ induces a map in $C(\mathrm{Var}(\mathbb{C})^{sm}/(S/\tilde{S}_I))$

$$\begin{aligned}(N_I(X/S)): (Q(X_I/\tilde{S}_I), I(p^*_{IJ}, p_{IJ*})(-,-)(H_{IJ})) \\ \to (Li_{I*}j^*_IF(X/S), I(p^*_{IJ}, p_{IJ*})(-,-)(T^q(D_{IJ})(j^*_IF(X/S)))).\end{aligned}$$

Denote $\bar{X}_I := \bar{X} \cap (\bar{Y} \times S_I) \subset \bar{Y} \times \tilde{S}_I$ the closure of $X_I \subset \bar{Y} \times \tilde{S}_I$, and $Z_I := Z \cap (\bar{Y} \times S_I) = \bar{X}_I \backslash X_I$. Consider for $I \subset [1, \cdots l]$ and $I \subset J$ the following commutative diagrams in $\mathrm{Var}(\mathbb{C})$

$$\begin{tikzcd}
X_I \arrow[r, "l_I"] \arrow[d] & Y \times \tilde{S}_I \arrow[d, "(n \times I)"] \arrow[dr, "p_{\tilde{S}_I}"] & \\
\bar{X}_I \arrow[r, "l_I"] & \bar{Y} \times \tilde{S}_I \arrow[r, "\bar{p}_{\tilde{S}_I}"] & \tilde{S}_I \\
Z_I = \bar{X}_I \backslash X_I \arrow[u] \arrow[ur, "l_{Z_I}"] \arrow[r] & D \times \tilde{S}_I \arrow[u, "(k \times I)"] \arrow[ur] &
\end{tikzcd}
, \quad
\begin{tikzcd}
\bar{Y} \times \tilde{S}_J \arrow[r, "p_{\tilde{S}_J}"] \arrow[d, "p'_{IJ}"] & \tilde{S}_J \arrow[d, "p_{IJ}"] \\
\bar{Y} \times \tilde{S}_I \arrow[r, "p_{\tilde{S}_I}"] & \tilde{S}_I
\end{tikzcd}$$

Let $\epsilon_1 : ((\bar{Y} \times \tilde{S}_I)_1, E_1) \to (\bar{Y} \times \tilde{S}_I, Z_I)$ a strict desingularization of the pair $(\bar{Y} \times \tilde{S}_I, Z_I)$, $\epsilon_2 : ((\bar{Y} \times \tilde{S}_I)_2, E_2) \to (\bar{Y} \times \tilde{S}_I, \bar{X}_I)$ a strict desingularization of the pair $(\bar{Y} \times \tilde{S}_I, \bar{X}_I)$ and a morphism $\epsilon_{12} : (\bar{Y} \times \tilde{S}_I)_2 \to (\bar{Y} \times \tilde{S}_I)_1$ such that the following diagram commutes (see definition-proposition 12) :

$$\begin{tikzcd}
(\bar{Y} \times \tilde{S}_I)_2 \arrow[r, "\epsilon_{12}"] \arrow[d, "\epsilon_2"] & (\bar{Y} \times \tilde{S}_I)_1 \arrow[d, "\epsilon_1"] \\
\bar{Y} \times \tilde{S}_I \arrow[r, "="] & \bar{Y} \times \tilde{S}_I
\end{tikzcd}$$

We have then the two canonical maps in $C(\mathrm{Var}(\mathbb{C})^{2,smpr}/(\tilde{S}_I))$

$$\begin{aligned}
&L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R_{((\bar{Y} \times \tilde{S}_I)^*, E^*)/\tilde{S}_I}(\rho^*_{\tilde{S}_I} Q(X_I/\tilde{S}_I)) \\
&:= L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}\mathrm{Cone}(p_{\tilde{S}_I*}E_{et}(\mathrm{Cone}((\mathbb{Z}((E_{1\bullet} \times \tilde{S}_I, E_{1\bullet})/(\bar{Y} \times \tilde{S}_I)_1 \times \tilde{S}_I, u_{IJ}) \to \\
&\mathbb{Z}(((\bar{Y} \times \tilde{S}_I)_1 \times \tilde{S}_I, (\bar{Y} \times \tilde{S}_I)_1)/(\bar{Y} \times \tilde{S}_I)_1) \times \tilde{S}_I)) \\
&\to \mathrm{Cone}(p_{\tilde{S}_I*}E_{et}((\mathbb{Z}((E_{2\bullet} \times \tilde{S}_I, E_{2\bullet})/(\bar{Y} \times \tilde{S}_I)_2 \times \tilde{S}_I), u_{IJ}) \to \\
&\mathbb{Z}(((\bar{Y} \times \tilde{S}_I)_2 \times \tilde{S}_I, (\bar{Y} \times \tilde{S}_I)_2)/(\bar{Y} \times \tilde{S}_I)_2 \times \tilde{S}_I))))) \\
&\xleftarrow{(T^{\mu,q}(\bar{p}_{\tilde{S}_I\sharp}, p_{\tilde{S}_I*})(-,-), T^{\mu,q}(\bar{p}_{\tilde{S}_I\sharp}, p_{\tilde{S}_I*})(-,-))} \\
&\mathrm{Cone}(\mathrm{Cone}((\mathbb{Z}((E_{1\bullet} \times \tilde{S}_I, E_{1\bullet})/\tilde{S}_I), u_{IJ}) \to \mathbb{Z}(((\bar{Y} \times \tilde{S}_I)_1 \times \tilde{S}_I, (\bar{Y} \times \tilde{S}_I)_1)/\tilde{S}_I)) \\
&\to \mathrm{Cone}((\mathbb{Z}((E_{2\bullet} \times \tilde{S}_I/E_{2\bullet})/\tilde{S}_I), u_{IJ}) \to \mathbb{Z}(((\bar{Y} \times \tilde{S}_I)_2 \times \tilde{S}_I/(\bar{Y} \times \tilde{S}_I)_2)/\tilde{S}_I))(-d_Y - d_{\tilde{S}_I})[-2d_Y - 2d_{\tilde{S}_I}] \\
&\xrightarrow{I_\delta((\bar{X}_I, Z_I)/\tilde{S}_I) := (\mathbb{Z}(l_{Z_I} \circ \epsilon_1 \times I), \mathbb{Z}(\epsilon_1 \times I), \mathbb{Z}(l_I \circ \epsilon_2 \times I), \mathbb{Z}(\epsilon_2 \times I))} \\
&\mathrm{Cone}(\mathbb{Z}((\bar{Y} \times \tilde{S}_I, Z_I)/\tilde{S}_I) \to \mathbb{Z}((\bar{Y} \times \tilde{S}_I, \bar{X}_I)/\tilde{S}_I))(-d_Y - d_{\tilde{S}_I})[-2d_Y - 2d_{\tilde{S}_I}]
\end{aligned}$$

where, the maps in $C(\mathrm{Var}(\mathbb{C})^{2,smpr}/(\tilde{S}_I))$

$$\begin{aligned}
&T^{\mu,q}(\bar{p}_{\tilde{S}_I\sharp}, p_{\tilde{S}_I*})(-,-) : \\
&\mathrm{Cone}((\mathbb{Z}((E_{1\bullet} \times \tilde{S}_I, E_{1\bullet})/\tilde{S}_I), u_{IJ}) \to \mathbb{Z}(((\bar{Y} \times \tilde{S}_I)_1 \times \tilde{S}_I, (\bar{Y} \times \tilde{S}_I)_1)/\tilde{S}_I)) \to \\
&p_{\tilde{S}_I*}E_{et}(\mathrm{Cone}((\mathbb{Z}((E_{1\bullet} \times \tilde{S}_I, E_{1\bullet})/(\bar{Y} \times \tilde{S}_I)_1 \times \tilde{S}_I, u_{IJ}) \to \\
&\mathbb{Z}(((\bar{Y} \times \tilde{S}_I)_1 \times \tilde{S}_I, (\bar{Y} \times \tilde{S}_I)_1)/(\bar{Y} \times \tilde{S}_I)_1) \times \tilde{S}_I))(d_Y + d_{\tilde{S}_I})[2d_Y + 2d_{\tilde{S}_I}]
\end{aligned}$$

and

$$\begin{aligned}
&T^{\mu,q}(\bar{p}_{\tilde{S}_I\sharp}, p_{\tilde{S}_I*})(-,-) : \\
&\mathrm{Cone}((\mathbb{Z}((E_{2\bullet} \times \tilde{S}_I, E_{2\bullet})/\tilde{S}_I), u_{IJ}) \to \mathbb{Z}(((\bar{Y} \times \tilde{S}_I)_2 \times \tilde{S}_I, (\bar{Y} \times \tilde{S}_I)_2)/\tilde{S}_I)) \to \\
&p_{\tilde{S}_I*}E_{et}(\mathrm{Cone}((\mathbb{Z}((E_{2\bullet} \times \tilde{S}_I, E_{2\bullet})/(\bar{Y} \times \tilde{S}_I)_2 \times \tilde{S}_I, u_{IJ}) \to \\
&\mathbb{Z}(((\bar{Y} \times \tilde{S}_I)_2 \times \tilde{S}_I, (\bar{Y} \times \tilde{S}_I)_2)/(\bar{Y} \times \tilde{S}_I)_1) \times \tilde{S}_I))(d_Y + d_{\tilde{S}_I})[2d_Y + 2d_{\tilde{S}_I}]
\end{aligned}$$

given in definition 35 are $(\mathbb{A}^1, et)$ local equivalence by proposition 36. We denote by $v^q_{IJ}(F(X/S))$ the composite

$$v^q_{IJ}(F(X/S))[d_{\tilde{S}_J}] : e'(\tilde{S}_I)_*\mathcal{H}om(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R_{((\bar{Y}\times\tilde{S}_I)^*,E^*)/\tilde{S}_I}(\rho^*_{\tilde{S}_I}Q(X_I/\tilde{S}_I)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))$$
$$\xrightarrow{p_{IJ*}T(p_{IJ},\Omega^{\Gamma,pr}_{\cdot})(-)\circ\mathrm{ad}(p^{*mod}_{IJ},p_{IJ})(-)}$$
$$e'(\tilde{S}_J)_*\mathcal{H}om(L\rho_{\tilde{S}_J*}\mu_{\tilde{S}_J*}p^*_{IJ}R_{((\bar{Y}\times\tilde{S}_I)^*,E^*)/\tilde{S}_I}(\rho^*_{\tilde{S}_I}Q(X_I/\tilde{S}_I), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR})))$$
$$\xrightarrow{\mathcal{H}om(T(p_{IJ},R^{CH})(Q(X_I/\tilde{S}_I))^{-1},E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I},F_{DR}))}$$
$$p_{IJ*}e'(\tilde{S}_J)_*\mathcal{H}om(L\rho_{\tilde{S}_J*}\mu_{\tilde{S}_J*}R_{((\bar{Y}\times\tilde{S}_I)^*\times\tilde{S}_{J\backslash I},E^*\times\tilde{S}_{J\backslash I})/\tilde{S}_J}(\rho^*_{\tilde{S}_J}p^*_{IJ}Q(X_I/\tilde{S}_I)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_J}, F_{DR}))$$
$$\xrightarrow{\mathcal{H}om(L\rho_{\tilde{S}_J*}\mu_{\tilde{S}_J*}R^{CH}_{\tilde{S}_J}(H_{IJ}),E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_J},F_{DR}))}$$
$$p_{IJ*}e'(\tilde{S}_J)_*\mathcal{H}om(L\rho_{\tilde{S}_J*}\mu_{\tilde{S}_J*}R_{((\bar{Y}\times\tilde{S}_J)^*,E'^*)/\tilde{S}_J}(\rho^*_{\tilde{S}_J}Q(X_J/\tilde{S}_J)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_J}, F_{DR})).$$

On the other hand, we have in $\pi_{\bar{X}}(C(MHM(\bar{X}))) \subset C_{\mathcal{D}fil}(\bar{X}/(\bar{Y}\times\tilde{S}_I))$

$$(\mathrm{Cone}(T(Z_I/\bar{X}_I,\gamma^{\vee,Hdg})(-))) : (\Gamma^{\vee,Hdg}_{\bar{X}_I}(O_{\bar{Y}\times\tilde{S}_I}, F_b), x_{IJ}(\bar{X}/S)) \to (\Gamma^{\vee,Hdg}_{Z_I}(O_{\bar{Y}\times\tilde{S}_I}, F_b), x_{IJ}(Z/S))$$
$$\xrightarrow{=} (n\times I)^{Hdg}_!(\Gamma^{\vee,Hdg}_{X_I}(O_{Y\times\tilde{S}_I}, F_b), x_{IJ}(X/S))$$

with

- for the closed embedding $\bar{X}\subset\bar{Y}\times S$ we consider the map in $\pi_{\bar{Y}\times\tilde{S}_J}(C(MHM(\bar{Y}\times\tilde{S}_J)))$

$$x_{IJ}(\bar{X}/S) : \Gamma^{\vee,Hdg}_{\bar{X}_I}(O_{\bar{Y}\times\tilde{S}_I}, F_b) \xrightarrow{\mathrm{ad}(p'^{*mod}_{IJ},p'_{IJ*})(-)} p'_{IJ*}p'^{*mod}_{IJ}\Gamma^{\vee,Hdg}_{\bar{X}_I}(O_{\bar{Y}\times\tilde{S}_I}, F_b)$$
$$\xrightarrow{T(\bar{X}_J/p'^{-1}_{IJ}(\bar{X}_I),\gamma^{\vee})(-)\circ T(p'_{IJ},\gamma^{\vee})(-)} \Gamma^{\vee,Hdg}_{\bar{X}_J}(O_{\bar{Y}\times\tilde{S}_J}, F_b),$$

- for the closed embedding $Z\subset\bar{Y}\times S$ we consider the map in $\pi_{\bar{Y}\times\tilde{S}_J}(C(MHM(\bar{Y}\times\tilde{S}_J)))$

$$x_{IJ}(Z/S) : \Gamma^{\vee,Hdg}_{Z_I}(O_{\bar{Y}\times\tilde{S}_I}, F_b) \xrightarrow{\mathrm{ad}(p'^{*mod}_{IJ},p'_{IJ*})(-)} p'_{IJ*}p'^{*mod}_{IJ}\Gamma^{\vee,Hdg}_{Z_I}(O_{\bar{Y}\times\tilde{S}_I}, F_b)$$
$$\xrightarrow{T(Z_J/p'^{-1}_{IJ}(Z_I),\gamma^{\vee})(-)\circ T(p'_{IJ},\gamma^{\vee})(-)} \Gamma^{\vee,Hdg}_{Z_J}(O_{\bar{Y}\times\tilde{S}_J}, F_b),$$

- for the closed embedding $X\subset Y\times S$ we consider the map in $\pi_{Y\times\tilde{S}_J}(C(MHM(Y\times\tilde{S}_J)))$

$$x_{IJ}(X/S)(-d_Y)[-2d_Y] : \Gamma^{\vee,Hdg}_{X_I}(O_{Y\times\tilde{S}_I}, F_b) \xrightarrow{\mathrm{ad}(p'^{*mod}_{IJ},p'_{IJ*})(-)} p'_{IJ*}p'^{*mod}_{IJ}\Gamma^{\vee,Hdg}_{X_I}(O_{Y\times\tilde{S}_I}, F_b)$$
$$\xrightarrow{T(X_J/X_I\times\tilde{S}_{J\backslash I},\gamma^{\vee,Hdg})(-)\circ T(p'_{IJ},\gamma^{\vee})(-)} \Gamma^{\vee,Hdg}_{X_J}(O_{Y\times\tilde{S}_J}, F_b).$$

The maps $x_{IJ}(X/S)$ gives the following maps in $C_{\mathcal{D}fil,S_J}(\tilde{S}_J)$

$$w_{IJ}(X/S)(-d_Y)[-2d_Y] : p_{\tilde{S}_I*}E_{zar}((\Omega^{\bullet}_{\bar{Y}\times\tilde{S}_I/\tilde{S}_I}, F_b)\otimes_{O_{\bar{Y}\times\tilde{S}_I}}(n\times I)^{Hdg}_!\Gamma^{\vee,Hdg}_{X_I}(O_{Y\times\tilde{S}_I}, F_b))$$
$$\xrightarrow{\mathrm{ad}(p^{*mod}_{IJ},p_{IJ})(-)} p_{IJ*}p^{*mod}_{IJ}p_{\tilde{S}_I*}E_{zar}((\Omega^{\bullet}_{\bar{Y}\times\tilde{S}_I/\tilde{S}_I}, F_b)\otimes_{O_{Y\times\tilde{S}_I}}(n\times I)^{Hdg}_!\Gamma^{\vee,Hdg}_{X_I}(O_{Y\times\tilde{S}_I}, F_b)$$
$$\xrightarrow{p_{IJ*}T^O_w(p_{IJ},p_{\tilde{S}_I})(-)} p_{\tilde{S}_J*}E_{zar}((\Omega^{\bullet}_{\bar{Y}\times\tilde{S}_J/\tilde{S}_J}, F_b)\otimes_{O_{\bar{Y}\times\tilde{S}_J}}p'^{*mod}_{IJ}(n\times I)^{Hdg}_!\Gamma^{\vee,Hdg}_{X_I}(O_{Y\times\tilde{S}_I}, F_b))$$
$$\xrightarrow{(p_{\tilde{S}_J*}E(DR(-)(x_{IJ}(X/S))} p_{\tilde{S}_J*}E_{zar}((\Omega^{\bullet}_{\bar{Y}\times\tilde{S}_J/\tilde{S}_J}, F_b)\otimes_{O_{\bar{Y}\times\tilde{S}_J}}(n\times I)^{Hdg}_!\Gamma^{\vee,Hdg}_{X_J}(O_{\bar{Y}\times\tilde{S}_J}, F_b)).$$

We have then the following lemma

Lemma 12. *(i) The map in $C(\mathrm{Var}(\mathbb{C})^{sm}/(S/\tilde{S}_I))$*

$$(N_I(X/S)) : (Q(X_I/\tilde{S}_I), H_{IJ}) \to (L(i_{I*}j_I^*F(X/S)), T^q(D_{IJ})(F(X/S))).$$

is an equivalence $(\mathbb{A}^1, et)$ local.

(ii) The maps $(N_I(X/S))$ induces an ∞-filtered quasi-isomorphism in $C_{\mathcal{D}fil}(S/(\tilde{S}_I))$

$$(\mathcal{H}om(\mathrm{Gr}^{12*}_{\tilde{S}_I} R^{CH}_{\tilde{S}_I}(N_I(X/S)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_b))) :$$
$$(e'(\tilde{S}_I)_*\mathcal{H}om(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho^*_{\tilde{S}_I}L(i_{I*}j_I^*F(X/S))), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u^q_{IJ}(F(X/S)))$$
$$\to (e'(\tilde{S}_I)_*\mathcal{H}om(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R_{((\bar{Y}\times\tilde{S}_I)^*,E^*)/\tilde{S}_I}(\rho^*_{\tilde{S}_I}Q(X_I/\tilde{S}_I)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], v^q_{IJ}(F(X/S)))$$

(iii) The maps $(I_\delta((\bar{X}_I, Z_I)/\tilde{S}_I))$ induce an ∞-filtered Zariski local equivalence in $C_{\mathcal{D}fil}(S/(\tilde{S}_I))$

$$(\mathcal{H}om(I_\delta((\bar{X}_I, Z_I)/\tilde{S}_I), k) :$$
$$(p_{\tilde{S}_I*}E_{zar}((\Omega^\bullet_{\bar{Y}\times\tilde{S}_I/\tilde{S}_I}, F_b)\otimes_{O_{\bar{Y}\times\tilde{S}_I}}(n\times I)^{Hdg}_!(\Gamma^{\vee,Hdg}_{X_I}(O_{Y\times\tilde{S}_I}, F_b))(d_Y)[2d_Y + d_{\tilde{S}_I}], w_{IJ}(X/S))$$
$$\to (e'(\tilde{S}_I)_*\mathcal{H}om(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R_{((\bar{Y}\times\tilde{S}_I)^*,E^*)/\tilde{S}_I}(\rho^*_{\tilde{S}_I}Q(X_I/\tilde{S}_I)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], v^q_{IJ}(F(X/S)))$$

Proof. (i): See lemma 9(i)
(ii): These maps induce a morphism in $C_{\mathcal{D}}(S/(\tilde{S}_I))$ by construction. It is an ∞-filtered quasi-isomorphism by (i) and proposition 111.
(iii): The fact that these maps define a morphism in $C_{\mathcal{D}}(S/(\tilde{S}_I))$ follows from the commutative diagrams in $C(\mathrm{Var}(\mathbb{C})^{2,smpr}/\tilde{S}_J)$ for $I \subset J$

$$\begin{tikzcd}
p^*_{IJ}L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R_{((\bar{Y}\times\tilde{S}_I)^*,E^*)/\tilde{S}_J}(\rho^*_{\tilde{S}_I}Q(X_I/\tilde{S}_I)) \arrow[r, "p^*_{IJ}I_\delta((\bar{X}_I,Z_I)/\tilde{S}_I)"] \arrow[d, "T(p_{IJ},R^{CH})(-)"] & p^*_{IJ}\mathrm{Cone}(\mathbb{Z}((\bar{Y}\times\tilde{S}_I, Z_I)/\tilde{S}_I) \to \mathbb{Z}((\bar{Y}\times\tilde{S}_I, \bar{X}_I)/\tilde{S}_I))(-d \arrow[d, "="] \\
L\rho_{\tilde{S}_J*}\mu_{\tilde{S}_J*}R_{((\bar{Y}\times\tilde{S}_J)^*,E^*)/\tilde{S}_J}(\rho^*_{\tilde{S}_J}p^*_{IJ}Q(X_I/\tilde{S}_I)) \arrow[r, "I_\delta((p'^{-1}_{IJ}(\bar{X}_I),p'^{-1}_{IJ}(Z_I))/\tilde{S}_J)"] & \mathrm{Cone}(\mathbb{Z}((\bar{Y}\times\tilde{S}_J, p'^{-1}_{IJ}(Z_I))/\tilde{S}_J) \to \mathbb{Z}((\bar{Y}\times\tilde{S}_J, p'^{-1}_{IJ}(\bar{X}_I)/\tilde{S}_J)) \\
L\rho_{\tilde{S}_J*}\mu_{\tilde{S}_J*}R_{((\bar{Y}\times\tilde{S}_J)^*,E^*)/\tilde{S}_J}(\rho^*_{\tilde{S}_J}Q(X_J/\tilde{S}_J)) \arrow[r, "I_\delta((\bar{X}_J,Z_J)/\tilde{S}_J)"] \arrow[u, "R^{CH}(H_{IJ})"] & \mathrm{Cone}(\mathbb{Z}((\bar{Y}\times\tilde{S}_J, Z_J)/\tilde{S}_J) \to \mathbb{Z}((\bar{Y}\times\tilde{S}_J, \bar{X}_J)/\tilde{S}_J))(-d_Y \arrow[u, "(\mathbb{Z}(\bar{Y}\times\tilde{S}_J),\mathbb{Z}(\bar{Y}\times\tilde{S}_J))"']
\end{tikzcd}$$

On the other hand, it is an ∞-filtered quasi-isomorphism by proposition 99 since we have by Yoneda lemma the following commutative diagram

$$\begin{tikzcd}
\begin{array}{c} p_{\tilde{S}_I*}E_{zar}((\Omega^\bullet_{\bar{Y}\times\tilde{S}_I/\tilde{S}_I}, F_b)\otimes_{O_{\bar{Y}\times\tilde{S}_I}} \\ (n\times I)^{Hdg}_!\Gamma^{\vee,Hdg}_{X_I}(O_{Y\times\tilde{S}_I}, F_b)))(d_Y + d_{\tilde{S}_I})[2d_Y + 2d_{\tilde{S}_I}] \end{array} \arrow[r, "\sim"] \arrow[dr, "'\mathcal{H}om(I_\delta((\bar{X}_I,Z_I)/\tilde{S}_I),k)"'] & \begin{array}{c} \mathrm{Cone}(p_{\tilde{S}_I*}E_{zar}(\Omega^\bullet_{(E_{2\bullet}\times\tilde{S}_I/\tilde{S}_I}, F_b)\otimes_{O_{E_{2\bullet}\times\tilde{S}_I}}(\Gamma^{\vee,Hdg}_{E_{2\bullet}}(O \\ p_{\tilde{S}_I*}E_{zar}(\Omega^\bullet_{(E_{1\bullet}\times\tilde{S}_I/\tilde{S}_I}, F_b)\otimes_{O_{E_{1\bullet}\times\tilde{S}_I}}(\Gamma^{\vee,Hdg}_{E_{2\bullet}}(O_{E_{1\bullet}\times\tilde{S}_I}, F_b))))(\end{array} \arrow[d] \\
& \begin{array}{c} (e'(\tilde{S}_I)_*\mathcal{H}om(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R_{((\bar{Y}\times\tilde{S}_I)^*,E^*)/\tilde{S}_J}(\rho^*_{\tilde{S}_I}Q(\\ E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))) \end{array}
\end{tikzcd}$$

and on the other hand by proposition 111,

$$e'(\tilde{S}_I)_*\mathcal{H}om((T^{\mu,q}(p_{\tilde{S}_I\sharp}, p_{\tilde{S}_I*})(-,-), T^{\mu,q}(p_{\tilde{S}_I\sharp}, p_{\tilde{S}_I*})(-,-)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))$$

is an equivalence $(\mathbb{A}^1, et)$ local. Moreover

$$k : E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}) \to E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR})$$

are 2-filtered equivalence Zariski local by proposition 111 and theorem 12. □

Proposition 114. *Let $f : X \to S$ a morphism with $S, X \in \mathrm{Var}(\mathbb{C})$. Assume there exist a factorization*

$$f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$$

of f with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $\bar{Y} \in \mathrm{PSmVar}(\mathbb{C})$ a compactification of Y with $\bar{Y} \backslash Y = D$ a normal crossing divisor, denote $k : D \hookrightarrow \bar{Y}$ the closed embedding and $n : Y \hookrightarrow \bar{Y}$ the open embedding. Denote $\bar{X} \subset \bar{Y} \times S$ the closure of $X \subset \bar{Y} \times S$. We have then the following commutative diagram in $\mathrm{Var}(\mathbb{C})$

$$\begin{tikzcd}
X \arrow[r, "l"] \arrow[d] & Y \times S \arrow[d, "(n\times I)"] \arrow[dr, "p_S"] & \\
\bar{X} \arrow[r, "l"] & \bar{Y} \times S \arrow[r, "\bar{p}_S"] & S \\
Z := \bar{X} \backslash X \arrow[u] \arrow[ur, "l_Z"] \arrow[r] & D \times S \arrow[u, "(k\times I)"] \arrow[ur] &
\end{tikzcd}.$$

Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. Then $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. Denote, for $I \subset [1, \cdots l]$, $S_I = \cap_{i \in I} S_i$ and $X_I = \cap_{i \in I} X_i$. Denote $\bar{X}_I := \bar{X} \cap (\bar{Y} \times S_I) \subset \bar{Y} \times \tilde{S}_I$ the closure of $X_I \subset \bar{Y} \times \tilde{S}_I$, and $Z_I := Z \cap (\bar{Y} \times S_I) = \bar{X}_I \backslash X_I \subset \bar{Y} \times \tilde{S}_I$. We have then for $I \subset [1, \cdots l]$, the following commutative diagram in $\mathrm{Var}(\mathbb{C})$

$$\begin{tikzcd}
X_I \arrow[r, "l_I"] \arrow[d] & Y \times \tilde{S}_I \arrow[d, "(n\times I)"] \arrow[dr, "p_{\tilde{S}_I}"] & \\
\bar{X}_I \arrow[r, "l_I"] & \bar{Y} \times \tilde{S}_I \arrow[r, "\bar{p}_{\tilde{S}_I}"] & \tilde{S}_I \\
Z_I = \bar{X}_I \backslash X_I \arrow[u] \arrow[ur, "l_{Z_I}"] \arrow[r] & D \times \tilde{S}_I \arrow[u, "(k\times I)"] \arrow[ur] &
\end{tikzcd}.$$

Let $F(X/S) := p_{S,\sharp}\Gamma^\vee_X \mathbb{Z}(Y \times S/Y \times S) \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$. We have then the following isomorphism in $D_{\mathcal{D}fil,\infty}(S/(\tilde{S}_I))$

$$\begin{aligned}
&I(X/S) : \mathcal{F}_S^{FDR}(M(X/S)) \xrightarrow{:=} \\
&(e'(\tilde{S}_I)_* \mathcal{H}om(L\rho_{\tilde{S}_I *}\mu_{\tilde{S}_I *} R^{CH}(\rho^*_{\tilde{S}_I} L(i_{I*} j_I^* F(X/S))), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u^q_{IJ}(F(X/S))) \\
&\xrightarrow{(\mathcal{H}om(L\rho_{\tilde{S}_I *}\mu_{\tilde{S}_I *} R^{CH}_{\tilde{S}_I}(N_I(X/S)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_b)))} \\
&(e'(\tilde{S}_I)_* \mathcal{H}om(L\rho_{\tilde{S}_I *}\mu_{\tilde{S}_I *} R_{((\bar{Y} \times \tilde{S}_I)^*, E^*)/\tilde{S}_J}(\rho^*_{\tilde{S}_I} Q(X_I/\tilde{S}_I)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], v^q_{IJ}(F(X/S))) \\
&\xrightarrow{(\mathcal{H}om(\rho_{\tilde{S}_I *} I_\delta((\bar{X}_I, Z_I)/\tilde{S}_I), k)[-d_{\tilde{S}_I}])^{-1}} \\
&(p_{\tilde{S}_I *} E_{zar}((\Omega^\bullet_{\bar{Y} \times \tilde{S}_I/\tilde{S}_I}, F_b) \otimes_{O_{\bar{Y} \times \tilde{S}_I}} (n \times I)^{Hdg}_! \Gamma^{\vee,Hdg}_{X_I}(O_{Y \times \tilde{S}_I}, F_b))(d_Y + d_{\tilde{S}_I})[2d_Y + d_{\tilde{S}_I}], w_{IJ}(X/S)) \\
&\xrightarrow{=:} \iota_S Rf^{Hdg}_! (\Gamma^{\vee,Hdg}_{X_I}(O_{Y \times \tilde{S}_I}, F_b)(d_Y)[2d_Y], x_{IJ}(X/S)). \xrightarrow{=:} \iota_S Rf^{Hdg}_! f^{*mod}_{Hdg} \mathbb{Z}^{Hdg}_S.
\end{aligned}$$

Proof. Follows from lemma 12. □

Corollary 5. *Let $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^{l} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. For $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $D(\mathbb{A}^1, et)(F) \in \mathrm{DA}_c(S)$, $u_{IJ}^q(F)$ are ∞-filtered Zariski local equivalence.*

Proof. Follows from definition by proposition 111 and the direct image of a D-module in the singular case (see section 4.3). □

Corollary 6. *(i) Let $S \in \mathrm{SmVar}(\mathbb{C})$. Then, for $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1, et)(F) \in \mathrm{DA}_c(S)$,*

$$H^i \mathcal{F}_S^{FDR}(M, W) := a_{zar} H^i (e'(S)_* \mathcal{H}om^\bullet (L\rho_{S*}\mu_{S*} R^{CH}(\rho_S^* L(F, W)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR}))[-d_S]) \in \pi_S(MHM(S))$$

for all $i \in \mathbb{Z}$, and for all $p \in \mathbb{Z}$, $\mathcal{F}_S^{FDR}(M, W) \in D_{\mathcal{D}(1,0)fil,\infty}(S/(\tilde{S}_I))$ is the class of a complex $\mathcal{F}_S^{FDR}(M, W)^t \in C_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))$ such that the differentials of $\mathrm{Gr}_W^p \mathcal{F}_S^{FDR}(M, W)^t$ are strict for the filtration F.

(ii) Let $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^{l} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. Then, for $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1, et)(F) \in \mathrm{DA}_c(S)$,

$$H^i \mathcal{F}_S^{FDR}(M, W) := (a_{zar} H^i e'(\tilde{S}_I)_* \mathcal{H}om^\bullet (L\rho_{\tilde{S}_I *}\mu_{\tilde{S}_I *} R^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^*(F, W))), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], H^i u_{IJ}^q(F, W)) \in \pi_S(MHM(S))$$

for all $i \in \mathbb{Z}$, and for all $p \in \mathbb{Z}$, $\mathcal{F}_S^{FDR}(M, W) \in D_{\mathcal{D}(1,0)fil,\infty}(S/(\tilde{S}_I))$ is the class of a complex $\mathcal{F}_S^{FDR}(M, W)^t \in C_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))$ such that the differentials of $\mathrm{Gr}_W^p \mathcal{F}_S^{FDR}(M, W)^t$ are strict for the filtration F.

Proof. (i):Follows from definition by proposition 111 and theorem 30. Indeed, let us prove that for $g : U'/S \to U/S$ a morphism with $U/S = (U, h), U'/S = (U', h') \in \mathrm{Var}(\mathbb{C})^{sm}/S$, U,U' connected, hence irreducible by smoothness, the complex

$$\mathcal{F}_S^{FDR}(g) := e'(S)_* \mathcal{H}om^\bullet (L\rho_{S*}\mu_{S*} R^{CH}(\mathbb{Z}(U'/S)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR}))[-d_S]$$
$$\xrightarrow{\mathcal{H}om(L\rho_{S*}\mu_{S*}R^{CH}(g),-)} e'(S)_* \mathcal{H}om^\bullet (L\rho_{S*}\mu_{S*} R^{CH}(\mathbb{Z}(U/S)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR})[-d_S]) \in C_{\mathcal{D}fil}(S)$$

satisfy $H^i \mathcal{F}_S^{FDR}(g) \in \pi_S(MHM(S))$ and is the class of a complex such that the differentials are strict for F. Let $U \subset \bar{X}$ a compactification of U and $U' \subset \bar{X}'$ a compactification of U', $S \subset \bar{S}$ a compactification of S with $\bar{X}, \bar{X}' \in \mathrm{PSmVar}(\mathbb{C})$, $i : \bar{D} = \cup_i \bar{D}_i := \bar{X} \backslash U \hookrightarrow \bar{X}$, $i : \bar{D}' = \cup_i \bar{D}'_i := \bar{X}' \backslash U' \hookrightarrow \bar{X}'$ normal crossing divisors, such that $g : U'/S \to U/S$ extend to $\bar{g} : \bar{X}'/\bar{S} \to \bar{X}/\bar{S}$: see section 2. Denote as in section 2, $X := \bar{h}^{-1}(S) \subset \bar{X}$ and $X' := \bar{h}'^{-1}(S) \subset \bar{X}'$, $D = \bar{D} \cap X$, $D' = \bar{D}' \cap X'$ the open subsets over $S \subset \bar{S}$. Denote $d = \dim(U)$, $d' = \dim(U')$, hence $d + d' = \dim(U \times_S U')$. Consider

$$[\Gamma_g]^t \in \mathrm{Hom}(C_*\mathbb{Z}^{tr}((\bar{X} \times S, X)/S)(-d)[-2d], C_*\mathbb{Z}^{tr}(\bar{X}' \times S, X'/S)(-d')[-2d'])$$

the morphism given by the transpose of the graph $\Gamma_g \subset X' \times_S X$ of $\bar{g}$. Since $\bar{g}^{-1}(\bar{D}_\bullet) \subset \bar{D}'_\bullet$, it induces the morphism

$$[\Gamma_g]^t \in \mathrm{Hom}(C_*\mathbb{Z}^{tr}((\bar{D}_\bullet \times S, D_\bullet)/S)(-d)[-2d], C_*\mathbb{Z}^{tr}(\bar{D}'_\bullet \times S, D'_\bullet/S)(-d')[-2d']).$$

We have then the following map in $C(\mathrm{Cor}(\mathrm{Var}(\mathbb{C})^{2,smpr}/S))$

$$(C_*T^{\mu,q}(p_{S\sharp}, p_{S*})(-,-) := ([\Delta_{D_\bullet} \times \square^*], [\Delta_X \times \square^*]), C_*T^{\mu,q}(p_{S\sharp}, p_{S*})(-,-) := ([\Delta_{D'_\bullet} \times \square^*], [\Delta_{X'} \times \square^*])) :$$
$$(\mathrm{Cone}(C_*\mathbb{Z}(i_\bullet \times I) : C_*\mathbb{Z}^{tr}((\bar{D}_\bullet \times S, D_\bullet)/S) \to C_*\mathbb{Z}^{tr}((\bar{X} \times S, X)/S))(-d)[-2d]$$
$$\xrightarrow{[\Gamma_g]^t} \mathrm{Cone}(C_*\mathbb{Z}(i'_\bullet \times I) : C_*\mathbb{Z}^{tr}((\bar{D}'_\bullet \times S, D'_\bullet)/S) \to C_*\mathbb{Z}^{tr}((\bar{X}' \times S, X')/S))(-d')[-2d'])$$
$$\to (L\rho_{S*}\mu_{S*} R^{CH}(\mathbb{Z}(U/S)) \xrightarrow{L\rho_{S*}\mu_{S*}R^{CH}(g)} L\rho_{S*}\mu_{S*} R^{CH}(\mathbb{Z}(V/S)))$$

where the columns are $(\mathbb{A}^1, et)$ local equivalence. We get the following map in $C_{\mathcal{D}fil}(S)$

$$\mathcal{H}om((C_*T^{\mu,q}(p_{S\sharp}, p_{S*})(-,-), C_*T^{\mu,q}(p_{S\sharp}, p_{S*})(-,-)), -) : \mathcal{F}_S^{FDR}(g) \to$$

$$\Omega(g) := e'(S)_*\mathcal{H}om((\mathrm{Cone}(C_*\mathbb{Z}(i_\bullet \times I) : C_*\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/S) \to C_*\mathbb{Z}((\bar{X} \times S, X)/S))(-d)[-2d] \xrightarrow{[\Gamma_g]^t}$$

$$\mathrm{Cone}(C_*\mathbb{Z}(i_\bullet \times I) : C_*\mathbb{Z}((\bar{D}'_\bullet \times S, D'_\bullet)/S) \to C_*\mathbb{Z}((\bar{X}' \times S, X')/S))(-d')[-2d']), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR}))$$

$$\xrightarrow{\mathcal{H}om(c,-)} (p_{S*}E(\Omega^\bullet_{\bar{X}\times S/S} \otimes_O (n' \times I)_{!Hdg}\Gamma^{\vee,Hdg}_{U'}(O_{U'\times S}, F_b))(d')[2d']$$

$$\xrightarrow{\Omega^{\Gamma,pr}([\Gamma_g^t])} p_{S*}E(\Omega^\bullet_{\bar{X}'\times S/S} \otimes_O (n \times I)_{!Hdg}\Gamma^{\vee,Hdg}_{U}(O_{U\times S}, F_b))(d)[2d]) =: H^0(\Omega(g))$$

which is a 2-filtered quasi-isomorphism by proposition 111.
(ii): Follows from (i) and for $g : U'/\tilde{S}_I \to U/\tilde{S}_I$ the commutative diagram

$$\begin{tikzcd}
p_{IJ}^{*mod}\mathcal{F}^{FDR}_{\tilde{S}_I}(g) \arrow[r, "T(p_{IJ},\Omega^{\gamma,pr})(-)"] \arrow[d, "p_{IJ}^{*mod}\mathcal{H}om((C_*T^{\mu,q}(p_{\tilde{S}_I\sharp},p_{\tilde{S}_I*})(-,-),C_*T^{\mu,q}(p_{\tilde{S}_I\sharp},p_{\tilde{S}_I*})(-,-)),-)"'] & \mathcal{F}^{FDR}_{\tilde{S}_J}(g \times I) \arrow[d, "(\mathcal{H}om(C_*T^{\mu,q}(p_{\tilde{S}_I\sharp},p_{\tilde{S}_I*})(-,-)),-)"] \\
p_{IJ}^{*mod}\Omega(g) \arrow[r, "T^O_w(p_{IJ},p_{\tilde{S}_I})(-)"] & \Omega(g \times I)
\end{tikzcd}$$

with $g \times I : U' \times \tilde{S}_{J\setminus I}/\tilde{S}_J \to U \times \tilde{S}_{J\setminus I}/\tilde{S}_J$. □

Proposition 115. *For $S \in \mathrm{Var}(\mathbb{C})$ not smooth, the functor (see corollary 6)*

$$\iota_S^{-1}\mathcal{F}_S^{FDR} : \mathrm{DA}_c^-(S)^{op} \to \pi_S(D(MHM(S))$$

does not depend on the choice of the open cover $S = \cup_i S_i$ and the closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$.

Proof. Let $S = \cup_{i=l+1}^{i=l'} S_i$ is an other open cover together with closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$ for $l+1 \leq i \leq l'$. Then, for $J' \subset I' \subset [l+1, \ldots, l'] = L'$ and $J \subset I \subset L = [1, \ldots, l]$,

$$T_S^{L/L'}(\iota_S^{-1}(e'(\tilde{S}_I)_*\mathcal{H}om(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho^*_{\tilde{S}_I}L(i_{I*}j_I^*F)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}(F)))$$

$$:= ((\mathrm{ho}\lim_{I\in L} p_{I'(I\sqcup I')*}\Gamma^{\vee,Hdg}_{S_{I\sqcup I'}} p^{*mod[-]}_{I(I\sqcup I')}(\iota_S^{-1}(e'(\tilde{S}_I)_*\mathcal{H}om(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho^*_{\tilde{S}_I}L(i_{I*}j_I^*F)),$$

$$E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], T_S^{L/L'}(u_{IJ}(F))))_I) \xrightarrow{(\mathrm{ho}\lim_{I\in L} u_{I(I\sqcup I')}(F))}$$

$$(\mathrm{ho}\lim_{I\in L} p_{I'(I\sqcup I')*}\Gamma^{\vee,Hdg}_{S_{I\sqcup I'}} p^{*mod[-]}_{I(I\sqcup I')} p_{I(I\sqcup I')*}e'(\tilde{S}_{(I\sqcup I')})_*\mathcal{H}om(L\rho_{\tilde{S}_{I\sqcup I'}*}\mu_{\tilde{S}_{I\sqcup I'}*}R^{CH}(\rho^*_{\tilde{S}_I}L(i_{(I\sqcup I')*}j^*_{(I\sqcup I')}F)),$$

$$E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_{I\sqcup I'}}, F_{DR}))[-d_{\tilde{S}_{I\sqcup I'}}], u_{(I\sqcup I')(I\sqcup J')}(F)) \xrightarrow{\mathrm{ad}(p^{*mod}_{I(I\sqcup I')}, p_{I(I\sqcup I')*})(-)\circ\gamma^{\vee,Hdg}_{S_{I\sqcup I'}}(-)}$$

$$(\mathrm{ho}\lim_{I\in L} p_{I'(I\sqcup I')*}e'(\tilde{S}_{(I\sqcup I')})_*\mathcal{H}om(L\rho_{\tilde{S}_{I\sqcup I'}*}\mu_{\tilde{S}_{I\sqcup I'}*}R^{CH}(\rho^*_{\tilde{S}_I}L(i_{(I\sqcup I')*}j^*_{(I\sqcup I')}F)),$$

$$E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_{I\sqcup I'}}, F_{DR}))[-d_{\tilde{S}_{I\sqcup I'}}], u_{(I\sqcup I')(I\sqcup J')}(F))$$

$$\xleftarrow{(\mathrm{ho}\lim_{I\in L} u_{I'(I\sqcup I')}(F))} (e'(\tilde{S}_{I'})_*\mathcal{H}om(L\rho_{\tilde{S}_{I'}*}\mu_{\tilde{S}_{I'}*}R^{CH}(\rho^*_{\tilde{S}_{I'}}L(i_{I'*}j^*_{I'}F)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_{I'}}, F_{DR}))[-d_{\tilde{S}_{I'}}], u_{I'J'}(F))$$

is an ∞-filtered Zariski local equivalence, since all the morphisms are ∞-filtered Zariski local equivalences by corollary 5 and proposition 97. □

We have the canonical transformation map between the filtered De Rham realization functor and the Gauss-Manin realization functor :

Definition 118. *Let $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^{l} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. Let $M \in \mathrm{DA}_c(S)$ and $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1, et)(F)$. We have, using definition 112(ii), definition 36, proposition 1 and proposition 111, the canonical map in $D_{O_S fil,\mathcal{D},\infty}(S/(\tilde{S}_I))$*

$$\begin{aligned}
&T(\mathcal{F}_S^{GM}, \mathcal{F}_S^{FDR})(M) : \\
&\mathcal{F}_S^{GM}(L\mathbb{D}_S M) := (e(\tilde{S}_I)_* \mathcal{H}om^\bullet(L(i_{I*} j_I^* \mathbb{D}_S LF), E_{et}(\Omega^\bullet_{/\tilde{S}_I}, F_b))[-d_{\tilde{S}_I}], u_{IJ}^q(F)) \\
&\xrightarrow{\sim} (e(\tilde{S}_I)_* \mathcal{H}om^\bullet(L\mathbb{D}^0_{\tilde{S}_I} L(i_{I*} j_I^* F), E_{et}(\Omega^\bullet_{/\tilde{S}_I}, F_b))[-d_{\tilde{S}_I}], u_{IJ}^{q,d}(F)) \\
&\xrightarrow{\mathcal{H}om(-,\mathrm{Gr}(\Omega_{\tilde{S}_I}))^{-1}} (e(\tilde{S}_I)_* \mathcal{H}om^\bullet(L\mathbb{D}^0_{\tilde{S}_I} L(i_{I*} j_I^* F), \mathrm{Gr}^{12}_{\tilde{S}_I *} E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^{q,d}(F)) \\
&\xrightarrow{(\mathcal{H}om^\bullet(T^{CH}_{\tilde{S}_I}(L(i_{I*} j_I^* F)), \mathrm{Gr}^{12}_{\tilde{S}_I *} E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}])} \\
&(e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\mathrm{Gr}^{12}_{\tilde{S}_I *} L\rho_{\tilde{S}_I *} \mu_{\tilde{S}_I *} R^{CH}(\rho^*_{\tilde{S}_I} L(i_{I*} j_I^* F)), \mathrm{Gr}^{12}_{\tilde{S}_I *} E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^{q,d}(F)) \\
&\xrightarrow{I(\mathrm{Gr}^{12*}_{\tilde{S}_I}, \mathrm{Gr}^{12}_{\tilde{S}_I *})(-,-) \circ \mathrm{Hom}(q,-)} \\
&(e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\mathrm{Gr}^{12*}_{\tilde{S}_I} L \, \mathrm{Gr}^{12}_{\tilde{S}_I *} L\rho_{\tilde{S}_I *} \mu_{\tilde{S}_I *} R^{CH}(\rho^*_{\tilde{S}_I} L(i_{I*} j_I^* F)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^{q,d}(F)) \\
&\xrightarrow{\mathcal{H}om(\mathrm{ad}(\mathrm{Gr}^{12*}_{\tilde{S}_I}, \mathrm{Gr}^{12}_{\tilde{S}_I *})(-) \circ q, -)^{-1}} \\
&(e'(\tilde{S}_I)_* \mathcal{H}om(L\rho_{\tilde{S}_I *} \mu_{\tilde{S}_I *} R^{CH}(\rho^*_{\tilde{S}_I} L(i_{I*} j_I^* F)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F)) =: \mathcal{F}_S^{FDR}(M)
\end{aligned}$$

Proposition 116. *Let $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^{l} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$.*

(i) For $M \in \mathrm{DA}_c(S)$ the map in $D_{O_S,\mathcal{D}}(S/(\tilde{S}_I)) = D_{O_S,\mathcal{D}}(S)$

$$o_{fil} T(\mathcal{F}_S^{GM}, \mathcal{F}_S^{FDR})(M) : o_{fil} \mathcal{F}_S^{GM}(L\mathbb{D}_S M) \xrightarrow{\sim} o_{fil} \mathcal{F}_S^{FDR}(M)$$

given in definition 118 is an isomorphism if we forgot the Hodge filtration F.

(ii) For $M \in \mathrm{DA}_c(S)$ and all $n, p \in \mathbb{Z}$, the map in $\mathrm{PSh}_{O_S,\mathcal{D}}(S/(\tilde{S}_I))$

$$F^p H^n T(\mathcal{F}_S^{GM}, \mathcal{F}_S^{FDR})(M) : F^p H^n \mathcal{F}_S^{GM}(L\mathbb{D}_S M) \hookrightarrow F^p H^n \mathcal{F}_S^{FDR}(M)$$

given in definition 118 is a monomorphism. Note that $F^p H^n T(\mathcal{F}_S^{GM}, \mathcal{F}_S^{FDR})(M)$ is NOT an isomorphism in general : take for example $M(S^o/S)^\vee = D(\mathbb{A}^1, et)(j_* E_{et}(\mathbb{Z}(S^o/S)))$ for an open embedding $j : S^o \hookrightarrow S$, then

$$H^n \mathcal{F}_S^{GM}(L\mathbb{D}_S M(S^o/S)^\vee) = \mathcal{F}_S^{GM}(\mathbb{Z}(S^o/S)) = j_* E(O_{S^o}, F_b) \notin \pi_S(MHM(S))$$

and hence is NOT isomorphic to $H^n \mathcal{F}_S^{FDR}(L\mathbb{D}_S M(S^o/S)^\vee) \in \pi_S(MHM(S))$, as filtered D_S-modules (see remark 9). It is an isomorphism in the very particular cases where $M = D(\mathbb{A}^1, et)(\mathbb{Z}(X/S))$ or $M = D(\mathbb{A}^1, et)(\mathbb{Z}(X^o/S))$ for $f : X \to S$ is a smooth proper morphism and $n : X^o \hookrightarrow X$ is an open subset such that $X \backslash X^o = \cup D_i$ is a normal crossing divisor and such that $f_{|D_i} = f \circ i_i : D_i \to X$ are SMOOTH morphism with $i_i : D_i \hookrightarrow X$ the closed embedding and considering $f_{|X^o} = f \circ n : X^o \to S$ (see proposition 109).

Proof. (i):Follows from the computation for a Borel-Moore motive.
(ii):Follows from (i). ∎

We now define the functorialities of $\mathcal{F}_S^{FDR}$ with respect to S which makes $\mathcal{F}^-_{FDR}$ a morphism of 2 functor.

Definition 119. *Let $S \in \mathrm{Var}(\mathbb{C})$. Let $Z \subset S$ a closed subset. Let $S = \cup_{i=1}^{l} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. Denote $Z_I := Z \cap S_I$. We then have closed embeddings $Z_I \hookrightarrow S_I \hookrightarrow \tilde{S}_I$.*

(i) For $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$, we will consider the following canonical map in $\pi_S(D(MHM(S))) \subset D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))$

$$T(\Gamma_Z^{\vee,Hdg}, \Omega_{/S}^{\Gamma,pr})(F,W) :$$
$$\Gamma_Z^{\vee,Hdg}\iota_S^{-1}(e'_*\mathcal{H}om^\bullet(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*}j_I^*(F,W))), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet,\Gamma,pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F,W))$$
$$\xrightarrow{\mathcal{H}om^\bullet(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R_{\tilde{S}_I}^{CH}(\gamma^{\vee,Z_I}(L(i_{I*}j_I^*(F,W)))), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet,\Gamma,pr}, F_{DR}))}$$
$$\Gamma_Z^{\vee,Hdg}\iota_S^{-1}(e'_*\mathcal{H}om^\bullet(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^* \Gamma_{Z_I}^\vee L(i_{I*}j_I^*(F,W))), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet,\Gamma,pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^{q,Z}(F,W))$$
$$\xrightarrow{=} \iota_S^{-1}(e'_*\mathcal{H}om^\bullet(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^* \Gamma_{Z_I}^\vee L(i_{I*}j_I^*(F,W))), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet,\Gamma,pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^{q,Z}(F,W)).$$

with

$$u_{IJ}^{q,Z}(F)[d_{\tilde{S}_I}] : e'(\tilde{S}_I)_*\mathcal{H}om^\bullet(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^* \Gamma_{Z_I}^\vee L(i_{I*}j_I^*F)), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet,\Gamma,pr}, F_{DR}))$$
$$\xrightarrow{p_{IJ*}T(p_{IJ},\Omega_{\cdot}^{\gamma,pr})(-)\circ\mathrm{ad}(p_{IJ}^{*mod},p_{IJ})(-)}$$
$$p_{IJ*}e'(\tilde{S}_J)_*\mathcal{H}om^\bullet(L\rho_{\tilde{S}_J*}\mu_{\tilde{S}_J*}p_{IJ}^*R^{CH}(\rho_{\tilde{S}_I}^* \Gamma_{Z_I}^\vee L(i_{I*}j_I^*F)), E_{et}(\Omega_{/\tilde{S}_J}^{\bullet,\Gamma,pr}, F_{DR}))$$
$$\xrightarrow{\mathcal{H}om(L\rho_{\tilde{S}_J*}\mu_{\tilde{S}_J*}T(p_{IJ},R^{CH})(Li_{I*}j_I^*F)^{-1}, E_{et}(\Omega_{/\tilde{S}_J}^{\bullet,\Gamma,pr}, F_{DR}))}$$
$$p_{IJ*}e'(\tilde{S}_J)_*\mathcal{H}om^\bullet(L\rho_{\tilde{S}_J*}\mu_{\tilde{S}_J*}R^{CH}(\rho_{\tilde{S}_J}^* p_{IJ}^*\Gamma_{Z_I}^\vee L(i_{I*}j_I^*F)), E_{et}(\Omega_{/\tilde{S}_J}^{\bullet,\Gamma,pr}, F_{DR}))$$
$$\xrightarrow{\mathcal{H}om(L\rho_{\tilde{S}_J*}\mu_{\tilde{S}_J*}R_{\tilde{S}_J}^{CH}(T^q(D_{IJ})(j_I^*F)\circ T(Z_J/Z_I\times\tilde{S}_{J\setminus I},\gamma^\vee)(-)\circ T(p_{IJ},\gamma)(-)), E_{et}(\Omega_{/\tilde{S}_J}^{\bullet,\Gamma,pr}, F_{DR}))}$$
$$p_{IJ*}e'(\tilde{S}_J)_*\mathcal{H}om^\bullet(L\rho_{\tilde{S}_J*}\mu_{\tilde{S}_J*}R^{CH}(\rho_{\tilde{S}_J}^* \Gamma_{Z_J}^\vee L(i_{J*}j_J^*F)), E_{et}(\Omega_{/\tilde{S}_J}^{\bullet,\Gamma,pr}, F_{DR})).$$

(ii) For $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$, we have also the following canonical map in $\pi_S(D(MHM(S))) \subset D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))$

$$T(\Gamma_Z^{Hdg}, \Omega_{/S}^{\Gamma,pr})(F,W) :$$
$$\iota_S^{-1}(e'_*\mathcal{H}om^\bullet(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^* L\Gamma_{Z_I}E(i_{I*}j_I^*\mathbb{D}_S(F,W))), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet,\Gamma,pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^{q,Z,d}(F,W)) \xrightarrow{=}$$
$$\Gamma_Z^{Hdg}\iota_S^{-1}(e'_*\mathcal{H}om^\bullet(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^* L\Gamma_{Z_I}E(i_{I*}j_I^*\mathbb{D}_S(F,W))), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet,\Gamma,pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^{q,Z,d}(F,W))$$
$$\xrightarrow{\mathcal{H}om^\bullet(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R_{\tilde{S}_I}^{CH}(\gamma^{Z_I}(-)), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet,\Gamma,pr}, F_{DR}))}$$
$$\Gamma_Z^{Hdg}\iota_S^{-1}(e'_*\mathcal{H}om^\bullet(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*}j_I^*\mathbb{D}_S(F,W))), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet,\Gamma,pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F,W))$$

with

$$u_{IJ}^{q,Z}(F)[d_{\tilde{S}_I}] : e'(\tilde{S}_I)_*\mathcal{H}om^\bullet(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^* L\Gamma_{Z_I}E(i_{I*}j_I^*\mathbb{D}_S LF)), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet,\Gamma,pr}, F_{DR}))$$
$$\xrightarrow{p_{IJ*}T(p_{IJ},\Omega_{\cdot}^{\gamma,pr})(-)\circ\mathrm{ad}(p_{IJ}^{*mod},p_{IJ})(-)}$$
$$p_{IJ*}e'(\tilde{S}_J)_*\mathcal{H}om^\bullet(L\rho_{\tilde{S}_J*}\mu_{\tilde{S}_J*}p_{IJ}^*R^{CH}(\rho_{\tilde{S}_I}^* L\Gamma_{Z_I}E(i_{I*}j_I^*\mathbb{D}_S F)), E_{et}(\Omega_{/\tilde{S}_J}^{\bullet,\Gamma,pr}, F_{DR}))$$
$$\xrightarrow{\mathcal{H}om(L\rho_{\tilde{S}_J*}\mu_{\tilde{S}_J*}T(p_{IJ},R^{CH})(Li_{I*}j_I^*F)^{-1}, E_{et}(\Omega_{/\tilde{S}_J}^{\bullet,\Gamma,pr}, F_{DR}))}$$
$$p_{IJ*}e'(\tilde{S}_J)_*\mathcal{H}om^\bullet(L\rho_{\tilde{S}_J*}\mu_{\tilde{S}_J*}R^{CH}(\rho_{\tilde{S}_J}^* Lp_{IJ}^*\Gamma_{Z_I}E(i_{I*}j_I^*\mathbb{D}_S LF)), E_{et}(\Omega_{/\tilde{S}_J}^{\bullet,\Gamma,pr}, F_{DR}))$$
$$\xrightarrow{\mathcal{H}om(L\rho_{\tilde{S}_J*}\mu_{\tilde{S}_J*}R_{\tilde{S}_J}^{CH}(\mathbb{D}_{\tilde{S}_J}S^q(D_{IJ})(\mathbb{D}_S LF)), E_{et}(\Omega_{/\tilde{S}_J}^{\bullet,\Gamma,pr}, F_{DR}))}$$
$$p_{IJ*}e'(\tilde{S}_J)_*\mathcal{H}om^\bullet(L\rho_{\tilde{S}_J*}\mu_{\tilde{S}_J*}R^{CH}(\rho_{\tilde{S}_J}^* \Gamma_{Z_J}E(i_{J*}j_J^*\mathbb{D}_S LF)), E_{et}(\Omega_{/\tilde{S}_J}^{\bullet,\Gamma,pr}, F_{DR})).$$

This transformation map will, with the projection case, gives the transformation between the pullback functor :

Definition 120. *Let $g : T \to S$ a morphism with $T, S \in \mathrm{SmVar}(\mathbb{C})$. Consider the factorization $g : T \xrightarrow{l} T \times S \xrightarrow{p_S} S$ where l is the graph embedding and p_S the projection. Let $M \in \mathrm{DA}_c(S)^-$ and $(F, W) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $(M, W) = D(\mathbb{A}^1_S, et)(F, W)$. Then, $D(\mathbb{A}^1_T, et)(g^*F) = g^*M$ and there exist $(F', W) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/T \times S)$ and an equivalence $(\mathbb{A}^1, et)$ local $e : \Gamma^\vee p_S^*(F, W) \to (F', W)$ such that $D(\mathbb{A}^1_{T\times S}, et)(F', W) = (\Gamma^\vee p_S^* M, W)$. We have then the canonical transformation in $\pi_T(D(MHM(T))$ using definition 114 and definition 119(i) :*

$$\begin{aligned}
&T(g, \mathcal{F}^{FDR})(M) : g^{\hat{*}mod,Hdg}\mathcal{F}_S^{FDR}(M) := \\
&\Gamma_T^{\vee,Hdg}\iota_T^{-1}(p_S^{*mod[-]}(e(S)_* \operatorname{Gr}^{12}_{S*} \mathcal{H}om^\bullet(L\rho_{S*}\mu_{S*}R^{CH}(\rho_S^* L(F, W)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR}))[-d_S])) \\
&\xrightarrow{T(p_S,\Omega^{\Gamma,pr}_{/\cdot})(-)} \\
&\Gamma_T^{\vee,Hdg}(e'(T\times S)_*\mathcal{H}om^\bullet(L\rho_{T\times S*}\mu_{T\times S*}p_S^* R^{CH}(\rho_S^* L(F)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/T\times S}, F_{DR}))[-d_S])) \\
&\xrightarrow{\mathcal{H}om(T(p_S,R^{CH})(L(F,W))^{-1}, E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR}))[-d_S]} \\
&\Gamma_T^{\vee,Hdg}\iota_T^{-1}(e'(T\times S)_*\mathcal{H}om^\bullet(L\rho_{T\times S*}\mu_{T\times S*}R^{CH}(\rho_{T\times S}^* p_S^* L(F, W)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/T\times S}, F_{DR}))[-d_S]) \xrightarrow{=} \\
&\Gamma_T^{\vee,Hdg}\iota_T^{-1}(e'(T\times S)_*\mathcal{H}om^\bullet(L\rho_{T\times S*}\mu_{T\times S*}R^{CH}(\rho_{T\times S}^* p_S^* L(F, W)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/T\times S}, F_{DR}))[-d_S]) \\
&\xrightarrow{T(\Gamma_T^{\vee,Hdg},\Omega^{\Gamma,pr}_{/T\times S})(F,W)} \\
&(e'(T\times S)_*\mathcal{H}om^\bullet(L\rho_{T\times S*}\mu_{T\times S*}R^{CH}(\rho_{T\times S}^* \Gamma_T^\vee p_S^* L(F, W)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/T\times S}, F_{DR}))[-d_S]) \\
&\xrightarrow{\mathcal{H}om(R^{CH}_{T\times S}(e),-)} \\
&(e'(T\times S)_*\mathcal{H}om^\bullet(L\rho_{T\times S*}\mu_{T\times S*}R^{CH}(\rho_{T\times S}^* L(F', W)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/T\times S}, F_{DR}))[-d_S]) \\
&=: \mathcal{F}^{FDR}_{T\times S}(l_* g^* M) = \mathcal{F}^{FDR}_T(g^* M)
\end{aligned}$$

where the last equality follows from proposition 115.

We give now the definition in the non smooth case Let $g : T \to S$ a morphism with $T, S \in \mathrm{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$ Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$, Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \to \tilde{S}_I$ is a lift of $g_I := g_{|T_I} : T_I \to S_I$. We recall the commutative diagram :

$$E_{IJg} = \begin{tikzcd} (Y\times\tilde{S}_I)\backslash T_I \arrow[r, "m'_I"] \arrow[d, "p_{\tilde{S}_I}"] & Y\times\tilde{S}_J \arrow[d, "\tilde{g}_I"] \\ \tilde{S}_I\backslash S_I \arrow[r, "m_I"] & \tilde{S}_I \end{tikzcd}, \; E_{IJ} = \begin{tikzcd} \tilde{S}_J\backslash S_J \arrow[r, "m_J"] \arrow[d, "p_{IJ}"] & \tilde{S}_J \arrow[d, "p_{IJ}"] \\ \tilde{S}_I\backslash(S_I\backslash S_J) \arrow[r, "m=m_{IJ}"] & \tilde{S}_I \end{tikzcd} \; E'_{IJ} = \begin{tikzcd} (Y\times\tilde{S}_J)\backslash T_J \arrow[r, "m'_J"] \arrow[d, "p'_{IJ}"] & Y\times\tilde{S}_J \arrow[d, "p'_{IJ}"] \\ (Y\times\tilde{S}_I)\backslash(T_I\backslash T_J) \arrow[r, "m'=m'_{IJ}"] & Y\times\tilde{S}_I \end{tikzcd}$$

For $I \subset J$, denote by $p_{IJ} : \tilde{S}_J \to \tilde{S}_I$ and $p'_{IJ} := I_Y \times p_{IJ} : Y \times \tilde{S}_J \to Y \times \tilde{S}_I$ the projections, so that $\tilde{g}_I \circ p'_{IJ} = p_{IJ} \circ \tilde{g}_J$. Consider, for $I \subset J \subset [1, \ldots, l]$, resp. for each $I \subset [1, \ldots, l]$, the following commutative diagrams in $\mathrm{Var}(\mathbb{C})$

$$D_{IJ} = \begin{tikzcd} S_I \arrow[r, "i_I"] & \tilde{S}_I \\ S_J \arrow[u, "j_{IJ}"] \arrow[r, "i_J"] & \tilde{S}_J \arrow[u, "p_{IJ}"] \end{tikzcd}, \; D'_{IJ} = \begin{tikzcd} T_I \arrow[r, "i'_I"] & Y\times\tilde{S}_I \\ T_J \arrow[u, "j'_{IJ}"] \arrow[r, "i'_J"] & Y\times\tilde{S}_J \arrow[u, "p'_{IJ}"] \end{tikzcd} \; D_{gI} = \begin{tikzcd} S_I \arrow[r, "i_I"] & \tilde{S}_I \\ T_I \arrow[u, "g_I"] \arrow[r, "i'_I"] & Y\times\tilde{S}_I \arrow[u, "\tilde{g}_I"] \end{tikzcd},$$

and $j_{IJ}: S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. Let $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$. Recall (see section 2) that since $j_I^{'*} i'_{I*} j_I^{'*} g^* F = 0$, the morphism $T(D_{gI})(j_I^* F) : \tilde{g}_I^* i_{I*} j_I^* F \to i'_{I*} j_I^{'*} g^* F$ factors trough

$$T(D_{gI})(j_I^*F) : \tilde{g}_I^* i_{I*} j_I^* F \xrightarrow{\gamma^{\vee}_{X_I}(-)} \Gamma^{\vee}_{X_I} \tilde{g}_I^* i_{I*} j_I^* F \xrightarrow{T^{\gamma}(D_{gI})(j_I^*F)} i'_{I*} j_I^{'*} g^* F.$$

and that the fact that the diagrams (55) commutes says that the maps $T^{q,\gamma}(D_{gI})(j_I^*F)$ define a morphism in $C(\mathrm{Var}(\mathbb{C})^{sm}/(T/(Y \times \tilde{S}_I)))$

$$(T^{q,\gamma}(D_{gI})(j_I^*F)) : (\Gamma^{\vee}_{T_I} \tilde{g}_I^* L(i_{I*} j_I^* F), T^q(D_{IJ})(j_I^*F) \circ T(T_I/T_I \times \tilde{S}_{J\setminus I}, \gamma^{\vee})(-) \circ T(p'_{IJ}, \gamma^{\vee})(-))) \\ \to (L(i'_{I*} j_I^{'*} g^* F), T^q(D'_{IJ})(j_I^{'*} g^* F))$$

Denote for short $d_{YI} := -d_Y - d_{\tilde{S}_I}$. We denote by $\tilde{g}_J^* u^q_{IJ}(F)$ the composite

$$\begin{aligned}
&\tilde{g}_J^* u^q_{IJ}(F)[-d_{YJ}] : \\
&e'(Y \times \tilde{S}_I)_* \mathcal{H}om(L\rho_{Y \times \tilde{S}_I *} \mu_{Y \times \tilde{S}_I *} R^{CH}(\rho^*_{Y \times \tilde{S}_I} \tilde{g}_I^* L(i_{I*} j_I^* F)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/Y \times \tilde{S}_I}, F_{DR})) \\
&\xrightarrow{p'_{IJ*} T(p'_{IJ}, \Omega^{\Gamma,pr})(-) \circ \mathrm{ad}(p_{IJ}^{'*mod}, p'_{IJ*})(-)} \\
&p'_{IJ*} e'(Y \times \tilde{S}_I)_* \mathcal{H}om(L\rho_{Y \times \tilde{S}_J *} \mu_{Y \times \tilde{S}_J *} p_{IJ}^{'*} R^{CH}(\rho^*_{Y \times \tilde{S}_I} \tilde{g}_I^* L(i_{I*} j_I^* F)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/Y \times \tilde{S}_J}, F_{DR})) \\
&\xrightarrow{\mathcal{H}om(L\rho_{Y \times \tilde{S}_J *} \mu^{12*}_{Y \times \tilde{S}_J *} T(p'_{IJ}, R^{CH})()^{-1}, E_{et}(\Omega^{\bullet,\Gamma,pr}_{/Y \times \tilde{S}_J}, F_{DR})} \\
&p'_{IJ*} e'(Y \times \tilde{S}_I)_* \mathcal{H}om(L\rho_{Y \times \tilde{S}_J *} \mu_{Y \times \tilde{S}_J *} R^{CH}(\rho^*_{Y \times \tilde{S}_J} p_{IJ}^{'*} \tilde{g}_I^* L(i_{I*} j_I^* F)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/Y \times \tilde{S}_J}, F_{DR})) \\
&\xrightarrow{\mathcal{H}om(L\rho_{Y \times \tilde{S}_J *} \mu_{Y \times \tilde{S}_J *} R^{CH}_{Y \times \tilde{S}_J}(T^q(D_{IJ})(j_I^*F)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/Y \times \tilde{S}_J}, F_{DR}))} \\
&e'(Y \times \tilde{S}_J)_* \mathcal{H}om(L\rho_{Y \times \tilde{S}_J *} \mu_{Y \times \tilde{S}_J *} R^{CH}(\rho^*_{Y \times \tilde{S}_J} \tilde{g}_J^* L(i_{J*} j_J^* F)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/Y \times \tilde{S}_I}, F_{DR})).
\end{aligned}$$

We denote by $\tilde{g}_J^{*,\gamma} u^q_{IJ}(F)$ the composite

$$\begin{aligned}
&\tilde{g}_J^{*,\gamma} u^q_{IJ}(F)[-d_{YJ}] : \\
&e'(Y \times \tilde{S}_I)_* \mathcal{H}om(L\rho_{Y \times \tilde{S}_I *} \mu_{Y \times \tilde{S}_I *} R^{CH}(\rho^*_{Y \times \tilde{S}_I} \Gamma^{\vee}_{T_I} \tilde{g}_I^* L(i_{I*} j_I^* F)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/Y \times \tilde{S}_I}, F_{DR})) \\
&\xrightarrow{p'_{IJ*} T(p'_{IJ}, \Omega^{\Gamma,pr})(-) \circ \mathrm{ad}(p_{IJ}^{'*mod}, p'_{IJ*})(-)} \\
&p'_{IJ*} e'(Y \times \tilde{S}_I)_* \mathcal{H}om(L\rho_{Y \times \tilde{S}_J *} \mu_{Y \times \tilde{S}_J *} p_{IJ}^{'*} R^{CH}(\rho^*_{Y \times \tilde{S}_I} \Gamma^{\vee}_{T_I} \tilde{g}_I^* L(i_{I*} j_I^* F)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/Y \times \tilde{S}_J}, F_{DR})) \\
&\xrightarrow{\mathcal{H}om(L\rho_{Y \times \tilde{S}_J *} \mu_{Y \times \tilde{S}_J *} T(p'_{IJ}, R^{CH})()^{-1}, E_{et}(\Omega^{\bullet,\Gamma,pr}_{/Y \times \tilde{S}_J}, F_{DR})} \\
&p'_{IJ*} e'(Y \times \tilde{S}_I)_* \mathcal{H}om(L\rho_{Y \times \tilde{S}_J *} \mathrm{Gr}^{12*}_{Y \times \tilde{S}_J} LR^{CH}(\rho^*_{Y \times \tilde{S}_J} p_{IJ}^{'*} \Gamma^{\vee}_{T_I} \tilde{g}_I^* L(i_{I*} j_I^* F)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/Y \times \tilde{S}_J}, F_{DR})) \\
&\xrightarrow{\mathcal{H}om(L\rho_{Y \times \tilde{S}_J *} \mu_{Y \times \tilde{S}_J} R^{CH}_{Y \times \tilde{S}_J}(T^q(D_{IJ})(j_I^*F) \circ T(T_I/T_I \times \tilde{S}_{J\setminus I}, \gamma^{\vee})(-) \circ T(p'_{IJ}, \gamma^{\vee})(-)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/Y \times \tilde{S}_J}, F_{DR}))} \\
&e'(Y \times \tilde{S}_J)_* \mathcal{H}om(L\rho_{Y \times \tilde{S}_J *} \mu^{12*}_{Y \times \tilde{S}_J} R^{CH}(\rho^*_{Y \times \tilde{S}_J} \Gamma^{\vee}_{T_J} \tilde{g}_J^* L(i_{J*} j_J^* F)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/Y \times \tilde{S}_I}, F_{DR})).
\end{aligned}$$

We then have then the following lemma :

Lemma 13. *(i) The morphism in* $C(\mathrm{Var}(\mathbb{C})^{sm}/(T/(Y \times \tilde{S}_I)))$

$$(T^{q,\gamma}(D_{gI})(j_I^*F)) : (\Gamma^{\vee}_{T_I} L\tilde{g}_I^* i_{I*} j_I^* F, T^q(D_{IJ})(j_I^*F) \circ T(T_I/T_I \times \tilde{S}_{J\setminus I}, \gamma^{\vee})(-) \circ T(p'_{IJ}, \gamma^{\vee})(-)) \\ \to (Li'_{I*} j_I^{'*} g^* F, T^q(D'_{IJ})(j_I^{'*} g^* F))$$

is an equivalence $(\mathbb{A}^1, et)$ *local.*

(ii) *The maps* $\mathcal{H}om((T^{q,\gamma}(D_{gI})(j_I^*F)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/Y\times\tilde{S}_I}), F_{DR}))$ *induce an* ∞*-filtered quasi-isomorphism in* $C_{\mathcal{D}fil}(T/(Y\times\tilde{S}_I))$

$$(\mathcal{H}om(L\rho_{Y\times\tilde{S}_I*}\mu_{Y\times\tilde{S}_I*}R^{CH}_{Y\times\tilde{S}_I}(T^{q,\gamma}(D_{gI})(j_I^*F)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/Y\times\tilde{S}_I}, F_{DR}))[d_{YI}]):$$
$$(e'(-)_*\mathcal{H}om(L\rho_{Y\times\tilde{S}_I*}\mu_{Y\times\tilde{S}_I*}R^{CH}(\rho^*_{Y\times\tilde{S}_I}\Gamma^\vee_{T_I}\tilde{g}_I^*L(i_{I*}j_I^*F)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/Y\times\tilde{S}_I}, F_{DR}))[d_{YI}], \tilde{g}_J^{*,\gamma}u^q_{IJ}(F))$$
$$\to(e'(-)_*\mathcal{H}om(L\rho_{Y\times\tilde{S}_I*}\mu_{Y\times\tilde{S}_I*}R^{CH}(\rho^*_{Y\times\tilde{S}_I}L(i'_{I*}j_I^{'*}g^*F)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/Y\times\tilde{S}_I}, F_{DR}))[d_{YI}], u^q_{IJ}(g^*F))$$

(iii) *The maps* $T(\tilde{g}_I,\Omega^{\Gamma,pr}_{\cdot})(-)$ *(see definition 114), induce a morphism in* $C_{\mathcal{D}fil}(T/(Y\times\tilde{S}_I))$

$$T(\tilde{g}_I,\Omega^{\Gamma,pr}_{/\cdot})(-))[d_{YI}]:$$
$$(\tilde{g}_I^{*mod}e'(\tilde{S}_I)_*\mathcal{H}om^\bullet(\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho^*_{\tilde{S}_I}(L(i_{I*}j_I^*F)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))[d_{YI}], \tilde{g}_J^{*mod}u^q_{IJ}(F))$$
$$\to(e'(-)_*\mathcal{H}om(\rho_{Y\times\tilde{S}_I*}\mu_{Y\times\tilde{S}_I*}\tilde{g}_I^*R^{CH}(\rho^*_{\tilde{S}_I}L(i_{I*}j_I^*F)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/Y\times\tilde{S}_I}, F_{DR}))[d_{YI}], \tilde{g}_J^*u^q_{IJ}(F)).$$

Proof. (i):Follows from theorem 16.
(ii):These morphism induce a morphism in $C_{\mathcal{D}fil}(T/(Y\times\tilde{S}_I))$ by construction. The fact that this morphism is an ∞-filtered equivalence Zariski local follows from (i) and proposition 112.
(iii):These morphism induce a morphism in $C_{\mathcal{D}fil}(T/(Y\times\tilde{S}_I))$ by construction. □

Definition 121. *Let* $g:T\to S$ *a morphism with* $T,S\in\mathrm{Var}(\mathbb{C})$*. Assume we have a factorization* $g:T\xrightarrow{l}Y\times S\xrightarrow{p_S}S$ *with* $Y\in\mathrm{SmVar}(\mathbb{C})$*,* l *a closed embedding and* p_S *the projection. Let* $S=\cup_{i=1}^l S_i$ *be an open cover such that there exists closed embeddings* $i_i:S_i\hookrightarrow\tilde{S}_i$ *with* $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$ *Then,* $T=\cup_{i=1}^l T_i$ *with* $T_i:=g^{-1}(S_i)$ *and we have closed embeddings* $i'_i:=i_i\circ l:T_i\hookrightarrow Y\times\tilde{S}_i$*, Moreover* $\tilde{g}_I:=p_{\tilde{S}_I}:Y\times\tilde{S}_I\to\tilde{S}_I$ *is a lift of* $g_I:=g_{|T_I}:T_I\to S_I$*. Let* $M\in\mathrm{DA}_c(S)^-$ *and* $(F,W)\in C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ *such that* $(M,W)=D(\mathbb{A}^1_S,et)(F,W)$*. Then,* $D(\mathbb{A}^1_T,et)(g^*F)=g^*M$ *and there exist* $(F',W)\in C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ *and an equivalence* $(\mathbb{A}^1,et)$ *local* $e:g^*(F,W)\to(F',W)$ *such that* $D(\mathbb{A}^1_T,et)(F',W)=(g^*M,W)$*.Denote for short* $d_{YI}:=-d_Y-d_{\tilde{S}_I}$*. We have, using definition 114 and definition 119(i), by lemma 13, the canonical map in* $\pi_T(D(MHM(T)))\subset D_{\mathcal{D}(1,0)fil,\infty}(T/(Y\times\tilde{S}_I))$

$$T(g,\mathcal{F}^{FDR})(M):g^{\hat{*}mod}_{Hdg}\iota_S^{-1}\mathcal{F}^{FDR}_S(M):=$$
$$\Gamma^{\vee,Hdg}_T\iota_T^{-1}(\tilde{g}_I^{*mod}(e'_*\mathcal{H}om^\bullet(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho^*_{\tilde{S}_I}(L(i_{I*}j_I^*(F,W))), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR})))[d_{YI}], \tilde{g}_J^{*mod}u^q_{IJ}(F,W)))$$
$$\xrightarrow{(T(\tilde{g}_I,\Omega^{\Gamma,pr}_{/\cdot})(-))}$$
$$\Gamma^{\vee,Hdg}_T\iota_T^{-1}(e'_*\mathcal{H}om^\bullet(L\rho_{Y\times\tilde{S}_I*}\mu_{Y\times\tilde{S}_I*}\tilde{g}_I^*R^{CH}(\rho^*_{\tilde{S}_I}L(i_{I*}j_I^*(F,W))), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/Y\times\tilde{S}_I}, F_{DR}))[d_{YI}], \tilde{g}_J^*u^q_{IJ}(F,W))$$
$$\xrightarrow{\mathcal{H}om(T(\tilde{g}_I,R^{CH})(-)^{-1},-)}$$
$$\Gamma^{\vee,Hdg}_T\iota_T^{-1}(e'_*\mathcal{H}om^\bullet(L\rho_{Y\times\tilde{S}_I*}\mu_{Y\times\tilde{S}_I*}R^{CH}(\rho^*_{Y\times\tilde{S}_I}\tilde{g}_I^*L(i_{I*}j_I^*(F,W))), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/Y\times\tilde{S}_I}, F_{DR}))[d_{YI}], \tilde{g}_J^*u^q_{IJ}(F,W))$$
$$\xrightarrow{T(\Gamma^{\vee,Hdg}_T,\Omega^{\Gamma,pr}_{/S})(F,W)}$$
$$\iota_T^{-1}(e'_*\mathcal{H}om^\bullet(L\rho_{Y\times\tilde{S}_I*}\mu_{Y\times\tilde{S}_I*}R^{CH}(\rho^*_{Y\times\tilde{S}_I}\Gamma^\vee_{T_I}\tilde{g}_I^*L(i_{I*}j_I^*(F,W))), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/Y\times\tilde{S}_I}, F_{DR}))[d_{YI}], \tilde{g}_J^{*,\gamma}u^q_{IJ}(F,W))$$
$$\xrightarrow{(\mathcal{H}om(L\rho_{Y\times\tilde{S}_I*}\mu_{Y\times\tilde{S}_I*}R^{CH}_{Y\times\tilde{S}_I}(T^{q,\gamma}(D_{gI})(j_I^*(F,W))),E_{et}(\Omega^{\bullet,\Gamma,pr}_{/Y\times\tilde{S}_I},F_{DR}))[d_{YI}])}$$
$$\iota_T^{-1}(e'_*\mathcal{H}om^\bullet(L\rho_{Y\times\tilde{S}_I*}\mu_{Y\times\tilde{S}_I*}R^{CH}(\rho^*_{Y\times\tilde{S}_I}L(i'_{I*}j_I^{'*}g^*(F,W))), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/Y\times\tilde{S}_I}, F_{DR}))[d_{YI}], u^q_{IJ}(g^*(F,W)))$$
$$\xrightarrow{\mathcal{H}om(R^{CH}_{Y\times\tilde{S}_I}(Li'_{I*}j_I^{'*}(e)),-)}$$
$$\iota_T^{-1}(e'_*\mathcal{H}om^\bullet(L\rho_{Y\times\tilde{S}_I*}\mu_{Y\times\tilde{S}_I*}R^{CH}(\rho^*_{Y\times\tilde{S}_I}L(i'_{I*}j_I^{'*}(F',W))), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/Y\times\tilde{S}_I}, F_{DR}))[d_{YI}], u^q_{IJ}(F',W))$$
$$\xrightarrow{=:}\mathcal{F}^{FDR}_T(g^*M)$$

Proposition 117. *Let $g : T \to S$ a morphism with $T, S \in \mathrm{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y_2 \times S \xrightarrow{p_S} S$ with $Y_2 \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$ Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y_2 \times \tilde{S}_i$, Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \to \tilde{S}_I$ is a lift of $g_I := g_{|T_I} : T_I \to S_I$. Let $f : X \to S$ a morphism with $X \in \mathrm{Var}(\mathbb{C})$ such that there exists a factorization $f : X \xrightarrow{l} Y_1 \times S \xrightarrow{p_S} S$, with $Y_1 \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have then the following commutative diagram whose squares are cartesians*

$$\begin{tikzcd}
f' : X_T \arrow[rr] \arrow[dr] \arrow[dd, "g'"'] & & Y_1 \times T \arrow[rr] \arrow[dr] & & T \arrow[dr] \arrow[dd] & \\
& Y_1 \times X \arrow[rr] \arrow[dl] & & Y_1 \times Y_2 \times S \arrow[rr] \arrow[dl] & & Y_2 \times S \arrow[dl] \\
f : X \arrow[rr] & & Y_1 \times S \arrow[rr] & & S &
\end{tikzcd}$$

Take a smooth compactification $\bar{Y}_1 \in \mathrm{PSmVar}(\mathbb{C})$ of Y_1, denote $\bar{X}_I \subset \bar{Y}_1 \times \tilde{S}_I$ the closure of X_I, and $Z_I := \bar{X}_I \backslash X_I$. Consider $F(X/S) := p_{S,\sharp}\Gamma_X^\vee \mathbb{Z}(Y_1 \times S/Y_1 \times S) \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ and the isomorphism in $C(\mathrm{Var}(\mathbb{C})^{sm}/T)$

$$T(f, g, F(X/S)) : g^* F(X/S) := g^* p_{S,\sharp}\Gamma_X^\vee \mathbb{Z}(Y_1 \times S/Y_1 \times S) \xrightarrow{\sim} p_{T,\sharp}\Gamma_{X_T}^\vee \mathbb{Z}(Y_1 \times T/Y_1 \times T) =: F(X_T/T).$$

which gives in $\mathrm{DA}(T)$ the isomorphism $T(f, g, F(X/S)) : g^* M(X/S) \xrightarrow{\sim} (X_T/T)$. Then the following diagram in $\pi_T(D(MHM(T))) \subset D_{\mathcal{D}(1,0)fil,\infty}(T/(Y_2 \times \tilde{S}_I))$, where the horizontal maps are given by proposition 114, commutes

$$\begin{tikzcd}
g_{Hdg}^{\hat{*}mod}\iota_S^{-1}\mathcal{F}_S^{FDR}(M(X/S)) \arrow[r, "g_{Hdg}^{\hat{*}mod}I(X/S)"] \arrow[dd, "'T(g,\mathcal{F}^{FDR})(M(X/S))"'] & g_{Hdg}^{\hat{*}mod}Rf_!^{Hdg}(\Gamma_{X_I}^{\vee,Hdg}(O_{Y_1\times\tilde{S}_I}, F_b)(d_{Y_1})[2d_{Y_1}], x_{IJ}(X/S)) \arrow[d, "T(p_{\tilde{S}_I}{,}\gamma^{\vee,Hdg})(-)"] \\
& Rf_!^{'Hdg}g_{Hdg}^{'\hat{*}mod}(\Gamma_{X_I}^{\vee,Hdg}(O_{Y_1\times\tilde{S}_I}, F_b)(d_{Y_1})[2d_{Y_1}], x_{IJ}(X/S)) \arrow[d, "T(p_{Y_1\times Y_2\times\tilde{S}_I,Hdg}^{\hat{*}mod}{,}p_{Y_1\times Y_2\times\tilde{S}_I,Hdg}^{*mod})(-)"] \\
\iota_T^{-1}\mathcal{F}_T^{FDR}(M(X_T/T)) \arrow[r, "I(X_T/T)"] & Rf_!^{'Hdg}(\Gamma_{X_{T_I}}^{\vee,Hdg}(O_{Y_2\times Y_1\times\tilde{S}_I}, F_b)(d_{Y_{12}})[2d_{Y_{12}}], x_{IJ}(X_T/T)).
\end{tikzcd}$$

with $d_{Y_{12}} = d_{Y_1} + d_{Y_2}$.

Proof. Follows immediately from definition. □

Theorem 35. *Let $g : T \to S$ a morphism, with $S, T \in \mathrm{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $M \in \mathrm{DA}_c(S)$. Then map in $\pi_T(D(MHM(T)))$*

$$T(g, \mathcal{F}^{FDR})(M) : g_{Hdg}^{\hat{*}mod}\mathcal{F}_S^{FDR}(M) \xrightarrow{\sim} \mathcal{F}_T^{FDR}(g^*M)$$

given in definition 121 is an isomorphism.

Proof. Follows from proposition 117 and proposition 114. □

Definition 122. • *Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have, for $M \in \mathrm{DA}_c(X)$, the following transformation map in $\pi_S(D(MHM(S)))$*

$$T_*(f,\mathcal{F}^{FDR})(M) : \mathcal{F}_S^{FDR}(Rf_*M) \xrightarrow{\mathrm{ad}(f_{Hdg}^{\hat{*}mod}, Rf_*^{Hdg})(-)} Rf_*^{Hdg} f_{Hdg}^{\hat{*}mod}\mathcal{F}_S^{FDR}(Rf_*M)$$
$$\xrightarrow{T(f,\mathcal{F}^{FDR})(Rf_*M)} Rf_*^{Hdg}\mathcal{F}_X^{FDR}(f^*Rf_*M) \xrightarrow{\mathcal{F}_X^{FDR}(\mathrm{ad}(f^*,Rf_*)(M))} Rf_*^{Hdg}\mathcal{F}_X^{FDR}(M)$$

Clearly, for $p : Y \times S \to S$ a projection with $Y \in \mathrm{PSmVar}(\mathbb{C})$, we have, for $M \in \mathrm{DA}_c(Y \times S)$, $T_*(p,\mathcal{F}^{FDR})(M) = T_!(p,\mathcal{F}^{FDR})(M)[d_Y]$

- *Let $S \in \mathrm{Var}(\mathbb{C})$. Let $Y \in \mathrm{SmVar}(\mathbb{C})$ and $p : Y \times S \to S$ the projection. We have then, for $M \in \mathrm{DA}(Y \times S)$ the following transformation map in $\pi_S(D(MHM(S)))$*

$$T_!(p,\mathcal{F}^{FDR})(M) : p_!^{Hdg}\mathcal{F}_{Y\times S}^{FDR}(M) \xrightarrow{\mathcal{F}_{Y\times S}^{FDR}(\mathrm{ad}(Lp_\sharp,p^*)(M))} Rp_!^{Hdg}\mathcal{F}_{Y\times S}^{FDR}(p^*Lp_\sharp M)$$
$$\xrightarrow{T(p,\mathcal{F}^{FDR})(Lp_\sharp(M,W))} Rp_!^{Hdg}p^{\hat{*}mod[-]}\mathcal{F}_S^{FDR}(Lp_\sharp M) \xrightarrow{T(p^{*mod},p^{\hat{*}mod})(-)} p_!^{Hdg}p^{*mod[-]}$$
$$\mathcal{F}_S^{FDR}(Lp_\sharp M) \xrightarrow{\mathrm{ad}(Rp_!^{Hdg},p^{*mod[-]})(\mathcal{F}_S^{FDR}(Lp_\sharp M))} \mathcal{F}_S^{FDR}(Lp_\sharp M)$$

- *Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have then, using the second point, for $M \in \mathrm{DA}(X)$ the following transformation map in $\pi_S(D(MHM(S)))$*

$$T_!(f,\mathcal{F}^{FDR})(M) : Rp_!^{Hdg}\mathcal{F}_X^{FDR}(M,W) := Rp_!^{Hdg}\mathcal{F}_{Y\times S}^{FDR}(l_*M)$$
$$\xrightarrow{T_!(p,\mathcal{F}^{FDR})(l_*M)} \mathcal{F}_S^{FDR}(Lp_\sharp l_*M) \xrightarrow{=} \mathcal{F}_S^{FDR}(Rf_!M)$$

- *Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have, using the third point, for $M \in \mathrm{DA}(S)$, the following transformation map in in $\pi_X(D(MHM(X)))$*

$$T^!(f,\mathcal{F}^{FDR})(M) : \mathcal{F}_X^{FDR}(f^!M) \xrightarrow{\mathrm{ad}(Rf_!^{Hdg},f_{Hdg}^{*mod})(\mathcal{F}_X^{FDR}(f^!M))} f_{Hdg}^{*mod}Rf_!^{Hdg}\mathcal{F}_X^{FDR}(f^!M)$$
$$\xrightarrow{T_!(p_S,\mathcal{F}^{FDR})(\mathcal{F}^{FDR}(f^!M))} f_{Hdg}^{*mod}\mathcal{F}_S^{FDR}(Rf_!f^!M) \xrightarrow{\mathcal{F}_S^{FDR}(\mathrm{ad}(Rf_!,f^!)(M))} f_{Hdg}^{*mod}\mathcal{F}_S^{FDR}(M)$$

Proposition 118. *Let $S \in \mathrm{Var}(\mathbb{C})$. Let $Y \in \mathrm{SmVar}(\mathbb{C})$ and $p : Y \times S \to S$ the projection. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i^o : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. For $I \subset [1,\cdots l]$, we denote by $S_I = \cap_{i\in I}S_i$, $j_I^o : S_I \hookrightarrow S$ and $j_I : Y \times S_I \hookrightarrow Y \times S$ the open embeddings. We then have closed embeddings $i_I : Y \times S_I \hookrightarrow Y \times \tilde{S}_I$. and we denote by $p_{\tilde{S}_I} : Y \times \tilde{S}_I \to \tilde{S}_I$ the projections. Let $f' : X' \to Y \times S$ a morphism, with $X' \in \mathrm{Var}(\mathbb{C})$ such that there exists a factorization $f' : X' \xrightarrow{l'} Y' \times Y \times S \xrightarrow{p'} Y \times S$ with $Y' \in \mathrm{SmVar}(\mathbb{C})$, l' a closed embedding and p' the projection. Denoting $X'_I := f'^{-1}(Y \times S_I)$, we have closed embeddings $i'_I : X'_I \hookrightarrow Y' \times Y \times \tilde{S}_I$ Consider*

$$F(X'/Y\times S) := p_{Y\times S,\sharp}\Gamma_{X'}^{\vee}\mathbb{Z}(Y'\times Y\times S/Y'\times Y\times S) \in C(\mathrm{Var}(\mathbb{C})^{sm}/Y\times S)$$

and $F(X'/S) := p_\sharp F(X'/Y\times S) \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$, so that $Lp_\sharp M(X'/Y\times S)[-2d_Y] =: M(X'/S)$. Then, the following diagram in $\pi_S(D(MHM(S))) \subset D_{\mathcal{D}(1,0)fil,\infty}(S/(Y\times\tilde{S}_I))$, where the vertical maps are given by proposition 114, commutes

$$\begin{tikzcd}
Rp^{Hdg}!\mathcal{F}_{Y\times S}^{FDR}(M(X'/Y\times S)) \arrow[r, "T_!(p,\mathcal{F}^{FDR})(M(X'/Y\times S))"] & \mathcal{F}_S^{FDR}(M(X'/S)) \\
Rp^{Hdg}!Rf_!'^{Hdg}f_{Hdg}'^{*mod}\mathbb{Z}_{Y\times S}^{Hdg} \arrow[r, "="] \arrow[u, "T(p_{Hdg}^{\hat{*}mod},p_{Hdg}^{*mod})(-)\circ Rp^{Hdg}!(I(X'/Y\times S))"] & Rf_!^{Hdg}f_{Hdg}^{*mod}\mathbb{Z}_S^{Hdg} \arrow[u, "I(X'/S)"']
\end{tikzcd}$$

Proof. Immediate from definition. ☐

Theorem 36. *(i) Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Then, for $M \in \mathrm{DA}_c(X)$, the map given in definition 122*

$$T_!(f, \mathcal{F}^{FDR})(M) : Rf_!^{Hdg}\mathcal{F}_X^{FDR}(M) \xrightarrow{\sim} \mathcal{F}_S^{FDR}(Rf_! M)$$

is an isomorphism in $\pi_S(D(MHM(S))$.

(ii) Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$, S quasi-projective. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have, for $M \in \mathrm{DA}_c(X)$, the map given in definition 122

$$T_*(f, \mathcal{F}^{FDR})(M) : \mathcal{F}_S^{FDR}(Rf_* M) \xrightarrow{\sim} Rf_*^{Hdg}\mathcal{F}_X^{FDR}(M)$$

is an isomorphism in $\pi_S(D(MHM(S))$.

(iii) Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$, S quasi-projective. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Then, for $M \in \mathrm{DA}_c(S)$, the map given in definition 122

$$T^!(f, \mathcal{F}^{FDR})(M) : \mathcal{F}_X^{FDR}(f^! M) \xrightarrow{\sim} f_{Hdg}^{*mod}\mathcal{F}_S^{FDR}(M)$$

is an isomorphism in $\pi_X(D(MHM(X))$.

Proof. (i): By proposition 118 and proposition 114, for $S \in \mathrm{Var}(\mathbb{C})$, $Y \in \mathrm{SmVar}(\mathbb{C})$, $p : Y \times S \to S$ the projection and $M \in \mathrm{DA}_c(Y \times S)$,

$$T_!(p, \mathcal{F}^{FDR})(M) : Rp_!^{Hdg}\mathcal{F}_{Y\times S}^{FDR}(M) \to \mathcal{F}_S^{FDR}(Rp_! M)$$

is an isomorphism.
(ii): Consider first an open embedding $n : S^o \hookrightarrow S$ with $S \in \mathrm{Var}(\mathbb{C})$ so that there exist a closed embedding $i : S \hookrightarrow \tilde{S}$ with $\tilde{S} \in \mathrm{SmVar}(\mathbb{C})$. Then, since

$$n^* : C(\mathrm{Var}(\mathbb{C})^{sm}/S) \to C(\mathrm{Var}(\mathbb{C})^{sm}/S^o)$$

is surjective, $n^* : \mathrm{DA}(S) \to \mathrm{DA}(S^o)$ is surjective. Denote by $i : Z = S\backslash S^o \hookrightarrow S$ the complementary closed embedding. By [1], $\mathrm{DA}_c(S)$ is generated by motives of the form

$$\begin{aligned}\mathrm{DA}_c(S) =< M(X'/S) = f'_*E(\mathbb{Z}_{X'}),\ f' : X' \to S \text{ proper with } X' \in \mathrm{SmVar}(\mathbb{C}),\\ \text{s.t. } f'^{-1}(Z) = X' \text{ or } f'^{-1}(Z) = \cup D_i = D \subset X' > .\end{aligned}$$

If $f'^{-1}(Z) = X'$, $n^*M(X'/S) = 0$. So let consider the case $f'^{-1}(Z) = \cup_{i=1}^l D_i = D \subset X'$ is a normal crossing divisor. Denote $f'_D : f'_{|D} : D \to Z$, $D_I = \cap_{i\in I} D_i$ and $i'_I : D_I \hookrightarrow X'$, $n' : X'^o := X'\backslash D \hookrightarrow X'$ the complementary open embedding and $f'^o : f'_{|X'^o} : X'^o \to S^o$. Denote $L = [1, \ldots, l]$. We have then a generalized distinguish triangle in $\mathrm{DA}(X')$

$$\begin{aligned}
a(n', i') : n'_*n'^*E_{et}(\mathbb{Z}_{X'}) &\xrightarrow{\sim} \mathrm{Cone}(\gamma_D(-) : \Gamma_D E_{et}(\mathbb{Z}_{X'}) \to E_{et}(\mathbb{Z}_{X'}))\\
&\xrightarrow{\sim} \mathrm{Cone}(\Gamma_{D_L}E_{et}(\mathbb{Z}_{X'}) \to \cdots \to \bigoplus_{i=1}^{l} E_{et}(\mathbb{Z}_{X'}))\\
&\xrightarrow{\sim} \mathrm{Cone}(i'_{L*}i'^!_L E_{et}(\mathbb{Z}_{X'}) \to \cdots \to \bigoplus_{i=1}^{l} i'_{i*}i'^!_i E_{et}(\mathbb{Z}_{X'}) \xrightarrow{\mathrm{ad}(i'_{i*}, i'^!_i)(E_{et}(\mathbb{Z}_{X'}))} E_{et}(\mathbb{Z}_{X'}))\\
&\xrightarrow{\sim} \mathrm{Cone}(i'_{L*}\mathbb{Z}_{D_L}[-l] \to \cdots \bigoplus_{i=1}^{l} i'_{i*}\mathbb{Z}_{D_i}[-1] \to \mathbb{Z}_{X'})
\end{aligned}$$

where the first isomorphism is the image of an homotopy equivalence by definition, the second one is the image of an homotopy equivalence by definition-proposition 4(ii), the third one follows by the localization property (see section 3, theorem 16) and the last one follows from purity since the D_I and X' are smooth (see section 3, theorem 16). Similarly, we have a generalized distinguish triangle in $D(MHM(X'))$

$$\begin{aligned}
a^{mod}(n',i') : n'^{Hdg}_{*} n'^{*}(O_{X'},F_b) &\xrightarrow{\sim} \mathrm{Cone}(\gamma^{Hdg}_{D}(O_{X'},F_b) : \Gamma^{Hdg}_{D}(O_{X'},F_b) \to (O_{X'},F_b)) \\
&\xrightarrow{\sim} \mathrm{Cone}(\Gamma^{Hdg}_{D_L}(O_{X'},F_b) \to \cdots \to \bigoplus_{i=1}^{l} \Gamma^{Hdg}_{D_i}(O_{X'},F_b) \xrightarrow{\oplus_i \gamma^{Hdg}_{D_i}(-)} (O_{X'},F_b)) \\
&\xrightarrow{\sim} \mathrm{Cone}(i'_{L*mod}(O_{D_L},F_b)[-l] \to \cdots \to \bigoplus_{i=1}^{l} i'_{i*mod}(O_{D_i},F_b)[-1] \\
&\xrightarrow{\mathrm{ad}(i'_{i*mod},i'^{\sharp}_{i})(O_{X'},F_b)} \cdots \to (O_{X'},F_b))
\end{aligned}$$

where the first isomorphism is the image of an homotopy equivalence by definition, the second one is the image of an homotopy equivalence by definition-proposition 19, and the third one follows by the localization property of mixed Hodge modules (see section 5). Consider $n^*M(X'/S) = M(X'^{o}/S^o)$. We have then the following commutative diagram in $\pi_S(MHM(S))$

$$\begin{tikzcd}
\mathcal{F}^{FDR}_S(Rn_*n^*M(X'/S)) = \mathcal{F}^{FDR}_S(Rn_*n^*Rf'_*\mathbb{Z}_{X'}) \arrow[r, "T_*(n,\mathcal{F}^{FDR})(n^*M(X'/S))"] \arrow[d, "\mathcal{F}^{FDR}_S(Rn_*T(n,f')(\mathbb{Z}_{X'}))"'] & n_{*Hdg}\mathcal{F}^{FDR}_{S^o}(n^*M(X'/S)) = n_{*Hdg}\mathcal{F}^{FDR}_{S^o}(n^*Rf'_*\mathbb{Z}_{X'}) \arrow[d, "n_{*Hdg}\mathcal{F}^{FDR}_{S^o}(T(n,f')(\mathbb{Z}_{X'}))"] \\
\mathcal{F}^{FDR}_S(Rn_*Rf'^{o}_*n'^*\mathbb{Z}_{X'}) = \mathcal{F}^{FDR}_S(Rf'_*Rn'_*n'^*\mathbb{Z}_{X'}) \arrow[r, "T_*(n,\mathcal{F}^{FDR})(M(X'^{o}/S))"] \arrow[d, "T_*(f',\mathcal{F}^{FDR})(Rn'_*n'^*\mathbb{Z}_{X'})"'] & n_{*Hdg}\mathcal{F}^{FDR}_{S^o}(Rf'^{o}_*n'^*\mathbb{Z}_{X'}) \arrow[d, "n_{*Hdg}T_*(f'^{o},\mathcal{F}^{FDR})(n'^*\mathbb{Z}_{X'})"] \\
Rf'_{*Hdg}\mathcal{F}^{FDR}_{X'}(Rn'_*n'^*\mathbb{Z}_{X'}) \arrow[r, "T_*(n,\mathcal{F}^{FDR})(n'^*\mathbb{Z}_{X'})"] \arrow[d, "Rf'_{*Hdg}\mathcal{F}^{FDR}_{X'}(a(n',i'))"'] & n_{*Hdg}Rf'^{o}_{*Hdg}\mathcal{F}^{FDR}_{X'^{o}}(n'^*\mathbb{Z}_{X'}) = Rf'_{*Hdg}n'_{*Hdg}n'^*\mathbb{Z}^{Hdg}_{X'}) \arrow[d, "Rf'_{*Hdg}a^{mod}(n',i')"] \\
Rf'^{Hdg}_{*}\mathcal{F}^{FDR}_{X'}(\mathrm{Cone}(i'_{L*}\mathbb{Z}_{D_L}[-l] \to \cdots \to \mathbb{Z}_{X'})) \arrow[r, "(T_*(i'_I,\mathcal{F}^{FDR})(-)\circ T(i'_I,\mathcal{F}^{FDR}_{Hdg})(-))"] & Rf'^{Hdg}_{*}\mathrm{Cone}(i'_{L*mod}(O_{D_L},F_b)[-l] \to \cdots \to (O_{X'},F_b))
\end{tikzcd}$$

Since all the morphism involved are isomorphisms, $T_*(n,\mathcal{F}^{FDR})(n^*M(X'/S))$ is an isomorphism. Hence, $T_*(n,\mathcal{F}^{FDR})(M)$ is an isomorphism for all $M \in \mathrm{DA}(S^o)$. Consider now the case of a general morphism $f : X \to S$, $X,S \in \mathrm{Var}(\mathbb{C})$, S quasi-projective, which factors trough $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with some $Y \in \mathrm{SmVar}(\mathbb{C})$. By definition, for $M \in \mathrm{DA}_c(X)$

$$\begin{aligned}
T_*(f,\mathcal{F}^{FDR})(M) : \mathcal{F}^{FDR}_S(Rf_*M) &= \mathcal{F}^{FDR}_S(Rp_{S*}l_*M) \\
&\xrightarrow{T_*(p_S,\mathcal{F}^{FDR})(l_*M)} Rp^{Hdg}_{S*}\mathcal{F}^{FDR}_{Y\times S}(l_*M) =: Rf^{Hdg}_*\mathcal{F}^{FDR}_X(M).
\end{aligned}$$

Hence, we have to show that for $S \in \mathrm{Var}(\mathbb{C})$, $Y \in \mathrm{SmVar}(\mathbb{C})$, $p : Y \times S \to S$ the projection, and $M \in \mathrm{DA}_c(Y \times S)$,

$$T_*(p,\mathcal{F}^{FDR})(M) : \mathcal{F}^{FDR}_S(Rp_*M) \to Rp^{Hdg}_*\mathcal{F}^{FDR}_{Y\times S}(M)$$

is an isomorphism. Take a smooth compactification $\bar{Y} \in \mathrm{PSmVar}(\mathbb{C})$ of Y. Denote by $n_0 : Y \hookrightarrow \bar{Y}$ and $n := n_0 \times I_S : Y \times S \hookrightarrow \bar{Y} \times S$ the open embeddings and by $\bar{p} : \bar{Y} \times S \to S$ the projection. We have $p = \bar{p} \circ n : Y \times S \to S$, which gives the factorization

$$\begin{aligned}
T_*(p,\mathcal{F}^{FDR})(M) : \mathcal{F}^{FDR}_S(Rp_*M) = \mathcal{F}^{FDR}_S(R\bar{p}_*Rn_*M) &\xrightarrow{T_*(\bar{p},\mathcal{F}^{FDR})(Rn_*M)} R\bar{p}^{Hdg}_*\mathcal{F}^{FDR}_{\bar{Y}\times S}(Rn_*M) \\
&\xrightarrow{T_*(n,\mathcal{F}^{FDR})(M)} R\bar{p}^{Hdg}_* n^{Hdg}_* \mathcal{F}^{FDR}_{Y\times S}(M) = Rp^{Hdg}_*\mathcal{F}^{FDR}_{Y\times S}(M).
\end{aligned}$$

By the open embedding case $T_*(n,\mathcal{F}^{FDR})(M)$ is an isomorphism. On the other hand, since $\bar{p}$ is proper, $T_*(\bar{p},\mathcal{F}^{FDR})(Rn_*M)=T_!(\bar{p},\mathcal{F}^{FDR})(Rn_*M)$ is an isomorphism by (i).
(iii): Denote by $n:Y\times S\backslash X\hookrightarrow Y\times S$ the complementary open embedding. We have, for $M\in\mathrm{DA}_c(S)$, the factorization

$$\begin{aligned}T^!(f,\mathcal{F}^{FDR})(M):\mathcal{F}_X^{FDR}(f^!M)=\mathcal{F}_{Y\times S}^{FDR}(l_*l^!p_S^!M)&\xrightarrow{\mathcal{F}_{Y\times S}^{FDR}(a(n,l))}\mathcal{F}_{Y\times S}^{FDR}(\mathrm{Cone}(p_S^!M\to Rn_*n^*p_S^!M)[-1])\\&\xrightarrow{=}\mathrm{Cone}(\mathcal{F}_{Y\times S}^{FDR}(p_S^!M)\to\mathcal{F}_{Y\times S}^{FDR}(Rn_*n^*p_S^!M))[-1]\\&\xrightarrow{(T(n,\mathcal{F}^{FDR})(p_S^!M)\circ T^!(p_S,\mathcal{F}^{FDR}(M)),T^!(p_S,\mathcal{F}^{FDR}(M)))}\\\mathrm{Cone}(p_S^{*mod[-]}\mathcal{F}_S^{FDR}(M)\to n_*^{Hdg}n^*p_S^{*mod[-]}\mathcal{F}_S^{FDR}(M))[-1]&\xrightarrow{\sim}f_{Hdg}^{*mod}\mathcal{F}_S^{FDR}(M).\end{aligned}$$

By (ii),$T(n,\mathcal{F}^{FDR})(p_S^!M)$ is an isomorphism. On the other hand, since p_S is a smooth morphism, $T^!(p_S,\mathcal{F}^{FDR}(M))=T(p_S,\mathbb{D}\mathcal{F}^{FDR}(M))[d_Y]$; hence, $T^!(p_S,\mathcal{F}^{FDR}(M))$ is an isomorphism by theorem 35. □

Lemma 14. *Let $g:T\to S$ a morphism with $T,S\in\mathrm{Var}(\mathbb{C})$. Assume we have a factorization $g:T\xrightarrow{l}Y\times S\xrightarrow{p_S}S$ with $Y\in\mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S=\cup_{i=1}^lS_i$ be an open cover such that there exists closed embeddings $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$ Then, $T=\cup_{i=1}^lT_i$ with $T_i:=g^{-1}(S_i)$ and we have closed embeddings $i'_i:=i_i\circ l:T_i\hookrightarrow Y\times\tilde{S}_i$, Moreover $\tilde{g}_I:=p_{\tilde{S}_I}:Y\times\tilde{S}_I\to\tilde{S}_I$ is a lift of $g_I:=g_{|T_I}:T_I\to S_I$. Let $M\in\mathrm{DA}_c(S)^-$ and $(F,W)\in C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $(M,W)=D(\mathbb{A}_S^1,et)(F,W)$. Then, $g^!M=L\mathbb{D}_Sg^*L\mathbb{D}_SM$, $D(\mathbb{A}_T^1,et)(g^*\mathbb{D}_SLF)=g^*L\mathbb{D}_SM$ and there exist $(F',W)\in C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ and an equivalence $(\mathbb{A}^1,et)$ local $e:(F',W)\to(g^*\mathbb{D}_SL(F,W))$ such that $D(\mathbb{A}_T^1,et)(F',W)=g^*L\mathbb{D}_S(M,W)$ and, using definition 114 and definition 119(ii) and lemma 13, the map in $\pi_T(D(MHM(T)))\subset D_{\mathcal{D}(1,0)fil,\infty}(T/(Y\times\tilde{S}_I))$*

$$T^!(g,\mathcal{F}^{FDR})(M):\mathcal{F}_T^{FDR}(g^!M)\to g_{Hdg}^{*mod}\mathcal{F}_S^{FDR}(M)$$

given in definition 122 is the inverse of the following map

$$\begin{aligned}&T^{!,-1}(g,\mathcal{F}^{FDR})(M):g_{Hdg}^{*mod}\iota_S^{-1}\mathcal{F}_S^{FDR}(M)\xrightarrow{:=}\\&(\Gamma_T^{Hdg}\iota_T^{-1}(\tilde{g}_I^{*mod}(e'_*\mathcal{H}om^\bullet(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^*L(i_{I*}j_I^*\mathbb{D}_SL(F,W))),\\&E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I},F_{DR})))[-d_{YI}],\tilde{g}_J^{*mod}u_{IJ}^q(\mathbb{D}_SL(F,W)))\xrightarrow{(T(\tilde{g}_I,\Omega^{\Gamma,pr}_{/\cdot})(-))}\\&\Gamma_T^{Hdg}\iota_T^{-1}(e'_*\mathcal{H}om^\bullet(L\rho_{Y\times\tilde{S}_I*}\mu_{Y\times\tilde{S}_I*}\tilde{g}_I^*R^{CH}(\rho_{\tilde{S}_I}^*L(i_{I*}j_I^*\mathbb{D}_SL(F,W))),\\&E_{et}(\Omega^{\bullet,\Gamma,pr}_{/Y\times\tilde{S}_I},F_{DR})),\tilde{g}_J^*u_{IJ}^q(\mathbb{D}_SL(F,W)))\xrightarrow{\mathcal{H}om(T(\tilde{g}_I,R^{CH})(-)^{-1},-)}\\&\Gamma_T^{Hdg}\iota_T^{-1}(e'_*\mathcal{H}om^\bullet(L\rho_{Y\times\tilde{S}_I*}\mu_{Y\times\tilde{S}_I*}R^{CH}(\rho_{Y\times\tilde{S}_I}^*\tilde{g}_I^*L(i_{I*}j_I^*\mathbb{D}_SL(F,W))),\\&E_{et}(\Omega^{\bullet,\Gamma,pr}_{/Y\times\tilde{S}_I},F_{DR}))[d_{YI}],\tilde{g}_J^*u_{IJ}^q\mathbb{D}_SL(F,W))\xrightarrow{T(\Gamma_T^{Hdg},\Omega^{\Gamma,pr}_{/S})(F,W)^{-1}}\\&\iota_T^{-1}(e'_*\mathcal{H}om^\bullet(L\rho_{Y\times\tilde{S}_I*}\mu_{Y\times\tilde{S}_I*}R^{CH}(\rho_{Y\times\tilde{S}_I}^*L\Gamma_{T_I}E(\tilde{g}_I^*L(i_{I*}j_I^*\mathbb{D}_SL(F,W))),\\&E_{et}(\Omega^{\bullet,\Gamma,pr}_{/Y\times\tilde{S}_I},F_{DR}))[d_{YI}],\tilde{g}_J^{*,\gamma,d}u_{IJ}^q(F,W))\\&\xrightarrow{(\mathcal{H}om(L\rho_{Y\times\tilde{S}_I*}\mu_{Y\times\tilde{S}_I*}R^{CH}_{Y\times\tilde{S}_I}(\mathbb{D}_{Y\times\tilde{S}_I}T^{q,\gamma}(D_{gI})(j_I^*\mathbb{D}_SL(F,W))),E_{et}(\Omega^{\bullet,\Gamma,pr}_{/Y\times\tilde{S}_I},F_{DR}))[d_{YI}])^{-1}}\\&\iota_T^{-1}(e'_*\mathcal{H}om^\bullet(L\rho_{Y\times\tilde{S}_I*}\mu_{Y\times\tilde{S}_I*}R^{CH}(\rho_{Y\times\tilde{S}_I}^*L\mathbb{D}_{Y\times\tilde{S}_I}L(i'_{I*}j_I^{'*}g^*\mathbb{D}_SL(F,W))),\\&E_{et}(\Omega^{\bullet,\Gamma,pr}_{/Y\times\tilde{S}_I},F_{DR}))[d_{YI}],u_{IJ}^{q,d}(g^*\mathbb{D}_SL(F,W)))\xrightarrow{\mathcal{H}om(R^{CH}_{Y\times\tilde{S}_I}(\mathbb{D}_{Y\times\tilde{S}_I}Li'_{I*}j_I^{'*}(e)),-))}\\&\iota_T^{-1}(e'_*\mathcal{H}om(L\rho_{Y\times\tilde{S}_I*}\mu_{Y\times\tilde{S}_I*}R^{CH}(\rho_{Y\times\tilde{S}_I}^*L\mathbb{D}_{Y\times\tilde{S}_I}L(i'_{I*}j_I^{'*}g^*\mathbb{D}_SL(F,W))),\\&E_{et}(\Omega^{\bullet,\Gamma,pr}_{/Y\times\tilde{S}_I},F_{DR}))[d_{YI}],u_{IJ}^{q,d}(L(F',W)))\xrightarrow{=:}\mathcal{F}_T^{FDR}(g^!M)\end{aligned}$$

We have the following proposition :

Proposition 119. *Let $g : T \to S$ a morphism with $T, S \in \mathrm{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^{l} S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$ Then, $T = \cup_{i=1}^{l} T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$, Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \to \tilde{S}_I$ is a lift of $g_I := g_{|T_I} : T_I \to S_I$. Let $M \in \mathrm{DA}_c(S)$ and $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1_S, et)(F)$. Then, $D(\mathbb{A}^1_T, et)(g^* F) = g^* M$. Then the following diagram in $D_{Ofil,\mathcal{D},\infty}(T/(Y \times \tilde{S}_I))$ commutes*

$$\begin{tikzcd}
Rg^{*mod[-],\Gamma}\mathcal{F}_S^{GM}(L\mathbb{D}_S M) \arrow[r, "Rg^{*mod[-],\Gamma}T(\mathcal{F}_S^{GM}{,}\mathcal{F}_S^{FDR})(M)"] \arrow[d, "T(g{,}\mathcal{F}^{GM})(L\mathbb{D}_S M)"] & Rg^{*mod[-],\Gamma}\mathcal{F}_S^{FDR}(M) \arrow[dr] & g^{*mod}_{Hdg}\mathcal{F}_S^{FDR}(M) \arrow[l, "T(g^{*mod}_{Hdg}{,}Rg^{*mod[-],\Gamma})(\mathcal{F}_S^{FDR}(M))"'] \arrow[d, "T^!(g{,}\mathcal{F}^{FDR})(M)^{-1}"] \\
\mathcal{F}_T^{GM}(g^* L\mathbb{D}_S M = L\mathbb{D}_T g^! M) \arrow[rr, "T(\mathcal{F}_T^{GM}{,}\mathcal{F}_T^{FDR})(g^* M)"] & & \mathcal{F}_T^{FDR}(g^! M)
\end{tikzcd}$$

Proof. Follows from lemma 14. □

Definition 123. *Let $S \in \mathrm{SmVar}(\mathbb{C})$. We have, for $M, N \in \mathrm{DA}(S)$ and $(F, W), (G, W) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S))$ projective such that $(M, W) = D(\mathbb{A}^1, et)(F, W)$ and $(N, W) = D(\mathbb{A}^1, et)(G, W)$, the following transformation map in $\pi_S(D(MHM(S)))$*

$$\begin{aligned}
T(\mathcal{F}_S^{FDR}, \otimes)(M, N) : \mathcal{F}_S^{FDR}(M) \otimes^L_{O_S} \mathcal{F}_S^{FDR}(N) \xrightarrow{:=} \\
(e'(S)_* \mathcal{H}om(L\rho_{S*}\mu_{S*}R^{CH}(\rho_S^*(F, W)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR}))[-d_S]) \otimes^{[-]}_{O_S} \\
(e'(S)_* \mathcal{H}om(L\rho_{S*}\mu_{S*}R^{CH}(\rho_S^*(G, W)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR}))[-d_S]) \\
\xrightarrow{=} (e'(S)_* \mathcal{H}om(L\rho_{S*}\mu_{S*}R^{CH}(\rho_S^*(F, W)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR})) \otimes_{O_S} \\
e'(S)_* \mathcal{H}om(L\rho_{S*}\mu_{S*}R^{CH}(\rho_S^*(G, W)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR}))[-d_S]) \\
\xrightarrow{T(\otimes, \Omega^{\Gamma,pr}_{/S})(-,-)} \\
(e'(S)_* \mathcal{H}om(L\rho_{S*}\mu_{S*}R^{CH}(\rho_S^*(F, W)) \otimes L\rho_{S*}\mu_{S*}R^{CH}(\rho_S^*(G, W)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR}))[-d_S]) \\
\xrightarrow{\mathcal{H}om(T(\otimes, R_S^{CH})(-,-),-)^{-1}} \\
e'(S)_* \mathcal{H}om(L\rho_{S*}\mu_{S*}R^{CH}(\rho_S^*((F, W) \otimes (G, W))), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR}))[-d_S] \xrightarrow{=:} \mathcal{F}_S^{FDR}(M \otimes N)
\end{aligned}$$

We now give the definition in the non smooth case :

Definition 124. *Let $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^{l} S_i$ an open affine covering and denote, for $I \subset [1, \cdots l]$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. We have, for $M, N \in \mathrm{DA}(S)$ and $(F, W), (G, W) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S))$ such that $(M, W) = D(\mathbb{A}^1, et)(F, W)$ and $(N, W) = D(\mathbb{A}^1, et)(G, W)$, the following transformation map in*

$\pi_S(D(MHM(S))) \subset D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))$

$$T(\mathcal{F}_S^{FDR},\otimes)(M,N) : \mathcal{F}_S^{FDR}(M)\otimes^{L[-]}_{O_S}\mathcal{F}_S^{FDR}(N) \xrightarrow{:=}$$
$$(e'_*\mathcal{H}om^\bullet(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^*L(i_{I*}j_I^*(F,W))),E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I},F_{DR}))[-d_{\tilde{S}_I}],u_{IJ}(F,W))\otimes^{[-]}_{O_S}$$
$$(e'_*\mathcal{H}om^\bullet(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^*L(i_{I*}j_I^*(G,W))),E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I},F_{DR}))[-d_{\tilde{S}_I}],u_{IJ}(G,W))$$
$$\xrightarrow{=}(e'_*\mathcal{H}om^\bullet(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^*L(i_{I*}j_I^*(F,W))),E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I},F_{DR}))\otimes_{O_{\tilde{S}_I}}$$
$$e'_*\mathcal{H}om^\bullet(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^*L(i_{I*}j_I^*(F,W))),E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I},F_{DR}))[-d_{\tilde{S}_I}],u_{IJ}(F)\otimes u_{IJ}(G))$$
$$\xrightarrow{(T(\otimes,\Omega^{\Gamma,pr}_{/\tilde{S}_I})(-,-))}$$
$$(e'_*\mathcal{H}om^\bullet(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^*L(i_{I*}j_I^*(F,W)))\otimes R^{CH}(\rho_{\tilde{S}_I}^*L(i_{I*}j_I^*(F,W))),$$
$$E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I},F_{DR}))[-d_{\tilde{S}_I}],v_{IJ}(F\otimes G))$$
$$\xrightarrow{\mathcal{H}om(T(\otimes,R^{CH}_{\tilde{S}_I})(-,-),-)^{-1}}$$
$$(e'_*\mathcal{H}om^\bullet(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^*(L(i_{I*}j_I^*(F,W)))\otimes L(i_{I*}j_I^*(F,W))),E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I},F_{DR}))[-d_{\tilde{S}_I}],u_{IJ}(F\otimes G))$$
$$\xrightarrow{\mathcal{H}om(R_{(-,-)/-}(T(\otimes,L)(-,-)),-)}$$
$$(e'_*\mathcal{H}om^\bullet(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^*(L(i_{I*}j_I^*((F,W)\otimes(G,W))))),E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I},F_{DR}))[-d_{\tilde{S}_I}],u_{IJ}(F\otimes G))$$
$$\xrightarrow{=:}\mathcal{F}_S^{FDR}(M\otimes N)$$

Proposition 120. *Let $f_1 : X_1 \to S$, $f_2 : X_2 \to S$ two morphism with $X_1, X_2, S \in \mathrm{Var}(\mathbb{C})$. Assume that there exist factorizations $f_1 : X_1 \xrightarrow{l_1} Y_1\times S \xrightarrow{p_S} S$, $f_2 : X_2 \xrightarrow{l_2} Y_2\times S \xrightarrow{p_S} S$ with $Y_1, Y_2 \in \mathrm{SmVar}(\mathbb{C})$, l_1, l_2 closed embeddings and p_S the projections. We have then the factorization*

$$f_{12} := f_1\times f_2 : X_{12} := X_1\times_S X_2 \xrightarrow{l_1\times l_2} Y_1\times Y_2\times S \xrightarrow{p_S} S$$

Let $S = \cup_{i=1}^l S_i$ an open affine covering and denote, for $I \subset [1,\cdots l]$, $S_I = \cap_{i\in I}S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. We have then the following commutative diagram in $\pi_S(DMHM(S)) \subset D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))$ where the vertical maps are given by proposition 114

$$\begin{tikzcd}
\mathcal{F}_S^{FDR}(M(X_1/S))\otimes^{L[-]}_{O_S}\mathcal{F}_S^{FDR}(M(X_2/S)) \arrow[r, "I(X_1/S)\otimes I(X_2/S)"] \arrow[d, "T(\mathcal{F}_S^{FDR},\otimes)(M(X_1/S),M(X_2/S))"] & \begin{matrix} Rf_{1!}^{Hdg}(\Gamma^{\vee,Hdg}_{X_{1I}}(O_{Y_1\times\tilde{S}_I},F_b)(d_2)[2d_1],x_{IJ}(X_1/S))\otimes_{O_S} \\ Rf_{2!}^{Hdg}(\Gamma^{\vee,Hdg}_{X_{2I}}(O_{Y_2\times\tilde{S}_I},F_b)(d_1)[2d_2],x_{IJ}(X_2/S)) \end{matrix} \arrow[d, "(Ew_{(Y_1\times\tilde{S}_I,Y_2\times\tilde{S}_I)/\tilde{S}_I})"] \\
\mathcal{F}_S^{FDR}(M(X_1/S)\otimes M(X_2/S) = M(X_1\times_S X_2/S)) \arrow[r, "I(X_{12}/S)"] & Rf_{12!}^{Hdg}(\Gamma^{\vee,Hdg}_{X_{1I}\times_S X_{2I}}(O_{Y_1\times Y_2\times\tilde{S}_I},F_b)(d_{12})[2d_{12}],x_{IJ}(X_1/S))
\end{tikzcd}$$

with $d_1 = d_{Y_1}$, $d_2 = d_{Y_2}$ and $d_{12} = d_{Y_1}+d_{Y_2}$.

Proof. Immediate from definition. $\square$

Theorem 37. *Let $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open affine covering and denote, for $I \subset [1,\cdots l]$, $S_I = \cap_{i\in I}S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. Then, for $M, N \in \mathrm{DA}_c(S)$, the map in $\pi_S(D(MHM(S)))$*

$$T(\mathcal{F}_S^{FDR},\otimes)(M,N) : \mathcal{F}_S^{FDR}(M)\otimes^L_{O_S}\mathcal{F}_S^{FDR}(N) \xrightarrow{\sim} \mathcal{F}_S^{FDR}(M\otimes N)$$

given in definition 124 is an isomorphism.

Proof. Follows from proposition 120. □

We have the following easy proposition

Proposition 121. *Let $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^{l} S_i$ an open affine covering and denote, for $I \subset [1, \cdots l]$, $S_I = \cap_{i\in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. We have, for $M, N \in \mathrm{DA}(S)$ and $F, G \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1, et)(F)$ and $N = D(\mathbb{A}^1, et)(G)$, the following commutative diagram in $D_{O_S fil,\mathcal{D},\infty}(S/(\tilde{S}_I))$*

$$\begin{tikzcd}
\mathcal{F}_S^{GM}(L\mathbb{D}_S M) \otimes^L_{O_S} \mathcal{F}_S^{GM}(L\mathbb{D}_S N) \arrow[r, "T(\mathcal{F}_S^{GM},\mathcal{F}_S^{FDR})(M)\otimes T(\mathcal{F}_S^{GM},\mathcal{F}_S^{FDR})(N)"] \arrow[d, "T(\mathcal{F}_S^{GM},\otimes)(L\mathbb{D}_S M, L\mathbb{D}_S N)"] & \mathcal{F}_S^{FDR}(M) \otimes^L_{O_S} \mathcal{F}_S^{FDR}(N) \arrow[d, "T(\mathcal{F}_S^{FDR},\otimes)(M,N)"] \\
\mathcal{F}_S^{GM}(L\mathbb{D}_S(M \otimes N)) \arrow[r, "T(\mathcal{F}_S^{GM},\mathcal{F}_S^{FDR})(M\otimes N)"] & \mathcal{F}_S^{FDR}(M \otimes N)
\end{tikzcd}$$

Proof. Immediate from definition. □

6.2 The analytic filtered De Rahm realization functor

On $\mathrm{AnSp}(\mathbb{C})$ the usual topology is equivalent to the etale topology since a morphism $r : U' \to U$ is etale (which means non ramified and flat, see section 2) if and only if for all $x \in U'$ there exist an open neighborhood $U'_x \subset U$ of x such that r induces an isomorphism $r_{|U'_x} : U'_x \xrightarrow{\sim} r(U'_x)$. We note $\tau = et$ the etale topology.

6.2.1 The analytic Gauss-Manin filtered De Rham realization functor and its transformation map with pullbacks

Consider, for $S \in \mathrm{AnSp}(\mathbb{C})$, the following composition of morphism in RCat (see section 2)

$$\tilde{e}(S) : (\mathrm{AnSp}(\mathbb{C})/S, O_{\mathrm{AnSp}(\mathbb{C})/S}) \xrightarrow{\rho_S} (\mathrm{AnSp}(\mathbb{C})^{sm}/S, O_{\mathrm{AnSp}(\mathbb{C})^{sm}/S}) \xrightarrow{e(S)} (S, O_S)$$

with, for $X/S = (X, h) \in \mathrm{AnSp}(\mathbb{C})/S$,

- $O_{\mathrm{AnSp}(\mathbb{C})/S}(X/S) := O_X(X)$,
- $(\tilde{e}(S)^* O_S(X/S) \to O_{\mathrm{AnSp}(\mathbb{C})/S}(X/S)) := (h^* O_S \to O_X)$.

and $O_{\mathrm{AnSp}(\mathbb{C})^{sm}/S} := \rho_{S*} O_{\mathrm{AnSp}(\mathbb{C})/S}$, that is, for $U/S = (U, h) \in \mathrm{AnSp}(\mathbb{C})^{sm}/S$, $O_{\mathrm{AnSp}(\mathbb{C})^{sm}/S}(U/S) := O_{\mathrm{AnSp}(\mathbb{C})/S}(U/S) := O_U(U)$

Definition 125. *(i) For $S \in \mathrm{Var}(\mathbb{C})$, we consider the complexes of presheaves*

$$\Omega^\bullet_{/S} := \mathrm{coker}(\Omega_{O_{\mathrm{AnSp}(\mathbb{C})/S}/\tilde{e}(S)^* O_S} : \Omega^\bullet_{\tilde{e}(S)^* O_S} \to \Omega^\bullet_{O_{\mathrm{AnSp}(\mathbb{C})/S}}) \in C_{O_S}(\mathrm{AnSp}(\mathbb{C})/S)$$

which is by definition given by

- *for X/S a morphism $\Omega^\bullet_{/S}(X/S) = \Omega^\bullet_{X/S}(X)$*
- *for $g : X'/S \to X/S$ a morphism,*

$$\Omega^\bullet_{/S}(g) := \Omega_{(X'/X)/(S/S)}(X') : \Omega^\bullet_{X/S}(X) \to g^* \Omega_{X/S}(X') \to \Omega^\bullet_{X'/S}(X')$$
$$\omega \mapsto \Omega_{(X'/X)/(S/S)}(X')(\omega) := g^*(\omega) : (\alpha \in \wedge^k T_{X'}(X') \mapsto \omega(dg(\alpha)))$$

(ii) For $S \in \mathrm{AnSp}(\mathbb{C})$, we consider the complexes of presheaves

$$\Omega^\bullet_{/S} := \rho_{S*} \tilde{\Omega}^\bullet_{/S} = \mathrm{coker}(\Omega_{O_{\mathrm{AnSp}(\mathbb{C})^{sm}/S}/e(S)^* O_S} : \Omega^\bullet_{e(S)^* O_S} \to \Omega^\bullet_{O_{\mathrm{AnSp}(\mathbb{C})^{sm}/S}}) \in C_{O_S}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$$

which is by definition given by

– *for U/S a smooth morphism $\Omega^\bullet_{/S}(U/S) = \Omega^\bullet_{U/S}(U)$*

– *for $g : U'/S \to U/S$ a morphism,*

$$\Omega^\bullet_{/S}(g) := \Omega_{(U'/U)/(S/S)}(U') : \Omega^\bullet_{U/S}(U) \to g^*\Omega_{U/S}(U') \to \Omega^\bullet_{U'/S}(U')$$
$$\omega \mapsto \Omega_{(U'/U)/(S/S)}(U')(\omega) := g^*(\omega) : (\alpha \in \wedge^k T_{U'}(U') \mapsto \omega(dg(\alpha)))$$

Remark 12. *For $S \in \mathrm{AnSp}(\mathbb{C})$, $\Omega^\bullet_{/S} \in C(\mathrm{AnSp}(\mathbb{C})/S)$ is by definition a natural extension of $\Omega^\bullet_{/S} \in C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$. However $\Omega^\bullet_{/S} \in C(\mathrm{AnSp}(\mathbb{C})/S)$ does NOT satisfy cdh descent.*

For a smooth morphism $h : U \to S$ with $S, U \in \mathrm{AnSm}(\mathbb{C})$, the cohomology presheaves $H^n\Omega^\bullet_{U/S}$ of the relative De Rham complex

$$DR(U/S) := \Omega^\bullet_{U/S} := \mathrm{coker}(h^*\Omega_S \to \Omega_U) \in C_{h^*O_S}(U)$$

for all $n \in \mathbb{Z}$, have a canonical structure of a complex of $h^*D^\infty_S$ modules given by the Gauss Manin connexion : for $S^o \subset S$ an open subset, $U^o = h^{-1}(S^o)$, $\gamma \in \Gamma(S^o, T_S)$ a vector field and $\hat{\omega} \in \Omega^p_{U/S}(U^o)^c$ a closed form, the action is given by

$$\gamma \cdot [\hat{\omega}] = [\widehat{\iota(\tilde{\gamma})\partial\omega}],$$

$\omega \in \Omega^p_U(U^o)$ being a representative of $\hat{\omega}$ and $\tilde{\gamma} \in \Gamma(U^o, T_U)$ a relevement of γ (h is a smooth morphism), so that

$$DR(U/S) := \Omega^\bullet_{U/S} := \mathrm{coker}(h^*\Omega_S \to \Omega_U) \in C_{h^*O_S, h^*\mathcal{D}^\infty}(U)$$

with this $h^*D^\infty_S$ structure. Hence we get $h_*\Omega^\bullet_{U/S} \in C_{O_S,\mathcal{D}^\infty}(S)$ considering this structure. Since h is a smooth morphism, $\Omega^p_{U/S}$ are locally free O_U modules.

The point (ii) of the definition 134 above gives the object in $\mathrm{DA}(S)$ which will, for S smooth, represent the analytic Gauss-Manin De Rham realisation. It is the class of an explicit complex of presheaves on $\mathrm{AnSp}(\mathbb{C})^{sm}/S$.

Proposition 122. *Let $S \in \mathrm{Var}(\mathbb{C})$.*

(i) For $U/S = (U, h) \in \mathrm{AnSp}(\mathbb{C})^{sm}/S$, we have $e(U)_*h^*\Omega^\bullet_{/S} = \Omega^\bullet_{U/S}$.

(ii) The complex of presheaves $\Omega^\bullet_{/S} \in C_{O_S}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$ is $\mathbb{D}^1$ homotopic. Note that however, for $p > 0$, the complexes of presheaves $\Omega^{\bullet \geq p}$ are NOT $\mathbb{D}^1_S$ local. On the other hand, $\Omega^\bullet_{/S}$ admits transferts (recall that means $\mathrm{Tr}(S)_*\mathrm{Tr}(S)^*\Omega^p_{/S} = \Omega^p_{/S}$).

(iii) If S is smooth, we get $(\Omega^\bullet_{/S}, F_b) \in C_{O_S fil, D^\infty_S}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ with the structure given by the Gauss Manin connexion. Note that however the D^∞_S structure on the cohomology groups given by Gauss Main connexion does NOT comes from a structure of D^∞_S module structure on the filtered complex of O_S module. The D_S structure on the cohomology groups satisfy a non trivial Griffitz transversality (in the non projection cases), whereas the filtration on the complex is the trivial one.

Proof. Similar to the proof of proposition 103. □

We have the following canonical transformation map given by the pullback of (relative) differential forms:

Let $g : T \to S$ a morphism with $T, S \in \mathrm{AnSp}(\mathbb{C})$. Consider the following commutative diagram in RCat :

$$\begin{tikzcd} D(g,e) : (\mathrm{AnSp}(\mathbb{C})^{sm}/T, O_{\mathrm{AnSp}(\mathbb{C})^{sm}/T}) \arrow[r, "P(g)"] \arrow[d, "e(T)"] & (\mathrm{AnSp}(\mathbb{C})^{sm}/S, O_{\mathrm{AnSp}(\mathbb{C})^{sm}/S}) \arrow[d, "e(S)"] \\ (T, O_T) \arrow[r, "P(g)"] & (S, O_S) \end{tikzcd}$$

It gives (see section 2) the canonical morphism in $C_{g^*O_S fil}(\mathrm{Var}(\mathbb{C})^{sm}/T)$

$$\Omega_{/(T/S)} := \Omega_{(O_{\mathrm{AnSp}(\mathbb{C})^{sm}/T}/g^*O_{\mathrm{AnSp}(\mathbb{C})^{sm}/S})/(O_T/g^*O_S)} : \\ g^*(\Omega^\bullet_{/S}, F_b) = \Omega^\bullet_{g^*O_{\mathrm{AnSp}(\mathbb{C})^{sm}/S}/g^*e(S)^*O_S} \to (\Omega^\bullet_{/T}, F_b) = \Omega^\bullet_{O_{\mathrm{AnSp}(\mathbb{C})^{sm}/T}/e(T)^*O_T}$$

which is by definition given by the pullback on differential forms : for $(V/T) = (V, h) \in \mathrm{Var}(\mathbb{C})^{sm}/T$,

$$\Omega_{/(T/S)}(V/T) : g^*(\Omega^\bullet_{/S})(V/T) := \lim_{(h':U\to S\mathrm{sm}, g':V\to U, h, g)} \Omega^\bullet_{U/S}(U) \xrightarrow{\Omega_{(V/U)/(T/S)}(V/T)} \Omega^\bullet_{V/T}(V) =: \Omega^\bullet_{/T}(V/T)$$

$$\hat{\omega} \mapsto \Omega_{(V/U)/(T/S)}(V/T)(\omega) := \hat{g'^*\omega}.$$

If S and T are smooth, $\Omega_{/(T/S)} : g^*(\Omega^\bullet_{/S}, F_b) \to (\Omega^\bullet_{/T}, F_b)$ is a map in $C_{g^*O_S fil, g^*D_S^\infty}(\mathrm{AnSp}(\mathbb{C})^{sm}/T)$ It induces the canonical morphism in $C_{g^*O_S fil, g^*D_S^\infty}(\mathrm{AnSp}(\mathbb{C})^{sm}/T)$:

$$E\Omega_{/(T/S)} : g^*E_{usu}(\Omega^\bullet_{/S}, F_b) \xrightarrow{T(g, E_{usu})(\Omega^\bullet_{/S}, F_b)} E_{usu}(g^*(\Omega^\bullet_{/S}, F_b)) \xrightarrow{E_{usu}(\Omega_{/(T/S)})} E_{usu}(\Omega^\bullet_{/T}, F_b).$$

Definition 126. *(i) Let $g : T \to S$ a morphism with $T, S \in \mathrm{AnSp}(\mathbb{C})$. We have, for $F \in C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$, the canonical transformation in $C_{O_T fil}(T)$:*

$$\begin{aligned} T^O(g, \Omega_{/\cdot})(F) : g^{*mod} L_O e(S)_* \mathcal{H}om^\bullet(F, E_{usu}(\Omega^\bullet_{/S}, F_b)) \\ \xrightarrow{:=} (g^* L_O e(S)_* \mathcal{H}om^\bullet(F, E_{et}(\Omega^\bullet_{/S}, F_b))) \otimes_{g^*O_S} O_T \\ \xrightarrow{T(e,g)(-)\circ T(g,L_O)(-)} L_O(e(T)_* g^* \mathcal{H}om^\bullet(F, E_{usu}(\Omega^\bullet_{/S}, F)) \otimes_{g^*O_S} O_T) \\ \xrightarrow{T(g,hom)(F, E_{et}(\Omega^\bullet_{/S}))\otimes I} L_O(e(T)_* \mathcal{H}om^\bullet(g^*F, g^*E_{usu}(\Omega^\bullet_{/S}, F_b)) \otimes_{g^*O_S} O_T) \\ \xrightarrow{ev(hom,\otimes)(-,-,-)} L_O e(T)_* \mathcal{H}om^\bullet(g^*F, g^*E_{usu}(\Omega^\bullet_{/S}, F_b) \otimes_{g^*e(S)^*O_S} e(T)^*O_T) \\ \xrightarrow{\mathcal{H}om^\bullet(g^*F, E\Omega_{/(T/S)}\otimes I)} L_O e(T)_* \mathcal{H}om^\bullet(g^*F, E_{usu}(\Omega^\bullet_{/T}, F_b) \otimes_{g^*e(S)^*O_S} e(T)^*O_T) \\ \xrightarrow{m} L_O e(T)_* \mathcal{H}om^\bullet(g^*F, E_{usu}(\Omega^\bullet_{/T}, F_b) \end{aligned}$$

where $m(\alpha \otimes h) := h.\alpha$ is the multiplication map.

(ii) Let $g : T \to S$ a morphism with $T, S \in \mathrm{AnSp}(\mathbb{C})$, S smooth. Assume there is a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with l a closed embedding, $Y \in \mathrm{AnSm}(\mathbb{C})$ and p_S the projection. We have, for $F \in C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$, the canonical transformation in $C_{O_T fil, \mathcal{D}^\infty}(Y \times S)$:

$$\begin{aligned} T(g, \Omega_{/\cdot})(F) : g^{*mod,\Gamma} e(S)_* \mathcal{H}om^\bullet(F, E_{usu}(\Omega^\bullet_{/S}, F_b)) \\ \xrightarrow{:=} \Gamma_T E_{usu}(p_S^{*mod} e(S)_* \mathcal{H}om^\bullet(F, E_{usu}(\Omega^\bullet_{/S}, F_b))) \\ \xrightarrow{T^O(p_S, \Omega_{/\cdot})(F)} \Gamma_T E_{usu}(e(T \times S)_* \mathcal{H}om^\bullet(p_S^*F, E_{usu}(\Omega^\bullet_{/T\times S}, F_b))) \\ \xrightarrow{=} e(T \times S)_* \Gamma_T(\mathcal{H}om^\bullet(p_S^*F, E_{usu}(\Omega^\bullet_{/T\times S}, F_b))) \\ \xrightarrow{I(\gamma,\mathrm{hom})(-,-)} e(T \times S)_* \mathcal{H}om^\bullet(\Gamma_T^\vee p_S^*F, E_{usu}(\Omega^\bullet_{/T\times S}, F_b)). \end{aligned}$$

For $Q \in Proj\,\mathrm{PSh}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$,

$$T(g, \Omega_{/\cdot})(Q) : g^{*mod,\Gamma} e(S)_* \mathcal{H}om^\bullet(Q, E_{usu}(\Omega^\bullet_{/S}, F_b)) \to e(T \times S)_* \mathcal{H}om^\bullet(\Gamma_T^\vee p_S^*Q, E_{usu}(\Omega^\bullet_{/Y\times S}, F_b))$$

is a map in $C_{O_T fil, \mathcal{D}^\infty}(Y \times S)$.

The following easy lemma describe these transformation map on representable presheaves :

Lemma 15. *Let $g : T \to S$ a morphism with $T, S \in \mathrm{AnSp}(\mathbb{C})$ and $h : U \to S$ is a smooth morphism with $U \in \mathrm{AnSp}(\mathbb{C})$. Consider a commutative diagram whose square are cartesian :*

$$\begin{tikzcd} g : T \arrow[r, "l"] & S \times Y \arrow[r, "p_S"] & S \\ g' : U_T \arrow[u, "h'"] \arrow[r, "l'"] & U \times Y \arrow[u, "h'':=h\times I"] \arrow[r, "p_U"] & U \arrow[u, "h"] \end{tikzcd}$$

with l, l' the graph embeddings and p_S, p_U the projections. Then $g^*\mathbb{Z}(U/S) = \mathbb{Z}(U_T/T)$ and

(i) we have the following commutative diagram in $C_{O_T fil}(T)$ (see definition 1 and definition 126(i)) :

$$\begin{tikzcd} g^{*mod}L_O e(S)_*\mathcal{H}om^\bullet(\mathbb{Z}(U/S), E_{usu}(\Omega^\bullet_{/S}, F_b)) \arrow[r, "T(g,\Omega_{/.})(\mathbb{Z}(U/S))"] \arrow[d, "="] & e(T)_*\mathcal{H}om^\bullet(\mathbb{Z}(U_T/T), E_{usu}(\Omega^\bullet_{/T}, F_b)) \arrow[d, "="] \\ g^{*mod}L_O h_* E_{usu}(\Omega^\bullet_{U/S}, F_b) \arrow[r, "T^{mod}_\omega(g,h)"] & h'_* E_{usu}(\Omega^\bullet_{U_T/T}, F_b) \end{tikzcd}$$

(ii) if $Y, S \in \mathrm{AnSm}(\mathbb{C})$, we have the following commutative diagram in $C_{O_T fil,\mathcal{D}^\infty}(T)$ (see definition 1 and definition 126(ii)) :

$$\begin{tikzcd} g^{*mod,\Gamma}e(S)_*\mathcal{H}om^\bullet(\mathbb{Z}(U/S), E_{usu}(\Omega^\bullet_{/S}, F_b)) \arrow[r, "T(g,\Omega_{/.})(\mathbb{Z}(U/S))"] \arrow[d, "="] & e(T)_*\mathcal{H}om^\bullet(\mathbb{Z}(U_T/T), E_{usu}(\Omega^\bullet_{/T}, F_b)) \arrow[d, "="] \\ g^{*mod,\Gamma}h_* E_{usu}(\Omega^\bullet_{U/S}, F_b) \arrow[r, "T^O_\omega(\gamma,\otimes)(-)\circ T^O_\omega(p_S,h)(-)"] & h'_* E_{usu}(\Omega^\bullet_{U_T/T}, F_b) \end{tikzcd}$$

where $j : T\backslash T \times S \hookrightarrow T \times S$ is the open complementary embedding,

Proof. Obvious. □

Proposition 123. *Let $p : S_{12} \to S_1$ is a smooth morphism with $S_1, S_{12} \in \mathrm{AnSp}(\mathbb{C})$. Then if $Q \in C(\mathrm{AnSp}(\mathbb{C})^{sm}/S_1)$ is projective,*

$$T(p, \Omega_{/.})(Q) : p^{*mod}e(S_1)_*\mathcal{H}om^\bullet(Q, E_{usu}(\Omega^\bullet_{/S_1}, F_b)) \to e(S_{12})_*\mathcal{H}om^\bullet(p^*Q, E_{usu}(\Omega^\bullet_{/S_{12}}, F_b))$$

is an isomorphism.

Proof. Similar to the proof of proposition 104. □

Let $S \in \mathrm{AnSp}(\mathbb{C})$ and $h : U \to S$ a morphism with $U \in \mathrm{AnSp}(\mathbb{C})$. We then have the canonical map given by the wedge product

$$w_{U/S} : \Omega^\bullet_{U/S} \otimes_{O_S} \Omega^\bullet_{U/S} \to \Omega^\bullet_{U/S}; \alpha \otimes \beta \mapsto \alpha \wedge \beta.$$

Let $S \in \mathrm{Var}(\mathbb{C})$ and $h_1 : U_1 \to S$, $h_2 : U_2 \to S$ two morphisms with $U_1, U_2 \in \mathrm{AnSp}(\mathbb{C})$. Denote $h_{12} : U_{12} := U_1 \times_S U_2 \to S$ and $p_{112} : U_1 \times_S U_2 \to U_1$, $p_{212} : U_1 \times_S U_2 \to U_2$ the projections. We then have the canonical map given by the wedge product

$$w_{(U_1,U_2)/S} : p^*_{112}\Omega^\bullet_{U_1/S} \otimes_{O_S} p^*_{212}\Omega^\bullet_{U_2/S} \to \Omega^\bullet_{U_{12}/S}; \alpha \otimes \beta \mapsto p^*_{112}\alpha \wedge p^*_{212}\beta$$

which gives the map

$$\begin{aligned} Ew_{(U_1,U_2)/S} : h_{1*}E_{usu}(\Omega^\bullet_{U_1/S}) \otimes_{O_S} h_{2*}E_{usu}(\Omega^\bullet_{U_2/S}) \\ \xrightarrow{\mathrm{ad}(p^*_{112},p_{112*})(-)\otimes\mathrm{ad}(p^*_{212},p_{212*})(-)} (h_{1*}p_{112*}p^*_{112}E_{usu}(\Omega^\bullet_{U_1/S})) \otimes_{O_S} (h_{2*}p_{212*}p^*_{212}E_{usu}(\Omega^\bullet_{U_2/S})) \\ \xrightarrow{=} h_{12*}(p^*_{112}E_{usu}(\Omega^\bullet_{U_1/S}) \otimes_{h^*_{12}O_S} p^*_{212}E_{usu}(\Omega^\bullet_{U_2/S}) \\ \xrightarrow{T(\otimes,E)(-)\circ(T(p_{112},E)(-)\otimes T(p_{212},E)(-))} h_{12*}E_{zar}(p^*_{112}\Omega^\bullet_{U_1/S} \otimes_{O_S} p^*_{212}\Omega^\bullet_{U_2/S}) \end{aligned}$$

Let $S\in\mathrm{AnSp}(\mathbb{C})$. We have the canonical map in $C_{O_Sfil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$

$$w_S:(\Omega^\bullet_{/S},F_b)\otimes_{O_S}(\Omega^\bullet_{/S},F_b)\to(\Omega^\bullet_{/S},F_b)$$

given by for $h:U\to S\in\mathrm{AnSp}(\mathbb{C})^{sm}/S$

$$w_S(U/S):(\Omega^\bullet_{U/S},F_b)\otimes_{h^*O_S}(\Omega^\bullet_{U/S},F_b)(U)\xrightarrow{w_{U/S}(U)}(\Omega^\bullet_{U/S},F_b)(U)$$

It gives the map

$$Ew_S:E_{usu}(\Omega^\bullet_{/S},F_b)\otimes_{O_S}E_{usu}(\Omega^\bullet_{/S},F_b)\xrightarrow{=}E_{usu}((\Omega^\bullet_{/S},F_b)\otimes_{O_S}(\Omega^\bullet_{/S},F_b))\xrightarrow{E_{usu}(w_S)}E_{usu}(\Omega^\bullet_{/S},F_b)$$

If $S\in\mathrm{AnSm}(\mathbb{C})$,

$$w_S:(\Omega^\bullet_{/S},F_b)\otimes_{O_S}(\Omega^\bullet_{/S},F_b)\to(\Omega^\bullet_{/S},F_b)$$

is a map in $C_{O_Sfil,D_S^\infty}(\mathrm{Var}(\mathbb{C})^{sm}/S)$.

Definition 127. *Let $S\in\mathrm{AnSp}(\mathbb{C})$. We have, for $F,G\in C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$, the canonical transformation in $C_{O_Sfil}(S)$:*

$$\begin{aligned}T(\otimes,\Omega)(F,G):e(S)_*\mathcal{H}om(F,E_{usu}(\Omega^\bullet_{/S},F_b))\otimes_{O_S}e(S)_*\mathcal{H}om(G,E_{usu}(\Omega^\bullet_{/S},F_b))\\ \xrightarrow{=}e(S)_*(\mathcal{H}om(F,E_{usu}(\Omega^\bullet_{/S},F_b))\otimes_{O_S}\mathcal{H}om(G,E_{usu}(\Omega^\bullet_{/S},F_b)))\\ \xrightarrow{e(S)_*T(\mathcal{H}om,\otimes)(-)}e(S)_*\mathcal{H}om(F\otimes G,E_{usu}(\Omega^\bullet_{/S},F_b)\otimes_{O_S}E_{usu}(\Omega^\bullet_{/S},F_b))\\ \xrightarrow{\mathcal{H}om(F\otimes G,Ew_S)}e(S)_*\mathcal{H}om(F\otimes G,E_{usu}(\Omega^\bullet_{/S},F_b))\end{aligned}$$

If $S\in\mathrm{AnSm}(\mathbb{C})$, $T(\otimes,\Omega)(F,G)$ is a map in $C_{O_Sfil,\mathcal{D}^\infty}(S)$.

Lemma 16. *Let $S\in\mathrm{AnSp}(\mathbb{C})$ and $h_1:U_1\to S$, $h_2:U_2\to S$ two smooth morphisms with $U_1,U_2\in\mathrm{AnSp}(\mathbb{C})$. Denote $h_{12}:U_{12}:=U_1\times_SU_2\to S$ and $p_{112}:U_1\times_SU_2\to U_1$, $p_{212}:U_1\times_SU_2\to U_2$ the projections. We then have the following commutative diagram*

$$\begin{tikzcd}e(S)_*\mathcal{H}om(F,E_{usu}(\Omega^\bullet_{/S},F_b))\otimes_{O_S}e(S)_*\mathcal{H}om(G,E_{usu}(\Omega^\bullet_{/S},F_b)) \arrow[r,"T(\otimes,\Omega)(F,G)"] \arrow[d,"="] & e(S)_*\mathcal{H}om(F\otimes G,E_{usu}(\Omega^\bullet_{/S},F)) \arrow[d,"="] \\ h_{1*}E_{usu}(\Omega^\bullet_{U_1/S},F_b)\otimes_{O_S}h_{2*}E_{usu}(\Omega^\bullet_{U_2/S},F_b) \arrow[r,"Ew_{(U_1,U_2)/S}"] & h_{12*}E_{zar}(\Omega^\bullet_{U_{12}/S},F_b)\end{tikzcd}$$

Proof. Follows from Yoneda lemma. □

We now define the analytic Gauss Manin De Rahm realization functor.

Let $S\in\mathrm{AnSp}(\mathbb{C})$ and $S=\cup_{i=1}^lS_i$ an open cover such that there exist closed embeddings $i_i:S_i\hookrightarrow\tilde S_i$ with $\tilde S_i\in\mathrm{AnSm}(\mathbb{C})$ an affine space. For $I\subset[1,\cdots l]$, denote by $S_I:=\cap_{i\in I}S_i$ and $j_I:S_I\hookrightarrow S$ the open embedding. We then have closed embeddings $i_I:S_I\hookrightarrow\tilde S_I:=\Pi_{i\in I}\tilde S_i$. Consider, for $I\subset J$, the following commutative diagram

$$D_{IJ}=\quad\begin{tikzcd}S_I \arrow[r,"i_I"] & \tilde S_I \\ S_J \arrow[u,"j_{IJ}"] \arrow[r,"i_J"] & \tilde S_J \arrow[u,"p_{IJ}"']\end{tikzcd}$$

and $j_{IJ} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. Considering the factorization of the diagram D_{IJ} by the fiber product :

$$\begin{tikzcd}
D_{IJ} = \tilde{S}_J = \tilde{S}_I \times \tilde{S}_{J\setminus I} \arrow[rr, "p_{IJ}"] & & \tilde{S}_I \\
 & S_I \times \tilde{S}_{J\setminus I} \arrow[ul, "i_I \times I"'] \arrow[dr, "p^0_{IJ}"] & \\
S_J \arrow[uu, "i_J"] \arrow[ur, "l_J"] \arrow[rr, "j_{IJ}"] & & S_I \arrow[uu, "i_I"']
\end{tikzcd}$$

the square of this factorization being cartesian, we have for $F \in C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$ the canonical map in $C(\mathrm{AnSp}(\mathbb{C})^{sm}/\tilde{S}_J)$

$$S(D_{IJ})(F) : Li_{J*}j_J^*F \xrightarrow{q} i_{J*}j_J^*F = (i_I \times I) * l_{J*}j_J^*F \xrightarrow{(i_I \times I)_* \operatorname{ad}(p^o_{IJ\sharp}, p^{o*}_{IJ})(-)}$$
$$(i_I \times I)_* p^{o*}_{IJ} p^0_{IJ\sharp} l_{J*}j_J^*F \xrightarrow{T(p_{IJ}, i_I)(-)^{-1}} p^*_{IJ} i_{I*} p^0_{IJ\sharp} l_{J*} j_I^* F = p^*_{IJ} i_{I*} j_I^* F$$

which factors through

$$S(D_{IJ})(F) : Li_{J*}j_I^*F \xrightarrow{S^q(D_{IJ})(F)} p^*_{IJ} Li_{I*} j_I^* F \xrightarrow{q} p^*_{IJ} i_{I*} j_I^* F$$

Definition 128. *(i) Let $S \in \mathrm{AnSm}(\mathbb{C})$. We have the functor*

$$\mathcal{H}om^\bullet(\cdot, E_{usu}(\Omega^\bullet_{/S}, F_b)) : C(\mathrm{AnSp}(\mathbb{C})^{sm}/S) \to C_{O_S fil, D^\infty_S}(S), \; F \mapsto e(S)_*\mathcal{H}om^\bullet(LF, E_{usu}(\Omega^\bullet_{/S}, F_b)).$$

(ii) Let $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. For $I \subset [1, \cdots l]$, denote by $S_I := \cap_{i\in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \Pi_{i\in I}\tilde{S}_i$. We have the functor

$$C(\mathrm{Var}(\mathbb{C})^{sm}/S)^{op} \to C_{Ofil,\mathcal{D}^\infty}(S/(\tilde{S}_I)), \;\; F \mapsto (e(\tilde{S}_I)_*\mathcal{H}om^\bullet(\mathrm{An}^*_{\tilde{S}_I} L(i_{I*}j_I^*F), E_{usu}(\Omega^\bullet_{/\tilde{S}_I}, F_b))[-d_{\tilde{S}_I}], u^q_{IJ}(F))$$

where

$$\begin{aligned}
u^q_{IJ}(F)[d_{\tilde{S}_J}] : & \; e(\tilde{S}_I)_*\mathcal{H}om^\bullet(\mathrm{An}^*_{\tilde{S}_I} L(i_{I*}j_I^*F), E_{usu}(\Omega^\bullet_{/\tilde{S}_I}, F_b)) \\
& \xrightarrow{\operatorname{ad}(p^{*mod}_{IJ}, p_{IJ*})(-)} p_{IJ*}p^{*mod}_{IJ} e(\tilde{S}_I)_*\mathcal{H}om^\bullet(\mathrm{An}^*_{\tilde{S}_I} L(i_{I*}j_I^*F), E_{usu}(\Omega^\bullet_{/\tilde{S}_I}, F_b)) \\
& \xrightarrow{p_{IJ*}T(p_{IJ}, \Omega_\cdot)(L(i_{I*}j_I^*F))} p_{IJ*}e(\tilde{S}_J)_*\mathcal{H}om^\bullet(\mathrm{An}^*_{\tilde{S}_J} p^*_{IJ} L(i_{I*}j_I^*F), E_{usu}(\Omega^\bullet_{/\tilde{S}_J}, F_b)) \\
& \xrightarrow{p_{IJ*}e(\tilde{S}_J)_*\mathcal{H}om(\mathrm{An}^*_{\tilde{S}_J} S^q(D_{IJ})(F), E_{usu}(\Omega^{\bullet,\Gamma}_{/\tilde{S}_J}, F_b))} p_{IJ*}e(\tilde{S}_J)_*\mathcal{H}om^\bullet(\mathrm{An}^*_{\tilde{S}_J} L(i_{J*}j_J^*F), E_{usu}(\Omega^\bullet_{/\tilde{S}_J}, F_b)).
\end{aligned}$$

For $I \subset J \subset K$, we have obviously $p_{IJ*}u_{JK}(F) \circ u_{IJ}(F) = u_{IK}(F)$.

We will prove in corollary 7 below that $u_{IJ}(F)$ are ∞-filtered usu local equivalence.

Proposition 124. *Let $S \in \mathrm{AnSp}(\mathbb{C})$. Let $m : Q_1 \to Q_2$ be an equivalence $(\mathbb{D}^1, et)$ local in $C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$ with Q_1, Q_2 complexes of projective presheaves. Then,*

$$e(S)_*\mathcal{H}om(m, E_{usu}(\Omega^\bullet_{/S}, F_b)) : e(S)_*\mathcal{H}om^\bullet(Q_2, E_{usu}(\Omega^\bullet_{/S}, F_b)) \to e(S)_*\mathcal{H}om^\bullet(Q_1, E_{usu}(\Omega^\bullet_{/S}, F_b))$$

is a quasi-isomorphism. It is thus an isomorphism in $D_{O_S fil, \mathcal{D}^\infty, \infty}(S)$ if S is smooth.

Proof. Similar to the proof of proposition 105. $\square$

Definition 129. *(i) We define, according to proposition 124, the filtered analytic Gauss-Manin realization functor defined as*

$$\mathcal{F}_{S,an}^{GM} : \mathrm{DA}_c(S)^{op} \to D_{O_S fil,\mathcal{D}^\infty,\infty}(S), \quad M \mapsto$$
$$\mathcal{F}_{S,an}^{GM}(M) := e(S)_*\mathcal{H}om^\bullet(\mathrm{An}_S^* L(F), E_{usu}(\Omega^\bullet_{/S}, F_b))[-d_S]$$
$$= e(S)_*\mathcal{H}om^\bullet(L(F), \mathrm{An}_{S*} E_{usu}(\Omega^\bullet_{/S}, F_b))[-d_S]$$

where $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$,

(ii) Let $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. For $I \subset [1, \cdots l]$, denote by $S_I = \cap_{i\in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \Pi_{i\in I}\tilde{S}_i$. We define the filtered analytic Gauss-Manin realization functor defined as

$$\mathcal{F}_{S,an}^{GM} : \mathrm{DA}_c(S)^{op} \to D_{Ofil,\mathcal{D}^\infty,\infty}(S/(\tilde{S}_I)), \ M \mapsto$$
$$\mathcal{F}_{S,an}^{GM}(M) := ((e(\tilde{S}_I)_*\mathcal{H}om^\bullet(\mathrm{An}_{\tilde{S}_I}^* L(i_{I*}j_I^*F), E_{usu}(\Omega^\bullet_{/\tilde{S}_I}), F_b))[-d_{\tilde{S}_I}], u_{IJ}^q(F))$$
$$= ((e(\tilde{S}_I)_*\mathcal{H}om^\bullet(L(i_{I*}j_I^*F), \mathrm{An}_{\tilde{S}_I*} E_{usu}(\Omega^\bullet_{/\tilde{S}_I}), F_b))[-d_{\tilde{S}_I}], u_{IJ}^q(F))$$

where $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$, see definition 128 and corollary 7.

Proposition 125. *For $S \in \mathrm{Var}(\mathbb{C})$, the functor $\mathcal{F}_S^{GM}$ is well defined.*

Proof. Similar to the proof of proposition 106. $\square$

Proposition 126. *Let $f : X \to S$ a morphism with $S, X \in \mathrm{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. Then $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. Denote, for $I \subset [1, \cdots l]$, $S_I = \cap_{i\in I} S_i$ and $X_I = \cap_{i\in I} X_i$. Assume there exist a factorization*

$$f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$$

of f with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We then have, for $I \subset [1, \cdots l]$, the following commutative diagrams which are cartesian

$$\begin{tikzcd} f_I = f_{|X_I} : X_I \arrow[r, "l_I"] \arrow[dr] & Y \times S_I \arrow[r, "p_{S_I}"] \arrow[d, "i'_I"] & S_I \arrow[d, "i_I"] \\ & Y \times \tilde{S}_I \arrow[r, "p_{\tilde{S}_I}"] & \tilde{S}_I \end{tikzcd} \quad , \quad \begin{tikzcd} Y \times \tilde{S}_J \arrow[r, "p_{\tilde{S}_J}"] \arrow[d, "p'_{IJ}"'] & \tilde{S}_J \arrow[d, "p_{IJ}"] \\ Y \times \tilde{S}_I \arrow[r, "p_{\tilde{S}_I}"] & \tilde{S}_I \end{tikzcd}$$

Let $F(X/S) := p_{S,\sharp}\Gamma_X^\vee \mathbb{Z}(Y\times S/Y\times S)$. The transformations maps $(N_I(X/S) : Q(X_I/\tilde{S}_I) \to i_{I*}j_I^*F(X/S))$ and $(k \circ I(\gamma, \hom)(-,-))$, for $I \subset [1, \cdots, l]$, induce an isomorphism in $D_{Ofil,\mathcal{D}^\infty,\infty}(S/(\tilde{S}_I))$

$$I^{GM}(X/S) :$$
$$\mathcal{F}_{S,an}^{GM}(M(X/S)) := (e(\tilde{S}_I)_*\mathcal{H}om(\mathrm{An}_{\tilde{S}_I}^* L(i_{I*}j_I^*F(X/S)), E_{usu}(\Omega^\bullet_{/\tilde{S}_I}, F_b))[-d_{\tilde{S}_I}], u_{IJ}^q(F(X/S)))$$
$$\xrightarrow{(e(\tilde{S}_I)_*\mathcal{H}om(\mathrm{An}_{\tilde{S}_I}^* N_I(X/S), E_{usu}(\Omega^\bullet_{/\tilde{S}_I}, F_b)))} (e(\tilde{S}_I)_*\mathcal{H}om(\mathrm{An}_{\tilde{S}_I}^* Q(X_I/\tilde{S}_I), E_{usu}(\Omega^\bullet_{/\tilde{S}_I}, F_b))[-d_{\tilde{S}_I}], v_{IJ}^q(F(X/S)))$$
$$\xrightarrow{(e(\tilde{S}_I)_*\mathcal{H}om(T(\mathrm{An},\gamma^\vee)(-)^{-1}, E_{usu}(\Omega^\bullet_{/\tilde{S}_I}, F_b)))} (e(\tilde{S}_I)_*\mathcal{H}om(Q(X_I^{an}/\tilde{S}_I^{an}), E_{usu}(\Omega^\bullet_{/\tilde{S}_I}, F_b))[-d_{\tilde{S}_I}], v_{IJ}^q(F(X/S)))$$
$$\xrightarrow{(I(\gamma,\hom)(-,-))^{-1}} (p_{\tilde{S}_I*}\Gamma_{X_I} E_{usu}(\Omega^\bullet_{Y\times\tilde{S}_I/\tilde{S}_I}, F_b)[-d_{\tilde{S}_I}], w_{IJ}(X/S)).$$

Proof. Similar to the proof of proposition 107. □

Corollary 7. *Let $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^{l} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. For $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $D(\mathbb{A}^1, et)(F) \in \mathrm{DA}_c(S)$, $u^q_{IJ}(F)$ are ∞-filtered usu local equivalence.*

Proof. Similar to corollary 4. □

We now define the functorialities of $\mathcal{F}_S^{GM}$ with respect to S which makes $\mathcal{F}^-_{GM}$ a morphism of 2-functor.

Definition 130. *Let $g : T \to S$ a morphism with $T, S \in \mathrm{SmVar}(\mathbb{C})$. Consider the factorization $g : T \xrightarrow{l} T \times S \xrightarrow{p_S} S$ where l is the graph embedding and p_S the projection. Let $M \in \mathrm{DA}_c(S)$ and $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1_S, et)(F)$. Then, $D(\mathbb{A}^1_T, et)(g^*F) = g^*M$.*

(i) We have then the canonical transformation in $D_{\mathcal{D}^\infty fil,\infty}(T \times S)$ (see definition 126) :

$$\begin{aligned} T(g, \mathcal{F}^{GM})(M) : Rg^{*mod[-],\Gamma}\mathcal{F}^{GM}_{S,an}(M) := g^{*mod,\Gamma} e(S)_*\mathcal{H}om^\bullet(\mathrm{An}_S^* L(F), E_{usu}(\Omega^\bullet_{/S}, F_b)))[-d_T] \\ \xrightarrow{T(g,\Omega_{/\cdot})(\mathrm{An}_S^* L(F))} e(T \times S)_*\mathcal{H}om^\bullet(\Gamma^\vee_T p_S^* \mathrm{An}_S^* L(F), E_{usu}(\Omega^\bullet_{/Y\times S}, F_b))[-d_T] \\ \xrightarrow{\mathcal{H}om(T(\mathrm{An},\gamma^\vee)(p_S^* LF)^{-1},-)} \mathcal{F}^{GM}_{T\times S,an}(l_* g^* M). \end{aligned}$$

where the last isomorphism in the derived category comes from proposition 125.

(ii) We have then the canonical transformation in $D_{Ofil,\infty}(T)$ (see definition 126) :

$$\begin{aligned} T^O(g, \mathcal{F}^{GM})(M) : Lg^{*mod[-]}\mathcal{F}^{GM}_{S,an}(M) := g^{*mod} L_O e(S)_*\mathcal{H}om^\bullet(\mathrm{An}_S^* L(F), E_{usu}(\Omega^\bullet_{/S}, F_b)))[-d_T] \\ \xrightarrow{T(g,\Omega_{/\cdot})(\mathrm{An}_S^* L(F))} e(T \times S)_*\mathcal{H}om^\bullet(g^* \mathrm{An}_S^* L(F), E_{usu}(\Omega^\bullet_{/Y\times S}, F_b))[-d_T] =: \mathcal{F}^{GM}_{T,an}(g^*M). \end{aligned}$$

We give now the definition in the non smooth case Let $g : T \to S$ a morphism with $T, S \in \mathrm{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^{l} S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$ Then, $T = \cup_{i=1}^{l} T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$, Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \to \tilde{S}_I$ is a lift of $g_I := g_{|T_I} : T_I \to S_I$. We recall the commutative diagram :

$$E_{IJg} = \begin{tikzcd} (Y \times \tilde{S}_I)\backslash T_I \arrow[r, "m'_I"] \arrow[d, "p_{\tilde{S}_I}"] & Y \times \tilde{S}_J \arrow[d, "\tilde{g}_I"] \\ \tilde{S}_I\backslash S_I \arrow[r, "m_I"] & \tilde{S}_I \end{tikzcd}, \quad E_{IJ} = \begin{tikzcd} \tilde{S}_J\backslash S_J \arrow[r, "m_J"] \arrow[d, "p_{IJ}"] & \tilde{S}_J \arrow[d, "p_{IJ}"] \\ \tilde{S}_I\backslash(S_I\backslash S_J) \arrow[r, "m=m_{IJ}"] & \tilde{S}_I \end{tikzcd} \quad E'_{IJ} = \begin{tikzcd} (Y \times \tilde{S}_J)\backslash T_J \arrow[r, "m'_J"] \arrow[d, "p'_{IJ}"] & Y \times \tilde{S}_J \arrow[d, "p'_{IJ}"] \\ (Y \times \tilde{S}_I)\backslash(T_I\backslash T_J) \arrow[r, "m'=m'_{IJ}"] & Y \times \tilde{S}_I \end{tikzcd}$$

For $I \subset J$, denote by $p_{IJ} : \tilde{S}_J \to \tilde{S}_I$ and $p'_{IJ} := I_Y \times p_{IJ} : Y \times \tilde{S}_J \to Y \times \tilde{S}_I$ the projections, so that $\tilde{g}_I \circ p'_{IJ} = p_{IJ} \circ \tilde{g}_J$. Consider, for $I \subset J \subset [1, \ldots, l]$, resp. for each $I \subset [1, \ldots, l]$, the following commutative diagrams in $\mathrm{Var}(\mathbb{C})$

$$D_{IJ} = \begin{tikzcd} S_I \arrow[r, "i_I"] & \tilde{S}_I \\ S_J \arrow[u, "j_{IJ}"] \arrow[r, "i_J"] & \tilde{S}_J \arrow[u, "p_{IJ}"] \end{tikzcd}, \quad D'_{IJ} = \begin{tikzcd} T_I \arrow[r, "i'_I"] & Y \times \tilde{S}_I \\ T_J \arrow[u, "j'_{IJ}"] \arrow[r, "i'_J"] & Y \times \tilde{S}_J \arrow[u, "p'_{IJ}"] \end{tikzcd} \quad D_{gI} = \begin{tikzcd} S_I \arrow[r, "i_I"] & \tilde{S}_I \\ T_I \arrow[u, "g_I"] \arrow[r, "i'_I"] & Y \times \tilde{S}_I \arrow[u, "\tilde{g}_I"] \end{tikzcd},$$

and $j_{IJ}: S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. Let $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$. The fact that the diagrams (55) commutes says that the maps $T^{q,\gamma}(D_{gI})(j_I^* F)$ define a morphism in $C(\mathrm{Var}(\mathbb{C})^{sm}/(T/(Y \times \tilde{S}_I)))$

$$(T^{q,\gamma}(D_{gI})(j_I^* F)) : (\Gamma^{\vee}_{T_I} \tilde{g}_I^* L(i_{I*} j_I^* F), \tilde{g}_J^* S^q(D_{IJ})(F)) \to (L(i'_{I*} j_I^{'*} g^* F), S^q(D'_{IJ})(g^* F))$$

We then have then the following lemma :

Lemma 17. *(i) The morphism in* $C(\mathrm{Var}(\mathbb{C})^{sm}/(T/(Y \times \tilde{S}_I)))$

$$(T^{q,\gamma}(D_{gI})(j_I^* F)) : (\Gamma^{\vee}_{T_I} L \tilde{g}_I^* i_{I*} j_I^* F, \tilde{g}_J^* S^q(D_{IJ})(F)) \to (i'_{I*} j_I^{'*} g^* F, S^q(D'_{IJ})(g^* F))$$

is an equivalence $(\mathbb{A}^1, et)$ *local.*

(ii) Denote for short $d_{YI} := -d_Y - d_{\tilde{S}_I}$*. The maps* $\mathcal{H}om(\mathrm{An}^*_{Y \times \tilde{S}_I}(T^{q,\gamma}(D_{gI})(j_I^* F)), E_{usu}(\Omega^{\bullet}_{/Y \times \tilde{S}_I}, F_b))$ *induce an* ∞*-filtered quasi-isomorphism in* $C_{Ofil,\mathcal{D}^\infty}(T/(Y \times \tilde{S}_I))$

$$\begin{gathered}(\mathcal{H}om(\mathrm{An}^*_{Y \times \tilde{S}_I} T^{q,\gamma}(D_{gI})(j_I^* F), E_{usu}(\Omega^{\bullet}_{/Y \times \tilde{S}_I}, F_b))) : \\ (e(Y \times \tilde{T}_I)_* \mathcal{H}om(\mathrm{An}^*_{Y \times \tilde{S}_I} L(i'_{I*} j_I^{'*} g^* F), E_{usu}(\Omega^{\bullet}_{/Y \times \tilde{S}_I}, F_b))[d_{YI}], u^q_{IJ}(g^* F)) \to \\ (e(Y \times \tilde{T}_I)_* \mathcal{H}om(\mathrm{An}^*_{Y \times \tilde{S}_I} \Gamma^{\vee}_{T_I} L(\tilde{g}_I^* i_{I*} j_I^* F), E_{usu}(\Omega^{\bullet}_{/Y \times \tilde{S}_I}, F_b))[d_{YI}], \tilde{g}_J^* u^q_{IJ}(F)_2)\end{gathered}$$

(iii) The maps $T(\tilde{g}_I, \Omega_\cdot)(L(i_{I*} j_I^* F))$ *(see definition 126) induce a morphism in* $C_{Ofil,\mathcal{D}^\infty}(T/(Y \times \tilde{S}_I))$

$$\begin{gathered}(T(\tilde{g}_I, \Omega_{/\cdot})(L(i_{I*} j_I^* F))) : \\ (\Gamma_{T_I} E_{zar}(\tilde{g}_I^{*mod[-]} e(\tilde{S}_I)_* \mathcal{H}om^{\bullet}(\mathrm{An}^*_{\tilde{S}_I} L(i_{I*} j_I^* F), E_{usu}(\Omega^{\bullet}_{/\tilde{S}_I}, F_b)))[d_{YI}], \tilde{g}_J^{*mod} u^q_{IJ}(F)) \to \\ (\Gamma_{T_I}(e(Y \times \tilde{S}_I)_* \mathcal{H}om(\mathrm{An}^*_{Y \times \tilde{S}_I} \tilde{g}_I^* L(i_{I*} j_I^* F), E_{usu}(\Omega^{\bullet}_{/Y \times \tilde{S}_I}, F_b)))[d_{YI}], \tilde{g}_J^* u^q_{IJ}(F)_1).\end{gathered}$$

Proof. (i):Follows from theorem 16
(ii): Similar to lemma 10(ii).
(iii):Similar to lemma 10(iii).

□

Definition 131. *Let* $g : T \to S$ *a morphism with* $T, S \in \mathrm{Var}(\mathbb{C})$*. Assume we have a factorization* $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ *with* $Y \in \mathrm{SmVar}(\mathbb{C})$*,* l *a closed embedding and* p_S *the projection. Let* $S = \cup_{i=1}^l S_i$ *be an open cover such that there exists closed embeddings* $i_i : S_i \hookrightarrow \tilde{S}_i$ *with* $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$ *Then,* $T = \cup_{i=1}^l T_i$ *with* $T_i := g^{-1}(S_i)$ *and we have closed embeddings* $i'_i := i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$*, Moreover* $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \to \tilde{S}_I$ *is a lift of* $g_I := g_{|T_I} : T_I \to S_I$*. Denote for short* $d_{YI} := -d_Y - d_{\tilde{S}_I}$*. Let* $M \in \mathrm{DA}_c(S)$ *and* $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ *such that* $M = D(\mathbb{A}^1_S, et)(F)$*. Then,* $D(\mathbb{A}^1_T, et)(g^* F) = g^* M$*. We* *have, by lemma 10, the canonical transformation in* $D_{Ofil,\mathcal{D}^\infty,\infty}(T/(Y\times\tilde{S}_I))$

$$\begin{aligned}
&T(g,\mathcal{F}^{GM})(M):Rg^{*mod[-],\Gamma}\mathcal{F}^{GM}_{S,an}(M):=\\
&(\Gamma_{T_I}E_{zar}(\tilde{g}_I^{*mod}e(\tilde{S}_I)_*\mathcal{H}om^\bullet(\mathrm{An}^*_{\tilde{S}_I}L(i_{I*}j_I^*F),E_{usu}(\Omega^\bullet_{/\tilde{S}_I},F_b)))[d_{YI}],\tilde{g}_J^{*mod}u^q_{IJ}(F))\\
&\xrightarrow{(\Gamma_{T_I}E(T(\tilde{g}_I,\Omega_{/\cdot})(\mathrm{An}^*_{\tilde{S}_I}L(i_{I*}j_I^*F))))}\\
&(\Gamma_{T_I}e(Y\times\tilde{S}_I)_*\mathcal{H}om^\bullet(\mathrm{An}^*_{Y\times\tilde{S}_I}\tilde{g}_I^*L(i_{I*}j_I^*F),E_{usu}(\Omega^\bullet_{/Y\times\tilde{S}_I},F_b))[d_{YI}],\tilde{g}_J^*u^q_{IJ}(F)_1)\\
&\xrightarrow{(I(\gamma,\hom(-,-)))}\\
&(e(Y\times\tilde{S}_I)_*\mathcal{H}om^\bullet(\Gamma^\vee_{T_I}\mathrm{An}^*_{Y\times\tilde{S}_I}\tilde{g}_I^*L(i_{I*}j_I^*F),E_{usu}(\Omega^\bullet_{/Y\times\tilde{S}_I},F_b))[d_{YI}],\tilde{g}_J^*u^q_{IJ}(F)_2)\\
&\xrightarrow{(e(Y\times\tilde{S}_I)_*\mathcal{H}om^\bullet(T(\mathrm{An},\gamma^\vee)(\tilde{g}_I^*L(i_{I*}j_I^*F))^{-1},E_{usu}(\Omega^\bullet_{/Y\times\tilde{S}_I},F_b)))}\\
&(e(Y\times\tilde{S}_I)_*\mathcal{H}om^\bullet(\mathrm{An}^*_{Y\times\tilde{S}_I}\Gamma^\vee_{T_I}\tilde{g}_I^*L(i_{I*}j_I^*F),E_{usu}(\Omega^\bullet_{/Y\times\tilde{S}_I},F_b))[d_{YI}],\tilde{g}_J^*u^q_{IJ}(F)_2)\\
&\xrightarrow{(e(Y\times\tilde{S}_I)_*\mathcal{H}om(\mathrm{An}^*_{Y\times\tilde{S}_I}T^{q,\gamma}(D_{gI})(j_I^*F),E_{usu}(\Omega^\bullet_{/Y\times\tilde{S}_I},F_b)))^{-1}}\\
&(e(Y\times\tilde{S}_I)_*\mathcal{H}om^\bullet(\mathrm{An}^*_{Y\times\tilde{S}_I}L(i'_{I*}j_I^{'*}g^*F),E_{usu}(\Omega^\bullet_{/Y\times\tilde{S}_I},F_b))[d_{YI}],u^q_{IJ}(g^*F))=:\mathcal{F}^{GM}_{T,an}(g^*M).
\end{aligned}$$

Proposition 127. *(i) Let $g:T\to S$ a morphism with $T,S\in\mathrm{Var}(\mathbb{C})$. Assume we have a factorization $g:T\xrightarrow{l}Y_2\times S\xrightarrow{p_S}S$ with $Y_2\in\mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S=\cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$ Then, $T=\cup_{i=1}^l T_i$ with $T_i:=g^{-1}(S_i)$ and we have closed embeddings $i'_i:=i_i\circ l:T_i\hookrightarrow Y_2\times\tilde{S}_i$, Moreover $\tilde{g}_I:=p_{\tilde{S}_I}:Y\times\tilde{S}_I\to\tilde{S}_I$ is a lift of $g_I:=g_{|T_I}:T_I\to S_I$. Let $f:X\to S$ a morphism with $X\in\mathrm{Var}(\mathbb{C})$. Assume that there is a factorization $f:X\xrightarrow{l}Y_1\times S\xrightarrow{p_S}S$, with $Y_1\in\mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have then the following commutative diagram whose squares are cartesians*

$$\begin{tikzcd}
f':X_T \arrow[r] \arrow[d] & Y_1\times T \arrow[r] \arrow[d] & T \arrow[d]\\
f''=f\times I:Y_2\times X \arrow[r] \arrow[d] & Y_1\times Y_2\times S \arrow[r] \arrow[d] & Y_2\times S \arrow[d]\\
f:X \arrow[r] & Y_1\times S \arrow[r] & S
\end{tikzcd}$$

Consider $F(X/S):=p_{S,\sharp}\Gamma^\vee_X\mathbb{Z}(Y_1\times S/Y_1\times S)[d_{Y_1}]$ and the isomorphism in $C(\mathrm{Var}(\mathbb{C})^{sm}/S)$

$$\begin{aligned}T(f,g,F(X/S)):g^*F(X/S):=g^*p_{S,\sharp}\Gamma^\vee_X\mathbb{Z}(Y_1\times S/Y_1\times S)\xrightarrow{\sim}\\ p_{T,\sharp}\Gamma^\vee_{X_T}\mathbb{Z}(Y_1\times T/Y_1\times T)=:F(X_T/T).\end{aligned}$$

which gives in $\mathrm{DA}(S)$ the isomorphism $T(f,g,F(X/S)):g^*M(X/S)\xrightarrow{\sim}M(X_T/T)$. Then, the following diagram in $D_{Ofil,\mathcal{D}^\infty,\infty}(T/(Y_2\times\tilde{S}_I))$ commutes

$$\begin{tikzcd}
Rg^{*mod,\Gamma}\mathcal{F}^{GM}_{S,an}(M(X/S)) \arrow[r, "T(g,\mathcal{F}^{GM})(M(X/S))"] \arrow[d, "I^{GM}(X/S)"] & \mathcal{F}^{GM}_{T,an}(M(X_T/T)) \arrow[d, "I^{GM}(X_T/T)"]\\
g^{*mod[-],\Gamma}(p_{\tilde{S}_I*}\Gamma_{X_I}E_{usu}(\Omega^\bullet_{Y_1\times\tilde{S}_I/\tilde{S}_I},F_b)[-d_{\tilde{S}_I}],w_{IJ}(X/S)) \arrow[r, "(T(\tilde{g}_I\times I,\gamma)(-)\circ T^O_w(\tilde{g}_I,p_{\tilde{S}_I}))"] & (p_{Y_2\times\tilde{S}_I*}\Gamma_{X_{T_I}}E_{usu}(\Omega^\bullet_{Y_2\times Y_1\times\tilde{S}_I/Y_2\times\tilde{S}_I},F_b)[-d_{Y_2} -\ w_{IJ}(X_T/T))
\end{tikzcd}$$

(ii) Let $g : T \to S$ a morphism with $T, S \in \mathrm{SmVar}(\mathbb{C})$. Let $f : X \to S$ a morphism with $X \in \mathrm{Var}(\mathbb{C})$. Assume that there is a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$, with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Consider $F(X/S) := p_{S,\sharp}\Gamma^\vee_X\mathbb{Z}(Y \times S/Y \times S)$ and the isomorphism in $C(\mathrm{Var}(\mathbb{C})^{sm}/S)$

$$T(f, g, F(X/S)) : g^*F(X/S) := g^*p_{S,\sharp}\Gamma^\vee_X\mathbb{Z}(Y \times S/Y \times S) \xrightarrow{\sim} p_{T,\sharp}\Gamma^\vee_{X_T}\mathbb{Z}(Y \times T/Y \times T)[d_Y] =: F(X_T/T).$$

which gives in $\mathrm{DA}(S)$ the isomorphism $T(f, g, F(X/S)) : g^*M(X/S) \xrightarrow{\sim} M(X_T/T)$. Then, the following diagram in $D_{Ofil,\infty}(T)$ commutes

$$\begin{tikzcd}
Lg^{*mod[-]}\mathcal{F}^{GM}_{S,an}(M(X/S)) \arrow[r, "T^O(g,\mathcal{F}^{GM})(M(X/S))"] \arrow[d, "I^{GM}(X/S)"] & \mathcal{F}^{GM}_{T,an}(M(X_T/T)) \arrow[d, "I^{GM}(X_T/T)"] \\
g^{*mod}L_O(p_{S*}\Gamma_X E_{usu}(\Omega^\bullet_{Y\times S/S}, F_b)[-d_T] \arrow[r, "(T(g\times I,\gamma)(-)\circ T^O_w(g,p_S))"] \arrow[d, "T_w(\otimes,\gamma)(O_{Y\times S})"] & p_{Y\times T*}\Gamma_{X_T}E_{usu}(\Omega^\bullet_{Y\times T/T}, F_b)[-d_T] \arrow[d, "T_w(\otimes,\gamma)(O_{Y\times T})"] \\
Lg^{*mod}\int^{FDR}_{p_S}(\Gamma_X E_{usu}(O_{Y\times S}, F_b)[-d_Y - d_T] \arrow[r, "T^{\mathcal{D}mod}(p_S,f)(-)"] & \int^{FDR}_{p_T}(\Gamma_{X_T}E_{usu}(O_{Y\times T}, F_b))[-d_Y - d_T].
\end{tikzcd}$$

Proof. Follows immediately from definition. □

We have the following theorem:

Theorem 38. *(i) Let $g : T \to S$ is a morphism with $T, S \in \mathrm{Var}(\mathbb{C})$. Assume there exist a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S}$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup^l_{i=1}S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. Then, for $M \in \mathrm{DA}_c(S)$*

$$T(g, \mathcal{F}^{GM}_{an})(M) : Rg^{*mod[-],\Gamma}\mathcal{F}^{GM}_{S,an}(M) \to \mathcal{F}^{GM}_{T,an}(g^*M)$$

is an isomorphism in $D_{O_T fil,\mathcal{D}^\infty,\infty}(T/(Y \times \tilde{S}_I))$.

(ii) Let $g : T \to S$ is a morphism with $T, S \in \mathrm{SmVar}(\mathbb{C})$. Then, for $M \in \mathrm{DA}_c(S)$

$$T(g, \mathcal{F}^{GM}_{an})(M) : Lg^{*mod[-]}\mathcal{F}^{GM}_{S,an}(M) \to \mathcal{F}^{GM}_{T,an}(g^*M)$$

is an isomorphism in $D_{O_T}(T)$.

Proof. (i):Follows from proposition 123.
(ii): : **First proof** : Follows from proposition 127, proposition 133 and proposition 92.
: **Second proof** : In the analytic case only, we can give a direct proof of this proposition : Indeed, let $g : T \to S$ is a morphism with $T, S \in \mathrm{AnSp}(\mathbb{C})$ and let $h : U \to S$ a smooth morphism with $U \in \mathrm{AnSp}(\mathbb{C}$, then,

$$T^O_\omega(g, h) : g^{*mod}L_{D^\infty}h_*E(\Omega^\bullet_{U/S}, F) \to h'_*E(\Omega^\bullet_{U_T/T}, F)$$

is an equivalence usu local : consider the following commutative diagram

$$\begin{tikzcd}
g^{*mod}L_O(h_*E(\mathbb{Z}_U) \otimes O_S) \arrow[r, "T(g,h)(E(\mathbb{Z}_U))"] \arrow[d, "g^{*mod}L_O T(h,\otimes)(-,-)"'] & h'_*E(\mathbb{Z}_{U_T}) \otimes O_T \arrow[d, "T(h',\otimes)(-,-)"] \\
g^{*mod}L_O h_*E(h^*O_S) \arrow[r, "T^{mod}(g,h)(h^*O_S)"] \arrow[d, "g^{*mod}L_O h_*E(\iota_{U/S})"'] & h'_*E(h'^*O_T) \arrow[d, "h'_*E(\iota_{U_T/T})"] \\
g^{*mod}L_O h_*E(\Omega^\bullet_{U/S}) \arrow[r, "T^O_\omega(g,h)"] & h'_*E(\Omega^\bullet_{U_T/T})
\end{tikzcd} ,$$

then,

- the maps $T(h',\otimes)(-,-)$ and $T(h,\otimes)(-,-)$ are usu local equivalence by proposition 9,
- since $h:U\to S$ is a smooth morphism, the inclusion $\iota_{U/S}:h^*O_S\to\Omega^\bullet_{U/S}$ is a quasi-isomorphism,
- since $h':U_T\to T$ is a smooth morphism, the inclusion $\iota_{U_T/T}:h^*O_T\to\Omega^\bullet_{U_T/T}$ is a quasi-isomorphism,
- since U, U_T, S, T are paracompact topological spaces (in particular Hausdorf), $T(g,h)(E(\mathbb{Z}_U):g^*h_*E(\mathbb{Z}_U)\to h'_*E(\mathbb{Z}_{U_T})$ is a quasi-isomorphism.

This fact, together with lemma 15, proves the proposition. □

We finish this subsection by some remarks on the absolute case and on a particular case of the relative case:

Proposition 128. *(i) Let $X\in\mathrm{PSmVar}(\mathbb{C})$ and $D=\cup D_i\subset X$ a normal crossing divisor. Consider the open embedding $j:U:=X\backslash D\hookrightarrow X$. Then,*

- *The map in $D_{fil,\infty}(\mathbb{C})$*

$$\begin{aligned}\mathrm{Hom}((0,\mathrm{ad}(j^*,j_*)(\mathbb{Z}(X/X)),E_{usu}(\Omega^\bullet,F_b)):\\ \mathcal{F}^{GM}_{an}(\mathbb{D}(\mathbb{Z}(U))):=\mathrm{Hom}(L\mathbb{D}(\mathbb{Z}(U)),E_{usu}(\Omega^\bullet,F_b))\\ \xrightarrow{\sim}\mathrm{Hom}(\mathrm{Cone}(\mathbb{Z}(D)\to\mathbb{Z}(X)),E_{zar}(\Omega^\bullet,F_b))=\Gamma(X,E_{usu}(\Omega^\bullet_X(\mathrm{nul}\,D),F_b)).\end{aligned}$$

  is an isomorphism, where we recall $\mathbb{D}(\mathbb{Z}(U):=a_{X*}j_*E_{et}(\mathbb{Z}(U/U))=a_{U*}E_{et}(\mathbb{Z}(U/U))$,

- *$\mathcal{F}^{GM}_{an}(\mathbb{Z}(U))=\Gamma(U,E_{usu}\Omega^\bullet_U,F_b)\in D_{fil,\infty}(\mathbb{C})$ is NOT isomorphic to $\Gamma(X,E_{usu}(\Omega^\bullet_X(\log D),F_b)$ in $D_{fil,\infty}(\mathbb{C})$ in general. For exemple U is affine, then U^{an} is Stein so that $H^n(U,\Omega^p_U)=0$ for all $p\in\mathbb{N}$, $p\neq 0$, so that the $E^{p,q}_\infty(\Gamma(U,E_{usu}(\Omega^\bullet_U,F_b)))$ are NOT isomorphic to $E^{p,q}_\infty(\Gamma(X,E_{usu}(\Omega^\bullet_X(\log D),F_b)))$ in this case. In particular, the map,*

$$j^*:=\mathrm{ad}(j^*,j_*)(-):H^n\Gamma(X,E_{usu}(\Omega^\bullet_X(\log D)))\xrightarrow{\sim}H^n\Gamma(U,E_{zar}(\Omega^\bullet_U))$$

  which is an isomorphism in $D(\mathbb{C})$ (i.e. if we forgot filtrations), gives embeddings

$$j^*:=\mathrm{ad}(j^*,j_*)(-):F^pH^n(U,\mathbb{C}):=F^pH^n\Gamma(X,E_{usu}(\Omega^\bullet_X(\log D),F_b))\hookrightarrow F^pH^n\Gamma(U,E_{usu}(\Omega^\bullet_U,F_b))$$

  which are NOT an isomorphism in general for $n,p\in\mathbb{Z}$. Note that, since $a_U:U\to\{\mathrm{pt}\}$ is not proper,

$$[\Delta_U]:\mathbb{Z}(U)\to a_{U*}E_{et}(\mathbb{Z}(U/U))[2d_U]$$

  is NOT an equivalence $(\mathbb{A}^1,et)$ local.

- *Let $Z\subset X$ a smooth subvariety and denote $U:=X\backslash Z$ the open complementary. Denote $M_Z(X):=\mathrm{Cone}(M(U)\to M(X))\in\mathrm{DA}(\mathbb{C})$. The map in $D_{fil,\infty}(\mathbb{C})$*

$$\begin{aligned}\mathrm{Hom}(G(X,Z),E_{usu}(\Omega^\bullet,F_b))^{-1}:\mathcal{F}^{GM}_{an}(M_Z(X)):=\mathrm{Hom}(a_{X\sharp}\Gamma^\vee_Z\mathbb{Z}(X/X)),E_{usu}(\Omega^\bullet,F_b))\xrightarrow{\sim}\\ \Gamma(X,\Gamma_ZE_{usu}(\Omega^\bullet_X,F_b))=\Gamma_Z(X,E_{usu}(\Omega^\bullet_X,F_b))\\ \xrightarrow{\sim}\mathcal{F}^{GM}_{an}(M(Z)(c)[2c])=\Gamma(Z,E_{usu}(\Omega^\bullet_Z,F_b))(-c)[-2c]\end{aligned}$$

  is an isomorphism, where $c=\mathrm{codim}(Z,X)$ and $G(X,Z):a_{X\sharp}\Gamma^\vee_Z\mathbb{Z}(X/X)\to\mathbb{Z}(Z)(c)[2c]$ is the Gynsin morphism.

- *Let $D\subset X$ a smooth divisor and denote $U:=X\backslash Z$ the open complementary Note that the canonical distinguish triangle in $\mathrm{DA}(\mathbb{C})$*

$$M(U)\xrightarrow{\mathrm{ad}(j_\sharp,j^*)(\mathbb{Z}(X/X))}M(X)\xrightarrow{\gamma^\vee_Z(\mathbb{Z}(X/X))}M_D(X)\to M(U)[1]$$

give a canonical triangle in $D_{fil,\infty}(\mathbb{C})$

$$\mathcal{F}^{GM}(M_D(X)) \xrightarrow{\mathcal{F}^{GM}(\gamma_Z^\vee(\mathbb{Z}(X/X)))} \mathcal{F}^{GM}(M(X)) \xrightarrow{\mathcal{F}^{GM}(\mathrm{ad}(j_\sharp,j^*)(\mathbb{Z}(X/X))} \mathcal{F}^{GM}(M(U)) \to \mathcal{F}^{GM}(M_D(X))[1],$$

which is NOT the image of a distinguish triangle in $\pi(D(MHM(\mathbb{C})))$*, as* $\mathcal{F}^{GM}(M(U)) \notin \pi(D(MHM(\mathbb{C})))$ *since the morphism*

$$j^* := \mathrm{ad}(j^*, j_*)(-) : H^n\Gamma(X, E_{usu}(\Omega_X^\bullet(\log D), F_b)) \to H^n\Gamma(U, E_{usu}(\Omega_U^\bullet, F_b))$$

are not strict. Note that if $U := S \backslash D$ *is affine, then by the exact sequence in* $C(\mathbb{Z})$

$$0 \to \Gamma_Z(X, E_{usu}(\Omega_X^p)) \to \Gamma(X, E_{usu}(\Omega_X^p)) \to \Gamma(U, E_{usu}(\Omega_U^p)) \to 0$$

we have $H^q\Gamma_Z(X, E_{usu}(\Omega_X^p)) = H^q(\Gamma(X, E_{usu}(\Omega_X^p)))$.

In particular, the maps

$$j^* := \mathrm{ad}(j^*, j_*)(-) : \Gamma(X, E_{usu}(\Omega_X^\bullet(\log D), F_b)) \to \Gamma(U, E_{usu}(\Omega_U^\bullet, F_b))$$

and

$$\begin{aligned} j^* := \mathrm{ad}(j^*, j_*)(-) : \mathrm{Cone}(\Gamma(X, E_{usu}(\Omega_X^\bullet, F_b)) \to \Gamma(X, E_{usu}(\Omega_X^\bullet(\log D), F_b))) \to \\ \mathrm{Cone}(\Gamma(X, E_{usu}(\Omega_X^\bullet, F_b)) \to \Gamma(U, E_{usu}(\Omega_U^\bullet, F_b))) =: \Gamma(X, \Gamma_Z E_{usu}(\Omega_X^\bullet, F_b)) \end{aligned}$$

are quasi-isomorphism (i.e. if we forgot filtrations), but the first one is NOT an ∞*-filtered quasi-isomorphism whereas the second one is an* ∞*-filtered quasi-isomorphism (recall that for* $r > 1$ *the* r*-filtered quasi-isomorphism does NOT satisfy the 2 of 3 property for morphism of canonical triangles : see section 2.1).*

(ii) *More generally, let* $f : X \to S$ *a smooth projective morphism with* $S, X \in \mathrm{SmVar}(\mathbb{C})$*. Let* $D = \cup D_i \subset X$ *a normal crossing divisor such that* $f_{|D_I} := f \circ i_I : D_I \to S$ *are SMOOTH morphisms (note that it is a very special case), with* $i_I : D_I \hookrightarrow X$ *the closed embeddings. Consider the open embedding* $j : U := X \backslash D \hookrightarrow X$ *and* $h := f \circ j : U \to S$.

- *The map in* $D_{\mathcal{D}fil,\infty}(S)$

$$\begin{aligned} &\mathrm{Hom}((0, \mathrm{ad}(j^*, j_*)(\mathbb{Z}(X/X))), E_{et}(\Omega^\bullet_{/S}, F_b)) : \\ &\mathcal{F}^{GM}_{S,an}(\mathbb{D}(\mathbb{Z}(U/S))) := \mathrm{Hom}(L\mathbb{D}(\mathbb{Z}(U/S)), E_{usu}(\Omega^\bullet_{/S}, F_b)) \\ &\xrightarrow{\sim} \mathrm{Hom}(\mathrm{Cone}(\mathbb{Z}(D) \to \mathbb{Z}(X)), E_{zar}(\Omega^\bullet_{/S}, F_b)) = f_* E_{usu}(\Omega^\bullet_{X/S}(\mathrm{nul}\, D), F_b). \end{aligned}$$

  is an isomorphism, where we recall $\mathbb{D}(\mathbb{Z}(U) := f_* j_* E_{et}(\mathbb{Z}(U/U)) = h_* E_{usu}(\mathbb{Z}(U/U))$,
- $\mathcal{F}^{GM}_{S,an}(\mathbb{Z}(U/S)) = h_* E_{usu}\Omega^\bullet_{U/S}, F_b) \in D_{\mathcal{D}fil,\infty}(S)$ *is NOT isomorphic to* $f_* E_{usu}(\Omega^\bullet_{X/S}(\log D), F_b)$ *in* $D_{\mathcal{D}fil,\infty}(S)$ *in general. In particular, the map,*

$$j^* := \mathrm{ad}(j^*, j_*)(-) : H^n f_* E_{usu}(\Omega^\bullet_{X/S}(\log D)) \xrightarrow{\sim} H^n h_* E_{usu}(\Omega^\bullet_{U/S})$$

  which is an isomorphism in $D_{\mathcal{D}}(S)$ *(i.e. if we forgot filtrations), gives embeddings*

$$j^* := \mathrm{ad}(j^*, j_*)(-) : F^p H^n h_* \mathbb{C}_U := F^p H^n f_* E_{usu}(\Omega^\bullet_{X/S}(\log D), F_b) \hookrightarrow F^p H^n h_* E_{usu}(\Omega^\bullet_{U/S}, F_b)$$

  which are NOT an isomorphism in general for $n, p \in \mathbb{Z}$*. Note that, since* $a_U : U \to \{\mathrm{pt}\}$ *is not proper,*

$$[\Delta_U] : \mathbb{Z}(U/S) \to h_* E_{usu}(\mathbb{Z}(U/U))[2d_U]$$

  is NOT an equivalence $(\mathbb{A}^1, et)$ *local.*

– *Let $Z \subset X$ a subvariety and denote $U := X\backslash Z$ the open complementary. Denote $M_Z(X/S) :=$ $\mathrm{Cone}(M(U/S) \to M(X/S)) \in \mathrm{DA}(S)$. If $f_{|Z} := f \circ i_Z : Z \to S$ is a SMOOTH morphism, the map in $D_{\mathcal{D}fil,\infty}(S)$*

$$\begin{aligned}
&\mathrm{Hom}(G(X,Z), E_{usu}(\Omega^\bullet, F_b)) : \\
\mathcal{F}^{GM}_{S,an}(M_Z(X/S)) := \mathrm{Hom}(f_\sharp\Gamma^\vee_Z\mathbb{Z}(X/X)), E_{usu}(\Omega^\bullet_{/S}, F_b)) &\xrightarrow{\sim} f_*\Gamma_Z E_{usu}(\Omega^\bullet_{X/S}, F_b)) \\
&\xrightarrow{\sim} \mathcal{F}^{GM}_S(M(Z/S)(c)[2c]) = f_{Z*}E_{usu}(\Omega^\bullet_{Z/S}, F_b)(-c)[-2c]
\end{aligned}$$

is an isomorphism, where $c = \mathrm{codim}(Z,X)$ and $G(X,Z) : f_\sharp\Gamma^\vee_Z\mathbb{Z}(X/X) \to \mathbb{Z}(Z/S)(c)[2c]$ is the Gynsin morphism.

– *Let $D \subset X$ a smooth divisor and denote $U := X\backslash Z$ the open complementary Note that the canonical distinguish triangle in $\mathrm{DA}(S)$*

$$M(U/S) \xrightarrow{\mathrm{ad}(j_\sharp,j^*)(\mathbb{Z}(X/X))} M(X/S) \xrightarrow{\gamma^\vee_Z(\mathbb{Z}(X/X))} M_D(X/S) \to M(U/S)[1]$$

give a canonical triangle in $D_{\mathcal{D}fil,\infty}(S)$

$$\begin{aligned}
\mathcal{F}^{GM}_{S,an}(M_D(X/S)) \xrightarrow{\mathcal{F}^{GM}(\gamma^\vee_Z(\mathbb{Z}(X/X)))} \mathcal{F}^{GM}_{S,an}(M(X/S)) &\xrightarrow{\mathcal{F}^{GM}(\mathrm{ad}(j_\sharp,j^*)(\mathbb{Z}(X/X))} \mathcal{F}^{GM}_{S,an}(M(U/S)) \\
&\to \mathcal{F}^{GM}_{S,an}(M_D(X/S))[1],
\end{aligned}$$

which is NOT the image of a distinguish triangle in $\pi_S(D(MHM(S)))$.

Proof. Similar to the proof of theorem 109. □

Definition 132. *Let $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open affine covering and denote, for $I \subset [1, \cdots l]$, $S_I = \cap_{i\in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. We have, for $M, N \in \mathrm{DA}(S)$ and $F, G \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1, et)(F)$ and $N = D(\mathbb{A}^1, et)(G)$, the following transformation map in $D_{Ofil,\mathcal{D}^\infty}(S/(\tilde{S}_I))$*

$$\begin{aligned}
&T(\mathcal{F}^{GM}_{S,an}, \otimes)(M,N) : \\
\mathcal{F}^{GM}_{S,an}(M) \otimes^L_{O_S} \mathcal{F}^{GM}_{S,an}(N) &:= (e(\tilde{S}_I)_*\mathcal{H}om(\mathrm{An}^*_{\tilde{S}_I} L(i_{I*}j_I^*F), E_{usu}(\Omega^\bullet_{/\tilde{S}_I}, F_b)), u_{IJ}(F)) \otimes_{O_S} \\
&(e(\tilde{S}_I)_*\mathcal{H}om(\mathrm{An}^*_{\tilde{S}_I} L(i_{I*}j_I^*G), E_{usu}(\Omega^\bullet_{/\tilde{S}_I}, F_b)), u_{IJ}(G)) \\
&\xrightarrow{=} ((e(\tilde{S}_I)_*\mathcal{H}om(\mathrm{An}^*_{\tilde{S}_I} L(i_{I*}j_I^*F), E_{usu}(\Omega^\bullet_{/\tilde{S}_I}, F_b)) \otimes_{O_{\tilde{S}_I}} \\
&e(\tilde{S}_I)_*\mathcal{H}om(\mathrm{An}^*_{\tilde{S}_I} L(i_{I*}j_I^*G), E_{usu}(\Omega^\bullet_{/\tilde{S}_I}, F_b))), u_{IJ}(F) \otimes u_{IJ}(G)) \\
&\xrightarrow{(T(\otimes,\Omega_{/\tilde{S}_I})(L(i_{I*}j_I^*F), L(i_{I*}j_I^*G)))} \\
&(e(\tilde{S}_I)_*\mathcal{H}om(\mathrm{An}^*_{\tilde{S}_I} L(i_{I*}j_I^*F) \otimes \mathrm{An}^*_{\tilde{S}_I} L(i_{I*}j_I^*G), E_{et}(\Omega^\bullet_{/\tilde{S}_I}, F_b)), v_{IJ}(F \otimes G)) \\
&\xrightarrow{=} (e(\tilde{S}_I)_*\mathcal{H}om(\mathrm{An}^*_{\tilde{S}_I} L(i_{I*}j_I^*(F \otimes G), E_{usu}(\Omega^\bullet_{/\tilde{S}_I}, F_b))), u_{IJ}(F \otimes G)) =: \mathcal{F}^{GM}_{S,an}(M \otimes N)
\end{aligned}$$

We have in the analytical case the following :

Proposition 129. *Let $S \in \mathrm{Var}(\mathbb{C})$. Then, for $M, N \in \mathrm{DA}_c(S)$*

$$T(\otimes, \mathcal{F}^{GM}_{S,an})(M,N) : \mathcal{F}^{GM}_{S,an}(M \otimes N) \xrightarrow{\sim} \mathcal{F}^{GM}_{S,an}(M) \otimes^L_{O_S} \mathcal{F}^{GM}_{S,an}(N)$$

is an isomorphism.

Proof. Asumme first that S is smooth. Let $h_1 : U_1 \to S$ and $h_2 : U_2 \to S$ smooth morphisms with $U_1, U_2 \in \mathrm{Var}(\mathbb{C})$ and consider $h_{12} : U_1 \times_S U_2 \to S$. We then have by lemma 16 the following commutative diagram

$$\begin{tikzcd}
e(S)_*\mathcal{H}om(\mathbb{Z}(U_1/S)\otimes\mathbb{Z}(U_2/S),E(\Omega^{\bullet}_{/S},F_b)) \arrow[r, "T(\otimes{,}\mathcal{F}_S^{GM}(M(U_1/S){,}M(U_2/S)))"] \arrow[d, "="] & e(S)_*\mathcal{H}om(\mathbb{Z}(U_1/S),E(\Omega^{\bullet}_{/S},F_b))\otimes_{O_S}e(S)_*\mathcal{H}om(\mathbb{Z}(U_2/S),E(\Omega^{\bullet}_{/S},F_b)) \arrow[d, "="] \\
h_{12*}E(\Omega_{U_1\times_S U_2/S},F) \arrow[r, "Ew_{(U_1,U_2)/S}"] \arrow[d, "\iota\otimes\iota"] & h_{1*}E(\omega_{U_1/S},F_b)\otimes_{O_S}h_{2*}E(\Omega_{U_2/S},F_b) \arrow[d, "\iota"] \\
h_{12*}E(h_{12}^*O_S) \arrow[r, "Ew_{(U_1,U_2)/S}"] \arrow[d, "T(h_{12},\otimes)(O_S{,}\mathbb{Z}_{U_{12}})"] & h_{1*}E(h_1^*O_S)\otimes_{O_S}h_{2*}E(h_2^*O_S) \arrow[d, "T(h_1,\otimes)(O_S{,}\mathbb{Z}_{U_1})\otimes T(h_2,\otimes)(O_S{,}\mathbb{Z}_{U_1})"] \\
h_{12*}E(\mathbb{Z}_{U_{12}})\otimes O_S \arrow[r, "Ew_{(U_1,U_2)/S}"] & (h_{1*}E(\mathbb{Z}_{U_1})\otimes_{O_S}h_{2*}E(\mathbb{Z}_{U_2}))
\end{tikzcd}$$

Since $U_1,U_2\in\mathrm{AnSp}(\mathbb{C})$ are locally contractible topological spaces, the lower row is an equivalence usu local by Kunneth formula for topological spaces (see section 2). This proves the proposition in the case S is smooth. Let $S\in\mathrm{Var}(\mathbb{C})$ and $S=\cup_{i=1}^{l}S_i$ an open cover such that there exist closed embeddings $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$. By definition, for $F,G\in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $M=D(\mathbb{A}^1,et)(F)$ and $N=D(\mathbb{A}^1,et)(G)$,

$$\begin{aligned}
&T(\otimes,\mathcal{F}_{S,an}^{GM}(M,N)):\\
&e(\tilde{S}_I)_*\mathcal{H}om(\mathrm{An}_{\tilde{S}_I}^*L(i_{I*}j_I^*(F\otimes G),E_{usu}(\Omega^{\bullet}_{/\tilde{S}_I},F_b))),u_{IJ}(F\otimes G))\\
&\xrightarrow{(T(\otimes,\Omega_{/\tilde{S}_I})(\mathrm{An}_{\tilde{S}_I}^*L(i_{I*}j_I^*F),\mathrm{An}_{\tilde{S}_I}^*L(i_{I*}j_I^*G)))}\\
&(e(\tilde{S}_I)_*\mathcal{H}om(\mathrm{An}_{\tilde{S}_I}^*L(i_{I*}j_I^*F),E_{usu}(\Omega^{\bullet}_{/\tilde{S}_I},F_b)),u_{IJ}(F))\otimes_{O_S}\\
&(e(\tilde{S}_I)_*\mathcal{H}om(\mathrm{An}_{\tilde{S}_I}^*L(i_{I*}j_I^*G),E_{usu}(\Omega^{\bullet}_{/\tilde{S}_I},F_b)),u_{IJ}(G))
\end{aligned}$$

Since $L(i_{I*}j_I^*F),L(i_{I*}j_I^*G)\in\mathrm{DA}_c(\tilde{S}_I)$, by the smooth case applied to $\tilde{S}_I$ for each I, $T(\otimes,\mathcal{F}_{S,an}^{FDR}(M,N))$ is an equivalence usu local. $\square$

6.2.2 The analytic filtered De Rham realization functor

Recall from section 2 that, for $S\in\mathrm{Var}(\mathbb{C})$ we have the following commutative diagrams of sites

$$\begin{tikzcd}
\mathrm{AnSp}(\mathbb{C})^2/S \arrow[rr, "\mu_S"] \arrow[dr, "\rho_S"] \arrow[dd, "\mathrm{An}_S"] & & \mathrm{AnSp}(\mathbb{C})^{2,pr}/S \arrow[dr, "\rho_S"] \arrow[dd, "\mathrm{An}_S"] & \\
& \mathrm{AnSp}(\mathbb{C})^{2,sm}/S \arrow[rr, "\mu_S"] \arrow[dd, "\mathrm{An}_S"] & & \mathrm{AnSp}(\mathbb{C})^{2,smpr}/S \arrow[dd, "\mathrm{An}_S"] \\
\mathrm{Var}(\mathbb{C})^2/S \arrow[rr, "\mu_S"] \arrow[dr, "\rho_S"] & & \mathrm{Var}(\mathbb{C})^{2,smpr}/S \arrow[dr, "\rho_S"] & \\
& \mathrm{Var}(\mathbb{C})^2/S \arrow[rr, "\mu_S"] & & \mathrm{Var}(\mathbb{C})^{2,smpr}/S
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{AnSp}(\mathbb{C})^{2,pr}/S \arrow[rr, "\mathrm{Gr}^{12}_S"] \arrow[dr, "\rho_S"] \arrow[dd, "\mathrm{An}_S"'] & & \mathrm{AnSp}(\mathbb{C})/S \arrow[dr, "\rho_S"] \arrow[dd, "\mathrm{An}_S"] & \\
& \mathrm{AnSp}(\mathbb{C})^{2,smpr}/S \arrow[rr, "\mathrm{Gr}^{12}_S"] \arrow[dd, "\mathrm{An}_S"] & & \mathrm{AnSp}(\mathbb{C})^{sm}/S \arrow[dd, "\mathrm{An}_S"] \\
\mathrm{Var}(\mathbb{C})^{2,pr}/S \arrow[rr, "\mathrm{Gr}^{12}_S"] \arrow[dr, "\rho_S"] & & \mathrm{Var}(\mathbb{C})/S \arrow[dr, "\rho_S"] & \\
& \mathrm{Var}(\mathbb{C})^{2,sm}/S \arrow[rr, "\mathrm{Gr}^{12}_S"] & & \mathrm{Var}(\mathbb{C})^{sm}/S
\end{tikzcd} , \tag{58}$$

and that for $f : T \to S$ a morphism with $T, S \in \mathrm{Var}(\mathbb{C})$ we have the following commutative diagrams of site,

$$\begin{tikzcd}
\mathrm{AnSp}(\mathbb{C})^2/T^{an} \arrow[rr, "\mathrm{An}_T"] \arrow[dr, "\rho_T"] \arrow[dd, "P(f)"] & & \mathrm{Var}(\mathbb{C})^2/T \arrow[dr, "\rho_T"] \arrow[dd, "P(f)"] & \\
& \mathrm{AnSp}(\mathbb{C})^{2,sm}/T^{an} \arrow[rr, "\mathrm{An}_T"] \arrow[dd, "P(f)"] & & \mathrm{Var}(\mathbb{C})^{2,sm}/T \arrow[dd, "P(f)"] \\
\mathrm{AnSp}(\mathbb{C})^2/S^{an} \arrow[rr, "\mathrm{An}_S"] \arrow[dr, "\rho_S"] & & \mathrm{Var}(\mathbb{C})^2/S \arrow[dr, "\rho_S"] & \\
& \mathrm{AnSp}(\mathbb{C})^{2,sm}/S^{an} \arrow[rr, "\mathrm{An}_S"] & & \mathrm{Var}(\mathbb{C})^{2,sm}/S
\end{tikzcd} .$$

Definition 133. *(i) For $S \in \mathrm{AnSp}(\mathbb{C})$, we consider the filtered complexes of presheaves*

$$(\Omega^{\bullet,\Gamma}_{/S}, F_b) \in C_{O_S fil}(\mathrm{AnSp}(\mathbb{C})^2/S)$$

given by

- *for $((X,Z),h) = (X,Z)/S \in \mathrm{AnSp}(\mathbb{C})^2/S$,*

$$\begin{aligned}(\Omega^{\bullet,\Gamma}_{/S}((X,Z)/S), F_b) : &= \Gamma^{\vee,h}_Z L_{h^*O}(\Omega^\bullet_{X/S}, F_b)(X) \\ : &= \mathbb{D}_{h^*O_S} L_{h^*O} \Gamma_Z E_{usu}(\mathbb{D}_{h^*O_S} L_{h^*O}(\Omega^\bullet_{X/S}, F_b))(X)\end{aligned}$$

- *for $g : (X_1, Z_1)/S = ((X_1, Z_1), h_1) \to (X,Z)/S = ((X,Z),h)$ a morphism in $\mathrm{AnSp}(\mathbb{C})^2/S$,*

$$\begin{aligned}\Omega^{\bullet,\Gamma}_{/S}(g) : \mathbb{D}_{h^*O_S} L_{h^*O} \Gamma_Z E_{usu}(\mathbb{D}_{h^*O_S} L_{h^*O}(\Omega^\bullet_{X/S}, F_b))(X) \\ \mathbb{D}_{h_1^*O_S} L_{h_1^*O} \Gamma_{Z_1} E_{usu}(\mathbb{D}_{h_1^*O_S} L_{h_1^*O}(\Omega^\bullet_{X_1/S}, F_b))(X_1)\end{aligned}$$

is given as in definition 110(i). For $S \in \mathrm{AnSm}(\mathbb{C})$, we consider the complexes of presheaves

$$(\Omega^{\bullet,\Gamma}_{/S}, F_b) := \rho_{S*}(\Omega^{\bullet,\Gamma}_{/S}, F_b) \in C_{O_S fil, D_S^\infty}(\mathrm{AnSp}(\mathbb{C})^{2,sm}/S)$$

(ii) For $S \in \mathrm{AnSm}(\mathbb{C})$, we have the canonical map $C_{O_S fil, D_S^\infty}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$

$$\mathrm{Gr}^O(\Omega_{/S}) : \mathrm{Gr}^{12}_{S*}\,\mu_{S*}(\Omega^{\bullet,\Gamma}_{/S}, F_b) \to (\Omega^\bullet_{/S}, F_b)$$

given as in definition 110(ii).

Definition 134. *(i) For $S \in \mathrm{SmVar}(\mathbb{C})$, we consider, using definition 112(i), the filtered complexes of presheaves*

$$(\Omega^{\bullet,\Gamma,pr}_{/S^{an}}, F_{DR}) \in C_{D_S^\infty fil}(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$$

given by,

- *for $(Y \times S, Z)/S = ((Y \times S, Z), p) \in \mathrm{Var}(\mathbb{C})^{2,smpr}/S$,*

$$(\Omega^{\bullet,\Gamma,pr}_{/S^{an}}((Y \times S, Z)/S), F_{DR}) := ((\Omega^{\bullet}_{(Y\times S)^{an}/S^{an}}, F_b) \otimes_{O_{(Y\times S)^{an}}} (\Gamma^{\vee,Hdg}_Z(O_{Y\times S}, F_b))^{an})((Y \times S)^{an})$$

with the structure of p^*D_S module given by proposition 60.

- *for $g : (Y_1 \times S, Z_1)/S = ((Y_1 \times S, Z_1), p_1) \to (Y \times S, Z)/S = ((Y \times S, Z), p)$ a morphism in $\mathrm{Var}(\mathbb{C})^{2,smpr}/S$,*

$$\Omega^{\bullet,\Gamma,pr}_{/S^{an}}(g) := (\Omega^{\bullet,\Gamma,pr}_{/S}(g))^{an} : ((\Omega^{\bullet}_{(Y\times S)^{an}/S^{an}}, F_b) \otimes_{O_{(Y\times S)^{an}}} (\Gamma^{\vee,Hdg}_Z(O_{Y\times S}, F_b))^{an})((Y \times S)^{an}) \to ((\Omega^{\bullet}_{(Y_1\times S)^{an}/S^{an}}, F_b) \otimes_{O_{(Y_1\times S)^{an}}} (\Gamma^{\vee,Hdg}_{Z_1}(O_{Y_1\times S}, F_b))^{an})((Y_1 \times S)^{an}).$$

For $S \in \mathrm{SmVar}(\mathbb{C})$, we get the filtered complexes of presheaves

$$(\Omega^{\bullet,\Gamma,pr,an}_{/S^{an}}, F_{DR}) := \mathrm{An}_S^{*mod}(\Omega^{\bullet,\Gamma,pr}_{/S^{an}}, F_{DR}) := \mathrm{An}_S^*(\Omega^{\bullet,\Gamma,pr}_{/S^{an}}, F_{DR}) \otimes_{O_S} O_{S^{an}} \in C_{D_S^\infty fil}(\mathrm{AnSp}(\mathbb{C})^{2,smpr}/S).$$

(ii) For $S \in \mathrm{SmVar}(\mathbb{C})$, we have the canonical map $C_{O_S fil, D_S^\infty}(\mathrm{Var}(\mathbb{C})^{sm}/S)$

$$\mathrm{Gr}(\Omega_{/S^{an}}) : \mathrm{Gr}^{12}_{S*}(\Omega^{\bullet,\Gamma,pr}_{/S^{an}}, F_b) \to \mathrm{An}_{S*}(\Omega^{\bullet}_{/S}, F_b)$$

given by

$$\mathrm{Gr}(\Omega_{/S^{an}})(U/S) := (\mathrm{Gr}(\Omega_{/S})(U/S))^{an} \otimes m : J_S((\Omega^{\bullet}_{(U\times S)^{an}/S^{an}}, F_b) \otimes_{O_{(U\times S)^{an}}} (\Gamma^{\vee,Hdg}_U(O_{U\times S}, F_b))^{an})((U \times S)^{an}) \to (\Omega^{\bullet}_{U^{an}/S^{an}}, F_b),$$

where $\mathrm{Gr}(\Omega_{/S^{an}})(U/S)(\omega \otimes m \otimes P) := P(\mathrm{Gr}(\Omega_{/S})(U/S)(\omega \otimes m))$ with $P \in \Gamma(S, D_S^\infty)$, see definition 112(ii), which gives by adjonction

$$\mathrm{Gr}(\Omega_{/S^{an}}) := I(\mathrm{An}_S^{*mod}, \mathrm{An}_S)(\mathrm{Gr}(\Omega_{/S^{an}})) : J_S(\mathrm{Gr}^{12}_{S*}(\Omega^{\bullet,\Gamma,pr,an}_{/S^{an}}, F_b)) \to (\Omega^{\bullet}_{/S}, F_b)$$

in $C_{O_S fil, D_S^\infty}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$.

Definition 135. *For $S \in \mathrm{SmVar}(\mathbb{C})$, we have the canonical map in $C_{O_S fil, D_S^\infty}(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$*

$$T(\Omega^{\Gamma}_{/S^{an}}) : \mathrm{An}_{S*}\,\mu_{S*}(\Omega^{\bullet,\Gamma}_{/S^{an}}, F_b) \to (\Omega^{\bullet,\Gamma,pr}_{/S^{an}}, F_{DR})$$

given by, for $(Y \times S, X)/S = ((Y \times S, Z), p) \in \mathrm{Var}(\mathbb{C})^{2,smpr}/S$

$$T(\Omega^{\Gamma}_{/S^{an}})((Y \times S, Z)/S) := (T(\Omega^{\Gamma}_{/S})((Y \times S, Z)/S))^{an} : (\Omega^{\bullet,\Gamma}_{/S}, F_b)((((Y \times S)^{an}, Z^{an})/S) := \mathbb{D}_{p^*O_S} L_{p^*O} \Gamma_Z E_{usu}(\mathbb{D}_{p^*O_S} L_{p^*O}(\Omega^{\bullet}_{(Y\times S)^{an}/S^{an}}, F_b))((Y \times S)^{an}) \to ((\Omega^{\bullet}_{(Y\times S)^{an}/S^{an}}, F_b) \otimes_{O_{(Y\times S)^{an}}} (\Gamma^{\vee,Hdg}_Z(O_{Y\times S}, F_b))^{an})((Y \times S)^{an}) =: (\Omega^{\bullet,\Gamma,pr}_{/S^{an}}, F_{DR})((Y \times S, Z)/S),$$

see definition 113. By definition we have $\mathrm{Gr}^O(\Omega_{/S^{an}}) = \mathrm{Gr}(\Omega_{/S^{an}}) \circ T(\Omega^{\Gamma}_{/S^{an}})$.

We have the following canonical transformation map given by the pullback of (relative) differential forms:

Let $g : T \to S$ a morphism with $T, S \in \mathrm{AnSm}(\mathbb{C})$.

- We have the canonical morphism in $C_{g^*O_Sfil,g^*D_S^\infty}(\mathrm{AnSp}(\mathbb{C})^{2,sm}/T)$

$$\Omega^\Gamma_{/(T/S)}:g^*(\Omega^{\bullet,\Gamma}_{/S},F_b)\to(\Omega^{\bullet,\Gamma}_{/T},F_b)$$

induced by the pullback of differential forms : for $((V,Z_1)/T)=((V,Z_1),h)\in\mathrm{AnSp}(\mathbb{C})^{2,sm}/T$,

$$\begin{aligned}\Omega^\Gamma_{/(T/S)}((V,Z_1)/T):\\ g^*\Omega^{\bullet,\Gamma}_{/S}((V,Z_1)/T):=\lim_{(h:(U,Z)\to S\mathrm{sm},g_1:(V,Z_1)\to(U_T,Z_T),h,g)}\Omega^{\bullet,\Gamma}_{/S}((U,Z)/S)\\ \xrightarrow{\Omega^{\bullet,\Gamma}_{/S}(g'\circ g_1)}\Omega^{\bullet,\Gamma}_{/S}((V,Z_1)/S)\xrightarrow{\Gamma^{\vee,h}_{Z_1}q(Y_1\times T)}\Omega^{\bullet,\Gamma}_{/T}((V,Z_1)/T),\end{aligned}$$

where $g':U_T:=U\times_S T\to U$ is the base change map and $q:\Omega^\bullet_{Y_1\times T/S}\to\Omega^\bullet_{Y_1\times T/T}$ is the quotient map. It induces the canonical morphisms in $C_{g^*O_Sfil,g^*D_S^\infty}(\mathrm{AnSp}(\mathbb{C})^{2,sm}/T)$:

$$E\Omega^\Gamma_{/(T/S)}:g^*E_{et}(\Omega^{\bullet,\Gamma}_{/S},F_b)\xrightarrow{T(g,E_{et})(\Omega^{\bullet,\Gamma}_{/S},F_b)}E_{et}(g^*(\Omega^{\bullet,\Gamma}_{/S},F_b))\xrightarrow{E_{et}(\Omega^\Gamma_{/(T/S)})}E_{et}(\Omega^{\bullet,\Gamma}_{/T},F_b)$$

- We have the canonical morphism in $C_{g^*D_S^\infty fil}(\mathrm{Var}(\mathbb{C})^{2,smpr}/T)$

$$\Omega^{\Gamma,pr}_{/(T/S)^{an}}:g^*(\Omega^{\bullet,\Gamma,pr}_{/S^{an}},F_{DR})\to(\Omega^{\bullet,\Gamma,pr}_{/T^{an}},F_{DR})$$

induced by the pullback of differential forms : for $((Y_1\times T,Z_1)/T)=((Y_1\times T,Z_1),p)\in\mathrm{Var}(\mathbb{C})^{2,smpr}/T$,

$$\begin{aligned}\Omega^{\Gamma,pr}_{/(T/S)^{an}}((Y_1\times T,Z_1)/T):\\ g^*\Omega^{\bullet,\Gamma,pr}_{/S^{an}}((Y_1\times T,Z_1)/T):=\lim_{(h:(Y\times S,Z)\to S,\,g_1:(Y_1\times T,Z_1)\to(Y\times T,Z_T),h,g)}\Omega^{\bullet,\Gamma,pr}_{/S^{an}}((Y\times T,Z)/S)\\ \xrightarrow{\Omega^{\bullet,\Gamma,pr}_{/S^{an}}(g'\circ g_1)}\Omega^{\bullet,\Gamma,pr}_{/S^{an}}((Y_1\times T,Z_1)/S)\xrightarrow{q(-)((Y_1\times T)^{an})}\Omega^{\bullet,\Gamma,pr}_{/T^{an}}((Y_1\times T,Z_1)/T),\end{aligned}$$

where $g'=(I_Y\times g):Y\times T\to Y\times S$ is the base change map and

$$q(M):\Omega_{(Y_1\times T)^{an}/S^{an}}\otimes_{O_{(Y_1\times T)^{an}}}(M,F)\to\Omega_{(Y_1\times T)^{an}/T^{an}}\otimes_{O_{(Y_1\times T)^{an}}}(M,F)$$

is the quotient map. It induces the canonical morphisms in $C_{g^*D_Sfil}(\mathrm{Var}(\mathbb{C})^{2,smpr}/T)$:

$$E\Omega^{\Gamma,pr}_{/(T/S)}:g^*E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S^{an}},F_{DR})\xrightarrow{T(g,E)(-)}E_{et}(g^*(\Omega^{\bullet,\Gamma,pr}_{/S^{an}},F_{DR}))\xrightarrow{E_{et}(\Omega^{\Gamma,pr}_{/(T/S)^{an}})}E_{et}(\Omega^{\bullet,\Gamma,pr}_{/T^{an}},F_{DR})$$

and

$$E\Omega^{\Gamma,pr}_{/(T/S)^{an}}:g^*E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/S^{an}},F_{DR})\xrightarrow{T(g,E)(-)}E_{zar}(g^*(\Omega^{\bullet,\Gamma,pr}_{/S^{an}},F_{DR}))\xrightarrow{E_{zar}(\Omega^{\Gamma,pr}_{/(T/S)^{an}})}E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/T^{an}},F_{DR}).$$

Definition 136. *Let $g:T\to S$ a morphism with $T,S\in\mathrm{SmVar}(\mathbb{C})$. We have, for $F\in C(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$, the canonical transformation in $C_{\mathcal{D}^\infty fil}(T)$:*

$$\begin{aligned}T(g,\Omega^{\Gamma,pr}_{/.})(F):g^{*mod}L_De(S)_*\operatorname{Gr}^{12}_{S*}\mathcal{H}om^\bullet(\mathrm{An}_S^*F,E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/S^{an}},F_{DR}))\\ \xrightarrow{:=}(g^*L_De(S)_*\mathcal{H}om^\bullet(F,E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/S^{an}},F_{DR})))\otimes_{g^*O_S}O_T\\ \xrightarrow{T(g,\mathrm{Gr}^{12})(-)\circ T(e,g)(-)\circ q}e(T)_*\operatorname{Gr}^{12}_{T*}g^*\mathcal{H}om^\bullet(\mathrm{An}_S^*F,E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S^{an}},F_{DR}))\otimes_{g^*O_S}O_T\\ \xrightarrow{(T(g,hom)(-,-)\otimes I)}e(T)_*\operatorname{Gr}^{12}_{T*}\mathcal{H}om^\bullet(\mathrm{An}_T^*g^*F,g^*E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/S^{an}},F_{DR}))\otimes_{g^*O_S}O_T\\ \xrightarrow{ev(hom,\otimes)(-,-,-)}e(T)_*\operatorname{Gr}^{12}_{T*}\mathcal{H}om^\bullet(\mathrm{An}_T^*g^*F,g^*E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/S^{an}},F_{DR}))\otimes_{g^*e(S)^*O_S}e(T)^*O_T\\ \xrightarrow{\mathcal{H}om^\bullet(\mathrm{An}_T^*g^*F,(E\Omega^{\Gamma,pr}_{/(T/S)}\otimes m))}e(T)_*\operatorname{Gr}^{12}_{T*}\mathcal{H}om^\bullet(\mathrm{An}_T^*g^*F,E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/T^{an}},F_{DR}))\end{aligned}$$

- Let $S\in\mathrm{AnSm}(\mathbb{C})$. We have the map in $C_{O_Sfil,D_S}(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$:
$$w_S:(\Omega^{\bullet,\Gamma}_{/S},F_b)\otimes_{O_S}(\Omega^{\bullet,\Gamma}_{/S},F_b)\to(\Omega^{\bullet,\Gamma}_{/S},F_b):$$
given by for $h:(U,Z)\to S\in\mathrm{Var}(\mathbb{C})^{2,sm}/S$,
$$w_S((U,Z)/S):(\Gamma^{\vee,h}_ZL_{h^*O_S}(\Omega^\bullet_{U/S},F_b)\otimes_{p^*O_S}\Gamma^{\vee,h}_ZL_{h^*O_S}(\Omega^\bullet_{U/S},F_b))(U)$$
$$\xrightarrow{(DR(-)(\gamma^{\vee,h}_Z(-))\circ w_{U/S})^\gamma(U)}\Gamma^{\vee,h}_ZL_{h^*O_S}(\Omega^\bullet_{U/S},F_b)(U)$$
which induces the map in $C_{O_Sfil,D_S}(\mathrm{Var}(\mathbb{C})^{2,sm}/S)$
$$Ew_S:E_{et}(\Omega^{\bullet,\Gamma}_{/S},F_b)\otimes_{O_S}E_{et}(\Omega^{\bullet,\Gamma}_{/S},F_b)\xrightarrow{=}E_{et}((\Omega^{\bullet,\Gamma}_{/S},F_b)\otimes_{O_S}(\Omega^{\bullet,\Gamma}_{/S},F_b))\xrightarrow{E_{et}(w_S)}E_{et}(\Omega^{\bullet,\Gamma}_{/S},F_b).$$
- Let $S\in\mathrm{SmVar}(\mathbb{C})$. We have the map in $C_{D_S^\infty fil}(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$:
$$w_S:(\Omega^{\bullet,\Gamma,pr}_{/S^{an}},F_{DR})\otimes_{O_S}(\Omega^{\bullet,\Gamma,pr}_{/S^{an}},F_{DR})\to(\Omega^{\bullet,\Gamma,pr}_{/S^{an}},F_{DR})$$
given by for $p:(Y\times S,Z)\to S\in\mathrm{Var}(\mathbb{C})^{2,smpr}/S$,
$$w_S((Y\times S,Z)/S):$$
$$(((\Omega^\bullet_{Y\times S/S}\otimes_{O_{Y\times S}}\Gamma^{\vee,Hdg}_Z)(O_{Y\times S},F_b))\otimes_{p^*O_S}(\Omega^\bullet_{Y\times S/S}\otimes_{O_{Y\times S}}\Gamma^{\vee,Hdg}_Z(O_{Y\times S},F_b)))(Y\times S)$$
$$\xrightarrow{(DR(-)(\gamma^{\vee,Hdg}_Z(-))\circ w_{Y\times S/S})^\gamma(Y\times S)}(\Omega^\bullet_{Y\times S/S}\otimes_{O_{Y\times S}}\Gamma^{\vee,Hdg}_Z(O_{Y\times S},F_b))(Y\times S)$$
which induces the map in $C_{D_S^\infty fil}(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$
$$Ew_S:E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S^{an}},F_{DR})\otimes_{O_S}E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S^{an}},F_{DR})\xrightarrow{=}$$
$$E_{et}((\Omega^{\bullet,\Gamma,pr}_{/S^{an}},F_{DR})\otimes_{O_S}(\Omega^{\bullet,\Gamma,pr}_{/S^{an}},F_{DR}))\xrightarrow{E_{et}(w_S)}E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S^{an}},F_{DR})$$
by the functoriality of the Godement resolution (see section 2).

Definition 137. *Let $S\in\mathrm{SmVar}(\mathbb{C})$. We have, for $F,G\in C(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$, the canonical transformation in $C_{\mathcal{D}^\infty fil}(S^{an})$:*
$$T(\otimes,\Omega)(F,G):$$
$$e(S)_*\operatorname{Gr}^{12}_{S*}\mathcal{H}om(\mathrm{An}_S^*F,E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/S},F_{DR}))\otimes_{O_S}e(S)_*\operatorname{Gr}^{12}_{S*}\mathcal{H}om(\mathrm{An}_S^*G,E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/S},F_{DR}))$$
$$\xrightarrow{=}e(S)_*\operatorname{Gr}^{12}_{S*}(\mathcal{H}om(\mathrm{An}_S^*F,E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/S},F_{DR}))\otimes_{O_S}\mathcal{H}om(\mathrm{An}_S^*G,E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/S},F_{DR})))$$
$$\xrightarrow{T(\mathcal{H}om,\otimes)(-)}e(S)_*\operatorname{Gr}^{12}_{S*}\mathcal{H}om(\mathrm{An}_S^*F\otimes\mathrm{An}_S^*G,E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})\otimes_{O_S}E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})))$$
$$\xrightarrow{=}e(S)_*\operatorname{Gr}^{12}_{S*}\mathcal{H}om(\mathrm{An}_S^*(F\otimes G),E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})\otimes_{O_S}E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))$$
$$\xrightarrow{\mathcal{H}om(F\otimes G,\mathrm{An}_S^{*mod}Ew_S)}e(S)_*\operatorname{Gr}^{12}_{S*}\mathcal{H}om(F\otimes G,E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/S},F_{DR})).$$

We have the following key proposition :

Proposition 130. *(i) Let $S\in\mathrm{AnSp}(\mathbb{C})$. The complex of presheaves $\Omega^{\bullet,\Gamma}_{/S}\in C_{O_S}(\mathrm{AnSp}(\mathbb{C})^{2,sm}/S)$ is $\mathbb{D}^1$ homotopic and admits transferts (i.e. $\mathrm{Tr}(S)_*\mathrm{Tr}(S)^*\Omega^{\bullet,\Gamma}_{/S}=\Omega^{\bullet,\Gamma}_{/S}$).*

(ii) Let $S\in\mathrm{SmVar}(\mathbb{C})$. The complex of presheaves $(\Omega^{\bullet,\Gamma,pr,an}_{/S^{an}},F_{DR})\in C_{D_S^\infty fil}(\mathrm{AnSp}(\mathbb{C})^{2,smpr}/S)$ 2-filtered $\mathbb{D}^1$ homotopic and admits transferts.

(iii) *Let $m : Q_1 \to Q_2$ be an equivalence $(\mathbb{D}^1, et)$ local with $Q_1, Q_2 \in C(\mathrm{AnSp}(\mathbb{C})^{smpr}/S)$ complexes of representable presheaves. Then,*

$$e(S)_* \operatorname{Gr}^{12}_{S*} \mathcal{H}om^\bullet(m, E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR})) : e(S)_* \operatorname{Gr}^{12}_{S*} \mathcal{H}om^\bullet(Q_2, E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/S^{an}}, F_{DR})) \to e(S)_* \operatorname{Gr}^{12}_{S*} \mathcal{H}om^\bullet(Q_1, E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/S^{an}}, F_{DR}))$$

is an 2-filtered quasi-isomorphism. It is thus an isomorphism in $D_{\mathcal{D}fil,\infty}(S)$.

Proof. Similar to proposition 111. □

We now define the filtered analytic De Rahm realization functor.

Definition 138. (i) *Let $S \in \mathrm{SmVar}(\mathbb{C})$. We have, using definition 134 and definition 34, the functor*

$$\mathcal{F}^{FDR}_{S,an} : C(\mathrm{Var}(\mathbb{C})^{sm}/S) \to C_{\mathcal{D}^\infty fil}(S^{an}),\ F \mapsto$$
$$\mathcal{F}^{FDR}_{S,an}(F) := e'(S)_* \mathcal{H}om^\bullet(\mathrm{An}^*_S L\rho_{S*}\mu_{S*}R^{CH}(\rho^*_S L(F)), E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/S}, F_{DR}))[-d_S]$$
$$= e'(S)_* \mathcal{H}om^\bullet(L\rho_{S*}\mu_{S*}R^{CH}(\rho^*_S L(F)), \mathrm{An}_{S*} E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/S}, F_{DR}))[-d_S]$$

denoting for short $e'(S) = e(S) \circ \operatorname{Gr}^{12}_S$.

(ii) *Let $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. For $I \subset [1, \cdots l]$, denote by $S_I := \cap_{i\in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \Pi_{i\in I}\tilde{S}_i$. Consider, for $I \subset J$, the following commutative diagram*

$$D_{IJ} = \begin{tikzcd} S_I \arrow[r, "i_I"] & \tilde{S}_I \\ S_J \arrow[u, "j_{IJ}"] \arrow[r, "i_J"] & \tilde{S}_J \arrow[u, "p_{IJ}"'] \end{tikzcd}$$

and $j_{IJ} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. We have, using definition 134 and definition 34, the functor

$$\mathcal{F}^{FDR}_{S,an} : C(\mathrm{Var}(\mathbb{C})^{sm}/S) \to C_{\mathcal{D}^\infty fil}(S^{an}/(\tilde{S}^{an}_I)),\ F \mapsto$$
$$\mathcal{F}^{FDR}_{S,an}(F) := (e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\mathrm{An}^*_{\tilde{S}_I} L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho^*_{\tilde{S}_I} L(i_{I*}j^*_I F)), E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u^q_{IJ}(F))$$
$$= (e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho^*_{\tilde{S}_I} L(i_{I*}j^*_I F)), \mathrm{An}_{\tilde{S}_I*} E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u^q_{IJ}(F))$$

where we have denoted for short $e'(\tilde{S}_I) = e(\tilde{S}_I) \circ \operatorname{Gr}^{12}_{\tilde{S}_I}$, and

$$u^q_{IJ}(F)[d_{\tilde{S}_J}] : e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\mathrm{An}^*_{\tilde{S}_I} L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho^*_{\tilde{S}_I} L(i_{I*}j^*_I F)), E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_I}, F_{DR}))$$
$$\xrightarrow{\mathrm{ad}(p^{*mod}_{IJ}, p_{IJ})(-)} p_{IJ*}p^{*mod}_{IJ} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\mathrm{An}^*_{\tilde{S}_I} L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho^*_{\tilde{S}_I} L(i_{I*}j^*_I F)), E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_I}, F_{DR}))$$
$$\xrightarrow{p_{IJ*}T(p_{IJ}, \Omega^{\gamma,pr}_{\cdot})(-)} p_{IJ*} e'(\tilde{S}_J)_* \mathcal{H}om^\bullet(\mathrm{An}^*_{\tilde{S}_I} L\rho_{\tilde{S}_J*}\mu_{\tilde{S}_J*}p^*_{IJ}R^{CH}(\rho^*_{\tilde{S}_I} L(i_{I*}j^*_I F)), E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_J}, F_{DR}))$$
$$\xrightarrow{\mathcal{H}om(\operatorname{Gr}^{12*}_{\tilde{S}_J} T(p_{IJ}, R^{CH})(Li_{I*}j^*_I F)^{-1}, E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_J}, F_{DR}))} p_{IJ*} e'(\tilde{S}_J)_* \mathcal{H}om^\bullet(\mathrm{An}^*_{\tilde{S}_I} L\rho_{\tilde{S}_J*}\mu_{\tilde{S}_J*}R^{CH}(\rho^*_{\tilde{S}_J} p^*_{IJ} L(i_{I*}j^*_I F)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_J}, F_{DR}))$$
$$\xrightarrow{\mathcal{H}om(\operatorname{Gr}^{12*}_{\tilde{S}_J} R^{CH}_{\tilde{S}_J}(T^q(D_{IJ})(j^*_I F)), E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_J}, F_{DR}))} p_{IJ*} e'(\tilde{S}_J)_* \mathcal{H}om^\bullet(\mathrm{An}^*_{\tilde{S}_I} L\rho_{\tilde{S}_J*}\mu_{\tilde{S}_J*}R^{CH}(\rho^*_{\tilde{S}_J} L(i_{J*}j^*_J F)), E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_J}, F_{DR})).$$

For $I \subset J \subset K$, we have obviously $p_{IJ*}u_{JK}(F) \circ u_{IJ}(F) = u_{IK}(F)$. We will prove in corollary 8 below that $u_{IJ}(F)$ are ∞-filtered Zariski local equivalence.

We have the following key proposition :

Proposition 131. *Let $S \in \mathrm{SmVar}(\mathbb{C})$.*

(i) Let $m : Q_1 \to Q_2$ be an etale local equivalence local with $Q_1, Q_2 \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ complexes of projective presheaves. Then,

$$e'(S)_*\mathcal{H}om^\bullet(\mathrm{An}_S^* L\rho_{S*}\mu_{S*}R_S^{CH}(\rho_S^*(m)), E_{et}(\Omega_{/S}^{\bullet,\Gamma,pr,an}, F_{DR}))[-d_S] :$$
$$e'(S)_*\mathcal{H}om^\bullet(\mathrm{An}_S^* L\rho_{S*}\mu_{S*}R^{CH}(\rho_S^*Q_1), E_{et}(\Omega_{/S}^{\bullet,\Gamma,pr,an}, F_{DR}))[-d_S]$$
$$\to e'(S)_*\mathcal{H}om^\bullet(\mathrm{An}_S^* L\rho_{S*}\mu_{S*}R^{CH}(\rho_S^*Q_2), E_{et}(\Omega_{/S}^{\bullet,\Gamma,pr,an}, F_{DR}))[-d_S]$$

is an ∞-filtered quasi-isomorphism. It is thus an isomorphism in $D_{\mathcal{D}fil,\infty}(S)$.

(ii) Let $m : Q_1 \to Q_2$ be an equivalence $(\mathbb{A}^1, et)$ local with $Q_1, Q_2 \in C(\mathrm{Proj}\,\mathrm{PSh}(\mathrm{Var}(\mathbb{C})^{sm}/S))$ complexes of representable presheaves. Then,

$$e'(S)_*\mathcal{H}om^\bullet(\mathrm{An}_S^* L\rho_{S*}\mu_{S*}R_S^{CH}(\rho_S^*(m)), E_{et}(\Omega_{/S}^{\bullet,\Gamma,pr,an}, F_{DR}))[-d_S] :$$
$$e'(S)_*\mathcal{H}om^\bullet(\mathrm{An}_S^* L\rho_{S*}\mu_{S*}R^{CH}(\rho_S^*Q_1), E_{et}(\Omega_{/S}^{\bullet,\Gamma,pr,an}, F_{DR}))[-d_S]$$
$$\to e'(S)_*\mathcal{H}om^\bullet(\mathrm{An}_S^* L\rho_{S*}\mu_{S*}R^{CH}(\rho_S^*Q_2), E_{et}(\Omega_{/S}^{\bullet,\Gamma,pr,an}, F_{DR}))[-d_S]$$

is an ∞-filtered quasi-isomorphism. It is thus an isomorphism in $D_{\mathcal{D}fil,\infty}(S)$.

Proof. Similar to the proof of proposition 112. □

Definition 139. *(i) Let $S \in \mathrm{SmVar}(\mathbb{C})$. We define using definition 138(i) and proposition 131(ii) the filtered algebraic De Rahm realization functor defined as*

$$\mathcal{F}_{S,an}^{FDR} : \mathrm{DA}_c(S) \to D_{\mathcal{D}^\infty fil,\infty}(S^{an}), M \mapsto \mathcal{F}_S^{FDR}(M) :=$$
$$e'(S)_*\mathcal{H}om^\bullet(\mathrm{An}_S^* L\rho_{S*}\mu_{S*}R^{CH}(\rho_S^*L(F)), E_{et}(\Omega_{/S}^{\bullet,\Gamma,pr,an}, F_{DR}))[-d_S]$$

where $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$.

(i)' For the Corti-Hanamura weight structure W on $\mathrm{DA}_c(S)^-$, we define using definition 138(i) and proposition 131(ii)

$$\mathcal{F}_{S,an}^{FDR} : \mathrm{DA}_c^-(S) \to D^-_{\mathcal{D}^\infty(1,0)fil,\infty}(S^{an}), M \mapsto \mathcal{F}_S^{FDR}((M, W)) :=$$
$$e'(S)_*\mathcal{H}om^\bullet(\mathrm{An}_S^* L\rho_{S*}\mu_{S*}R^{CH}(\rho_S^*L(F, W)), E_{et}(\Omega_{/S}^{\bullet,\Gamma,pr,an}, F_{DR}))[-d_S]$$

where $(F, W) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)((F, W))$ using corollary 1. Note that the filtration induced by W is a filtration by sub D_S module, which is a stronger property then Griffitz transversality. Of course, the filtration induced by F satisfy only Griffitz transversality in general.

(ii) Let $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. For $I \subset [1, \cdots l]$, denote by $S_I = \cap_{i\in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \Pi_{i\in I}\tilde{S}_i$. We define, using definition 138(ii), proposition 131(ii) and corollary 8, the filtered algebraic De Rahm realization functor defined as

$$\mathcal{F}_{S,an}^{FDR} : \mathrm{DA}_c(S) \to D_{\mathcal{D}^\infty fil,\infty}(S^{an}/(\tilde{S}_I^{an})), M \mapsto \mathcal{F}_S^{FDR}(M) :=$$
$$(e'(\tilde{S}_I)_*\mathcal{H}om^\bullet(\mathrm{An}_{\tilde{S}_I}^* L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*}j_I^*F)), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet,\Gamma,pr,an}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F))$$

where $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$, see definition 116 .

(ii)' *For the Corti-Hanamura weight structure W on $\mathrm{DA}_c^-(S)$, using definition 116(ii), proposition 112(ii) and corollary 5,*

$$\mathcal{F}_{S,an}^{FDR} : \mathrm{DA}_c^-(S) \to D^-_{\mathcal{D}^\infty(1,0)fil,\infty}(S^{an}/(\tilde{S}_I^{an})), M \mapsto \mathcal{F}_S^{FDR}((M,W)) :=$$
$$(e'(\tilde{S}_I)_*\mathcal{H}om^\bullet(\mathrm{An}_{\tilde{S}_I}^* \, L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*}j_I^*(F,W))), E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F,W))$$

where $(F,W) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is such that $(M,W) = D(\mathbb{A}^1,et)(F,W)$ using corollary 1. Note that the filtration induced by W is a filtration by sub $D_{\tilde{S}_I}$-modules, which is a stronger property then Griffitz transversality. Of course, the filtration induced by F satisfy only Griffitz transversality in general.

Proposition 132. *For $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$, the functor $\mathcal{F}_{S,an}^{FDR}$ is well defined.*

Proof. Similar to the proof of proposition 113. □

Proposition 133. *Let $f : X \to S$ a morphism with $S, X \in \mathrm{Var}(\mathbb{C})$. Assume there exist a factorization*

$$f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$$

of f with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $\bar{Y} \in \mathrm{PSmVar}(\mathbb{C})$ a compactification of Y with $\bar{Y}\backslash Y = D$ a normal crossing divisor, denote $k : D \hookrightarrow \bar{Y}$ the closed embedding and $n : Y \hookrightarrow \bar{Y}$ the open embedding. Denote $\bar{X} \subset \bar{Y} \times S$ the closure of $X \subset \bar{Y} \times S$. We have then the following commutative diagram in $\mathrm{Var}(\mathbb{C})$

$$\begin{tikzcd}
X \arrow[r, "l"] \arrow[d] & Y \times S \arrow[d, "(n\times I)"] \arrow[dr, "p_S"] & \\
\bar{X} \arrow[r, "l"] & \bar{Y} \times S \arrow[r, "\bar{p}_S"] & S \\
Z := \bar{X}\backslash X \arrow[u] \arrow[r] \arrow[ur, "l_Z"] & D \times S \arrow[u, "(k\times I)"'] \arrow[ur] &
\end{tikzcd} .$$

Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. Then $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. Denote, for $I \subset [1, \cdots l]$, $S_I = \cap_{i\in I} S_i$ and $X_I = \cap_{i\in I} X_i$. Denote $\bar{X}_I := \bar{X} \cap (\bar{Y} \times S_I) \subset \bar{Y} \times \tilde{S}_I$ the closure of $X_I \subset \bar{Y} \times \tilde{S}_I$, and $Z_I := Z \cap (\bar{Y} \times S_I) = \bar{X}_I\backslash X_I \subset \bar{Y} \times \tilde{S}_I$. We have then for $I \subset [1, \cdots l]$, the following commutative diagram in $\mathrm{Var}(\mathbb{C})$

$$\begin{tikzcd}
X_I \arrow[r, "l_I"] \arrow[d] & Y \times \tilde{S}_I \arrow[d, "(n\times I)"] \arrow[dr, "p_{\tilde{S}_I}"] & \\
\bar{X}_I \arrow[r, "l_I"] & \bar{Y} \times \tilde{S}_I \arrow[r, "\bar{p}_{\tilde{S}_I}"] & \tilde{S}_I \\
Z_I = \bar{X}_I\backslash X_I \arrow[u] \arrow[r] \arrow[ur, "l_{Z_I}"] & D \times \tilde{S}_I \arrow[u, "(k\times I)"'] \arrow[ur] &
\end{tikzcd} .$$

Let $F(X/S) := p_{S,\sharp}\Gamma^\vee_X\mathbb{Z}(X\times S/X\times S)$. *We have then the following isomorphism in* $D_{\mathcal{D}fil,\infty}(S/(\tilde{S}_I))$

$$I(X/S):\mathcal{F}^{FDR}_{S,an}(M(X/S))\xrightarrow{:=}$$
$$(e'_*\mathcal{H}om(\mathrm{An}^*_{\tilde{S}_I}L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho^*_{\tilde{S}_I}L(i_{I*}j^*_IF(X/S))),E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_I},F_{DR}))[-d_{\tilde{S}_I}],u^q_{IJ}(F(X/S)))$$
$$\xrightarrow{(\mathcal{H}om(\mathrm{An}^*_{\tilde{S}_I}L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}_{\tilde{S}_I}(N_I(X/S)),E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_I},F_{DR})))}$$
$$(e'_*\mathcal{H}om(\mathrm{An}^*_{\tilde{S}_I}L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R_{((\bar{Y}\times\tilde{S}_I)^*,E^*)/\tilde{S}_J}(\rho^*_{\tilde{S}_I}Q(X_I/\tilde{S}_I)),E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_I},F_{DR}))[-d_{\tilde{S}_I}],v^q_{IJ}(F(X/S)))$$
$$\xrightarrow{(\mathcal{H}om(\mathrm{An}^*_{\tilde{S}_I}I_\delta((\bar{X}_I,Z_I)/\tilde{S}_I),k)[-d_{\tilde{S}_I}])^{-1}}$$
$$(\bar{p}_{\tilde{S}_I*}E_{usu}((\Omega^\bullet_{\bar{Y}\times\tilde{S}_I/\tilde{S}_I},F_b)\otimes_{O_{Y\times\tilde{S}_I}}(n\times I)^{Hdg}_!\Gamma^{\vee,Hdg}_{X_I}(O_{(Y\times\tilde{S}_I)^{an}},F_b))(d_Y+d_{\tilde{S}_I})[2d_Y+d_{\tilde{S}_I}],w_{IJ}(X/S))$$
$$\xrightarrow{=:}\iota_SRf^{Hdg}_!(\Gamma^{\vee,Hdg}_{X_I}(O_{(Y\times\tilde{S}_I)^{an}},F_b)(d_Y)[2d_Y],x_{IJ}(X/S))\xrightarrow{=:}\iota_SRf^{Hdg}_!f^{*mod}_{Hdg}\mathbb{Z}^{Hdg}_{S^{an}}$$

Proof. Similar to the proof of proposition 114. □

Corollary 8. *Let* $S\in\mathrm{Var}(\mathbb{C})$ *and* $S=\cup_{i=1}^lS_i$ *an open cover such that there exist closed embeddings* $i_i:S_i\hookrightarrow\tilde{S}_i$ *with* $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$. *For* $F\in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ *such that* $D(\mathbb{A}^1,et)(F)\in\mathrm{DA}_c(S)$, $u^q_{IJ}(F)$ *are* ∞*-filtered usu local equivalence.*

Proof. Similar to the proof of corollary 5. □

Corollary 9. *Let* $S\in\mathrm{Var}(\mathbb{C})$ *and* $S=\cup_{i=1}^lS_i$ *an open cover such that there exist closed embeddings* $i_i:S_i\hookrightarrow\tilde{S}_i$ *with* $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$. *Then, for* $F\in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ *such that* $M=D(\mathbb{A}^1,et)(F)\in\mathrm{DA}_c(S)$,

$$H^i\mathcal{F}^{FDR}_{S,an}(M,W):=((e'(\tilde{S}_I)_*\mathcal{H}om^\bullet(\mathrm{An}^*_{\tilde{S}_I}L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho^*_{\tilde{S}_I}L(i_{I*}j^*_I(F,W)))),$$
$$E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I},F_{DR}))[-d_{\tilde{S}_I}],u^q_{IJ}(F,W))\in\pi_S(MHM(S^{an}))$$

for all $i\in\mathbb{Z}$, *and for all* $p\in\mathbb{Z}$, $\mathcal{F}^{FDR}_{S,an}(M,W)\in D_{\mathcal{D}(1,0)fil,\infty}(S/(\tilde{S}_I))$ *is the class of a complex* $\mathcal{F}^{FDR}_{S,an}(M,W)^t\in C_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))$ *such that the differentials of* $\mathrm{Gr}^p_W\mathcal{F}^{FDR}_{S,an}(M,W)^t$ *are strict for the filtration* F.

Proof. Similar to the proof of corollary 6. □

Proposition 134. *For* $S\in\mathrm{Var}(\mathbb{C})$ *not smooth, the functor (see corollary 6)*

$$\iota^{-1}_S\mathcal{F}^{FDR}_{S,an}:\mathrm{DA}^-_c(S)^{op}\to\pi_S(D(MHM(S^{an}))$$

does not depend on the choice of the open cover $S=\cup_iS_i$ *and the closed embeddings* $i_i:S_i\hookrightarrow\tilde{S}_i$ *with* $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$.

Proof. Similar to the proof of proposition 115. □

We have the canonical transformation map between the filtered analytic De Rham realization functor and the analytic Gauss-Manin realization functor :

Definition 140. *Let* $S\in\mathrm{Var}(\mathbb{C})$ *and* $S=\cup_{i=1}^lS_i$ *an open cover such that there exist closed embeddings* $i_i:S_i\hookrightarrow\tilde{S}_i$ *with* $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$. *Let* $M\in\mathrm{DA}_c(S)$ *and* $F\in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ *such that*

$M = D(\mathbb{A}^1, et)(F)$. *We have, using definition 134(ii), the canonical map in* $D_{O_S fil,\mathcal{D}^\infty,\infty}(S^{an}/(\tilde{S}_I^{an}))$

$$\begin{aligned}
&T(\mathcal{F}_{S,an}^{GM}, \mathcal{F}_{S,an}^{FDR})(M) : \\
&\mathcal{F}_{S,an}^{GM}(L\mathbb{D}_S M) := (e(\tilde{S}_I)_*\mathcal{H}om^\bullet(\mathrm{An}_{\tilde{S}_I}^* L(i_{I*}j_I^*\mathbb{D}_S LF), E_{et}(\Omega^\bullet_{/\tilde{S}_I}, F_b)), u_{IJ}^q(F)) \\
&\xrightarrow{\sim} (e(\tilde{S}_I)_*\mathcal{H}om^\bullet(\mathrm{An}_{\tilde{S}_I}^* L\mathbb{D}_{\tilde{S}_I}^0(L(i_{I*}j_I^*F)), E_{et}(\Omega^\bullet_{/\tilde{S}_I}, F_b)), u_{IJ}^{q,d}(F)) \\
&\xrightarrow{\mathcal{H}om(-,\mathrm{Gr}(\Omega_{\tilde{S}_I^{an}}))^{-1}} J_S(e(\tilde{S}_I)_*\mathcal{H}om^\bullet(\mathrm{An}_{\tilde{S}_I}^* L\mathbb{D}_{\tilde{S}_I}^0(L(i_{I*}j_I^*F)), \mathrm{Gr}_{\tilde{S}_I*}^{12} E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_I}, F_{DR})), u_{IJ}^{q,d}(F)) \\
&\xrightarrow{(\mathcal{H}om^\bullet(\mathrm{An}_{\tilde{S}_I}^* T_{\tilde{S}_I}^{CH}(L(i_{I*}j_I^*F)), E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_I}, F_{DR})))} \\
&J_S(e'(\tilde{S}_I)_*\mathcal{H}om(\mathrm{An}_{\tilde{S}_I}^* \mathrm{Gr}_{\tilde{S}_I*}^{12} L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*}j_I^*F)), \mathrm{Gr}_{\tilde{S}_I*}^{12} E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F)) \\
&\xrightarrow{\mathcal{H}om(\mathrm{ad}(\mathrm{Gr}^*,\mathrm{Gr}_*)(-)\circ q,-)\circ I(\mathrm{Gr}_{\tilde{S}_I}^{12*},\mathrm{Gr}_{\tilde{S}_I*}^{12})(-,-)\circ\mathcal{H}om(T(\mathrm{An},\mathrm{Gr})(-),-)^{-1}} \\
&J_S(e'(\tilde{S}_I)_*\mathcal{H}om(\mathrm{An}_{\tilde{S}_I}^* L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*}j_I^*F)), E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F)) \\
&=: J_S(\mathcal{F}_{S,an}^{FDR}(M))
\end{aligned}$$

We now define the functorialities of $\mathcal{F}_S^{FDR}$ with respect to S which makes $\mathcal{F}_{FDR}^-$ a morphism of 2 functor.

Definition 141. *Let* $S \in \mathrm{Var}(\mathbb{C})$. *Let* $Z \subset S$ *a closed subset. Let* $S = \cup_{i=1}^l S_i$ *an open cover such that there exist closed embeddings* $i_i : S_i \hookrightarrow \tilde{S}_i$ *with* $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. *Denote* $Z_I := Z \cap S_I$. *We then have closed embeddings* $Z_I \hookrightarrow S_I \hookrightarrow \tilde{S}_I$.

(i) For $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$, *we will consider the following canonical map in* $\pi_S(D(MHM(S^{an}))) \subset D_{\mathcal{D}(1,0)fil}(S^{an}/(\tilde{S}_I^{an}))$

$$\begin{aligned}
&T(\Gamma_Z^{\vee,Hdg}, \Omega^{\Gamma,pr,an}_{/S})(F,W) : \\
&\Gamma_Z^{\vee,Hdg}\iota_S^{-1}(e'(\tilde{S}_I)_*\mathcal{H}om^\bullet(\mathrm{An}_{\tilde{S}_I}^* L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*}j_I^*(F,W))), \\
&E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F,W)) \\
&\xrightarrow{\mathcal{H}om^\bullet(\mathrm{An}_{\tilde{S}_I}^* L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R_{\tilde{S}_I}^{CH}(\gamma^{\vee,Z_I}(L(i_{I*}j_I^*(F,W)))), E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_I}, F_{DR}))} \\
&\Gamma_Z^{\vee,Hdg}\iota_S^{-1}(e'(\tilde{S}_I)_*\mathcal{H}om^\bullet(\mathrm{An}_{\tilde{S}_I}^* L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^*\Gamma_{Z_I}^\vee L(i_{I*}j_I^*(F,W))), \\
&E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^{q,Z}(F,W)) \\
&\xrightarrow{=} \iota_S^{-1}(e'(\tilde{S}_I)_*\mathcal{H}om^\bullet(\mathrm{An}_{\tilde{S}_I}^* L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^*\Gamma_{Z_I}^\vee L(i_{I*}j_I^*(F,W))), \\
&E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^{q,Z}(F,W)),
\end{aligned}$$

with $u_{IJ}^{q,Z}(F)$ *given as in definition 119(i).*

(ii) For $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$, *we have also the following canonical map in* $\pi_S(D(MHM(S^{an}))) \subset$

$D_{\mathcal{D}(1,0)fil}(S^{an}/(\tilde{S}_I^{an}))$

$$
\begin{aligned}
&T(\Gamma^{Hdg}_Z,\Omega^{\Gamma,pr}_{/S})(F,W):\\
&\iota_S^{-1}(e'(\tilde{S}_I)_*\mathcal{H}om^\bullet(\mathrm{An}^*_{\tilde{S}_I}L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho^*_{\tilde{S}_I}L\Gamma_{Z_I}E(i_{I*}j_I^*\mathbb{D}_S(F,W))),\\
&E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I},F_{DR}))[-d_{\tilde{S}_I}],u^{q,Z,d}_{IJ}(F,W))\\
&\xrightarrow{=}\Gamma^{Hdg}_Z\iota_S^{-1}(e'(\tilde{S}_I)_*\mathcal{H}om^\bullet(\mathrm{An}^*_{\tilde{S}_I}L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho^*_{\tilde{S}_I}L\Gamma_{Z_I}E(i_{I*}j_I^*\mathbb{D}_S(F,W))),\\
&E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I},F_{DR}))[-d_{\tilde{S}_I}],u^{q,Z,d}_{IJ}(F,W))\\
&\xrightarrow{\mathcal{H}om^\bullet(\mathrm{An}^*_{\tilde{S}_I}L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}_{\tilde{S}_I}(\gamma^{Z_I}(-)),E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I},F_{DR}))}\\
&\Gamma^{Hdg}_Z\iota_S^{-1}(e'(\tilde{S}_I)_*\mathcal{H}om^\bullet(\mathrm{An}^*_{\tilde{S}_I}L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho^*_{\tilde{S}_I}L(i_{I*}j_I^*\mathbb{D}_S(F,W))),\\
&E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I},F_{DR}))[-d_{\tilde{S}_I}],u^q_{IJ}(F,W))
\end{aligned}
$$

with $u^{q,Z}_{IJ}(F)$ given as in definition 119(ii).

Definition 142. *Let $g:T\to S$ a morphism with $T,S\in\mathrm{Var}(\mathbb{C})$. Assume we have a factorization $g:T\xrightarrow{l}Y\times S\xrightarrow{p_S}S$ with $Y\in\mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S=\cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$ Then, $T=\cup_{i=1}^l T_i$ with $T_i:=g^{-1}(S_i)$ and we have closed embeddings $i'_i:=i_i\circ l:T_i\hookrightarrow Y\times\tilde{S}_i$, Moreover $\tilde{g}_I:=p_{\tilde{S}_I}:Y\times\tilde{S}_I\to\tilde{S}_I$ is a lift of $g_I:=g_{|T_I}:T_I\to S_I$. Let $M\in\mathrm{DA}_c(S)$ and $(F,W)\in C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $(M,W)=D(\mathbb{A}^1_S,et)(F,W)$. Then, $D(\mathbb{A}^1_T,et)(g^*F)=g^*M$ and there exist $(F',W)\in C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ and an equivalence $(\mathbb{A}^1,et)$ local $e:g^*(F,W)\to(F',W)$ such that $D(\mathbb{A}^1_T,et)(F',W)=(g^*M,W)$. We have, using definition 136 and definition 141(i), the canonical map in $\pi_T(D(MHM(T^{an})))\subset D_{\mathcal{D}(1,0)fil,\infty}(T^{an}/(Y^{an}\times\tilde{S}_I^{an}))$*

$$
\begin{aligned}
&T(g,\mathcal{F}^{FDR}_{an})(M):g^{\hat{*}mod}_{Hdg}\iota_S^{-1}\mathcal{F}^{FDR}_{S,an}(M):=\\
&(\Gamma^{\vee,Hdg}_T\iota_T^{-1}(\tilde{g}_I^{*mod}(e'(\tilde{S}_I)_*\mathcal{H}om^\bullet(L\,\mathrm{An}^*_{\tilde{S}_I}\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho^*_{\tilde{S}_I}(L(i_{I*}j_I^*(F,W))),\\
&E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_I},F_{DR})))[d_{YI}],\tilde{g}_J^{*mod}u^q_{IJ}(F,W))\xrightarrow{(T(\tilde{g}_I,\Omega^{\Gamma,pr,an}_{/\cdot})(-)}\\
&\Gamma^{\vee,Hdg}_T\iota_T^{-1}(e'(-)_*\mathcal{H}om(\mathrm{An}^*_{Y\times\tilde{S}_I}L\rho_{Y\times\tilde{S}_I*}\mu_{Y\times\tilde{S}_I*}\tilde{g}_I^*R^{CH}(\rho^*_{\tilde{S}_I}L(i_{I*}j_I^*(F,W))),\\
&E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/Y\times\tilde{S}_I},F_{DR}))[d_{YI}],\tilde{g}_J^*u^q_{IJ}(F,W))\xrightarrow{\mathcal{H}om(T(\tilde{g}_I,R^{CH})(-)^{-1},-)}\\
&\Gamma^{\vee,Hdg}_T\iota_T^{-1}(e'_*\mathcal{H}om(\mathrm{An}^*_{Y\times\tilde{S}_I}L\rho_{Y\times\tilde{S}_I*}\mu_{Y\times\tilde{S}_I*}R^{CH}(\rho^*_{Y\times\tilde{S}_I}\tilde{g}_I^*L(i_{I*}j_I^*(F,W))),\\
&E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/Y\times\tilde{S}_I},F_{DR}))[d_{YI}],\tilde{g}_J^*u^q_{IJ}(F,W))\xrightarrow{T(\Gamma^{\vee,Hdg}_T,\Omega^{\Gamma,pr,an}_{/S})(F,W)}\\
&\iota_T^{-1}(e'_*\mathcal{H}om(\mathrm{An}^*_{Y\times\tilde{S}_I}L\rho_{Y\times\tilde{S}_I*}\mu_{Y\times\tilde{S}_I*}R^{CH}(\rho^*_{Y\times\tilde{S}_I}\Gamma^\vee_{T_I}\tilde{g}_I^*L(i_{I*}j_I^*(F,W))),\\
&E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/Y\times\tilde{S}_I},F_{DR}))[d_{YI}],\tilde{g}_J^{*,\gamma}u^q_{IJ}(F,W))\\
&\xrightarrow{(\mathcal{H}om(\mathrm{An}^*_{Y\times\tilde{S}_I}L\rho_{Y\times\tilde{S}_I*}\mu^{12*}_{Y\times\tilde{S}_I*}R^{CH}_{Y\times\tilde{S}_I}(T^{q,\gamma}(D_{gI})(j_I^*(F,W))),E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/Y\times\tilde{S}_I},F_{DR}))[d_{YI}])}\\
&\iota_T^{-1}(e'_*\mathcal{H}om(\mathrm{An}^*_{Y\times\tilde{S}_I}L\rho_{Y\times\tilde{S}_I*}\mu_{Y\times\tilde{S}_I*}R^{CH}(\rho^*_{Y\times\tilde{S}_I}L(i'_{I*}j_I^{'*}g^*(F,W))),\\
&E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/Y\times\tilde{S}_I},F_{DR}))[d_{YI}],u^q_{IJ}(g^*(F,W)))\xrightarrow{\mathcal{H}om(R^{CH}_{Y\times\tilde{S}_I}(Li'_{I*}j_I^{'*}(e)),)}\\
&\iota_T^{-1}(e'_*\mathcal{H}om(\mathrm{An}^*_{Y\times\tilde{S}_I}L\rho_{Y\times\tilde{S}_I*}\mu_{Y\times\tilde{S}_I*}R^{CH}(\rho^*_{Y\times\tilde{S}_I}L(i'_{I*}j_I^{'*}(F',W))),\\
&E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/Y\times\tilde{S}_I},F_{DR}))[d_{YI}],u^q_{IJ}(F',W))\xrightarrow{=:}\mathcal{F}^{FDR}_{T,an}(g^*M)
\end{aligned}
$$

Proposition 135. *Let $g : T \to S$ a morphism with $T, S \in \mathrm{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y_2 \times S \xrightarrow{p_S} S$ with $Y_2 \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^{l} S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$ Then, $T = \cup_{i=1}^{l} T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i_i' := i_i \circ l : T_i \hookrightarrow Y_2 \times \tilde{S}_i$, Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \to \tilde{S}_I$ is a lift of $g_I := g_{|T_I} : T_I \to S_I$. Let $f : X \to S$ a morphism with $X \in \mathrm{Var}(\mathbb{C})$ such that there exists a factorization $f : X \xrightarrow{l} Y_1 \times S \xrightarrow{p_S} S$, with $Y_1 \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have then the following commutative diagram whose squares are cartesians*

$$\begin{tikzcd}
f' : X_T \arrow[r] \arrow[dd, "g'"'] \arrow[dr] & Y_1 \times T \arrow[rr] \arrow[dr] & & T \arrow[dd, "g"] \arrow[dr] & \\
& Y_1 \times X \arrow[r] \arrow[dl] & Y_1 \times Y_2 \times S \arrow[rr] \arrow[dl] & & Y_2 \times S \arrow[dl] \\
f : X \arrow[r] & Y_1 \times S \arrow[rr] & & S &
\end{tikzcd}$$

Take a smooth compactification $\bar{Y}_1 \in \mathrm{PSmVar}(\mathbb{C})$ of Y_1, denote $\bar{X}_I \subset \bar{Y}_1 \times \tilde{S}_I$ the closure of X_I, and $Z_I := \bar{X}_I \backslash X_I$. Consider $F(X/S) := p_{S,\sharp}\Gamma_X^{\vee}\mathbb{Z}(Y_1 \times S/Y_1 \times S) \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ and the isomorphism in $C(\mathrm{Var}(\mathbb{C})^{sm}/T)$

$$T(f, g, F(X/S)) : g^* F(X/S) := g^* p_{S,\sharp}\Gamma_X^{\vee}\mathbb{Z}(Y_1 \times S/Y_1 \times S) \xrightarrow{\sim} p_{T,\sharp}\Gamma_{X_T}^{\vee}\mathbb{Z}(Y_1 \times T/Y_1 \times T) =: F(X_T/T).$$

which gives in $\mathrm{DA}(T)$ the isomorphism $T(f, g, F(X/S)) : g^* M(X/S) \xrightarrow{\sim} M(X_T/T)$. Then the following diagram in $\pi_T(D(MHM(T))) \subset D_{\mathcal{D}(1,0)fil,\infty}(T/(Y_2 \times \tilde{S}_I))$, where the horizontal maps are given by proposition 133, commutes

$$\begin{tikzcd}
g_{Hdg}^{\hat{*}mod}\iota_S^{-1}\mathcal{F}_{S,an}^{FDR}(M(X/S)) \arrow[r, "g_{Hdg}^{\hat{*}mod}I(X/S)"] \arrow[dd, "'T(g,\mathcal{F}^{FDR})(M(X/S))"'] & g_{Hdg}^{\hat{*}mod}Rf_!^{Hdg}(\Gamma_{X_I}^{\vee,Hdg}(O_{(Y_1\times\tilde{S}_I)^{an}}, F_b)(d_{Y_1})[2d_{Y_1}], x_{IJ}(X/S)) \arrow[d, "T(p_{\tilde{S}_I},\gamma^{\vee,Hdg})(-)"] \\
& Rf_!'^{Hdg}g_{Hdg}'^{\hat{*}mod}(\Gamma_{X_I}^{\vee,Hdg}(O_{(Y_1\times\tilde{S}_I)^{an}}, F_b)(d_{Y_1})[2d_{Y_1}], x_{IJ}(X/S)) \arrow[d, "T(p_{Y_1\times Y_2\times\tilde{S}_I,Hdg}^{\hat{*}mod},p_{Y_1\times Y_2\times\tilde{S}_I,Hdg}^{*mod})(-)"] \\
\iota_T^{-1}\mathcal{F}_{T,an}^{FDR}(M(X_T/T)) \arrow[r, "I(X_T/T)"] & Rf_!'^{Hdg}(\Gamma_{X_{T_I}}^{\vee,Hdg}(O_{(Y_2\times Y_1\times\tilde{S}_I)^{an}}, F_b)(d_{Y_{12}})[2d_{Y_{12}}], x_{IJ}(X_T/T)).
\end{tikzcd}$$

with $d_{Y_{12}} = d_{Y_1} + d_{Y_2}$.

Proof. Follows immediately from definition. □

Theorem 39. *Let $g : T \to S$ a morphism, with $S, T \in \mathrm{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $M \in \mathrm{DA}_c(S)$. Then map in $\pi_T(D(MHM(T^{an})))$*

$$T(g, \mathcal{F}_{an}^{FDR})(M) : g_{Hdg}^{\hat{*}mod}\mathcal{F}_{S,an}^{FDR}(M) \xrightarrow{\sim} \mathcal{F}_{T,an}^{FDR}(g^* M)$$

given in definition 142 is an isomorphism.

Proof. Follows from proposition 135 and proposition 133. □

Definition 143. • *Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have, for $M \in \mathrm{DA}_c(X)$, the following transformation map in $\pi_S(D(MHM(S^{an})))$*

$$T_*(f, \mathcal{F}^{FDR}_{an})(M) : \mathcal{F}^{FDR}_{S,an}(Rf_*M) \xrightarrow{\mathrm{ad}(f^{\hat{*}mod}_{Hdg}, Rf^{Hdg}_*)(-)} Rf^{Hdg}_* f^{\hat{*}mod}_{Hdg}\mathcal{F}^{FDR}_{S,an}(Rf_*M)$$
$$\xrightarrow{T(f,\mathcal{F}^{FDR}_{an})(Rf_*M)} Rf^{Hdg}_*\mathcal{F}^{FDR}_{X,an}(f^*Rf_*M) \xrightarrow{\mathcal{F}^{FDR}_X(\mathrm{ad}(f^*, Rf_*)(M))} Rf^{Hdg}_*\mathcal{F}^{FDR}_{X,an}(M)$$

Clearly, for $p : Y \times S \to S$ a projection with $Y \in \mathrm{PSmVar}(\mathbb{C})$, we have, for $M \in \mathrm{DA}_c(Y \times S)$, $T_*(p, \mathcal{F}^{FDR})(M) = T_!(p, \mathcal{F}^{FDR})(M)[2d_Y]$

- *Let $S \in \mathrm{Var}(\mathbb{C})$. Let $Y \in \mathrm{SmVar}(\mathbb{C})$ and $p : Y \times S \to S$ the projection. We have then, for $M \in \mathrm{DA}(Y \times S)$ the following transformation map in $\pi_S(D(MHM(S^{an})))$*

$$T_!(p, \mathcal{F}^{FDR}_{an})(M) : p^{Hdg}_!\mathcal{F}^{FDR}_{Y\times S,an}(M) \xrightarrow{\mathcal{F}^{FDR}_{Y\times S,an}(\mathrm{ad}(Lp_\sharp, p^*)(M))} Rp^{Hdg}_!\mathcal{F}^{FDR}_{Y\times S,an}(p^*Lp_\sharp(M))$$
$$\xrightarrow{T(p,\mathcal{F}^{FDR})(Lp_\sharp(M,W))} Rp^{Hdg}_! p^{\hat{*}mod[-]}\mathcal{F}^{FDR}_{S,an}(Lp_\sharp M) \xrightarrow{T(p^{*mod}, p^{\hat{*}mod})(-)} p^{Hdg}_! p^{*mod[-]}$$
$$\mathcal{F}^{FDR}_{S,an}(Lp_\sharp M) \xrightarrow{\mathrm{ad}(Rp^{Hdg}_!, p^{*mod[-]})(\mathcal{F}^{FDR}_{S,an}(Lp_\sharp M))} \mathcal{F}^{FDR}_{S,an}(Lp_\sharp M)$$

- *Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have then, using the second point, for $M \in \mathrm{DA}(X)$ the following transformation map in $\pi_S(D(MHM(S^{an})))$*

$$T_!(f, \mathcal{F}^{FDR}_{an})(M) : Rp^{Hdg}_!\mathcal{F}^{FDR}_X(M) := Rp^{Hdg}_!\mathcal{F}^{FDR}_{Y\times S,an}(l_*M)$$
$$\xrightarrow{T_!(p,\mathcal{F}^{FDR}_{an})(l_*M)} \mathcal{F}^{FDR}_{S,an}(Lp_\sharp l_*M) \xrightarrow{=} \mathcal{F}^{FDR}_{S,an}(Rf_!M)$$

- *Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have, using the third point, for $M \in \mathrm{DA}(S)$, the following transformation map in in $\pi_X(D(MHM(X^{an})))$*

$$T^!(f, \mathcal{F}^{FDR}_{an})(M) : \mathcal{F}^{FDR}_{X,an}(f^!(M,W)) \xrightarrow{\mathrm{ad}(Rf^{Hdg}_!, f^{*mod}_{Hdg})(\mathcal{F}^{FDR}_{X,an}(f^!M))} f^{*mod}_{Hdg}Rf^{Hdg}_!\mathcal{F}^{FDR}_{X,an}(f^!M)$$
$$\xrightarrow{T_!(p_S,\mathcal{F}^{FDR}_{an})(\mathcal{F}^{FDR}_{an}(f^!M))} f^{*mod}_{Hdg}\mathcal{F}^{FDR}_{S,an}(Rf_!f^!(M,W)) \xrightarrow{\mathcal{F}^{FDR}_{S,an}(\mathrm{ad}(Rf_!, f^!)(M))} f^{*mod}_{Hdg}\mathcal{F}^{FDR}_{S,an}(M)$$

Proposition 136. *Let $S \in \mathrm{Var}(\mathbb{C})$. Let $Y \in \mathrm{SmVar}(\mathbb{C})$ and $p : Y \times S \to S$ the projection. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i^o_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. For $I \subset [1, \cdots l]$, we denote by $S_I = \cap_{i\in I}S_i$, $j^o_I : S_I \hookrightarrow S$ and $j_I : Y \times S_I \hookrightarrow Y \times S$ the open embeddings. We then have closed embeddings $i_I : Y \times S_I \hookrightarrow Y \times \tilde{S}_I$. and we denote by $p_{\tilde{S}_I} : Y \times \tilde{S}_I \to \tilde{S}_I$ the projections. Let $f' : X' \to Y \times S$ a morphism, with $X' \in \mathrm{Var}(\mathbb{C})$ such that there exists a factorization $f' : X' \xrightarrow{l'} Y' \times Y \times S \xrightarrow{p'} Y \times S$ with $Y' \in \mathrm{SmVar}(\mathbb{C})$, l' a closed embedding and p' the projection. Denoting $X'_I := f'^{-1}(Y \times S_I)$, we have closed embeddings $i'_I : X'_I \hookrightarrow Y' \times Y \times \tilde{S}_I$ Consider*

$$F(X'/Y \times S) := p_{Y\times S,\sharp}\Gamma^\vee_{X'}\mathbb{Z}(Y' \times Y \times S/Y' \times Y \times S) \in C(\mathrm{Var}(\mathbb{C})^{sm}/Y \times S)$$

and $F(X'/S) := p_\sharp F(X'/Y \times S) \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$, so that $Lp_\sharp M(X'/Y \times S)[-2d_Y] =: M(X'/S)$. Then, the following diagram in $\pi_S(D(MHM(S^{an}))) \subset D_{\mathcal{D}(1,0)fil,\infty}(S^{an}/(Y^{an} \times \tilde{S}^{an}_I))$, where the vertical maps are given by proposition 133, commutes

$$\begin{tikzcd}
Rp^{Hdg}!\mathcal{F}^{FDR}_{Y\times S,an}(M(X'/Y \times S)) \arrow[r, "T_!(p,\mathcal{F}^{FDR}_{an})(M(X'/Y\times S))"] & \mathcal{F}^{FDR}_{S,an}(M(X'/S)) \\
Rp^{Hdg}!Rf'^{Hdg}_! f'^{*mod}_{Hdg}\mathbb{Z}^{Hdg}_{(Y\times S)^{an}} \arrow[u, "T(p^{\hat{*}mod}_{Hdg}, p^{*mod}_{Hdg})(-)\circ Rp^{Hdg}!(I(X'/Y\times S))"] \arrow[r, "="] & Rf^{Hdg}_! f^{*mod}_{Hdg}\mathbb{Z}^{Hdg}_{S^{an}} \arrow[u, "I(X'/S)"']
\end{tikzcd}$$

Proof. Immediate from definition. □

Theorem 40. *(i) Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Then, for $M \in \mathrm{DA}_c(X)$,*

$$T_!(f, \mathcal{F}^{FDR}_{an})(M) : Rf^{Hdg}_! \mathcal{F}^{FDR}_{X,an}(M) \xrightarrow{\sim} \mathcal{F}^{FDR}_{S,an}(Rf_! M)$$

is an isomorphism in $\pi_S(D(MHM(S^{an}))$

(ii) Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$, S quasi-projective. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have, for $M \in \mathrm{DA}_c(X)$,

$$T_*(f, \mathcal{F}^{FDR}_{an})(M) : \mathcal{F}^{FDR}_{S,an}(Rf_* M) \xrightarrow{\sim} Rf^{Hdg}_* \mathcal{F}^{FDR}_{X,an}(M)$$

is an isomorphism in $\pi_S(D(MHM(S^{an}))$.

(iii) Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$, S quasi-projective. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Then, for $M \in \mathrm{DA}_c(S)$

$$T^!(f, \mathcal{F}^{FDR}_{an})(M) : \mathcal{F}^{FDR}_{X,an}(f^! M) \xrightarrow{\sim} f^{*mod}_{Hdg} \mathcal{F}^{FDR}_{S,an}(M)$$

is an isomorphism in $\pi_X(D(MHM(X^{an}))$.

Proof. Similar to the proof of theorem 36. □

Proposition 137. *Let $g : T \to S$ a morphism with $T, S \in \mathrm{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$ Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$, Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \to \tilde{S}_I$ is a lift of $g_I := g_{|T_I} : T_I \to S_I$. Let $M \in \mathrm{DA}_c(S)$ and $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1_S, et)(F)$. Then, $D(\mathbb{A}^1_T, et)(g^* F) = g^* M$. Then the following diagram in $D_{Ofil,\mathcal{D}^\infty,\infty}(T^{an}/(Y^{an} \times \tilde{S}^{an}_I))$ commutes*

$$\begin{tikzcd}
Rg^{*mod[-],\Gamma} \mathcal{F}^{GM}_{S,an}(L\mathbb{D}_S M) \arrow[r, "Rg^{*mod[-],\Gamma} T(\mathcal{F}^{GM}_{S,an}{,}\mathcal{F}^{FDR}_{S,an})(M)"] \arrow[d, "T(g{,}\mathcal{F}^{GM}_{an})(L\mathbb{D}_S M)"] & Rg^{*mod[-],\Gamma} \mathcal{F}^{FDR}_{S,an}(M) \arrow[dr] & g^{*mod}_{Hdg} \mathcal{F}^{FDR}_{S,an}(M) \arrow[l, "T(g^{*mod}_{Hdg}{,}Rg^{*mod[-],\Gamma})(\mathcal{F}^{FDR}_{S,an}(M))"'] \arrow[d, "T^!(g{,}\mathcal{F}^{FDR}_{an})(M)^{-1}"] \\
\mathcal{F}^{GM}_{T,an}(g^* L\mathbb{D}_S M = L\mathbb{D}_T g^! M) \arrow[rr, "T(\mathcal{F}^{GM}_{T,an}{,}\mathcal{F}^{FDR}_{T,an})(g^* M)"] & & \mathcal{F}^{FDR}_{T,an}(g^! M)
\end{tikzcd}$$

Proof. Similar to the proof of proposition 119. □

Definition 144. *Let $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open affine covering and denote, for $I \subset [1, \cdots l]$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. We have, for $M, N \in \mathrm{DA}(S)$ and $(F, W), (G, W) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S))$ such that $(M, W) = D(\mathbb{A}^1, et)(F, W)$ and $(N, W) = D(\mathbb{A}^1, et)(G, W)$, the following transformation map in*

$\pi_S(D(MHM(S^{an}))) \subset D_{\mathcal{D}(1,0)fil}(S^{an}/(\tilde{S}_I^{an}))$

$$\begin{aligned}
&T(\mathcal{F}_{S,an}^{FDR},\otimes)(M,N) : \mathcal{F}_{S,an}^{FDR}(M) \otimes_{O_S}^{L[-]} \mathcal{F}_{S,an}^{FDR}(N) \\
&\xrightarrow{:=} (e'_*\mathcal{H}om^\bullet(\mathrm{An}_{\tilde{S}_I}^* L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*}j_I^*(F,W))), E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}(F,W)) \otimes_{O_S}^{[-]} \\
&(e'_*\mathcal{H}om^\bullet(\mathrm{An}_{\tilde{S}_I}^* L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*}j_I^*(G,W))), E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}(G,W)) \\
&\xrightarrow{=} (e'_*\mathcal{H}om^\bullet(\mathrm{An}_{\tilde{S}_I}^* L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*}j_I^*(F,W))), E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_I}, F_{DR})) \otimes_{O_{\tilde{S}_I}} \\
&e'_*\mathcal{H}om^\bullet(\mathrm{An}_{\tilde{S}_I}^* L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*}j_I^*(F,W))), E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}(F) \otimes u_{IJ}(G)) \\
&\xrightarrow{(T(\otimes,\Omega^{\Gamma,pr,an}_{/\tilde{S}_I})(-,-))} \\
&(e'_*\mathcal{H}om(\mathrm{An}_{\tilde{S}_I}^* L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}(R^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*}j_I^*(F,W))) \otimes R^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*}j_I^*(F,W)))), \\
&E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], v_{IJ}(F \otimes G)) \\
&\xrightarrow{\mathcal{H}om(T(\otimes,R^{CH}_{\tilde{S}_I})(-,-)^{-1},-)} \\
&(e'_*\mathcal{H}om(\mathrm{An}_{\tilde{S}_I}^* L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^*(L(i_{I*}j_I^*(F,W))) \otimes L(i_{I*}j_I^*(F,W))), E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}(F \otimes G)) \\
&\xrightarrow{\mathcal{H}om(R_{(-,-)/-}(T(\otimes,L)(-,-)),-)} \\
&(e'_*\mathcal{H}om(\mathrm{An}_{\tilde{S}_I}^* L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^*(L(i_{I*}j_I^*((F,W) \otimes (G,W))))), \\
&E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}(F \otimes G)) \xrightarrow{=:} \mathcal{F}_{S,an}^{FDR}(M \otimes N)
\end{aligned}$$

Proposition 138. *Let $f_1 : X_1 \to S$, $f_2 : X_2 \to S$ two morphism with $X_1, X_2, S \in \mathrm{Var}(\mathbb{C})$. Assume that there exist factorizations $f_1 : X_1 \xrightarrow{l_1} Y_1 \times S \xrightarrow{p_S} S$, $f_2 : X_2 \xrightarrow{l_2} Y_2 \times S \xrightarrow{p_S} S$ with $Y_1, Y_2 \in \mathrm{SmVar}(\mathbb{C})$, l_1, l_2 closed embeddings and p_S the projections. We have then the factorization*

$$f_{12} := f_1 \times f_2 : X_{12} := X_1 \times_S X_2 \xrightarrow{l_1 \times l_2} Y_1 \times Y_2 \times S \xrightarrow{p_S} S$$

Let $S = \cup_{i=1}^l S_i$ an open affine covering and denote, for $I \subset [1, \cdots l]$, $S_I = \cap_{i\in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. We have then the following commutative diagram in $\pi_S(DMHM(S^{an})) \subset D_{\mathcal{D}(1,0)fil}(S^{an}/(\tilde{S}_I^{an}))$ where the vertical maps are given by proposition 133

$$\begin{array}{ccc}
\mathcal{F}_{S,an}^{FDR}(M(X_1/S)) \otimes_{O_S}^{L[-]} \mathcal{F}_{S,an}^{FDR}(M(X_2/S)) & \xrightarrow{I(X_1/S)\otimes I(X_2/S)} & \begin{array}{c} Rf_{1!}^{Hdg}(\Gamma^{\vee,Hdg}_{X_{1I}}(O_{(Y_1\times\tilde{S}_I)^{an}}, F_b)(d_1)[2d_1], x_{IJ}(X_1/S)) \otimes_{O} \\ Rf_{2!}^{Hdg}(\Gamma^{\vee,Hdg}_{X_{2I}}(O_{(Y_2\times\tilde{S}_I)^{an}}, F_b)(d_2)[2d_2], x_{IJ}(X_2/S)) \end{array} \\
\downarrow{\scriptstyle T(\mathcal{F}_{S,an}^{FDR},\otimes)(M(X_1/S),M(X_2/S))} & & \downarrow{\scriptstyle (Ew_{(Y_1\times\tilde{S}_I,Y_2\times\tilde{S}_I)/\tilde{S}_I})} \\
\mathcal{F}_{S,an}^{FDR}(M(X_1/S) \otimes M(X_2/S) = M(X_1 \times_S X_2/S)) & \xrightarrow{I(X_{12}/S)} & Rf_{12!}^{Hdg}(\Gamma^{\vee,Hdg}_{X_{1I}\times_S X_{2I}}(O_{(Y_1\times Y_2\times\tilde{S}_I)^{an}}, F_b)(d_{12})[2d_{12}], x_{IJ}(X_1/
\end{array}$$

Proof. Immediate from definition. □

Theorem 41. *Let $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open affine covering and denote, for $I \subset [1, \cdots l]$, $S_I = \cap_{i\in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. Then, for $M, N \in \mathrm{DA}_c(S)$, the map in $\pi_S(D(MHM(S^{an})))$*

$$T(\mathcal{F}_{S,an}^{FDR},\otimes)(M,N) : \mathcal{F}_{S,an}^{FDR}(M) \otimes_{O_S}^{L} \mathcal{F}_{S,an}^{FDR}(N) \xrightarrow{\sim} \mathcal{F}_{S,an}^{FDR}(M \otimes N)$$

given in definition 144 is an isomorphism.

Proof. Follows from proposition 138. □

We have the following easy proposition

Proposition 139. *Let $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^{l} S_i$ an open affine covering and denote, for $I \subset [1, \cdots l]$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. We have, for $M, N \in \mathrm{DA}(S)$ and $F, G \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1, et)(F)$ and $N = D(\mathbb{A}^1, et)(G)$, the following commutative diagram in $D_{O_S fil,\mathcal{D}^{\infty},\infty}(S^{an}/(\tilde{S}_I^{an}))$*

$$\begin{tikzcd}
\mathcal{F}^{GM}_{S,an}(L\mathbb{D}_S M) \otimes^L_{O_S} \mathcal{F}^{GM}_{S,an}(L\mathbb{D}_S N) \arrow[rr, "T(\mathcal{F}^{GM}_{S,an},\mathcal{F}^{FDR}_{S,an})(M) \otimes T(\mathcal{F}^{GM}_{S,an},\mathcal{F}^{FDR}_{S,an})(N)"] \arrow[d, "T(\mathcal{F}^{GM}_{S,an},\otimes)(L\mathbb{D}_S M, L\mathbb{D}_S N)"] & & \mathcal{F}^{FDR}_{S,an}(M) \otimes^L_{O_S} \mathcal{F}^{FDR}_{S,an}(N) \arrow[d, "T(\mathcal{F}^{FDR}_{S,an},\otimes)(M,N)"] \\
\mathcal{F}^{GM}_{S,an}(\mathbb{D}_S L(M \otimes N)) \arrow[rr, "T(\mathcal{F}^{GM}_{S,an},\mathcal{F}^{FDR}_{S,an})(M \otimes N)"] & & \mathcal{F}^{FDR}_{S,an}(M \otimes N)
\end{tikzcd}$$

Proof. Immediate from definition. □

6.3 The transformation map between the analytic De Rahm functor and the analytification of the algebraic De Rahm functor

6.3.1 The transformation map between the analytic Gauss Manin realization functor and the analytification of the algebraic Gauss Manin realization functor

Recall from section 2 that, for $f : T \to S$ a morphism with $T, S \in \mathrm{Var}(\mathbb{C})$, we have the following commutative diagram of sites (38)

$$\begin{tikzcd}
\mathrm{AnSp}(\mathbb{C})/T^{an} \arrow[rr, "\mathrm{An}_T"] \arrow[dr, "\rho_T"] \arrow[dd, "P(f)"] & & \mathrm{Var}(\mathbb{C})/T \arrow[dr, "\rho_T"] \arrow[dd, "P(f)"] & \\
& \mathrm{AnSp}(\mathbb{C})^{sm}/T^{an} \arrow[rr, "\mathrm{An}_T"] \arrow[dd, "P(f)"] & & \mathrm{Var}(\mathbb{C})^{sm}/T \arrow[dd, "P(f)"] \\
\mathrm{AnSp}(\mathbb{C})/S^{an} \arrow[rr, "\mathrm{An}_S"] \arrow[dr, "\rho_S"] & & \mathrm{Var}(\mathbb{C})/S \arrow[dr, "\rho_S"] & \\
& \mathrm{AnSp}(\mathbb{C})^{sm}/S^{an} \arrow[rr, "\mathrm{An}_S"] & & \mathrm{Var}(\mathbb{C})^{sm}/S
\end{tikzcd} .$$

We have the following canonical transformation map given by the pullback of (relative) differential forms: Let $S \in \mathrm{Var}(\mathbb{C})$. Consider the following commutative diagram in RCat :

$$\begin{tikzcd}
D(an,e) : (\mathrm{AnSp}(\mathbb{C})^{sm}/S^{an}, O_{\mathrm{AnSp}(\mathbb{C})^{sm}/T}) \arrow[r, "\mathrm{An}_S"] \arrow[d, "e(T)"] & (\mathrm{Var}(\mathbb{C})^{sm}/S, O_{\mathrm{Var}(\mathbb{C})^{sm}/S}) \arrow[d, "e(S)"] \\
(S^{an}, O_{S^{an}}) \arrow[r, "\mathrm{an}_S"] & (S, O_S)
\end{tikzcd}$$

It gives (see section 2) the canonical morphism in $C_{\mathrm{an}_S^* O_S}(\mathrm{AnSp}(\mathbb{C})^{sm}/S^{an})$

$$\Omega_{/(S^{an}/S)} := \Omega_{(O_{\mathrm{AnSp}(\mathbb{C})^{sm}/S^{an}} / \mathrm{An}_S^* O_{\mathrm{Var}(\mathbb{C})^{sm}/S})/(O_{S^{an}} / \mathrm{an}_S^* O_S)} :$$
$$\mathrm{An}_S^*(\Omega^{\bullet}_{/S}, F_b) = (\Omega^{\bullet}_{\mathrm{An}_S^* O_{\mathrm{Var}(\mathbb{C})^{sm}/S} / \mathrm{An}_S^* e(S)^* O_S}, F_b) \to (\Omega^{\bullet}_{/S^{an}}, F_b) = (\Omega^{\bullet}_{O_{\mathrm{AnSp}(\mathbb{C})^{sm}/S^{an}}/e(S^{an})^* O_{S^{an}}}, F_b)$$

which is by definition given by the analytification on differential forms : for $(V/S^{an}) = (V, h) \in \mathrm{AnSp}(\mathbb{C})^{sm}/S^{an}$,

$$\Omega_{/(S^{an}/S)}(V/S^{an}) : \hat{\omega} \in \mathrm{An}_S^*(\Omega^r_{/S})(V/S^{an}) := \lim_{(h':U \to S \mathrm{Ssm}, g':V \to U^{an}, h, g)} \Omega^r_{/S}(U/S)$$
$$\mapsto \Omega_{(V/U)/(S^{an}/S)}(V/S^{an})(\omega) := \widehat{\mathrm{an}_S^*(\omega)} \in \Omega^r_{S^{an}}(V/S^{an});$$

with $\omega \in \Gamma(U, \Omega^r_U)$ is such that $q(\omega) = \hat{\omega}$. If $S \in \mathrm{SmVar}(\mathbb{C})$, the map $\Omega_{/(T/S)} : \mathrm{An}_S^* \Omega^\bullet_{/S} \to \Omega^\bullet_{/S^{an}}$ is a map in $C_{O_S fil,\mathcal{D}}(\mathrm{AnSp}(\mathbb{C})^{sm}/S^{an})$. It induces the canonical morphism in $C_{O_S fil,\mathcal{D}}(\mathrm{AnSp}(\mathbb{C})^{sm}/S^{an})$:

$$E\Omega_{/(S^{an}/S)} : \mathrm{An}_S^* E_{et}(\Omega^\bullet_{/S}, F_b) \xrightarrow{T(\mathrm{An}_S,E)(\Omega^\bullet_{/S},F_b)} E_{et}(\mathrm{An}_S^*(\Omega^\bullet_{/S}, F_b)) \xrightarrow{E(\Omega_{/(S^{an}/S)})} E_{et}(\Omega^\bullet_{/S^{an}}, F_b)$$

We have the following canonical transformation map given by the analytical functor:

Definition 145. *Let $S \in \mathrm{SmVar}(\mathbb{C})$.*

(i) For $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$, we have the canonical transformation map in $C_{Ofil,\mathcal{D}}(S^{an})$

$$\begin{aligned}
&T(an, \Omega_{/\cdot})(F) : \\
&((e(S)_*\mathcal{H}om^\bullet(F, E_{et}(\Omega^\bullet_{/S}, F)))^{an}) := O_{S^{an}} \otimes_{\mathrm{an}_S^* O_S} \mathrm{an}_S^*(e(S)_*\mathcal{H}om^\bullet(F, E_{et}(\Omega^\bullet_{/S}, F_b))) \\
&\xrightarrow{T(an,e)(-)} O_{S^{an}} \otimes_{\mathrm{an}_S^* O_S} (e(S^{an})_* \mathrm{An}_S^* \mathcal{H}om^\bullet(F, E_{et}(\Omega^\bullet_{/S}, F_b))) \\
&\xrightarrow{T(\mathrm{An},hom)(F,E_{et}(\Omega_{/S},F_b))} O_{S^{an}} \otimes_{\mathrm{an}_S^* O_S} (e(S^{an})_*\mathcal{H}om^\bullet(\mathrm{An}_S^* F, \mathrm{An}_S^* E_{et}(\Omega^\bullet_{/S}, F_b))) \\
&\xrightarrow{\mathcal{H}om(\mathrm{An}_S^* F, E\Omega_{/(S^{an}/S)} \otimes m)} e(S^{an})_*\mathcal{H}om^\bullet(\mathrm{An}_S^* F, E_{et}(\Omega^\bullet_{/S^{an}}, F_b))
\end{aligned}$$

(ii) We get from (i), for $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$, the canonical transformation map in $\mathrm{PSh}_{\mathcal{D}^\infty}(S^{an})$

$$\begin{aligned}
&T^n(an, \Omega_{/\cdot})(F) : J_S H^n((e(S)_*\mathcal{H}om^\bullet(F, E_{et}(\Omega^\bullet_{/S}, F_b)))^{an}) \\
&\xrightarrow{J_S(H^n T(an,\Omega_{/\cdot})(F))} J_S H^n(e(S^{an})_*\mathcal{H}om^\bullet(\mathrm{An}_S^* F, E_{et}(\Omega^\bullet_{/S^{an}}, F_b))) \\
&\xrightarrow{\mathcal{J}_S(-)} H^n e(S^{an})_*\mathcal{H}om^\bullet(\mathrm{An}_S^* F, E_{et}(\Omega^\bullet_{/S^{an}}, F_b))
\end{aligned}$$

Lemma 18. *Let $S \in \mathrm{SmVar}(\mathbb{C})$.*

(i) For $h : U \to S$ a smooth morphism with $U \in \mathrm{SmVar}(\mathbb{C})$, the following diagram commutes

$$\begin{tikzcd}
e(S)_*\mathcal{H}om^\bullet(\mathbb{Z}(U/S), E_{et}(\Omega^\bullet_{/S}, F_b))^{an} \arrow[r, "T(\Omega_{/\cdot},an)(\mathbb{Z}(U/S))"] & e(S^{an})_*\mathcal{H}om^\bullet(\mathbb{Z}(U^{an}/S^{an}), E_{et}(\Omega^\bullet_{/S^{an}}, F_b)) . \\
(h_* E_{zar}(\Omega_{U/S}, F_b))^{an} \arrow[u, "k"] \arrow[r, "T^O_\omega(an,h)"] & h_{an*} E_{usu}(\Omega_{U^{an}/S^{an}}, F_b) \arrow[u, "="']
\end{tikzcd}$$

(ii) For $h : U \to S$ a smooth morphism with $U \in \mathrm{SmVar}(\mathbb{C})$, the following diagram commutes

$$\begin{tikzcd}
J_S H^n((e(S)_*\mathcal{H}om^\bullet(\mathbb{Z}(U/S), E_{et}(\Omega^\bullet_{/S}, F)))^{an}) \arrow[r, "T^n(\Omega_{/\cdot},an)(\mathbb{Z}(U/S))"] & H^n e(S^{an})_*\mathcal{H}om^\bullet(\mathbb{Z}(U^{an}/S^{an}), E_{et}(\Omega^\bullet_{/S^{an}}, F_b)) . \\
J_S H^n((h_* E_{zar}(\Omega_{U/S}, F_b))^{an}) \arrow[u, "k"] \arrow[r, "\mathcal{J}_S(-) \circ J_S T^O_\omega(an,h)(O_U,F)"] & H^n h_{an*} E_{usu}(\Omega_{U^{an}/S^{an}}, F_b) \arrow[u, "="']
\end{tikzcd}$$

Proof. Follows from Yoneda lemma. □

By definition of the algebraic an analytic De Rahm realization functor, we have a natural transformation between them :

Definition 146. *Let $S \in \mathrm{SmVar}(\mathbb{C})$. Let $M \in \mathrm{DA}_c(S)$ and $Q \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ projectively cofibrant such that $M = D(\mathbb{A}^1_S, et)(Q)$. We have the canonical transformation in $D_{Ofil,\mathcal{D}}(S^{an})$*

$$\begin{aligned}
&T(An, \mathcal{F}^{GM}_{an})(M) : (\mathcal{F}^S_{GM}(M))^{an} := (e(S)_*\mathcal{H}om^\bullet(Q, E_{et}(\Omega^\bullet_{/S}, F_b)))^{an}[-d_S] \\
&\xrightarrow{T(an,\Omega_{/\cdot})(Q)} e(S)_*\mathcal{H}om^\bullet(\mathrm{An}_S^* Q, E_{et}(\Omega^\bullet_{/S^{an}}, F_b))[-d_S] \\
&\xrightarrow{=} e(S)_*\mathcal{H}om^\bullet(\mathrm{An}_S^* Q, E_{usu}(\Omega^\bullet_{/S^{an}}, F_b))[-d_S] =: \mathcal{F}^{GM}_{S,an}(M)
\end{aligned}$$

We give now the definition in the non smooth case : Let $S\in\mathrm{Var}(\mathbb{C})$. Let $S=\cup_{i=1}^l S_i$ be an open cover such that there exist closed embeddings $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$. For $I\subset J$, denote by $p_{IJ}:\tilde{S}_J\to\tilde{S}_I$ the projection. Consider, for $I\subset J\subset[1,\ldots,l]$, resp. for each $I\subset[1,\ldots,l]$, the following commutative diagrams in $\mathrm{Var}(\mathbb{C})$

$$D_{IJ}=\begin{tikzcd} S_I \arrow[r, "i_I"] & \tilde{S}_I \\ S_J \arrow[u, "j_{IJ}"] \arrow[r, "i_J"] & \tilde{S}_J \arrow[u, "p_{IJ}"'] \end{tikzcd}.$$

We then have the following lemma

Lemma 19. *The maps $T(an,\Omega.)(L(i_{I*}j_I^*F))$ induce a morphism in $C_{Ofil,\mathcal{D}}(S/(\tilde{S}_I))$*

$$\begin{aligned}(T(an,\Omega_{/.})(L(i_{I*}j_I^*F))):(e(\tilde{S}_I)_*\mathcal{H}om^\bullet(L(i_{I*}j_I^*F),E_{et}(\Omega^\bullet_{/\tilde{S}_I},F_b)))^{an}[-d_{\tilde{S}_I}],(u^q_{IJ}(F))^{an})\\ \to(e(\tilde{S}_I)_*\mathcal{H}om(\mathrm{An}(\tilde{S}_I)^*L(i_{I*}j_I^*F),E_{et}(\Omega^\bullet_{/\tilde{S}_I},F_b))[-d_{\tilde{S}_I}],u^q_{IJ}(F))\end{aligned}$$

Proof. Obvious. □

Definition 147. *Let $S\in\mathrm{Var}(\mathbb{C})$. Let $S=\cup_{i=1}^l S_i$ be an open cover such that there exist closed embeddings $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$. For $I\subset[1,\ldots,l]$, denote $S_I=\cap_{i\in I}S_i$. We have then closed embeddings $i_I:S_I\hookrightarrow\tilde{S}_I=\Pi_{i\in I}\tilde{S}_i$. Let $M\in\mathrm{DA}_c(S)$ and $F\in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $M=D(\mathbb{A}^1_S,et)(F)$. We have, by lemma 19, the canonical transformation in $D_{Ofil,\mathcal{D},\infty}(S^{an})$*

$$\begin{aligned}T(An,\mathcal{F}^{GM})(M):(\mathcal{F}_S^{GM}(M))^{an}:=(e(\tilde{S}_I)_*\mathcal{H}om^\bullet(L(i_{I*}j_I^*F),E_{et}(\Omega^\bullet_{/\tilde{S}_I},F_b)))^{an}[-d_{\tilde{S}_I}],(u^q_{IJ}(F))^{an})\\ \xrightarrow{T(an,\Omega_{/.})(L(i_{I*}j_I^*F)))}(e(\tilde{S}_I)_*\mathcal{H}om(\mathrm{An}(\tilde{S}_I)^*L(i_{I*}j_I^*F),E_{et}(\Omega^\bullet_{/\tilde{S}_I},F))[-d_{\tilde{S}_I}],u^q_{IJ}(F))\\ \xrightarrow{=}(e(\tilde{S}_I)_*\mathcal{H}om(\mathrm{An}(\tilde{S}_I)^*L(i_{I*}j_I^*F),E_{usu}(\Omega^\bullet_{/\tilde{S}_I},F))[-d_{\tilde{S}_I}],u^q_{IJ}(F))=:\mathcal{F}^{GM}_{S,an}(M)\end{aligned}$$

The following proposition says this transformation map between $\mathcal{F}^{S,an}$ and $(\mathcal{F}_S^{FDR})^{an}$ is functorial in $S\in\mathrm{Var}(\mathbb{C})$, hence define a commutative diagram of morphism of 2-functor :

Proposition 140. *(i) Let $g:T\to S$ a morphism with $T,S\in\mathrm{Var}(\mathbb{C})$. Assume there exist a factorization $g:T\xrightarrow{l}Y\times S\xrightarrow{p_S}S$ with, $Y\in\mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S=\cup_{i=1}^l S_i$ be an open cover such that there exist closed embeddings $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$. We then have closed embedding $i_i\circ l:T_i\hookrightarrow Y\times\tilde{S}_i$ and $\tilde{g}_I:=p_{\tilde{S}_I}:Y\times\tilde{S}_I\to\tilde{S}_I$ is a lift of $g_I:=g_{|T_I}:T_I\to S_I$. Then, for $M\in\mathrm{DA}_c(S)$, the following diagram in $D_{Ofil,\mathcal{D},\infty}(T^{an}/(Y^{an}\times\tilde{S}_I^{an}))$ commutes*

$$\begin{tikzcd} Rg^{*mod[-],\Gamma}(\mathcal{F}_S^{GM}(M))^{an} \arrow[r, "Lg^{*mod[-],\Gamma}_{an}(T(An,\mathcal{F}_S^{FDR})(M))"] \arrow[d, "(T(g,\mathcal{F}^{GM})(M))^{an}"'] & Rg^{*mod[-],\Gamma}(\mathcal{F}^{GM}_{S,an}(M)) \arrow[d, "T(g,\mathcal{F}^{GM}_{an})(M)"] \\ (\mathcal{F}_T^{GM}(g^*M))^{an} \arrow[r, "(T(An,\mathcal{F}_T^{GM})(g^*M))"] & (\mathcal{F}^{GM}_{T,an}(g^*M)) \end{tikzcd}$$

(ii) Let $S\in\mathrm{Var}(\mathbb{C})$. Let $S=\cup_{i=1}^l S_i$ be an open cover such that there exist closed embeddings $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$. Then, for $M,N\in\mathrm{DA}_c(S)$, the following diagram in $D_{Ofil,\mathcal{D},\infty}(S^{an}/(\tilde{S}_I^{an}))$

commutes

$$\begin{tikzcd}
(\mathcal{F}_S^{GM}(M) \otimes_{O_S} \mathcal{F}_S^{GM}(N))^{an} \arrow[r, "(T(\otimes,\mathcal{F}^{GM})(M,N))^{an}"] \arrow[d, "="'] & (\mathcal{F}_S^{GM}(M\otimes N))^{an} \arrow[dd, "T(An,\mathcal{F}_S^{GM})(M\otimes N)"] \\
(\mathcal{F}_S^{GM}(M))^{an} \otimes O_{S^{an}} \mathcal{F}_S^{GM}(N)^{an} \arrow[d, "T(An,\mathcal{F}_S^{GM})(M)\otimes_{O_{S^{an}}} T(An,\mathcal{F}_S^{GM})(M)"'] & \\
\mathcal{F}_{S,an}^{GM}(M) \otimes_{O_{S^{an}}} \mathcal{F}_{S,an}^{GM}(N) \arrow[r, "(T(\otimes,\mathcal{F}_{an}^{GM})(M,N))"] & \mathcal{F}_{S,an}^{GM}(M\otimes N)
\end{tikzcd}$$

Proof. Immediate from definition. □

Proposition 141. *Let $f : X \to S$ a morphism with $S, X \in \mathrm{Var}(\mathbb{C})$. Assume there exist a factorization*

$$f : X \xrightarrow{l} Y \times S \xrightarrow{p} S$$

with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p the projection. Let $S = \cup_i S_i$ an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$.

(i) We have then the following commutative diagram in $D_{Ofil,\mathcal{D},\infty}(S^{an}/(\tilde{S}_I^{an}))$,

$$\begin{tikzcd}
(\mathcal{F}_S^{GM}(M(X/S)))^{an} \arrow[r, "T(\mathrm{An},\mathcal{F}_{an}^{FDR})(M(X/S))"] \arrow[d, "I^{GM}(X/S)^{an}"'] & \mathcal{F}_{S,an}^{GM}(M(X/S)) \arrow[d, "I^{GM}(X/S)"] \\
((p_{\tilde{S}_I *}\Gamma_{X_I} E_{zar}(\Omega^{\bullet}_{Y\times\tilde{S}_I/\tilde{S}_I}))^{an}[-d_{\tilde{S}_I}], w_{IJ}(X/S)^{an}) \arrow[r, "(T_\omega^O(an,p_{\tilde{S}_I})^\gamma)", "{,}"'] \arrow[d, "((p_* T_\omega^O(\otimes,\gamma)(-))^{an})"'] & (p_{\tilde{S}_I *}\Gamma_X E_{usu}(\Omega^{\bullet}_{(Y\times\tilde{S}_I)^{an}/\tilde{S}_I^{an}})[-d_{\tilde{S}_I}], w_I \arrow[d, "(p_* T_\omega^O(\otimes,\gamma)(-"] \\
(\int_f^{FDR}(\Gamma_{X_I} E_{zar}(O_{Y\times\tilde{S}_I}, F_b)[-d_Y - d_{\tilde{S}_I}], x_{IJ}(X/S)))^{an} \arrow[r, "(T(\mathrm{an},\gamma)(-))\circ T^{\mathcal{D}mod}(an,f)(-)", "{,}"'] & \int_f^{FDR}(\Gamma_{X_I} E_{usu}(O_{(Y\times\tilde{S}_I)^{an}}, F_b)[-d_Y - d
\end{tikzcd}$$

(ii) We have then the following commutative diagram in $\mathrm{PSh}_{\mathcal{D}^\infty fil}(S^{an}/(\tilde{S}_I^{an}))$,

$$\begin{tikzcd}
J_S H^n(\mathcal{F}_S^{GM}(M^{BM}(X/S)))^{an} \arrow[r, "\mathcal{J}_S(-)\circ H^n T(\mathrm{An},\mathcal{F}_{an}^{FDR})(M(X/S))"] \arrow[d, "H^n(I^{GM}(X/S)^{an})"'] & H^n\mathcal{F}_{S,an}^{GM}(M(X \arrow[d, "H^n I"] \\
J_S H^n((p_{\tilde{S}_I *}\Gamma_{X_I} E_{zar}(\Omega^{\bullet}_{Y\times\tilde{S}_I/\tilde{S}_I}))^{an}[-d_{\tilde{S}_I}], w_{IJ}(X/S)^{an}) \arrow[r, "H^n(T_\omega^O(an,p_{\tilde{S}_I})^\gamma))", "{,}"'] \arrow[d, "H^n((p_* T_\omega^O(\otimes,\gamma)(-))^{an})"'] & H^n(p_{\tilde{S}_I *}\Gamma_X E_{usu}(\Omega^{\bullet}_{(Y\times\tilde{S}_I)^{an}/\tilde{S}} \arrow[d, "H^n("] \\
J_S H^n(\int_f^{FDR}(\Gamma_{X_I} E_{zar}(O_{Y\times\tilde{S}_I}, F_b)[-d_Y - d_{\tilde{S}_I}], x_{IJ}(X/S)))^{an} \arrow[r, "H^n(T(\mathrm{an},\gamma)(-))\circ H^n T^{\mathcal{D}mod}(an,f)(-)", "{,}"'] & \int_f^{FDR}(\Gamma_{X_I} E_{usu}(O_{(Y\times\tilde{S}_I)^{an}}, F_b)[-
\end{tikzcd}$$

Proof. (i):Immediate from definition.
(ii):Follows from (i). □

We deduce from proposition 141 and theorem 23 (GAGA for D-modules) the following :

Theorem 42. *(i) Let $S \in \mathrm{Var}(\mathbb{C})$. Then, for $M \in \mathrm{DA}_c(S)$*

$$\mathcal{J}_S(-) \circ H^n T(\mathrm{An}, \mathcal{F}_{an}^{GM})(M) : J_S(H^n(\mathcal{F}_S^{GM}(M))^{an}) \xrightarrow{\sim} H^n \mathcal{F}_{S,an}^{GM}(M)$$

is an isomorphism in $\mathrm{PSh}_{\mathcal{D}}(S^{an}/(\tilde{S}_I^{an}))$.

(ii) *A relative version of Grothendieck GAGA theorem for De Rham cohomology Let $h : U \to S$ a smooth morphism with $S, U \in \mathrm{SmVar}(\mathbb{C})$. Then,*

$$\mathcal{J}_S(-) \circ J_S T^O_\omega(an, h) : J_S((R^n h_* \Omega^\bullet_{U/S})^{an}) \xrightarrow{\sim} R^n h_* \Omega^\bullet_{U^{an}/S^{an}}$$

is an isomorphism in $\mathrm{PSh}_{\mathcal{D}}(S^{an})$.

Proof. (i):Follows from proposition 141(ii) and theorem 23 using a resolution by Corti-Hanamura motives. (ii):Follows from (i) and lemma 18(ii). $\square$

6.3.2 The transformation map between the analytic filtered De Rham realization functor and the analytification of the filtered algebraic De Rham realization functor

Recall from section 2 that, for $S \in \mathrm{Var}(\mathbb{C})$ we have the following commutative diagrams of sites

$$\begin{tikzcd}
\mathrm{AnSp}(\mathbb{C})^2/S \arrow[rr, "\mu_S"] \arrow[dr, "\rho_S"] \arrow[dd, "\mathrm{An}_S"'] & & \mathrm{AnSp}(\mathbb{C})^{2,pr}/S \arrow[dr, "\rho_S"] \arrow[dd, "\mathrm{An}_S"] & \\
& \mathrm{AnSp}(\mathbb{C})^{2,sm}/S \arrow[rr, "\mu_S"] \arrow[dd, "\mathrm{An}_S"] & & \mathrm{AnSp}(\mathbb{C})^{2,smpr}/S \arrow[dd, "\mathrm{An}_S"] \\
\mathrm{Var}(\mathbb{C})^2/S \arrow[rr, "\mu_S"] \arrow[dr, "\rho_S"] & & \mathrm{Var}(\mathbb{C})^{2,smpr}/S \arrow[dr, "\rho_S"] & \\
& \mathrm{Var}(\mathbb{C})^2/S \arrow[rr, "\mu_S"] & & \mathrm{Var}(\mathbb{C})^{2,smpr}/S
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{AnSp}(\mathbb{C})^{2,pr}/S \arrow[rr, "\mathrm{Gr}^{12}_S"] \arrow[dr, "\rho_S"] \arrow[dd, "\mathrm{An}_S"'] & & \mathrm{AnSp}(\mathbb{C})/S \arrow[dr, "\rho_S"] \arrow[dd, "\mathrm{An}_S"] & \\
& \mathrm{AnSp}(\mathbb{C})^{2,smpr}/S \arrow[rr, "\mathrm{Gr}^{12}_S"] \arrow[dd, "\mathrm{An}_S"] & & \mathrm{AnSp}(\mathbb{C})^{sm}/S \arrow[dd, "\mathrm{An}_S"] \\
\mathrm{Var}(\mathbb{C})^{2,pr}/S \arrow[rr, "\mathrm{Gr}^{12}_S"] \arrow[dr, "\rho_S"] & & \mathrm{Var}(\mathbb{C})/S \arrow[dr, "\rho_S"] & \\
& \mathrm{Var}(\mathbb{C})^{2,sm}/S \arrow[rr, "\mathrm{Gr}^{12}_S"] & & \mathrm{Var}(\mathbb{C})^{sm}/S
\end{tikzcd}, \tag{59}$$

and that for $f : T \to S$ a morphism with $T, S \in \mathrm{Var}(\mathbb{C})$ we have the following commutative diagrams of site,

$$\begin{tikzcd}
\mathrm{AnSp}(\mathbb{C})^2/T^{an} \arrow[rr, "\mathrm{An}_T"] \arrow[dr, "\rho_T"] \arrow[dd, "P(f)"] & & \mathrm{Var}(\mathbb{C})^2/T \arrow[dr, "\rho_T"] \arrow[dd, "P(f)"] & \\
& \mathrm{AnSp}(\mathbb{C})^{2,sm}/T^{an} \arrow[rr, "\mathrm{An}_T"] \arrow[dd, "P(f)"] & & \mathrm{Var}(\mathbb{C})^{2,sm}/T \arrow[dd, "P(f)"] \\
\mathrm{AnSp}(\mathbb{C})^2/S^{an} \arrow[rr, "\mathrm{An}_S"] \arrow[dr, "\rho_S"] & & \mathrm{Var}(\mathbb{C})^2/S \arrow[dr, "\rho_S"] & \\
& \mathrm{AnSp}(\mathbb{C})^{2,sm}/S^{an} \arrow[rr, "\mathrm{An}_S"] & & \mathrm{Var}(\mathbb{C})^{2,sm}/S
\end{tikzcd}.$$

Let $S \in \mathrm{SmVar}(\mathbb{C})$. We have the canonical map in $C(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$

$$\Omega^{\Gamma,pr}_{/(S^{an}/S)} : (\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR}) \to (\Omega^{\bullet,\Gamma,pr}_{/S^{an}}, F_{DR})$$

given by for $p : (Y \times S, Z) \to S$ the projection with $Y \in \mathrm{SmVar}(\mathbb{C})$,

$$\Omega^{\Gamma,pr}_{/(S^{an}/S)}((Y \times S, Z)/S) : \Omega^{\bullet}_{Y\times S/S} \otimes_{O_{Y\times S}} \Gamma^{\vee,Hdg}_{Z}(O_{Y\times S}, F_b)$$
$$\xrightarrow{\Omega_{((Y\times S)^{an}/Y\times S)/(S^{an}/S)}(-)} \Omega^{\bullet}_{(Y\times S)^{an}/S^{an}} \otimes_{O_{(Y\times S)^{an}}} (\Gamma^{\vee,Hdg}_{Z}(O_{Y\times S}, F_b))^{an}$$

We have the following canonical transformation map given by the analytical functor:

Definition 148. *Let $S \in \mathrm{SmVar}(\mathbb{C})$. For $F \in C(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)$, we have the canonical transformation map in $C_{\mathcal{D}^\infty fil}(S^{an})$*

$$T(an, \Omega^{\Gamma,pr}_{/S})(F) :$$
$$(e(S)_*\mathcal{H}om^\bullet(F, E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR})))^{an} := O_{S^{an}} \otimes_{\mathrm{an}_S^* O_S} \mathrm{an}_S^*(e(S)_*\mathcal{H}om^\bullet(F, E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR})))$$
$$\xrightarrow{T(an,e)(-)} O_{S^{an}} \otimes_{\mathrm{an}_S^* O_S} (e(S^{an})_* \mathrm{An}_S^* \mathcal{H}om^\bullet(F, E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR})))$$
$$\xrightarrow{T(\mathrm{An},hom)(-,-)} O_{S^{an}} \otimes_{\mathrm{an}_S^* O_S} (e(S^{an})_*\mathcal{H}om^\bullet(\mathrm{An}_S^* F, \mathrm{An}_S^* E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR})))$$
$$\xrightarrow{\mathcal{H}om(\mathrm{An}_S^* F, \mathrm{An}_S^* E_{et}(\Omega^{\Gamma,pr}_{/(S^{an}/S)})\otimes m)} e(S^{an})_*\mathcal{H}om^\bullet(\mathrm{An}_S^* F, E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/S^{an}}, F_{DR}))$$

By definition of the algebraic an analytic De Rahm realization functor, we have a natural transformation between them :

Definition 149. *Let $S \in \mathrm{SmVar}(\mathbb{C})$. Let $M \in \mathrm{DA}_c(S)$ and $(F, W) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $(M, W) = D(\mathbb{A}^1_S, et)(F, W)$. We have the canonical transformation map $\pi_S(D(MHM(S))) \subset D_{\mathcal{D}(1,0)fil,\infty}(S^{an})$*

$$T(An, \mathcal{F}^{FDR}_{an})(M) : (\mathcal{F}^S_{FDR}(M))^{an} := (e(S)_*\mathcal{H}om^\bullet(L\rho_{S*}\mu_{S*}R^{CH}(\rho_S^* L(F, W)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR})))^{an}$$
$$\xrightarrow{T(an,\Omega^{\Gamma,pr}_{/S})(-)} e(S)_*\mathcal{H}om^\bullet(\mathrm{An}_S^* L\rho_{S*}\mu_{S*}R^{CH}(\rho_S^* L(F, W)), E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/S^{an}}, F_{DR}))$$
$$\xrightarrow{=} e(S)_*\mathcal{H}om^\bullet(\mathrm{An}_S^* L\rho_{S*}\mu_{S*}R^{CH}(\rho_S^* L(F, W)), E_{usu}(\Omega^{\bullet,\Gamma,pr,an}_{/S^{an}}, F_{DR})) =: \mathcal{F}^{FDR}_{S,an}(M)$$

We give now the definition in the non smooth case : Let $S \in \mathrm{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. For $I \subset J$, denote by $p_{IJ} : \tilde{S}_J \to \tilde{S}_I$ the projection. Consider, for $I \subset J \subset [1, \ldots, l]$, resp. for each $I \subset [1, \ldots, l]$, the following commutative diagrams in $\mathrm{Var}(\mathbb{C})$

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ \uparrow{\scriptstyle j_{IJ}} & & \uparrow{\scriptstyle p_{IJ}} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array} .$$

We then have the following lemma

Lemma 20. *The maps $T(an, \Omega.)(-)$ induce a morphism in $C_{\mathcal{D}^\infty(fil)}(S/(\tilde{S}_I))$*

$$(T(an, \Omega^{\Gamma,pr}_{/\tilde{S}_I})(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*}j_I^* F))) :$$
$$((e'(\tilde{S}_I)_*\mathcal{H}om^\bullet(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*}j_I^* F)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR})))^{an}, (u^q_{IJ}(F))^{an})$$
$$\to (e'(\tilde{S}_I)_*\mathcal{H}om^\bullet(\mathrm{An}(\tilde{S}_I)^* L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*}j_I^* F)), E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_I}, F_{DR})), u^q_{IJ}(F))$$

Proof. Obvious. □

Definition 150. *Let $S \in \mathrm{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^{l} S_i$ be an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. For $I \subset [1, \ldots, l]$, denote $S_I = \cap_{i \in I} S_i$. We have then closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I = \Pi_{i \in I} \tilde{S}_i$. Let $M \in \mathrm{DA}_c(S)$ and $(F, W) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $(M, W) = D(\mathbb{A}^1_S, et)(F, W)$. We have, by lemma 20, the canonical transformation map in $\pi_S(D(MHM(S))) \subset D_{\mathcal{D}(1,0)fil,\infty}(S^{an})$*

$$\begin{aligned}
&T(An, \mathcal{F}_S^{FDR})(M) : (\mathcal{F}_S^{FDR}(M))^{an} \xrightarrow{:=} \\
&((e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(L\rho_{\tilde{S}_I *}\mu_{\tilde{S}_I *} R^{CH}(\rho^*_{\tilde{S}_I} L(i_{I*} j_I^*(F, W))), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR})))^{an}, (u^q_{IJ}(F, W))^{an}) \\
&\xrightarrow{(T(an,\Omega^{\Gamma,pr}_{/\tilde{S}_I})(L\rho_{\tilde{S}_I *}\mu_{\tilde{S}_I *} R^{CH}(\rho^*_{\tilde{S}_I} L(i_{I*} j_I^*(F,W)))))} \\
&(e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\mathrm{An}(\tilde{S}_I)^* L\rho_{\tilde{S}_I *}\mu_{\tilde{S}_I *} R^{CH}(\rho^*_{\tilde{S}_I} L(i_{I*} j_I^*(F, W))), E_{et}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_I}, F_{DR})), u^q_{IJ}(F, W)) \\
&\xrightarrow{=} (e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\mathrm{An}(\tilde{S}_I)^* L\rho_{\tilde{S}_I *}\mu_{\tilde{S}_I *} R^{CH}(\rho^*_{\tilde{S}_I} L(i_{I*} j_I^*(F, W))), E_{usu}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_I}, F_{DR})), u^q_{IJ}(F, W)) \\
&\xrightarrow{=:} \mathcal{F}^{FDR}_{S,an}(M)
\end{aligned}$$

The following proposition says this transformation map between $\mathcal{F}^{S,an}$ and $(\mathcal{F}_S^{FDR})^{an}$ is functorial in $S \in \mathrm{Var}(\mathbb{C})$, hence define a commutative diagram of morphism of 2-functor :

Proposition 142. *(i) Let $g : T \to S$ a morphism with $T, S \in \mathrm{Var}(\mathbb{C})$. Assume there exist a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with, $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^{l} S_i$ be an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. We then have closed embedding $i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$ and $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \to \tilde{S}_I$ is a lift of $g_I := g_{|T_I} : T_I \to S_I$.*

(i0) Then, for $M \in \mathrm{DA}_c(S)$, the following diagram in $\pi_T(D(MHM(T^{an})) \subset D_{\mathcal{D}(1,0)fil,\infty}(T^{an}/(Y^{an} \times \tilde{S}_I^{an}))$, see definition 121 and definition 142 commutes

$$\begin{array}{ccc}
g^{\hat{*}mod}_{Hdg}((\mathcal{F}_S^{FDR}(M))^{an}) = (g^{\hat{*}mod}_{Hdg}(\mathcal{F}_S^{FDR}(M)))^{an} & \xrightarrow{g^{\hat{*}mod}_{Hdg}(T(An,\mathcal{F}_S^{FDR})(M))} & g^{\hat{*}mod}_{Hdg}(\mathcal{F}^{FDR}_{S,an}(M)) \\
\downarrow {'(T(g,\mathcal{F}^{FDR})(M))^{an}} & & \downarrow T(g,\mathcal{F}^{FDR}_{an})(M) \\
(\mathcal{F}_T^{FDR}(g^*M))^{an} & \xrightarrow{T(An,\mathcal{F}_T^{FDR})(g^*M)} & \mathcal{F}^{FDR}_{T,an}(g^*M)
\end{array}$$

(i1) Then, for $M \in \mathrm{DA}_c(S)$, the following diagram in $\pi_T(D(MHM(T)) \subset D_{\mathcal{D}(1,0)fil,\infty}(T^{an}/(Y^{an} \times \tilde{S}_I^{an}))$ commutes

$$\begin{array}{ccc}
g^{*mod}_{Hdg}((\mathcal{F}_S^{FDR}(M))^{an}) = (g^{*mod}_{Hdg}(\mathcal{F}_S^{FDR}(M)))^{an} & \xrightarrow{g^{\hat{*}mod}_{Hdg}(T(An,\mathcal{F}_S^{FDR})(M))} & g^{*mod}_{Hdg}(\mathcal{F}^{FDR}_{S,an}(M)) \\
\uparrow {(T(g,\mathcal{F}^{FDR})(M))^{an}} & & \uparrow {'T^!(g,\mathcal{F}^{FDR}_{an})(M)} \\
(\mathcal{F}_T^{FDR}(g^*M))^{an} & \xrightarrow{T(An,\mathcal{F}_T^{FDR})(g^*M)} & \mathcal{F}^{FDR}_{T,an}(g^*M)
\end{array}$$

(i2) Then, for $M \in \mathrm{DA}_c(T)$, the following diagram in $\pi_S(D(MHM(S^{an})) \subset D_{\mathcal{D}(1,0)fil,\infty}(S^{an}/(\tilde{S}_I^{an}))$ commutes

$$\begin{array}{ccc}
Rg^{Hdg}_*((\mathcal{F}_T^{FDR}(M))^{an}) = (Rg^{Hdg} * (\mathcal{F}_T^{FDR}(M)))^{an} & \xrightarrow{Rg^{Hdg}_*(T(An,\mathcal{F}_S^{FDR})(M))} & Rg^{Hdg}_*(\mathcal{F}^{FDR}_{S,an}(M)) \\
\uparrow {(T_*(g,\mathcal{F}^{FDR})(M))^{an}} & & \uparrow {'T_*(g,\mathcal{F}^{FDR}_{an})(M)} \\
(\mathcal{F}_S^{FDR}(Rg_*M))^{an} & \xrightarrow{T(An,\mathcal{F}_S^{FDR})(Rg_*M)} & \mathcal{F}^{FDR}_{S,an}(Rg_*M)
\end{array}$$

(i3) *Then, for* $M\in\mathrm{DA}_c(T)$*, the following diagram in* $\pi_S(D(MHM(S^{an}))\subset D_{\mathcal{D}(1,0)fil,\infty}(S^{an}/(\tilde{S}_I^{an}))$ *commutes*

$$\begin{tikzcd} Rg_!^{Hdg}((\mathcal{F}_T^{FDR}(M))^{an})=(Rg^{Hdg}!(\mathcal{F}_T^{FDR}(M)))^{an} \arrow[r, "Rg_!^{Hdg}(T(An,\mathcal{F}_S^{FDR})(M))"] \arrow[d, "'(T_!(g,\mathcal{F}^{FDR})(M))^{an}"'] & Rg_!^{Hdg}(\mathcal{F}_{S,an}^{FDR}(M)) \arrow[d, "T_!(g,\mathcal{F}_{an}^{FDR})(M)"] \\ (\mathcal{F}_S^{FDR}(Rg_!M))^{an} \arrow[r, "T(An,\mathcal{F}_S^{FDR})(Rg_!M)"] & \mathcal{F}_{S,an}^{FDR}(Rg_!M) \end{tikzcd}$$

(ii) *Let* $S\in\mathrm{Var}(\mathbb{C})$*. Let* $S=\cup_{i=1}^l S_i$ *be an open cover such that there exist closed embeddings* $i_i:S_i\hookrightarrow\tilde{S}_i$ *with* $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$*. Then, for* $M,N\in\mathrm{DA}_c(S)$*, the following diagram in* $\pi_S(D(MHM(S)))\subset D_{\mathcal{D}(1,0)fil,\infty}(S^{an}/(\tilde{S}_I^{an}))$ *commutes*

$$\begin{tikzcd} (\mathcal{F}_S^{FDR}(M)\otimes_{O_S}\mathcal{F}_S^{FDR}(N))^{an} \arrow[r, "(T(\otimes,\mathcal{F}^{FDR})(M,N))^{an}"] \arrow[d, "="'] & (\mathcal{F}_S^{FDR}(M\otimes N))^{an} \arrow[dd, "T(An,\mathcal{F}_S^{FDR})(M\otimes N)"] \\ (\mathcal{F}_S^{FDR}(M))^{an}\otimes O_{S^{an}}(\mathcal{F}_S^{FDR}(N))^{an}) \arrow[d, "T(An,\mathcal{F}_S^{FDR})(M)\otimes_{O_{S^{an}}}T(An,\mathcal{F}_S^{FDR})(M)"'] & \\ \mathcal{F}_{S,an}^{FDR}(M)\otimes_{O_{S^{an}}}\mathcal{F}_{S,an}^{FDR}(N) \arrow[r, "(T(\otimes,\mathcal{F}_{an}^{FDR})(M,N))"] & \mathcal{F}_{S,an}^{FDR}(M\otimes N) \end{tikzcd}$$

Proof. Immediate from definition. ☐

Proposition 143. *Let* $f:X\to S$ *a morphism with* $S,X\in\mathrm{Var}(\mathbb{C})$*. Assume there exist a factorization*

$$f:X\xrightarrow{l}Y\times S\xrightarrow{p}S$$

with $Y\in\mathrm{SmVar}(\mathbb{C})$*,* l *a closed embedding and* p *the projection. Let* $S=\cup_i S_i$ *an open affine cover and* $i_i:S_i\hookrightarrow\tilde{S}_i$ *closed embeddings with* $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$*. We have then the following commutative diagram in* $D_{\mathcal{D}^\infty fil,\infty}(S^{an})$*,*

$$\begin{tikzcd} (\mathcal{F}_S^{FDR}(M(X/S)))^{an} \arrow[r, "T(\mathrm{An},\mathcal{F}_S^{FDR})(M(X/S))"] \arrow[d, "I(X/S)"] & \mathcal{F}_{S,an}^{FDR}(M(X/S)) \arrow[d, "I(X/S)"] \\ (Rf_!^{Hdg}(\Gamma_{X_I}^{\vee,Hdg}(O_{Y\times\tilde{S}_I},F_b)(d_Y)[2d_Y],x_{IJ}(X/S)))^{an} \arrow[r, "T^{\mathcal{D}mod}(an,p_{\tilde{S}_I})(-)"] & Rf_!^{Hdg}((\Gamma_{X_I}^{\vee,Hdg}(O_{(Y\times\tilde{S}_I)^{an}},F_b)(d_Y)[2d_Y],x_{IJ}(X/S))) \end{tikzcd}.$$

Proof. Immediate from definition. ☐

We deduce from proposition 143 and theorem 23 (GAGA for D-modules) the following :

Theorem 43. *Let* $S\in\mathrm{Var}(\mathbb{C})$*. For* $M\in\mathrm{DA}_c(S)$*, the map in* $\pi_S(D(MHM(S^{an})))\subset D_{\mathcal{D}(1,0)fil,\infty}(S^{an})$

$$T(An,\mathcal{F}^{FDR})(M):(\mathcal{F}_S^{FDR}(M))^{an}\xrightarrow{\sim}\mathcal{F}_{S,an}^{FDR}(M)$$

given in definition 150 is an isomorphism.

Proof. Follows from proposition 143 and theorem 23. ☐

We finish this subsection by the following easy proposition :

Proposition 144. *Let $S \in \mathrm{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^{l} S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$ Let $M \in \mathrm{DA}_c(S)$ and $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1_S, et)(F)$. Then the following diagram in $D_{Ofil,\mathcal{D},\infty}(S/\tilde{S}_I)$ commutes*

$$\begin{tikzcd}
(F_S^{GM}(L\mathbb{D}_S M))^{an} \arrow[r, "\mathcal{J}\circ(T(\mathcal{F}_S^{GM},\mathcal{F}_S^{FDR})(M))^{an}"] \arrow[d, "T(\mathrm{An},\mathcal{F}_S^{GM})(\mathbb{D}_S M)"'] & J_S((\mathcal{F}_S^{FDR}(M))^{an}) \arrow[d, "J_S(T(\mathrm{An},\mathcal{F}_S^{FDR})(M))"] \\
\mathcal{F}_{S,an}^{GM}(L\mathbb{D}_S M) \arrow[r, "T(\mathcal{F}_{S,an}^{GM},\mathcal{F}_{S,an}^{FDR})(M)"] & J_S(\mathcal{F}_{S,an}^{FDR}(M))
\end{tikzcd}$$

Proof. Immediate from definition. □

7 The Hodge realization functor for relative motives

7.1 The Betti realization functor

We have two definition of the Betti realization functor which coincide at least for constructible motives, one given by [1] using the analytical functor and one given in [7] by composing the analytical functor with the forgetfull functor to the topological space of a complex analytic space wich is a CW complex (see also [20] for the absolute case) .

Definition 151. *Let $S \in \mathrm{Var}(\mathbb{C})$.*

(i) The Ayoub's Betti realization functor is

$$\mathrm{Bti}_S^* : \mathrm{DA}(S) \to D(S^{an}) \ , \ M \in \mathrm{DA}(S) \mapsto \mathrm{Bti}_S^* M = Re(S^{an})_* \mathrm{An}_S^* M = e(S^{an})_* \underline{\mathrm{sing}}_{\mathbb{D}^*} \mathrm{An}_S^* F$$

where $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$.

(ii) In [7], we define the Betti realization functor as

$$\widetilde{\mathrm{Bti}}_S^* : \mathrm{DA}(S) \to D(S^{an}) = D(S^{cw}) \ , \ M \mapsto \widetilde{\mathrm{Bti}}_S^* M = Re(S^{cw})_* \widetilde{\mathrm{Cw}}_S^* M = e(S^{cw})_* \underline{\mathrm{sing}}_{\mathbb{I}^*} \widetilde{\mathrm{Cw}}_S^* F$$

where $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$.

(iii) For the Corti-Hanamura weight structure on $\mathrm{DA}^-(S)$, we have by functoriality of (i) the functor

$$\mathrm{Bti}_S^* : \mathrm{DA}^-(S) \to D_{fil,\infty}(S^{an}) \ , \ M \mapsto \mathrm{Bti}_S^* M = e(S^{an})_* \underline{\mathrm{sing}}_{\mathbb{D}^*} \mathrm{An}_S^*(F, W)$$

where $(F, W) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is such that $(M, W) = D(\mathbb{A}^1, et)(F, W)$.

Note that by [7], An_S^* and $\widetilde{\mathrm{Cw}}_S^*$ derive trivially.

Note that, by considering the explicit $\mathbb{D}^1_S$ local model for presheaves on $\mathrm{AnSp}(\mathbb{C})^{sm}/S^{an}$, $\mathrm{Bti}_S^*(\mathrm{DA}^-(S)) \subset D^-(S^{an})$; by considering the explicit $\mathbb{I}^1_S$ local model for presheaves on CW^{sm}/S^{cw}, $\widetilde{\mathrm{Bti}}_S^*(\mathrm{DA}^-(S)) \subset D^-(S^{an})$.

Definition 152. *Let $f : T \to S$ a morphism, with $T, S \in \mathrm{Var}(\mathbb{C})$. We have, for $M \in \mathrm{DA}(S)$, $(F, W) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $(M, W) = D(\mathbb{A}^1, et)(F, W)$, and an equivalence $(\mathbb{A}^1, et)$ local $e : f^*(F, W) \to (F', W)$ with $(F', W) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $(f^*M, W) = D(\mathbb{A}^1, et)(F', W)$ the following canonical transformation map in $D_{fil}(T)$:*

$$T(f, \mathrm{Bti})(M) : f^* \mathrm{Bti}_S^* M := f^* e(S^{an})_* \underline{\mathrm{sing}}_{\mathbb{D}^*} \mathrm{An}_S^*(F, W) \xrightarrow{T(f,e)(-)} e(T^{an})_* f^* \underline{\mathrm{sing}}_{\mathbb{D}^*} \mathrm{An}_S^*(F, W) \quad (60)$$

$$\xrightarrow{e(T^{an})_* T(f,c)(F,W)} e(T^{an})_* \underline{\mathrm{sing}}_{\mathbb{D}^*} f^* \mathrm{An}_T^*(F, W) \xrightarrow{=} e(T^{an})_* \underline{\mathrm{sing}}_{\mathbb{D}^*} \mathrm{An}_T^* f^*(F, W) \quad (61)$$

$$\xrightarrow{e(T^{an})_* \underline{\mathrm{sing}}_{\mathbb{D}^*} \mathrm{An}_T^* e} e(T^{an})_* \underline{\mathrm{sing}}_{\mathbb{D}^*} \mathrm{An}_T^*(F', W) =: \mathrm{Bti}_T^* f^* M \quad (62)$$

Theorem 44. *Let $f : X \to S$ a morphism, with $X, S \in \mathrm{Var}(\mathbb{C})$. For $M \in \mathrm{DA}_c(S)$,*

$$T(f, \mathrm{Bti})(M) : f^* \mathrm{Bti}_S^*(M) \xrightarrow{\sim} \mathrm{Bti}_X^* f^*(M)$$

is an isomorphism in $D_{fil}(X)$.

Proof. See [1]. □

Definition 153. • *Let $f : X \to S$ a morphism, with $X, S \in \mathrm{Var}(\mathbb{C})$. We have, for $M \in \mathrm{DA}_c(X)$, the following transformation map*

$$T_*(f, \mathrm{Bti})(M) : \mathrm{Bti}_S^*(Rf_*M) \xrightarrow{\mathrm{ad}(f^*, Rf_*)(\mathrm{Bti}_S^*(f_*M))} Rf_*f^* \mathrm{Bti}_S^*(Rf_*M)$$
$$\xrightarrow{T(f,\mathrm{Bti})(f_*M)} Rf_* \mathrm{Bti}_X^*(f^*Rf_*M) \xrightarrow{\mathrm{Bti}_X^*(\mathrm{ad}(f^*,Rf_*)(M))} Rf_* \mathrm{Bti}_X^*(M)$$

Clearly, if $l : Z \hookrightarrow S$ is a closed embedding, then $T_*(l, \mathrm{Bti})(M)$ is an isomorphism by theorem 44.

- *Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have then, for $M \in \mathrm{DA}_c(X)$, using theorem 44 for closed embeddings, the following transformation map*

$$T_!(f, \mathcal{F}_{an}^{FDR})(M) : Rf_! \mathrm{Bti}_X^*(M) = Rp_{S!}l_* \mathrm{Bti}_X^*(M) \xrightarrow{T_*(l,\mathrm{Bti})(M)^{-1}} Rp_{S!} \mathrm{Bti}(Y \times S)^*(l_*M)$$
$$\xrightarrow{\mathrm{Bti}(Y\times S)^* \mathrm{ad}(Lp_{S\sharp}, p_S^*)(l_*M)} Rp_{S!} \mathrm{Bti}(Y \times S)^*(p_S^* Lp_{S\sharp}l_*M) \xrightarrow{T(p_S,\mathrm{Bti})(p_{S\sharp}l_*M)}$$
$$Rp_{S!}p_S^* \mathrm{Bti}(Y \times S)^*(Lp_{S\sharp}l_*M) = Rp_{S!}p_S^! \mathrm{Bti}(Y \times S)^*(f_!M) \xrightarrow{\mathrm{ad}(Rp_{S!},p_S^!)(-)} \mathrm{Bti}(Y \times S)^*(f_!M)$$

Clearly, for $f : X \to S$ a proper morphism, with $X, S \in \mathrm{Var}(\mathbb{C})$ we have, for $M \in \mathrm{DA}_c(Y \times S)$, $T_!(f, \mathrm{Bti})(M) = T_*(f, \mathrm{Bti})(M)$.

- *Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$. We have, using the second point, for $M \in \mathrm{DA}(S)$, the following transformation map*

$$T^!(f, \mathrm{Bti})(M) : \mathrm{Bti}_X^*(f^!M) \xrightarrow{\mathrm{ad}(f_!,Rf^!)(\mathrm{Bti}_X^*(f^!M))} f^!Rf_! \mathrm{Bti}_X^*(f^!M)$$
$$\xrightarrow{T_!(f,\mathrm{Bti})((f^!M))} f^! \mathrm{Bti}_S^*(f_!f^!M) \xrightarrow{\mathrm{Bti}_S^*(\mathrm{ad}(f_!,f^!)(M))} f^! \mathrm{Bti}_S^*(M)$$

Definition 154. *Let $S \in \mathrm{Var}(\mathbb{C})$. We have, for $M, N \in \mathrm{DA}(S)$ and $F, G \in C_(\mathrm{Var}(\mathbb{C})^{sm}/S))$ such that $M = D(\mathbb{A}^1, et)(F)$ and $N = D(\mathbb{A}^1, et)(G)$, the following transformation map in $D_{\mathcal{D}fil}(S)$*

$$\mathrm{Bti}_S^* M \otimes \mathrm{Bti}_S^* N := (e(S)_*\underline{\mathrm{sing}}_{\mathbb{D}^*} \mathrm{An}_S^* F) \otimes (e(S)_*\underline{\mathrm{sing}}_{\mathbb{D}^*} \mathrm{An}_S^* G)$$
$$\xrightarrow{T(\mathrm{sing}_{D^*},\otimes)(\mathrm{An}_S^* F,\mathrm{An}_S^* G)} e(S)_*\underline{\mathrm{sing}}_{\mathbb{D}^*} \mathrm{An}_S^*(F \otimes G) =: \mathrm{Bti}_S^*(M \otimes N)$$

Theorem 45. *(i) Let $f : X \to S$ a morphism, with $X, S \in \mathrm{Var}(\mathbb{C})$. For $M \in \mathrm{DA}_c(X)$,*

$$T_!(f, \mathrm{Bti})(M, W) : f_! \mathrm{Bti}_X^*(M) \xrightarrow{\sim} \mathrm{Bti}_S^* f_!M$$

is an isomorphism.

(ii) Let $f : X \to S$ a morphism, with $X, S \in \mathrm{Var}(\mathbb{C})$. For $M \in \mathrm{DA}_c(X)$,

$$T_*(f, \mathrm{Bti})(M, W) : f_* \mathrm{Bti}_X^* M \xrightarrow{\sim} \mathrm{Bti}_S^* Rf_*M$$

is an isomorphism.

(iii) *Let $f : X \to S$ a morphism, with $X, S \in \mathrm{Var}(\mathbb{C})$. For $M \in \mathrm{DA}_c(S)$,*

$$T^!(f, \mathrm{Bti})(M) : f^! \,\mathrm{Bti}_S^* M \xrightarrow{\sim} \mathrm{Bti}_X^* f^! M$$

is an isomorphism.

(iv) *Let $S \in \mathrm{Var}(\mathbb{C})$. For $M, N \in \mathrm{DA}_c(S)$,*

$$T(\otimes, \mathrm{Bti})(M) : \mathrm{Bti}_S^* M \otimes \mathrm{Bti}_S^* N \xrightarrow{\sim} \mathrm{Bti}_X^*(M \otimes N)$$

is an isomorphism.

Proof. See [1]. □

The main result on the Betti realization functor is the following

Theorem 46. (i) *We have $\mathrm{Bti}_S^* = \widetilde{\mathrm{Bti}}_S^*$ on $\mathrm{DA}^-(S)$*

(ii) *The canonical transformations $T(f, \mathrm{Bti})$, for $f : T \to S$ a morphism in $\mathrm{Var}(\mathbb{C})$, define a morphism of 2 functor*

$$\mathrm{Bti}_\cdot^* : \mathrm{DA}(\cdot) \to D(\cdot^{an}), \; S \in \mathrm{Var}(\mathbb{C}) \mapsto \mathrm{Bti}_S^* : \mathrm{DA}(S) \to D(S^{an})$$

which is a morphism of homotopic 2 functor.

Proof. (i): See [7]
(ii):Follows from theorem 44 and theorem 45. □

Remark 13. *For $X \in \mathrm{Var}(\mathbb{C})$, the quasi-isomorphisms*

$$\mathbb{Z}\,\mathrm{Hom}(\bar{\mathbb{D}}_{et}^\bullet, X) \xrightarrow{\mathrm{An}^*} \mathbb{Z}\,\mathrm{Hom}(\bar{\mathbb{D}}^n(0,1), X^{an}) \xrightarrow{\mathrm{Hom}(i, X^{cw})} \mathbb{Z}\,\mathrm{Hom}([0,1]^n, X^{cw}),$$

where,

$$\bar{\mathbb{D}}_{et}^n := (e : U \to \mathbb{A}^n, \bar{\mathbb{D}}^n(0,1) \subset e(U)) \in \mathrm{Fun}(\mathcal{V}_{\mathbb{A}^n}^{et}(\bar{\mathbb{D}}^n(0,1)), \mathrm{Var}(\mathbb{C}))$$

is the system of etale neighborhood of the closed ball $\bar{\mathbb{D}}^n(0,1) \subset \mathbb{A}^n$, and $i : [0,1]^n \hookrightarrow \bar{\mathbb{D}}^n(0,1)$ is the closed embedding, shows that a closed singular chain $\alpha \in \mathbb{Z}\,\mathrm{Hom}^n([0,1]^n, X^{cw})$, is homologue to a closed singular chain

$$\beta = \alpha + \partial\gamma = \tilde{\beta}_{|[0,1]^n} \in \mathbb{Z}\,\mathrm{Hom}^n(\Delta^n, X^{cw})$$

which is the restriction by the closed embedding $[0,1]^n \hookrightarrow U^{cw} \xrightarrow{e} \mathbb{A}^n$, where $e : U \to \mathbb{A}^n$ an etale morphism with $U \in \mathrm{Var}(\mathbb{C})$, of a complex algebraic morphism $\tilde{\beta} : U \to X$. Hence $\beta([0,1]^n) = \tilde{\beta}([0,1]^n) \subset X$ is the restriction of a real algebraic subset of dimension n in $Res_{\mathbb{R}}(X)$ (after restriction a scalar that is under the identification $\mathbb{C} \simeq \mathbb{R}^2$).

Definition 155. *Let $S \in \mathrm{Var}(\mathbb{C})$ The cohomological Betti realization functor is*

$$\mathrm{Bti}_S^\vee : \mathrm{DA}(S) \to D(S^{cw}),$$
$$M \mapsto \mathrm{Bti}_S^\vee(M) := R\mathcal{H}om(\mathrm{Bti}_S^* M, \mathbb{Z}_{S^{cw}}) = R\mathcal{H}om(M, \mathrm{Bti}_{S*}\mathbb{Z}_{S^{cw}})$$

where for $\mathrm{Bti}_{S*} : K \in D(S^{cw}) \mapsto R\,\mathrm{An}_{S*}\, e(S^{an})^* K \in \mathrm{DA}(S)$ is the right ajoint to Bti_S^*.

7.2 The Hodge realization functor for relative motives

Recall (see section 2) that for $S\in\mathrm{Var}(\mathbb{C})$, we consider the dual functor

$$\mathbb{D}_S:C(\mathrm{Var}(\mathbb{C})^{sm}/S)\to C(\mathrm{Var}(\mathbb{C})^{sm}/S),F\mapsto\mathbb{D}_SF:=\mathcal{H}om(F,E_{et}(\mathbb{Z}_S)).$$

Similarly, for $S\in\mathrm{AnSp}(\mathbb{C})$, we consider the dual functor

$$\mathbb{D}_S:C(\mathrm{AnSp}(\mathbb{C})^{sm}/S)\to C(\mathrm{AnSp}(\mathbb{C})^{sm}/S),F\mapsto\mathbb{D}_SF:=\mathcal{H}om(F,E_{usu}(\mathbb{Z}_S))$$

The filtered De Rham algebraic realization functor constructed in section 6 and on the other hand the Betti realization functor (see section 7.1) give the Hodge realization functor :

Definition 156. *Let $S\in\mathrm{SmVar}(\mathbb{C})$. We define the Hodge realization functor as*

$$\begin{aligned}\mathcal{F}_S^{Hdg}:=(\mathcal{F}_S^{FDR},\mathrm{Bti}_S^*):C(\mathrm{Var}(\mathbb{C})^{sm}/S)\to C_{\mathcal{D}(1,0)fil}(S)\times_I C(S^{an}),\\ F\mapsto\mathcal{F}_S^{Hdg}(F):=(\mathcal{F}_S^{FDR}(F),\mathrm{Bti}_S^*F,\alpha(F)),\end{aligned}$$

inducing by the results of section 6

$$\begin{aligned}\mathcal{F}_S^{Hdg}:=(\mathcal{F}_S^{FDR},\mathrm{Bti}_S^*):\mathrm{DA}_c(S)\to D_{\mathcal{D}(1,0)fil,\infty}(S)\times_I D_{fil}(S^{an}),\\ M\mapsto\mathcal{F}_S^{Hdg}(M):=(\mathcal{F}_S^{FDR}(M),\mathrm{Bti}_S^*M,\alpha(M)),\end{aligned}$$

where $\alpha(M)=\alpha(F)$ is the map in $D_{\mathbb{C}fil}(S^{an})$

$$\begin{aligned}&\alpha(M):(\mathrm{Bti}_S^*M)\otimes\mathbb{C}_S:=(e(S)_*\underline{\mathrm{sing}}_{\mathbb{D}^*}\mathrm{An}_S^*L(F,W))\otimes\mathbb{C}_S\\
&\xrightarrow{s(e(S)_*\underline{\mathrm{sing}}_{\mathbb{D}^*}\mathrm{An}_S^*L(F,W))}DR(S)^{[-]}(\mathcal{H}om(L\mathbb{D}_SL(e(S)_*\underline{\mathrm{sing}}_{\mathbb{D}^*}\mathrm{An}_S^*L(F,W)),E_{usu}(O_S)))\\
&\xrightarrow{=}DR(S)^{[-]}(\mathcal{H}om(L\mathbb{D}_SL(e(S)_*\underline{\mathrm{sing}}_{\mathbb{D}^*}\mathrm{An}_S^*L(F,W)),e(S)_*E_{usu}(\Omega^\bullet_{/S})))\\
&\xrightarrow{DR(S)^{[-]}(\mathcal{H}om(T(e,hom)(-,E(\mathbb{Z}))\circ q,-))}DR(S)^{[-]}(\mathcal{H}om(Le(S)_*L\mathbb{D}_SL\underline{\mathrm{sing}}_{\mathbb{D}^*}\mathrm{An}_S^*L(F,W)),e(S)_*E_{usu}(\Omega^\bullet_{/S})))\\
&\xrightarrow{DR(S)^{[-]}(\mathcal{H}om(Le(S)_*\mathbb{D}_Sc(\mathrm{An}_S^*L(F,W)),e(S)_*E_{usu}(\Omega^\bullet_{/S})))}\\
&DR(S)^{[-]}(\mathcal{H}om(Le(S)_*L\mathbb{D}_S\mathrm{An}_S^*L(F,W)),e(S)_*E_{usu}(\Omega^\bullet_{/S})))\\
&\xrightarrow{DR(S)^{[-]}(q\circ\mathcal{H}om(\mathcal{H}om(T(\mathrm{An},hom)(-,E_{et}(\mathbb{Z})),e(S)_*E_{usu}(\Omega^\bullet_{/S}))))}\\
&DR(S)^{[-]}(\mathcal{H}om(Le(S)_*\mathrm{An}_S^*L\mathbb{D}_SL(F,W),e(S)_*E_{usu}(\Omega^\bullet_{/S})))\\
&\xrightarrow{DR(S)^{[-]}(\mathcal{H}om(-,e(S)_*E_{usu}(\mathrm{Gr}(\Omega_{/S^{an}})))^{-1})}\\
&DR(S)^{[-]}(\mathcal{H}om(Le(S)_*\mathrm{An}_S^*L\mathbb{D}_SL(F,W)),e(S)_*\mathrm{Gr}_{S*}^{12}E_{usu}(\Omega^{\bullet,\Gamma,pr,an}_{/S^{an}})))\\
&\xrightarrow{DR(S)^{[-]}((\mathcal{H}om(q,-)\circ T(e,hom)(-,-))^{-1})}DR(S)^{[-]}(e(S)_*\mathcal{H}om(\mathrm{An}_S^*L\mathbb{D}_SL(F,W)),\mathrm{Gr}_{S*}^{12}E_{usu}(\Omega^{\bullet,\Gamma,pr,an}_{/S^{an}})))\\
&\xrightarrow{=}DR(S)^{[-]}(e(S)_*\mathcal{H}om(\mathrm{An}_S^*L\mathbb{D}_S^0(L(F,W)),\mathrm{Gr}_{S*}^{12}E_{usu}(\Omega^{\bullet,\Gamma,pr,an}_{/S^{an}})))\\
&\xrightarrow{DR(S)^{[-]}(\mathcal{H}om(\mathrm{An}_S^*LT_S^{CH}(L(F,W)),E_{usu}(\Omega^{\bullet,\Gamma,pr,an}_{/S^{an}})))}\\
&DR(S)^{[-]}(e'(S)_*\mathcal{H}om(\mathrm{An}_S^*L\,\mathrm{Gr}_{S*}^{12}L\rho_{S*}\mu_{S*}R^{CH}(\rho_S^*L(F,W)),\mathrm{Gr}_{S*}^{12}E_{usu}(\Omega^{\bullet,\Gamma,pr,an}_{/S^{an}})))\\
&\xrightarrow{DR(S)^{[-]}(I(\mathrm{Gr}_S^{12*},\mathrm{Gr}_{S*}^{12})(-,-)\circ I(\mathrm{An}_S^*,\mathrm{An}_{S*})(-,-))}\\
&DR(S)^{[-]}(e'(S)_*\mathcal{H}om(\mathrm{Gr}_S^{12*}L\,\mathrm{Gr}_{S*}^{12}L\rho_{S*}\mu_{S*}R^{CH}(\rho_S^*L(F,W)),\mathrm{An}_{S*}E_{usu}(\Omega^{\bullet,\Gamma,pr,an}_{/S^{an}})))\\
&\xrightarrow{DR(S)^{[-]}(\mathcal{H}om(\mathrm{ad}(\mathrm{Gr}_S^{12*},\mathrm{Gr}_{S*}^{12})(-)\circ q,-)^{-1}}\\
&DR(S)^{[-]}(e'(S)_*\mathcal{H}om(L\rho_{S*}\mu_{S*}R^{CH}(\rho_S^*L(F,W)),\mathrm{An}_{S*}E_{usu}(\Omega^{\bullet,\Gamma,pr,an}_{/S^{an}})))\xrightarrow{=:}DR(S)^{[-]}(\mathcal{F}_{S,an}^{DR}(M,W))\\
&\xrightarrow{DR(S)^{[-]}(T(An,\mathcal{F}_S^{FDR})(M,W)^{-1})}DR(S)^{[-]}((\mathcal{F}_S^{DR}(M,W))^{an})\end{aligned}$$

where $(F,W)\in C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ *is such that* $(M,W)=D(\mathbb{A}^1,et)(F,W)$

- $s(K):K\to DR(S)^{[-]}(\mathcal{H}om(\mathbb{D}_SLK,E(O_S))$ *is the isomorphism of theorem 26,*
- *the map*

$$\begin{aligned}\mathrm{Hom}(q,-)\circ T(e,hom)(-,-):e(S)_*\mathcal{H}om(L\,\mathrm{An}_S^*\,\mathbb{D}_SL(F,W)),\mathrm{Gr}_{S*}^{12}\,E_{usu}(\Omega^{\bullet,\Gamma,pr,an}_{/S^{an}}))) \\ \to\mathcal{H}om(Le(S)_*\,\mathrm{An}_S^*\,\mathbb{D}_SL(F,W)),e(S)_*\,\mathrm{Gr}_{S*}^{12}\,E_{usu}(\Omega^{\bullet,\Gamma,pr,an}_{/S^{an}})))\end{aligned}$$

  is a filtered equivalence usu local by proposition 39 and proposition 130,
- $T_S^{CH}(L(F,W)):\mathrm{Gr}_{S*}^{12}\,L\rho_{S*}\mu_{S*}R^{CH}(\rho_S^*L(F,W))\to\mathbb{D}_S^0(L(F,W))$ *is given in definition 36,*
- $\mathrm{ad}(\mathrm{Gr}_S^{12*},\mathrm{Gr}_{S*}^{12})(L\rho_{S*}\mu_{S*}R^{CH}(\rho_S^*L(F,W)))\circ q$ *is an equivalence* $(\mathbb{A}^1,et)$ *local by lemma 1,*
- $\mathcal{F}_S^{DR}(M):=o_{fil}\mathcal{F}_S^{FDR}(M)\in D_{\mathcal{D}0fil}(S)$, $\mathcal{F}_{S,an}^{DR}(M):=o_{fil}\mathcal{F}_{S,an}^{FDR}(M)\in D_{\mathcal{D}0fil}(S^{an})$,
- $T(An,\mathcal{F}_S^{FDR})(M):(\mathcal{F}_S^{DR}(M))^{an}\xrightarrow{\sim}\mathcal{F}_{S,an}^{DR}(M)$ *is an isomorphism by theorem 43.*

We now give the definition in the non smooth case :

Definition 157. *Let* $S\in\mathrm{Var}(\mathbb{C})$. *Let* $S=\cup_iS_i$ *an open cover such that there exists closed embedding* $i_i:S\hookrightarrow\tilde{S}_i$ *with* $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$. *We define the Hodge realization functor as*

$$\begin{aligned}\mathcal{F}_S^{Hdg}:=(\mathcal{F}_S^{FDR},\mathrm{Bti}_S^*):C(\mathrm{Var}(\mathbb{C})^{sm}/S)\to C_{\mathcal{D}fil}(S/(\tilde{S}_I))\times_I C(S^{an}), \\ F\mapsto\mathcal{F}_S^{Hdg}(F):=(\mathcal{F}_S^{FDR}(F),\mathrm{Bti}_S^*\,F,\alpha(F)),\end{aligned}$$

inducing by the results of section 6

$$\begin{aligned}\mathcal{F}_S^{Hdg}:=(\mathcal{F}_S^{FDR},\mathrm{Bti}_S^*):\mathrm{DA}(S)\to D_{\mathcal{D}(1,0)fil,\infty}(S/(\tilde{S}_I))\times_I D_{fil}(S^{an}), \\ M\mapsto\mathcal{F}_S^{Hdg}(M):=(\mathcal{F}_S^{FDR}(M),\mathrm{Bti}_S^*\,M,\alpha(M)),\end{aligned}$$

where $\alpha(M)=\alpha(F)$ *is the map in* $D_{\mathbb{C}fil}(S^{an}/(\tilde{S}_I^{an}))$

$$\alpha(M):T(S/(\tilde{S}_I))((\mathrm{Bti}_S^*(M,W))\otimes\mathbb{C}_S):=(i_{I*}j_I^*((e(S)_*\underline{\mathrm{sing}}_{\mathbb{D}^*}\mathrm{An}_S^*L(F,W))\otimes C_S),,T^q(D_{IJ})(-))$$
$$\xrightarrow{s(e(S)_*\underline{\mathrm{sing}}_{\mathbb{D}^*}\mathrm{An}_S^*L(F,W))}$$
$$DR(S)^{[-]}(\mathcal{H}om(L\mathbb{D}_{\tilde{S}_I}L(i_{I*}j_I^*e(S)_*\underline{\mathrm{sing}}_{\mathbb{D}^*}\mathrm{An}_S^*L(F,W)),E_{usu}(O_{\tilde{S}_I})),u_{IJ}(-))$$
$$\xrightarrow{=}DR(S)^{[-]}(\mathcal{H}om(L\mathbb{D}_{\tilde{S}_I}L(i_{I*}j_I^*(e(S)_*\underline{\mathrm{sing}}_{\mathbb{D}^*}\mathrm{An}_S^*L(F,W)),e(\tilde{S}_I)_*E_{usu}(\Omega^\bullet_{/\tilde{S}_I}))),u_{IJ}(-))$$
$$\xrightarrow{DR(S)^{[-]}(\mathcal{H}om(T(e,hom)(-,E(\mathbb{Z}))\circ q,-))}$$
$$DR(S)^{[-]}(\mathcal{H}om(Le(\tilde{S}_I)_*L\mathbb{D}_{\tilde{S}_I}L(i_{I*}j_I^*\underline{\mathrm{sing}}_{\mathbb{D}^*}\mathrm{An}_S^*L(F,W)),e(\tilde{S}_I)_*E_{usu}(\Omega^\bullet_{/\tilde{S}_I})))$$
$$\xrightarrow{DR(S)^{[-]}(L\mathcal{H}om(e(\tilde{S}_I)_*L\mathbb{D}_{\tilde{S}_I}Li_{I*}j_I^*c(\mathrm{An}_S^*L(F,W)),e(\tilde{S}_I)_*E_{usu}(\Omega^\bullet_{/\tilde{S}_I})))}$$
$$DR(S)^{[-]}(\mathcal{H}om(Le(\tilde{S}_I)_*L\mathbb{D}_{\tilde{S}_I}L(i_{I*}j_I^*\mathrm{An}_S^*L(F,W)),e(\tilde{S}_I)_*E_{usu}(\Omega^\bullet_{/\tilde{S}_I})),u_{IJ}(-))$$
$$\xrightarrow{DR(S)^{[-]}(\mathcal{H}om(\mathcal{H}om(T(\mathrm{An},hom)(-,E_{et}(\mathbb{Z}))\circ\mathbb{D}_{\tilde{S}_I}T(\mathrm{An},i_I)(-),e(\tilde{S}_I)_*E_{usu}(\Omega^\bullet_{/\tilde{S}_I}))))}$$
$$DR(S)^{[-]}(\mathcal{H}om(Le(\tilde{S}_I)_*\mathrm{An}_{\tilde{S}_I}^*L\mathbb{D}_{\tilde{S}_I}L(i_{I*}j_I^*(F,W)),e(\tilde{S}_I)_*E_{usu}(\Omega^\bullet_{/\tilde{S}_I})),u_{IJ}(-))$$
$$DR(S)^{[-]}(\mathcal{H}om(Le(\tilde{S}_I)_*\mathrm{An}_{\tilde{S}_I}^*L\mathbb{D}^0_{\tilde{S}_I}(L(i_{I*}j_I^*(F,W))),e(\tilde{S}_I)_*E_{usu}(\Omega^\bullet_{/\tilde{S}_I})),u_{IJ}(-))$$
$$\xrightarrow{DR(S)^{[-]}((\mathcal{H}om(-,e(\tilde{S}_I)_*E_{usu}(\mathrm{Gr}(\Omega_{/S^{an}}))))^{-1})}$$
$$DR(S)^{[-]}(\mathcal{H}om(Le(\tilde{S}_I)_*\mathrm{An}_{\tilde{S}_I}^*L\mathbb{D}^0_{\tilde{S}_I}L(i_{I*}j_I^*(F,W)),e(\tilde{S}_I)_*\mathrm{Gr}^{12}_{\tilde{S}_I*}E_{usu}(\Omega^{\bullet,\Gamma,pr,an}_{/S^{an}})),u_{IJ}(-))$$
$$\xrightarrow{DR(S)^{[-]}(\mathcal{H}om(q,-)\circ T(e,hom)(-,-)^{-1})}$$
$$DR(S)^{[-]}(e(\tilde{S}_I)_*\mathcal{H}om(\mathrm{An}_{\tilde{S}_I}^*L\mathbb{D}^0_{\tilde{S}_I}L(i_{I*}j_I^*(F,W)),\mathrm{Gr}^{12}_{\tilde{S}_I*}E_{usu}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_I^{an}})),u_{IJ}(-))$$
$$\xrightarrow{DR(S)^{[-]}(\mathcal{H}om(\mathrm{An}_{\tilde{S}_I}^*LT^{CH}_{\tilde{S}_I}(L(i_{I*}j_I^*(F,W)))\circ q,\mathrm{Gr}^{12}_{\tilde{S}_I*}E_{usu}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_I^{an}})))}$$
$$DR(S)^{[-]}(e'(\tilde{S}_I)_*\mathcal{H}om(\mathrm{An}^{12}_{\tilde{S}_I*}\mathrm{Gr}^{12}_{\tilde{S}_I*}L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho^*_{\tilde{S}_I}L(i_{I*}j_I^*(F,W)),\mathrm{Gr}^{12}_{\tilde{S}_I*}E_{usu}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_I^{an}})),u_{IJ}(-))$$
$$\xrightarrow{DR(S)^{[-]}(I(\mathrm{Gr}^{12*}_{\tilde{S}_I},\mathrm{Gr}^{12}_{\tilde{S}_I*})(-,-))\circ I(\mathrm{An}^{12*}_{\tilde{S}_I},\mathrm{An}^{12}_{\tilde{S}_I*})(-,-))}$$
$$DR(S)^{[-]}(e'(\tilde{S}_I)_*\mathcal{H}om(\mathrm{Gr}^{12*}_{\tilde{S}_I}L\,\mathrm{Gr}^{12}_{\tilde{S}_I*}\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho^*_{\tilde{S}_I}L(i_{I*}j_I^*(F,W)),\mathrm{An}^{12}_{\tilde{S}_I*}E_{usu}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_I^{an}})),u_{IJ}(-))$$
$$\xrightarrow{\mathcal{H}om(\mathcal{H}om(\mathrm{ad}(\mathrm{Gr}^{12*}_{\tilde{S}_I},\mathrm{Gr}^{12}_{\tilde{S}_I*})(-)\circ q,-)}$$
$$DR(S)^{[-]}(e'(\tilde{S}_I)_*\mathcal{H}om(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho^*_{\tilde{S}_I}L(i_{I*}j_I^*(F,W)),\mathrm{An}^{12}_{\tilde{S}_I*}E_{usu}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_I^{an}}))),u_{IJ}(F))$$
$$\xrightarrow{=:}DR(S)^{[-]}(\mathcal{F}^{DR}_{S,an}(M,W))$$
$$\xrightarrow{DR(S)^{[-]}(T(An,\mathcal{F}_S^{FDR})(M,W)^{-1})}DR(S)^{[-]}((\mathcal{F}^{DR}_S(M,W))^{an})$$

where $(F,W)\in C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ *is such that* $(M,W)=D(\mathbb{A}^1,et)(F,W)$

- $s(K):K\to DR(S)^{[-]}(\mathcal{H}om(\mathbb{D}_SLK,E(O_S))$ *is the isomorphism of theorem 26,*
- *the map* $\mathcal{H}om(q,-)\circ T(e,hom)(-,-)$ *is a filtered equivalence usu local by proposition 39 and proposition 130,*
- $T^{CH}_{\tilde{S}_I}(L(F,W)):\mathrm{Gr}^{12}_{\tilde{S}_I*}L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho^*_{\tilde{S}_I}Li_{I*}j_I^*(F,W))\to\mathbb{D}^0_S(Li_{I*}j_I^*(F,W))$ *is given in definition 36,*
- $\mathrm{ad}(\mathrm{Gr}^{12*}_{\tilde{S}_I},\mathrm{Gr}^{12}_{\tilde{S}_I*})(L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho^*_{\tilde{S}_I}Li_{I*}j_I^*(F,W)))\circ q$ *is an equivalence* $(\mathbb{A}^1,et)$ *local by lemma 1,*
- $\mathcal{F}^{DR}_S(M):=o_{fil}\mathcal{F}^{FDR}_S(M)\in D_{\mathcal{D}0fil}(S)$, $\mathcal{F}^{DR}_{S,an}(M):=o_{fil}\mathcal{F}^{FDR}_{S,an}(M)\in D_{\mathcal{D}0fil}(S^{an})$,

- $T(An,\mathcal{F}_S^{FDR})(M):(\mathcal{F}_S^{DR}(M))^{an}\xrightarrow{\sim}\mathcal{F}_{S,an}^{DR}(M)$ *is an isomorphism by theorem 43.*

We now give the functoriality with respect to the five operation using the De Rahm realization case and the Betti realization case :

Proposition 145. *(i) Let $g:T\to S$ a morphism with $T,S\in\mathrm{Var}(\mathbb{C})$. Assume there exists a factorization $g:T\xrightarrow{l}Y\times S\xrightarrow{p}S$, with $Y\in\mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p the projection. Let $S=\cup_{i\in I}S_i$ an open cover and $i:S_i\hookrightarrow\tilde{S}_i$ closed embeddings with $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$. Then, $\tilde{g}_I:Y\times\tilde{S}_I\to\tilde{S}_I$ is a lift of $g_I=g_{|T_I}:T_I\to S_I$. Then, for $M\in DA_c(S)$, the following diagram commutes :*

$$\begin{tikzcd}
g^*(\alpha(M)):g^*\mathrm{Bti}_S^*M \arrow[r] \arrow[d, "T(g,bti)(M)"'] & DR(T)^{[-]}(g_{Hdg}^{\hat{*}mod}(\mathcal{F}_{S,an}^{DR}(M))) \arrow[r] \arrow[d, "DR(T)^{[-]}(T(g,\mathcal{F}_{an}^{FDR})(M))"] & DR(T)^{[-]}((g_{Hdg}^{\hat{*}mod}\mathcal{F}_S^{DR}(M))^{an}) \arrow[d, "DR(T)^{[-]}((T(g,\mathcal{F}^{FDR})(M))^{an})"] \\
\alpha(g^*M):\mathrm{Bti}_T^*g^*M \arrow[r] & DR(T)^{[-]}(\mathcal{F}_{T,an}^{DR}(g^*M)) \arrow[r] & DR(T)^{[-]}((\mathcal{F}_T^{DR}(g^*M))^{an})
\end{tikzcd},$$

see definition 100, definition 121, definition 142 and definition 152

(ii) Let $f:T\to S$ a morphism with $T,S\in\mathrm{QPVar}(\mathbb{C})$. Then, for $M\in DA_c(T)$,the following diagram commutes :

$$\begin{tikzcd}
f_*(\alpha(M)):Rf_*\mathrm{Bti}_T^*M \arrow[r] & DR(S)^{[-]}(Rf_*^{Hdg}\mathcal{F}_{T,an}^{DR}(M)) \arrow[r] & DR(S)^{[-]}((Rf_*^{Hdg}\mathcal{F}_T^{DR}(M))^{an}) \\
\alpha(Rf_*M):\mathrm{Bti}_S^*Rf_*M \arrow[r] \arrow[u, "T_*(f,bti)(M)"] & DR(S)^{[-]}(\mathcal{F}_{S,an}^{DR}(Rf_*M)) \arrow[r] \arrow[u] & DR(S)^{[-]}((\mathcal{F}_S^{DR}(Rf_*M))^{an}) \arrow[u, "DR(S)^{[-]}((T_*(f,\mathcal{F}^{FDR})(M))^{an})"']
\end{tikzcd}$$

see definition 100, definition 122, definition 143 and definition 153

(iii) Let $f:T\to S$ a morphism with $T,S\in\mathrm{QPVar}(\mathbb{C})$. Then, for $M\in DA_c(T)$,the following diagram commutes :

$$\begin{tikzcd}
f_!(\alpha(M)):Rf_!\mathrm{Bti}_T^*M \arrow[r] \arrow[d, "T_!(f,bti)(M)"'] & DR(S)^{[-]}(Rf_!^{Hdg}\mathcal{F}_{DR}^{T,an}(M)) \arrow[r] \arrow[d] & DR(S)^{[-]}((Rf_!^{Hdg}\mathcal{F}_{DR}^{T}(M))^{an}) \arrow[d, "DR(S)^{[-]}((T_!(f,\mathcal{F}_{DR})(M))^{an})"] \\
\alpha(Rf_!M):\mathrm{Bti}_S^*f_!M \arrow[r] & DR(S)^{[-]}((\mathcal{F}_{DR}^{S,an}(Rf_!M))) \arrow[r] & DR(S)^{[-]}((\mathcal{F}_{DR}^{S}(Rf_!M))^{an})
\end{tikzcd}$$

see definition 100, definition 122, definition 143 and definition 153

(iv) Let $f:T\to S$ a morphism with $T,S\in\mathrm{QPVar}(\mathbb{C})$. Then, for $M\in DA_c(S)$,the following diagram commutes :

$$\begin{tikzcd}
f^!(\alpha(M)):f^!\mathrm{Bti}_S^*M \arrow[r] & DR(T)^{[-]}(Rf_{Hdg}^{*mod}(\mathcal{F}_{S,an}^{DR}(M))) \arrow[r] & DR(T)^{[-]}((f_{Hdg}^{*mod}\mathcal{F}_S^{DR}(M))^{an}) \\
\alpha(f^!M):\mathrm{Bti}_T^*f^!M \arrow[r] \arrow[u, "T^!(f,bti)(M)"] & DR(T)^{[-]}(\mathcal{F}_{T,an}^{DR}(f^!M)) \arrow[r] \arrow[u, "T^!(f,\mathcal{F}_{an}^{FDR})(M))"] & DR(T)^{[-]}((\mathcal{F}_T^{DR}(f^!M))^{an}) \arrow[u, "DR^{[-]}(T)((T^!(g,\mathcal{F}^{FDR})(M))^{an})"']
\end{tikzcd}$$

see definition 100, definition 122,definition 143 and definition 153

(v) Let $S\in\mathrm{Var}(\mathbb{C})$. Then, for $M,N\in DA_c(S)$,the following diagram commutes :

$$\begin{tikzcd}
\mathrm{Bti}_S^*M\otimes\mathrm{Bti}_S^*N \arrow[r, "\alpha(M)\otimes\alpha(N)"] \arrow[d, "T(\otimes,bti)(M,N)"'] & DR(S)((\mathcal{F}_S^{DR}(M)\otimes_{O_S}\mathcal{F}_S^{DR}(N))^{an}) \arrow[d, "DR(S)((T(\otimes,\mathcal{F}^{DR})(M,N))^{an})"] \\
\mathrm{Bti}_S^*(M\otimes N) \arrow[r, "(\alpha(M\otimes N))"] & DR(S)((\mathcal{F}_{DR}^S(M\otimes N))^{an})
\end{tikzcd}$$

see definition 124 and definition 154.

Proof. (i): The commutativity of the right square is given by applying the functor $DR(T)^{[-]}$ to the commutative diagram

$$\begin{tikzcd}
g_{Hdg}^{\hat{*}mod}((\mathcal{F}_S^{FDR}(M))^{an}) = (g_{Hdg}^{\hat{*}mod}(\mathcal{F}_S^{FDR}(M)))^{an} \arrow[r, "g_{Hdg}^{\hat{*}mod}(T(An,\mathcal{F}_S^{FDR})(M))"] \arrow[d, "'(T(g,\mathcal{F}^{FDR})(M))^{an}"'] & g_{Hdg}^{\hat{*}mod}(\mathcal{F}_{S,an}^{FDR}(M)) \arrow[d, "T(g,\mathcal{F}_{an}^{FDR})(M)"] \\
(\mathcal{F}_T^{FDR}(g^*M))^{an} \arrow[r, "T(An,\mathcal{F}_T^{FDR})(g^*M)"] & \mathcal{F}_{T,an}^{FDR}(g^*M)
\end{tikzcd}$$

given in proposition 142(i0). On the other hand, the commutativity of the left square follows from the following commutative diagram :

$$\begin{tikzcd}
(\tilde{g}_I^* i_{I*} j_I^* e(S)_* \mathrm{An}_S^* LF, \tilde{g}_J^* u_{IJ}(-)) \arrow[r] \arrow[d, "T(e,g)(F)"] & DR(T)^{[-]}(\Gamma_{T_I}^{\vee,Hdg} \tilde{g}_I^{*mod} e'(\tilde{S}_I)_* \mathcal{H}om(\mathrm{An}_{\tilde{S}_I}^* L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^* Li_{I*}j_I^*F), E_{usu}(\Omega^{\bullet,\Gamma,pr,an}_{/\tilde{S}_I^{an}})), \tilde{g}_I^{\hat{*}mod}(u_{IJ}(F))) \arrow[d, "DR(T)^{[-]}(T(g,\mathcal{F}_{an}^{DR})(M))"] \\
(i'_{I*} j_I^{'*} e(T)_* \mathrm{An}_T^* g^*F, u_{IJ}(-)) \arrow[r] & DR(T)^{[-]}(e'(Y \times \tilde{S}_I)_* \mathcal{H}om(\mathrm{An}_{Y\times\tilde{S}_I}^* L\rho_{\tilde{S}_I*}\mu_{\tilde{S}_I*}R^{CH}(\rho_{Y\times\tilde{S}_I}^* \tilde{g}_I^* Li_{I*}j_I^*F), E_{usu}(\Omega^{\bullet}_{/Y\times\tilde{S}_I})), \tilde{g}_J^* u_{IJ}^q(F))
\end{tikzcd}$$

(ii): Follows from (i) by adjonction.
(iii): The closed embedding case is given by (ii) and the smooth projection case follows from (i) by adjonction.
(iv): Follows from (iii) by adjonction.
(v):Obvious

□

We can now state the following key proposition and the main theorem:

Proposition 146. *Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$, X quasi-projective. Consider a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y = \mathbb{P}^{N,o} \subset \mathbb{P}^N$ an open subset, l a closed embedding and p_S the projection. Let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in$ $\mathrm{SmVar}(\mathbb{C})$. Recall that $S_I := \cap_{i\in I} S_i$, $X_I = f^{-1}(S_I)$, and $\tilde{S}_I := \Pi_{i\in I}\tilde{S}_i$. Then, using proposition 145(iii), the maps of definition 122 and definition 153 gives an isomorphism in $D_{\mathcal{D}(1,0)fil,\infty}(S/(\tilde{S}_I)) \times_I D(S^{an})$*

$$\begin{aligned}
&(T_!(f,\mathcal{F}^{FDR})(\mathbb{Z}(X/X)), T_!(f,\mathrm{Bti})(\mathbb{Z}(X/X))) : \\
\mathcal{F}_S^{Hdg}(M^{BM}(X/S)) &:= (\mathcal{F}_S^{FDR}(Rf_!\mathbb{Z}(X/X)), \mathrm{Bti}_S^* Rf_!\mathbb{Z}(X/X), \alpha(Rf_!\mathbb{Z}(X/X))) \\
&\xrightarrow{\sim} (Rf_{Hdg!}(\Gamma_{X_I}^{\vee,Hdg}(O_{Y\times\tilde{S}_I}, F_b), x_{IJ}(X/S)), Rf_!\mathbb{Z}_{X^{an}}, f_!(\alpha(X/(Y\times\tilde{S}_I)))) =: Rf_{!Hdg}\mathbb{Z}_X^{Hdg}
\end{aligned}$$

Proof. Follows from proposition 145(iii),theorem 36(i) and theorem 45(i). □

The main theorem of this article is the following :

Theorem 47. *(i) For $S \in \mathrm{Var}(\mathbb{C})$, we have $\mathcal{F}_S^{Hdg}(\mathrm{DA}_c(S)) \subset D(MHM(S))$.*

(ii) The Hodge realization functor $\mathcal{F}_{Hdg}(-)$ define a morphism of 2-functor on $\mathrm{Var}(\mathbb{C})$

$$\mathcal{F}_-^{Hdg} : \mathrm{Var}(\mathbb{C}) \to (\mathrm{DA}_c(-) \to D(MHM(-)))$$

whose restriction to $\mathrm{QPVar}(\mathbb{C})$ *is an homotopic 2-functor in sense of Ayoub. More precisely,* *(ii0) for $g : T \to S$ a morphism, with $T, S \in \mathrm{QPVar}(\mathbb{C})$, and $M \in \mathrm{DA}_c(S)$, the the maps of definition 121 and of definition 152 induce an isomorphism in $D(MHM(T))$*

$$\begin{aligned} T(g, \mathcal{F}^{Hdg})(M) &:= (T(g, \mathcal{F}^{FDR})(M), T(g, bti)(M)) : \\ g^{\hat{*}Hdg}\mathcal{F}_S^{Hdg}(M) &:= (g^{\hat{*}mod}_{Hdg}\mathcal{F}_S^{FDR}(M), g^*\,\mathrm{Bti}_S(M), g^*(\alpha(M))) \\ &\xrightarrow{\sim} (\mathcal{F}_T^{FDR}(g^*M), \mathrm{Bti}_T^*(g^*M), \alpha(g^*M)) =: \mathcal{F}_T^{Hdg}(g^*M), \end{aligned}$$

(ii1) for $f : T \to S$ a morphism, with $T, S \in \mathrm{QPVar}(\mathbb{C})$, and $M \in \mathrm{DA}_c(T)$, the maps of definition 122 and of definition 153 induce an isomorphism in $D(MHM(S))$

$$\begin{aligned} T_*(f, \mathcal{F}^{Hdg})(M) &:= (T_*(f, \mathcal{F}^{FDR})(M), T_*(f, bti)(M)) : \\ Rf_{Hdg*}\mathcal{F}_T^{Hdg}(M) &:= (Rf_*^{Hdg}\mathcal{F}_T^{FDR}(M), Rf_*\,\mathrm{Bti}_S(M), f_*(\alpha(M))) \\ &\xrightarrow{\sim} (\mathcal{F}_S^{FDR}(Rf_*M), \mathrm{Bti}_S^*(Rf_*M), \alpha(Rf_*M)) =: \mathcal{F}_S^{Hdg}(Rf_*M), \end{aligned}$$

(ii2) for $f : T \to S$ a morphism, with $T, S \in \mathrm{QPVar}(\mathbb{C})$, and $M \in \mathrm{DA}_c(T)$, the maps of definition 122 and of definition 153 induce an isomorphism in $D(MHM(S))$

$$\begin{aligned} T_!(f, \mathcal{F}^{Hdg})(M) &:= (T_!(f, \mathcal{F}^{FDR})(M), T_!(f, bti)(M)) : \\ Rf_{!Hdg}\mathcal{F}_T^{Hdg}(M) &:= (Rf_!^{Hdg}\mathcal{F}_T^{FDR}(M), Rf_!\,\mathrm{Bti}_S^*(M), f_!(\alpha(M))) \\ &\xrightarrow{\sim} (\mathcal{F}_S^{FDR}(Rf_!M), \mathrm{Bti}_S^*(Rf_!M), \alpha(f_!M)) =: \mathcal{F}_T^{Hdg}(f_!M), \end{aligned}$$

(ii3) for $f : T \to S$ a morphism, with $T, S \in \mathrm{QPVar}(\mathbb{C})$, and $M \in \mathrm{DA}_c(S)$, the maps of definition 122 and of definition 153 induce an isomorphism in $D(MHM(T))$

$$\begin{aligned} T^!(f, \mathcal{F}^{Hdg})(M) &:= (T^!(f, \mathcal{F}^{FDR})(M), T^!(f, bti)(M)) : \\ f^{*Hdg}\mathcal{F}_S^{Hdg}(M) &:= (f^{*mod}_{Hdg}\mathcal{F}_S^{FDR}(M), f^!\,\mathrm{Bti}_S(M), f^!(\alpha(M))) \\ &\xrightarrow{\sim} (\mathcal{F}_T^{FDR}(f^!M), \mathrm{Bti}_T^*(f^!M), \alpha(f^!M)) =: \mathcal{F}_T^{Hdg}(f^!M), \end{aligned}$$

(ii4) for $S \in \mathrm{Var}(\mathbb{C})$, and $M, N \in \mathrm{DA}_c(S)$, the maps of definition 124 and of definition 154 induce an isomorphism in $D(MHM(S))$

$$\begin{aligned} &T(\otimes, \mathcal{F}^{Hdg})(M, N) := (T(\otimes, \mathcal{F}_S^{FDR})(M, N), T(\otimes, bti)(M, N)) : \\ &(\mathcal{F}_S^{FDR}(M) \otimes^L_{O_S} \mathcal{F}_S^{FDR}(N), \mathrm{Bti}_S(M) \otimes \mathrm{Bti}_S(N), \alpha(M) \otimes \alpha(N)) \\ &\xrightarrow{\sim} \mathcal{F}_S^{Hdg}(M \otimes N) := (\mathcal{F}_S^{FDR}(M \otimes N), \mathrm{Bti}_S(M \otimes N), \alpha(M \otimes N)). \end{aligned}$$

(iii) For $S \in \mathrm{Var}(\mathbb{C})$, the following diagram commutes :

$$\begin{tikzcd} \mathrm{Var}(\mathbb{C})/S \arrow[r, "MH(/S)"] \arrow[d, "M(/S)"'] & D(MHM(S)) \arrow[d, "\iota^S"] \\ \mathrm{DA}(S) \arrow[r, "\mathcal{F}_S^{Hdg}"] & D_{\mathcal{D}fil,\infty}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}) \end{tikzcd}$$

Proof. (i): By corollary 6, the Hodge realization functor

$$\begin{aligned} \mathcal{F}_S^{Hdg} := (\mathcal{F}_S^{FDR}, \mathrm{Bti}_S^*) : \mathrm{DA}(S) &\to D_{\mathcal{D}(1,0)fil,\infty}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}), \\ M &\mapsto \mathcal{F}_S^{Hdg}(M) := (\mathcal{F}_S^{FDR}(M), \mathrm{Bti}_S^* M, \alpha(M)), \end{aligned}$$

factors trough

$$\begin{aligned} &\mathcal{F}_S^{Hdg} := (\mathcal{F}_S^{FDR}, \mathrm{Bti}_S^*) : \\ &\mathrm{DA}(S) \to D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}) \to D_{\mathcal{D}(1,0)fil,\infty}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}). \end{aligned}$$

Let $M \in \mathrm{DA}_c(S)$. There exist a generalized distinguish triangle in $\mathrm{DA}(S)$

$$M \to M(X_0/S)[d_0] \to \cdots \to M(X_m/S)[d_m],$$

with $f_n : X_n \to S$ morphisms and $X_n \in \mathrm{QPVar}(\mathbb{C})$. This gives the following generalized distinguish triangle in $D_{\mathcal{D}fil}(S/(\tilde{S}_I)) \times_I D(S^{an})$

$$\mathcal{F}_S^{Hdg}(M) \to \mathcal{F}_S^{Hdg}(M(X_0/S))[d_0] \to \cdots \to \mathcal{F}_S^{Hdg}(M(X_m/S))[d_m],$$

On the other hand, by proposition 146, we have

$$\mathcal{F}_S^{Hdg}(M(X_n/S)) \xrightarrow{\sim} Rf_{*Hdg}\mathbb{Z}_X^{Hdg} \in D(MHM(S))$$

(ii0): Follows from theorem 35, proposition 145(i) and theorem 44.
(ii1): Follows from theorem 36(ii), proposition 145(ii), and theorem 45(ii).
(ii2):Follows from theorem 36(i), proposition 145(iii), and theorem 45(i).
(ii3): Follows from theorem 36(iii), proposition 145(iv), and theorem 45(iii).
(ii4):Follows from theorem 37, proposition 145(v) and theorem 45(iv).
(iii): By (ii), for $g : X'/S \to X/S$ a morphism, with $X', X, S \in \mathrm{Var}(\mathbb{C})$ and $X/S = f : X \to S$, $X'/S = f' : X' \to S$, we have by adjonction the following commutative diagram

$$\begin{tikzcd}
\mathcal{F}_S^{Hdg}(M(X'/S) = f'_! f'^! \mathbb{Z}_S = f_! g_! g^! f^! \mathbb{Z}_S) \arrow[r, "\mathcal{F}_S^{Hdg}(M(/S)(g) = f_! \mathrm{ad}(g_!, g^!)(f^! \mathbb{Z}_S))"] \arrow[d, "\sim"'] & \mathcal{F}_S^{Hdg}(M(X/S) = f_! f^! \mathbb{Z}_S) . \arrow[d, "\sim"] \\
MH(X'/S) = f'_! f'^! \mathbb{Z}_S^{Hdg} = f_! g_! g^! f^! \mathbb{Z}_S^{dg} \arrow[r, "f_! \mathrm{ad}(g_!, g^!)(f^! \mathbb{Z}_S^{Hdg})"] & MH(X/S) = f_! f^! \mathbb{Z}_S^{Hdg}
\end{tikzcd}$$

This proves (iii). $\square$

The theorem 47 gives immediately the following :

Corollary 10. *Let $f : X \to S$, $f' : X' \to S$ morphisms, with $X, X', S \in \mathrm{Var}(\mathbb{C})$. Let $\bar{S} \in \mathrm{PVar}(\mathbb{C})$ a compactification of S. Let $\bar{X}, \bar{X}' \in \mathrm{PVar}(\mathbb{C})$ compactifiaction of X and X' respectively, such that f (resp. f') extend to a morphism $\bar{f} : \bar{X} \to \bar{S}$, resp. $\bar{f}' : \bar{X}' \to \bar{S}$. Denote $D = \bar{X} \backslash X$ and $D' = \bar{X}' \backslash X'$ and $E = (D \times_{\bar{S}} \bar{X}') \cup (\bar{X} \times_{\bar{S}} D')$. We have the following commutative digram*

$$\begin{tikzcd}
RHom^{\bullet}(M((\bar{X}, D)/\bar{S}), M((\bar{X}', D')/\bar{S})^{\vee}(d)) \arrow[r, "\mathcal{F}_S^{FDR}"] \arrow[d, "RI()"] & RHom^{\bullet}(f_! \mathbb{Z}_X^{Hdg}, f'_! \mathbb{Z}_{X'}^{Hdg}) \arrow[d, "RI()"] \\
RHom^{\bullet}(M(\mathrm{pt}), M(\bar{X} \times_{\bar{S}} \bar{X}', E)(d)) \arrow[r, "\mathcal{F}_{FDR}^{\mathrm{pt}}"] \arrow[d, "l"] & RHom^{\bullet}(\mathbb{Z}_{\mathrm{pt}}^{Hdg}, a_{X \times_S X'!} \mathbb{Z}_{X \times_S X'}^{Hdg}) \arrow[d, "l"] \\
\mathcal{Z}^d(\bar{X} \times_{\bar{S}} \bar{X}', E, \bullet) \arrow[r, "\mathcal{R}^d_{\bar{X} \times_{\bar{S}} \bar{X}'}"] & C^{\mathcal{D}}_{2d_X - 2d + \bullet}(\bar{X} \times_{\bar{S}} \bar{X}', E, Z(d))
\end{tikzcd}$$

Proof. The upper square of this diagram follows from theorem 47(ii) and the following isomorphism :

- $\mathrm{ad}(j_!, j^*)(\mathbb{Z}_{\bar{X}}) : M(X/\bar{S}) \xrightarrow{\sim} \mathrm{Cone}(M(\bar{X}/\bar{S}) \xrightarrow{\mathrm{ad}(i^*, i_*)(\mathbb{Z}_{\bar{X}})} M(D/\bar{S}))[-1] =: M((\bar{X}, D)/\bar{S})$
- $\mathrm{ad}(j'_!, j'^*)(\mathbb{Z}_{\bar{X}}) : M(X'/\bar{S}) \xrightarrow{\sim} \mathrm{Cone}(M(\bar{X}'/\bar{S}) \xrightarrow{\mathrm{ad}(i'^*, i'_*)(\mathbb{Z}_{\bar{X}'})} M(D'/\bar{S}))[-1] =: M((\bar{X}', D')/\bar{S})$

where $i : D \hookrightarrow \bar{X}$, $i' : D \hookrightarrow \bar{X}$ denote the closed embeddings and $j : X \hookrightarrow \bar{X}$, $j' : X' \hookrightarrow \bar{X}'$ the open embeddings. On the other side, the lower square follows from the absolute case.

$\square$

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