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The Hodge realization functor on the derived category of relative motives

Johann Bouali

► **To cite this version:**

Johann Bouali. The Hodge realization functor on the derived category of relative motives. 2020.
hal-02888285v1

HAL Id: hal-02888285

<https://hal.science/hal-02888285v1>

Preprint submitted on 2 Jul 2020 (v1), last revised 24 Jan 2022 (v11)

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The Hodge realization functor on the derived category of relative motives

Johann Bouali

July 2, 2020

Abstract

We give, for a complex algebraic variety S , a Hodge realization functor \mathcal{F}_S^{Hdg} from the (unbounded) derived category of constructible motives $DA_c(S)$ over S to the (unbounded) derived category $D(MHM(S))$ of algebraic mixed Hodge modules over S . Moreover, for $f : T \rightarrow S$ a morphism of complex quasi-projective algebraic varieties, \mathcal{F}_-^{Hdg} commutes with the four operations $f^*, f_*, f_!, f^!$ on $DA_c(-)$ and $D(MHM(-))$, making in particular the Hodge realization functor a morphism of 2-functor on the category of complex quasi-projective algebraic varieties which for a given S sends $DA_c(S)$ to $D(MHM(S))$, moreover \mathcal{F}_S^{Hdg} commutes with tensor product. We also give an algebraic and analytic Gauss-Manin realization functor from which we obtain a base change theorem for algebraic De Rham cohomology and for all smooth morphisms a relative version of the comparison theorem of Grothendieck between the algebraic De Rham cohomology and the analytic De Rham cohomology.

Contents

1	Introduction	3
2	Generalities and Notations	7
2.1	Notations	7
2.2	Additive categories, abelian categories and tensor triangulated categories	15
2.3	Presheaves on a site and on a ringed topos	17
2.3.1	Functorialities	17
2.3.2	Canonical flasque resolution of a presheaf on a site or a presheaf of module on a ringed topos	24
2.3.3	Canonical projective resolution of a presheaf of module on a ringed topos	25
2.3.4	The De Rham complex of a ringed topos and functorialities	26
2.4	Presheaves on diagrams of sites or on diagrams of ringed topos	28
2.5	Presheaves on topological spaces and presheaves of modules on a ringed spaces	31
2.6	Presheaves on the big Zariski site or on the big etale site	44
2.7	Presheaves on the big Zariski site or the big etale site of pairs	57
2.8	Presheaves on the big analytical site	66
2.9	Presheaves on the big analytical site of pairs	76
2.10	The analytical functor for presheaves on the big Zariski or etale site and on the big Zariski or etale site of pairs	84
2.11	The De Rham complexes of algebraic varieties and analytical spaces	87
2.12	Resolution of a representable presheaf on $\text{Var}(\mathbb{C})/S$ by Corti-Hanamura presheaves and the functorialities of these resolutions	89
2.13	The derived categories of filtered complexes of presheaves on a site or of filtered complexes of presheaves of modules on a ringed topos	94

3	Triangulated category of motives	96
3.1	Definition and the six functor formalism	96
3.2	Constructible motives and resolution of a motive by Corti-Hanamura motives	100
3.3	The restriction of relative motives to their Zariski sites	102
3.4	Motives of complex analytic spaces	102
4	The category of filtered D modules on commutative ringed topos, on commutative ringed spaces, complex algebraic varieties complex analytic spaces and the functorialities	103
4.1	The The category of filtered D modules on commutative ringed topos, on commutative ringed spaces, and the functorialities	103
4.1.1	Definitions et functorialities	103
4.1.2	The De Rham complex of a (left) filtered D-module and the Spencer complex of a right filtered D-module	109
4.1.3	The support section functor for D module on ringed spaces	113
4.2	The D-modules on smooth complex algebraic varieties and on complex analytic manifold and their functorialities in the filtered case	116
4.2.1	Functorialities	122
4.2.2	The (relative) De Rahm of a (filtered) complex of a D-module and the filtered De Rham direct image	138
4.2.3	The support section functors for D modules and the graph inverse image	148
4.2.4	The 2 functors and transformations maps for D modules on the smooth complex algebraic varieties and the complex analytic manifolds	153
4.3	The D modules on singular algebraic varieties and singular complex analytic spaces	164
4.3.1	Definition	164
4.3.2	Duality in the singular case	169
4.3.3	Inverse image in the singular case	170
4.3.4	Direct image functor in the singular case	172
4.3.5	Tensor product in the singular case	173
4.3.6	The 2 functors of D modules on the category of complex algebraic varieties and on the category of complex analytic spaces, and the transformation maps	174
4.4	The category of complexes of quasi-coherent sheaves whose cohomology sheaves has a structure of D-modules	178
4.4.1	Definition on a smooth complex algebraic variety or smooth complex analytic space and the functorialities	178
4.4.2	Definition on a singular complex algebraic variety or singular complex analytic space and the functorialities	180
5	The category of mixed Hodge modules on complex algebraic varieties and complex analytic sapces and the functorialities	182
5.1	The De Rahm functor for D modules on a complex analytic space	182
5.2	The filtered Hodge direct image, the filtered Hodge inverse image, and the hodge support section functors for mixed hodge modules	186
6	The algebraic and analytic filtered De Rham realizations for Voevodsky relative motives	207
6.1	The algebraic filtered De Rham realization functor	207
6.1.1	The algebraic Gauss-Manin filtered De Rham realization functor and its transformation map with pullbacks	207
6.1.2	The algebraic filtered De Rham realization functor and the commutativity with the six operation	224
6.2	The analytic filtered De Rahm realization functor	260

6.2.1	The analytic Gauss-Manin filtered De Rham realization functor and its transformation map with pullbacks	260
6.2.2	The analytic filtered De Rham realization functor	272
6.3	The transformation map between the analytic De Rham functor and the analytification of the algebraic De Rham functor	288
6.3.1	The transformation map between the analytic Gauss Manin realization functor and the analytification of the algebraic Gauss Manin realization functor	288
6.3.2	The transformation map between the analytic filtered De Rham realization functor and the analytification of the filtered algebraic De Rham realization functor	292
7	The Hodge realization functor for relative motives	296
7.1	The Betti realization functor	296
7.2	The Hodge realization functor for relative motives	299

1 Introduction

Saito's theory of mixed Hodge modules associate to each complex algebraic variety S a category $MHM(S)$ which is a full subcategory of $\text{PSh}_{\mathcal{D}(1,0)fil, rh}(S/(\tilde{S}_I)) \times_I C_{fil}(S)$ which extend variations of mixed Hodge structure and admits a canonical monoidal structure given by tensor product, and associate to each morphism of complex algebraic varieties $f : X \rightarrow S$, four functor $Rf_{Hdg!}, Rf_{Hdg*}, f^*Hdg, f^*Hdg$. In the case of a smooth proper morphism $f : X \rightarrow S$ with S and X smooth, $H^n Rf_{Hdg*} \mathbb{Z}_X^{Hdg}$ is the variation of Hodge structure given by the Gauss-Manin connexion and the local system $H^n Rf_* \mathbb{Z}_X$. Moreover, these functors induce the six functor formalism of Grothendieck. We thus have, for a complex algebraic variety S a canonical functor

$$MH(/S) : \text{Var}(\mathbb{C})/S \rightarrow D(MHM(S)), (f : X \rightarrow S) \mapsto Rf_{!Hdg} \mathbb{Z}_X^{Hdg}$$

and

$$MH(/-) : \text{Var}(\mathbb{C}) \rightarrow \text{TriCat}, S \mapsto (MH(S) : \text{Var}(\mathbb{C})/S \rightarrow D(MHM(S))),$$

is a morphism of 2-functor. In this work, we extend $MH(/-)$ to motives by constructing, for each complex algebraic variety S , a canonical functor $\mathcal{F}_S^{Hdg} : \text{DA}(S) \rightarrow D(MHM(S))$ which is monoidal, that is commutes with tensor product, together with, for each morphism of complex algebraic varieties $g : T \rightarrow S$ a canonical transformation map $T(g, \mathcal{F}^{Hdg})$, which make

$$\mathcal{F}_-^{Hdg} : \text{Var}(\mathbb{C}) \rightarrow \text{TriCat}, S \mapsto (\mathcal{F}_S^{Hdg} : \text{DA}(S) \rightarrow D(MHM(S))),$$

is a morphism of 2-functor : this is the contain of theorem 45. A partial result in this direction has been obtained by Ivorra in [17] using a different approach. We already have a Betti realization functor

$$\text{Bti}_- : \text{Var}(\mathbb{C}) \rightarrow \text{TriCat}, S \mapsto (\text{Bti}_S^* : \text{DA}(S) \rightarrow D(S)),$$

which extend the Betti realization. The functor $\mathcal{F}_-^{Hdg} := (\mathcal{F}_-^{FDR}, \text{Bti}_-)$ is obtained by constructing the De Rham part

$$\mathcal{F}_-^{FDR} : \text{Var}(\mathbb{C}) \rightarrow \text{TriCat}, S \mapsto (\mathcal{F}_S^{FDR} : \text{DA}(S) \rightarrow D_{\mathcal{D}(1,0)fil, \infty}(S)),$$

which takes values in the derived category of filtered algebraic D -modules obtained by inverting ∞ -filtered Zariski local equivalence. This method can be seen as a relative version of the construction of F.Lecompte and N.Wach in [20].

In section 6.1.1 and 6.2.1, we construct an algebraic and analytic Gauss-Manin realization functor, but this functor does NOT give a complex of filtered D -module, BUT a complex of filtered O -modules whose cohomology sheaves have a structure of filtered D_S modules. Hence, it does NOT get to the

desired category. However this functor gives some interesting results. Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$ smooth. For $I \subset [1, \dots, l]$, denote by $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. We define the filtered algebraic Gauss-Manin realization functor defined as

$$\mathcal{F}_S^{GM} : \text{DA}_c(S)^{op} \rightarrow D_{O_{fil}, \mathcal{D}, \infty}(S/(\tilde{S}_I))^\vee, M \mapsto \\ \mathcal{F}_S^{GM}(M) := ((e(\tilde{S}_I)_* \mathcal{H}om^\bullet(L(i_{I*} j_I^* F), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)))[-d_{\tilde{S}_I}], u_{I,J}^q(F))$$

where $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$, see definition 104 and corollary 4. We then prove (theorem 30):

Theorem 1. (i) *Let $g : T \rightarrow S$ is a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{ps} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and ps the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, for $M \in \text{DA}_c(S)$*

$$T(g, \mathcal{F}_S^{GM})(M) : Rg^{*mod[-], \Gamma} \mathcal{F}_S^{GM}(M) \rightarrow \mathcal{F}_T^{GM}(g^* M)$$

is an isomorphism in $D_{O_T, fil, \mathcal{D}, \infty}(T/(Y \times \tilde{S}_I))$.

(ii) *Let $g : T \rightarrow S$ is a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Then, for $M \in \text{DA}_c(S)$*

$$T^O(g, \mathcal{F}_S^{GM})(M) : Lg^{*mod} \mathcal{F}_S^{GM}(M) \rightarrow \mathcal{F}_T^{GM}(g^* M)$$

is an isomorphism in $D_{O_T}(T)$.

(iii) *A base change theorem for algebraic De Rham cohomology : Let $g : T \rightarrow S$ is a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Let $h : U \rightarrow S$ a smooth morphism with $U \in \text{Var}(\mathbb{C})$. Then the map (see definition 1)*

$$T_w^O(g, h) : Lg^{*mod} Rh_*(\Omega_{U/S}^\bullet, F_b) \rightarrow Rh'_*(\Omega_{U/T}^\bullet, F_b)$$

is an isomorphism in $D_{O_T}(T)$.

Another consequence is the following (theorem 31):

Theorem 2. *Let $X \in \text{PSmVar}(\mathbb{C})$ and $D = \cup D_i \subset X$ a normal crossing divisor. Consider the open embedding $j : U := X \setminus D \hookrightarrow X$. Then, the inclusion*

$$j^* : \Gamma(X, E_{zar}(\Omega_X^\bullet(\log D), F_b)) \hookrightarrow \Gamma(U, E_{zar}(\Omega_U^\bullet, F_b))$$

is an ∞ -filtered quasi-isomorphism, that is

$$j^* : F^p H^n(U, \mathbb{C}) := F^p H^n \Gamma(X, E_{zar}(\Omega_X^\bullet(\log D), F_b)) \xrightarrow{\sim} F^p H^n \Gamma(U, E_{zar}(\Omega_U^\bullet, F_b)) =: F_b^p H^n(U, \mathbb{C})$$

is an isomorphism for all $n, p \in \mathbb{Z}$ (note that it is obviously injective since $j^ : H^n \Gamma(X, E_{zar}(\Omega_X^\bullet(\log D))) \xrightarrow{\sim} H^n \Gamma(U, E_{zar}(\Omega_U^\bullet))$ is an isomorphism if we forgot filtrations). Note that however, it is NOT a filtered quasi-isomorphism (for example if U is affine $H^q(U, \Omega_U^p) = 0$ for $q > 0$) that it is not an isomorphism on the E_1 terms of the spectral sequences in general.*

Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote by $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. We define the filtered analytic Gauss-Manin realization functor defined as

$$\mathcal{F}_S^{GM} : \text{DA}_c(S)^{op} \rightarrow D_{O_{fil}, \mathcal{D}, \infty}(S/(\tilde{S}_I))^\vee, M \mapsto \\ \mathcal{F}_S^{GM}(M) := ((e(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^* F), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)))[-d_{\tilde{S}_I}], u_{I,J}^q(F))$$

where $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$, see definition 128 and corollary 7. We then prove (theorem 35):

Theorem 3. (i) Let $g : T \rightarrow S$ is a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{ps} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \bigcup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, for $M \in \text{DA}_c(S)$

$$T(g, \mathcal{F}_{an}^{GM})(M) : Rg^{*mod[-], \Gamma} \mathcal{F}_{S, an}^{GM}(M) \rightarrow \mathcal{F}_{T, an}^{GM}(g^* M)$$

is an isomorphism in $D_{O_T \text{fil}, \mathcal{D}^\infty, \infty}(T/(Y \times \tilde{S}_I))$.

(ii) Let $g : T \rightarrow S$ is a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Then, for $M \in \text{DA}_c(S)$

$$T(g, \mathcal{F}_{an}^{GM})(M) : Lg^{*mod} \mathcal{F}_{S, an}^{GM}(M) \rightarrow \mathcal{F}_{T, an}^{GM}(g^* M)$$

is an isomorphism in $D_{O_T}(T)$.

A consequence of the construction of the transformation map between the algebraic and analytic Gauss-Manin realization functor is the following (theorem 40)

Theorem 4. (i) Let $S \in \text{Var}(\mathbb{C})$. Then, for $M \in \text{DA}_c(S)$

$$\mathcal{J}_S(-) \circ H^n T(\text{An}, \mathcal{F}_{an}^{GM})(M) : J_S(H^n(\mathcal{F}_S^{GM}(M))^{an}) \xrightarrow{\sim} H^n \mathcal{F}_{S, an}^{GM}(M)$$

is an isomorphism in $\text{PSh}_{\mathcal{D}}(S^{an}/(\tilde{S}_I^{an}))$.

(ii) A relative version of Grothendieck GAGA theorem for De Rham cohomology Let $h : U \rightarrow S$ a smooth morphism with $S, U \in \text{SmVar}(\mathbb{C})$. Then,

$$\mathcal{J}_S(-) \circ J_S T_\omega^O(an, h) : J_S((R^n h_* \Omega_{U/S}^\bullet)^{an}) \xrightarrow{\sim} R^n h_* \Omega_{U^{an}/S^{an}}^\bullet$$

is an isomorphism in $\text{PSh}_{\mathcal{D}}(S^{an})$.

In section 6.1.2, using results of sections 2, 4 and 5, we construct the algebraic filtered De Rham realization functor \mathcal{F}_S^{FDR} . Let $S \in \text{Var}(\mathbb{C})$ and $S = \bigcup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote by $S_I = \bigcap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. We define, by definition 116(ii) which use definition 112 and definition 36, proposition 105(ii) and corollary 5, the filtered algebraic De Rham realization functor defined as

$$\begin{aligned} \mathcal{F}_S^{FDR} : \text{DA}_c(S) &\rightarrow D_{\mathcal{D} \text{fil}, \infty}(S/(\tilde{S}_I)), M \mapsto \mathcal{F}_S^{FDR}(M) := \\ &(\varinjlim_{r-(-)} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\rho_{\tilde{S}_I}^* \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{et}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F)) \end{aligned}$$

where $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$. By computing this functor an homological motive, we get by proposition 107, for $S \in \text{Var}(\mathbb{C})$ and $M \in \text{DA}_c(S)$, $\mathcal{F}_S^{FDR}(M) \in \pi_S(D(\text{MHM}(S)))$, and the following (theorem 32, theorem 33 and theorem 34):

Theorem 5. (i) Let $g : T \rightarrow S$ a morphism, with $S, T \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{ps} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $M \in \text{DA}_c(S)$. Then map in $\pi_T(D(\text{MHM}(T)))$

$$T(g, \mathcal{F}^{FDR})(M) : g_{Hdg}^{*mod} \mathcal{F}_S^{FDR}(M) \xrightarrow{\sim} \mathcal{F}_T^{FDR}(g^* M)$$

given in definition 121 is an isomorphism.

(ii) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{ps} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and ps the projection. Then, for $M \in \text{DA}_c(X)$, the map given in definition 122

$$T_!(f, \mathcal{F}^{FDR})(M) : Rf_!^{Hdg} \mathcal{F}_X^{FDR}(M) \xrightarrow{\sim} \mathcal{F}_S^{FDR}(Rf_!M)$$

is an isomorphism in $\pi_S(D(MHM(S)))$.

(iii) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$, S quasi-projective. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{ps} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and ps the projection. We have, for $M \in \text{DA}_c(X)$, the map given in definition 122

$$T_*(f, \mathcal{F}^{FDR})(M) : \mathcal{F}_S^{FDR}(Rf_*M) \xrightarrow{\sim} Rf_*^{Hdg} \mathcal{F}_X^{FDR}(M)$$

is an isomorphism in $\pi_S(D(MHM(S)))$.

(iv) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$, S quasi-projective. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{ps} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and ps the projection. Then, for $M \in \text{DA}_c(S)$, the map given in definition 122

$$T^!(f, \mathcal{F}^{FDR})(M) : \mathcal{F}_X^{FDR}(f^!M) \xrightarrow{\sim} f_{Hdg}^{*mod} \mathcal{F}_S^{FDR}(M)$$

is an isomorphism in $\pi_X(D(MHM(X)))$.

(v) Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open affine covering and denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, for $M, N \in \text{DA}_c(S)$, the map in $\pi_S(D(MHM(S)))$

$$T(\mathcal{F}_S^{FDR}, \otimes)(M, N) : \mathcal{F}_S^{FDR}(M) \otimes_{\mathcal{O}_S}^L \mathcal{F}_S^{FDR}(N) \xrightarrow{\sim} \mathcal{F}_S^{FDR}(M \otimes N)$$

given in definition 124 is an isomorphism.

Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_i S_i$ an open cover such that there exists closed embedding $i_i : S \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We define the Hodge realization functor as

$$\begin{aligned} \mathcal{F}_S^{Hdg} &:= (\mathcal{F}_S^{FDR}, \text{Bti}_S^*) : \text{DA}(S) \rightarrow D_{\mathcal{D}fil, \infty}(S/(\tilde{S}_I)) \times_I D(S^{an}), \\ M &\mapsto \mathcal{F}_S^{Hdg}(M) := (\mathcal{F}_S^{FDR}(M), \text{Bti}_S^* M, \alpha(M)), \end{aligned}$$

where $\alpha(M)$ is given in definition 157. The main theorem of this article is the following (theorem 45):

Theorem 6. (i) For $S \in \text{Var}(\mathbb{C})$, we have $\mathcal{F}_S^{Hdg}(\text{DA}_c(S)) \subset D(MHM(S))$.

(ii) The Hodge realization functor \mathcal{F}_-^{Hdg} define a morphism of 2-functor on $\text{Var}(\mathbb{C})$

$$\mathcal{F}_-^{Hdg} : \text{Var}(\mathbb{C}) \rightarrow (\text{DA}_c(-) \rightarrow D(MHM(-)))$$

whose restriction to $\text{QPVar}(\mathbb{C})$ is an homotopic 2-functor in sense of Ayoub. More precisely,

(ii0) for $g : T \rightarrow S$ a morphism, with $T, S \in \text{QPVar}(\mathbb{C})$, and $M \in \text{DA}_c(S)$, the the maps of definition 121 and of definition 152 induce an isomorphism in $D(MHM(T))$

$$\begin{aligned} T(g, \mathcal{F}^{Hdg})(M) &:= (T(g, \mathcal{F}^{FDR})(M), T(g, \text{bti})(M)) : \\ g^{*Hdg} \mathcal{F}_S^{Hdg}(M) &:= (g_{Hdg}^{*mod} \mathcal{F}_S^{FDR}(M), g^* \text{Bti}_S(M), g^*(\alpha(M))) \\ &\xrightarrow{\sim} (\mathcal{F}_T^{FDR}(g^*M), \text{Bti}_T^*(g^*M), \alpha(g^*M)) =: \mathcal{F}_T^{Hdg}(g^*M), \end{aligned}$$

(ii1) for $f : T \rightarrow S$ a morphism, with $T, S \in \text{QPVar}(\mathbb{C})$, and $M \in \text{DA}_c(T)$, the maps of definition 122 and of definition 153 induce an isomorphism in $D(\text{MHM}(S))$

$$\begin{aligned} T_*(f, \mathcal{F}^{Hdg})(M) &:= (T_*(f, \mathcal{F}^{FDR})(M), T_*(f, \text{bti})(M)) : \\ Rf_{Hdg*} \mathcal{F}_T^{Hdg}(M) &:= (Rf_*^{Hdg} \mathcal{F}_T^{FDR}(M), Rf_* \text{Bti}_S(M), f_*(\alpha(M))) \\ &\xrightarrow{\sim} (\mathcal{F}_S^{FDR}(Rf_*M), \text{Bti}_S^*(Rf_*M), \alpha(Rf_*M)) =: \mathcal{F}_S^{Hdg}(Rf_*M), \end{aligned}$$

(ii2) for $f : T \rightarrow S$ a morphism, with $T, S \in \text{QPVar}(\mathbb{C})$, and $M \in \text{DA}_c(T)$, the maps of definition 122 and of definition 153 induce an isomorphism in $D(\text{MHM}(S))$

$$\begin{aligned} T_!(f, \mathcal{F}^{Hdg})(M) &:= (T_!(f, \mathcal{F}^{FDR})(M), T_!(f, \text{bti})(M)) : \\ Rf_{!Hdg} \mathcal{F}_T^{Hdg}(M) &:= (Rf_{!}^{Hdg} \mathcal{F}_T^{FDR}(M), Rf_{!} \text{Bti}_S^*(M), f_!(\alpha(M))) \\ &\xrightarrow{\sim} (\mathcal{F}_S^{FDR}(Rf_{!}M), \text{Bti}_S^*(Rf_{!}M), \alpha(f_{!}M)) =: \mathcal{F}_T^{Hdg}(f_{!}M), \end{aligned}$$

(ii3) for $f : T \rightarrow S$ a morphism, with $T, S \in \text{QPVar}(\mathbb{C})$, and $M \in \text{DA}_c(S)$, the maps of definition 122 and of definition 153 induce an isomorphism in $D(\text{MHM}(T))$

$$\begin{aligned} T^!(f, \mathcal{F}^{Hdg})(M) &:= (T^!(f, \mathcal{F}^{FDR})(M), T^!(f, \text{bti})(M)) : \\ f^{*Hdg} \mathcal{F}_S^{Hdg}(M) &:= (f^{*mod} \mathcal{F}_S^{FDR}(M), f^! \text{Bti}_S(M), f^!(\alpha(M))) \\ &\xrightarrow{\sim} (\mathcal{F}_T^{FDR}(f^!M), \text{Bti}_T^*(f^!M), \alpha(f^!M)) =: \mathcal{F}_T^{Hdg}(f^!M), \end{aligned}$$

(ii4) for $S \in \text{Var}(\mathbb{C})$, and $M, N \in \text{DA}_c(S)$, the maps of definition 124 and of definition 154 induce an isomorphism in $D(\text{MHM}(S))$

$$\begin{aligned} T(\otimes, \mathcal{F}^{Hdg})(M, N) &:= (T(\otimes, \mathcal{F}_S^{FDR})(M, N), T(\otimes, \text{bti})(M, N)) : \\ (\mathcal{F}_S^{FDR}(M) \otimes_{O_S}^L \mathcal{F}_S^{FDR}(N), \text{Bti}_S(M) \otimes \text{Bti}_S(N), \alpha(M) \otimes \alpha(N)) \\ &\xrightarrow{\sim} \mathcal{F}_S^{Hdg}(M \otimes N) := (\mathcal{F}_S^{FDR}(M \otimes N), \text{Bti}_S(M \otimes N), \alpha(M \otimes N)). \end{aligned}$$

(iii) For $S \in \text{Var}(\mathbb{C})$, the following diagram commutes :

$$\begin{array}{ccc} \text{Var}(\mathbb{C})/S & \xrightarrow{MH(/S)} & D(\text{MHM}(S)) \\ \downarrow M(/S) & & \downarrow \iota^S \\ \text{DA}(S) & \xrightarrow{\mathcal{F}_S^{Hdg}} & D_{\mathcal{D}fil, \infty}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}) \end{array}$$

We obtain theorem 6 from theorem 5 and from the result on the Betti factor after checking the compatibility of these transformation maps with the isomorphisms $\alpha(M)$.

I thank F.Mokrane for his help and support during his work as well as J.Wildeshaus for the interest and remarks that he made on a first version of this text.

2 Generalities and Notations

2.1 Notations

- After fixing a universe, we denote by
 - Set the category of sets,
 - Top the category of topological spaces,
 - Ring the category of rings and $\text{cRing} \subset \text{Ring}$ the full subcategory of commutative rings,

- RTop the category of ringed spaces,
 - * whose set of objects is $\text{RTop} := \{(X, O_X), X \in \text{Top}, O_X \in \text{PSh}(X, \text{Ring})\}$
 - * whose set of morphism is $\text{Hom}((T, O_T), (S, O_S)) := \{((f : T \rightarrow S), (a_f : f^*O_S \rightarrow O_T))\}$
 and by $ts : \text{RTop} \rightarrow \text{Top}$ the forgetfull functor.
- Cat the category of small categories which comes with the forgetful functor $o : \text{Cat} \rightarrow \text{Fun}(\Delta^1, \text{Set})$, where $\text{Fun}(\Delta^1, \text{Set})$ is the category of simplicial sets,
- RCat the category of ringed topoi
 - * whose set of objects is $\text{RTop} := \{(\mathcal{X}, O_{\mathcal{X}}), \mathcal{X} \in \text{Cat}, O_{\mathcal{X}} \in \text{PSh}(\mathcal{X}, \text{Ring})\}$,
 - * whose set of morphism is $\text{Hom}((\mathcal{T}, O_{\mathcal{T}}), (\mathcal{S}, O_{\mathcal{S}})) := \{((f : \mathcal{T} \rightarrow \mathcal{S}), (a_f : f^*O_{\mathcal{S}} \rightarrow O_{\mathcal{T}})), \}$
 and by $tc : \text{RCat} \rightarrow \text{Cat}$ the forgetfull functor.
- Let $F : \mathcal{C} \rightarrow \mathcal{C}'$ be a functor with $\mathcal{C}, \mathcal{C}' \in \text{Cat}$. For $X \in \mathcal{C}$, we denote by $F(X) \in \mathcal{C}'$ the image of X , and for $X, Y \in \mathcal{C}$, we denote by $F^{X, Y} : \text{Hom}(X, Y) \rightarrow \text{Hom}(F(X), F(Y))$ the corresponding map.
- Let $\mathcal{C} \in \text{Cat}$. For $S \in \mathcal{C}$, we denote by \mathcal{C}/S the category
 - whose set of objects $(\mathcal{C}/S)^0 = \{X/S = (X, h)\}$ consist of the morphisms $h : X \rightarrow S$ with $X \in \mathcal{C}$,
 - whose set of morphism $\text{Hom}(X'/S, X/S)$ between $X'/S = (X', h'), X/S = (X, h) \in \mathcal{C}/S$ consists of the morphisms $(g : X' \rightarrow X) \in \text{Hom}(X', X)$ such that $h \circ g = h'$.

We have then, for $S \in \mathcal{C}$, the canonical forgetful functor

$$r(S) : \mathcal{C}/S \rightarrow \mathcal{C}, \quad X/S \mapsto r(S)(X/S) = X, \quad (g : X'/S \rightarrow X/S) \mapsto r(S)(g) = g$$

and we denote again $r(S) : \mathcal{C} \rightarrow \mathcal{C}/S$ the corresponding morphism of (pre)sites.

- Let $F : \mathcal{C} \rightarrow \mathcal{C}'$ be a functor with $\mathcal{C}, \mathcal{C}' \in \text{Cat}$. Then for $S \in \mathcal{C}$, we have the canonical functor

$$F_S : \mathcal{C}/S \rightarrow \mathcal{C}'/F(S), \quad X/S \mapsto F(X/S) = F(X)/F(S), \\ (g : X'/S \rightarrow X/S) \mapsto (F(g) : F(X')/F(S) \rightarrow F(X)/F(S))$$

- Let $\mathcal{S} \in \text{Cat}$. Then, for a morphism $f : X' \rightarrow X$ with $X, X' \in \mathcal{S}$ we have the functor

$$C(f) : \mathcal{S}/X' \rightarrow \mathcal{S}/X, \quad Y/X' = (Y, f_1) \mapsto C(f)(Y/X') := (Y, f \circ f_1) \in \mathcal{S}/X, \\ (g : Y_1/X' \rightarrow Y_2/X') \mapsto (C(f)(g) := g : Y_1/X \rightarrow Y_2/X)$$

- Let $\mathcal{S} \in \text{Cat}$ a category which admits fiber products. Then, for a morphism $f : X' \rightarrow X$ with $X, X' \in \mathcal{S}$, we have the pullback functor

$$P(f) : \mathcal{S}/X \rightarrow \mathcal{S}/X', \quad Y/X \mapsto P(f)(Y/X) := Y \times_X X'/X' \in \mathcal{S}/X', \\ (g : Y_1/X \rightarrow Y_2/X) \mapsto (P(f)(g) := (g \times I) : Y_1 \times_X X' \rightarrow Y_2 \times_X X')$$

which is right adjoint to $C(f) : \mathcal{S}/X' \rightarrow \mathcal{S}/X$, and we denote again $P(f) : \mathcal{S}/X' \rightarrow \mathcal{S}/X$ the corresponding morphism of (pre)sites.

- Let $\mathcal{C}, \mathcal{I} \in \text{Cat}$. Assume that \mathcal{C} admits fiber products. For $(S_{\bullet}) \in \text{Fun}(\mathcal{I}^{op}, \mathcal{C})$, we denote by $\mathcal{C}/(S_{\bullet}) \in \text{Fun}(\mathcal{I}, \text{Cat})$ the diagram of category given by
 - for $I \in \mathcal{I}$, $\mathcal{C}/(S_{\bullet})(I) := \mathcal{C}/S_I$,
 - for $r_{IJ} : I \rightarrow J$, $\mathcal{C}/(S_{\bullet})(r_{IJ}) := P(r_{IJ}) : \mathcal{C}/S_I \rightarrow \mathcal{C}/S_J$, where we denoted again $r_{IJ} : S_J \rightarrow S_I$ the associated morphism in \mathcal{C} .

- Let $(F, G) : \mathcal{C} \rightleftarrows \mathcal{C}'$ an adjunction between two categories.
 - For $X \in \mathcal{C}$ and $Y \in \mathcal{C}'$, we consider the adjunction isomorphisms
 - * $I(F, G)(X, Y) : \text{Hom}(F(X), Y) \rightarrow \text{Hom}(X, G(Y))$, $(u : F(X) \rightarrow Y) \mapsto (I(F, G)(X, Y)(u) : X \rightarrow G(Y))$
 - * $I(F, G)(X, Y) : \text{Hom}(X, G(Y)) \rightarrow \text{Hom}(F(X), Y)$, $(v : X \rightarrow G(Y)) \mapsto (I(F, G)(X, Y)(v) : F(X) \rightarrow Y)$.
 - For $X \in \mathcal{C}$, we denote by $\text{ad}(F, G)(X) := I(F, G)(X, F(X))(I_{F(X)}) : X \rightarrow G \circ F(X)$.
 - For $Y \in \mathcal{C}'$ we denote also by $\text{ad}(F, G)(Y) := I(F, G)(G(Y), Y)(I_{G(Y)}) : F \circ G(Y) \rightarrow Y$.

Hence,

- for $u : F(X) \rightarrow Y$ a morphism with $X \in \mathcal{C}$ and $Y \in \mathcal{C}'$, we have $I(F, G)(X, Y)(u) = G(u) \circ \text{ad}(F, G)(X)$,
- for $v : X \rightarrow G(Y)$ a morphism with $X \in \mathcal{C}$ and $Y \in \mathcal{C}'$, we have $I(F, G)(X, Y)(v) = \text{ad}(F, G)(Y) \circ F(v)$.
- Let \mathcal{C} a category.
 - We denote by (\mathcal{C}, F) the category of filtered objects : $(X, F) \in (\mathcal{C}, F)$ is a sequence $(F^\bullet X)_{\bullet \in \mathbb{Z}}$ indexed by \mathbb{Z} with value in \mathcal{C} together with monomorphisms $a_p : F^p X \hookrightarrow F^{p-1} X \hookrightarrow X$.
 - We denote by (\mathcal{C}, F, W) the category of bifiltered objects : $(X, F, W) \in (\mathcal{C}, F, W)$ is a sequence $(W^\bullet F^\bullet X)_{\bullet, \bullet \in \mathbb{Z}^2}$ indexed by \mathbb{Z}^2 with value in \mathcal{C} together with monomorphisms $W^q F^p X \hookrightarrow F^{p-1} X, W^q F^p X \hookrightarrow W^{q-1} F^p X$.
- For \mathcal{C} a category and $\Sigma : \mathcal{C} \rightarrow \mathcal{C}$ an endofunctor, we denote by (\mathcal{C}, Σ) the corresponding category of spectra, whose objects are sequence of objects of \mathcal{C} $(T_i)_{i \in \mathbb{Z}} \in \text{Fun}(\mathbb{Z}, \mathcal{C})$ together with morphisms $s_i : T_i \rightarrow \Sigma T_{i+1}$, and whose morphism from (T_i) to (T'_i) are sequence of morphisms $T_i \rightarrow T'_i$ which commutes with the s_i .
- Let \mathcal{A} an additive category.
 - We denote by $C(\mathcal{A}) := \text{Fun}(\mathbb{Z}, \mathcal{A})$ the category of (unbounded) complexes with value in \mathcal{A} , where we have denoted \mathbb{Z} the category whose set of objects is \mathbb{Z} , and whose set of morphism between $m, n \in \mathbb{Z}$ consists of one element (identity) if $n = m$, of one element if $n = m + 1$ and is \emptyset in the other cases.
 - We have the full subcategories $C^b(\mathcal{A}), C^-(\mathcal{A}), C^+(\mathcal{A})$ of $C(\mathcal{A})$ consisting of bounded, resp. bounded above, resp. bounded below complexes.
 - We denote by $K(\mathcal{A}) := \text{Ho}(C(\mathcal{A}))$ the homotopy category of $C(\mathcal{A})$ and by $Ho : C(\mathcal{A}) \rightarrow K(\mathcal{A})$ the full homotopy functor. The category $K(\mathcal{A})$ is in the standard way a triangulated category.
- Let \mathcal{A} an additive category.
 - We denote by $C_{fil}(\mathcal{A}) \subset (C(\mathcal{A}), F) = C(\mathcal{A}, F)$ the full additive subcategory of filtered complexes of \mathcal{A} such that the filtration is biregular : for $(A^\bullet, F) \in (C(\mathcal{A}), F)$, we say that F is biregular if $F^\bullet A^r$ is finite for all $r \in \mathbb{Z}$.
 - We denote by $C_{2fil}(\mathcal{A}) \subset (C(\mathcal{A}), F, W) = C(\mathcal{A}, F, W)$ the full subcategory of bifiltered complexes of \mathcal{A} such that the filtration is biregular.
 - For $A^\bullet \in C(\mathcal{A})$, we denote by $(A^\bullet, F_b) \in (C(\mathcal{A}), F)$ the complex endowed with the trivial filtration (filtration bete) : $F^p A^n = 0$ if $p \geq n + 1$ and $F^p A^n = A^n$ if $p \leq n$.

- Let \mathcal{A} be an abelian category. Then the additive category (\mathcal{A}, F) is an exact category which admits kernel and cokernel (but is NOT an abelian category). A morphism $\phi : (M, F) \rightarrow (N, F)$ with $(M, F) \in (\mathcal{A}, F)$ is strict if the inclusion $\phi(F^n M) \subset F^n N \cap \text{Im}(\phi)$ is an equality, i.e. if $\phi(F^n M) = F^n N \cap \text{Im}(\phi)$.

- Let \mathcal{A} be an abelian category.

- For $(A^\bullet, F) \in C(\mathcal{A}, F)$, considering $a_p : F^p A^\bullet \hookrightarrow A^\bullet$ the structural monomorphism of the filtration, we denote by, for $n \in \mathbb{N}$,

$$H^n(A^\bullet, F) \in (\mathcal{A}, F), \quad F^p H^n(A^\bullet, F) := \text{Im}(H^n(a_p) : H^n(F^p A^\bullet) \rightarrow H^n(A^\bullet)) \subset H^n(A^\bullet)$$

the filtration induced on the cohomology objects of the complex. In the case $(A^\bullet, F) \in C_{fil}(\mathcal{A})$, the spectral sequence $E_r^{p,q}(A^\bullet, F)$ associated to (A^\bullet, F) converge to $\text{Gr}_F^p H^{p+q}(A^\bullet, F)$, that is for all $p, q \in \mathbb{Z}$, there exist $r_{p,q} \in \mathbb{N}$, such that $E_s^{p,q}(A^\bullet, F) = \text{Gr}_F^p H^{p+q}(A^\bullet, F)$ for all $s \leq r_{p,q}$.

- Let $r \in \mathbb{N}$. A morphism $m : (A^\bullet, F) \rightarrow (B^\bullet, F)$ with $(A^\bullet, F), (B^\bullet, F) \in C(\mathcal{A}, F)$ is said to be an r -filtered quasi-isomorphism if for all $p, q \in \mathbb{Z}$,

$$E_r^{p,q}(m) : E_r^{p,q}(A^\bullet, F) \xrightarrow{\sim} E_r^{p,q}(B^\bullet, F)$$

is an isomorphism in \mathcal{A} , note that by definition this r does NOT depend on p and q . A filtered quasi-isomorphism is by definition an 1-filtered quasi-isomorphism.

- A morphism $m : (A^\bullet, F) \rightarrow (B^\bullet, F)$ with $(A^\bullet, F), (B^\bullet, F) \in C(\mathcal{A}, F)$ is said to be an ∞ -filtered quasi-isomorphism if there exist $r \in \mathbb{N}$ such that for all $p, q \in \mathbb{Z}$,

$$E_r^{p,q}(m) : E_r^{p,q}(A^\bullet, F) \xrightarrow{\sim} E_r^{p,q}(B^\bullet, F)$$

is an isomorphism in \mathcal{A} , note that by definition this r does NOT depend on p and q . If a morphism $m : (A^\bullet, F) \rightarrow (B^\bullet, F)$, with $(A^\bullet, F), (B^\bullet, F) \in C_{fil}(\mathcal{A})$ is an ∞ -filtered quasi-isomorphism, then for all $n \in \mathbb{Z}$

$$H^n(m) : H^n(A^\bullet, F) \xrightarrow{\sim} H^n(B^\bullet, F)$$

is a filtered isomorphism, i.e. an isomorphism in (\mathcal{A}, F) . Note that the converse is NOT true since the integer $r_{p,q} \in \mathbb{N}$ such that $E_s^{p,q}(A^\bullet, F) = \text{Gr}_F^p H^{p+q}(A^\bullet, F)$ and $E_s^{p,q}(B^\bullet, F) = \text{Gr}_F^p H^{p+q}(B^\bullet, F)$ for $s \leq r_{p,q}$ depends on p and q .

- Let \mathcal{A} be an abelian category.

- We denote by $D(\mathcal{A})$ the localization of $K(\mathcal{A})$ with respect to the quasi-isomorphisms and by $D : K(\mathcal{A}) \rightarrow D(\mathcal{A})$ the localization functor. The category $D(\mathcal{A})$ is a triangulated category in the unique way such that D a triangulated functor.

- We denote by $D_{fil,r}(\mathcal{A})$ the localization of $K_{fil}(\mathcal{A})$ with respect to the r -filtered quasi-isomorphisms and by $D : K_{fil}(\mathcal{A}) \rightarrow D_{fil,r}(\mathcal{A})$ the localization functor.

- We denote by $D_{fil,\infty}(\mathcal{A})$ the localization of $K_{fil}(\mathcal{A})$ with respect to the ∞ -filtered quasi-isomorphisms and by $D : K_{fil}(\mathcal{A}) \rightarrow D_{fil,\infty}(\mathcal{A})$ the localization functor.

- Let \mathcal{A} be an abelian category. We denote by $\text{Inj}(\mathcal{A}) \subset \mathcal{A}$ the full subcategory of injective objects, and by $\text{Proj}(\mathcal{A}) \subset \mathcal{A}$ the full subcategory of projective objects.

- For $\mathcal{S} \in \text{Cat}$ a small category, we denote by

- $\text{PSh}(\mathcal{S}) := \text{PSh}(\mathcal{S}, \text{Ab}) := \text{Fun}(\mathcal{S}, \text{Ab})$ the category of presheaves on \mathcal{S} , i.e. the category of presheaves of abelian groups on \mathcal{S} ,

- $\text{PSh}(\mathcal{S}, \text{Ring}) := \text{Fun}(\mathcal{S}, \text{Ring})$ the category of presheaves of ring on \mathcal{S} , and $\text{PSh}(\mathcal{S}, \text{cRing}) \subset \text{PSh}(\mathcal{S}, \text{Ring})$ the full subcategory of presheaves of commutative ring.
- for $F \in \text{PSh}(\mathcal{S})$ and $X \in \mathcal{S}$, $F(X) = \Gamma(X, F)$ the sections on X and for $h : X' \rightarrow X$ a morphism with $X, X' \in \mathcal{S}$, $F(h) := F^{X, Y}(h) : F(X) \rightarrow F(X')$ the morphism of abelian groups,
- $C(\mathcal{S}) = \text{PSh}(\mathcal{S}, C(\mathbb{Z})) = C(\text{PSh}(\mathcal{S})) = \text{PSh}(\mathcal{S} \times \mathbb{Z})$ the big abelian category of complexes of presheaves on \mathcal{S} with value in abelian groups,
- $K(\mathcal{S}) := K(\text{PSh}(\mathcal{S})) = \text{Ho}(C(\mathcal{S}))$ In particular, we have the full homotopy functor $Ho : C(\mathcal{S}) \rightarrow K(\mathcal{S})$,
- $C_{(2)\text{fil}}(\mathcal{S}) := C_{(2)\text{fil}}(\text{PSh}(\mathcal{S})) \subset C(\text{PSh}(\mathcal{S}), F, W)$ the big abelian category of (bi)filtered complexes of presheaves on \mathcal{S} with value in abelian groups such that the filtration is biregular, and $\text{PSh}_{(2)\text{fil}}(\mathcal{S}) = (\text{PSh}(\mathcal{S}), F, W)$,
- $K_{\text{fil}}(\mathcal{S}) := K(\text{PSh}_{\text{fil}}(\mathcal{S})) = \text{Ho}(C_{\text{fil}}(\mathcal{S}))$

For $f : \mathcal{T} \rightarrow \mathcal{S}$ a morphism a presite with $\mathcal{T}, \mathcal{S} \in \text{Cat}$, given by the functor $P(f) : \mathcal{S} \rightarrow \mathcal{T}$, we will consider the adjunctions given by the direct and inverse image functors :

- $(f^*, f_*) = (f^{-1}, f_*) : \text{PSh}(\mathcal{S}) \rightleftarrows \text{PSh}(\mathcal{T})$, which induces $(f^*, f_*) : C(\mathcal{S}) \rightleftarrows C(\mathcal{T})$, we denote, for $F \in C(\mathcal{S})$ and $G \in C(\mathcal{T})$ by

$$\text{ad}(f^*, f_*)(F) : F \rightarrow f_* f^* F, \text{ad}(f^*, f_*)(G) : f^* f_* G \rightarrow G$$

the adjunction maps,

- $(f_*, f^\perp) : \text{PSh}(\mathcal{T}) \rightleftarrows \text{PSh}(\mathcal{S})$, which induces $(f_*, f^\perp) : C(\mathcal{T}) \rightleftarrows C(\mathcal{S})$, we denote for $F \in C(\mathcal{S})$ and $G \in C(\mathcal{T})$ by

$$\text{ad}(f_*, f^\perp)(F) : G \rightarrow f^\perp f_* G, \text{ad}(f_*, f^\perp)(G) : f_* f^\perp F \rightarrow F$$

the adjunction maps.

- For $(\mathcal{S}, O_S) \in \text{RCat}$ a ringed topos, we denote by

- $\text{PSh}_{O_S}(\mathcal{S})$ the category of presheaves of O_S modules on \mathcal{S} , whose objects are $\text{PSh}_{O_S}(\mathcal{S})^0 := \{(M, m), M \in \text{PSh}(\mathcal{S}), m : M \otimes O_S \rightarrow M\}$, together with the forgetful functor $o : \text{PSh}(\mathcal{S}) \rightarrow \text{PSh}_{O_S}(\mathcal{S})$,
- $C_{O_S}(\mathcal{S}) = C(\text{PSh}_{O_S}(\mathcal{S}))$ the big abelian category of complexes of presheaves of O_S modules on \mathcal{S} ,
- $K_{O_S}(\mathcal{S}) := K(\text{PSh}_{O_S}(\mathcal{S})) = \text{Ho}(C_{O_S}(\mathcal{S}))$, in particular, we have the full homotopy functor $Ho : C_{O_S}(\mathcal{S}) \rightarrow K_{O_S}(\mathcal{S})$,
- $C_{O_S(2)\text{fil}}(\mathcal{S}) := C_{(2)\text{fil}}(\text{PSh}_{O_S}(\mathcal{S})) \subset C(\text{PSh}_{O_S}(\mathcal{S}), F, W)$, the big abelian category of (bi)filtered complexes of presheaves of O_S modules on \mathcal{S} such that the filtration is biregular and $\text{PSh}_{O_S(2)\text{fil}}(\mathcal{S}) = (\text{PSh}_{O_S}(\mathcal{S}), F, W)$,
- $K_{O_S(2)\text{fil}}(\mathcal{S}) := K(\text{PSh}_{O_S(2)\text{fil}}(\mathcal{S})) = \text{Ho}(C_{O_S(2)\text{fil}}(\mathcal{S}))$.

- For $\mathcal{S}_\bullet \in \text{Fun}(\mathcal{I}, \text{Cat})$ a diagram of (pre)sites, with $\mathcal{I} \in \text{Cat}$ a small category, we denote by

- $\text{PSh}(\mathcal{S}_\bullet) := \text{PSh}(\mathcal{S}_\bullet, \text{Ab})$ the category of presheaves on \mathcal{S}_\bullet ,
 - * whose objects are $\text{PSh}(\mathcal{S}_\bullet)^0 := \{(F_I, u_{IJ})_{I \in \mathcal{I}}\}$, with $F_I \in \text{PSh}(\mathcal{S}_I)$, and for $r_{IJ} : I \rightarrow J$ with $I, J \in \mathcal{I}$, $u_{IJ} : F_I \rightarrow r_{IJ*} F_J$ are morphism in $\text{PSh}(\mathcal{S}_I)$, noting again $r_{IJ} : \mathcal{S}_J \rightarrow \mathcal{S}_I$ the associated morphism of presite,
 - * whose morphism are $m = (m_I) : (F_I, u_{IJ}) \rightarrow (G_I, v_{IJ})$ satisfying $v_{IJ} \circ m_I = r_{IJ*} m_J \circ u_{IJ}$ in $\text{PSh}(\mathcal{S}_I)$,

- $\text{PSh}(\mathcal{S}_\bullet, \text{Ring})$ the category of presheaves of ring on \mathcal{S}_\bullet given in the same way, and $\text{PSh}(\mathcal{S}_\bullet, \text{cRing}) \subset \text{PSh}(\mathcal{S}_\bullet, \text{Ring})$ the full subcategory of presheaves of commutative ring.
- $C(\mathcal{S}_\bullet) := C(\text{PSh}(\mathcal{S}_\bullet))$ the big abelian category of complexes of presheaves on \mathcal{S}_\bullet with value in abelian groups,
- $K(\mathcal{S}_\bullet) := K(\text{PSh}(\mathcal{S}_\bullet)) = \text{Ho}(C(\mathcal{S}_\bullet))$, in particular, we have the full homotopy functor $Ho : C(\mathcal{S}_\bullet) \rightarrow K(\mathcal{S}_\bullet)$,
- $C_{(2)fil}(\mathcal{S}_\bullet) := C_{(2)fil}(\text{PSh}(\mathcal{S}_\bullet)) \subset C(\text{PSh}(\mathcal{S}_\bullet), F, W)$ the big abelian category of (bi)filtered complexes of presheaves on \mathcal{S}_\bullet with value in abelian groups such that the filtration is biregular, and $\text{PSh}_{(2)fil}(\mathcal{S}_\bullet) = (\text{PSh}(\mathcal{S}_\bullet), F, W)$, by definition $C_{(2)fil}(\mathcal{S}_\bullet)$ is the category
 - * whose objects are $C_{(2)fil}(\mathcal{S}_\bullet)^0 := \{(F_I, F, W), u_{IJ}\}_{I \in \mathcal{I}}$, with $(F_I, F, W) \in C_{(2)fil}(\mathcal{S}_I)$, and for $r_{IJ} : I \rightarrow J$ with $I, J \in \mathcal{I}$, $u_{IJ} : (F_I, F, W) \rightarrow r_{IJ*}(F_J, F, W)$ are morphism in $C_{(2)fil}(\mathcal{S}_I)$, noting again $r_{IJ} : \mathcal{S}_J \rightarrow \mathcal{S}_I$ the associated morphism of presite,
 - * whose morphism are $m = (m_I) : ((F_I, F, W), u_{IJ}) \rightarrow ((G_I, F, W), v_{IJ})$ satisfying $v_{IJ} \circ m_I = r_{IJ*}m_J \circ u_{IJ}$ in $C_{(2)fil}(\mathcal{S}_I)$,
- $K_{fil}(\mathcal{S}_\bullet) := K(\text{PSh}_{fil}(\mathcal{S}_\bullet)) = \text{Ho}(C_{fil}(\mathcal{S}_\bullet))$

Let $\mathcal{I}, \mathcal{I}' \in \text{Cat}$ be small categories. Let $(f_\bullet, s) : \mathcal{T}_\bullet \rightarrow \mathcal{S}_\bullet$ a morphism a diagrams of (pre)site with $\mathcal{T}_\bullet \in \text{Fun}(\mathcal{I}, \text{Cat})$, $\mathcal{S}_\bullet \in \text{Fun}(\mathcal{I}', \text{Cat})$, which is by definition given by a functor $s : \mathcal{I} \rightarrow \mathcal{I}'$ and morphism of functor $P(f_\bullet) : \mathcal{S}_{s(\bullet)} := \mathcal{S}_\bullet \circ s \rightarrow \mathcal{T}_\bullet$. Here, we denote for short, $\mathcal{S}_{s(\bullet)} := \mathcal{S}_\bullet \circ s \in \text{Fun}(\mathcal{I}, \text{Cat})$. We have then, for $r_{IJ} : I \rightarrow J$ a morphism, with $I, J \in \mathcal{I}$, a commutative diagram in Cat

$$D_{fIJ} := \begin{array}{ccc} \mathcal{S}_{s(J)} & \xrightarrow{r_{IJ}^s} & \mathcal{S}_{s(I)} \\ f_J \uparrow & & \uparrow f_I \\ \mathcal{T}_J & \xrightarrow{r_{IJ}^t} & \mathcal{T}_I \end{array} .$$

We will consider the the adjunction given by the direct and inverse image functors :

$$\begin{aligned} ((f_\bullet, s)^*, (f_\bullet, s)_*) &= ((f_\bullet, s)^{-1}, (f_\bullet, s)_*) : \text{PSh}(\mathcal{S}_{s(\bullet)}) \rightleftarrows \text{PSh}(\mathcal{T}_\bullet), \\ F &= (F_I, u_{IJ}) \mapsto (f_\bullet, s)^* F := (f_I^* F_I, T(D_{fIJ})(F_J) \circ f_I^* u_{IJ}), \\ G &= (G_I, v_{IJ}) \mapsto (f_\bullet, s)_* G := (f_{I*} G_I, f_{I*} v_{IJ}). \end{aligned}$$

It induces the adjunction $((f_\bullet, s)^*, (f_\bullet, s)_*) : C(\mathcal{S}_{s(\bullet)}) \rightleftarrows C(\mathcal{T}_\bullet)$. We denote, for $(F_I, u_{IJ}) \in C(\mathcal{S}_{s(\bullet)})$ and $(G_I, v_{IJ}) \in C(\mathcal{T}_\bullet)$ by

$$\begin{aligned} \text{ad}((f_\bullet, s)^*, (f_\bullet, s)_*)((F_I, u_{IJ})) &: (F_I, u_{IJ}) \rightarrow (f_\bullet, s)_*(f_\bullet, s)^*(F_I, u_{IJ}), \\ \text{ad}((f_\bullet, s)^*, (f_\bullet, s)_*)((G_I, v_{IJ})) &: (f_\bullet, s)^*(f_\bullet, s)_*(G_I, v_{IJ}) \rightarrow (G_I, v_{IJ}) \end{aligned}$$

the adjunction maps.

- Let $\mathcal{I} \in \text{Cat}$ a small category. For $(\mathcal{S}_\bullet, O_{\mathcal{S}_\bullet}) \in \text{Fun}(\mathcal{I}, \text{RCat})$ a diagram of ringed topos, we denote by
 - $\text{PSh}_{O_{\mathcal{S}_\bullet}}(\mathcal{S}_\bullet)$ the category of presheaves of modules on $(\mathcal{S}_\bullet, O_{\mathcal{S}_\bullet})$,
 - * whose objects are $\text{PSh}_{O_{\mathcal{S}_\bullet}}(\mathcal{S}_\bullet)^0 := \{(F_I, u_{IJ})\}_{I \in \mathcal{I}}$, with $F_I \in \text{PSh}_{O_{\mathcal{S}_I}}(\mathcal{S}_I)$, and for $r_{IJ} : I \rightarrow J$ with $I, J \in \mathcal{I}$, $u_{IJ} : F_I \rightarrow r_{IJ*}F_J$ are morphism in $\text{PSh}_{O_{\mathcal{S}_I}}(\mathcal{S}_I)$, noting again $r_{IJ} : \mathcal{S}_J \rightarrow \mathcal{S}_I$ the associated morphism of presite,
 - * whose morphism are $m = (m_I) : (F_I, u_{IJ}) \rightarrow (G_I, v_{IJ})$ satisfying $v_{IJ} \circ m_I = r_{IJ*}m_J \circ u_{IJ}$ in $\text{PSh}_{O_{\mathcal{S}_I}}(\mathcal{S}_I)$,
 - $C_{O_{\mathcal{S}_\bullet}}(\mathcal{S}_\bullet) := C(\text{PSh}_{O_{\mathcal{S}_\bullet}}(\mathcal{S}_\bullet))$,

- $K_{O_{S_\bullet}}(\mathcal{S}_\bullet) := K(\text{PSh}_{O_{S_\bullet}}(\mathcal{S}_\bullet)) = \text{Ho}(C_{O_{S_\bullet}}(\mathcal{S}_\bullet))$, in particular, we have the full homotopy functor $Ho : C(\mathcal{S}_\bullet) \rightarrow K(\mathcal{S}_\bullet)$,
 - $C_{O_{S_\bullet}(2)fil}(\mathcal{S}_\bullet) := C_{O_{S_\bullet}(2)fil}(\text{PSh}(\mathcal{S}_\bullet)) \subset C(\text{PSh}_{O_{S_\bullet}}(\mathcal{S}_\bullet), F, W)$ the big abelian category of (bi)filtered complexes of presheaves of modules on $(\mathcal{S}_\bullet, O_{S_\bullet})$ such that the filtration is biregular, and $\text{PSh}_{O_{S_\bullet}(2)fil}(\mathcal{S}_\bullet) = (\text{PSh}_{O_{S_\bullet}}(\mathcal{S}_\bullet), F, W)$, by definition $C_{O_{S_\bullet}(2)fil}(\mathcal{S}_\bullet)$ is the category
 - * whose objects are $C_{O_{S_\bullet}(2)fil}(\mathcal{S}_\bullet)^0 := \{(F_I, F, W), u_{IJ}\}_{I \in \mathcal{I}}$, with $(F_I, F, W) \in C_{O_{S_I}(2)fil}(\mathcal{S}_I)$, and for $r_{IJ} : I \rightarrow J$ with $I, J \in \mathcal{I}$, $u_{IJ} : (F_I, F, W) \rightarrow r_{IJ*}(F_J, F, W)$ are morphism in $C_{O_{S_I}(2)fil}(\mathcal{S}_I)$, noting again $r_{IJ} : \mathcal{S}_J \rightarrow \mathcal{S}_I$ the associated morphism of presite,
 - * whose morphism are $m = (m_I) : ((F_I, F, W), u_{IJ}) \rightarrow ((G_I, F, W), v_{IJ})$ satisfying $v_{IJ} \circ m_I = r_{IJ*}m_J \circ u_{IJ}$ in $C_{O_{S_I}(2)fil}(\mathcal{S}_I)$,
 - $K_{O_{S_\bullet}(2)fil}(\mathcal{S}_\bullet) := K(\text{PSh}_{O_{S_\bullet}(2)fil}(\mathcal{S}_\bullet)) = \text{Ho}(C_{O_{S_\bullet}(2)fil}(\mathcal{S}_\bullet))$.
- Let $\mathcal{S} \in \text{Cat}$. For $\Sigma : C(\mathcal{S}) \rightarrow C(\mathcal{S})$ an endofunctor, we denote by $C_\Sigma(\mathcal{S}) = (C(\mathcal{S}), \Sigma)$ the corresponding category of spectra.
 - Denote by $\text{Sch} \subset \text{RTop}$ the full subcategory of schemes. For a field k , we consider $\text{Sch}/k := \text{Sch}/\text{Spec } k$ the category of schemes over $\text{Spec } k$. We then denote by
 - $\text{Var}(k) \subset \text{Sch}/k$ the full subcategory consisting of algebraic varieties over k , i.e. schemes of finite type over k ,
 - $\text{PVar}(k) \subset \text{QPVar}(k) \subset \text{Var}(k)$ the full subcategories consisting of quasi-projective varieties and projective varieties respectively,
 - $\text{PSmVar}(k) \subset \text{SmVar}(k) \subset \text{Var}(k)$ the full subcategories consisting of smooth varieties and smooth projective varieties respectively.

- Denote by Top^2 the category whose set of objects is

$$(\text{Top}^2)^0 := \{(X, Z), Z \subset X \text{ closed}\} \subset \text{Top} \times \text{Top}$$

and whose set of morphism between $(X_1, Z_1), (X_2, Z_2) \in \text{Top}^2$ is

$$\text{Hom}_{\text{Top}^2}((X_1, Z_1), (X_2, Z_2)) := \{(f : X_1 \rightarrow X_2), \text{ s.t. } Z_1 \subset f^{-1}(Z_2)\} \subset \text{Hom}_{\text{Top}}(X_1, X_2)$$

For $S \in \text{Top}$, $\text{Top}^2/S := \text{Top}^2/(S, S)$ is then by definition the category whose set of objects is

$$(\text{Top}^2/S)^0 := \{((X, Z), h), h : X \rightarrow S, Z \subset X \text{ closed}\} \subset \text{Top}^2/S \times \text{Top}$$

and whose set of morphisms between $(X_1, Z_1)/S = ((X_1, Z_1), h_1), (X_2, Z_2)/S = ((X_2, Z_2), h_2) \in \text{Top}^2/S$ is the subset

$$\begin{aligned} & \text{Hom}_{\text{Top}^2/S}((X_1, Z_1)/S, (X_2, Z_2)/S) := \\ & \{(f : X_1 \rightarrow X_2), \text{ s.t. } h_1 \circ f = h_2 \text{ and } Z_1 \subset f^{-1}(Z_2)\} \subset \text{Hom}_{\text{RTop}}(X_1, X_2) \end{aligned}$$

We denote by

$$\mu_S : \text{Top}^{2,pr}/S := \{((Y \times S, Z), p), p : Y \times S \rightarrow S, Z \subset Y \times S \text{ closed}\} \hookrightarrow \text{Top}^2/S$$

the full subcategory whose objects are those with $p : Y \times S \rightarrow S$ a projection, and again $\mu_S : \text{Top}^2/S \rightarrow \text{Top}^{2,pr}/S$ the corresponding morphism of sites. We denote by

$$\begin{aligned} & \text{Gr}_S^{12} : \text{Top}^2/S \rightarrow \text{Top}^{2,pr}/S, X/S \mapsto \text{Gr}_S^{12}(X/S) := (X \times S, \bar{X})/S, \\ & (g : X/S \rightarrow X'/S) \mapsto \text{Gr}_S^{12}(g) := (g \times I_S : (X \times S, \bar{X}) \rightarrow (X' \times S, \bar{X}')) \end{aligned}$$

the graph functor, $X \hookrightarrow X \times S$ being the graph embedding (which is a closed embedding if X is separated), and again $\text{Gr}_S^{12} : \text{Top}^{2,pr}/S \rightarrow \text{Top}^2/S$ the corresponding morphism of sites.

- Denote by \mathbf{RTop}^2 the category whose set of objects is

$$(\mathbf{RTop}^2)^0 := \{((X, O_X), Z), Z \subset X \text{ closed}\} \subset \mathbf{RTop} \times \mathbf{Top}$$

and whose set of morphism between $((X_1, O_{X_1}), Z_1), ((X_2, O_{X_2}), Z_2) \in \mathbf{RTop}^2$ is

$$\begin{aligned} \mathbf{Hom}_{\mathbf{RTop}^2}(((X_1, O_{X_1}), Z_1), ((X_2, O_{X_2}), Z_2)) := \\ \{(f : (X_1, O_{X_1}) \rightarrow (X_2, O_{X_2})), \text{ s.t. } Z_1 \subset f^{-1}(Z_2)\} \subset \mathbf{Hom}_{\mathbf{RTop}}((X_1, O_{X_1}), (X_2, O_{X_2})) \end{aligned}$$

For $(S, O_S) \in \mathbf{RTop}$, $\mathbf{RTop}^2 / (S, O_S) := \mathbf{RTop}^2 / ((S, O_S), S)$ is then by definition the category whose set of objects is

$$\begin{aligned} (\mathbf{RTop}^2 / (S, O_S))^0 := \\ \{(((X, O_X), Z), h), h : (X, O_X) \rightarrow (S, O_S), Z \subset X \text{ closed}\} \subset \mathbf{RTop} / (S, O_S) \times \mathbf{Top} \end{aligned}$$

and whose set of morphisms between $((X_1, O_{X_1}), Z_1, h_1), ((X_2, O_{X_2}), Z_2, h_2) \in \mathbf{RTop}^2 / (S, O_S)$ is the subset

$$\begin{aligned} \mathbf{Hom}_{\mathbf{RTop}^2 / (S, O_S)}(((X_1, O_{X_1}), Z_1) / (S, O_S), ((X_2, O_{X_2}), Z_2) / (S, O_S)) := \\ \{(f : (X_1, O_{X_1}) \rightarrow (X_2, O_{X_2})), \text{ s.t. } h_1 \circ f = h_2 \text{ and } Z_1 \subset f^{-1}(Z_2)\} \\ \subset \mathbf{Hom}_{\mathbf{RTop}}((X_1, O_{X_1}), (X_2, O_{X_2})) \end{aligned}$$

We denote by

$$\mu_S : \mathbf{RTop}^{2, pr} / S := \{(((Y \times S, q^* O_Y \otimes p^* O_S), Z), p), p : Y \times S \rightarrow S, Z \subset Y \times S \text{ closed}\} \hookrightarrow \mathbf{RTop}^2 / S$$

the full subcategory whose objects are those with $p : Y \times S \rightarrow S$ is a projection, and again $\mu_S : \mathbf{RTop}^2 / S \rightarrow \mathbf{RTop}^{2, pr} / S$ the corresponding morphism of sites. We denote by

$$\begin{aligned} \text{Gr}_S^{12} : \mathbf{RTop} / S \rightarrow \mathbf{RTop}^{2, pr} / S, \\ (X, O_X) / (S, O_S) \mapsto \text{Gr}_S^{12}((X, O_X) / (S, O_S)) := ((X \times S, q^* O_X \otimes p^* O_S), \bar{X}) / (S, O_S), \\ (g : (X, O_X) / (S, O_S) \rightarrow (X', O_{X'}) / (S, O_S)) \mapsto \\ \text{Gr}_S^{12}(g) := (g \times I_S : ((X \times S, q^* O_X \otimes p^* O_S), \bar{X}) \rightarrow ((X' \times S, q^* O_X \otimes p^* O_S), \bar{X}')) \end{aligned}$$

the graph functor, $X \hookrightarrow X \times S$ being the graph embedding (which is a closed embedding if X is separated), $p : X \times S \rightarrow S$, $q : X \times S \rightarrow X$ the projections, and again $\text{Gr}_S^{12} : \mathbf{RTop}^{2, pr} / S \rightarrow \mathbf{RTop} / S$ the corresponding morphism of sites.

- We denote by $\mathbf{Sch}^2 \subset \mathbf{RTop}^2$ the full subcategory such that the first factors are schemes. For a field k , we denote by $\mathbf{Sch}^2 / k := \mathbf{Sch}^2 / (\text{Spec } k, \{\text{pt}\})$ and by

- $\mathbf{Var}(k)^2 \subset \mathbf{Sch}^2 / k$ the full subcategory such that the first factors are algebraic varieties over k , i.e. schemes of finite type over k ,
- $\mathbf{PVar}(k)^2 \subset \mathbf{QPVar}(k)^2 \subset \mathbf{Var}(k)^2$ the full subcategories such that the first factors are quasi-projective varieties and projective varieties respectively,
- $\mathbf{PSmVar}(k)^2 \subset \mathbf{SmVar}(k)^2 \subset \mathbf{Var}(k)^2$ the full subcategories such that the first factors are smooth varieties and smooth projective varieties respectively.

In particular we have, for $S \in \mathbf{Var}(k)$, the graph functor

$$\begin{aligned} \text{Gr}_S^{12} : \mathbf{Var}(k) / S \rightarrow \mathbf{Var}(k)^{2, pr} / S, X / S \mapsto \text{Gr}_S^{12}(X / S) := (X \times S, X) / S, \\ (g : X / S \rightarrow X' / S) \mapsto \text{Gr}_S^{12}(g) := (g \times I_S : (X \times S, X) \rightarrow (X' \times S, X')) \end{aligned}$$

the graph embedding $X \hookrightarrow X \times S$ is a closed embedding since X is separated in the subcategory of schemes $\mathbf{Sch} \subset \mathbf{RTop}$, and again $\text{Gr}_S^{12} : \mathbf{Var}(k)^{2, pr} / S \rightarrow \mathbf{Var}(k) / S$ the corresponding morphism of sites.

- Denote by $CW \subset \text{Top}$ the full subcategory of CW complexes, by $CS \subset CW$ the full subcategory of Δ complexes, by $\text{TM}(\mathbb{R}) \subset CW$ the full subcategory of topological (real) manifolds which admits a CW structure (a topological manifold admits a CW structure if it admits a differential structure) and by $\text{Diff}(\mathbb{R}) \subset \text{RTop}$ the full subcategory of differentiable (real) manifold. We denote by $CW^2 \subset \text{Top}^2$ the full subcategory such that the first factors are CW complexes, by $\text{TM}(\mathbb{R})^2 \subset CW^2$ the full subcategory such that the first factors are topological (real) manifolds and by $\text{Diff}(\mathbb{R})^2 \subset \text{RTop}^2$ the full subcategory such that the first factors are differentiable (real) manifold.
- Denote by $\text{AnSp}(\mathbb{C}) \subset \text{RTop}$ the full subcategory of analytic spaces over \mathbb{C} , and by $\text{AnSm}(\mathbb{C}) \subset \text{AnSp}(\mathbb{C})$ the full subcategory of smooth analytic spaces (i.e. complex analytic manifold). We denote by $\text{AnSp}(\mathbb{C})^2 \subset \text{RTop}^2$ the full subcategory such that the first factors are analytic spaces over \mathbb{C} , and by $\text{AnSm}(\mathbb{C})^2 \subset \text{AnSp}(\mathbb{C})^2$ the full subcategory such that the first factors are smooth analytic spaces (i.e. complex analytic manifold). In particular we have, for $S \in \text{AnSp}(\mathbb{C})$, the graph functor

$$\begin{aligned} \text{Gr}_S^{12} : \text{AnSp}(\mathbb{C})/S &\rightarrow \text{AnSp}(\mathbb{C})^{2,pr}/S, \quad X/S \mapsto \text{Gr}_S^{12}(X/S) := (X \times S, X)/S, \\ (g : X/S &\rightarrow X'/S) \mapsto \text{Gr}_S^{12}(g) := (g \times I_S : (X \times S, X) \rightarrow (X' \times S, X')) \end{aligned}$$

the graph embedding $X \hookrightarrow X \times S$ is a closed embedding since X is separated in RTop , and again $\text{Gr}_S^{12} : \text{AnSp}(\mathbb{C})^{2,pr}/S \rightarrow \text{AnSp}(\mathbb{C})/S$ the corresponding morphism of sites.

- For $V \in \text{Var}(\mathbb{C})$, we denote by $V^{an} \in \text{AnSp}(\mathbb{C})$ the complex analytic space associated to V with the usual topology induced by the usual topology of \mathbb{C}^N . For $W \in \text{AnSp}(\mathbb{C})$, we denote by $W^{cw} \in \text{AnSp}(\mathbb{C})$ the topological space given by W which is a CW complex. For simplicity, for $V \in \text{Var}(\mathbb{C})$, we denote by $V^{cw} := (V^{an})^{cw} \in CW$. We have then

- the analytical functor $\text{An} : \text{Var}(\mathbb{C}) \rightarrow \text{AnSp}(\mathbb{C})$, $\text{An}(V) = V^{an}$,
- the forgetful functor $\text{Cw} = tp : \text{AnSp}(\mathbb{C}) \rightarrow CW$, $\text{Cw}(W) = W^{cw}$,
- the composite of these two functors $\widetilde{\text{Cw}} = \text{Cw} \circ \text{An} : \text{Var}(\mathbb{C}) \rightarrow CW$, $\widetilde{\text{Cw}}(V) = V^{cw}$.

We have then

- the analytical functor $\text{An} : \text{Var}(\mathbb{C})^2 \rightarrow \text{AnSp}(\mathbb{C})^2$, $\text{An}((V, Z)) = (V^{an}, Z^{an})$,
- the forgetful functor $\text{Cw} = tp : \text{AnSp}(\mathbb{C})^2 \rightarrow CW^2$, $\text{Cw}((W, Z)) = (W^{cw}, Z^{cw})$,
- the composite of these two functors $\widetilde{\text{Cw}} = \text{Cw} \circ \text{An} : \text{Var}(\mathbb{C})^2 \rightarrow CW^2$, $\widetilde{\text{Cw}}((V, Z)) = (V^{cw}, Z^{cw})$.

2.2 Additive categories, abelian categories and tensor triangulated categories

Let \mathcal{A} an additive category.

- For $\phi : F^\bullet \rightarrow G^\bullet$ a morphism with $F^\bullet, G^\bullet \in C(\mathcal{A})$, we have the mapping cylinder $\text{Cyl}(\phi) := ((F^n \oplus F^{n+1} \oplus G^{n+1}, (\partial_F^n, \partial_F^{n+1}, \phi^{n+1} + \partial^n G) \in C(\mathcal{A}))$. and the mapping cone $\text{Cone}(\phi) := ((F^n \oplus G^{n+1}, (\partial_F^n, \phi^{n+1} + \partial^n G) \in C(\mathcal{A}))$.
- The category $K(\mathcal{A}) := \text{Ho}(C(\mathcal{A}))$ is a triangulated category with distinguish triangles $F^\bullet \xrightarrow{i_F} \text{Cyl}(\phi) \xrightarrow{q_F} \text{Cone}(\phi) \xrightarrow{r_F} F^\bullet[1]$.
- The category (\mathcal{A}, F) is obviously again an additive category.
- Let $\phi : F^\bullet \rightarrow G^\bullet$ a morphism with $F^\bullet, G^\bullet \in C(\mathcal{A})$. Then it is obviously a morphism of filtered complex $\phi : (F^\bullet, F_b) \rightarrow (G^\bullet, F_b)$, where we recall that F_b is the trivial filtration $(F^\bullet, F_b), (G^\bullet, F_b) \in C_{fil}(\mathcal{A})$.

We recall the following property of the internal hom functor if it exists of a tensor triangulated category and the definition of compact and cocompact object.

Proposition 1. *Let (\mathcal{T}, \otimes) a tensor triangulated category admitting countable direct sum and product compatible with the triangulation. Assume that \mathcal{T} has an internal hom (bi)functor $R\mathcal{H}om(.,.) : \mathcal{T}^2 \rightarrow \mathcal{T}$ which is by definition the right adjoint to $(\cdot \otimes \cdot) : \mathcal{T}^2 \rightarrow \mathcal{T}$. Then,*

- for $N \in \mathcal{T}$, the functor $R\mathcal{H}om(., N) : \mathcal{T} \rightarrow \mathcal{T}$ commutes with homotopy colimits : for $M = \text{holim}_{\rightarrow i \in I} M_i$, where I is a countable category, we have

$$R\mathcal{H}om(M, N) \xrightarrow{\sim} \text{ho} \lim_{\leftarrow i \in I} R\mathcal{H}om(M_i, N).$$

- dually, for $M \in \mathcal{T}$, the functor $R\mathcal{H}om(M, \cdot) : \mathcal{T} \rightarrow \mathcal{T}$ commutes with homotopy limits : for $N = \text{holim}_{\leftarrow i \in I} N_i$, where I is a countable category, we have

$$R\mathcal{H}om(M, N) \xrightarrow{\sim} \text{ho} \lim_{\leftarrow} R\mathcal{H}om(M, N_i).$$

Proof. Standard. □

Let (\mathcal{T}, \otimes) a tensor triangulated category admitting countable direct sum and product compatible with the triangulation. Assume that \mathcal{T} has an internal hom functor $R\mathcal{H}om(.,.) : \mathcal{T} \rightarrow \mathcal{T}$.

- For $N \in \mathcal{T}$, the functor $R\mathcal{H}om(., N) : \mathcal{T} \rightarrow \mathcal{T}$ does not commutes in general with homotopy limits : for $M = \text{holim}_{\leftarrow i \in I} M_i$, where I is a countable category, the canonical map

$$\text{holim}_{\rightarrow i \in I} R\mathcal{H}om(M_i, N) \rightarrow R\mathcal{H}om(M, N)$$

is not an isomorphism in general if I is infinite. It commutes if and only if N is compact.

- Dually, for $M \in \mathcal{T}$, the functor $R\mathcal{H}om(M, \cdot) : \mathcal{T} \rightarrow \mathcal{T}$ does not commutes in general with infinite homotopy colimits. It commutes if and only if M is cocompact.

Most triangulated category comes from the localization of the category of complexes of an abelian category with respect to quasi-isomorphisms. In the case where the abelian category have enough injective or projective object, the triangulated category is the homotopy category of the complexes of injective, resp. projective, objects.

Proposition 2. *Let \mathcal{A} an abelian category with enough injective and projective.*

- A quasi-isomorphism $\phi : Q^\bullet \rightarrow F^\bullet$, with $F^\bullet, Q^\bullet \in C^-(\mathcal{A})$ such that the Q^n are projective is an homotopy equivalence.
- Dually, a quasi-isomorphism $\phi : F^\bullet \rightarrow I^\bullet$, with $F^\bullet, I^\bullet \in C^+(\mathcal{A})$ such that the I^n are projective is an homotopy equivalence.

Proof. Standard. □

Proposition 3. *Let \mathcal{A} an abelian category with enough injective and projective satisfying AB3 (i.e. countable direct sum of exact sequences are exact sequence).*

- Let $K(P) \subset K(\mathcal{A})$ be the thick subcategory generated by (unbounded) complexes of projective objects. Then, $K(P) \hookrightarrow K(\mathcal{A}) \xrightarrow{D} D(\mathcal{A})$ is an equivalence of triangulated categories.
- Similarly, let $K(I) \subset K(\mathcal{A})$ be the thick subcategory generated by (unbounded) complexes of injective objects. Then $K(I) \hookrightarrow K(\mathcal{A}) \xrightarrow{D} D(\mathcal{A})$ is an equivalence of triangulated categories.

Proof. It follows from proposition 2 : see [22]. □

2.3 Presheaves on a site and on a ringed topos

2.3.1 Functorialities

Let $\mathcal{S} \in \text{Cat}$ a small category. For $X \in \mathcal{S}$ we denote by $\mathbb{Z}(X) \in \text{PSh}(\mathcal{S})$ the presheaf represented by X . By Yoneda lemma, a representable presheaf $\mathbb{Z}(X)$ is projective.

Proposition 4. • *Let $\mathcal{S} \in \text{Cat}$ a small category. The projective presheaves $\text{Proj}(\text{PSh}(\mathcal{S})) \subset \text{PSh}(\mathcal{S})$ are the direct summand of the representable presheaves $\mathbb{Z}(X)$ with $X \in \mathcal{S}$.*

- *More generally let $(\mathcal{S}, O_{\mathcal{S}}) \in \text{RCat}$ a ringed topos. The projective presheaves $\text{Proj}(\text{PSh}_{O_{\mathcal{S}}}(\mathcal{S})) \subset \text{PSh}_{O_{\mathcal{S}}}(\mathcal{S})$ of $O_{\mathcal{S}}$ modules are the direct summand of the representable presheaves $\mathbb{Z}(X) \otimes O_{\mathcal{S}}$ with $X \in \mathcal{S}$.*

Proof. Standard. □

Let $f : \mathcal{T} \rightarrow \mathcal{S}$ a morphism of presite with $\mathcal{T}, \mathcal{S} \in \text{Cat}$. For $h : U \rightarrow S$ a morphism with $U, S \in \mathcal{S}$, we have $f^*\mathbb{Z}(U/S) = \mathbb{Z}(P(f)(U/S))$.

We will consider in this article filtered complexes of presheaves on a site. Let $f : \mathcal{T} \rightarrow \mathcal{S}$ a morphism of presite with $\mathcal{T}, \mathcal{S} \in \text{Cat}$.

- The functor $f_* : C(\mathcal{T}) \rightarrow C(\mathcal{S})$ gives, by functoriality, the functor

$$f_* : C_{(2)\text{fil}}(\mathcal{T}) \rightarrow C_{(2)\text{fil}}(\mathcal{S}), (G, F) \mapsto f_*(G, F) := (f_*G, f_*F),$$

since f_* preserves monomorphisms.

- The functor $f^* : C(\mathcal{S}) \rightarrow C(\mathcal{T})$ gives, by functoriality, the functor

$$f^* : C_{(2)\text{fil}}(\mathcal{S}) \rightarrow C_{(2)\text{fil}}(\mathcal{T}), (G, F) \mapsto f^*(G, F), F^p(f^*(G, F)) := \text{Im}(f^*F^pG \rightarrow f^*G).$$

In the particular case where $f^* : \text{PSh}(\mathcal{S}) \rightarrow \text{PSh}(\mathcal{T})$ preserves monomorphisms, we have $f^*(G, F) = (f^*G, f^*F)$.

- The functor $f^\perp : C(\mathcal{S}) \rightarrow C(\mathcal{T})$ gives, by functoriality, the functor

$$f^\perp : C_{(2)\text{fil}}(\mathcal{T}) \rightarrow C_{(2)\text{fil}}(\mathcal{S}), (G, F) \mapsto f^\perp(G, F) := (f^\perp G, f^\perp F),$$

since $f^\perp : C(\mathcal{S}) \rightarrow C(\mathcal{T})$ preserves monomorphisms.

Let $f : \mathcal{T} \rightarrow \mathcal{S}$ a morphism of presite with $\mathcal{T}, \mathcal{S} \in \text{Cat}$.

- The adjunction $(f^*, f_*) = (f^{-1}, f_*) : C(\mathcal{S}) \rightleftarrows C(\mathcal{T})$, gives an adjunction

$$(f^*, f_*) : C_{(2)\text{fil}}(\mathcal{S}) \rightleftarrows C_{(2)\text{fil}}(\mathcal{T}), (G, F) \mapsto f^*(G, F), (G, F) \mapsto f_*(G, F),$$

with adjunction maps, for $(G_1, F) \in C_{(2)\text{fil}}(\mathcal{S})$ and $(G_2, F) \in C_{(2)\text{fil}}(\mathcal{T})$

$$\text{ad}(f^*, f_*)(G_1, F) : (G_1, F) \rightarrow f_*f^*(G_1, F), \text{ad}(f^*, f_*)(G_2, F) : f^*f_*(G_2, F) \rightarrow (G_2, F).$$

- The adjunction $(f_*, f^\perp) : C(\mathcal{S}) \rightleftarrows C(\mathcal{T})$, gives an adjunction

$$(f_*, f^\perp) : C_{(2)\text{fil}}(\mathcal{T}) \rightleftarrows C_{(2)\text{fil}}(\mathcal{S}), (G, F) \mapsto f_*(G, F), (G, F) \mapsto f^\perp(G, F),$$

with adjunction maps, for $(G_1, F) \in C_{(2)\text{fil}}(\mathcal{S})$ and $(G_2, F) \in C_{(2)\text{fil}}(\mathcal{S})$

$$\text{ad}(f_*, f^\perp)(G_2, F) : (G_2, F) \rightarrow f^\perp f_*(G_2, F), \text{ad}(f_*, f^\perp)(G_1, F) : f_*f^\perp(G_1, F) \rightarrow (G_1, F).$$

Remark 1. Let $\mathcal{T}, \mathcal{S} \in \text{Cat}$ small categories and $f : \mathcal{T} \rightarrow \mathcal{S}$ a morphism of presite. Then the functor $f^* : \text{PSh}(\mathcal{S}) \rightarrow \text{PSh}(\mathcal{T})$ preserve epimorphism but does NOT preserve monomorphism in general (the colimits involved are NOT filtered colimits). However it preserve monomorphism between projective presheaves by Yoneda and we thus set for $(Q, F) \in C_{fil}(\text{Proj}(\text{PSh}(\mathcal{S})))$, that is $F^p Q^n \in \text{Proj}(\text{PSh}(\mathcal{S}))$ for all $p, n \in \mathbb{Z}$, $f^*(Q, F) := (f^*Q, f^*F)$.

For a commutative diagram of presite :

$$D = \begin{array}{ccc} \mathcal{T}' & \xrightarrow{g_2} & \mathcal{S}' \\ \downarrow f_2 & & \downarrow f_1 \\ \mathcal{T} & \xrightarrow{g_1} & \mathcal{S} \end{array} ,$$

with $\mathcal{T}, \mathcal{T}'\mathcal{S}, \mathcal{S}' \in \text{Cat}$, we denote by, for $F \in C(\mathcal{S}')$,

$$T(D)(F) : g_1^* f_{1*} F \xrightarrow{g_1^* f_{1*} \text{ad}(g_2^*, g_{2*})(F)} g_1^* f_{1*} g_{2*} g_2^* F = g_1^* g_{1*} f_{2*} g_2^* F \xrightarrow{\text{ad}(g_1^* g_{1*})(f_{2*} g_2^* F)} f_{2*} g_2^* F$$

the canonical transformation map in $C(\mathcal{T})$, and for $(G, F) \in C_{fil}(\mathcal{S}')$,

$$T(D)(G, F) : g_1^* f_{1*}(G, F) \xrightarrow{g_1^* f_{1*} \text{ad}(g_2^*, g_{2*})(G, F)} g_1^* f_{1*} g_{2*} g_2^*(G, F) = g_1^* g_{1*} f_{2*} g_2^*(G, F) \xrightarrow{\text{ad}(g_1^* g_{1*})(f_{2*} g_2^*(G, F))} f_{2*} g_2^*(G, F).$$

the canonical transformation map in $C_{fil}(\mathcal{T})$ given by the adjunction maps.

We will use the internal hom functor and the tensor product for presheaves on a site or for presheaves of modules on a ringed topos. We recall the definition in the filtered case.

- Let $(\mathcal{S}, O_{\mathcal{S}}) \in \text{RCat}$. We have the tensor product bifunctor

$$(\cdot) \otimes (\cdot) : \text{PSh}(\mathcal{S})^2 \rightarrow \text{PSh}(\mathcal{S}), (F, G) \mapsto (X \in \mathcal{S} \mapsto (F \otimes G)(X) := F(X) \otimes G(X))$$

It induces a bifunctor :

$$(\cdot) \otimes (\cdot) : C(\mathcal{S}) \times C(\mathcal{S}) \rightarrow C(\mathcal{S}), (F, G) \mapsto F \otimes G := \text{Tot}(F^\bullet \otimes G^\bullet), (F \otimes G)^n = \bigoplus_{r \in \mathbb{Z}} F^r \otimes G^{n-r}$$

and a bifunctor

$$(\cdot) \otimes (\cdot) : C(\mathcal{S}) \times C_{O_{\mathcal{S}}}(\mathcal{S}) \rightarrow C_{O_{\mathcal{S}}}(\mathcal{S}), \alpha.(F \otimes G) := F \otimes (\alpha.G)$$

For $(G_1, F), (G_2, F) \in C_{fil}(\mathcal{S})$, $G_3 \in C(\mathcal{S})$, we define (note that tensor product preserve monomorphism only after tensoring with $\mathbb{Q}_{\mathcal{S}} \in \text{PSh}(\mathcal{S})$)

$$- F^p((G_1, F) \otimes G_3) := \text{Im}(F^p G_1 \otimes G_3 \rightarrow G_1 \otimes G_3) \text{ and } F^p(G_3 \otimes (G_1, F)) := \text{Im}(G_3 \otimes F^p G_3 \rightarrow G_3 \otimes G_1),$$

$$- F^p F^q((G_1, F) \otimes (G_2, F)) := \text{Im}(F^p G_1 \otimes F^q G_2 \rightarrow G_1 \otimes G_2) \text{ and}$$

$$F^k((G_1, F) \otimes (G_2, F)) := F^k \text{Tot}_{FF}((G_1, F) \otimes (G_2, F)) := \bigoplus_{p \in \mathbb{Z}} \text{Im}(F^p G_1 \otimes F^{k-q} G_2 \rightarrow G_1 \otimes G_2)$$

Note that in the case where $G_1^n = 0$ for $n < 0$, we have $(G_1, F_b) \otimes (G_2, F) = G_1 \otimes (G_2, F)$. We get the bifunctors

$$(-) \otimes (-) : C_{fil}(\mathcal{S})^2 \rightarrow C_{fil}(\mathcal{S}), (-) \otimes (-) : C_{fil}(\mathcal{S}) \times C_{O_{\mathcal{S}} fil}(\mathcal{S}) \rightarrow C_{O_{\mathcal{S}} fil}(\mathcal{S}).$$

We have the tensor product bifunctor

$$(\cdot) \otimes_{O_{\mathcal{S}}} (\cdot) : \text{PSh}_{O_{\mathcal{S}}}(\mathcal{S})^2 \rightarrow \text{PSh}(\mathcal{S}), (F, G) \mapsto (X \in \mathcal{S} \mapsto (F \otimes_{O_{\mathcal{S}}} G)(X) := F(X) \otimes_{O_{\mathcal{S}}(X)} G(X))$$

It induces a bifunctor :

$$(\cdot) \otimes_{O_S} (\cdot) : C_{O_S}(\mathcal{S}) \times C_{O_S}(\mathcal{S}) \rightarrow C(\mathcal{S}), (F, G) \mapsto F \otimes_{O_S} G := \text{Tot}(F^\bullet \otimes_{O_S} G^\bullet)$$

For $(G_1, F), (G_2, F) \in C_{O_S \text{fil}}(\mathcal{S}), G_3 \in C_{O_S}(\mathcal{S})$, we define similarly $(G_1, F) \otimes_{O_S} G_3, G_3 \otimes_{O_S} (G_1, F)$, and

$$F^k((G_1, F) \otimes_{O_S} (G_2, F)) := F^k \text{Tot}_{FF}((G_1, F) \otimes_{O_S} (G_2, F)) := \bigoplus_{p \in \mathbb{Z}} \text{Im}(F^p G_1 \otimes_{O_S} F^{k-q} G_2 \rightarrow G_1 \otimes_{O_S} G_2)$$

Note that in the case where $G_1^n = 0$ for $n < 0$, we have $(G_1, F_b) \otimes_{O_S} (G_2, F) = G_1 \otimes_{O_S} (G_2, F)$. This gives

- in all case it gives the bifunctor $(-) \otimes_{O_S} (-) : C_{O_S^{op} \text{fil}}(\mathcal{S}) \otimes C_{O_S \text{fil}}(\mathcal{S}) \rightarrow C_{\text{fil}}(\mathcal{S})$.
- in the case O_S is commutative, it gives the bifunctor $(-) \otimes_{O_S} (-) : C_{O_S \text{fil}}(\mathcal{S})^2 \rightarrow C_{O_S \text{fil}}(\mathcal{S})$.

- Let $(\mathcal{S}, O_S) \in \text{RCat}$. We have the internal hom bifunctor

$$\begin{aligned} \mathcal{H}om(\cdot, \cdot) : \text{PSh}(\mathcal{S})^2 &\rightarrow \text{PSh}(\mathcal{S}), \\ (F, G) &\mapsto (X \in \mathcal{S} \mapsto \mathcal{H}om(F, G)(X) := \text{Hom}(r(X)_* F, r(X)_* G) \end{aligned}$$

with $r(X) : \mathcal{S} \rightarrow \mathcal{S}/X$ (see subsection 2.1). It induces a bifunctors :

$$\mathcal{H}om(\cdot, \cdot) : C(\mathcal{S}) \times C(\mathcal{S}) \rightarrow C(\mathcal{S}), (F, G) \mapsto \mathcal{H}om^\bullet(F, G)$$

and a bifunctor

$$\mathcal{H}om(\cdot, \cdot) : C(\mathcal{S}) \times C_{O_S}(\mathcal{S}) \rightarrow C_{O_S}(\mathcal{S}), \alpha \cdot \mathcal{H}om(F, G) := \mathcal{H}om(F, \alpha \cdot G)$$

For $(G_1, F), (G_2, F) \in C_{\text{fil}}(\mathcal{S}), G_3 \in C(\mathcal{S})$, we define

- $F^p \mathcal{H}om(G_3, (G_1, F)) := \mathcal{H}om(G_3, F^p G_1) \hookrightarrow \mathcal{H}om(G_3, G_1)$, note that the functor $G \mapsto \mathcal{H}om(F, G)$ preserve monomorphism,
- the dual filtration $F^{-p} \mathcal{H}om((G_1, F), G_3) := \ker(\mathcal{H}om(G_1, G_3) \rightarrow \text{Hom}(F^p G_1, G_3))$
- $F^p F^q \mathcal{H}om((G_1, F), (G_2, F)) := \ker(\mathcal{H}om(G_1, F^p G_2) \rightarrow \text{Hom}(F^q G_1, F^p G_2))$, and

$$\begin{aligned} F^k \mathcal{H}om^\bullet((G_1, F), (G_2, F)) &:= \text{Tot}_{FF} \mathcal{H}om((G_1, F), (G_2, F)) := \\ &\bigoplus_{p \in \mathbb{Z}} \ker(\mathcal{H}om(G_1, F^{k+p} G_2) \rightarrow \mathcal{H}om(F^p G_1, F^{k+p} G_2)) \end{aligned}$$

We get the bifunctors

$$\mathcal{H}om(\cdot, \cdot) : C_{\text{fil}}(\mathcal{S}) \times C_{\text{fil}}(\mathcal{S}) \rightarrow C_{\text{fil}}(\mathcal{S}), \quad \mathcal{H}om(\cdot, \cdot) : C_{\text{fil}}(\mathcal{S}) \times C_{O_S \text{fil}}(\mathcal{S}) \rightarrow C_{O_S \text{fil}}(\mathcal{S}).$$

We have the internal hom bifunctor

$$\begin{aligned} \mathcal{H}om_{O_S}(\cdot, \cdot) : \text{PSh}_{O_S}(\mathcal{S}) \times \text{PSh}_{O_S}(\mathcal{S}) &\rightarrow \text{PSh}(\mathcal{S}) \\ (F, G) &\mapsto (X \in \mathcal{S} \mapsto \mathcal{H}om_{O_S}(F, G)(X) := \text{Hom}_{O_S}(r(X)_* F, r(X)_* G). \end{aligned}$$

It gives similarly

- in all case a bifunctor $\mathcal{H}om_{O_S}(\cdot, \cdot) : C_{\text{fil}_{O_S}}(\mathcal{S}) \times C_{\text{fil}_{O_S}}(\mathcal{S}) \rightarrow C_{\text{fil}}(\mathcal{S})$,
- the case O_S is commutative, a bifunctor $\mathcal{H}om_{O_S}(\cdot, \cdot) : C_{\text{fil}_{O_S}}(\mathcal{S}) \times C_{\text{fil}_{O_S}}(\mathcal{S}) \rightarrow C_{O_S \text{fil}}(\mathcal{S})$.

Let $\phi : A \rightarrow B$ of rings.

- Let M a A module. We say that M admits a B module structure if there exists a structure of B module on the abelian group M which is compatible with ϕ together with the A module structure on M .

- For N_1 a A -module and N_2 a B module. $I(A/B)(N_1, N_2) : \text{Hom}_A(N_1, N_2) \xrightarrow{\sim} \text{Hom}_B(N_1 \otimes_A B, N_2)$ is the adjunction between the restriction of scalars and the extension of scalars.
- For N', N'' a A -modules, $ev_A(\text{hom}, \otimes)(N', N'', B) : \text{Hom}_A(N', N'') \otimes_A B \rightarrow \text{Hom}_A(N', N'' \otimes_A B)$. is the evaluation classical map.

Let $\phi : (\mathcal{S}, O_1) \rightarrow (\mathcal{S}, O_2)$ a morphism of presheaves of ring on $\mathcal{S} \in \text{Cat}$.

- Let $M \in \text{PSh}_{O_1}(\mathcal{S})$. We say that M admits an O_2 module structure if there exists a structure of O_2 module on $M \in \text{PSh}(\mathcal{S})$ which is compatible with ϕ together with the O_1 module structure on M .
- For $N_1 \in C_{O_1}(\mathcal{S})$ and $N_2 \in C_{O_2}(\mathcal{S})$,

$$I(O_1/O_2)(N_1, N_2) : \text{Hom}_{O_1}(N_1, N_2) \xrightarrow{\sim} \text{Hom}_{O_2}(N_1 \otimes_{O_1} B, N_2)$$

is the adjunction between the restriction of scalars and the extension of scalars.

- For $N', N'' \in C_{O_1}(\mathcal{S})$,

$$ev_{O_1}(\text{hom}, \otimes)(N', N'', O_2) : \text{Hom}_{O_1}(N', N'') \otimes_{O_1} O_2 \rightarrow \text{Hom}_{O_1}(N', N'' \otimes_{O_1} O_2).$$

is the classical evaluation map.

Let $(\mathcal{S}, O_S) \in \text{RCat}$.

- For $F_1, F_2, G_1, G_2 \in C(\mathcal{S})$, we denote by

$$T(\otimes, \text{Hom})(F_1, F_2, G_1, G_2) : \text{Hom}(F_1, G_1) \otimes \text{Hom}(F_2, G_2) \rightarrow \text{Hom}(F_1 \otimes F_2, G_1 \otimes G_2)$$

the canonical map.

- For $G_3 \in C(\mathcal{S})$ and $G_1, G_2 \in C_{O_S}(\mathcal{S})$, we denote by

$$\begin{aligned} ev(\text{hom}, \otimes)(G_3, G_1, G_2) : \text{Hom}(G_3, G_1) \otimes_{O_S} G_2 &\rightarrow \text{Hom}(G_3, G_1 \otimes_{O_S} G_2) \\ \phi \otimes s &\mapsto (s' \mapsto \phi(s') \otimes s) \end{aligned}$$

- Let $\mathcal{S} \in \text{Cat}$ a small category. Let $(H_X : C(\mathcal{S}/X) \rightarrow C(\mathcal{S}/X))_{X \in \mathcal{S}}$ a family of functors which is functorial in X . We have by definition, for $F_1, F_2 \in C(\mathcal{S})$, the canonical transformation map

$$T(H, \text{hom})(F_1, F_2) : H(\text{Hom}^\bullet(F_1, F_2)) \rightarrow \text{Hom}^\bullet(H(F_1), H(F_2)), \quad (1)$$

$$\phi \in \text{Hom}(F_1|_X, F_2|_X) \mapsto H^{F_1|_X, F_2|_X}(\phi) \in \text{Hom}(H(F_1|_X), H(F_2|_X)) \quad (2)$$

in $C(\mathcal{S})$.

Let $\mathcal{T}, \mathcal{S} \in \text{Cat}$ small categories and $f : \mathcal{T} \rightarrow \mathcal{S}$ a morphism of presite.

- For $F_1, F_2 \in C(\mathcal{T})$ we have by definition $f_*(F_1 \otimes F_2) = f_*F_1 \otimes f_*F_2$. For $G_1, G_2 \in C(\mathcal{S})$, we have a canonical isomorphism $f^*G_1 \otimes f^*G_2 \xrightarrow{\sim} f^*(G_1 \otimes G_2)$ since the tensor product is a right exact functor, and a canonical map $f^\perp G_1 \otimes f^\perp G_2 \rightarrow f^\perp(G_1 \otimes G_2)$.
- We have for $F \in C(\mathcal{S})$ and $G \in C(\mathcal{T})$ the adjunction isomorphism,

$$I(f^*, f_*)(F, G) : f_*\text{Hom}^\bullet(f^*F, G) \xrightarrow{\sim} \text{Hom}^\bullet(F, f_*G). \quad (3)$$

- Let $O_S \in \text{PSh}(\mathcal{S}, \text{Ring})$ by a presheaf of ring so that $(\mathcal{S}, O_S), (\mathcal{T}, f^*O_S) \in \text{RCat}$. We have for $F \in C_{O_S}(\mathcal{S})$ and $G \in C_{f^*O_S}(\mathcal{T})$ the adjunction isomorphism,

$$I(f^*, f_*)(F, G) : f_*\text{Hom}_{f^*O_S}^\bullet(f^*F, G) \xrightarrow{\sim} \text{Hom}_{O_S}^\bullet(F, f_*G), \quad (4)$$

and

- the map $\text{ad}(f^*, f_*)(F) : F \rightarrow f_* f^* F$ in $C(\mathcal{S})$ is O_S linear, that is a map in $C_{O_S}(\mathcal{S})$,
- the map $\text{ad}(f^*, f_*)(G) : f^* f_* G \rightarrow G$ in $C(\mathcal{T})$ is $f^* O_S$ linear, that is a map in $C_{f^* O_S}(\mathcal{T})$.

- For $F_1, F_2 \in C(\mathcal{T})$, we have the canonical map

$$T_*(f, \text{hom})(F_1, F_2) := T(f_*, \text{hom}) : f_* \mathcal{H}om^\bullet(F_1, F_2) \rightarrow \mathcal{H}om^\bullet(f_* F_1, f_* F_2), \quad (5)$$

$$\text{for } X \in \mathcal{S}, \phi \in \text{Hom}(F_1|_{f^*(X)}, F_2|_{f^*(X)}) \mapsto f_*^{F_1|_{f^*(X)}, F_2|_{f^*(X)}}(\phi) \in \text{Hom}(f_* F_1|_{f^*(X)}, f_* F_2|_{f^*(X)}) \quad (6)$$

given by evaluation.

- For $G_1, G_2 \in C(\mathcal{S})$, we have the following canonical transformation in $C(\mathcal{T})$

$$T(f, \text{hom})(G_1, G_2) := T(f^*, \text{hom})(G_1, G_2) : \quad (7)$$

$$f^* \mathcal{H}om^\bullet(G_1, G_2) \xrightarrow{f^* \mathcal{H}om(G_1, \text{ad}(f^*, f_*)(G_2))} f^* \mathcal{H}om^\bullet(G_1, f_* f^* G_2) \xrightarrow{f^* I(f^*, f_*)(G_1, G_2)} \quad (8)$$

$$f^* f_* \mathcal{H}om^\bullet(f^* G_1, f^* G_2) \xrightarrow{\text{ad}(f^*, f_*)(\mathcal{H}om(f^* G_1, f^* G_2))} \mathcal{H}om^\bullet(f^* G_1, f^* G_2), \quad (9)$$

- Let $O_S \in \text{PSh}(\mathcal{S}, \text{Ring})$ by a presheaf of ring so that $(\mathcal{S}, O_S), (\mathcal{T}, f^* O_S) \in \text{RCat}$. For $G_1, G_2 \in C_{O_S}(\mathcal{S})$, we have the following canonical transformation in $C_{f^* O_S}(\mathcal{T})$

$$T(f, \text{hom})(G_1, G_2) := T(f^*, \text{hom})(G_1, G_2) : \quad (10)$$

$$f^* \mathcal{H}om_{O_S}^\bullet(G_1, G_2) \xrightarrow{f^* \mathcal{H}om_{O_S}(G_1, \text{ad}(f^*, f_*)(G_2))} f^* \mathcal{H}om_{O_S}^\bullet(G_1, f_* f^* G_2) \xrightarrow{f^* I(f^*, f_*)(G_1, G_2)} \quad (11)$$

$$f^* f_* \mathcal{H}om_{f^* O_S}^\bullet(f^* G_1, f^* G_2) \xrightarrow{\text{ad}(f^*, f_*)(\mathcal{H}om_{f^* O_S}(f^* G_1, f^* G_2))} \mathcal{H}om_{f^* O_S}^\bullet(f^* G_1, f^* G_2), \quad (12)$$

- Let $O_S \in \text{PSh}(\mathcal{S}, \text{Ring})$ by a presheaf of ring so that $(\mathcal{S}, O_S), (\mathcal{T}, f^* O_S) \in \text{RCat}$. For $M \in C_{O_S}(\mathcal{S})$ and $N \in C_{f^* O_S}(\mathcal{T})$, we denote by

$$T(f, \otimes)(M, N) : M \otimes_{O_S} f_* N \xrightarrow{\text{ad}(f^*, f_*)(M \otimes_{O_S} f_* N)} \quad (13)$$

$$f_* f^*(M \otimes_{O_S} f_* N) = f_*(f^* M \otimes_{f^* O_S} f^* f_* N) \xrightarrow{\text{ad}(f^*, f_*)(N)} f_*(f^* M \otimes_{f^* O_S} N) \quad (14)$$

the canonical transformation map.

Let $f : (\mathcal{T}, O_T) \rightarrow (\mathcal{S}, O_S)$ a morphism with $(\mathcal{S}, O_S), (\mathcal{T}, O_T) \in \text{RCat}$. We have the adjunction

$$(f^{*mod}, f_*) : C_{O_S}(\mathcal{S}) \rightleftarrows C_{O_T}(\mathcal{T})$$

with $f^{*mod} G := f^* G \otimes_{f^* O_S} O_T$. If $f^* : C(\mathcal{S}) \rightarrow C(\mathcal{T})$ preserve monomorphisms, it induces the adjunction

$$(f^{*mod}, f_*) : C_{O_S \text{fil}}(\mathcal{S}) \rightleftarrows C_{O_T \text{fil}}(\mathcal{T})$$

with $f^{*mod}(G, F) := f^*(G, F) \otimes_{f^* O_S} O_T$.

For a commutative diagram in RCat :

$$D = \begin{array}{ccc} (\mathcal{T}', O_2') & \xrightarrow{g_2} & (\mathcal{S}', O_1') \\ \downarrow f_2 & & \downarrow f_1 \\ (\mathcal{T}, O_2) & \xrightarrow{g_1} & (\mathcal{S}, O_1) \end{array}$$

we denote by, for $F \in C_{O_1'}(\mathcal{S}')$,

$$T^{mod}(D)(F) : g_1^{*mod} f_{1*} F \xrightarrow{g_1^{*mod} f_{1*} \text{ad}(g_2^{*mod}, g_{2*})(F)} g_1^{*mod} f_{1*} g_{2*} g_2^{*mod} F = g_1^{*mod} g_{1*} f_{2*} g_2^{*mod} F \xrightarrow{\text{ad}(g_1^{*mod} g_{1*})(f_{2*} g_2^{*mod} F)} f_{2*} g_2^{*mod} F$$

the canonical transformation map in $C_{O_2}(\mathcal{T})$ and, for $(G, F) \in C_{O'_1 \text{fil}}(\mathcal{S}')$,

$$T^{\text{mod}}(D)(G, F) : g_1^{*\text{mod}} f_{1*}(G, F) \xrightarrow{g_1^{*\text{mod}} f_{1*} \text{ad}(g_2^{*\text{mod}}, g_{2*})(G, F)} g_1^{*\text{mod}} f_{1*} g_{2*} g_2^{*\text{mod}}(G, F) = g_1^{*\text{mod}} g_{1*} f_{2*} g_2^{*\text{mod}}(G, F) \xrightarrow{\text{ad}(g_1^{*\text{mod}} g_{1*})(f_{2*} g_2^{*\text{mod}}(G, F))} f_{2*} g_2^{*\text{mod}}(G, F)$$

the canonical transformation map in $C_{O_2 \text{fil}}(\mathcal{T})$ given by the adjonction maps.

Let $f : (\mathcal{T}, O_T) \rightarrow (\mathcal{S}, O_S)$ a morphism with $(\mathcal{S}, O_S), (\mathcal{T}, O_T) \in \text{RCat}$.

- We have, for $M, N \in C_{O_S}(\mathcal{S})$ the canonical transformation map in $C_{O_T}(\mathcal{T})$

$$T^{\text{mod}}(f, \text{hom})(M, N) : f^{*\text{mod}} \text{Hom}_{O_1}(M, N) \xrightarrow{T(f, \text{hom})(M, N) \otimes_{f^* O_1} O_2} \text{Hom}_{f^* O_1}(f^* M, f^* N) \otimes_{f^* O_1} O_2 \xrightarrow{e(\text{hom}, \otimes)(f^* M, f^* N)} \text{Hom}_{f^* O_1}(f^* M, f^{*\text{mod}} N) \xrightarrow{I(f^* O_1 / O_2)(f^* M, f^{*\text{mod}} N)} \text{Hom}_{O_2}(f^{*\text{mod}} M, f^{*\text{mod}} N)$$

- We have, for $M \in C_{O_S}(\mathcal{S})$ and $N \in C_{O_T}(\mathcal{T})$, the canonical transformation map in $C_{O_T}(\mathcal{T})$

$$T^{\text{mod}}(f, \otimes)(M, N) : M \otimes_{O_S} f_* N \xrightarrow{\text{ad}(f^{*\text{mod}}, f_*)(M \otimes_{O_S} f_* N)} \quad (15)$$

$$f_* f^{*\text{mod}}(M \otimes_{O_S} f_* N) = f_*(f^{*\text{mod}} M \otimes_{O_T} f^{*\text{mod}} f_* N) \xrightarrow{\text{ad}(f^{*\text{mod}}, f_*)(N)} f_*(f^{*\text{mod}} M \otimes_{O_T} N) \quad (16)$$

the canonical transformation map.

We now give some properties of the tensor product functor and hom functor given above

Proposition 5. *Let $(\mathcal{S}, O_S) \in \text{RCat}$. Then, the functors*

- $(-) \otimes (-) : C(\mathcal{S})^2 \rightarrow C(\mathcal{S}), C(\mathcal{S}) \times C_{O_S}(\mathcal{S}) \rightarrow C_{O_S}(\mathcal{S})$
- $(-) \otimes_{O_S} (-) : C_{O_S^{\text{op}}}(\mathcal{S}) \times C_{O_S}(\mathcal{S}) \rightarrow C(\mathcal{S})$ and in case O_S is commutative $(-) \otimes_{O_S} (-) : C_{O_S}(\mathcal{S})^2 \rightarrow C_{O_S}(\mathcal{S})$

are left Quillen functor for the projective model structure. In particular,

- for $L \in C(\mathcal{S})$ is such that $L^n \in \text{PSh}(\mathcal{S})$ are projective for all $n \in \mathbb{Z}$, and $\phi : F \rightarrow G$ is a quasi-isomorphism with $F, G \in C(\mathcal{S})$, then $\phi \otimes I : F \otimes L \rightarrow G \otimes L$ is a quasi-isomorphism,
- for $L \in C_{O_S}(\mathcal{S})$ is such that $L^n \in \text{PSh}_{O_S}(\mathcal{S})$ are projective for all $n \in \mathbb{Z}$, and $\phi : F \rightarrow G$ is a quasi-isomorphism with $F, G \in C_{O_S}(\mathcal{S})$, then $\phi \otimes I : F \otimes_{O_S} L \rightarrow G \otimes_{O_S} L$ is a quasi-isomorphism.

Proof. Standard. □

Proposition 6. *Let $(\mathcal{S}, O_S) \in \text{RCat}$. Then, the functors*

- $\text{Hom}(\cdot, \cdot) : C(\mathcal{S}) \times C_{O_S}(\mathcal{S}) \rightarrow C_{O_S}(\mathcal{S}), C(\mathcal{S}) \times C_{O_S}(\mathcal{S}) \rightarrow C_{O_S}(\mathcal{S}),$
- $\text{Hom}_{O_S}(\cdot, \cdot) : C_{O_S^{\text{op}}}(\mathcal{S}) \times C_{O_S}(\mathcal{S}) \rightarrow C(\mathcal{S})$ and in the case O_S is commutative $\text{Hom}_{O_S}(\cdot, \cdot) : C_{O_S}(\mathcal{S}) \times C_{O_S}(\mathcal{S}) \rightarrow C_{O_S}(\mathcal{S}),$

are on the left hand side left Quillen functor for the projective model structure. In particular,

- for $L \in C(\mathcal{S})$ is such that $L^n \in \text{PSh}(\mathcal{S})$ are projective for all $n \in \mathbb{Z}$, and $\phi : F \rightarrow G$ is a quasi-isomorphism with $F, G \in C(\mathcal{S})$, then $\text{Hom}(L, \phi) : \text{Hom}^\bullet(L, F) \rightarrow \text{Hom}^\bullet(L, G)$ is a quasi-isomorphism,
- for $L \in C_{O_S}(\mathcal{S})$ is such that $L^n \in \text{PSh}_{O_S}(\mathcal{S})$ are projective for all $n \in \mathbb{Z}$, and $\phi : F \rightarrow G$ is a quasi-isomorphism with $F, G \in C_{O_S}(\mathcal{S})$, then $\text{Hom}_{O_S}(L, \phi) : \text{Hom}_{O_S}^\bullet(L, F) \rightarrow \text{Hom}_{O_S}^\bullet(L, G)$ is a quasi-isomorphism.

Proof. Standard. □

Let $\mathcal{S} \in \text{Cat}$ a site endowed with topology τ . Denote by $a_\tau : \text{PSh}(\mathcal{S}) \rightarrow \text{Sh}(\mathcal{S})$ the sheafification functor. A morphism $\phi : F^\bullet \rightarrow G^\bullet$ with $F^\bullet, G^\bullet \in C(\mathcal{S})$ is said to be a τ local equivalence if

$$a_\tau H^n(\phi) : a_\tau H^n(F^\bullet) \rightarrow a_\tau H^n(G^\bullet)$$

is an isomorphism for all $n \in \mathbb{Z}$, where a_τ is the sheafification functor. Recall that $C_{fil}(\mathcal{S}) \subset (C(\mathcal{S}), F) = C(\text{PSh}(\mathcal{S}), F)$ denotes the category of filtered complexes of abelian presheaves on \mathcal{S} whose filtration is biregular.

- A morphism $\phi : (F^\bullet, F) \rightarrow (G^\bullet, F)$ with $(F^\bullet, F), (G^\bullet, F) \in C_{fil}(\mathcal{S})$ is said to be a filtered τ local equivalence or an 1-filtered τ local equivalence if

$$a_\tau H^n(\phi) : a_\tau H^n(\text{Gr}_F^p F^\bullet) \xrightarrow{\sim} a_\tau H^n(\text{Gr}_F^p G^\bullet)$$

is an isomorphism for all $n, p \in \mathbb{Z}$.

- Let $r \in \mathbb{N}$. More generally, a morphism $\phi : (F^\bullet, F) \rightarrow (G^\bullet, F)$ with $(F^\bullet, F), (G^\bullet, F) \in C_{fil}(\mathcal{S})$ is said to be an r -filtered τ local equivalence if for all $p, q \in \mathbb{Z}$,

$$a_\tau E_r^{p,q}(\phi) : a_\tau E_r^{p,q}(F^\bullet, F) \xrightarrow{\sim} a_\tau E_r^{p,q}(G^\bullet, F)$$

is an isomorphism. Note that if ϕ is an r -filtered τ local equivalence, that it is an s -filtered τ local equivalence for all $s \geq r$.

- A morphism $\phi : (F^\bullet, F) \rightarrow (G^\bullet, F)$ with $(F^\bullet, F), (G^\bullet, F) \in C_{fil}(\mathcal{S})$ is said to be a ∞ -filtered τ local equivalence if there exists $r \in \mathbb{N}$ such that ϕ is an r -filtered τ local equivalence. If a morphism $\phi : (F^\bullet, F) \rightarrow (G^\bullet, F)$ with $(F^\bullet, F), (G^\bullet, F) \in C_{fil}(\mathcal{S})$ is an ∞ -filtered τ local equivalence then, for all $n \in \mathbb{Z}$,

$$a_\tau H^n(\phi) : a_\tau H^n(F^\bullet, F) \rightarrow a_\tau H^n(G^\bullet, F)$$

is an isomorphism of filtered sheaves on \mathcal{S} . Recall the converse is NOT true (see section 2.1).

Let (\mathcal{S}, O) a ringed topos where $\mathcal{S} \in \text{Cat}$ is a site endowed with topology τ . Let $r \in \mathbb{N}$. A morphism $\phi : (F^\bullet, F) \rightarrow (G^\bullet, F)$ with $(F^\bullet, F), (G^\bullet, F) \in C_{O_S fil}(\mathcal{S})$ is said to be an r -filtered τ local equivalence if $o\phi$ is one, where $o : C_{O_S fil}(\mathcal{S}) \rightarrow C_{fil}(\mathcal{S})$ is the forgetful functor, that is if

$$a_\tau E_r^{p,q}(\phi) : a_\tau E_r^{p,q}(F^\bullet, F) \rightarrow a_\tau E_r^{p,q}(G^\bullet, F)$$

is an isomorphism for all $p, q \in \mathbb{Z}$.

Let $\mathcal{S} \in \text{Cat}$ a site which admits fiber product, endowed with topology τ . A complex of presheaves $F^\bullet \in C(\mathcal{S})$ is said to be τ fibrant if it satisfy descent for covers in \mathcal{S} , i.e. if for all $X \in \mathcal{S}$ and all τ covers $(c_i : U_i \rightarrow X)_{i \in I}$ of X , denoting $U_J := (U_{i_0} \times_S U_{i_1} \times_S \cdots U_{i_r})_{i_k \in J}$ and for $I \subset J$, $p_{IJ} : U_J \rightarrow U_I$ is the projection,

$$F^\bullet(c_i) : F^\bullet(X) \rightarrow \text{Tot}(\oplus_{\text{card} I = \bullet} F^\bullet(U_I))$$

is a quasi-isomorphism of complexes of abelian groups.

- A complex of filtered presheaves $(F^\bullet, F) \in C_{fil}(\mathcal{S})$ is said to be filtered τ fibrant or 1-filtered τ fibrant if it satisfy descent for covers in \mathcal{S} , i.e. if for all $X \in \mathcal{S}$ and all τ covers $(c_i : U_i \rightarrow X)_{i \in I}$ of X ,

$$(F^\bullet, F)(c_i) : (F^\bullet, F)(X) \rightarrow \text{Tot}(\oplus_{\text{card} I = \bullet} (F^\bullet, F)(U_I))$$

is a filtered quasi-isomorphism of filtered complexes of abelian groups.

- Let $r \in \mathbb{N}$. More generally, a complex of filtered presheaves $(F^\bullet, F) \in C_{fil}(\mathcal{S})$ is said to be r -filtered τ fibrant if it satisfy descent for covers in \mathcal{S} , i.e. if for all $X \in \mathcal{S}$ and all τ covers $(c_i : U_i \rightarrow X)_{i \in I}$ of X ,

$$E_r^{p,q}(F^\bullet, F)(c_i) : E_r^{p,q}(F^\bullet, F)(X) \rightarrow E_r^{p,q}(\text{Tot}(\bigoplus_{card I = \bullet} (F^\bullet, F)(U_I)))$$

is an isomorphism for all $p, q \in \mathbb{Z}$. Note that if (F^\bullet, F) is r -filtered τ fibrant, then it is s -filtered τ fibrant for all $s \geq r$.

- A complex of filtered presheaves $(F^\bullet, F) \in C_{fil}(\mathcal{S})$ is said to be ∞ -filtered τ fibrant if there exist $r \in \mathbb{N}$ such that (F^\bullet, F) is r -filtered τ fibrant. If a complex of filtered presheaves $(F^\bullet, F) \in C_{fil}(\mathcal{S})$ is ∞ -filtered τ fibrant, then for all $X \in \mathcal{S}$ and all τ covers $(c_i : U_i \rightarrow X)_{i \in I}$ of X ,

$$H^n(F^\bullet, F)(c_i) : H^n(F^\bullet, F)(X) \rightarrow H^n \text{Tot}(\bigoplus_{card I = \bullet} (F^\bullet, F)(U_I))$$

is a filtered isomorphism for all $n \in \mathbb{Z}$.

2.3.2 Canonical flasque resolution of a presheaf on a site or a presheaf of module on a ringed topos

Let $\mathcal{S} \in \text{Cat}$ a site with topology τ . Denote $a_\tau : \text{PSh}(\mathcal{S}) \rightarrow \text{Shv}(\mathcal{S})$ the sheafification functor. There is for $F \in C(\mathcal{S})$ an explicit τ fibrant replacement :

- $k : F^\bullet \rightarrow E_\tau^\bullet(F^\bullet) := \text{Tot}(E_\tau^\bullet(F^\bullet))$, if $F^\bullet \in C^+(\mathcal{S})$,
- $k : F^\bullet \rightarrow E_\tau^\bullet(F^\bullet) := \text{holim Tot}(E_\tau^\bullet(F^{\bullet \geq n}))$, if $F^\bullet \in C(\mathcal{S})$ is not bounded below.

The bicomplex $E^\bullet(F^\bullet) := E_\tau^\bullet(F^\bullet)$ together with the map $k : F^\bullet \rightarrow E^\bullet(F^\bullet)$ is given inductively by

- considering $p_S : \mathcal{S}^\delta \rightarrow \mathcal{S}$ the morphism of site from the discrete category \mathcal{S}^τ whose objects are the points of the topos \mathcal{S} and we take

$$k_0 := \text{ad}(p_S^*, p_{S*})(F^\bullet) \rightarrow E^0(F^\bullet) := p_{S*} p_S^* F^\bullet := \bigoplus_{s \in \mathcal{S}^\tau} \lim_{X \in \mathcal{S}, s \in X} F^\bullet(X),$$

then $a_\tau k_0 : a_\tau F^\bullet \rightarrow E^0(F^\bullet)$ is injective and $E^0(F^\bullet)$ is τ fibrant,

- denote $Q^0(F^\bullet) := a_\tau \text{coker}(k_0 : F^\bullet \rightarrow E^0(F^\bullet))$ and take the composite

$$E^0(F^\bullet) \rightarrow Q^0(F^\bullet) \rightarrow E^1(F^\bullet) := E^0(Q^0(F^\bullet)).$$

Note that $k : F^\bullet \rightarrow E^\bullet(F^\bullet)$ is a τ local equivalence and that $a_\tau k : a_\tau F^\bullet \rightarrow E^\bullet(F^\bullet)$ is injective by construction.

Since E^0 is functorial, E is functorial: for $m : F^\bullet \rightarrow G^\bullet$ a morphism, with $F^\bullet, G^\bullet \in C(\mathcal{S})$, we have a canonical morphism $E(m) : E(F) \rightarrow E(G)$ such that $E(m) \circ k = k' \circ m$, with $k : F \rightarrow E(F)$ and $k' : G \rightarrow E(G)$. Note that E^0 , hence E preserve monomorphisms. This gives, for $(F^\bullet, F) \in C_{fil}(\mathcal{S})$, a filtered τ local equivalence $k : (F^\bullet, F) \rightarrow E^\bullet(F^\bullet, F)$ with $E^\bullet(F^\bullet, F)$ filtered τ fibrant.

Moreover, we have a canonical morphism $E(F) \otimes E(G) \rightarrow E(F \otimes G)$.

There is, for $g : \mathcal{T} \rightarrow \mathcal{S}$ a morphism of presite with $\mathcal{T}, \mathcal{S} \in \text{Cat}$ two site, and $F^\bullet \in C(\mathcal{S})$, a canonical transformation

$$T(g, E)(F^\bullet) : g^* E(F^\bullet) \rightarrow E(g^* F^\bullet) \tag{17}$$

given inductively by

- $T(g, E^0)(F) := T(g, p_S)(p_S^* F) : g^* E^0(F) = g^* p_{S*} p_S^* F \rightarrow p_{T*} g^* p_S^* F = p_{T*} p_T^* g^* F = E^0(g^* F)$,
- $T(g, Q^0)(F) := \overline{T(g, E^0)(F)} : g^* Q^0(F) = \text{coker}(g^* F \rightarrow g^* E^0(F)) \rightarrow Q^0(g^* F) = \text{coker}(g^* F \rightarrow E^0(g^* F))$

- $T(g, Q^1)(F) : g^* E^1(F) = g^* E^0(Q^0(F)) \xrightarrow{T(g, E^0)(Q^0(F))} E^0(g^* Q^0(F)) \xrightarrow{E^0(T(g, Q^0)(F))} E^0(Q^0(g^* F)) = E^1(g^* F)$.

Let $(\mathcal{S}, O_{\mathcal{S}}) \in \text{RCat}$ with topology τ . Then, for $F^\bullet \in C_{O_{\mathcal{S}}}(\mathcal{S})$, $E_\tau(F^\bullet)$ is naturally a complex of $O_{\mathcal{S}}$ modules such that $k : F^\bullet \rightarrow E_\tau(F^\bullet)$ is $O_{\mathcal{S}}$ linear, that is a morphism in $C_{O_{\mathcal{S}}}(\mathcal{S})$.

We will use the following :

Proposition 7. *Let $\mathcal{S} \in \text{Cat}$ an small category endowed with a topology τ . Let $\phi : G_1^\bullet \rightarrow G_2^\bullet$ a morphism with $G_1^\bullet, G_2^\bullet \in C(\mathcal{S})$. It is a morphism of filtered complex $\phi : (G_1^\bullet, F_b) \rightarrow (G_2^\bullet, F_b)$, where we recall that F_b is the trivial filtration. If $\phi : G_1^\bullet \rightarrow G_2^\bullet$ is an homotopy equivalence, then $E(\phi) : E^\bullet(G_1^\bullet, F_b) \rightarrow E^\bullet(G_2^\bullet, F_b)$ is a 2-filtered quasi-isomorphism that is for all $p, q \in \mathbb{Z}$*

$$E_2^{p,q}(E(\phi)) : E_2^{p,q}(E^\bullet(G_1^\bullet, F_b)) \xrightarrow{\sim} E_2^{p,q}(E^\bullet(G_2^\bullet, F_b))$$

is an isomorphism

Proof. Let $X \in \mathcal{S}$. Consider the cohomological functors

$$T_X^n : K(\mathcal{S}) \rightarrow \text{Ab}, G^\bullet \mapsto H^n E^\bullet(G^\bullet)(X) =: \mathbb{H}^n(X, G^\bullet)$$

and denote, for $G^\bullet \in C(\mathcal{S})$ by $E_r^{p,q}(T_X(G^\bullet))$ the associated spectral sequence. The result then follows (see [30]) from the following commutative diagram of complexes for each $r \in \mathbb{N}$

$$\begin{array}{ccc} E_r^{\bullet,\bullet}(E^\bullet(G_1^\bullet, F_b)) & \xrightarrow{E_r^{\bullet,\bullet}E(\phi)} & E_r^{\bullet,\bullet}E^\bullet(G_2^\bullet, F_b) \\ \downarrow = & & \downarrow = \\ E_r^{\bullet,\bullet}(T_X(G_1^\bullet)) & \xrightarrow{E_r^{\bullet,\bullet}(T_X(\phi))} & E_r^{\bullet,\bullet}(T_X(G_2^\bullet)) \end{array}$$

□

2.3.3 Canonical projective resolution of a presheaf of module on a ringed topos

Let $(\mathcal{S}, O_{\mathcal{S}}) \in \text{RCat}$. We recall that we denote by, for $U \in \mathcal{S}$, $\mathbb{Z}(U) \in \text{PSh}(\mathcal{S})$ the presheaf represented by U : for $V \in \mathcal{S}$ $\mathbb{Z}(U)(V) = \mathbb{Z} \text{Hom}(V, U)$, and for $h : V_1 \rightarrow V_2$ a morphism in \mathcal{S} , and $h_1 : V_1 \rightarrow U$ $\mathbb{Z}(U)(h) : h_1 \rightarrow h \circ h_1$, and s is the morphism of presheaf given by $s(V_1)(h_1) = F(h_1)(s) \in F(V_1)$. There is for $F \in C_{O_{\mathcal{S}}}(\mathcal{S})$ a complex of $O_{\mathcal{S}}$ module an explicit projective replacement :

- $q : L_O^\bullet(F^\bullet) := \text{Tot}(L_O^\bullet(F^\bullet)) \rightarrow F^\bullet$, if $F^\bullet \in C^-(\mathcal{S})$,
- $q : L_O^\bullet(F^\bullet) := \text{holim Tot}(L_O^\bullet(F^{\bullet \leq n}))$ if $F^\bullet \in C(\mathcal{S})$ is not bounded above.

For $O_{\mathcal{S}} = \mathbb{Z}_{\mathcal{S}}$, we denote $L_{\mathbb{Z}_{\mathcal{S}}}^\bullet(F^\bullet) =: L(F^\bullet)$. The bicomplex $L_O^\bullet(F^\bullet)$ together with the map $q : L_O^\bullet(F^\bullet) \rightarrow F^\bullet$ is given inductively by

- considering the pairs $\{U \in \mathcal{S}, s \in F(U)\}$, where U is an object of \mathcal{S} and s a section of F over U we take

$$q_0 : L_O^0(F) := \bigoplus_{(U \in \mathcal{S}, s \in F(U))} \mathbb{Z}(U) \otimes O_{\mathcal{S}} \xrightarrow{s} F,$$

then q_0 is surjective and $L_O^0(F)$ is projective, this construction is functorial : for $m : F \rightarrow G$ a morphism in $\text{PSh}(\mathcal{S})$ the following diagram commutes

$$\begin{array}{ccc} \bigoplus_{(U \in \mathcal{S}, s \in F(U))} \mathbb{Z}(U) \otimes O_{\mathcal{S}} & \xrightarrow{q_0} & F \\ L_O(m) \downarrow & & \downarrow m \\ \bigoplus_{(U \in \mathcal{S}, s' \in G(U))} \mathbb{Z}(U) \otimes O_{\mathcal{S}} & \xrightarrow{q_0} & G \end{array}$$

where $(L_O(m))_{|(U,s)}(U, m(U)(s)) = I_{\mathbb{Z}(U)}$ and $(L_O(m))_{|(U,s)}(U, s') = 0$ if $s' \neq m(U)(s)$,

- denote $K_O^0(F) := \ker(q_0 : L_O^0(F) \rightarrow F)$ and take the composite

$$q_1 : L_O^1(F^\bullet) := L_O^0(K_O^0(F^\bullet)) \xrightarrow{q_0(K_O^0(F))} K_O^0(F^\bullet) \hookrightarrow L_O^0(F^\bullet).$$

Note that $q = q(F) : L(F^\bullet) \rightarrow F^\bullet$ is a surjective quasi-isomorphism by construction. Since L_O^0 is functorial, L_O is functorial : for $m : F^\bullet \rightarrow G^\bullet$ a morphism, with $F^\bullet, G^\bullet \in C(\mathcal{S})$, we have a canonical morphism $L_O(m) : L_O(F) \rightarrow L_O(G)$ such that $q' \circ L_O(m) = m \circ q'$, with $q : L_O(F) \rightarrow F$ and $q' : L_O(G) \rightarrow G$. Note that L_O^0 and hence L_O preserve monomorphisms. In particular, it gives for $(F^\bullet, F) \in C_{O_S}(\mathcal{S})$, a filtered quasi-isomorphism $q : L_O(F^\bullet, F) \rightarrow (F^\bullet, F)$. Moreover, we have a canonical morphism $L_O(F) \otimes L_O(G) \rightarrow L_O(F \otimes G)$.

Let $g : \mathcal{T} \rightarrow \mathcal{S}$ a morphism of presite with $\mathcal{T}, \mathcal{S} \in \text{Cat}$ two sites.

- Let $F^\bullet \in C(\mathcal{S})$. Since $g^*L(F^\bullet)$ is projective and $q(g^*F) : L(g^*F) \rightarrow g^*F^\bullet$ is a surjective quasi-isomorphism, there is a canonical transformation

$$T(g, L)(F^\bullet) : g^*L(F^\bullet) \rightarrow L(g^*F) \quad (18)$$

unique up to homotopy such that $q(g^*F) \circ T(g, L)(F^\bullet) = g^*q(F)$.

- Let $F^\bullet \in C(\mathcal{S})$. Since $L(g^*F)$ is projective and $g^*q(F) : g^*L(F) \rightarrow g^*F^\bullet$ is a surjective quasi-isomorphism, there is a canonical transformation

$$T(g, L)(F^\bullet) : L(g^*F) \rightarrow g^*L(F^\bullet) \quad (19)$$

unique up to homotopy such that $g^*q(F) \circ T(g, L)(F^\bullet) = q(g^*F)$.

- Let $F^\bullet \in C(\mathcal{T})$. Since $L(g_*F)$ is projective and $g_*q(F) : g_*L(F) \rightarrow g_*F^\bullet$ is a surjective quasi-isomorphism, there is a canonical transformation

$$T_*(g, L)(F^\bullet) : L(g_*F) \rightarrow g_*L(F^\bullet) \quad (20)$$

unique up to homotopy such that $g_*q(F) \circ T_*(g, L)(F^\bullet) = q(g_*F)$.

Let $g : (\mathcal{T}, O_{\mathcal{T}}) \rightarrow (\mathcal{S}, O_{\mathcal{S}})$ a morphism with $(\mathcal{T}, O_{\mathcal{T}}), (\mathcal{S}, O_{\mathcal{S}}) \in \text{RCat}$. Let $F^\bullet \in C_{O_S}(\mathcal{S})$. Since $g^{*mod}L_O(F^\bullet)$ is projective and $q(g^{*mod}F) : L_O(g^{*mod}F) \rightarrow g^{*mod}F^\bullet$ is a surjective quasi-isomorphism, there is a canonical transformation

$$T(g, L_O)(F^\bullet) : g^{*mod}L_O(F^\bullet) \rightarrow L_O(g^{*mod}F) \quad (21)$$

unique up to homotopy such that $q(g^{*mod}F) \circ T(g, L_O)(F^\bullet) = g^{*mod}q(F)$.

Let $p : (\mathcal{S}_{12}, O_{\mathcal{S}_{12}}) \rightarrow (\mathcal{S}_1, O_{\mathcal{S}_1})$ a morphism with $(\mathcal{S}_{12}, O_{\mathcal{S}_{12}}), (\mathcal{S}_1, O_{\mathcal{S}_1}) \in \text{RCat}$, such that the structural morphism $p^*O_{\mathcal{S}_1} \rightarrow O_{\mathcal{S}_{12}}$ is flat. Let $F^\bullet \in C_{O_S}(\mathcal{S})$. Since $L_O(p^{*mod}F^\bullet)$ is projective and $p^{*mod}q(F) : p^{*mod}L_O(F^\bullet) \rightarrow p^{*mod}F^\bullet$ is a surjective quasi-isomorphism, there is also in this case a canonical transformation

$$T(p, L_O)(F^\bullet) : L_O(p^{*mod}F^\bullet) \rightarrow p^{*mod}L_O(F^\bullet) \quad (22)$$

unique up to homotopy such that $p^{*mod}q(F) \circ T(p, L_O)(F^\bullet) = q(p^{*mod}F)$.

2.3.4 The De Rham complex of a ringed topos and functorialities

Let $A \in \text{cRing}$ a commutative ring. For $M \in \text{Mod}(A)$, we denote by

$$\text{Der}_A(A, M) \subset \text{Hom}(A, M) = \text{Hom}_{\text{Ab}}(A, M)$$

the abelian subgroup of derivation. Denote by $I_A = \ker(s_A : A \otimes A \rightarrow A) \subset A \otimes A$ the diagonal ideal with $s_A(a_1, a_2) = a_1 - a_2$. Let $\Omega_A := I_A/I_A^2 \in \text{Mod}(A)$ together with its derivation map $d = d_A : A \rightarrow \Omega_A$. Then, for $M \in \text{Mod}(A)$ the canonical map

$$w(M) : \text{Hom}_A(\Omega_A, M) \xrightarrow{\sim} \text{Der}_A(A, M), \psi \mapsto \phi \circ d$$

is an isomorphism, that is Ω_A is the universal derivation. In particular, its dual $T_A := \mathbb{D}^A(\Omega_A) = D^A(I_A/I_A^2)$ is isomorphic to the derivations group : $w(A) : T_A \xrightarrow{\sim} \text{Der}_A(A, A)$. Also note that $\text{Der}_A(A, A) \subset \text{Hom}(A, A)$ is a Lie subalgebra. If $\phi : A \rightarrow B$ is a morphism of commutative ring, we have a canonical morphism of abelian group $\Omega_{(B/A)}\phi : \Omega_A \rightarrow \Omega_B$.

Let $(\mathcal{S}, O_S) \in \text{RCat}$, with $O_S \in \text{PSh}(\mathcal{S}, \text{cRing})$ commutative. For $G \in \text{PSh}_{O_S}(\mathcal{S})$, we denote by

$$\text{Der}_{O_S}(O_S, G) \subset \mathcal{H}om(O_S, G) = \mathcal{H}om_{\text{Ab}}(O_S, G)$$

the abelian subpresheaf of derivation. Denote by $\mathcal{I}_S = \ker(s_S : O_S \otimes O_S \rightarrow O_S) \in \text{PSh}_{O_S \times O_S}(\mathcal{S})$ the diagonal ideal with $s_S(X) = s_{O_S(X)}$ for $X \in \mathcal{S}$. Then $\Omega_{O_S} := \mathcal{I}_S/\mathcal{I}_S^2 \in \text{PSh}_{O_S}(\mathcal{S})$ together with its derivation map $d : O_S \rightarrow \Omega_{O_S}$ is the universal derivation O_S -module : the canonical map

$$w(G) : \mathcal{H}om_{O_S}(\Omega_{O_S}, G) \xrightarrow{\sim} \text{Der}_{O_S}(O_S, G), \phi \mapsto \phi \circ d$$

is an isomorphism. In particular, its dual $T_{O_S} := \mathbb{D}_{O_S}^O(\Omega_{O_S}) = \mathbb{D}_S^O(\mathcal{I}_S/\mathcal{I}_S^2)$ is isomorphic to the presheaf of derivations : $w(O_S) : T_{O_S} \xrightarrow{\sim} \text{Der}_{O_S}(O_S, O_S)$ and $\text{Der}_{O_S}(O_S, O_S) \subset \text{Hom}(O_S, O_S)$ is a Lie subalgebra. The universal derivation $d = d_{O_S} : O_S \rightarrow \Omega_{O_S}$ induces the De Rham complex

$$DR(O_S) : \Omega_S^\bullet := \wedge^\bullet \Omega_{O_S} \in C(\mathcal{S})$$

A morphism $\phi : O'_S \rightarrow O_S$ with $O_S, O'_S \in \text{PSh}(\mathcal{S}, \text{cRing})$ induces by the universal property canonical morphisms

$$\Omega_{O'_S/O_S} : \Omega_{O'_S} \rightarrow \Omega_{O_S}, \mathbb{D}_{O'_S}^O \Omega_{O'_S/O_S} : T_{O_S} \rightarrow T_{O'_S}$$

in $\text{PSh}_{O_S}(\mathcal{S})$.

- In the particular cases where $S = (S, O_S) \in \text{Var}(\mathbb{C})$ or $S = (S, O_S) \in \text{AnSp}(\mathbb{C})$, we denote as usual $\Omega_S := \Omega_{O_S/\mathbb{C}_S}$, $T_S := T_{O_S/\mathbb{C}_S}$ and $DR(S) := DR(O_S/\mathbb{C}_S) : \Omega_S^\bullet \in C(S)$.
- In the particular cases where $S = (S, O_S) \in \text{Diff}(\mathbb{R})$ is a differential manifold, we denote as usual $\mathcal{A}_S := \Omega_{O_S/\mathbb{R}_S}$, $T_S := T_{O_S/\mathbb{R}_S}$ and $DR(S) := DR(O_S/\mathbb{R}_S) : \mathcal{A}_S^\bullet \in C(S)$.

For $f : (\mathcal{X}, O_X) \rightarrow (\mathcal{S}, O_S)$ with $(\mathcal{X}, O_X), (\mathcal{S}, O_S) \in \text{RCat}$ such that O_X and O_S are commutative, we denote by

$$\Omega_{O_X/f^*O_S} := \text{coker}(\Omega_{O_X/f^*O_S} : \Omega_{f^*O_S} \rightarrow \Omega_{O_X}) \in \text{PSh}_{f^*O_S}(\mathcal{X})$$

the relative cotangent sheaf. The surjection $q = q_{O_X/f} : \Omega_{O_X} \rightarrow \Omega_{O_X/f^*O_S}$ gives the derivation $w(\Omega_{O_X/f^*O_S})(q) = d_{O_X/f} : O_X \rightarrow \Omega_{O_X/f^*O_S}$. It induces the surjections $q^p := \wedge^p q : \Omega_{O_X}^p \rightarrow \Omega_{O_X/f^*O_S}^p$. We then have the relative De Rham complex

$$DR(O_X/f^*O_S) := \Omega_{O_X/f^*O_S}^\bullet := \wedge^\bullet \Omega_{O_X/f^*O_S} \in C_{f^*O_S}(\mathcal{X}).$$

whose differentials are given by

$$\text{for } X^o \in \mathcal{X} \text{ and } \omega \in \Gamma(X^o, \Omega_{O_X}^p) \quad d(q^p(\omega)) := q^{p+1}(d(\omega))$$

Note that $\Omega_{O_X/f^*O_S}^\bullet \in C_{f^*O_S}(\mathcal{S})$ is a complex of f^*O_S modules, but is NOT a complex of O_X module since the differential is a derivation hence NOT O_X linear. On the other hand, the canonical map in $\text{PSh}_{f^*O_S}(\mathcal{S})$

$$T(f, \text{hom})(O_S, O_S) : f^*\mathcal{H}om(O_S, O_S) \rightarrow \mathcal{H}om(f^*O_S, f^*O_S)$$

induces morphisms

$$T(f, \text{hom})(O_S, O_S) : f^*T_{O_S} \rightarrow T_{f^*O_S} \text{ and } \mathbb{D}_{f^*O_S}^O T(f, \text{hom})(O_S, O_S) : \Omega_{f^*O_S} \rightarrow f^*\Omega_{O_S}.$$

In this article, We will be interested in the following particular cases :

- In the particular case where $O_S \text{PSh}(\mathcal{S}, \text{cRing})$ is a sheaf, $\Omega_{O_S}, T_{O_S} \in \text{PSh}_{O_S}(\mathcal{S})$ are sheaves. Hence,

$$T(f, \text{hom})(O_S, O_S) : a_\tau f^* T_{O_S} \xrightarrow{\sim} T_{a_\tau f^* O_S} \quad \text{and} \quad \mathbb{D}_{f^* O_S}^O T(f, \text{hom})(O_S, O_S) : \Omega_{a_\tau f^* O_S} \xrightarrow{\sim} a_\tau f^* \Omega_{O_S}$$

are isomorphisms, where $a_\tau : \text{PSh}(\mathcal{S}) \rightarrow \text{Shv}(\mathcal{S})$ is the sheafification functor. We will note again in this case by abuse (as usual) $f^* O_S := a_\tau f^* O_S$, $f^* \Omega_{O_S} := a_\tau f^* \Omega_{O_S}$ and $f^* T_{O_S} := a_\tau f^* T_{O_S}$, so that

$$\Omega_{f^* O_S} = f^* \Omega_{O_S} \quad \text{and} \quad f^* T_{O_S} = T_{f^* O_S}$$

- In the particular cases where $S = (S, O_S), X = (X, O_X) \in \text{Var}(\mathbb{C})$ or $S = (S, O_S), X = (X, O_X) \in \text{AnSp}(\mathbb{C})$, we denote as usual $\Omega_{X/S} := \Omega_{O_X/f^* O_S}$, $q_{X/S} := q_{O_X/S} : \Omega_X \rightarrow \Omega_{X/S}$ and $DR(X/S) := DR(O_X/f^* O_S) : \Omega_{X/S}^\bullet \in C_{f^* O_S}(S)$.
- In the particular cases where $S = (S, O_S), X = (X, O_X) \in \text{Diff}(\mathbb{R})$, we denote as usual $\mathcal{A}_{X/S} := \Omega_{O_X/f^* O_S}$, $q_{X/S} := q_{O_X/S} : \mathcal{A}_X \rightarrow \mathcal{A}_{X/S}$ and $DR(X/S) := DR(O_X/f^* O_S) : \mathcal{A}_{X/S}^\bullet \in C_{f^* O_S}(S)$.

Definition 1. For a commutative diagram in RCat

$$D = \begin{array}{ccc} (\mathcal{X}, O_X) & \xrightarrow{f} & (S, O_S) \\ g' \uparrow & & \uparrow g \\ (\mathcal{X}', O_{X'}) & \xrightarrow{f'} & (\mathcal{T}, O_T) \end{array}$$

whose structural presheaves are commutative sheaves, the map in $C_{g'^* O_X \text{fil}}(\mathcal{X}')$

$$\Omega_{O_{X'}/g'^* O_X} : g'^*(\Omega_{O_X}^\bullet, F_b) = (\Omega_{g'^* O_X}^\bullet, F_b) \rightarrow (\Omega_{O_{X'}}^\bullet, F_b)$$

pass to quotient to give the map in $C_{g'^* O_X \text{fil}}(\mathcal{X}')$

$$\begin{aligned} \Omega_{(O_{X'}/g'^* O_X)/(O_T/g^* O_S)} &:= (\Omega_{O_{X'}/g'^* O_X})^q : \\ g'^*(\Omega_{O_X/f^* O_S}^\bullet, F_b) &= (\Omega_{g'^* O_X/g'^* f^* O_S}^\bullet, F_b) \rightarrow (\Omega_{O_{X'}/f'^* O_T}^\bullet, F_b) \end{aligned}$$

It is in particular given for $X'^o \in \mathcal{X}'$, $g'^*(X^o) \leftarrow X'^o$ and $\hat{\omega} \in \Gamma(X^o, \Omega_{O_X/f^* O_S}^p)$,

$$\Omega_{(O_{X'}/g'^* O_X)/(O_T/g^* O_S)}(X'^o)(\omega) := q_{O_{X'}/f'}(\Omega_{O_{X'}/g'^* O_X}(\omega)) \in \Gamma(X'^o, \Omega_{O_{X'}/f'^* O_T}^p).$$

where $\omega \in \Gamma(X^o, \Omega_{O_X}^p)$ such that $q_{O_X/f}(\omega) = \hat{\omega}$. We then have the following canonical transformation map in $C_{O_T \text{fil}}(\mathcal{T})$

$$\begin{aligned} T_\omega^O(D) : g^{* \text{mod}} L_O f_* E(\Omega_{O_X/f^* O_S}^\bullet, F_b) &\xrightarrow{q} g^* f_* E(\Omega_{O_X/f^* O_S}^\bullet, F_b) \otimes_{g^* O_S} O_T \\ \xrightarrow{T(g', E)(-) \circ T(D)(E(\Omega_{O_X/f^* O_S}^\bullet))} & (f'_* E(g'^*(\Omega_{O_X/f^* O_S}^\bullet, F_b))) \otimes_{g^* O_S} O_T \\ \xrightarrow{E(\Omega_{(O_{X'}/g'^* O_X)/(O_T/g^* O_S)})} & f'_* E(\Omega_{O_{X'}/f'^* O_T}^\bullet, F_b) \otimes_{g^* O_S} O_T \xrightarrow{m} f'_* E(\Omega_{O_{X'}/f'^* O_T}^\bullet, F_b), \end{aligned}$$

with $m(n \otimes s) = s.n$.

2.4 Presheaves on diagrams of sites or on diagrams of ringed tops

Let $\mathcal{I}, \mathcal{I}' \in \text{Cat}$ and $(f_\bullet, s) : \mathcal{T}_\bullet \rightarrow \mathcal{S}_\bullet$ a morphism of diagrams of presites with $\mathcal{T}_\bullet \in \text{Fun}(\mathcal{I}, \text{Cat})$, $\mathcal{S}_\bullet \in \text{Fun}(\mathcal{I}', \text{Cat})$. Recall it is by definition given by a functor $s : \mathcal{I} \rightarrow \mathcal{I}'$ and morphism of functor $P(f_\bullet) :$

$\mathcal{S}_s(\bullet) := \mathcal{S}_\bullet \circ s \rightarrow \mathcal{T}_\bullet$. and that we denote for short, $\mathcal{S}_s(\bullet) := \mathcal{S}_\bullet \circ s \in \text{Fun}(\mathcal{I}, \text{Cat})$. Recall that, for $r_{IJ} : I \rightarrow J$ a morphism, with $I, J \in \mathcal{I}$, D_{fIJ} is the commutative diagram in Cat

$$D_{fIJ} := \begin{array}{ccc} \mathcal{S}_{s(J)} & \xrightarrow{r_{IJ}^s} & \mathcal{S}_{s(I)} \\ f_J \uparrow & & \uparrow f_I \\ \mathcal{T}_J & \xrightarrow{r_{IJ}^t} & \mathcal{T}_I \end{array} .$$

The adjunction

$$\begin{aligned} ((f_\bullet, s)^*, (f_\bullet, s)_*) &= ((f_\bullet, s)^{-1}, (f_\bullet, s)_*) : C(\mathcal{S}_s(\bullet)) \rightleftarrows C(\mathcal{T}_\bullet), \\ G = (G_I, u_{IJ}) &\mapsto (f_\bullet, s)^*(G) := (f_I^*(G_I), T(D_{fIJ})(G_J) \circ f_I^* u_{IJ}) \\ G = (G_I, u_{IJ}) &\mapsto (f_\bullet, s)_*(G) := (f_{I*}(G_I), f_{I*} u_{IJ}) \end{aligned}$$

gives an adjunction

$$\begin{aligned} ((f_\bullet, s)^*, (f_\bullet, s)_*) &: C_{(2)fil}(\mathcal{S}_s(\bullet)) \rightleftarrows C_{(2)fil}(\mathcal{T}_\bullet), \\ (G, F) = ((G_I, F), u_{IJ}) &\mapsto (f_\bullet, s)^*(G, F) := (f_I^*(G_I, F), T(D_{fIJ})(G_J, F) \circ f_I^* u_{IJ}) \\ (G, F) = ((G_I, F), u_{IJ}) &\mapsto (f_\bullet, s)_*(G, F) := (f_{I*}(G_I, F), f_{I*} u_{IJ}). \end{aligned}$$

For a commutative diagram of diagrams of presite :

$$D = \begin{array}{ccc} \mathcal{T}'_\bullet & \xrightarrow{(g_2, s'_2)} & \mathcal{S}'_{s'_2(\bullet)} \\ \downarrow (f_2, s_2) & & \downarrow (f_1, s_1) \\ \mathcal{T}_{s_2(\bullet)} & \xrightarrow{(g_1, s'_1)} & \mathcal{S}_{s(\bullet)} \end{array} ,$$

with $\mathcal{I}, \mathcal{I}', \mathcal{J}, \mathcal{J}' \in \text{Cat}$ and $\mathcal{T}_\bullet \in \text{Fun}(\mathcal{I}, \text{Cat}), \mathcal{T}'_\bullet \in \text{Fun}(\mathcal{I}', \text{Cat}), \mathcal{S}_\bullet \in \text{Fun}(\mathcal{J}, \text{Cat}), \mathcal{S}'_\bullet \in \text{Fun}(\mathcal{J}', \text{Cat})$, and $s = s_1 \circ s'_2 = s_2 \circ s'_1 : \mathcal{I}' \rightarrow \mathcal{J}$, we denote by, for $F = (F_I, u_{IJ}) \in C(\mathcal{S}'_{s'_2(\bullet)})$,

$$T(D)(F) : g_1^* f_{1*} F \xrightarrow{g_1^* f_{1*} \text{ad}(g_2^*, g_{2*})(F)} g_1^* f_{1*} g_{2*} g_2^* F = g_1^* g_{1*} f_{2*} g_2^* F \xrightarrow{\text{ad}(g_1^* g_{1*})(f_{2*} g_2^* F)} f_{2*} g_2^* F$$

the canonical transformation map in $C(\mathcal{T}_{s_2(\bullet)})$, and for $(G, F) = ((G_I, F), u_{IJ}) \in C_{fil}(\mathcal{S}'_{s'_2(\bullet)})$,

$$\begin{aligned} T(D)(G, F) : g_1^* f_{1*}(G, F) &\xrightarrow{g_1^* f_{1*} \text{ad}(g_2^*, g_{2*})(G, F)} g_1^* f_{1*} g_{2*} g_2^*(G, F) = g_1^* g_{1*} f_{2*} g_2^*(G, F) \\ &\xrightarrow{\text{ad}(g_1^* g_{1*})(f_{2*} g_2^*(G, F))} f_{2*} g_2^*(G, F) \end{aligned}$$

the canonical transformation map in $C_{fil}(\mathcal{T}_{s_2(\bullet)})$ given by the adjunction maps.

Let $\mathcal{S}_\bullet \in \text{Fun}(\mathcal{I}, \text{RCat})$ a diagram of ringed topos with $\mathcal{I} \in \text{Cat}$. We have the tensor product bifunctor

$$\begin{aligned} (\cdot) \otimes (\cdot) &: \text{PSh}(\mathcal{S}_\bullet)^2 \rightarrow \text{PSh}(\mathcal{S}_\bullet), \\ ((F_I, u_{IJ}, (G_I, u_{IJ})) &\mapsto (F_I, u_{IJ}) \otimes (G_I, u_{IJ}) := (F_I \otimes G_I, u_{IJ} \otimes v_{IJ}) \end{aligned}$$

We get the bifunctors

$$(-) \otimes (-) : C_{fil}(\mathcal{S}_\bullet)^2 \rightarrow C_{fil}(\mathcal{S}_\bullet), \quad (-) \otimes (-) : C_{fil}(\mathcal{S}_\bullet) \times C_{O_S fil}(\mathcal{S}_\bullet) \rightarrow C_{O_S fil}(\mathcal{S}_\bullet).$$

We have the tensor product bifunctor

$$\begin{aligned} (\cdot) \otimes_{O_S} (\cdot) &: \text{PSh}_{O_S}(\mathcal{S}_\bullet)^2 \rightarrow \text{PSh}(\mathcal{S}_\bullet), \\ ((F_I, u_{IJ}, (G_I, u_{IJ})) &\mapsto (F_I, u_{IJ}) \otimes_{O_S} (G_I, u_{IJ}) := (F_I \otimes_{O_S} G_I, u_{IJ} \otimes v_{IJ}) \end{aligned}$$

which gives,

- in all case it gives the bifunctor $(-) \otimes_{O_S} (-) : C_{O_S^{op} fil}(\mathcal{S}_\bullet) \otimes C_{O_S fil}(\mathcal{S}_\bullet) \rightarrow C_{fil}(\mathcal{S}_\bullet)$.
- in the case O_S is commutative, it gives the bifunctor $(-) \otimes_{O_S} (-) : C_{O_S fil}(\mathcal{S}_\bullet)^2 \rightarrow C_{O_S fil}(\mathcal{S}_\bullet)$.

Let $(f_\bullet, s) : (\mathcal{T}_\bullet, O_T) \rightarrow (\mathcal{S}_\bullet, O_S)$ a morphism with $(\mathcal{S}_\bullet, O_S) \in \text{Fun}(\mathcal{I}', \text{RCat})$, $(\mathcal{T}_\bullet, O_T) \in \text{Fun}(\mathcal{I}, \text{RCat})$ and $\mathcal{I}, \mathcal{I}' \in \text{Cat}$. which is by definition given by a functor $s : \mathcal{I} \rightarrow \mathcal{I}'$ and morphism of ringed topos $f_\bullet : (\mathcal{T}_\bullet, O_T) \rightarrow (\mathcal{S}_{s(\bullet)}, O_S)$. As before, we denote for short, $(\mathcal{S}_{s(\bullet)}, O_S) := (\mathcal{S}_\bullet, O_S) \circ s \in \text{Fun}(\mathcal{I}, \text{RCat})$. Denote as before, for $r_{IJ} : I \rightarrow J$ a morphism, with $I, J \in \mathcal{I}$, D_{fIJ} the commutative diagram in RCat

$$D_{fIJ} := \begin{array}{ccc} \mathcal{S}_{s(J)} & \xrightarrow{r_{IJ}^s} & \mathcal{S}_{s(I)} \\ f_J \uparrow & & \uparrow f_I \\ \mathcal{T}_J & \xrightarrow{r_{IJ}^t} & \mathcal{T}_I \end{array} .$$

We have then the adjunction

$$\begin{aligned} ((f_\bullet, s)^{*mod}, (f_\bullet, s)_*) : C_{O_S}(\mathcal{S}_{s(\bullet)}) &\rightleftarrows C_{O_T}(\mathcal{T}_\bullet), \\ (G_I, u_{IJ}) \mapsto (f_\bullet, s)^{*mod}(G_I, u_{IJ}) &:= (f_I^{*mod} G_I, T^{mod}(D_{fIJ})(G_J) \circ f_I^{*mod} u_{IJ}), \\ (G_I, u_{IJ}) \mapsto (f_\bullet, s)_*(G_I, u_{IJ}) &:= (f_{I*} G_I, f_{I*} u_{IJ}). \end{aligned}$$

which induces the adjunction

$$\begin{aligned} ((f_\bullet, s)^{*mod}, (f_\bullet, s)_*) : C_{O_S fil}(\mathcal{S}_{s(\bullet)}) &\rightleftarrows C_{O_T fil}(\mathcal{T}_\bullet), \\ ((G_I, F), u_{IJ}) \mapsto (f_\bullet, s)^{*mod}((G_I, F), u_{IJ}) &:= (f_I^{*mod}(G_I, F), T^{mod}(D_{fIJ})(G_J) \circ f_I^{*mod} u_{IJ}), \\ ((G_I, F), u_{IJ}) \mapsto (f_\bullet, s)_*((G_I, F), u_{IJ}) &:= (f_{I*}(G_I, F), f_{I*} u_{IJ}). \end{aligned}$$

For a commutative diagram of diagrams of ringed topos, :

$$D = \begin{array}{ccc} (\mathcal{T}'_\bullet, O'_2) & \xrightarrow{(g_2, s'_2)} & (\mathcal{S}'_{s'_2(\bullet)}, O'_1) \\ \downarrow (f_2, s_2) & & \downarrow (f_1, s_1) \\ (\mathcal{T}_\bullet, O_2) & \xrightarrow{(g_1, s'_1)} & (\mathcal{S}_{s(\bullet)}, O_1) \end{array} ,$$

with $\mathcal{I}, \mathcal{I}', \mathcal{J}, \mathcal{J}' \in \text{Cat}$ and $\mathcal{T}_\bullet \in \text{Fun}(\mathcal{I}, \text{Cat})$, $\mathcal{T}'_\bullet \in \text{Fun}(\mathcal{I}', \text{Cat})$, $\mathcal{S}_\bullet \in \text{Fun}(\mathcal{J}, \text{Cat})$, $\mathcal{S}'_\bullet \in \text{Fun}(\mathcal{J}', \text{Cat})$, and $s = s_1 \circ s'_2 = s'_1 \circ s_2 : \mathcal{I}' \rightarrow \mathcal{J}$, we denote by, for $F = (F_I, u_{IJ}) \in C_{O'_1}(\mathcal{S}'_{s'_2(\bullet)})$,

$$T^{mod}(D)(F) : g_1^{*mod} f_{1*} F \xrightarrow{g_1^{*mod} f_{1*} \text{ad}(g_2^{*mod}, g_{2*})(F)} g_1^{*mod} f_{1*} g_{2*} g_2^{*mod} F = g_1^{*mod} g_{1*} f_{2*} g_2^{*mod} F \xrightarrow{\text{ad}(g_1^{*mod} g_{1*})(f_{2*} g_2^{*mod} F)} f_{2*} g_2^{*mod} F$$

the canonical transformation map in $C_{O_2}(\mathcal{T}_{s_2(\bullet)})$, and for $G = ((G_I, F), u_{IJ}) \in C_{O'_1 fil}(\mathcal{S}'_{s'_2(\bullet)})$,

$$T^{mod}(D)(G, F) : g_1^{*mod} f_{1*}(G, F) \xrightarrow{g_1^{*mod} f_{1*} \text{ad}(g_2^{*mod}, g_{2*})(G, F)} g_1^{*mod} f_{1*} g_{2*} g_2^{*mod}(G, F) = g_1^{*mod} g_{1*} f_{2*} g_2^{*mod}(G, F) \xrightarrow{\text{ad}(g_1^{*mod} g_{1*})(f_{2*} g_2^{*mod}(G, F))} f_{2*} g_2^{*mod}(G, F)$$

the canonical transformation map in $C_{O_2 fil}(\mathcal{T}_{s_2(\bullet)})$ given by the adjunction maps.

Let $(\mathcal{S}_\bullet, O_S) \in \text{Fun}(\mathcal{I}, \text{RCat})$ a diagram of ringed topos with $\mathcal{I} \in \text{Cat}$ and, for $I \in \mathcal{I}$, \mathcal{S}_I is endowed with topology τ_I and for $r : I \rightarrow J$ a morphism with $I, J \in \mathcal{I}$, $r_{IJ} : \hat{S}_J \rightarrow \hat{S}_I$ is continous. Let $r \in \mathbb{N}$.

- A morphism $\phi = (\phi_I) : ((F_I, F), u_{IJ}) \rightarrow ((G_I, F), u_{IJ})$ with $((F_I, F), u_{IJ}), ((G_I, F), u_{IJ}) \in C_{O_S fil}(\mathcal{S}_\bullet)$ is said to be an r -filtered τ local equivalence if the ϕ_I are r -filtered τ local equivalences for all $I \in \mathcal{I}$.
- $((G_I, F), u_{IJ}) \in C_{O_S fil}(\mathcal{S}_\bullet)$ is said to be r -filtered τ fibrant if the $(G_I, F) \in C_{O_S fil}(\mathcal{S}_I)$ are r -filtered τ fibrant for all $I \in \mathcal{I}$.

2.5 Presheaves on topological spaces and presheaves of modules on a ringed spaces

In this subsection, we will consider the particular case of presheaves on topological spaces.

Let $f : T \rightarrow S$ a continuous map with $S, T \in \text{Top}$. We denote as usual the adjunction

$$(f^*, f_*) := (P(f)^*, P(f)_*) : \text{PSh}(S) \rightleftarrows \text{PSh}(T)$$

induced by the morphism of site given by the pullback functor

$$P(f) : \text{Ouv}(S) \rightarrow \text{Ouv}(T), (S^o \subset S) \mapsto P(f)(S^o) := S^o \times_S T \xrightarrow{\sim} f^{-1}(S^o) \subset T$$

Since the colimits involved in the definition of $f^* = P(f)^*$ are filtered, f^* also preserve monomorphism. Hence, we get an adjunction

$$(f^*, f_*) : \text{PSh}_{\text{fil}}(S) \rightleftarrows \text{PSh}_{\text{fil}}(T), f^*(G, F) := (f^*G, f^*F)$$

Let $f : (T, O_T) \rightarrow (S, O_S)$ a morphism with $(S, O_S), (T, O_T) \in \text{Top}$. We have then the adjunction

$$(f^{*\text{mod}}, f_*) := (P(f)^{\text{mod}}, P(f)_*) : \text{PSh}_{O_S \text{fil}}(S) \rightleftarrows \text{PSh}_{O_T \text{fil}}(T), f^{*\text{mod}}(G, F) := f^*(G, F) \otimes_{f^*O_S} O_T$$

Recall $\text{CW} \subset \text{Top}$ is the full subcategory whose objects consists of CW complexes. Denote, for $n \in \mathbb{N}$, $\mathbb{I}^n := [0, 1]^n$, $S^n := \mathbb{I}^n / \partial \mathbb{I}^n \in \text{CW}$ and $\Delta^n \subset \mathbb{I}^n$ the n dimensional simplex. We get $\mathbb{I}^*, \Delta^* \in \text{Fun}(\Delta, \text{CW})$. Denote for $S \in \text{Top}$, $\Sigma_1 S := S \times \mathbb{I}^1 / ((\{0\} \times S) \cup (\{1\} \times S)) \in \text{Top}$.

- Let $f : T \rightarrow S$ a morphism with $T, S \in \text{Top}$. We have the mapping cylinder $\text{Cyl}(f) := (T \times \mathbb{I}^1) \sqcup_f S \in \text{Top}$ and the mapping cone $\text{Cone}(f) := (T \times \mathbb{I}^1) \sqcup_f S \in \text{Top}$. We have then the quotient map $q_f : \text{Cyl}(f) \rightarrow \text{Cone}(f)$ and a canonical retraction $r_f : \text{Cone}(f) \rightarrow \Sigma^1 T$
- Recall two morphisms $f, g : T \rightarrow S$ with $T, S \in \text{Top}$ are homotopic if there exist $H : T \times \mathbb{I}^1 \rightarrow S$ continuous such that $H \circ (I \times i_0) = f$ and $H \circ (I \times i_1) = g$. Then $K(\text{Top}) := \text{Ho}_{O_T^1}(\text{Top})$ is a triangulated category with distinguish triangle

$$T \xrightarrow{i_T} \text{Cyl}(f) \xrightarrow{q_f} \text{Cone}(f) \xrightarrow{r_f} \Sigma^1 T.$$

- For $X \in \text{Top}$, denote for $n \in \mathbb{N}$, $\pi_n(X) : \text{Hom}_{K(\text{Top})}(S^n, X)$ the homotopy groups. For $f : T \rightarrow S$ a morphism with $T, S \in \text{Top}$, we have for $n \in \mathbb{N}$ the morphisms of abelian groups

$$f_* : \pi_n(T) \rightarrow \pi_n(S), h \mapsto f \circ h$$

Recall two morphisms $f, g : T \rightarrow S$ with $T, S \in \text{Top}$ are weakly homotopic if $f_* = g_* : \pi_n(T) \rightarrow \pi_n(S)$ for all $n \in \mathbb{N}$.

- For $X \in \text{Top}$, denote by $C_*^{\text{sing}}(X) := \mathbb{Z} \text{Hom}(\Delta^*, X) \in C^-(\mathbb{Z})$ the complex of singular chains and by $C_{\text{sing}}^*(X) := \mathbb{D}^{\mathbb{Z}} C_*^{\text{sing}}(X) := \mathbb{D}^{\mathbb{Z}} \text{Hom}(\Delta^*, X) \in C^-(\mathbb{Z})$ the complex of singular cochains. For $f : T \rightarrow S$ a morphism with $T, S \in \text{Top}$, we have

– the morphism of complexes of abelian groups

$$f_* : C_*^{\text{sing}}(T) \rightarrow C_*^{\text{sing}}(S), \sigma \mapsto f \circ \sigma,$$

– the morphism of complexes of abelian groups

$$f^* := \mathbb{D}^{\mathbb{Z}} f_* : C_{\text{sing}}^*(T) \rightarrow C_{\text{sing}}^*(S), \alpha \mapsto f^* \alpha : (\sigma \mapsto f^* \alpha(\sigma) := \alpha(f \circ \sigma))$$

We denote by $C_{X,\text{sing}}^* \in C^+(X)$ the complex of presheaves of singular cochains given by,

$$(U \subset X) \mapsto C_{X,\text{sing}}^*(U) := C_{X,\text{sing}}^*(U) := C_{\text{sing}}^*(U) := \mathbb{D}^{\mathbb{Z}} \mathbb{Z} \text{Hom}(\Delta^*, U),$$

$$(j : U_2 \hookrightarrow U_1) \mapsto (j^* : C_{\text{sing}}^*(U_1) \rightarrow C_{\text{sing}}^*(U_2))$$

and by $c_X : \mathbb{Z}_X \rightarrow C_{X,\text{sing}}^*$ the inclusion map. For $f : T \rightarrow S$ a morphism with $T, S \in \text{Top}$, we have the morphism of complexes of presheaves

$$f^* : C_{S,\text{sing}}^* \rightarrow f_* C_{T,\text{sing}}^*$$

in $C(S)$.

Theorem 7. (i) *If two morphisms $f, g : T \rightarrow S$ with $T, S \in \text{Top}$ are weakly homotopic, then*

$$H^n(f_*) = H^n(g_*) : H_{n,\text{sing}}(T, \mathbb{Z}) := H^n C_*^{\text{sing}}(T) \rightarrow H_{n,\text{sing}}(S, \mathbb{Z}) := H^n C_*^{\text{sing}}(S).$$

(ii) *For $S \in \text{Top}$ there exists $CW(S) \in \text{CW}$ together with a morphism $L_S : CW(S) \rightarrow S$ which is a weakly homotopic equivalence, that is $L_{S*} : \pi_n(CW(S)) \xrightarrow{\sim} \pi_n(S)$ are isomorphisms of abelian groups for all $n \in \mathbb{N}$.*

(ii)' *For $f : T \rightarrow S$ a morphism, with $T, S \in \text{Top}$, and $L_S : CW(S) \rightarrow S$, $L_T : CW(T) \rightarrow T$ weakly homotopy equivalence with $CW(S), CW(T) \in \text{CW}$ there exist a morphism $L(f) : CW(T) \rightarrow CW(S)$ unique up to homotopy such that the following diagram in Top commutes*

$$\begin{array}{ccc} CW(S) & \xrightarrow{L_S} & S \\ L(f) \uparrow & & \uparrow f \\ CW(T) & \xrightarrow{L_T} & T \end{array}$$

In particular, for $S \in \text{Top}$, $CW(S)$ is unique up to homotopy.

Proof. See [14]. □

We have Kunneth formula for singular cohomology :

Proposition 8. *Let $X_1, X_2 \in \text{Top}$. Denote by $p_1 : X_1 \times X_2 \rightarrow X_1$ and $p_2 : X_1 \times X_2 \rightarrow X_2$ the projections. Then*

$$p_1^* \otimes p_2^* : C_{\text{sing}}^*(X_1) \otimes C_{\text{sing}}^*(X_2) \rightarrow C_{\text{sing}}^*(X_1 \times X_2)$$

is a quasi-isomorphism.

Proof. Standard (see [14] for example): follows from the fact that for all $p \in \mathbb{N}$, $H^n C_{\text{sing}}^*(\Delta^p) = 0$ for all $n \in \mathbb{Z}$. □

Remark 2. *By definition, $X \in \text{Top}$ is locally contractile if and only if the inclusion map $c_X : \mathbb{Z}_X \rightarrow C_{X,\text{sing}}^*$ is an equivalence top local. In this case it induce, by taking injective resolutions, for $n \in \mathbb{Z}$ isomorphisms*

$$H^n c_X^k : H^n(X, \mathbb{Z}_X) \xrightarrow{\sim} \mathbb{H}^n(X, C_{X,\text{sing}}^*) = H^n C_{\text{sing}}^*(X) =: H_{\text{sing}}^n(X, \mathbb{Z}).$$

We will use the following easy propositions :

Proposition 9. (i) *Let $(S, O_S) \in \text{RTop}$. Then, if $K^\bullet \in C_{O_S}^-(S)$ is a bounded above complex such that $K^n \in \text{PSh}_{O_S}(S)$ are locally free for all $n \in \mathbb{Z}$, and $\phi : F^\bullet \rightarrow G^\bullet$ is a top local equivalence with $F, G \in C_{O_S}(S)$, then $\phi \otimes I : F^\bullet \otimes_{O_S} L^\bullet \rightarrow G^\bullet \otimes_{O_S} L^\bullet$ is an equivalence top local.*

(ii) Let $f : (T, O_T) \rightarrow (S, O_S)$ a morphism with $(T, O_T), (S, O_S) \in \text{RTop}$. Then, if $K \in C_{O_S}^b(S)$ is a bounded complex such that $K^n \in \text{PSh}_{O_S}(S)$ are locally free for all $n \in \mathbb{Z}$, and $N \in C_{O_T}(T)$

$$k \circ T^{\text{mod}}(f, \otimes)(M, E(N)) : K \otimes_{O_S} f_* E(N) \rightarrow f_*((f^{*\text{mod}} K) \otimes_{O_T} E(N)) \rightarrow f_* E((f^{*\text{mod}} K) \otimes_{O_T} E(N))$$

is an equivalence top local.

Proof. Standard. \square

Proposition 10. Let $i : (Z, O_Z) \hookrightarrow (S, O_S)$ a closed embedding of ringed spaces, with $Z, S \in \text{Top}$. Then for $M \in C_{O_S}(S)$ and $N \in C_{i^*O_S}(Z)$,

$$T(i, \otimes)(M, N) : M \otimes_{O_S} i_* N \rightarrow i_*(i^* M \otimes_{i^*O_S} N)$$

is an equivalence top local.

Proof. Standard. Follows from the fact that $j^* i_* N = 0$. \square

We note the following :

Proposition 11. Let $(S, O_S) \in \text{Sch}$ such that $O_{S,s}$ are reduced local rings for all $s \in S$. For $s \in S$ consider $q : L_{O_{S,s}}(k(s)) \rightarrow k(s)$ the canonical projective resolution of the $O_{S,s}$ module $k(s) := O_{S,s}/m_s$ (the residual field) of $s \in S$. For $s \in S$ denote by $i_s : \{s\} \hookrightarrow S$ the embedding. Let $\phi : F \rightarrow G$ a morphism with $F, G \in C_{O_{S,c}}(S)$ i.e. such that $a_{\text{zar}} H^n F, a_{\text{zar}} H^n G \in \text{Coh}(S)$. If

$$i_s^* \phi \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s)) : i_s^* F \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s)) \rightarrow i_s^* G \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s))$$

is a quasi-isomorphism for all $s \in S$, then $\phi : F \rightarrow G$ is an equivalence top local.

Proof. Let $s \in S$. Since tensorizing with $L_{i_s^* O_S}(k(s))$ is an exact functor, we have canonical isomorphism $\alpha(F), \alpha(G)$ fitting in a commutative diagram

$$\begin{array}{ccc} H^n(i_s^* F \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s))) & \xrightarrow{H^n(i_s^* \phi \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s)))} & H^n(i_s^* G \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s))) \\ \downarrow \alpha(F) & & \downarrow \alpha(G) \\ i_s^*(H^n F) \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s)) & \xrightarrow{i_s^*(H^n \phi) \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s))} & i_s^*(H^n G) \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s)) \end{array}$$

Let $n \in \mathbb{Z}$. By hypothesis

$$H^n(i_s^* \phi \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s))) : H^n(i_s^* F \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s))) \xrightarrow{\sim} H^n(i_s^* G \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s)))$$

is an isomorphism. Hence, the diagram 2.5 implies that

$$i_s^*(H^n \phi) \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s)) : i_s^*(H^n F) \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s)) \xrightarrow{\sim} i_s^*(H^n G) \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s))$$

is an isomorphism. We conclude on the one hand that $i_s^* H^n \phi : i_s^* H^n F \rightarrow i_s^* H^n G$ is surjective by Nakayama lemma since $i_s^* H^n F, i_s^* H^n G$ are $O_{S,s}$ modules of finite type as $F, G \in C_{O_{S,c}}(S)$ has coherent cohomology sheaves, and on the other hand that the rows of the following commutative diagram are isomorphism

$$\begin{array}{ccc} H^0(i_s^*(H^n F) \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s))) & \xrightarrow{H^0(i_s^*(H^n \phi) \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s))) \sim} & H^0(i_s^*(H^n G) \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s))) \\ \downarrow = & & \downarrow = \\ i_s^*(H^n F) \otimes_{i_s^* O_S} k(s) & \xrightarrow{i_s^*(H^n \phi) \otimes_{i_s^* O_S} k(s) \sim} & i_s^*(H^n G) \otimes_{i_s^* O_S} k(s) \end{array}$$

Since

$$i_s^*(H^n \phi) \otimes_{i_s^* O_S} k(s) : i_s^*(H^n F) \otimes_{i_s^* O_S} k(s) \xrightarrow{\sim} i_s^*(H^n G) \otimes_{i_s^* O_S} k(s)$$

is an isomorphism for all $s \in S$, $O_{S,s} =: i_s^* O_S$ are reduced, and $a_{\text{zar}} H^n F, a_{\text{zar}} H^n G$ are coherent, $i_s^* H^n \phi : i_s^* H^n F \rightarrow i_s^* H^n G$ are injective. \square

Let $i : Z \hookrightarrow S$ a closed embedding, with $S, Z \in \text{Top}$. Denote by $j : S \setminus Z \hookrightarrow S$ the open embedding of the complementary subset. We have the adjunction

$$(i_*, i^!) := (i_*, i^\perp) : C(Z) \rightarrow C(S), \text{ with in this case } i^!F := \ker(F \rightarrow j_*j^*F).$$

It induces the adjunction $(i_*, i^!) : C_{(2)fil}(Z) \rightarrow C_{(2)fil}(S)$ (we recall that $i^! := i^\perp$ preserve monomorphisms).

Let $i : Z \hookrightarrow S$ a closed embedding, with $S, Z \in \text{Top}$. Denote by $j : S \setminus Z \hookrightarrow S$ the open embedding of the complementary subset. We have the support section functors :

- We have the functor

$$\Gamma_Z : C(S) \rightarrow C(S), F \mapsto \Gamma_Z(F) := \text{Cone}(\text{ad}(j^*, j_*)(F) : F \rightarrow j_*j^*F)[-1],$$

together with the canonical map $\gamma_Z(F) : \Gamma_Z F \rightarrow F$. We have the factorization

$$\text{ad}(i_*, i^!)(F) : i_*i^!F \xrightarrow{\text{ad}(i_*, i^!)(F)^\gamma} \Gamma_Z F \xrightarrow{\gamma_Z(F)} F,$$

and $\text{ad}(i_*, i^!)(F)^\gamma : i_*i^!F \rightarrow \Gamma_Z F$ is an homotopy equivalence. Since Γ_Z preserve monomorphisms, it induce the functor

$$\Gamma_Z : C_{fil}(S) \rightarrow C_{fil}(S), (G, F) \mapsto \Gamma_Z(G, F) := (\Gamma_Z G, \Gamma_Z F),$$

together with the canonical map $\gamma_Z((G, F)) : \Gamma_Z(G, F) \rightarrow (G, F)$.

- We have also the functor

$$\Gamma_Z^\vee : C(S) \rightarrow C(S), F \mapsto \Gamma_Z^\vee F := \text{Cone}(\text{ad}(j_!, j^*)(F) : j_!j^*F \rightarrow F),$$

together with the canonical map $\gamma_Z^\vee(F) : F \rightarrow \Gamma_Z^\vee F$. We have the factorization

$$\text{ad}(i^*, i_*)(F) : F \xrightarrow{\gamma_Z^\vee(F)} \Gamma_Z^\vee F \xrightarrow{\text{ad}(i^*, i_*)(F)^\gamma} i_*i^*F,$$

and $\text{ad}(i^*, i_*)(F)^\gamma : \Gamma_Z^\vee F \rightarrow i_*i^*F$ is an homotopy equivalence. Since Γ_Z^\vee preserve monomorphisms, it induce the functor

$$\Gamma_Z : C_{fil}(S) \rightarrow C_{fil}(S), (G, F) \mapsto \Gamma_Z^\vee(G, F) := (\Gamma_Z^\vee G, \Gamma_Z^\vee F),$$

together with the canonical map $\gamma_Z^\vee(G, F) : (G, F) \rightarrow \Gamma_Z^\vee(G, F)$.

Definition-Proposition 1. (i) Let $g : S' \rightarrow S$ a morphism and $i : Z \hookrightarrow S$ a closed embedding with $S', S, Z \in \text{Top}$. Then, for $(G, F) \in C_{fil}(S)$, there is a canonical map in $C_{fil}(S')$

$$T(g, \gamma)(G, F) : g^*\Gamma_Z(G, F) \rightarrow \Gamma_{Z \times_S S'} g^*(G, F)$$

unique up to homotopy such that $\gamma_{Z \times_S S'}(g^*(G, F)) \circ T(g, \gamma)(G, F) = g^*\gamma_Z(G, F)$.

(ii) Let $i_1 : Z_1 \hookrightarrow S$, $i_2 : Z_2 \hookrightarrow Z_1$ be closed embeddings with $S, Z_1, Z_2 \in \text{Top}$. Then, for $(G, F) \in C_{fil}(S)$,

- there is a canonical map $T(Z_2/Z_1, \gamma)(G, F) : \Gamma_{Z_2}(G, F) \rightarrow \Gamma_{Z_1}(G, F)$ in $C_{fil}(S)$ unique up to homotopy such that $\gamma_{Z_1}(G, F) \circ T(Z_2/Z_1, \gamma)(G, F) = \gamma_{Z_2}(G, F)$ together with a distinguish triangle in $K_{fil}(S) := K(\text{PSh}_{fil}(S))$

$$\Gamma_{Z_2}(G, F) \xrightarrow{T(Z_2/Z_1, \gamma)(G, F)} \Gamma_{Z_1}(G, F) \xrightarrow{\text{ad}(j_2^*, j_2^*)(\Gamma_{Z_1}(G, F))} \Gamma_{Z_1 \setminus Z_2}(G, F) \rightarrow \Gamma_{Z_2}(G, F)[1]$$

- there is a canonical map $T(Z_2/Z_1, \gamma^\vee)(G, F) : \Gamma_{Z_1}^\vee(G, F) \rightarrow \Gamma_{Z_2}^\vee(G, F)$ in $C_{fil}(S)$ unique up to homotopy such that $\gamma_{Z_2}^\vee(G, F) = T(Z_2/Z_1, \gamma^\vee)(G, F) \circ \gamma_{Z_1}^\vee(G, F)$. together with a distinguished triangle in $K_{fil}(S)$

$$\Gamma_{Z_1 \setminus Z_2}^\vee(G, F) \xrightarrow{\text{ad}(j_{21}, j_2^*)(G, F)} \Gamma_{Z_1}^\vee(G, F) \xrightarrow{T(Z_2/Z_1, \gamma^\vee)(G, F)} \Gamma_{Z_2}^\vee(G, F) \rightarrow \Gamma_{Z_2 \setminus Z_1}^\vee(G, F)[1]$$

- (iii) Consider a morphism $g : (S', Z') \rightarrow (S, Z)$ with $(S', Z'), (S, Z) \in \text{Top}^2$. We denote, for $G \in C(S)$ the composite

$$T(D, \gamma^\vee)(G) : g^* \Gamma_Z^\vee G \xrightarrow{\sim} \Gamma_{Z \times_S S'}^\vee g^* G \xrightarrow{T(Z'/Z \times_S S', \gamma^\vee)(G)} \Gamma_{Z'}^\vee g^* G$$

$$\text{and we have then the factorization } \gamma_{Z'}^\vee(g^* G) : g^* G \xrightarrow{g^* \gamma_Z^\vee(G)} g^* \Gamma_Z^\vee G \xrightarrow{T(D, \gamma^\vee)(G)} \Gamma_{Z'}^\vee g^* G.$$

Proof. (i): We have the cartesian square

$$\begin{array}{ccc} S \setminus Z & \xrightarrow{j} & S \\ g' \uparrow & & \uparrow g \\ S' \setminus Z \times_S S' & \xrightarrow{j'} & S' \end{array}$$

and the map is given by

$$(I, T(g, j)(j^* G)) : \text{Cone}(g^* G \rightarrow g^* j_* j^* G) \rightarrow \text{Cone}(g^* G \rightarrow j'_* j'^* g^* G = j'_* g'^* j^* G).$$

- (ii): Follows from the fact that $j_1^* \Gamma_{Z_2} G = 0$ and $j_1^* \Gamma_{Z_2}^\vee G = 0$, with $j_1 : S \setminus Z_1 \hookrightarrow S$ the closed embedding.
- (iii): Obvious. \square

Let $(S, O_S) \in \text{RTop}$. Let $Z \subset S$ a closed subset. Denote by $j : S \setminus Z \hookrightarrow S$ the open complementary embedding,

- For $G \in C_{O_S}(S)$, $\Gamma_Z G := \text{Cone}(\text{ad}(j^*, j_*)(G) : F \rightarrow j_* j^* G)[-1]$ has a (unique) structure of O_S module such that $\gamma_Z(G) : \Gamma_Z G \rightarrow G$ is a map in $C_{O_S}(S)$. This gives the functor

$$\Gamma_Z : C_{O_S fil}(S) \rightarrow C_{fil O_S}(S), (G, F) \mapsto \Gamma_Z(G, F) := (\Gamma_Z G, \Gamma_Z F),$$

together with the canonical map $\gamma_Z((G, F)) : \Gamma_Z(G, F) \rightarrow (G, F)$. Let $Z_2 \subset Z$ a closed subset. Then, for $G \in C_{O_S}(S)$, $T(Z_2/Z, \gamma)(G) : \Gamma_{Z_2} G \rightarrow \Gamma_Z G$ is a map in $C_{O_S}(S)$ (i.e. is O_S linear).

- For $G \in C_{O_S}(S)$, $\Gamma_Z^\vee G := \text{Cone}(\text{ad}(j_!, j^*)(G) : j_! j^* G \rightarrow G)$ has a unique structure of O_S module, such that $\gamma_Z^\vee(G) : G \rightarrow \Gamma_Z^\vee G$ is a map in $C_{O_S}(S)$. This gives the functor

$$\Gamma_Z^\vee : C_{O_S fil}(S) \rightarrow C_{fil O_S}(S), (G, F) \mapsto \Gamma_Z^\vee(G, F) := (\Gamma_Z^\vee G, \Gamma_Z^\vee F),$$

together with the canonical map $\gamma_Z^\vee((G, F)) : (G, F) \rightarrow \Gamma_Z^\vee(G, F)$. Let $Z_2 \subset Z$ a closed subset. Then, for $G \in C_{O_S}(S)$, $T(Z_2/Z, \gamma^\vee)(G) : \Gamma_{Z_2}^\vee G \rightarrow \Gamma_Z^\vee G$ is a map in $C_{O_S}(S)$ (i.e. is O_S linear).

- For $G \in C_{O_S}(S)$, we will use

$$\begin{aligned} \Gamma_Z^{\vee, h} G &:= \mathbb{D}_S^O L_O \Gamma_Z E(\mathbb{D}_S^O G) \\ &:= \text{Cone}(\mathbb{D}_S^O L_O \text{ad}(j_*, j^*)(E(\mathbb{D}_S^O G)) : \mathbb{D}_S^O L_O j_* j^* E(\mathbb{D}_S^O G) \rightarrow \mathbb{D}_S^O L_O E(\mathbb{D}_S^O G)) \end{aligned}$$

and we have the canonical map $\gamma_Z^{\vee, h}(G) : M \rightarrow \Gamma_Z^{\vee, h} G$ of O_S module. The factorization

$$\begin{aligned} \text{ad}(j_!, j^*)(L_O M) : j_! j^* L_O G &\xrightarrow{(k \circ \mathbb{D}^O I(j_!, j^*)(\mathbb{D}^O j^* L_O G) \circ \text{ad}(j_! j^* L_O G))^q} \\ &\xrightarrow{\text{ad}(j_*, j^*)(E(\mathbb{D}_S^O L_O G))} \mathbb{D}_S^O L_O E(\mathbb{D}_S^O L_O G) \end{aligned}$$

gives the factorization $\gamma_Z^{\vee,h}(L_O G) : L_O G \xrightarrow{\gamma_Z^{\vee}(L_O G)} \Gamma_Z^{\vee} L_O G \xrightarrow{(k \circ \mathbb{D}^O I(j!, j^*))(\mathbb{D}^O j^* L_O G) \circ d(j! j^* L_O G))^q} \Gamma_Z^{\vee,h} L_O G$. We get the functor

$$\Gamma_Z^{\vee,h} : C_{O_S \text{fil}}(S) \rightarrow C_{O_S \text{fil}}(S), (G, F) \mapsto \Gamma_Z^{\vee,h}(G, F) := \mathbb{D}_S^O L_O \Gamma_Z E(\mathbb{D}_S^O(G, F)),$$

together with the factorization

$$\begin{aligned} \gamma_Z^{\vee,h}(L_O(G, F)) : L_O(G, F) &\xrightarrow{\gamma_Z^{\vee}(L_O(G, F))} \Gamma_Z^{\vee} L_O(G, F) \\ &\xrightarrow{(k \circ \mathbb{D}^O I(j!, j^*))(\mathbb{D}_S^O j^* L_O(G, F)) \circ d(j! j^* L_O(G, F)))^q} \Gamma_Z^{\vee,h} L_O(G, F), \end{aligned}$$

- Consider $\mathcal{I} \subset O_S$ a right ideal of O_S such that $\mathcal{I}_Z^o \subset \mathcal{I}$, where $\mathcal{I}_Z^o \subset O_S$ is the left and right ideal consisting of section which vanish on Z .
 - For $G \in \text{PSh}_{O_S}(S)$, we consider, $S^o \subset S$ being an open subset,

$$\mathcal{I}G(S^o) = \langle \{f.m, m \in G(S^o), f \in \mathcal{I}(S^o)\} \rangle \subset G(S^o)$$

since \mathcal{I} is a right ideal, and we denote by $b_I(G) : \mathcal{I}G \rightarrow G$ the injective morphism of O_S modules and by $c_Z(G) : G \rightarrow G/\mathcal{I}G$ the quotient map. The adjunction map $\text{ad}(j!, j^*)(G) : j! j^* G \rightarrow G$ factors through $b_I(G)$:

$$\text{ad}(j!, j^*)(G) : j! j^* G \xrightarrow{b_{Z/S}^I(G)} \mathcal{I}G \xrightarrow{b_I(G)} G$$

We have then the support section functor,

$$\Gamma_Z^{\vee,O,I} : C_{O_S}(S) \rightarrow C_{O_S}(S), G \mapsto \Gamma_Z^{\vee,O,I} G := \text{Cone}(b_I(G) : \mathcal{I}G \rightarrow G)$$

together with the canonical map $\gamma_Z^{\vee,O}(G) : G \rightarrow \Gamma_Z^{\vee,O} G$ which factors through

$$\gamma_Z^{\vee,O,I}(G) : G \xrightarrow{\gamma_Z^{\vee}(G)} \Gamma_Z^{\vee} G \xrightarrow{b_{S/Z}^I(G)} \Gamma_Z^{\vee,O} G.$$

By the exact sequence $0 \rightarrow \mathcal{I}G \xrightarrow{b_I(G)} G \xrightarrow{c_I(G)} G/\mathcal{I}G \rightarrow 0$, we have an homotopy equivalence $c_I(G) : \Gamma_Z^{\vee,O,I} G \rightarrow G/\mathcal{I}G$.

- For $G \in \text{PSh}_{O_S}(S)$, we consider

$$b'_I(G) : G \rightarrow G \otimes_{O_S} \mathbb{D}_S^O(\mathcal{I}) := G \otimes_{O_S} \mathcal{H}om(\mathcal{I}, O_S)$$

The adjunction map $\text{ad}(j^*, j_*)(G) : G \rightarrow j_* j^* G$ factors through $b'_I(G)$:

$$\text{ad}(j^*, j_*)(G) : G \xrightarrow{b'_I(G)} G \otimes_{O_S} \mathbb{D}_S^O(\mathcal{I}) \xrightarrow{b'_{Z/S}(G)} j_* j^* G$$

We have then the support section functor,

$$\Gamma_Z^{O,I} : C_{O_S}(S) \rightarrow C_{O_S}(S), G \mapsto \Gamma_Z^{O,I} G := \text{Cone}(b'_I(G) : G \rightarrow G \otimes_{O_S} \mathbb{D}_S^O(\mathcal{I}))[-1]$$

together with the canonical map $\gamma_Z^O(G) : \Gamma_Z^O G \rightarrow G$ which factors through

$$\gamma_Z^{O,I}(G) : \Gamma_Z^O G \xrightarrow{b'_{S/Z}(G)} \Gamma_Z G \xrightarrow{\gamma_Z(G)} G.$$

– By definition, we have for a canonical isomorphism

$$I(D, \gamma^O)(G) : \mathbb{D}_S^O \Gamma^{\vee, O, I} G \xrightarrow{\sim} \Gamma_Z^{O, I} \mathbb{D}_S^O G$$

which gives the transformation map in $C_{O_S}(S)$

$$\begin{aligned} T(D, \gamma^O)(G) : \Gamma^{\vee, O, I} \mathbb{D}_S^O G &\xrightarrow{d(-)} \mathbb{D}_S^{O, 2} \Gamma^{\vee, O, I} \mathbb{D}_S^O G \xrightarrow{\mathbb{D}_S^O I(D, \gamma^O)(\mathbb{D}_S^O G)^{-1}} \\ &\mathbb{D}_S^O \Gamma_Z^{O, I} \mathbb{D}_S^{O, 2} G \xrightarrow{\mathbb{D}_S^O \Gamma_Z^{O, I} d(G)} \mathbb{D}_S^O \Gamma_Z^{O, I} G \end{aligned}$$

Definition-Proposition 2. (i) Let $g : (S', O_{S'}) \rightarrow (S, O_S)$ a morphism and $i : Z \hookrightarrow S$ a closed embedding with $(S', O_{S'}, (S, O_S)) \in \text{RTop}$. Then, for $(G, F) \in C_{O_S \text{fil}}(S)$, there is a canonical map in $C_{O_{S'}, \text{fil}}(S')$

$$T^{\text{mod}}(g, \gamma)(G, F) : g^{*\text{mod}} \Gamma_Z(G, F) \rightarrow \Gamma_{Z \times_S S'} g^{*\text{mod}}(G, F)$$

unique up to homotopy, such that $\gamma_{Z \times_S S'}(g^{*\text{mod}} G) \circ T^{\text{mod}}(g, \gamma)(G) = g^{*\text{mod}} \gamma_Z G$.

(ii) Let $i_1 : (Z_1, O_{Z_1}) \hookrightarrow (S, O_S)$, $i_2 : (Z_2, O_{Z_2}) \hookrightarrow (Z_1, O_{Z_1})$ be closed embeddings with $S, Z_1, Z_2 \in \text{Top}$. Then, for $(G, F) \in C_{O_S \text{fil}}(S)$, there is a canonical map in $C_{O_S \text{fil}}(S)$

$$T(Z_2/Z_1, \gamma^{\vee, O})(G, F) : \Gamma_{Z_1}^{\vee, O}(G, F) \rightarrow \Gamma_{Z_2}^{\vee, O}(G, F)$$

unique up to homotopy such that $\gamma_{Z_2}^{\vee, O}(G, F) = T(Z_2/Z_1, \gamma^{\vee, O})(G, F) \circ \gamma_{Z_1}^{\vee, O}(G, F)$.

(iii) Consider a morphism $g : ((S', O_{S'}), Z') \rightarrow ((S, O_S), Z)$ with $((S', O_{S'}), Z') \rightarrow ((S, O_S), Z) \in \text{RTop}^2$. We denote, for $M \in C_{O_S}(S)$ the composite

$$T^{\text{mod}}(D, \gamma^{\vee, O})(G) : g^{*\text{mod}} \Gamma_Z^{\vee, O} G \xrightarrow{\sim} \Gamma_{Z \times_S S'}^{\vee, O} g^{*\text{mod}} G \xrightarrow{T(Z'/Z \times_S S', \gamma^{\vee, O})(G)} \Gamma_{Z'}^{\vee, O} g^{*\text{mod}} G$$

and we have then the factorization

$$\gamma_{Z'}^{\vee, O}(g^{*\text{mod}} M) : g^{*\text{mod}} G \xrightarrow{g^{*\text{mod}} \gamma_Z^{\vee, O}(G)} g^{*\text{mod}} \Gamma_Z^{\vee, O} G \xrightarrow{T^{\text{mod}}(D, \gamma^{\vee, O})(G)} \Gamma_{Z'}^{\vee, O} g^{*\text{mod}} G$$

Proof. (i): We have the cartesian square

$$\begin{array}{ccc} S \setminus Z & \xrightarrow{j} & S \\ g' \uparrow & & \uparrow g \\ S' \setminus Z \times_S S' & \xrightarrow{j'} & S' \end{array}$$

and the map is given by

$$(I, T^{\text{mod}}(g, j)(j^* G)) : \text{Cone}(g^{*\text{mod}} G \rightarrow g^{*\text{mod}} j_* j^* G) \rightarrow \text{Cone}(g^{*\text{mod}} G \rightarrow j'_* j'^* g^{*\text{mod}} G = j'_* g'^* j^* G).$$

(ii): Obvious.

(iii): Obvious. □

Definition-Proposition 3. Consider a commutative diagram in RTop

$$\begin{array}{ccccc} D_0 = f : (X, O_X) & \xrightarrow{i} & (Y, O_Y) & \xrightarrow{p} & (S, O_S) \\ g' \uparrow & & g'' \uparrow & & \uparrow g \\ f' : (X', O_{X'}) & \xrightarrow{i'} & (Y', O_{Y'}) & \xrightarrow{p'} & (T, O_T) \end{array}$$

with i, i' being closed embeddings. Denote by D the right square of D_0 . The closed embedding $i' : X' \hookrightarrow Y'$ factors through $i' : X' \xrightarrow{i'_1} X \times_Y Y' \xrightarrow{i'_0} Y'$ where i'_1, i'_0 are closed embeddings.

(i) We have the canonical map,

$$E(\Omega_{((O_{Y'}/g''^*O_Y)/(O_T/g^*O_S))}) \circ T(g'', E)(-) \circ T(g'', \gamma)(-) : \\ g''^* \Gamma_X E(\Omega_{O_Y/p^*O_S}^\bullet, F_b) \rightarrow \Gamma_{X \times_Y Y'} E(\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b)$$

unique up to homotopy such that the following diagram in $C_{g''^*p^*O_S \text{ fil}}(Y') = C_{p'^*g^*O_S \text{ fil}}(Y')$ commutes

$$\begin{array}{ccc} g''^* \Gamma_X E(\Omega_{O_Y/p^*O_S}^\bullet, F_b) & \xrightarrow{E(\Omega_{((O_{Y'}/g''^*O_Y)/(O_T/g^*O_S))}) \circ T(g'', E)(-) \circ T(g'', \gamma)(-)} & \Gamma_{X \times_Y Y'} E(\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b) . \\ \gamma_X(-) \downarrow & & \downarrow \gamma_{X \times_Y Y'}(-) \\ g''^* E(\Omega_{O_Y/p^*O_S}^\bullet, F_b) & \xrightarrow{E(\Omega_{((O_{Y'}/g''^*O_Y)/(O_T/g^*O_S))}) \circ T(g'', E)(-)} & E(\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b) \end{array}$$

(ii) There is a canonical map,

$$T_\omega^O(D)^\gamma : g^{*\text{mod}} L_{Op^*} \Gamma_X E(\Omega_{O_Y/p^*O_S}^\bullet, F_b) \rightarrow p'_* \Gamma_{X \times_Y Y'} E(\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b)$$

unique up to homotopy such that the following diagram in $C_{O_T \text{ fil}}(T)$ commutes

$$\begin{array}{ccc} g^{*\text{mod}} L_{Op^*} \Gamma_X E(\Omega_{O_Y/p^*O_S}^\bullet, F_b) & \xrightarrow{T_\omega^O(D)^\gamma} & p'_* \Gamma_{X \times_Y Y'} E(\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b) . \\ \gamma_X(-) \downarrow & & \downarrow \gamma_{X \times_Y Y'}(-) \\ g^{*\text{mod}} L_{Op^*} E(\Omega_{O_Y/p^*O_S}^\bullet, F_b) & \xrightarrow{T_\omega^O(D)} & p'_* E(\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b) \end{array}$$

(iii) We have the canonical map in $C_{f'^*O_T}(Y')$

$$T(X'/X \times_Y Y', \gamma)(E(\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b)) : \Gamma_{X'} E(\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b) \rightarrow \Gamma_{X \times_Y Y'} E(\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b)$$

unique up to homotopy such that $\gamma_{X \times_Y Y'}(-) \circ T(X'/X \times_Y Y', \gamma)(-) = \gamma_{X'}(-)$.

Proof. Immediate from definition. We take for the map of point (ii) the composite

$$\begin{aligned} & T_\omega^O(D)^\gamma : g^{*\text{mod}} L_{Op^*} \Gamma_X E(\Omega_{O_Y/p^*O_S}^\bullet, F_b) \xrightarrow{q} g^* p^* \Gamma_X E(\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{g^*O_S} O_T \\ & \xrightarrow{T(g'', E)(-) \circ T(g'', \gamma)(-) \circ T(D)(E(\Omega_{O_X/p^*O_S}^\bullet))} (p'_* \Gamma_{X \times_Y Y'} E(g''^* \Omega_{O_Y/p^*O_S}^\bullet, F_b)) \otimes_{g^*O_S} O_T \\ & \xrightarrow{E(\Omega_{(O_{Y'}/g''^*O_Y)/(O_T/g^*O_S)})} p'_* \Gamma_{X \times_Y Y'} E(\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b) \otimes_{g^*O_S} O_T \xrightarrow{m} p'_* \Gamma_{X \times_Y Y'} E(\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b), \end{aligned}$$

with $m(n \otimes s) = s.n$. □

Definition 2. (i) Let $S \in \text{Top}$. For $Z \subset S$ a closed subset, we denote by $C_Z(S) \subset C(S)$ the full subcategory consisting of complexes of presheaves $F \in C(S)$ such that $a_{\text{top}} H^n(j^*F) = 0$ for all $n \in \mathbb{Z}$, where $j : S \setminus Z \hookrightarrow S$ is the complementary open embedding and a_{top} is the sheafification functor.

(i)' More generally, let $(S, O_S) \in \text{RTop}$. For $Z \subset S$ a closed subset, we denote by

$$C_{O_S, Z}(S) \subset C_{O_S}(S), \quad \mathcal{QCoh}_Z(S) \subset \mathcal{QCoh}(S)$$

the full subcategories consisting of complexes of presheaves $G \in C_{O_S}(S)$ such that $a_{\text{top}} H^n(j^*F) = 0$ for all $n \in \mathbb{Z}$, resp. quasi-coherent sheaves $G \in \mathcal{QCoh}(S)$ such that $j^*F = 0$.

(ii) Let $S \in \text{Top}$. For $Z \subset S$ a closed subset, we denote by $C_{\text{fil},Z}(S) \subset C_{\text{fil}}(S)$ the full subcategory consisting of filtered complexes of presheaves $(G, F) \in C_{\text{fil}}(S)$ such that there exist $r \in \mathbb{N}$ such that $a_{\text{top}} j^* E_r^{p,q}(G, F) = 0$ for all $p, q \in \mathbb{Z}$, where $j : S \setminus Z \hookrightarrow S$ is the complementary open embedding and a_{top} is the sheafification functor. Note that this definition say that this r does NOT depend on p and q .

(ii)' More generally, let $(S, O_S) \in \text{RTop}$. For $Z \subset S$ a closed subset, we denote by

$$C_{O_S, \text{fil}, Z}(S) \subset C_{O_S, \text{fil}}(S), \mathcal{Q}Coh_{\text{fil}, Z}(S) \subset \mathcal{Q}Coh(S)$$

the full subcategories consisting of filtered complexes of presheaves $(G, F) \in C_{O_S, \text{fil}}(S)$ such that there exist $r \in \mathbb{N}$ such that $a_{\text{top}} j^* E_r^{p,q}(G, F) = 0$ for all $p, q \in \mathbb{Z}$, resp. filtered quasi-coherent sheaves $(G, F) \in \mathcal{Q}Coh(S)$ such that there exist $r \in \mathbb{N}$ such that $j^* E_r^{p,q} F = 0$ for all $p, q \in \mathbb{Z}$. Note that this definition say that this r does NOT depend on p and q .

Let $(S, O_S) \in \text{RTop}$ and $Z \subset S$ a closed subset.

- For $(G, F) \in C_{\text{fil}}(S)$, we have $\Gamma_Z(G, F), \Gamma_Z^\vee(G, F) \in C_{\text{fil}, Z}(S)$.
- For $(G, F) \in C_{O_S, \text{fil}}(S)$, we have $\Gamma_Z(G, F), \Gamma_Z^\vee(G, F), \Gamma_Z^{\vee, h}(G, F), \Gamma_Z^{\vee, O}(G, F) \in C_{O_S, \text{fil}, Z}(S)$.

Proposition 12. Let $S \in \text{Top}$ and $Z \subset S$ a closed subspace. Denote by $i : Z \hookrightarrow S$ the closed embedding.

- (i) The functor $i^* : \text{Shv}_Z(S) \rightarrow \text{Shv}(Z)$ is an equivalence of category whose inverse is $i_* : \text{Shv}(Z) \rightarrow \text{Shv}_Z(S)$. More precisely $\text{ad}(i_*, i^*)(H) : i^* i_* H \rightarrow H$ is an isomorphism if $H \in \text{Shv}(Z)$ and $\text{ad}(i_*, i^*)(G) : G \rightarrow i_* i^* G$ is an isomorphism if $G \in \text{Shv}_Z(S)$.
- (ii) : The functor $i^* : \text{Shv}_{\text{fil}, Z}(S) \rightarrow \text{Shv}_{\text{fil}}(Z)$ is an equivalence of category whose inverse is $i_* : \text{Shv}_{\text{fil}}(Z) \rightarrow \text{Shv}_{\text{fil}, Z}(S)$. More precisely $\text{ad}(i_*, i^*)(H, F) : i^* i_*(H, F) \rightarrow (H, F)$ is an isomorphism if $(H, F) \in \text{Shv}(Z)$ and $\text{ad}(i_*, i^*)(G, F) : (G, F) \rightarrow i_* i^*(G, F)$ is an isomorphism if $(G, F) \in \text{Shv}_Z(S)$.
- (iii) : The functor $i^* : D_{\tau, \text{fil}, Z}(S) \rightarrow D_{\tau, \text{fil}}(Z)$ is an equivalence of category whose inverse is $i_* : D_{\tau, \text{fil}}(Z) \rightarrow D_{\tau, \text{fil}, Z}(S)$. More precisely $\text{ad}(i_*, i^*)(H, F) : i^* i_*(H, F) \rightarrow (H, F)$ is an equivalence top local if $(H, F) \in C_{\text{fil}}(Z)$ and $\text{ad}(i_*, i^*)(G, F) : (G, F) \rightarrow i_* i^*(G, F)$ is an equivalence top local if $(G, F) \in C_{\text{fil}, Z}(S)$.

Proof. (i): Standard.

(ii): Follows from (i).

(iii): Follows from (ii). □

Let $S \in \text{Top}$ and $Z \subset S$ a closed subspace. By proposition 12, if $G \in C(S)$, $\text{ad}(i_*, i^*)(\Gamma_Z G) : \Gamma_Z G \rightarrow i_* i^* \Gamma_Z G$ is an equivalence top local since $\Gamma_Z G \in C_Z(S)$.

Let $(S, O_S) \in \text{RTop}$. Let $D = \cup_i D_i \subset X$ a normal crossing divisor, denote by $j : S \setminus D \hookrightarrow S$ the open embedding, and consider $\mathcal{I}_D \subset O_S$ the ideal of vanishing function on D which is invertible. We set, for $M \in C_{O_S}(S)$,

$$M(*D) := \lim_n \mathcal{H}om_{O_S}(\mathcal{I}^n, M),$$

and we denote by $a_D(F) : F \rightarrow F(*D)$ the surjective morphism of presheaves. The adjunction map $\text{ad}(j^*, j_*)(F) : F \rightarrow j_* j^* F$ factors through $a_D(F) :$

$$\text{ad}(j^*, j_*)(F) : F \xrightarrow{a_D(F)} F(*D) \xrightarrow{a_{S/D}(F)} j_* j^* F$$

Remark 3. • Let $j : U \hookrightarrow X$ an open embedding, with $(X, O_X) \in \text{RTop}$. Then if $F \in \text{Coh}_{O_U}(U)$ is a coherent sheaf of O_U module, $j_* F$ is quasi-coherent but NOT coherent in general. In particular for $F \in C_{O_U}(U)$ whose cohomology sheaves $a_{\text{tau}} H^n F$ are coherent for all $n \in \mathbb{Z}$, the cohomology sheaves $R^n j_* F := a_\tau H^n j_* E(F)$ of $Rj_* F = j_* E(F)$ are quasi-coherent but NOT coherent.

- Let $j : U \hookrightarrow X$ an open embedding, with $X \in \text{Sch}$. Then if $F \in \text{Coh}(U)$ is a coherent sheaf of O_U module, j_*F is quasi-coherent but NOT coherent. However, there exist an O_X submodule $\tilde{F} \subset j_*F$ such that $j^*\tilde{F} = F$ and $\tilde{F} \in \text{Coh}(X)$.

The following propositions are true for schemes but NOT for arbitrary ringed spaces like analytic spaces :

Proposition 13. (i) Let $X = (X, O_X) \in \text{Sch}$ a noetherien scheme and $D \subset X$ a closed subset. Denote by $j : U = X \setminus D \hookrightarrow X$ an open embedding. Then for $F \in \mathcal{QCoh}_{O_U}(U)$ a quasi coherent sheaf, $j_*F \in \mathcal{QCoh}_{O_X}(X)$ is quasi-coherent and is the direct limit of its coherent subsheaves.

(ii) Let $X = (X, O_X)$ a noetherien scheme and $D = \cup D_i \subset X$ a normal crossing divisor. Denote by $j : U = X \setminus D \hookrightarrow X$ an open embedding. Then for $F \in \mathcal{QCoh}_{O_U}(U)$ a quasi coherent sheaf, the canonical map $a_{X/D}(F) : F(*D) \xrightarrow{\sim} j_*F$ is an isomorphism.

Proof. Standard. □

Proposition 14. Let $S = (S, O_S) \in \text{Sch}$ and $Z \subset S$ a closed subscheme. Denote by $i : Z \hookrightarrow S$ the closed embedding.

(i) For $G \in \mathcal{QCoh}_Z(S)$, i^*G has a canonical structure of O_Z module. Moreover, the functor $i^* : \mathcal{QCoh}_Z(S) \rightarrow \mathcal{QCoh}(Z)$ is an equivalence of category whose inverse is $i_* : \mathcal{QCoh}(Z) \rightarrow \mathcal{QCoh}_Z(S)$.

(ii) : The functor $i^* : \mathcal{QCoh}_{\text{fil},Z}(S) \rightarrow \mathcal{QCoh}_{\text{fil}}(Z)$ is an equivalence of category whose inverse is $i_* : \mathcal{QCoh}_{\text{fil}}(Z) \rightarrow \mathcal{QCoh}_{\text{fil},Z}(S)$.

(iii) : The functor $i^* : D_{O_S \text{fil},Z,\text{qc}}(S) \rightarrow D_{O_Z \text{fil},\text{qc}}(Z)$ is an equivalence of category whose inverse is $i_* : D_{O_Z \text{fil},\text{qc}}(Z) \rightarrow D_{O_S \text{fil},Z,\text{qc}}(S)$.

Proof. (i):Standard.

(ii): Follows from (i).

(iii): Follows from (ii) since i^* and i_* are exact functors. □

Definition 3. Let $(S, O_S) \in \text{RTop}$ a locally ringed space with O_S commutative. Consider an $\kappa_S \in C_{O_S}(S)$. Let $\mathcal{I} \subset O_S$ an ideal subsheaf and $Z = V(\mathcal{I}) \subset S$ the associated closed subset. For $G \in \text{PSh}_{O_S}(S)$, we denote by $\hat{G}_Z := \hat{G}_{\mathcal{I}} := \lim_k G/\mathcal{I}^k G$ the completion with respect to the ideal \mathcal{I} and by $c_Z^\infty(G) : G \rightarrow \hat{G}_Z$ the quotient map. Then, the canonical map

$$d_{\kappa_S, Z}(G) : G \xrightarrow{d(G)} \mathbb{D}_S^{O,2} G \xrightarrow{T^{\text{mod}}(\otimes \kappa_S, \text{hom})(\mathbb{D}_S^O G, O_S)} \mathcal{H}om_{O_S}(\mathbb{D}_S^O G \otimes_{O_S} \kappa_S, \kappa_S) \xrightarrow{T^{\text{mod}}(\Gamma_Z E, \text{hom})(-, -)} \mathcal{H}om_{O_S}(\Gamma_Z E(\mathbb{D}_S^O G \otimes_{O_S} \kappa_S), \Gamma_Z E(\kappa_S))$$

factors through

$$d_{\kappa_S, Z}(G) : G \xrightarrow{c_Z^\infty(G)} \hat{G}_Z \xrightarrow{d_{\kappa_S, Z}(G)} \mathcal{H}om_{O_S}(\Gamma_Z E(\mathbb{D}_S^O G \otimes_{O_S} \kappa_S), \Gamma_Z E(\kappa_S))$$

Clearly if $G \in C_{O_S}(S)$ then $d_{\kappa_S, Z}(G)$ is a map in $C_{O_S}(S)$. On the other hand, we have a commutative diagram

$$\begin{array}{ccc} \Omega_{O_S}^p & \xrightarrow{d_{\kappa_S, Z}(\Omega_{O_S}^p)} & \mathcal{H}om_{O_S}(\Gamma_Z E(\mathbb{D}_S^O \Omega_{O_S}^p \otimes_{O_S} \kappa_S), \Gamma_Z E(\kappa_S)) \\ \downarrow d & & \downarrow d^{\kappa_S, \gamma} \\ \Omega_{O_S}^{p+1} & \xrightarrow{d_{\kappa_S, Z}(\Omega_{O_S}^{p+1})} & \mathcal{H}om_{O_S}(\Gamma_Z E(\mathbb{D}_S^O \Omega_{O_S}^{p+1} \otimes_{O_S} \kappa_S), \Gamma_Z E(\kappa_S)) \end{array}$$

so that $d_{\kappa_S, Z}(\Omega_{O_S}^\bullet) \in C(S)$.

The following theorem is the from [13]

Theorem 8. *Let $S \in \text{Var}(\mathbb{C})$. Let $Z = V(\mathcal{I}) \subset S$ a closed subset. Denote by $K_S \in \text{PSh}_{O_S}(S)$ the canonical bundle. Then, for $G \in C_{O_S, c}(S)$,*

$$d_{K_S, Z}(G) : \hat{G}_Z \rightarrow \mathcal{H}om_{O_S}(\Gamma_Z E(\mathbb{D}_S^0 G \otimes_{O_S} K_S), \Gamma_Z E(K_S))$$

is an equivalence Zariski local.

Let $f : (X, O_X) \rightarrow (S, O_S)$ a morphism with $(S, O_S) \in \text{RTop}$. In the particular case where O_S is a commutative sheaf of ring, $T_{O_S} \in \text{PSh}_{O_S}(S)$ and $\Omega_{O_S} = \mathbb{D}_{O_S} T_{O_S} \in \text{PSh}_{O_S}(S)$ are sheaves and the morphism in $\text{PSh}(X)$

$$T(f, \text{hom})(O_S, O_S) : f^* \mathcal{H}om(O_S, O_S) \rightarrow \mathcal{H}om(f^* O_S, f^* O_S)$$

induces isomorphisms $T(f, \text{hom})(O_S, O_S) : f^* T_{O_S} \xrightarrow{\sim} T_{f^* O_S}$ and $\mathbb{D}_{f^* O_S} T(f, \text{hom})(O_S, O_S) : \Omega_{f^* O_S} \rightarrow f^* \Omega_{O_S}$ where for $F \in \text{Shv}(S)$, we denote again (as usual) by abuse $f^* F := a_\tau f^* F \in \text{Shv}(S)$, $a_\tau : \text{PSh}(S) \rightarrow \text{Shv}(S)$ being the sheafification functor.

Definition 4. (i) *Let $(X, O_X) \in \text{RTop}$. A foliation $(X, O_X)/\mathcal{F}$ on (X, O_X) is an O_X module $\Omega_{O_X/\mathcal{F}} \in \text{PSh}_{O_X}(X)$ together with a derivation map $d := d_{\mathcal{F}} : O_X \rightarrow \Omega_{O_X/\mathcal{F}}$ such that*

- the associated map $q := q_{\mathcal{F}} := \omega_X(d) : \Omega_{O_X} \rightarrow \Omega_{O_X/\mathcal{F}}$ is surjective
- satisfy the integrability condition $d(\ker q) \subset \ker q$ which implies that the map $d : \Omega_{O_X}^p \rightarrow \Omega_{O_X}^{p+1}$ induce factors trough

$$\begin{array}{ccc} \Omega_{O_X}^p & \xrightarrow{d} & \Omega_{O_X}^{p+1} \\ q^p := \wedge^p q \downarrow & & \downarrow q^{p+1} := \wedge^{p+1} q \\ \Omega_{O_X/\mathcal{F}}^p & \xrightarrow{d} & \Omega_{O_X/\mathcal{F}}^{p+1} \end{array}$$

and $d : \Omega_{O_X/\mathcal{F}}^p \rightarrow \Omega_{O_X/\mathcal{F}}^{p+1}$ is necessary unique by the surjectivity of $q^p : \Omega_{O_X}^p \rightarrow \Omega_{O_X/\mathcal{F}}^p$.

In the particular case where $\Omega_{O_X/\mathcal{F}} \in \text{PSh}_{O_X}(X)$ is a locally free sheaf of O_X module, $\mathbb{D}_{O_X} q : T_{O_X/\mathcal{F}} := \mathbb{D}_{O_X} \Omega_{O_X/\mathcal{F}} \rightarrow T_{O_X}$ is injective and the second condition is then equivalent to the fact that the sub O_X module $T_{O_X/\mathcal{F}} \subset T_{O_X}$ is a Lie subalgebra, that is $[T_{O_X/\mathcal{F}}, T_{O_X/\mathcal{F}}] \subset T_{O_X/\mathcal{F}}$.

(ii) *A piece of leaf a foliation $(X, O_X)/\mathcal{F}$ with $(X, O_X) \in \text{RTop}$ such that O_X is a commutative sheaf of ring is an injective morphism of ringed spaces $l : (Z, O_Z) \hookrightarrow (X, O_X)$ such that $\Omega_{i^* O_X/O_Z} :: \Omega_{i^* O_X} \rightarrow \Omega_{O_Z}$ factors trough an isomorphism*

$$\Omega_{i^* O_X/O_Z} : \Omega_{i^* O_X} \xrightarrow{\mathbb{D}_{i^* O_X} T(i, \text{hom})(O_X, O_X)} i^* \Omega_{O_X} \xrightarrow{i^* q} i^* \Omega_{O_X/\mathcal{F}} \rightarrow \Omega_{O_Z}.$$

(iii) *If $f : (X, O_X) \rightarrow (S, O_S)$ is a morphism with $(X, O_X), (S, O_S) \in \text{RTop}$, we have the foliation $(X, O_X)/(S, O_S) := ((X, O_X), f)$ on (X, O_X) given by the surjection*

$$q : \Omega_{O_X} \rightarrow \Omega_{O_X/f^* O_S} := \text{coker}(\Omega_{O_X/f^* O_S} : \Omega_{f^* O_S} \rightarrow \Omega_{O_X}).$$

The fibers $i_{X_s} : (X_s, O_{X_s}) \hookrightarrow (X, O_X)$ for each $s \in S$ are the leaves of the foliation.

(iv) *We have the category FolRTop*

- whose objects are foliated ringed spaces $(X, O_X)/\mathcal{F}$ with O_X a commutatif sheaf of ring and

- whose morphisms $f : (X, O_X)/\mathcal{F} \rightarrow (S, O_S)/\mathcal{G}$ are morphisms of ringed spaces $f : (X, O_X) \rightarrow (S, O_S)$ such that $\Omega_{O_X/f^*O_S} : \Omega_{f^*O_S} \rightarrow \Omega_{O_X}$ factors through

$$\begin{array}{ccc} f^*\Omega_{O_S} & \xrightarrow{\mathbb{D}_{f^*O_S} T(f, \text{hom})(O_S, O_S)^{-1}} \Omega_{f^*O_S} & \xrightarrow{\Omega_{O_X/f^*O_S}} \Omega_{O_X} \\ f^*q_{\mathcal{G}} \downarrow & & \downarrow q_{\mathcal{F}} \\ f^*\Omega_{O_S/\mathcal{G}} & \xrightarrow{\Omega_{O_X/f^*O_S}^q} & \Omega_{O_X/\mathcal{F}} \end{array}$$

This category admits inverse limits with $(X, O_X)/\mathcal{F} \times (Y, O_Y)/\mathcal{G} = (X \times Y, p_X^*O_X \otimes p_Y^*O_Y)/p_X^*\mathcal{F} \otimes p_Y^*\mathcal{G}$ and

$$(X, O_X)/\mathcal{F} \times_{(S, O_S)/\mathcal{H}} (Y, O_Y)/\mathcal{G} = (X \times_S Y, \delta_S^*(p_X^*O_X \otimes p_Y^*O_Y))/p_X^*\mathcal{F} \otimes p_Y^*\mathcal{G}$$

with $\delta_S : X \times_S Y \hookrightarrow X \times Y$ the embedding given by the diagonal $\delta_S : S \hookrightarrow S \times S$.

Let $S \in \text{Top}$. Let $S = \bigcup_{i=1}^l S_i$ an open cover and denote by $S_I = \bigcap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{Top}$. For $I \subset [1, \dots, l]$, denote by $\tilde{S}_I = \prod_{i \in I} \tilde{S}_i$. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I$ and, for $J \subset I$, the following commutative diagram

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ j_{IJ} \uparrow & & \uparrow p_{IJ} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}$$

where $j_{IJ} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. and $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ the projection. This gives the diagram of topological spaces $(\tilde{S}_I) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{Top})$ which gives the diagram $(\tilde{S}_I) \in (\text{Ouv}(\tilde{S}_I)) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{Cat})$ Denote $m : \tilde{S}_I \setminus (S_I \setminus S_J) \hookrightarrow \tilde{S}_I$ the open embedding.

Definition 5. Let $S \in \text{Top}$. Let $S = \bigcup_{i=1}^l S_i$ an open cover and denote by $S_I = \bigcap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{Top}$. We denote by $C_{\text{fil}}(S/(\tilde{S}_I)) \subset C_{\text{fil}}(\tilde{S}_I)$ the full subcategory

- whose objects are $(G, F) = ((G_I, F)_{I \subset [1, \dots, l]}, u_{IJ})$, with $(G_I, F) \in C_{\text{fil}, S_I}(\tilde{S}_I)$, and $u_{IJ} : m^*(G_I, F) \rightarrow m^*p_{IJ*}(G_J, F)$ are ∞ -filtered top local equivalences satisfying for $I \subset J \subset K$, $p_{IJ*}u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{\text{fil}}(\tilde{S}_I)$,
- the morphisms $m : ((G_I, F), u_{IJ}) \rightarrow ((H_I, F), v_{IJ})$ being (see section 2.1) a family of morphism of complexes,

$$m = (m_I : (G_I, F) \rightarrow (H_I, F))_{I \subset [1, \dots, l]}$$

such that $v_{IJ} \circ m_I = p_{IJ*}m_J \circ u_{IJ}$ in $C_{\text{fil}}(\tilde{S}_I)$.

A morphism $m : ((G_I, F), u_{IJ}) \rightarrow ((H_I, F), v_{IJ})$ is said to be an r -filtered top local equivalence if all the m_I are r -filtered top local equivalences.

Denote $L = [1, \dots, l]$ and for $I \subset L$, $p_{0(0I)} : S \times \tilde{S}_I \rightarrow S$, $p_{I(0I)} : S \times \tilde{S}_I \rightarrow S_I$ the projections. By definition, we have functors

- $T(S/(\tilde{S}_I)) : C_{\text{fil}}(S) \rightarrow C_{\text{fil}}(S/(\tilde{S}_I))$, $(G, F) \mapsto (i_{I*}j_I^*(G, F), I)$
- $T((\tilde{S}_I)/S) : C_{\text{fil}}(S/(\tilde{S}_I)) \rightarrow C_{\text{fil}}(S)$, $((G_I, F), u_{IJ}) \mapsto \text{ho} \lim_{I \subset L} p_{0(0I)*} \Gamma_{\tilde{S}_I}^{\vee} p_{I(0I)}^*(G_I, F)$.

Note that the functors $T(S/(\tilde{S}_I))$ are embedding, since

$$\text{ad}(i_I^*, i_{I*})(j_I^*F) : i_I^*i_{I*}j_I^*F \rightarrow j_I^*F$$

are top local equivalence.

Let $f : X \rightarrow S$ a morphism, with $X, S \in \text{Top}$. Let $S = \cup_{i=1}^l S_i$ and $X = \cup_{i=1}^l X_i$ be open covers and $i_i : S_i \hookrightarrow \tilde{S}_i$, $i'_i : X_i \hookrightarrow \tilde{X}_i$ be closed embeddings, such that, for each $i \in [1, l]$, $f_i := f|_{X_i} : X_i \rightarrow S_i$ lift to a morphism $\tilde{f}_i : \tilde{X}_i \rightarrow \tilde{S}_i$. Then, $f_I = f|_{X_I} : X_I = \cap_{i \in I} X_i \rightarrow S_I = \cap_{i \in I} S_i$ lift to the morphism

$$\tilde{f}_I = \prod_{i \in I} \tilde{f}_i : \tilde{X}_I = \prod_{i \in I} \tilde{X}_i \rightarrow \tilde{S}_I = \prod_{i \in I} \tilde{S}_i$$

Denote by $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ and $p'_{IJ} : \tilde{X}_J \rightarrow \tilde{X}_I$ the projections. Consider for $J \subset I$ the following commutative diagrams

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ j_{IJ} \uparrow & & \uparrow p_{IJ} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}, \quad D'_{IJ} = \begin{array}{ccc} X_I & \xrightarrow{i'_I} & \tilde{X}_I \\ j'_{IJ} \uparrow & & \uparrow p'_{IJ} \\ X_J & \xrightarrow{i'_J} & \tilde{X}_J \end{array}, \quad D_{fI} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ f_I \uparrow & & \uparrow \tilde{f}_I \\ X_I & \xrightarrow{i'_I} & \tilde{X}_I \end{array}$$

We have then following commutative diagram

$$\begin{array}{ccccc} & & X_I & \xrightarrow{n'_I} & \tilde{X}_I & \longleftarrow & \tilde{X}_I \setminus X_I \\ & \nearrow j'_{IJ} & \uparrow p'_{IJ} & & \uparrow p'_{IJ} & & \uparrow p'_{IJ} \\ i'_J : X_J & \xrightarrow{l_{IJ}} & X_I \times X_I \times \tilde{X}_J & \xrightarrow{n'_I \times I} & \tilde{X}_J & \longleftarrow & \tilde{X}_J \setminus X_J \\ & & & & \uparrow p'_{IJ} & & \uparrow p'_{IJ} \\ & & & & & & \tilde{X}_I \setminus X_I \end{array}$$

whose square are cartesian. We then have the pullback functor

$$f^* : C_{(2)fil}(S/(\tilde{S}_I)) \rightarrow C_{(2)fil}(X/(\tilde{X}_I)), \\ ((G_I, F), u_{IJ}) \mapsto f^*((G_I, F), u_{IJ}) := (\Gamma_{X_I}^\vee \tilde{f}_I^*(G_I, F), \tilde{f}_I^* u_{IJ})$$

with

$$\tilde{f}_J^* u_{IJ} : \Gamma_{X_I}^\vee \tilde{f}_I^*(G_I, F) \xrightarrow{\text{ad}(p'_{IJ}, p'_{IJ*})(-)} p'_{IJ*} p'_{IJ}^* \Gamma_{X_I}^\vee \tilde{f}_I^*(G_I, F) \xrightarrow{T_{\sharp}(p_{IJ}, n'_I)(-)^{-1}} p'_{IJ*} \Gamma_{X_I \times \tilde{X}_J}^\vee p'_{IJ}^* \tilde{f}_I^*(G_I, F) \\ \xrightarrow{p'_{IJ*} \gamma_{X_J}^\vee(-)} p'_{IJ*} \Gamma_{X_J}^\vee p'_{IJ}^* \tilde{f}_I^*(G_I, F) = p'_{IJ*} \Gamma_{X_J}^\vee \tilde{f}_I^* p'_{IJ}^*(G_I, F) \xrightarrow{\Gamma_{X_J}^\vee \tilde{f}_I^*(p'_{IJ}, p'_{IJ*})(-, -)(u_{IJ})} \Gamma_{X_J}^\vee \tilde{f}_I^*(G_J, F)$$

Let $(G, F) \in C_{fil}(S)$. Since, $j'_I{}^* i'_I{}^* j'_I{}^* f^*(G, F) = 0$, the morphism $T(D_{fI})(j'_I{}^*(G, F)) : \tilde{f}_I^* i'_I{}^* j'_I{}^*(G, F) \rightarrow i'_I{}^* j'_I{}^* f^*(G, F)$ factors trough

$$T(D_{fI})(j'_I{}^*(G, F)) : \tilde{f}_I^* i'_I{}^* j'_I{}^*(G, F) \xrightarrow{\gamma_{X_I}^\vee(-)} \Gamma_{X_I}^\vee \tilde{f}_I^* i'_I{}^* j'_I{}^*(G, F) \xrightarrow{T^\gamma(D_{fI})(j'_I{}^*(G, F))} i'_I{}^* j'_I{}^* f^*(G, F)$$

We have then, for $(G, F) \in C_{fil}(S)$, the canonical transformation map

$$\begin{array}{ccc} f^* T(S/(\tilde{S}_I))(G, F) & \xrightarrow{T(f, T(0/I))(G, F)} & T(X/(\tilde{X}_I))(f^*(G, F)) \\ \downarrow = & & \downarrow = \\ (\Gamma_{X_I}^\vee \tilde{f}_I^* i'_I{}^* j'_I{}^*(G, F), \tilde{f}_I^* I) & \xrightarrow{T^\gamma(D_{fI})(j'_I{}^*(G, F))} & (i'_I{}^* j'_I{}^* f^*(G, F), I) \end{array}$$

Proposition 15. Let $S \in \text{Top}$. Let $S = \cup_{i=1}^l S_i$ an open cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{Top}$. Denote by $D_{(2)fil, \infty}((S/(\tilde{S}_I))) = \text{Ho}_{\text{top}, \infty}(C_{(2)fil}((S/(\tilde{S}_I))))$ the localization of $C_{(2)fil}((S/(\tilde{S}_I)))$ with respect to top local equivalences. The functor $T(S/(\tilde{S}_I))$ induces an equivalence of category

$$T(S/(\tilde{S}_I)) : D_{(2)fil, \infty}(S) \xrightarrow{\sim} D_{(2)fil, \infty}((S/(\tilde{S}_I)))$$

with inverse $T((\tilde{S}_I)/S)$

Proof. Follows from the fact that for $(G, F) \in C_{fil}(S)$,

$$\text{ho} \lim_{ICL} p_{0(0I)*} \Gamma_{S_I}^\vee p_{I(0I)}^* (i_{I*} j_I^* (G, F)) \rightarrow p_{0(0I)*} \Gamma_{S_I}^\vee j_I^* (G, F)$$

is an equivalence top local. \square

For $f : T \rightarrow S$ a morphism with $T, S \in \text{Top}$ locally compact (in particular Hausdorff), e.g. $T, S \in \text{CW}$, there is also a functor $f_! : C(T) \rightarrow C(S)$ given by the section which have compact support over f , and, for $K_1, K_2 \in C(T)$, we have a canonical map

$$T_!(f, \text{hom}) : f_* \mathcal{H}om(K_1, K_2) \rightarrow \mathcal{H}om(f_! K_1, f_! K_2)$$

The main result on presheaves on locally compact spaces is the following :

Theorem 9. *Let $f : T \rightarrow S$ a morphism with $T, S \in \text{Top}$ locally compact.*

(i) *The derived functor $Rf_! : D(T) \rightarrow D(S)$ has a right adjoint $f^!$ (Verdier duality) and, for $K_1, K_2 \in D(T)$ and $K_3, K_4 \in D(S)$, we have canonical isomorphisms*

$$\begin{aligned} & - Rf_* R\mathcal{H}om^\bullet(Rf_! K_1, K_3) \xrightarrow{\sim} R\mathcal{H}om^\bullet(K_1, f^! K_3) \\ & - f^! R\mathcal{H}om^\bullet(K_3, K_4) \xrightarrow{\sim} R\mathcal{H}om^\bullet(f^* K_3, f^! K_4) \end{aligned}$$

(ii) *Denote, for $K \in D(S)$, $D(K) = R\mathcal{H}om^\bullet(K, a_S^! \mathbb{Z}) \in D(S)$ the Verdier dual of K . Then, if $K \in D_c(S)$, the evaluation map $ev^c(S)(K) : K \rightarrow D(D(K))$ is an isomorphism.*

(iii) *Assume we have a factorization $f : T \xrightarrow{l} Y \xrightarrow{p} S$ of f with l a closed embedding and p a smooth morphism of relative dimension d . Then $f^! K = i^! p^* K[d]$*

Proof. (i): Standard, the proof is formal (see [29]).

(ii): See [29].

(iii): The fact that $p^! K = p^* K[d]$ follows by Poincare duality for topological manifold. \square

We have by theorem 9 a pair of adjoint functor

$$(Rf_!, f^!) : D(T) \rightleftarrows D(S)$$

- with $f_! = f_*$ if f is proper,
- with $f^! = f^*[d]$ if f is smooth of relative dimension d .

2.6 Presheaves on the big Zariski site or on the big etale site

For $S \in \text{Var}(\mathbb{C})$, we denote by $\rho_S : \text{Var}(\mathbb{C})^{sm}/S \hookrightarrow \text{Var}(\mathbb{C})/S$ be the full subcategory consisting of the objects $U/S = (U, h) \in \text{Var}(\mathbb{C})/S$ such that the morphism $h : U \rightarrow S$ is smooth. That is, $\text{Var}(\mathbb{C})^{sm}/S$ is the category

- whose objects are smooth morphisms $U/S = (U, h)$, $h : U \rightarrow S$ with $U \in \text{Var}(\mathbb{C})$,
- whose morphisms $g : U/S = (U, h_1) \rightarrow V/S = (V, h_2)$ is a morphism $g : U \rightarrow V$ of complex algebraic varieties such that $h_2 \circ g = h_1$.

We denote again $\rho_S : \text{Var}(\mathbb{C})/S \rightarrow \text{Var}(\mathbb{C})^{sm}/S$ the associated morphism of site. We will consider

$$r^s(S) : \text{Var}(\mathbb{C}) \xrightarrow{r(S)} \text{Var}(\mathbb{C})/S \xrightarrow{\rho_S} \text{Var}(\mathbb{C})^{sm}/S$$

the composite morphism of site. For $S \in \text{Var}(\mathbb{C})$, we denote by $\mathbb{Z}_S := \mathbb{Z}(S/S) \in \text{PSh}(\text{Var}(\mathbb{C})^{sm}/S)$ the constant presheaf. By Yoneda lemma, we have for $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$, $\mathcal{H}om(\mathbb{Z}_S, F) = F$. For $f : T \rightarrow S$ a morphism, with $T, S \in \text{Var}(\mathbb{C})$, we have the following commutative diagram of sites

$$\begin{array}{ccc} \text{Var}(\mathbb{C})/T & \xrightarrow{\rho_T} & \text{Var}(\mathbb{C})^{sm}/T \\ \downarrow P(f) & & \downarrow P(f) \\ \text{Var}(\mathbb{C})/S & \xrightarrow{\rho_S} & \text{Var}(\mathbb{C})^{sm}/S \end{array} \quad (23)$$

We denote, for $S \in \text{Var}(\mathbb{C})$, the obvious morphism of sites

$$\tilde{e}(S) : \text{Var}(\mathbb{C})/S \xrightarrow{\rho_S} \text{Var}(\mathbb{C})^{sm}/S \xrightarrow{e(S)} \text{Ouv}(S)$$

where $\text{Ouv}(S)$ is the set of the Zariski open subsets of S , given by the inclusion functors $\tilde{e}(S) : \text{Ouv}(S) \hookrightarrow \text{Var}(\mathbb{C})^{sm}/S \hookrightarrow \text{Var}(\mathbb{C})/S$. By definition, for $f : T \rightarrow S$ a morphism with $S, T \in \text{Var}(\mathbb{C})$, the commutative diagram of sites (23) extend a commutative diagram of sites :

$$\begin{array}{ccccc} \tilde{e}(T) : \text{Var}(\mathbb{C})/T & \xrightarrow{\rho_T} & \text{Var}(\mathbb{C})^{sm}/T & \xrightarrow{e(T)} & \text{Ouv}(T) \\ \downarrow P(f) & & \downarrow P(f) & & \downarrow P(f) \\ \tilde{e}(S) : \text{Var}(\mathbb{C})/S & \xrightarrow{\rho_S} & \text{Var}(\mathbb{C})^{sm}/S & \xrightarrow{e(S)} & \text{Ouv}(S) \end{array} \quad (24)$$

- As usual, we denote by

$$(f^*, f_*) := (P(f)^*, P(f)_*) : C(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{sm}/T)$$

the adjunction induced by $P(f) : \text{Var}(\mathbb{C})^{sm}/T \rightarrow \text{Var}(\mathbb{C})^{sm}/S$. Since the colimits involved in the definition of $f^* = P(f)^*$ are filtered, f^* also preserve monomorphism. Hence, we get an adjunction

$$(f^*, f_*) : C_{fil}(\text{Var}(\mathbb{C})^{sm}/S) \rightleftarrows C_{fil}(\text{Var}(\mathbb{C})^{sm}/T), f^*(G, F) := (f^*G, f^*F)$$

- As usual, we denote by

$$(f^*, f_*) := (P(f)^*, P(f)_*) : C(\text{Var}(\mathbb{C})/S) \rightarrow C(\text{Var}(\mathbb{C})/T)$$

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$$(f^*, f_*) : C_{fil}(\text{Var}(\mathbb{C})/S) \rightleftarrows C_{fil}(\text{Var}(\mathbb{C})/T), f^*(G, F) := (f^*G, f^*F)$$

- For $h : U \rightarrow S$ a smooth morphism with $U, S \in \text{Var}(\mathbb{C})$, the pullback functor $P(h) : \text{Var}(\mathbb{C})^{sm}/S \rightarrow \text{Var}(\mathbb{C})^{sm}/U$ admits a left adjoint $C(h)(X \rightarrow U) = (X \rightarrow U \rightarrow S)$. Hence, $h^* : C(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{sm}/U)$ admits a left adjoint

$$h_{\sharp} : C(\text{Var}(\mathbb{C})^{sm}/U) \rightarrow C(\text{Var}(\mathbb{C})^{sm}/S), F \mapsto ((V, h_0) \mapsto \lim_{(V', h_0 h') \rightarrow (V, h_0)} F(V', h'))$$

Note that we have $\mathbb{Z}(V/S) = h_{\sharp}\mathbb{Z}_V$. More generally for $h' : V' \rightarrow V$ a smooth morphism, $V', V \in \text{Var}(\mathbb{C})$, we have $h_{\sharp}(\mathbb{Z}(V'/V)) = \mathbb{Z}(V'/S)$ with $V'/S = (V', h_0 \circ h')$. Hence, since projective presheaves are the direct summands of the representable presheaves, h_{\sharp} sends projective presheaves to projective presheaves. For $F^{\bullet} \in C(\text{Var}(\mathbb{C})^{sm}/S)$ and $G^{\bullet} \in C(\text{Var}(\mathbb{C})^{sm}/U)$, we have the adjunction maps

$$\text{ad}(h_{\sharp}, h^*)(G^{\bullet}) : G^{\bullet} \rightarrow h^*h_{\sharp}G^{\bullet}, \text{ad}(h_{\sharp}, h^*)(F^{\bullet}) : h_{\sharp}h^*F^{\bullet} \rightarrow F^{\bullet}.$$

For a smooth morphism $h : U \rightarrow S$, with $U, S \in \text{Var}(\mathbb{C})$, we have the adjunction isomorphism, for $F \in C(\text{Var}(\mathbb{C})^{sm}/U)$ and $G \in C(\text{Var}(\mathbb{C})^{sm}/S)$,

$$I(h_{\sharp}, h^*)(F, G) : \mathcal{H}om^{\bullet}(h_{\sharp}F, G) \xrightarrow{\sim} h_*\mathcal{H}om^{\bullet}(F, h^*G). \quad (25)$$

- For $f : T \rightarrow S$ any morphism with $T, S \in \text{Var}(\mathbb{C})$, the pullback functor $P(f) \text{Var}(\mathbb{C})/T \rightarrow \text{Var}(\mathbb{C})/S$ admits a left adjoint $C(f)(X \rightarrow T) = (X \rightarrow T \rightarrow S)$. Hence, $f^* : C(\text{Var}(\mathbb{C})/S) \rightarrow C(\text{Var}(\mathbb{C})/T)$ admits a left adjoint

$$f_{\sharp} : C(\text{Var}(\mathbb{C})/T) \rightarrow C(\text{Var}(\mathbb{C})/S), F \mapsto ((V, h_0) \mapsto \lim_{(V', h_0 h') \rightarrow (V, h_0)} F(V', h'))$$

Note that we have $\mathbb{Z}(V/S) = h_{\sharp} \mathbb{Z}_V$. More generally for $h' : V' \rightarrow V$ a morphism, $V', V \in \text{Var}(\mathbb{C})$, we have $h_{\sharp}(\mathbb{Z}(V'/V)) = \mathbb{Z}(V'/S)$ with $V'/S = (V', h \circ h')$. Hence, since projective presheaves are the direct summands of the representable presheaves, h_{\sharp} sends projective presheaves to projective presheaves. For $F^{\bullet} \in C(\text{Var}(\mathbb{C})/S)$ and $G^{\bullet} \in C(\text{Var}(\mathbb{C})/T)$, we have the adjunction maps

$$\text{ad}(f_{\sharp}, f^*)(G^{\bullet}) : G^{\bullet} \rightarrow f^* f_{\sharp} G^{\bullet}, \text{ad}(f_{\sharp}, f^*)(F^{\bullet}) : f_{\sharp} f^* F^{\bullet} \rightarrow F^{\bullet}.$$

For a morphism $f : T \rightarrow S$, with $T, S \in \text{Var}(\mathbb{C})$, we have the adjunction isomorphism, for $F \in C(\text{Var}(\mathbb{C})/T)$ and $G \in C(\text{Var}(\mathbb{C})/S)$,

$$I(f_{\sharp}, f^*)(F, G) : \mathcal{H}om^{\bullet}(f_{\sharp} F, G) \xrightarrow{\sim} f_* \mathcal{H}om^{\bullet}(F, f^* G). \quad (26)$$

- For a commutative diagram in $\text{Var}(\mathbb{C})$:

$$D = \begin{array}{ccc} V & \xrightarrow{g_2} & U \\ \downarrow h_2 & & \downarrow h_1 \\ T & \xrightarrow{g_1} & S \end{array},$$

where h_1 and h_2 are smooth, we denote by, for $F^{\bullet} \in C(\text{Var}(\mathbb{C})^{sm}/U)$,

$$T_{\sharp}(D)(F^{\bullet}) : h_{2\sharp} g_2^* F^{\bullet} \rightarrow g_1^* h_{1\sharp} F^{\bullet}$$

the canonical map in $C(\text{Var}(\mathbb{C})^{sm}/T)$ given by adjunction. If D is cartesian with $h_1 = h$, $g_1 = g$, $f_2 = h' : U_T \rightarrow T$, $g' : U_T \rightarrow U$,

$$T_{\sharp}(D)(F^{\bullet}) =: T_{\sharp}(g, h)(F^{\bullet}) : h'_{\sharp} g'^* F^{\bullet} \xrightarrow{\sim} g^* h_{\sharp} F^{\bullet}$$

is an isomorphism and for $G^{\bullet} \in C(\text{Var}(\mathbb{C})^{sm}/T)$

$$T(D)(G^{\bullet}) =: T(g, h)(G^{\bullet}) : g^* h_* G^{\bullet} \xrightarrow{\sim} h'_* g'^* G^{\bullet}$$

is an isomorphism.

- For a commutative diagram in $\text{Var}(\mathbb{C})$:

$$D = \begin{array}{ccc} V & \xrightarrow{g_2} & X \\ \downarrow f_2 & & \downarrow f_1 \\ T & \xrightarrow{g_1} & S \end{array},$$

we denote by, for $F^{\bullet} \in C(\text{Var}(\mathbb{C})/X)$,

$$T_{\sharp}(D)(F^{\bullet}) : f_{2\sharp} g_2^* F^{\bullet} \rightarrow g_1^* f_{1\sharp} F^{\bullet}$$

the canonical map in $C(\text{Var}(\mathbb{C})/T)$ given by adjunction. If D is cartesian with $h_1 = h$, $g_1 = g$, $f_2 = h' : X_T \rightarrow T$, $g' : X_T \rightarrow X$,

$$T_{\sharp}(D)(F^{\bullet}) =: T_{\sharp}(g, f)(F^{\bullet}) : f'_{\sharp} g'^* F^{\bullet} \xrightarrow{\sim} g^* f_{\sharp} F^{\bullet}$$

is an isomorphism and for $G^{\bullet} \in C(\text{Var}(\mathbb{C})/T)$

$$T(D)(G^{\bullet}) =: T(g, h)(G^{\bullet}) : f^* g_* G^{\bullet} \xrightarrow{\sim} g'_* f'^* G^{\bullet}$$

is an isomorphism.

For $f : T \rightarrow S$ a morphism with $S, T \in \text{Var}(\mathbb{C})$,

- we get for $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ from the a commutative diagram of sites (24) the following canonical transformation

$$T(e, f)(F^\bullet) : f^*e(S)_*F^\bullet \rightarrow e(T)_*f^*F^\bullet,$$

which is NOT a quasi-isomorphism in general. However, for $h : U \rightarrow S$ a smooth morphism with $S, U \in \text{Var}(\mathbb{C})$, $T(e, h)(F^\bullet) : h^*e(S)_*F^\bullet \xrightarrow{\sim} e(T)_*h^*F^\bullet$ is an isomorphism.

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Let $S \in \text{Var}(\mathbb{C})$,

- We have for $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$,
 - $e(S)_*(F \otimes G) = (e(S)_*F) \otimes (e(S)_*G)$ by definition
 - the canonical forgetfull map

$$T(S, \text{hom})(F, G) : e(S)_*\mathcal{H}om^\bullet(F, G) \rightarrow \mathcal{H}om^\bullet(e(S)_*F, e(S)_*G).$$

which is NOT a quasi-isomorphism in general.

By definition, we have for $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$, $e(S)_*E_{zar}(F) = E_{zar}(e(S)_*F)$.

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which is NOT a quasi-isomorphism in general.

By definition, we have for $F \in C(\text{Var}(\mathbb{C})/S)$, $e(S)_*E_{zar}(F) = E_{zar}(e(S)_*F)$.

Let $S \in \text{Var}(\mathbb{C})$.

- We have the dual functor

$$\mathbb{D}_S : C(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{sm}/S), F \mapsto \mathbb{D}_S(F) := \mathcal{H}om(F, E_{et}(\mathbb{Z}_S))$$

It induces the functor

$$L\mathbb{D}_S : C(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{sm}/S), F \mapsto L\mathbb{D}_S(F) := \mathbb{D}_S(LF) := \mathcal{H}om(LF, E_{et}(\mathbb{Z}_S))$$

- We have the dual functor

$$\mathbb{D}_S : C(\text{Var}(\mathbb{C})/S) \rightarrow C(\text{Var}(\mathbb{C})/S), F \mapsto \mathbb{D}_S(F) := \mathcal{H}om(F, E_{et}(\mathbb{Z}_S))$$

It induces the functor

$$L\mathbb{D}_S : C(\text{Var}(\mathbb{C})/S) \rightarrow C(\text{Var}(\mathbb{C})/S), F \mapsto L\mathbb{D}_S(F) := \mathbb{D}_S(LF) := \mathcal{H}om(LF, E_{et}(\mathbb{Z}_S))$$

The adjonctions

$$(\tilde{e}(S)^*, \tilde{e}(S)_*) : C(\text{Var}(\mathbb{C})/S) \rightleftarrows C(S), \quad (e(S)^*, e(S)_*) : C(\text{Var}(\mathbb{C})^{sm}/S) \rightleftarrows C(S)$$

induce adjonctions

$$(\tilde{e}(S)^*, \tilde{e}(S)_*) : C_{fil}(\text{Var}(\mathbb{C})/S) \rightleftarrows C_{fil}(S), \quad (e(S)^*, e(S)_*) : C_{fil}(\text{Var}(\mathbb{C})^{sm}/S) \rightleftarrows C_{fil}(S)$$

given by $e(S)_*(G, F) := (e(S)_*G, e(S)_*F)$, since $e(S)_*$ and $e(S)^*$ preserve monomorphisms. Note that

- for $F \in \text{PSh}(\text{Var}(\mathbb{C})^{sm}/S)$, $e(S)_*F$ is simply the restriction of F to the small Zariski site of X ,
- for $F \in \text{PSh}(\text{Var}(\mathbb{C})/S)$, $\tilde{e}(S)_*F = e(S)_*\rho_S^*F$ is simply the restriction of F to the small Zariski site of X , ρ_S^*F being the restriction of F to $\text{Var}(\mathbb{C})^{sm}/S$.

Together with the internal hom functor, we get the bifunctor,

$$e(S)_*\mathcal{H}om(\cdot, \cdot) : C_{fil}(\text{Var}(\mathbb{C})^{sm}/S) \times C_{fil}(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C_{2fil}(S), \quad (27)$$

$$((F, W), (G, F)) \mapsto e(S)_*\mathcal{H}om^\bullet((F^\bullet, W), (G^\bullet, F)). \quad (28)$$

For $i : Z \hookrightarrow S$ a closed embedding, with $Z, S \in \text{Var}(\mathbb{C})$, we denote by

$$(i_*, i^!) := (P(i)_*, P(i)^\perp) : C(\text{Var}(\mathbb{C})^{sm}/Z) \rightleftarrows C(\text{Var}(\mathbb{C})^{sm}/S)$$

the adjunction induced by the morphism of site $P(i) : \text{Var}(\mathbb{C})^{sm}/Z \rightarrow \text{Var}(\mathbb{C})^{sm}/S$. For $i : Z \hookrightarrow S$ a closed embedding, $Z, S \in \text{Var}(\mathbb{C})$, we denote

$$\mathbb{Z}_{Z,S} := \text{Cone}(\text{ad}(i^*, i_*)(\mathbb{Z}_S) : \mathbb{Z}_S \rightarrow i_*\mathbb{Z}_Z)$$

We have the support section functors of a closed embedding $i : Z \hookrightarrow S$ for presheaves on the big Zariski site.

Definition 6. *Let $i : Z \hookrightarrow S$ be a closed embedding with $S, Z \in \text{Var}(\mathbb{C})$ and $j : S \setminus Z \hookrightarrow S$ be the open complementary subset.*

(i) *We define the functor*

$$\Gamma_Z : C(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{sm}/S), \quad G^\bullet \mapsto \Gamma_Z G^\bullet := \text{Cone}(\text{ad}(j^*, j_*)(G^\bullet) : G^\bullet \rightarrow j_*j^*G^\bullet)[-1],$$

so that there is then a canonical map $\gamma_Z(G^\bullet) : \Gamma_Z G^\bullet \rightarrow G^\bullet$.

(ii) *We have the dual functor of (i) :*

$$\Gamma_Z^\vee : C(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{sm}/S), \quad F \mapsto \Gamma_Z^\vee(F^\bullet) := \text{Cone}(\text{ad}(j_\#^*, j^*)(G^\bullet) : j_\#j^*G^\bullet \rightarrow G^\bullet),$$

together with the canonical map $\gamma_Z^\vee(G) : F \rightarrow \Gamma_Z^\vee(G)$.

(iii) *For $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$, we denote by*

$$I(\gamma, \text{hom})(F, G) := (I, I(j_\#^*, j^*)(F, G)^{-1}) : \Gamma_Z \mathcal{H}om(F, G) \xrightarrow{\sim} \mathcal{H}om(\Gamma_Z^\vee F, G)$$

the canonical isomorphism given by adjunction.

Let $i : Z \hookrightarrow S$ be a closed embedding with $S, Z \in \text{Var}(\mathbb{C})$ and $j : S \setminus Z \hookrightarrow S$ be the open complementary.

- For $G \in C(\text{Var}(\mathbb{C})^{sm}/S)$, the adjonction map $\text{ad}(i_*, i^!)(G) : i_* i^! G \rightarrow G$ factor through $\gamma_Z(G) :$

$$\text{ad}(i_*, i^!)(G) : i_* i^! G \xrightarrow{\text{ad}(i_*, i^!)(G)^\gamma} \Gamma_Z(G) \xrightarrow{\gamma_Z(G)} G.$$

However, note that when dealing with the big sites $P(i) : \text{Var}(\mathbb{C})^{sm}/Z \rightarrow \text{Var}(\mathbb{C})^{sm}/S$, if $G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is not \mathbb{A}_S^1 local and Zariski fibrant,

$$\text{ad}(i_*, i^!)(G)^\gamma : i_* i^! G \rightarrow \Gamma_Z(G)$$

is NOT and homotopy equivalence, and $\Gamma_Z G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is NOT in general in the image of the functor $i_* : C(\text{Var}(\mathbb{C})^{sm}/Z) \rightarrow C(\text{Var}(\mathbb{C})^{sm}/S)$.

- For $G \in C(\text{Var}(\mathbb{C})^{sm}/S)$, the adjonction map $\text{ad}(i^*, i_*)(G) : G \rightarrow i_* i^* G$ factor through $\gamma_Z^\vee(G) :$

$$\text{ad}(i^*, i_*)(G) : G \xrightarrow{\gamma_Z^\vee(G)} \Gamma_Z^\vee G \xrightarrow{\text{ad}(i^*, i_*)(G)^\gamma} i_* i^* G,$$

and as in (i), $\text{ad}(i^*, i_*)(G)^\gamma : \Gamma_Z^\vee(G) \rightarrow i_* i^* G$ is NOT an homotopy equivalence but

Let $i : Z \hookrightarrow S$ be a closed embedding with $S, Z \in \text{Var}(\mathbb{C})$.

- Since $\Gamma_Z : C(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{sm}/S)$ preserve monomorphism, it induces a functor

$$\Gamma_Z : C_{fil}(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C_{fil}(\text{Var}(\mathbb{C})^{sm}/S), (G, F) \mapsto \Gamma_Z(G, F) := (\Gamma_Z G, \Gamma_Z F)$$

- Since $\Gamma_Z^\vee : C(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{sm}/S)$ preserve monomorphism, it induces a functor

$$\Gamma_Z^\vee : C_{fil}(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C_{fil}(\text{Var}(\mathbb{C})^{sm}/S), (G, F) \mapsto \Gamma_Z^\vee(G, F) := (\Gamma_Z^\vee G, \Gamma_Z^\vee F)$$

Definition-Proposition 4. (i) Let $g : S' \rightarrow S$ a morphism and $i : Z \hookrightarrow S$ a closed embedding with $S', S, Z \in \text{Var}(\mathbb{C})$. Then, for $(G, F) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$, there exist a map in $C_{fil}(\text{Var}(\mathbb{C})^{sm}/S')$

$$T(g, \gamma)(G, F) : g^* \Gamma_Z(G, F) \rightarrow \Gamma_{Z \times_S S'} g^*(G, F)$$

unique up to homotopy such that $\gamma_{Z \times_S S'}(g^*(G, F)) \circ T(g, \gamma)(G, F) = g^* \gamma_Z(G, F)$.

(ii) Let $i_1 : Z_1 \hookrightarrow S$, $i_2 : Z_2 \hookrightarrow Z_1$ be closed embeddings with $S, Z_1, Z_2 \in \text{Var}(\mathbb{C})$. Then, for $(G, F) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$,

- there exist a canonical map $T(Z_2/Z_1, \gamma)(G, F) : \Gamma_{Z_2}(G, F) \rightarrow \Gamma_{Z_1}(G, F)$ in $C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ unique up to homotopy such that $\gamma_{Z_1}(G, F) \circ T(Z_2/Z_1, \gamma)(G, F) = \gamma_{Z_2}(G, F)$, together with a distinguish triangle

$$\Gamma_{Z_2}(G, F) \xrightarrow{T(Z_2/Z_1, \gamma)(G, F)} \Gamma_{Z_1}(G, F) \xrightarrow{\text{ad}(j_2^*, j_{2*})(\Gamma_{Z_1}(G, F))} \Gamma_{Z_1 \setminus Z_2}(G, F) \rightarrow \Gamma_{Z_2}(G, F)[1]$$

in $K_{fil}(\text{Var}(\mathbb{C})^{sm}/S) := K(\text{PSh}_{fil}(\text{Var}(\mathbb{C})^{sm}/S))$,

- there exist a map $T(Z_2/Z_1, \gamma^\vee)(G, F) : \Gamma_{Z_1}^\vee(G, F) \rightarrow \Gamma_{Z_2}^\vee(G, F)$ in $C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ unique up to homotopy such that $\gamma_{Z_2}^\vee(G, F) = T(Z_2/Z_1, \gamma^\vee)(G, F) \circ \gamma_{Z_1}^\vee(G, F)$, together with a distinguish triangle

$$\Gamma_{Z_1 \setminus Z_2}^\vee(G, F) \xrightarrow{\text{ad}(j_{2\#}, j_2^*)(\Gamma_{Z_1}^\vee(G, F))} \Gamma_{Z_1}^\vee(G, F) \xrightarrow{T(Z_2/Z_1, \gamma^\vee)(G, F)} \Gamma_{Z_2}^\vee(G, F) \rightarrow \Gamma_{Z_1 \setminus Z_2}^\vee(G, F)[1]$$

in $K_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$.

(iii) Consider a morphism $g : (S', Z') \rightarrow (S, Z)$ with $(S', Z'), (S, Z) \in \text{Var}^2(\mathbb{C})$. We denote, for $G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ the composite

$$T(D, \gamma^\vee)(G) : g^* \Gamma_{Z'}^\vee G \xrightarrow{\sim} \Gamma_{Z \times_S S'}^\vee g^* G \xrightarrow{T(Z'/Z \times_S S', \gamma^\vee)(G)} \Gamma_{Z'}^\vee g^* G$$

and we have then the factorization $\gamma_{Z'}^\vee(g^* G) : g^* G \xrightarrow{g^* \gamma_Z^\vee(G)} g^* \Gamma_Z^\vee G \xrightarrow{T(D, \gamma^\vee)(G)} \Gamma_{Z'}^\vee g^* G$.

Proof. (i): We have the cartesian square

$$\begin{array}{ccc} S \setminus Z & \xrightarrow{j} & S \\ g' \uparrow & & \uparrow g \\ S' \setminus Z \times_S S' & \xrightarrow{j'} & S' \end{array}$$

and the map is given by

$$(I, T(g, j)(j^* G)) : \text{Cone}(g^* G \rightarrow g^* j_* j^* G) \rightarrow \text{Cone}(g^* G \rightarrow j'_* j'^* g^* G = j'_* g'^* j^* G).$$

(ii): Follows from the fact that $j_1^* \Gamma_{Z_2} G = 0$ and $j_1^* \Gamma_{Z_2}^\vee G = 0$, with $j_1 : S \setminus Z_1 \hookrightarrow S$ the closed embedding.

(iii): Obvious. \square

The following easy proposition concern the restriction from the big Zariski site to the small site Zariski site :

Proposition 16. For $f : T \rightarrow S$ a morphism and $i : Z \hookrightarrow S$ a closed embedding, with $Z, S, T \in \text{Var}(\mathbb{C})$, we have

$$(i) \quad e(S)_* f_* = f_* e(T)_* \text{ and } e(S)^* f_* = f_* e(T)^*$$

$$(ii) \quad e(S)_* \Gamma_Z = \Gamma_Z e(S)_*$$

Proof. (i): The first equality $e(S)_* f_* = f_* e(T)_*$ is given by the diagram (24). The second equality is immediate from definition after a direct computation.

(ii) For $G^\bullet \in C(\text{Var}(\mathbb{C})^{sm}/S)$, we have the canonical equality

$$\begin{aligned} e(S)_* \Gamma_Z(G^\bullet) = e(S)_* \text{Cone}(G \rightarrow j_* j^* G^\bullet)[-1] &= \text{Cone}(e(S)_* G^\bullet \rightarrow e(S)_* j_* j^* G^\bullet)[-1] \\ &= \text{Cone}(e(S)_* G^\bullet \rightarrow j_* j^* e(S)_* G^\bullet)[-1] \\ &= \Gamma_Z e(S)_* G^\bullet \end{aligned}$$

by (i) and since $j : S \setminus Z \hookrightarrow S$ is a smooth morphism. \square

Definition 7. For $S \in \text{Var}(\mathbb{C})$, we denote by

$$C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S) := C_{e(S)^* O_S}(\text{Var}(\mathbb{C})^{sm}/S)$$

the category of complexes of presheaves on $\text{Var}(\mathbb{C})^{sm}/S$ endowed with a structure of $e(S)^* O_S$ module, and by

$$C_{O_S \text{ fil}}(\text{Var}(\mathbb{C})^{sm}/S) := C_{e(S)^* O_S \text{ fil}}(\text{Var}(\mathbb{C})^{sm}/S)$$

the category of filtered complexes of presheaves on $\text{Var}(\mathbb{C})^{sm}/S$ endowed with a structure of $e(S)^* O_S$ module.

Let $S \in \text{Var}(\mathbb{C})$. Let $Z \subset S$ a closed subset. Denote by $j : S \setminus Z \hookrightarrow S$ the open complementary embedding,

- For $G \in C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S)$, $\Gamma_Z G := \text{Cone}(\text{ad}(j^*, j_*)(G) : F \rightarrow j_* j^* G)[-1]$ has a (unique) structure of $e(S)^* O_S$ module such that $\gamma_Z(G) : \Gamma_Z G \rightarrow G$ is a map in $C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S)$. This gives the functor

$$\Gamma_Z : C_{O_S \text{fil}}(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C_{\text{fil}O_S}(\text{Var}(\mathbb{C})^{sm}/S), (G, F) \mapsto \Gamma_Z(G, F) := (\Gamma_Z G, \Gamma_Z F),$$

together with the canonical map $\gamma_Z((G, F) : \Gamma_Z(G, F) \rightarrow (G, F)$. Let $Z_2 \subset Z$ a closed subset. Then, for $G \in C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S)$, $T(Z_2/Z, \gamma)(G) : \Gamma_{Z_2} G \rightarrow \Gamma_Z G$ is a map in $C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S)$ (i.e. is $e(S)^* O_S$ linear).

- For $G \in C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S)$, $\Gamma_Z^\vee G := \text{Cone}(\text{ad}(j_\sharp, j^*)(G) : j_\sharp j^* G \rightarrow G)$ has a unique structure of $e(S)^* O_S$ module, such that $\gamma_Z^\vee(G) : G \rightarrow \Gamma_Z^\vee G$ is a map in $C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S)$. This gives the the functor

$$\Gamma_Z^\vee : C_{O_S \text{fil}}(S) \rightarrow C_{\text{fil}O_S}(S), (G, F) \mapsto \Gamma_Z^\vee(G, F) := (\Gamma_Z^\vee G, \Gamma_Z^\vee F),$$

together with the canonical map $\gamma_Z^\vee((G, F) : (G, F) \rightarrow \Gamma_Z^\vee(G, F)$. Let $Z_2 \subset Z$ a closed subset. Then, for $G \in C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S)$, $T(Z_2/Z, \gamma^\vee)(G) : \Gamma_{Z_2}^\vee G \rightarrow \Gamma_Z^\vee G$ is a map in $C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S)$ (i.e. is $e(S)^* O_S$ linear).

Definition 8. Let $S \in \text{Var}(\mathbb{C})$. Let $Z \subset S$ a closed subset.

(i) We denote by

$$C_Z(\text{Var}(\mathbb{C})^{sm}/S) \subset C(\text{Var}(\mathbb{C})^{sm}/S)$$

the full subcategory consisting of complexes of presheaves $F^\bullet \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $a_{et} H^n(j^* F^\bullet) = 0$ for all $n \in \mathbb{Z}$, where $j : S \setminus Z \hookrightarrow S$ is the complementary open embedding and a_{et} is the sheafification functor.

(i)' We denote by

$$C_{O_S, Z}(\text{Var}(\mathbb{C})^{sm}/S) \subset C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S),$$

the full subcategory consisting of complexes of presheaves $F^\bullet \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $a_{et} H^n(j^* F^\bullet) = 0$ for all $n \in \mathbb{Z}$, where $j : S \setminus Z \hookrightarrow S$ is the complementary open embedding and a_{et} is the sheafification functor.

(ii) We denote by

$$C_{\text{fil}, Z}(\text{Var}(\mathbb{C})^{sm}/S) \subset C_{\text{fil}}(\text{Var}(\mathbb{C})^{sm}/S)$$

the full subcategory consisting of filtered complexes of presheaves $(F^\bullet, F) \in C_{\text{fil}}(\text{Var}(\mathbb{C})^{sm}/S)$ such that there exist $r \in \mathbb{N}$ such that $a_{et} j^* E_r^{p, q}(F^\bullet, F) = 0$ for all $p, q \in \mathbb{Z}$, where $j : S \setminus Z \hookrightarrow S$ is the complementary open embedding and a_{et} is the sheafification functor. Note that by definition this r does NOT depend on p and q .

(ii)' We denote by

$$C_{O_S \text{fil}, Z}(\text{Var}(\mathbb{C})^{sm}/S) \subset C_{O_S \text{fil}}(\text{Var}(\mathbb{C})^{sm}/S)$$

the full subcategory consisting of filtered complexes of presheaves $(F^\bullet, F) \in C_{O_S \text{fil}}(\text{Var}(\mathbb{C})^{sm}/S)$ such that there exist $r \in \mathbb{N}$ such that $a_{et} j^* E_r^{p, q}(F^\bullet, F) = 0$ for all $p, q \in \mathbb{Z}$, where $j : S \setminus Z \hookrightarrow S$ is the complementary open embedding and a_{et} is the sheafification functor. Note that by definition this r does NOT depend on p and q .

Let $S \in \text{Var}(\mathbb{C})$ and $Z \subset S$ a closed subset.

- For $(G, F) \in C_{\text{fil}}(\text{Var}(\mathbb{C})^{sm}/S)$, we have $\Gamma_Z(G, F), \Gamma_Z^\vee(G, F) \in C_{\text{fil}, Z}(\text{Var}(\mathbb{C})^{sm}/S)$.
- For $(G, F) \in C_{O_S \text{fil}}(\text{Var}(\mathbb{C})^{sm}/S)$, we have $\Gamma_Z(G, F), \Gamma_Z^\vee(G, F) \in C_{O_S \text{fil}, Z}(\text{Var}(\mathbb{C})^{sm}/S)$.

Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open affine cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{Var}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote by $\tilde{S}_I = \prod_{i \in I} \tilde{S}_i$. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I$ and for $J \subset I$ the following commutative diagram

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ j_{IJ} \uparrow & & \uparrow p_{IJ} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}$$

where $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ is the projection and $j_{IJ} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. This gives the diagram of algebraic varieties $(\tilde{S}_I) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{Var}(\mathbb{C}))$ which the diagram of sites $\text{Var}(\mathbb{C})^{sm}/(\tilde{S}_I) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{Cat})$. Denote by $m : \tilde{S}_I \setminus (S_I \setminus S_J) \hookrightarrow \tilde{S}_I$ the open embedding.

Definition 9. Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{Var}(\mathbb{C})$. We will denote by $C_{fil}(\text{Var}(\mathbb{C})^{sm}/(S/(\tilde{S}_I))) \subset C_{fil}(\text{Var}(\mathbb{C})^{sm}/(\tilde{S}_I))$ the full subcategory

- whose objects $(G, F) = ((G_I, F)_{I \subset [1, \dots, l]}, u_{IJ})$, with $(G_I, F) \in C_{fil, S_I}(\text{Var}(\mathbb{C})^{sm}/\tilde{S}_I)$, and $u_{IJ} : m^*(G_I, F) \rightarrow m^*p_{IJ*}(G_J, F)$ for $I \subset J$, are ∞ -filtered Zariski local equivalence, satisfying for $I \subset J \subset K$, $p_{IJ*}u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{fil}(\text{Var}(\mathbb{C})^{sm}/\tilde{S}_I)$,
- the morphisms $m : ((G, F), u_{IJ}) \rightarrow ((H, F), v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,

$$m = (m_I : (G_I, F) \rightarrow (H_I, F))_{I \subset [1, \dots, l]}$$

such that $v_{IJ} \circ m_I = p_{IJ*}m_J \circ u_{IJ}$ in $C_{fil}(\text{Var}(\mathbb{C})^{sm}/\tilde{S}_I)$.

A morphism $m : ((G_I, F), u_{IJ}) \rightarrow ((H_I, F), v_{IJ})$ is said to an r -filtered Zariski, resp. etale local, equivalence, if all the m_I are r -filtered Zariski, resp. etale, local equivalences.

Denote $L = [1, \dots, l]$ and for $I \subset L$, $p_{0(I)} : S \times \tilde{S}_I \rightarrow S$, $p_{I(0I)} : S \times \tilde{S}_I \rightarrow S_I$ the projections. By definition, we have functors

- $T(S/(\tilde{S}_I)) : C_{fil}(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C_{fil}(\text{Var}(\mathbb{C})^{sm}/(S/(\tilde{S}_I)))$, $(G, F) \mapsto (i_{I*}j_I^*(G, F), T(D_{IJ})(j_I^*(G, F)))$,
- $T((\tilde{S}_I)/S) : C_{fil}(\text{Var}(\mathbb{C})^{sm}/(S/(\tilde{S}_I))) \rightarrow C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$, $((G_I, F), u_{IJ}) \mapsto \text{ho} \lim_{I \subset L} p_{0(0I)*} \Gamma_{S_I}^\vee p_{I(0I)}^*(G_I, F)$.

Note that the functors $T(S/(\tilde{S}_I))$ are NOT embedding, since

$$\text{ad}(i_I^*, i_{I*})(j_I^*F) : i_I^*i_{I*}j_I^*F \rightarrow j_I^*F$$

are Zariski local equivalence but NOT isomorphism since we are dealing with the morphism of big sites $P(i_I) : \text{Var}(\mathbb{C})^{sm}/S_I \rightarrow \text{Var}(\mathbb{C})^{sm}/\tilde{S}_I$. However, by theorem 14, these functors induce full embeddings

$$T(S/(\tilde{S}_I)) : D_{fil}(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow D_{fil}(\text{Var}(\mathbb{C})^{sm}/(S/(\tilde{S}_I)))$$

since for $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$,

$$\text{ho} \lim_{I \subset L} p_{0(0I)*} \Gamma_{S_I}^\vee p_{I(0I)}^*(i_{I*}j_I^*F) \rightarrow p_{0(0I)*} \Gamma_{S_I}^\vee j_I^*F$$

is an equivalence Zariski local.

Let $f : X \rightarrow S$ a morphism, with $X, S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ and $X = \cup_{i=1}^l X_i$ be affine open covers and $i_i : S_i \hookrightarrow \tilde{S}_i$, $i'_i : X_i \hookrightarrow \tilde{X}_i$ be closed embeddings. Let $f_i : \tilde{X}_i \rightarrow \tilde{S}_i$ be a lift of the morphism $f_i = f|_{X_i} : X_i \rightarrow S_i$. Then, $f_I = f|_{X_I} : X_I = \cap_{i \in I} X_i \rightarrow S_I = \cap_{i \in I} S_i$ lift to the morphism

$$\tilde{f}_I = \prod_{i \in I} \tilde{f}_i : \tilde{X}_I = \prod_{i \in I} \tilde{X}_i \rightarrow \tilde{S}_I = \prod_{i \in I} \tilde{S}_i$$

Denote by $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ and $p'_{IJ} : \tilde{X}_J \rightarrow \tilde{X}_I$ the projections. Consider for $J \subset I$ the following commutative diagrams

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ \uparrow j_{IJ} & & \uparrow p_{IJ} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}, \quad D'_{IJ} = \begin{array}{ccc} X_I & \xrightarrow{i'_I} & \tilde{X}_I \\ \uparrow j'_{IJ} & & \uparrow p'_{IJ} \\ X_J & \xrightarrow{i'_J} & \tilde{X}_J \end{array}, \quad D_{fI} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ \uparrow f_I & & \uparrow \tilde{f}_I \\ X_I & \xrightarrow{i'_I} & \tilde{X}_I \end{array}$$

We have then following commutative diagram

$$\begin{array}{ccccc} & & X_I & \xrightarrow{n'_I} & \tilde{X}_I & \longleftarrow & \tilde{X}_I \setminus X_I \\ & \nearrow j'_{IJ} & \uparrow p'_{IJ} & & \uparrow p'_{IJ} & & \uparrow p'_{IJ} \\ i'_J : X_J & \xrightarrow{l_{IJ}} & X_I \times X_I \times \tilde{X}_J^{n'_I \times I} & \longrightarrow & \tilde{X}_J & \longleftarrow & \tilde{X}_J \setminus X_J \\ & & & & \uparrow p'_{IJ} & & \uparrow n'_J \end{array}$$

whose square are cartesian. We then have the pullback functor

$$f^* : C_{(2)fil}(\text{Var}(\mathbb{C})^{sm}/S/(\tilde{S}_I)) \rightarrow C_{(2)fil}(\text{Var}(\mathbb{C})^{sm}/X/(\tilde{X}_I)), \\ ((G_I, F), u_{IJ}) \mapsto f^*((G_I, F), u_{IJ}) := (\Gamma_{X_I}^\vee \tilde{f}_I^*(G_I, F), \tilde{f}_J^* u_{IJ})$$

with

$$\begin{aligned} \tilde{f}_J^* u_{IJ} : \Gamma_{X_I}^\vee \tilde{f}_I^*(G_I, F) &\xrightarrow{\text{ad}(p'_{IJ}, p'_{IJ*})(-)} p'_{IJ*} p'_{IJ}^* \Gamma_{X_I}^\vee \tilde{f}_I^*(G_I, F) \xrightarrow{T_{\sharp}(p_{IJ}, n'_I)(-)^{-1}} p'_{IJ*} \Gamma_{X_I \times \tilde{X}_J \setminus I}^\vee p'_{IJ}^* \tilde{f}_I^*(G_I, F) \\ &\xrightarrow{p'_{IJ*} \gamma_{X_J}^\vee(-)} p'_{IJ*} \Gamma_{X_J}^\vee p'_{IJ}^* \tilde{f}_I^*(G_I, F) = p'_{IJ*} \Gamma_{X_J}^\vee \tilde{f}_J^* p'_{IJ}^*(G_I, F) \xrightarrow{\Gamma_{X_J}^\vee \tilde{f}_J^* I(p'_{IJ}, p_{IJ*})(-, -)(u_{IJ})} \Gamma_{X_J}^\vee \tilde{f}_J^*(G_J, F) \end{aligned}$$

Let $(G, F) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$. Since, $j'_I{}^* i'_{I*} j'_I{}^* f^*(G, F) = 0$, the morphism $T(D_{fI})(j'_I{}^*(G, F)) : \tilde{f}_I^* i_{I*} j'_I{}^*(G, F) \rightarrow i'_{I*} j'_I{}^* f^*(G, F)$ factors through

$$T(D_{fI})(j'_I{}^*(G, F)) : \tilde{f}_I^* i_{I*} j'_I{}^*(G, F) \xrightarrow{\gamma_{\tilde{X}_I}^\vee(-)} \Gamma_{\tilde{X}_I}^\vee \tilde{f}_I^* i_{I*} j'_I{}^*(G, F) \xrightarrow{T^\gamma(D_{fI})(j'_I{}^*(G, F))} i'_{I*} j'_I{}^* f^*(G, F)$$

We have then, for $(G, F) \in C_{fil}(S)$, the canonical transformation map

$$\begin{array}{ccc} f^* T(S/(\tilde{S}_I))(G, F) & \xrightarrow{T(f, T(0/I))(G, F)} & T(X/(\tilde{X}_I))(f^*(G, F)) \\ \downarrow = & & \downarrow = \\ (\Gamma_{X_I}^\vee \tilde{f}_I^* i_{I*} j'_I{}^*(G, F), \tilde{f}_J^* I) & \xrightarrow{T^\gamma(D_{fI})(j'_I{}^*(G, F))} & (i'_{I*} j'_I{}^* f^*(G, F), I) \end{array}$$

To show that the cohomology sheaves of the filtered De Rham realization functor of constructible motives are mixed hodge modules, we will need to take presheaves of the following form

Definition 10. (i) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume that there exist a factorization $f : X \xrightarrow{i} Y \times S \xrightarrow{p} S$, with $Y \in \text{SmVar}(\mathbb{C})$, $i : X \hookrightarrow Y$ is a closed embedding and p the projection. We then consider

$$Q(X/S) := p_{\sharp} \Gamma_X^\vee \mathbb{Z}_{Y \times S}[d_Y] \in C(\text{Var}(\mathbb{C})^{sm}/S).$$

By definition $Q(X/S)$ is projective.

(ii) Let $f : X \rightarrow S$ and $g : T \rightarrow S$ two morphism with $X, S, T \in \text{Var}(\mathbb{C})$. Assume that there exist a factorization $f : X \xrightarrow{i} Y \times S \xrightarrow{p} S$, with $Y \in \text{SmVar}(\mathbb{C})$, $i : X \hookrightarrow Y$ is a closed embedding and p the projection. We then have the following commutative diagram whose squares are cartesian

$$\begin{array}{ccccc} f : X & \xrightarrow{i} & Y \times S & \xrightarrow{p} & S \\ g' \uparrow & & g'' := (I \times g) \uparrow & & g \uparrow \\ f' : X_T & \xrightarrow{i'} & Y \times T & \xrightarrow{p'} & T \end{array}$$

We then have the canonical isomorphism in $C(\text{Var}(\mathbb{C})^{sm}/T)$

$$\begin{aligned} T(f, g, Q) : g^*Q(X/S) &:= g^*p_{\sharp}\Gamma_X^{\vee}\mathbb{Z}_{Y \times S} \xrightarrow{T_{\sharp}(g,p)(-)^{-1}} p'_{\sharp}g''^*\Gamma_X^{\vee}\mathbb{Z}_{Y \times S} \\ &\xrightarrow{p'_{\sharp}T(g'', \gamma^{\vee})(-)^{-1}} p'_{\sharp}\Gamma_{X_T}^{\vee}\mathbb{Z}_{Y \times T} =: Q(X_T/T). \end{aligned}$$

(iii) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume that there exist a factorization $f : X \xrightarrow{i} Y \times S \xrightarrow{p} S$, with $Y \in \text{SmVar}(\mathbb{C})$, $i : X \hookrightarrow Y$ is a closed embedding and p the projection. We then consider

$$Q^h(X/S) := p_{\sharp}L\Gamma_X E_{et}(\mathbb{Z}_{Y \times S})[d_Y] \in C(\text{Var}(\mathbb{C})^{sm}/S).$$

(iv) Let $f : X \rightarrow S$ and $g : T \rightarrow S$ two morphism with $X, S, T \in \text{Var}(\mathbb{C})$. Assume that there exist a factorization $f : X \xrightarrow{i} Y \times S \xrightarrow{p} S$, with $Y \in \text{SmVar}(\mathbb{C})$, $i : X \hookrightarrow Y$ is a closed embedding and p the projection. We then have the following commutative diagram whose squares are cartesian

$$\begin{array}{ccccc} f : X & \xrightarrow{i} & Y \times S & \xrightarrow{p} & S \\ g' \uparrow & & g'' := (I \times g) \uparrow & & g \uparrow \\ f' : X_T & \xrightarrow{i'} & Y \times T & \xrightarrow{p'} & T \end{array}$$

We then have the canonical morphism in $C(\text{Var}(\mathbb{C})^{sm}/T)$

$$\begin{aligned} T(f, g, Q^h) : g^*Q^h(X/S) &:= g^*p_{\sharp}L\Gamma_X E_{et}(\mathbb{Z}_{Y \times S}) \xrightarrow{T_{\sharp}(g,p)(-)^{-1}} p'_{\sharp}Lg''^*\Gamma_X E_{et}(\mathbb{Z}_{Y \times S}) \\ &\xrightarrow{p'_{\sharp}T(g'', \gamma)(-)} p'_{\sharp}L\Gamma_{X_T} E_{et}(\mathbb{Z}_{Y \times T}) =: Q^h(X_T/T). \end{aligned}$$

We now give the definition of the \mathbb{A}^1 local property :

Definition 11. Let $S \in \text{Var}(\mathbb{C})$. Denote for short $\text{Var}(\mathbb{C})^{(sm)}/S$ either the category $\text{Var}(\mathbb{C})/S$ or the category $\text{Var}(\mathbb{C})^{sm}/S$.

(i) A complex $F \in C(\text{Var}(\mathbb{C})^{(sm)}/S)$ is said to be \mathbb{A}^1 invariant if for all $U/S \in \text{Var}(\mathbb{C})^{(sm)}/S$,

$$F(p_U) : F(U/S) \rightarrow F(U \times \mathbb{A}^1/S)$$

is a quasi-isomorphism, where $p_U : U \times \mathbb{A}^1 \rightarrow U$ is the projection.

(ii) Let τ a topology on $\text{Var}(\mathbb{C})$. A complex $F \in C(\text{Var}(\mathbb{C})^{(sm)}/S)$ is said to be \mathbb{A}^1 local for the topology τ , if for a (hence every) τ local equivalence $k : F \rightarrow G$ with k injective and $G \in C(\text{Var}(\mathbb{C})^{(sm)}/S)$ τ fibrant, e.g. $k : F \rightarrow E_{\tau}(F)$, G is \mathbb{A}^1 invariant for all $n \in \mathbb{Z}$.

(iii) A morphism $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{(sm)}/S)$ is said to an (\mathbb{A}^1, et) local equivalence if for all $H \in C(\text{Var}(\mathbb{C})^{(sm)}/S)$ which is \mathbb{A}^1 local for the etale topology

$$\text{Hom}(L(m), E_{et}(H)) : \text{Hom}(L(G), E_{et}(H)) \rightarrow \text{Hom}(L(F), E_{et}(H))$$

is a quasi-isomorphism. Equivalently, $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{(sm)}/S)$ is an (\mathbb{A}^1, et) local equivalence if and only if there exists $X_1/S, \dots, X_r/S \in \text{Var}(\mathbb{C})^{(sm)}/S$ such that we have in $\text{Ho}_{et}(C(\text{Var}(\mathbb{C})^{(sm)}/S))$

$$\text{Cone}(m) \xrightarrow{\sim} \text{Cone}(\text{Cone}(\mathbb{Z}(X_1 \times \mathbb{A}^1/S) \rightarrow \mathbb{Z}(X_1/S)) \rightarrow \dots \rightarrow \text{Cone}(\mathbb{Z}(X_r \times \mathbb{A}^1/S) \rightarrow \mathbb{Z}(X_r/S)))$$

Proposition 17. *With the weak equivalence the (\mathbb{A}_S^1, et) equivalence and the fibration the epimorphism with \mathbb{A}_S^1 local and etale fibrant kernels gives a model structure on $C(\text{Var}(\mathbb{C})^{(sm)}/S)$: the left bousfield localization of the projective model structure of $C(\text{Var}(\mathbb{C})^{(sm)}/S)$.*

Proof. See [10]. □

We will consider for the construction of the filtered De Rham realization functor the filtered case :

Definition 12. *Let $S \in \text{Var}(\mathbb{C})$. Denote for short $\text{Var}(\mathbb{C})^{(sm)}/S$ either the category $\text{Var}(\mathbb{C})/S$ or the category $\text{Var}(\mathbb{C})^{sm}/S$.*

(i) *Let $r \in \mathbb{N}$. A filtered complex $(G, F) \in C_{fil}(\text{Var}(\mathbb{C})^{(sm)}/S)$ is said to be an r -filtered \mathbb{A}^1 invariant if for all $U/S \in \text{Var}(\mathbb{C})^{(sm)}/S$, and all $p, q \in \mathbb{Z}$,*

$$E_r^{p,q}G(p_U) : E_r^{p,q}(G, F)(U/S) \xrightarrow{\sim} E_r^{p,q}(G, F)(U \times \mathbb{A}^1/S)$$

is an isomorphism of abelian groups, where $p_U : U \times \mathbb{A}^1 \rightarrow U$ is the projection. Note that this definition say that this r does NOT depend on p and q .

(ii) *A filtered complex $(G, F) \in C_{fil}(\text{Var}(\mathbb{C})^{(sm)}/S)$ is said to be ∞ -filtered \mathbb{A}^1 invariant if for all $U/S \in \text{Var}(\mathbb{C})^{(sm)}/S$, there exist $r \in \mathbb{N}$ such that for all $p, q \in \mathbb{Z}$,*

$$E_r^{p,q}G(p_U) : E_r^{p,q}(G, F)(U/S) \xrightarrow{\sim} E_r^{p,q}(G, F)(U \times \mathbb{A}^1/S)$$

is an isomorphism of abelian groups, where $p_U : U \times \mathbb{A}^1 \rightarrow U$ is the projection (note that this definition say that this r does NOT depend on p and q but may depends on U/S). This implies that for all $U/S \in \text{Var}(\mathbb{C})^{(sm)}/S$,

$$H^n G(p_U) : H^n(G, F)(U/S) \xrightarrow{\sim} H^n(G, F)(U \times \mathbb{A}^1/S)$$

is a filtered isomorphism of filtered abelian groups for all $n \in \mathbb{Z}$.

(iii) *Let τ a topology on $\text{Var}(\mathbb{C})^{(sm)}$. A filtered complex $(G, F) \in C_{fil}(\text{Var}(\mathbb{C})^{(sm)}/S)$ is said to be ∞ -filtered \mathbb{A}^1 local for the topology τ if for a (hence every) ∞ -filtered τ local equivalence $k : (G, F) \rightarrow (H, F)$ with k injective and $(H, F) \in C_{fil}(\text{Var}(\mathbb{C})^{(sm)}/S)$ ∞ -filtered τ fibrant, e.g. $k : (G, F) \rightarrow E_\tau(G, F)$, (H, F) is ∞ -filtered \mathbb{A}^1 invariant.*

Lemma 1. *Let $S \in \text{Var}(\mathbb{C})$.*

(i) *Let $(G, F) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ Then, if $m : F_1 \rightarrow F_2$ with $F_1, F_2 \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is an etale local equivalence,*

$$\text{Hom}(m, E_{et}(G, F)) : \text{Hom}^\bullet(F_2, E_{et}(G, F)) \rightarrow \text{Hom}^\bullet(F_1, E_{et}(G, F))$$

is a filtered quasi-isomorphism.

(ii) Let $(G, F) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ be ∞ -filtered \mathbb{A}^1 local for the etale topology. Then, if $m : F_1 \rightarrow F_2$ with $F_1, F_2 \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is an (\mathbb{A}^1, et) local equivalence,

$$\text{Hom}(m, E_{et}(G, F)) : \text{Hom}^\bullet(F_2, E_{et}(G, F)) \rightarrow \text{Hom}^\bullet(F_1, E_{et}(G, F))$$

is an ∞ -filtered quasi-isomorphism.

Proof. (i): Follows from the fact that $E_{et}(G, F)$ is (1-)filtered etale fibrant.

(ii): By definition of an (\mathbb{A}^1, et) local equivalence, we have there exists $X_1/S, \dots, X_r/S \in \text{Var}(\mathbb{C})^{sm}/S$ such that we have in $\text{Ho}_{et}(C(\text{Var}(\mathbb{C})^{sm}/S))$

$$\text{Cone}(m) \xrightarrow{\sim} \text{Cone}(\text{Cone}(\mathbb{Z}(X_1 \times \mathbb{A}^1/S) \rightarrow \mathbb{Z}(X_1/S)) \rightarrow \dots \rightarrow \text{Cone}(\mathbb{Z}(X_r \times \mathbb{A}^1/S) \rightarrow \mathbb{Z}(X_r/S)))$$

This gives in $D_{fil,1}(\mathbb{Z}) := \text{Ho}_{fil,1}(\mathbb{Z})$,

$$\begin{aligned} \text{Cone}(\text{Hom}(m, E_{et}(G, F))) &\xrightarrow{\sim} \text{Cone}(\text{Cone}(E_{et}(G, F)(X_1/S) \rightarrow E_{et}(G, F)(X_1 \times \mathbb{A}^1/S)) \\ &\rightarrow \dots \rightarrow \text{Cone}(E_{et}(G, F)(X_r/S) \rightarrow E_{et}(G, F)(X_r \times \mathbb{A}^1/S))) \end{aligned}$$

Since $(G, F) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ is ∞ -filtered \mathbb{A}^1 local for the etale topology, there exist $s_i \in \mathbb{Z}$ such that for all $p, q \in \mathbb{Z}$,

$$E_{s_i}^{p,q} \text{Cone}(E_{et}(G, F)(X_i/S) \rightarrow E_{et}(G, F)(X_i \times \mathbb{A}^1/S)) = 0$$

Hence for $s := \max((s_i)_{1 \leq i \leq r})$, $E_s^{p,q} \text{Cone}(\text{Hom}(m, E_{et}(G, F))) = 0$. □

Let $S \in \text{Var}(\mathbb{C})$. For $U/S = (U, h) \in \text{Var}(\mathbb{C})^{sm}/S$, we consider

$$\mathbb{A}^* \times U/S = (\mathbb{A}^* \times U, h \circ p) \in \text{Fun}(\Delta, \text{Var}(\mathbb{C})^{sm}/S).$$

For $F \in C^-(\text{Var}(\mathbb{C})^{sm}/S)$, it gives the complex

$$C_*F \in C^-(\text{Var}(\mathbb{C})^{sm}/S), U/S = (U, h) \mapsto C_*F(U/S) := \text{Tot } F(\mathbb{A}^* \times U/S)$$

together with the canonical map $c_F := (0, I_F) : C_*F \rightarrow F$. For $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$, we get

$$C_*F := \text{holim}_n C_*F^{\leq n} \in C(\text{Var}(\mathbb{C})^{sm}/S),$$

together with the canonical map $c_F := (0, I_F) : C_*F \rightarrow F$. For $m : F \rightarrow G$ a morphism, with $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$, we get by functoriality the morphism $C_*m : C_*F \rightarrow C_*G$.

Proposition 18. *Let $S \in \text{Var}(\mathbb{C})$. Then for $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$, C_*F is \mathbb{A}^1 local for the etale topology and $c_F : C_*F \rightarrow F$ is an equivalence (\mathbb{A}^1, et) local.*

Proof. Standard : see [10] for example. □

Proposition 19. *Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$.*

- (i) *The functor $g^* : C(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{sm}/T)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends (\mathbb{A}^1, et) local equivalence to (\mathbb{A}^1, et) local equivalence.*
- (ii) *The functor $g^* : C(\text{Var}(\mathbb{C})/S) \rightarrow C(\text{Var}(\mathbb{C})/T)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends (\mathbb{A}^1, et) local equivalence to (\mathbb{A}^1, et) local equivalence.*

Proof. Standard : see [10] for example. □

Proposition 20. *Let $S \in \text{Var}(\mathbb{C})$. The functor $\rho_{S*} : C(\text{Var}(\mathbb{C})/S) \rightarrow C(\text{Var}(\mathbb{C})^{sm}/S)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends (\mathbb{A}^1, et) local equivalence to (\mathbb{A}^1, et) local equivalence.*

Proof. Standard : see [10] for example. \square

For $S \in \text{Var}(\mathbb{C})$, let $\text{Cor}(\text{Var}(\mathbb{C})^{sm}/S)$ be the category

- whose objects are smooth morphisms $U/S = (U, h)$, $h : U \rightarrow S$ with $U \in \text{Var}(\mathbb{C})$,
- whose morphisms $\alpha : U/S = (U, h_1) \rightarrow V/S = (V, h_2)$ is finite correspondence that is $\alpha \in \bigoplus_i \mathcal{Z}^{fs}(U_i \times_S V)$, where $U = \sqcup_i U_i$, with U_i connected, and $\mathcal{Z}^{fs}(U_i \times_S V)$ is the abelian group of cycle finite and surjective over U_i .

We denote by $\text{Tr}(S) : \text{Cor}(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow \text{Var}(\mathbb{C})^{sm}/S$ the morphism of site given by the inclusion functor $\text{Tr}(S) : \text{Var}(\mathbb{C})^{sm}/S \hookrightarrow \text{Cor}(\text{Var}(\mathbb{C})^{sm}/S)$ It induces an adjunction

$$(\text{Tr}(S)^* \text{Tr}(S)_*) : C(\text{Var}(\mathbb{C})^{sm}/S) \rightleftarrows C(\text{Cor}(\text{Var}(\mathbb{C})^{sm}/S))$$

A complex of preheaves $G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is said to admit transferts if it is in the image of the embedding

$$\text{Tr}(S)_* : C(\text{Cor}(\text{Var}(\mathbb{C})^{sm}/S) \hookrightarrow C(\text{Var}(\mathbb{C})^{sm}/S),$$

that is $G = \text{Tr}(S)_* \text{Tr}(S)^* G$.

We will use to define the algebraic De Rahm realization functor the following

Theorem 10. *Let $\phi : F^\bullet \rightarrow G^\bullet$ an etale local equivalence with $F^\bullet, G^\bullet \in C(\text{Var}(\mathbb{C})^{sm}/S)$. If F^\bullet and G^\bullet are \mathbb{A}^1 local and admit transferts then $\phi : F^\bullet \rightarrow G^\bullet$ is a Zariski local equivalence. Hence if $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is \mathbb{A}^1 local and admits transfert*

$$k : E_{zar}(F) \rightarrow E_{et}(E_{zar}(F)) = E_{et}(F)$$

is a Zariski local equivalence.

Proof. See [10]. \square

2.7 Presheaves on the big Zariski site or the big etale site of pairs

We recall the definition given in subsection 5.1 : For $S \in \text{Var}(\mathbb{C})$, $\text{Var}(\mathbb{C})^2/S := \text{Var}(\mathbb{C})^2/(S, S)$ is by definition (see subsection 2.1) the category whose set of objects is

$$(\text{Var}(\mathbb{C})^2/S)^0 := \{((X, Z), h), h : X \rightarrow S, Z \subset X \text{ closed}\} \subset \text{Var}(\mathbb{C})/S \times \text{Top}$$

and whose set of morphisms between $(X_1, Z_1)/S = ((X_1, Z_1), h_1), (X_1, Z_1)/S = ((X_2, Z_2), h_2) \in \text{Var}(\mathbb{C})^2/S$ is the subset

$$\begin{aligned} & \text{Hom}_{\text{Var}(\mathbb{C})^2/S}((X_1, Z_1)/S, (X_2, Z_2)/S) := \\ & \{(f : X_2 \rightarrow X_1), \text{ s.t. } h_1 \circ f = h_2 \text{ and } Z_1 \subset f^{-1}(Z_2)\} \subset \text{Hom}_{\text{Var}(\mathbb{C})}(X_1, X_2) \end{aligned}$$

The category $\text{Var}(\mathbb{C})^2$ admits fiber products : $(X_1, Z_1) \times_{(S, Z)} (X_2, Z_2) = (X_1 \times_S X_2, Z_1 \times_Z Z_2)$. In particular, for $f : T \rightarrow S$ a morphism with $S, T \in \text{Var}(\mathbb{C})$, we have the pullback functor

$$P(f) : \text{Var}(\mathbb{C})^2/S \rightarrow \text{Var}(\mathbb{C})^2/T, P(f)((X, Z)/S) := (X_T, Z_T)/T, P(f)(g) := (g \times_S f)$$

and we note again $P(f) : \text{Var}(\mathbb{C})^2/T \rightarrow \text{Var}(\mathbb{C})^2/S$ the corresponding morphism of sites.

We will consider in the construction of the filtered De Rham realization functor the full subcategory $\text{Var}(\mathbb{C})^{2,sm}/S \subset \text{Var}(\mathbb{C})^2/S$ such that the first factor is a smooth morphism : We will also consider, in order to obtain a complex of D modules in the construction of the filtered De Rham realization functor, the restriction to the full subcategory $\text{Var}(\mathbb{C})^{2,pr}/S \subset \text{Var}(\mathbb{C})^2/S$ such that the first factor is a projection :

Definition 13. (i) Let $S \in \text{Var}(\mathbb{C})$. We denote by

$$\rho_S : \text{Var}(\mathbb{C})^{2,sm}/S \hookrightarrow \text{Var}(\mathbb{C})^2/S$$

the full subcategory consisting of the objects $(U, Z)/S = ((U, Z), h) \in \text{Var}(\mathbb{C})^2/S$ such that the morphism $h : U \rightarrow S$ is smooth. That is, $\text{Var}(\mathbb{C})^{2,sm}/S$ is the category

- whose objects are $(U, Z)/S = ((U, Z), h)$, with $U \in \text{Var}(\mathbb{C})$, $Z \subset U$ a closed subset, and $h : U \rightarrow S$ a smooth morphism,
- whose morphisms $g : (U, Z)/S = ((U, Z), h_1) \rightarrow (U', Z')/S = ((U', Z'), h_2)$ is a morphism $g : U \rightarrow U'$ of complex algebraic varieties such that $Z \subset g^{-1}(Z')$ and $h_2 \circ g = h_1$.

We denote again $\rho_S : \text{Var}(\mathbb{C})^2/S \rightarrow \text{Var}(\mathbb{C})^{2,sm}/S$ the associated morphism of site. We have

$$r^s(S) : \text{Var}(\mathbb{C})^2 \xrightarrow{r(S) := r(S, S)} \text{Var}(\mathbb{C})^2/S \xrightarrow{\rho_S} \text{Var}(\mathbb{C})^{2,sm}/S$$

the composite morphism of site.

(ii) Let $S \in \text{Var}(\mathbb{C})$. We will consider the full subcategory

$$\mu_S : \text{Var}(\mathbb{C})^{2,pr}/S \hookrightarrow \text{Var}(\mathbb{C})^2/S$$

whose subset of object consist of those whose morphism is a projection to S :

$$(\text{Var}(\mathbb{C})^{2,pr}/S)^0 := \{((Y \times S, X), p), Y \in \text{Var}(\mathbb{C}), p : Y \times S \rightarrow S \text{ the projection}\} \subset (\text{Var}(\mathbb{C})^2/S)^0.$$

(iii) We will consider the full subcategory

$$\mu_S : (\text{Var}(\mathbb{C})^{2,smpr}/S) \hookrightarrow \text{Var}(\mathbb{C})^{2,sm}/S$$

whose subset of object consist of those whose morphism is a smooth projection to S :

$$(\text{Var}(\mathbb{C})^{2,smpr}/S)^0 := \{((Y \times S, X), p), Y \in \text{SmVar}(\mathbb{C}), p : Y \times S \rightarrow S \text{ the projection}\} \subset (\text{Var}(\mathbb{C})^2/S)^0$$

For $f : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$, we have by definition, the following commutative diagram of sites

$$\begin{array}{ccccc}
\text{Var}(\mathbb{C})^2/T & \xrightarrow{\mu_T} & \text{Var}(\mathbb{C})^{2,pr}/T & & \\
\downarrow P(f) & \searrow \rho_T^2 & \downarrow P(f) & \searrow \rho_T & \\
& & \text{Var}(\mathbb{C})^{2,sm}/T & \xrightarrow{\mu_T} & \text{Var}(\mathbb{C})^{2,smpr}/T \\
& & \downarrow P(f) & & \downarrow P(f) \\
\text{Var}(\mathbb{C})^2/S & \xrightarrow{\mu_S} & \text{Var}(\mathbb{C})^{2,pr}/S & & \\
& \searrow \rho_S & \downarrow P(f) & \searrow \rho_S & \\
& & \text{Var}(\mathbb{C})^{2,sm}/S & \xrightarrow{\mu_S} & \text{Var}(\mathbb{C})^{2,smpr}/S
\end{array} \tag{29}$$

Recall we have (see subsection 2.1), for $S \in \text{Var}(\mathbb{C})$, the graph functor

$$\begin{aligned}
\text{Gr}_S^{12} : \text{Var}(\mathbb{C})/S &\rightarrow \text{Var}(\mathbb{C})^{2,pr}/S, \quad X/S \mapsto \text{Gr}_S^{12}(X/S) := (X \times S, X)/S, \\
(g : X/S \rightarrow X'/S) &\mapsto \text{Gr}_S^{12}(g) := (g \times I_S : (X \times S, X) \rightarrow (X' \times S, X'))
\end{aligned}$$

For $f : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$, we have by definition, the following commutative diagram of sites

$$\begin{array}{ccccc}
\text{Var}(\mathbb{C})^{2,pr}/T & \xrightarrow{\text{Gr}_T^{12}} & \text{Var}(\mathbb{C})/T & & \\
\downarrow P(f) & \searrow \rho_T & \downarrow & \searrow \rho_T & \\
& & \text{Var}(\mathbb{C})^{2,smpr}/T & \xrightarrow{\text{Gr}_T^{12}} & \text{Var}(\mathbb{C})^{sm}/T \\
& & \downarrow & \downarrow P(f) & \downarrow P(f) \\
\text{Var}(\mathbb{C})^{2,pr}/S & \xrightarrow{P(f)} & \text{Var}(\mathbb{C})/S & & \\
& \searrow \rho_S & \downarrow & \searrow \rho_S & \\
& & \text{Var}(\mathbb{C})^{2,sm}/S & \xrightarrow{\text{Gr}_S^{12}} & \text{Var}(\mathbb{C})^{sm}/S
\end{array} \quad . \quad (30)$$

where we recall that $P(f)((X, Z)/S) := ((X_T, Z_T)/T)$, since smooth morphisms are preserved by base change.

As usual, we denote by

$$(f^*, f_*) := (P(f)^*, P(f)_*) : C(\text{Var}(\mathbb{C})^{2,sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{2,sm}/T)$$

the adjunction induced by $P(f) : \text{Var}(\mathbb{C})^{2,sm}/T \rightarrow \text{Var}(\mathbb{C})^{2,sm}/S$. Since the colimits involved in the definition of $f^* = P(f)^*$ are filtered, f^* also preserve monomorphism. Hence, we get an adjunction

$$(f^*, f_*) : C_{fil}(\text{Var}(\mathbb{C})^{2,sm}/S) \rightleftarrows C_{fil}(\text{Var}(\mathbb{C})^{2,sm}/T), \quad f^*(G, F) := (f^*G, f^*F)$$

For $S \in \text{Var}(\mathbb{C})$, we denote by $\mathbb{Z}_S := \mathbb{Z}((S, S)/(S, S)) \in \text{PSh}(\text{Var}(\mathbb{C})^{2,sm}/S)$ the constant presheaf. By Yoneda lemma, we have for $F \in C(\text{Var}(\mathbb{C})^{2,sm}/S)$, $\mathcal{H}om(\mathbb{Z}_S, F) = F$.

For $h : U \rightarrow S$ a smooth morphism with $U, S \in \text{Var}(\mathbb{C})$, $P(h) : \text{Var}(\mathbb{C})^{2,sm}/S \rightarrow \text{Var}(\mathbb{C})^{2,sm}/U$ admits a left adjoint

$$C(h) : \text{Var}(\mathbb{C})^{2,sm}/U \rightarrow \text{Var}(\mathbb{C})^{2,sm}/S, \quad C(h)((U', Z'), h') = ((U', Z'), h' \circ h).$$

Hence $h^* : C(\text{Var}(\mathbb{C})^{2,sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{2,sm}/U)$ admits a left adjoint

$$h_{\#} : C(\text{Var}(\mathbb{C})^{2,sm}/U) \rightarrow C(\text{Var}(\mathbb{C})^{2,sm}/S), \quad F \mapsto (h_{\#}F : ((U, Z), h_0) \mapsto \lim_{((U', Z'), h' \circ h) \rightarrow ((U, Z), h_0)} F((U', Z')/U))$$

For $F^\bullet \in C(\text{Var}(\mathbb{C})^{2,sm}/S)$ and $G^\bullet \in C(\text{Var}(\mathbb{C})^{2,sm}/U)$, we have the adjunction maps

$$\text{ad}(h_{\#}, h^*)(G^\bullet) : G^\bullet \rightarrow h^*h_{\#}G^\bullet, \quad \text{ad}(h_{\#}, h^*)(F^\bullet) : h_{\#}h^*F^\bullet \rightarrow F^\bullet.$$

For a smooth morphism $h : U \rightarrow S$, with $U, S \in \text{Var}(\mathbb{C})$, we have the adjunction isomorphism, for $F \in C(\text{Var}(\mathbb{C})^{2,sm}/U)$ and $G \in C(\text{Var}(\mathbb{C})^{2,sm}/S)$,

$$I(h_{\#}, h^*)(F, G) : \mathcal{H}om^\bullet(h_{\#}F, G) \xrightarrow{\sim} h_*\mathcal{H}om^\bullet(F, h^*G). \quad (31)$$

For a commutative diagram in $\text{Var}(\mathbb{C})$:

$$D = \begin{array}{ccc}
V & \xrightarrow{g_2} & U \\
\downarrow h_2 & & \downarrow h_1 \\
T & \xrightarrow{g_1} & S
\end{array} ,$$

where h_1 and h_2 are smooth, we denote by, for $F^\bullet \in C(\text{Var}(\mathbb{C})^{2,sm}/U)$,

$$T_{\#}(D)(F^\bullet) : h_{2\#}g_2^*F^\bullet \rightarrow g_1^*h_{1\#}F^\bullet$$

the canonical map given by adjunction. If D is cartesian with $h_1 = h$, $g_1 = g$, $f_2 = h' : U_T \rightarrow T$, $g' : U_T \rightarrow U$,

$$T_{\sharp}(D)(F^{\bullet}) =: T_{\sharp}(g, h)(F) : h'_{\sharp} g'^{*} F^{\bullet} \xrightarrow{\sim} g^* h_{\sharp} F^{\bullet}$$

is an isomorphism.

We have the support section functors of a closed embedding $i : Z \hookrightarrow S$ for presheaves on the big Zariski site of pairs.

Definition 14. Let $i : Z \hookrightarrow S$ be a closed embedding with $S, Z \in \text{Var}(\mathbb{C})$ and $j : S \setminus Z \hookrightarrow S$ be the open complementary subset.

(i) We define the functor

$$\Gamma_Z : C(\text{Var}(\mathbb{C})^{2,sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{2,sm}/S), G^{\bullet} \mapsto \Gamma_Z G^{\bullet} := \text{Cone}(\text{ad}(j^*, j_*)(G^{\bullet}) : G^{\bullet} \rightarrow j_* j^* G^{\bullet})[-1],$$

so that there is then a canonical map $\gamma_Z(G^{\bullet}) : \Gamma_Z G^{\bullet} \rightarrow G^{\bullet}$.

(ii) We have the dual functor of (i) :

$$\Gamma_Z^{\vee} : C(\text{Var}(\mathbb{C})^{2,sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{2,sm}/S), F \mapsto \Gamma_Z^{\vee}(F^{\bullet}) := \text{Cone}(\text{ad}(j_{\sharp}, j^*)(G^{\bullet}) : j_{\sharp} j^* G^{\bullet} \rightarrow G^{\bullet}),$$

together with the canonical map $\gamma_Z^{\vee}(G) : F \rightarrow \Gamma_Z^{\vee}(G)$.

(iii) For $F, G \in C(\text{Var}(\mathbb{C})^{2,sm}/S)$, we denote by

$$I(\gamma, \text{hom})(F, G) := (I, I(j_{\sharp}, j^*)(F, G)^{-1}) : \Gamma_Z \mathcal{H}om(F, G) \xrightarrow{\sim} \mathcal{H}om(\Gamma_Z^{\vee} F, G)$$

the canonical isomorphism given by adjunction.

Note that we have similarly for $i : Z \hookrightarrow S$, $i' : Z' \hookrightarrow Z$ closed embeddings, $g : T \rightarrow S$ a morphism with $T, S, Z \in \text{Var}(\mathbb{C})$ and $F \in C(\text{Var}(\mathbb{C})^{2,sm}/S)$, the canonical maps in $C(\text{Var}(\mathbb{C})^{2,sm}/S)$

- $T(g, \gamma)(F) : g^* \Gamma_Z F \xrightarrow{\sim} \Gamma_{Z \times_S T} g^* F$, $T(g, \gamma^{\vee})(F) : \Gamma_{Z \times_S T}^{\vee} g^* F \xrightarrow{\sim} g^* \Gamma_Z F$
- $T(Z'/Z, \gamma)(F) : \Gamma_{Z'} F \rightarrow \Gamma_Z F$, $T(Z'/Z, \gamma^{\vee})(F) : \Gamma_{Z'}^{\vee} F \rightarrow \Gamma_Z^{\vee} F$

but we will not use them in this article.

We now define the Zariski and the etale topology on $\text{Var}(\mathbb{C})^2/S$.

Definition 15. Let $S \in \text{Var}(\mathbb{C})$.

(i) Denote by τ a topology on $\text{Var}(\mathbb{C})$, e.g. the Zariski or the etale topology. The τ covers in $\text{Var}(\mathbb{C})^2/S$ of $(X, Z)/S$ are the families of morphisms

$$\{(c_i : (U_i, Z \times_X U_i)/S \rightarrow (X, Z)/S)_{i \in I}, \text{ with } (c_i : U_i \rightarrow X)_{i \in I} \text{ } \tau \text{ cover of } X \text{ in } \text{Var}(\mathbb{C})\}$$

(ii) Denote by τ the Zariski or the etale topology on $\text{Var}(\mathbb{C})$. The τ covers in $\text{Var}(\mathbb{C})^{2,sm}/S$ of $(U, Z)/S$ are the families of morphisms

$$\{(c_i : (U_i, Z \times_U U_i)/S \rightarrow (U, Z)/S)_{i \in I}, \text{ with } (c_i : U_i \rightarrow U)_{i \in I} \text{ } \tau \text{ cover of } U \text{ in } \text{Var}(\mathbb{C})\}$$

(iii) Denote by τ the Zariski or the etale topology on $\text{Var}(\mathbb{C})$. The τ covers in $\text{Var}(\mathbb{C})^{2,(sm)pr}/S$ of $(Y \times S, Z)/S$ are the families of morphisms

$$\{(c_i \times I_S : (U_i \times S, Z \times_{Y \times S} U_i \times S)/S \rightarrow (Y \times S, Z)/S)_{i \in I}, \text{ with } (c_i : U_i \rightarrow Y)_{i \in I} \text{ } \tau \text{ cover of } Y \text{ in } \text{Var}(\mathbb{C})\}$$

Let $S \in \text{Var}(\mathbb{C})$. Denote by τ the Zariski or the etale topology on $\text{Var}(\mathbb{C})$. In particular, denoting $a_{\tau} : \text{PSh}(\text{Var}(\mathbb{C})^{2,(sm)}/S) \rightarrow \text{Shv}(\text{Var}(\mathbb{C})^{2,(sm)}/S)$ and $a_{\tau} : \text{PSh}(\text{Var}(\mathbb{C})^{2,(sm)pr}/S) \rightarrow \text{Shv}(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$ the sheafification functors,

- a morphism $\phi : F \rightarrow G$, with $F, G \in C(\text{Var}(\mathbb{C})^{2,(sm)}/S)$, is a τ local equivalence if $a_\tau H^n \phi : a_\tau H^n F \rightarrow a_\tau H^n G$ is an isomorphism, a morphism $\phi : F \rightarrow G$, with $F, G \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$, is a τ local equivalence if $a_\tau H^n \phi : a_\tau H^n F \rightarrow a_\tau H^n G$ is an isomorphism,
- a morphism $\phi : (G_1, F) \rightarrow (G_2, F)$, with $(G_1, F), (G_2, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/S)$, is an r -filtered τ local equivalence if for all $p, q \in \mathbb{Z}$, $a_\tau E_r^{p,q} \phi : a_\tau E_r^{p,q}(G_1, F) \rightarrow a_\tau E_r^{p,q}(G_2, F)$ is an isomorphism of sheaves on $\text{Var}(\mathbb{C})^{2,(sm)}/S$, a morphism $\phi : (G_1, F) \rightarrow (G_2, F)$, with $(G_1, F), (G_2, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$, is an r -filtered τ local equivalence if for all $p, q \in \mathbb{Z}$, $a_\tau E_r^{p,q} \phi : a_\tau E_r^{p,q}(G_1, F) \rightarrow a_\tau E_r^{p,q}(G_2, F)$ is an isomorphism of sheaves on $\text{Var}(\mathbb{C})^{2,(sm)pr}/S$,
- $F^\bullet \in C(\text{Var}(\mathbb{C})^{2,(sm)}/S)$ is τ fibrant if for all $(U, Z)/S \in \text{Var}(\mathbb{C})^{2,(sm)}/S$ and all τ covers $(c_i : (U_i, Z \times_U U_i)/S \rightarrow (U, Z)/S)_{i \in I}$ of $(U, Z)/S$,

$$F^\bullet(c_i) : F^\bullet((U, Z)/S) \rightarrow \text{Tot}(\bigoplus_{card I = \bullet} F^\bullet((U_i, Z \times_U U_i)/S))$$

is a quasi-isomorphism of complexes of abelian groups, $F^\bullet \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$ is τ fibrant if for all $(Y \times S, Z)/S \in \text{Var}(\mathbb{C})^{2,(sm)pr}/S$ and all τ covers $(c_i \times I_S : (U_i \times S, Z \times_{Y \times S} U_i \times S)/S \rightarrow (Y \times S, Z)/S)_{i \in I}$ of $(Y \times S, Z)/S$,

$$F^\bullet(c_i \times I_S) : F^\bullet((Y \times S, Z)/S) \rightarrow \text{Tot}(\bigoplus_{card I = \bullet} F^\bullet((U_i \times S, Z_I \times_Y U_j)/S))$$

is a quasi-isomorphism of complexes of abelian groups,

- $(F^\bullet, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/S)$ is r -filtered τ fibrant if for all $(U, Z)/S \in \text{Var}(\mathbb{C})^{2,(sm)}/S$ and all τ covers $(c_i : (U_i, Z \times_U U_i)/S \rightarrow (U, Z)/S)_{i \in I}$ of $(U, Z)/S$,

$$E_r^{p,q}(F^\bullet, F)(c_i) : E_r^{p,q}(F^\bullet, F)((U, Z)/S) \rightarrow E_r^{p,q}(\text{Tot}(\bigoplus_{card I = \bullet} (F^\bullet, F)((U_i, Z \times_U U_i)/S)))$$

is an isomorphism of abelian groups for all $p, q \in \mathbb{Z}$, $(F^\bullet, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$ is r -filtered τ fibrant if for all $(Y \times S, Z)/S \in \text{Var}(\mathbb{C})^{2,(sm)pr}/S$ and all τ covers $(c_i \times I_S : (U_i \times S, Z \times_{Y \times S} U_i \times S)/S \rightarrow (Y \times S, Z)/S)_{i \in I}$ of $(Y \times S, Z)/S$,

$$E_r^{p,q}(F^\bullet, F)(c_i \times I_S) : E_r^{p,q}(F^\bullet, F)((Y \times S, Z)/S) \xrightarrow{\sim} E_r^{p,q}(\text{Tot}(\bigoplus_{card I = \bullet} (F^\bullet, F)((U_i \times S, Z \times_Y U_i)/S)))$$

is an isomorphism of abelian groups for all $p, q \in \mathbb{Z}$.

Will now define the \mathbb{A}^1 local property on $\text{Var}(\mathbb{C})^2/S$.

Definition 16. Let $S \in \text{Var}(\mathbb{C})$. Denote for short $\text{Var}(\mathbb{C})^{2,(sm)}/S$ either the category $\text{Var}(\mathbb{C})^2/S$ or the category $\text{Var}(\mathbb{C})^{2,sm}/S$. Denote for short $\text{Var}(\mathbb{C})^{2,(sm)pr}/S$ either the category $\text{Var}(\mathbb{C})^{2,pr}/S$ or the category $\text{Var}(\mathbb{C})^{2,smpr}/S$.

- (i) A complex $F \in C(\text{Var}(\mathbb{C})^{2,(sm)}/S)$, is said to be \mathbb{A}^1 invariant if for all $(X, Z)/S \in \text{Var}(\mathbb{C})^{2,(sm)}/S$

$$F(p_X) : F((X, Z)/S) \rightarrow F((X \times \mathbb{A}^1, (Z \times \mathbb{A}^1))/S)$$

is a quasi-isomorphism, where $p_X : (X \times \mathbb{A}^1, (Z \times \mathbb{A}^1)) \rightarrow (X, Z)$ is the projection.

- (i)' A complex $G \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$, is said to be \mathbb{A}^1 invariant if for all $(Y \times S, Z)/S \in \text{Var}(\mathbb{C})^{2,(sm)pr}/S$

$$G(p_{Y \times S}) : G((Y \times S, Z)/S) \rightarrow G((Y \times \mathbb{A}^1 \times S, (Z \times \mathbb{A}^1))/S)$$

is a quasi-isomorphism of abelian group for all $p, q \in \mathbb{Z}$.

- (ii) Let τ a topology on $\text{Var}(\mathbb{C})$. A complex $F \in C(\text{Var}(\mathbb{C})^{2,(sm)}/S)$ is said to be \mathbb{A}^1 local for the τ topology induced on $\text{Var}(\mathbb{C})^2/S$, if for an (hence every) τ local equivalence $k : F \rightarrow G$ with k injective and $G \in C(\text{Var}(\mathbb{C})^{2,(sm)}/S)$ τ fibrant, e.g. $k : F \rightarrow E_\tau(F)$, G is \mathbb{A}^1 invariant.

(ii)' Let τ a topology on $\text{Var}(\mathbb{C})$. A complex $F \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$ is said to be \mathbb{A}^1 local for the τ topology induced on $\text{Var}(\mathbb{C})^{2,pr}/S$, if for an (hence every) τ local equivalence $k : F \rightarrow G$ with k injective and $G \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$ τ fibrant, e.g. $k : F \rightarrow E_\tau(F)$, G is \mathbb{A}^1 invariant.

(iii)' A morphism $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{2,(sm)}/S)$ is said to an (\mathbb{A}^1, et) local equivalence if for all $H \in C(\text{Var}(\mathbb{C})^{2,(sm)}/S)$ which is \mathbb{A}^1 local for the etale topology

$$\text{Hom}(L(m), E_{et}(H)) : \text{Hom}(L(G), E_{et}(H)) \rightarrow \text{Hom}(L(F), E_{et}(H))$$

is a quasi-isomorphism.

(iii)' A morphism $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$ is said to an (\mathbb{A}^1, et) local equivalence if for all $H \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$ which is \mathbb{A}^1 local for the etale topology

$$\text{Hom}(L(m), E_{et}(H)) : \text{Hom}(L(G), E_{et}(H)) \rightarrow \text{Hom}(L(F), E_{et}(H))$$

is a quasi-isomorphism.

Equivalently, $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{2,(sm)}/S)$ is an (\mathbb{A}^1, et) local equivalence if and only if there exists $(X_1, Z_1)/S, \dots, (X_r, Z_r)/S \in \text{Var}(\mathbb{C})^{2,sm}/S$ such that we have in $\text{Ho}_{et}(C(\text{Var}(\mathbb{C})^{2,sm}/S))$

$$\begin{aligned} \text{Cone}(m) &\xrightarrow{\sim} \text{Cone}(\text{Cone}(\mathbb{Z}((X_1 \times \mathbb{A}^1, Z_1 \times \mathbb{A}^1)/S) \rightarrow \mathbb{Z}((X_1, Z_1)/S)) \\ &\rightarrow \dots \rightarrow \text{Cone}(\mathbb{Z}((X_1 \times \mathbb{A}^1, Z_1 \times \mathbb{A}^1)/S) \rightarrow \mathbb{Z}((X_1, Z_1)/S)) \end{aligned}$$

In the filtered case, we consider :

Definition 17. Let $S \in \text{Var}(\mathbb{C})$. Denote for short $\text{Var}(\mathbb{C})^{2,(sm)}/S$ either the category $\text{Var}(\mathbb{C})^2/S$ or the category $\text{Var}(\mathbb{C})^{2,sm}/S$. Denote for short $\text{Var}(\mathbb{C})^{2,(sm)pr}/S$ either the category $\text{Var}(\mathbb{C})^{2,pr}/S$ or the category $\text{Var}(\mathbb{C})^{2,smpr}/S$.

(i) Let $r \in \mathbb{N}$. A filtered complex $(G, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/S)$, is said to be r -filtered \mathbb{A}^1 invariant if for all $(X, Z)/S \in \text{Var}(\mathbb{C})^{2,(sm)}/S$

$$E_r^{p,q}(G, F)(p_X) : (G, F)((X, Z)/S) \rightarrow E_r^{p,q}(G, F)((X \times \mathbb{A}^1, (Z \times \mathbb{A}^1))/S)$$

is an isomorphism of abelian groups for all $p, q \in \mathbb{Z}$, where $p_X : (X \times \mathbb{A}^1, (Z \times \mathbb{A}^1)) \rightarrow (X, Z)$ is the projection. Note that this definition say that this r does NOT depend on p and q .

(i)' Let $r \in \mathbb{N}$. A filtered complex $(G, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$, is said to be r -filtered \mathbb{A}^1 invariant if for all $(Y \times S, Z)/S \in \text{Var}(\mathbb{C})^{2,(sm)pr}/S$

$$E_r^{p,q}(G, F)(p_{Y \times S}) : E_r^{p,q}(G, F)((Y \times S, Z)/S) \rightarrow E_r^{p,q}(G, F)((Y \times \mathbb{A}^1 \times S, (Z \times \mathbb{A}^1))/S)$$

is an isomorphism of abelian group for all $p, q \in \mathbb{Z}$. Note that this definition say that this r does NOT depend on p and q .

(ii) A filtered complex $(G, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/S)$, is said to be ∞ -filtered \mathbb{A}^1 invariant if for all $(X, Z)/S \in \text{Var}(\mathbb{C})^{2,(sm)}/S$ there exist $r \in \mathbb{N}$ such that

$$E_r^{p,q}(G, F)(p_X) : E_r^{p,q}(G, F)((X, Z)/S) \rightarrow E_r^{p,q}(G, F)((X \times \mathbb{A}^1, (Z \times \mathbb{A}^1))/S)$$

is an isomorphism of abelian groups for all $p, q \in \mathbb{Z}$, where $p_X : (X \times \mathbb{A}^1, (Z \times \mathbb{A}^1)) \rightarrow (X, Z)$ is the projection (note that this definition say that this r does NOT depend on p and q but may depends on $(X, Z)/S$). This implies that, for all $(X, Z)/S \in \text{Var}(\mathbb{C})^{2,(sm)}/S$,

$$H^n(G, F)(p_X) : H^n(G, F)((X, Z)/S) \rightarrow H^n(G, F)((X \times \mathbb{A}^1, (Z \times \mathbb{A}^1))/S)$$

is a filtered isomorphism of filtered abelian groups for all $n \in \mathbb{Z}$.

(ii)' Similarly, a filtered complex $(G, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$, is said to be ∞ -filtered \mathbb{A}^1 invariant if for all $(Y \times S, Z)/S \in \text{Var}(\mathbb{C})^{2,(sm)pr}/S$ there exist $r \in \mathbb{N}$ such that

$$E_r^{p,q}(G, F)(p_{Y \times S}) : E_r^{p,q}(G, F)((Y \times S, Z)/S) \rightarrow E_r^{p,q}(G, F)((Y \times \mathbb{A}^1 \times S, (Z \times \mathbb{A}^1))/S)$$

is an isomorphism of abelian group for all $p, q \in \mathbb{Z}$ (note that this definition say that this r does NOT depend on p and q but may depends on $(Y \times \mathbb{A}^1, Z)/S$. This implies that, for all $(Y \times S, Z)/S \in \text{Var}(\mathbb{C})^{2,(sm)pr}/S$,

$$H^n(G, F)(p_{Y \times S}) : H^n(G, F)((Y \times S, Z)/S) \rightarrow H^n(G, F)((Y \times \mathbb{A}^1 \times S, (Z \times \mathbb{A}^1))/S)$$

is an filtered isomorphism of filtered abelian groups for all $n \in \mathbb{Z}$.

(iii) Let τ a topology on $\text{Var}(\mathbb{C})$. A filtered complex $(G, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/S)$, is said to be ∞ -filtered \mathbb{A}^1 local for the τ topology induced on $\text{Var}(\mathbb{C})^2/S$, if for an (hence every) ∞ -filtered τ local equivalence $k : (G, F) \rightarrow (H, F)$ with k injective and $(H, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/S)$, ∞ -filtered τ fibrant, e.g. $k : (G, F) \rightarrow E_\tau(G, F)$, (H, F) is ∞ -filtered \mathbb{A}^1 invariant.

(iii)' Similarly, a filtered complex $(G, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$, is said to be ∞ -filtered \mathbb{A}^1 local for the τ topology induced on $\text{Var}(\mathbb{C})^2/S$, if for an (hence every) ∞ -filtered τ local equivalence $k : (G, F) \rightarrow (H, F)$ with k injective and $(H, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$, ∞ -filtered τ fibrant, e.g. $k : (G, F) \rightarrow E_\tau(G, F)$, (H, F) is ∞ -filtered \mathbb{A}^1 invariant.

Lemma 2. Let $S \in \text{Var}(\mathbb{C})$.

(i) Let $(G, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,smpr}/S)$ Then, if $m : F_1 \rightarrow F_2$ with $F_1, F_2 \in C(\text{Var}(\mathbb{C})^{2,smpr}/S)$ is an etale local equivalence,

$$\text{Hom}(m, E_{et}(G, F)) : \text{Hom}^\bullet(F_2, E_{et}(G, F)) \rightarrow \text{Hom}^\bullet(F_1, E_{et}(G, F))$$

is a filtered quasi-isomorphism.

(ii) Let $(G, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,smpr}/S)$ be ∞ -filtered \mathbb{A}^1 local for the etale topology. Then, if $m : F_1 \rightarrow F_2$ with $F_1, F_2 \in C(\text{Var}(\mathbb{C})^{2,smpr}/S)$ is an (\mathbb{A}^1, et) local equivalence,

$$\text{Hom}(m, E_{et}(G, F)) : \text{Hom}^\bullet(F_2, E_{et}(G, F)) \rightarrow \text{Hom}^\bullet(F_1, E_{et}(G, F))$$

is an ∞ -filtered quasi-isomorphism.

Proof. (i): Follows from the fact that $E_{et}(G, F)$ is (1-)filtered etale fibrant.

(ii): By definition of an (\mathbb{A}^1, et) local equivalence, we have there exists $(X_1, Z_1)/S, \dots, (X_r, Z_r)/S \in \text{Var}(\mathbb{C})^{2,smpr}/S$ such that we have in $\text{Ho}_{et}(C(\text{Var}(\mathbb{C})^{2,smpr}/S))$

$$\text{Cone}(m) \xrightarrow{\sim} \text{Cone}(\text{Cone}(\mathbb{Z}((X_1 \times \mathbb{A}^1, Z_1 \times \mathbb{A}^1)/S) \rightarrow \mathbb{Z}((X_1, Z_1)/S)) \rightarrow \dots \rightarrow \text{Cone}(\mathbb{Z}((X_r \times \mathbb{A}^1, Z_r \times \mathbb{A}^1)/S) \rightarrow \mathbb{Z}((X_r, Z_r)/S))$$

This gives in $D_{fil,1}(\mathbb{Z}) := \text{Ho}_{fil,1}(\mathbb{Z})$,

$$\begin{aligned} \text{Cone}(\text{Hom}(m, E_{et}(G, F))) &\xrightarrow{\sim} \text{Cone}(\text{Cone}(E_{et}(G, F)((X_1, Z_1)/S) \rightarrow E_{et}(G, F)((X_1 \times \mathbb{A}^1, Z_1 \times \mathbb{A}^1)/S)) \\ &\rightarrow \dots \rightarrow \text{Cone}(E_{et}(G, F)((X_r, Z_r)/S) \rightarrow E_{et}(G, F)((X_r \times \mathbb{A}^1, Z_r \times \mathbb{A}^1)/S)) \end{aligned}$$

Since $(G, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,smpr}/S)$ is ∞ -filtered \mathbb{A}^1 local for the etale topology, there exist $s_i \in \mathbb{Z}$ such that for all $p, q \in \mathbb{Z}$,

$$E_{s_i}^{p,q} \text{Cone}(E_{et}(G, F)((X_i, Z_i)/S) \rightarrow E_{et}(G, F)((X_i \times \mathbb{A}^1, Z_i \times \mathbb{A}^1)/S)) = 0$$

Hence for $s := \max((s_i)_{1 \leq i \leq r})$, $E_s^{p,q} \text{Cone}(\text{Hom}(m, E_{et}(G, F))) = 0$. \square

We have, similarly to the case of single varieties the following :

Proposition 21. *Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$.*

- (i) *The functor $g^* : C(\text{Var}(\mathbb{C})^{2,(sm)}/S) \rightarrow C(\text{Var}(\mathbb{C})^{2,(sm)}/T)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends (\mathbb{A}^1, et) local equivalence to (\mathbb{A}^1, et) local equivalence.*
- (ii) *The functor $g^* : C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S) \rightarrow C(\text{Var}(\mathbb{C})^{2,(sm)pr}/T)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends (\mathbb{A}^1, et) local equivalence to (\mathbb{A}^1, et) local equivalence.*

Proof. (i): Since the functor g^* preserve epimorphism and also monomorphism (the colimits involved being filtered), g^* sends quasi-isomorphism to quasi-isomorphism. Hence it preserve Zariski and etale local equivalence. The fact that it preserve (\mathbb{A}^1, et) local equivalence then follows similarly to the single case by the fact that g_* preserve by definition \mathbb{A}^1 equivariant presheaves.

(ii): Similar to (i). □

Proposition 22. *Let $S \in \text{Var}(\mathbb{C})$.*

- (i) *The functor $\rho_{S*} : C(\text{Var}(\mathbb{C})^2/S) \rightarrow C(\text{Var}(\mathbb{C})^{2,sm}/S)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends (\mathbb{A}^1, et) local equivalence to (\mathbb{A}^1, et) local equivalence.*
- (ii) *The functor $\rho_{S*} : C(\text{Var}(\mathbb{C})^{2,pr}/S) \rightarrow C(\text{Var}(\mathbb{C})^{2,smpr}/S)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends (\mathbb{A}^1, et) local equivalence to (\mathbb{A}^1, et) local equivalence.*

Proof. Similar to the proof of proposition 20. □

We also have

Proposition 23. *Let $S \in \text{Var}(\mathbb{C})$.*

- (i) *The functor $\text{Gr}_S^{12*} : C(\text{Var}(\mathbb{C})/S) \rightarrow C(\text{Var}(\mathbb{C})^{2,pr}/S)$ sends etale local equivalence to etale local equivalence.*
- (i)' *The functor $\text{Gr}_S^{12*} : C(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{2,smpr}/S)$ sends etale local equivalence to etale local equivalence.*
- (ii) *The functor $\text{Gr}_S^{12*} : C(\text{Var}(\mathbb{C})/S) \rightarrow C(\text{Var}(\mathbb{C})^{2,pr}/S)$ sends (\mathbb{A}^1, et) local equivalence to (\mathbb{A}^1, et) local equivalence.*
- (ii)' *The functor $\text{Gr}_S^{12*} : C(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{2,smpr}/S)$ sends (\mathbb{A}^1, et) local equivalence to (\mathbb{A}^1, et) local equivalence.*

Proof. (i): Follows from the fact that $\text{Gr}_{S*}^{12} : C(\text{Var}(\mathbb{C})^{2,pr}/S) \rightarrow C(\text{Var}(\mathbb{C})/S)$ sends etale fibrant complex of presheaves to etale fibrant complex of presheaves.

(i)': Follows from the fact that $\text{Gr}_{S*}^{12} : C(\text{Var}(\mathbb{C})^{2,smpr}/S) \rightarrow C(\text{Var}(\mathbb{C})^{sm}/S)$ sends etale fibrant complex of presheaves to etale fibrant complex of presheaves.

(ii): Follows from (i) and the fact that $\text{Gr}_{S*}^{12} : C(\text{Var}(\mathbb{C})^{2,pr}/S) \rightarrow C(\text{Var}(\mathbb{C})/S)$ sends \mathbb{A}^1 invariant complex of presheaves to \mathbb{A}^1 invariant complex of presheaves.

(ii)': Follows from (i)' and the fact that $\text{Gr}_{S*}^{12} : C(\text{Var}(\mathbb{C})^{2,smpr}/S) \rightarrow C(\text{Var}(\mathbb{C})^{sm}/S)$ sends \mathbb{A}^1 invariant complex of presheaves to \mathbb{A}^1 invariant complex of presheaves. □

We have the following canonical functor :

Definition 18. (i) For $S \in \text{Var}(\mathbb{C})$, we have the functor

$$\begin{aligned} (-)^\Gamma &: C(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{2,sm}/S), \\ F &\longmapsto F^\Gamma : (((U, Z)/S) = ((U, Z), h) \mapsto F^\Gamma((U, Z)/S) := (\Gamma_Z^\vee h^* LF)(U/U), \\ &\quad (g : ((U', Z'), h') \rightarrow ((U, Z), h)) \mapsto \\ &\quad (F^\Gamma(g) : (\Gamma_Z^\vee h^* LF)(U/U) \xrightarrow{i_{(\Gamma_Z^\vee h^* LF)(U/U)}} (g^*(\Gamma_Z^\vee h^* LF))(U'/U') \\ &\quad \xrightarrow{T(g, \gamma^\vee)(h^* LF)(U'/U')} (\Gamma_{Z \times_U U'}^\vee g^* h^* LF)(U'/U') \\ &\quad \xrightarrow{T(Z'/Z \times_U U', \gamma^\vee)(g^* h^* LF)(U'/U')} (\Gamma_{Z'}^\vee g^* h^* LF)(U'/U')) \end{aligned}$$

where $i_{(\Gamma_Z^\vee h^* LF)(U/U)}$ is the canonical arrow of the inductive limit. Similarly, we have, for $S \in \text{Var}(\mathbb{C})$, the functor

$$\begin{aligned} (-)^\Gamma &: C(\text{Var}(\mathbb{C})/S) \rightarrow C(\text{Var}(\mathbb{C})^2/S), \\ F &\longmapsto F^\Gamma : (((X, Z)/S) = ((X, Z), h) \mapsto F^\Gamma((X, Z)/S) := (\Gamma_Z^\vee h^* F)(X/X), \\ &\quad (g : ((X', Z'), h') \rightarrow ((X, Z), h)) \mapsto (F^\Gamma(g) : (\Gamma_Z^\vee h^* LF)(X/X) \rightarrow (\Gamma_{Z'}^\vee h'^* LF)(X'/X')) \end{aligned}$$

Note that for $S \in \text{Var}(\mathbb{C})$, $I(S/S) : \mathbb{Z}((S, S)/S) \rightarrow \mathbb{Z}(S/S)^\Gamma$ given by

$$I(S/S)((U, Z), h) : \mathbb{Z}((S, S)/S)((U, Z), h) \xrightarrow{\gamma^\vee(\mathbb{Z}(U/U))(U/U)} \mathbb{Z}(S/S)^\Gamma((U, Z), h) := (\Gamma_Z^\vee \mathbb{Z}(U/U))(U/U)$$

is an isomorphism.

(ii) Let $f : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. For $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$, we have the canonical morphism in $C(\text{Var}(\mathbb{C})^{2,sm}/T)$

$$\begin{aligned} T(f, \Gamma)(F) &:= T^*(f, \Gamma)(F) : f^*(F^\Gamma) \rightarrow (f^*F)^\Gamma, \\ T(f, \Gamma)(F)((U', Z')/T = ((U', Z'), h')) &: \\ f^*(F^\Gamma)((U', Z'), h') &:= \lim_{((U', Z'), h') \xrightarrow{l} ((U_T, Z_T), h_T) \xrightarrow{f_U} ((U, Z), h)} (\Gamma_Z^\vee h^* LF)(U/U) \\ &\xrightarrow{F^\Gamma(f_U \circ l)} (\Gamma_{Z'}^\vee l^* f_U^* h^* LF)(U'/U') = (\Gamma_{Z'}^\vee h'^* f^* LF)(U'/U') \\ &\xrightarrow{(\Gamma_{Z'}^\vee h'^* T(f, L)(F))(U'/U')} (\Gamma_{Z'}^\vee h'^* L f^* F)(U'/U') =: (f^*F)^\Gamma((U', Z'), h') \end{aligned}$$

where $f_U : U_T : U \times_S T \rightarrow U$ and $h_T : U_T := U \times_S T \rightarrow T$ are the base change maps, the equality following from the fact that $h \circ f_U \circ l = f \circ h_T \circ l = f \circ h'$. For $F \in C(\text{Var}(\mathbb{C})/S)$, we have similarly the canonical morphism in $C(\text{Var}(\mathbb{C})^2/T)$

$$T(f, \Gamma)(F) : f^*(F^\Gamma) \rightarrow (f^*F)^\Gamma.$$

(iii) Let $h : U \rightarrow S$ a smooth morphism with $U, S \in \text{Var}(\mathbb{C})$. We have, for $F \in C(\text{Var}(\mathbb{C})^{sm}/U)$, the canonical morphism in $C(\text{Var}(\mathbb{C})^{2,sm}/S)$

$$\begin{aligned} T_\sharp(h, \Gamma)(F) &: h_\sharp(F^\Gamma) \rightarrow (h_\sharp LF)^\Gamma, \\ T_\sharp(h, \Gamma)(F)((U', Z'), h') &: h_\sharp(F^\Gamma)((U', Z'), h') := \lim_{((U', Z'), h') \xrightarrow{l} ((U, U), h)} (\Gamma_Z^\vee l^* LF)(U'/U') \\ &\xrightarrow{(\Gamma_{Z'}^\vee l^* \text{ad}(h_\sharp, h^*)(LF))(U'/U')} (\Gamma_{Z'}^\vee l^* h^* h_\sharp LF)(U'/U') =: (h_\sharp LF)^\Gamma((U', Z')/h') \end{aligned}$$

(iv) Let $i : Z_0 \hookrightarrow S$ a closed embedding with $Z_0, S \in \text{Var}(\mathbb{C})$. We have the canonical morphism in $C(\text{Var}(\mathbb{C})^{2,sm}/S)$

$$\begin{aligned} T_*(i, \Gamma)(\mathbb{Z}(Z_0/Z_0)) &: i_*(\mathbb{Z}(Z_0/Z_0))^\Gamma \rightarrow (i_*\mathbb{Z}(Z/Z))^\Gamma, \\ T_*(i, \Gamma)(\mathbb{Z}(Z_0/Z_0))((U, Z), h) &: i_*((\mathbb{Z}(Z_0/Z_0))^\Gamma((U, Z), h)) := (\Gamma_{Z \times_S Z_0}^\vee \mathbb{Z}(Z_0/Z_0))(U \times_S Z_0) \\ &\xrightarrow{T(i_*, \gamma^\vee)(\mathbb{Z}(Z_0/Z_0))(U \times_S Z_0)} (\Gamma_Z^\vee i_* \mathbb{Z}(Z_0/Z_0))(U \times_S Z_0) =: (i_*\mathbb{Z}(Z/Z))^\Gamma((U, Z), h) \end{aligned}$$

Definition 19. Let $S \in \text{Var}(\mathbb{C})$. We have for $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ the canonical map in $C(\text{Var}(\mathbb{C})^{sm}/S)$

$$\begin{aligned} \text{Gr}(F) &: \text{Gr}_{S*}^{12} \mu_{S*} F^\Gamma \rightarrow F, \\ \text{Gr}(F)(U/S) &: \Gamma_U^\vee p^* F(U \times S/U \times S) \xrightarrow{\text{ad}(l^*, l_*)(p^* F)(U \times S/U \times S)} h^* F(U/U) = F(U/S) \end{aligned}$$

where $h : U \rightarrow S$ is a smooth morphism with $U \in \text{Var}(\mathbb{C})$ and $h : U \xrightarrow{l} U \times S \xrightarrow{p} S$ is the graph factorization with l the graph embedding and p the projection.

Proposition 24. Let $S \in \text{Var}(\mathbb{C})$.

(i) Then,

- if $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is a quasi-isomorphism, $m^\Gamma : F^\Gamma \rightarrow G^\Gamma$ is a quasi-isomorphism in $C(\text{Var}(\mathbb{C})^{2,sm}/S)$,
- if $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is a Zariski local equivalence, $m^\Gamma : F^\Gamma \rightarrow G^\Gamma$ is a Zariski local equivalence in $C(\text{Var}(\mathbb{C})^{2,sm}/S)$, if $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is an étale local equivalence, $m^\Gamma : F^\Gamma \rightarrow G^\Gamma$ is an étale local equivalence in $C(\text{Var}(\mathbb{C})^{2,sm}/S)$,
- if $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is an $(\mathbb{A}^1, \text{ét})$ local equivalence, $m^\Gamma : F^\Gamma \rightarrow G^\Gamma$ is an $(\mathbb{A}^1, \text{ét})$ local equivalence in $C(\text{Var}(\mathbb{C})^{2,sm}/S)$.

(ii) Then,

- if $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})/S)$ is a quasi-isomorphism, $m^\Gamma : F^\Gamma \rightarrow G^\Gamma$ is a quasi-isomorphism in $C(\text{Var}(\mathbb{C})^2/S)$,
- if $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is a Zariski local equivalence, $m^\Gamma : F^\Gamma \rightarrow G^\Gamma$ is a Zariski local equivalence in $C(\text{Var}(\mathbb{C})^{2,sm}/S)$, if $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})/S)$ is an étale local equivalence, $m^\Gamma : F^\Gamma \rightarrow G^\Gamma$ is an étale local equivalence in $C(\text{Var}(\mathbb{C})^2/S)$,
- if $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is an $(\mathbb{A}^1, \text{ét})$ local equivalence, $m^\Gamma : F^\Gamma \rightarrow G^\Gamma$ is an $(\mathbb{A}^1, \text{ét})$ local equivalence in $C(\text{Var}(\mathbb{C})^2/S)$.

Proof. (i): Follows immediately from the fact that for $((U, Z), h) \in \text{Var}(\mathbb{C})^{2,sm}/S$,

- if $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is a quasi-isomorphism, $\Gamma_Z^\vee h^* LF(m) : \Gamma_Z^\vee h^* LF \rightarrow \Gamma_Z^\vee h^* LG$ is a quasi-isomorphism
- if $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is a Zariski or étale local equivalence, $\Gamma_Z^\vee h^* LF(m) : \Gamma_Z^\vee h^* LF \rightarrow \Gamma_Z^\vee h^* LG$ is a Zariski, resp. étale, local equivalence,
- if $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is an $(\mathbb{A}^1, \text{ét})$ local equivalence, $\Gamma_Z^\vee h^* LF(m) : \Gamma_Z^\vee h^* LF \rightarrow \Gamma_Z^\vee h^* LG$ is an $(\mathbb{A}^1, \text{ét})$ local equivalence.

(ii): Similar to (i). □

2.8 Presheaves on the big analytical site

For $S \in \text{AnSp}(\mathbb{C})$, we denote by $\rho_S : \text{AnSp}(\mathbb{C})^{sm}/S \hookrightarrow \text{AnSp}(\mathbb{C})/S$ be the full subcategory consisting of the objects $U/S = (U, h) \in \text{AnSp}(\mathbb{C})/S$ such that the morphism $h : U \rightarrow S$ is smooth. That is, $\text{AnSp}(\mathbb{C})^{sm}/S$ is the category

- whose objects are smooth morphisms $U/S = (U, h)$, $h : U \rightarrow S$ with $U \in \text{AnSp}(\mathbb{C})$,
- whose morphisms $g : U/S = (U, h_1) \rightarrow V/S = (V, h_2)$ is a morphism $g : U \rightarrow V$ of complex algebraic varieties such that $h_2 \circ g = h_1$.

We denote again $\rho_S : \text{AnSp}(\mathbb{C})/S \rightarrow \text{AnSp}(\mathbb{C})^{sm}/S$ the associated morphism of site. We will consider

$$r^s(S) : \text{AnSp}(\mathbb{C}) \xrightarrow{r(S)} \text{AnSp}(\mathbb{C})/S \xrightarrow{\rho_S} \text{AnSp}(\mathbb{C})^{sm}/S$$

the composite morphism of site. For $S \in \text{AnSp}(\mathbb{C})$, we denote by $\mathbb{Z}_S := \mathbb{Z}(S/S) \in \text{PSh}(\text{AnSp}(\mathbb{C})^{sm}/S)$ the constant presheaf. By Yoneda lemma, we have for $F \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$, $\mathcal{H}om(\mathbb{Z}_S, F) = F$. For $f : T \rightarrow S$ a morphism, with $T, S \in \text{AnSp}(\mathbb{C})$, we have the following commutative diagram of sites

$$\begin{array}{ccc} \text{AnSp}(\mathbb{C})/T & \xrightarrow{\rho_T} & \text{AnSp}(\mathbb{C})^{sm}/T \\ \downarrow P(f) & & \downarrow P(f) \\ \text{AnSp}(\mathbb{C})/S & \xrightarrow{\rho_S} & \text{AnSp}(\mathbb{C})^{sm}/S \end{array} \quad (32)$$

We denote, for $S \in \text{AnSp}(\mathbb{C})$, the obvious morphism of sites

$$\tilde{e}(S) : \text{AnSp}(\mathbb{C})/S \xrightarrow{\rho_S} \text{AnSp}(\mathbb{C})^{sm}/S \xrightarrow{e(S)} \text{Ouv}(S)$$

where $\text{Ouv}(S)$ is the set of the open subsets of S , given by the inclusion functors $\tilde{e}(S) : \text{Ouv}(S) \hookrightarrow \text{AnSp}(\mathbb{C})^{sm}/S \hookrightarrow \text{AnSp}(\mathbb{C})/S$. By definition, for $f : T \rightarrow S$ a morphism with $S, T \in \text{AnSp}(\mathbb{C})$, the commutative diagram of sites (32) extend a commutative diagram of sites :

$$\begin{array}{ccccc} \tilde{e}(T) : \text{AnSp}(\mathbb{C})/T & \xrightarrow{\rho_T} & \text{AnSp}(\mathbb{C})^{sm}/T & \xrightarrow{e(T)} & \text{Ouv}(T) \\ \downarrow P(f) & & \downarrow P(f) & & \downarrow P(f) \\ \tilde{e}(S) : \text{AnSp}(\mathbb{C})/S & \xrightarrow{\rho_S} & \text{AnSp}(\mathbb{C})^{sm}/S & \xrightarrow{e(S)} & \text{Ouv}(S) \end{array} \quad (33)$$

- As usual, we denote by

$$(f^*, f_*) := (P(f)^*, P(f)_*) : C(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{sm}/T)$$

the adjunction induced by $P(f) : \text{AnSp}(\mathbb{C})^{sm}/T \rightarrow \text{AnSp}(\mathbb{C})^{sm}/S$. Since the colimits involved in the definition of $f^* = P(f)^*$ are filtered, f^* also preserve monomorphism. Hence, we get an adjunction

$$(f^*, f_*) : C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S) \rightleftarrows C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/T), \quad f^*(G, F) := (f^*G, f^*F)$$

- As usual, we denote by

$$(f^*, f_*) := (P(f)^*, P(f)_*) : C(\text{AnSp}(\mathbb{C})/S) \rightarrow C(\text{AnSp}(\mathbb{C})/T)$$

the adjunction induced by $P(f) : \text{AnSp}(\mathbb{C})/T \rightarrow \text{AnSp}(\mathbb{C})/S$. Since the colimits involved in the definition of $f^* = P(f)^*$ are filtered, f^* also preserve monomorphism. Hence, we get an adjunction

$$(f^*, f_*) : C_{fil}(\text{AnSp}(\mathbb{C})/S) \rightleftarrows C_{fil}(\text{AnSp}(\mathbb{C})/T), \quad f^*(G, F) := (f^*G, f^*F)$$

- For $h : U \rightarrow S$ a smooth morphism with $U, S \in \text{AnSp}(\mathbb{C})$, the pullback functor $P(h) : \text{AnSp}(\mathbb{C})^{sm}/S \rightarrow \text{AnSp}(\mathbb{C})^{sm}/U$ admits a left adjoint $C(h)(X \rightarrow U) = (X \rightarrow U \rightarrow S)$. Hence, $h^* : C(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{sm}/U)$ admits a left adjoint

$$h_{\sharp} : C(\text{AnSp}(\mathbb{C})^{sm}/U) \rightarrow C(\text{AnSp}(\mathbb{C})^{sm}/S), \quad F \mapsto ((V, h_0) \mapsto \lim_{(V', h \circ h') \rightarrow (V, h_0)} F(V', h'))$$

Note that we have $\mathbb{Z}(V/S) = h_{\sharp}\mathbb{Z}_V$. More generally for $h' : V' \rightarrow V$ a smooth morphism, $V', V \in \text{AnSp}(\mathbb{C})$, we have $h_{\sharp}(\mathbb{Z}(V'/V)) = \mathbb{Z}(V'/S)$ with $V'/S = (V', h \circ h')$. Hence, since projective

presheaves are the direct summands of the representable presheaves, h_{\sharp} sends projective presheaves to projective presheaves. For $F^{\bullet} \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$ and $G^{\bullet} \in C(\text{AnSp}(\mathbb{C})^{sm}/U)$, we have the adjunction maps

$$\text{ad}(h_{\sharp}, h^*)(G^{\bullet}) : G^{\bullet} \rightarrow h^*h_{\sharp}G^{\bullet}, \quad \text{ad}(h_{\sharp}, h^*)(F^{\bullet}) : h_{\sharp}h^*F^{\bullet} \rightarrow F^{\bullet}.$$

For a smooth morphism $h : U \rightarrow S$, with $U, S \in \text{AnSp}(\mathbb{C})$, we have the adjunction isomorphism, for $F \in C(\text{AnSp}(\mathbb{C})^{sm}/U)$ and $G \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$,

$$I(h_{\sharp}, h^*)(F, G) : \mathcal{H}om^{\bullet}(h_{\sharp}F, G) \xrightarrow{\sim} h_*\mathcal{H}om^{\bullet}(F, h^*G). \quad (34)$$

- For $f : T \rightarrow S$ any morphism with $T, S \in \text{AnSp}(\mathbb{C})$, the pullback functor $P(f) : \text{AnSp}(\mathbb{C})/T \rightarrow \text{AnSp}(\mathbb{C})/S$ admits a left adjoint $C(f)(X \rightarrow T) = (X \rightarrow T \rightarrow S)$. Hence, $f^* : C(\text{Var AnSp}(\mathbb{C})/S) \rightarrow C(\text{AnSp}(\mathbb{C})/T)$ admits a left adjoint

$$f_{\sharp} : C(\text{AnSp}(\mathbb{C})/T) \rightarrow C(\text{AnSp}(\mathbb{C})/S), \quad F \mapsto ((V, h_0) \mapsto \lim_{(V', h_0 h') \rightarrow (V, h_0)} F(V', h'))$$

Note that we have $\mathbb{Z}(V/S) = h_{\sharp}\mathbb{Z}_V$. More generally for $h' : V' \rightarrow V$ a morphism, $V', V \in \text{AnSp}(\mathbb{C})$, we have $h_{\sharp}(\mathbb{Z}(V'/V)) = \mathbb{Z}(V'/S)$ with $V'/S = (V', h' \circ h)$. Hence, since projective presheaves are the direct summands of the representable presheaves, h_{\sharp} sends projective presheaves to projective presheaves. For $F^{\bullet} \in C(\text{AnSp}(\mathbb{C})/S)$ and $G^{\bullet} \in C(\text{AnSp}(\mathbb{C})/T)$, we have the adjunction maps

$$\text{ad}(f_{\sharp}, f^*)(G^{\bullet}) : G^{\bullet} \rightarrow f^*f_{\sharp}G^{\bullet}, \quad \text{ad}(f_{\sharp}, f^*)(F^{\bullet}) : f_{\sharp}f^*F^{\bullet} \rightarrow F^{\bullet}.$$

For a morphism $f : T \rightarrow S$, with $T, S \in \text{AnSp}(\mathbb{C})$, we have the adjunction isomorphism, for $F \in C(\text{AnSp}(\mathbb{C})/T)$ and $G \in C(\text{AnSp}(\mathbb{C})/S)$,

$$I(f_{\sharp}, f^*)(F, G) : \mathcal{H}om^{\bullet}(f_{\sharp}F, G) \xrightarrow{\sim} f_*\mathcal{H}om^{\bullet}(F, f^*G). \quad (35)$$

- For a commutative diagram in $\text{AnSp}(\mathbb{C})$:

$$D = \begin{array}{ccc} V & \xrightarrow{g_2} & U \\ \downarrow h_2 & & \downarrow h_1 \\ T & \xrightarrow{g_1} & S \end{array},$$

where h_1 and h_2 are smooth, we denote by, for $F^{\bullet} \in C(\text{AnSp}(\mathbb{C})^{sm}/U)$,

$$T_{\sharp}(D)(F^{\bullet}) : h_{2\sharp}g_2^*F^{\bullet} \rightarrow g_1^*h_{1\sharp}F^{\bullet}$$

the canonical map in $C(\text{AnSp}(\mathbb{C})^{sm}/T)$ given by adjunction. If D is cartesian with $h_1 = h$, $g_1 = g$, $f_2 = h' : U_T \rightarrow T$, $g' : U_T \rightarrow U$,

$$T_{\sharp}(D)(F^{\bullet}) =: T_{\sharp}(g, h)(F^{\bullet}) : h'_{\sharp}g'^*F^{\bullet} \xrightarrow{\sim} g^*h_{\sharp}F^{\bullet}$$

is an isomorphism and for $G^{\bullet} \in C(\text{AnSp}(\mathbb{C})^{sm}/T)$

$$T(D)(G^{\bullet}) =: T(g, h)(G^{\bullet}) : g^*h_*G^{\bullet} \xrightarrow{\sim} h'_*g'^*G^{\bullet}$$

is an isomorphism.

- For a commutative diagram in $\text{AnSp}(\mathbb{C})$:

$$D = \begin{array}{ccc} V & \xrightarrow{g_2} & X \\ \downarrow f_2 & & \downarrow f_1 \\ T & \xrightarrow{g_1} & S \end{array},$$

we denote by, for $F^\bullet \in C(\text{AnSp}(\mathbb{C})/X)$,

$$T_{\sharp}(D)(F^\bullet) : f_{2\sharp}g_2^*F^\bullet \rightarrow g_1^*f_{1\sharp}F^\bullet$$

the canonical map in $C(\text{AnSp}(\mathbb{C})/T)$ given by adjonction. If D is cartesian with $h_1 = h$, $g_1 = g$, $f_2 = h' : X_T \rightarrow T$, $g' : X_T \rightarrow X$,

$$T_{\sharp}(D)(F^\bullet) =: T_{\sharp}(g, f)(F^\bullet) : f'_{\sharp}f'^*F^\bullet \xrightarrow{\sim} g^*f_{\sharp}F^\bullet$$

is an isomorphism and for $G^\bullet \in C(\text{AnSp}(\mathbb{C})/T)$

$$T(D)(G^\bullet) =: T(g, h)(G^\bullet) : f^*g_*G^\bullet \xrightarrow{\sim} g'_*f'^*G^\bullet$$

is an isomorphism.

For $f : T \rightarrow S$ a morphism with $S, T \in \text{Var AnSp}(\mathbb{C})$,

- we get for $F \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$ from the a commutative diagram of sites (33) the following canonical transformation

$$T(e, f)(F^\bullet) : f^*e(S)_*F^\bullet \rightarrow e(T)_*f^*F^\bullet,$$

which is NOT a quasi-isomorphism in general. However, for $h : U \rightarrow S$ a smooth morphism with $S, U \in \text{AnSp}(\mathbb{C})$, $T(e, h)(F^\bullet) : h^*e(S)_*F^\bullet \xrightarrow{\sim} e(T)_*h^*F^\bullet$ is an isomorphism.

- we get for $F \in C(\text{AnSp}(\mathbb{C})/S)$ from the a commutative diagram of sites (33) the following canonical transformation

$$T(e, f)(F^\bullet) : f^*e(S)_*F^\bullet \rightarrow e(T)_*f^*F^\bullet,$$

which is NOT a quasi-isomorphism in general. However, for $h : U \rightarrow S$ a smooth morphism with $S, U \in \text{AnSp}(\mathbb{C})$, $T(e, h)(F^\bullet) : h^*e(S)_*F^\bullet \xrightarrow{\sim} e(T)_*h^*F^\bullet$ is an isomorphism.

Let $S \in \text{AnSp}(\mathbb{C})$,

- We have for $F, G \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$,
 - $e(S)_*(F \otimes G) = (e(S)_*F) \otimes (e(S)_*G)$ by definition
 - the canonical forgetfull map

$$T(S, \text{hom})(F, G) : e(S)_*\mathcal{H}om^\bullet(F, G) \rightarrow \mathcal{H}om^\bullet(e(S)_*F, e(S)_*G).$$

which is NOT a quasi-isomorphism in general.

By definition, we have for $F \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$, $e(S)_*E_{usu}(F) = E_{usu}(e(S)_*F)$.

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By definition, we have for $F \in C(\text{AnSp}(\mathbb{C})/S)$, $e(S)_*E_{usu}(F) = E_{usu}(e(S)_*F)$.

Let $S \in \text{AnSp}(\mathbb{C})$. We have the support section functor of a closed subset $Z \subset S$ for presheaves on the big analytical site.

Definition 20. Let $S \in \text{AnSp}(\mathbb{C})$. Let $Z \subset S$ a closed subset. Denote by $j : S \setminus Z \hookrightarrow S$ be the open complementary subset.

(i) We define the functor

$$\Gamma_Z : C(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{sm}/S), G^\bullet \mapsto \Gamma_Z G^\bullet := \text{Cone}(\text{ad}(j^*, j_*)(G^\bullet) : G^\bullet \rightarrow j_* j^* G^\bullet)[-1],$$

so that there is then a canonical map $\gamma_Z(G^\bullet) : \Gamma_Z G^\bullet \rightarrow G^\bullet$.

(ii) We have the dual functor of (i) :

$$\Gamma_Z^\vee : C(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{sm}/S), F \mapsto \Gamma_Z^\vee(F^\bullet) := \text{Cone}(\text{ad}(j_{\sharp}, j^*)(G^\bullet) : j_{\sharp} j^* G^\bullet \rightarrow G^\bullet),$$

together with the canonical map $\gamma_Z^\vee(G) : F \rightarrow \Gamma_Z^\vee(G)$.

(iii) For $F, G \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$, we denote by

$$I(\gamma, \text{hom})(F, G) := (I, I(j_{\sharp}, j^*)(F, G)) : \Gamma_Z \mathcal{H}om(F, G) \xrightarrow{\sim} \mathcal{H}om(\Gamma_Z^\vee F, G)$$

the canonical isomorphism given by adjunction.

Let $S \in \text{AnSp}(\mathbb{C})$ and $Z \subset S$ a closed subset.

- Since $\Gamma_Z : C(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{sm}/S)$ preserve monomorphism, it induces a functor

$$\Gamma_Z : C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S), (G, F) \mapsto \Gamma_Z(G, F) := (\Gamma_Z G, \Gamma_Z F)$$

- Since $\Gamma_Z^\vee : C(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{sm}/S)$ preserve monomorphism, it induces a functor

$$\Gamma_Z^\vee : C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S), (G, F) \mapsto \Gamma_Z^\vee(G, F) := (\Gamma_Z^\vee G, \Gamma_Z^\vee F)$$

Definition-Proposition 5. (i) Let $g : S' \rightarrow S$ a morphism and $i : Z \hookrightarrow S$ a closed embedding with $S', S, Z \in \text{AnSp}(\mathbb{C})$. Then, for $(G, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S)$, there exist a map in $C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S')$

$$T(g, \gamma)(G, F) : g^* \Gamma_Z(G, F) \rightarrow \Gamma_{Z \times_S S'} g^*(G, F)$$

unique up to homotopy, such that $\gamma_{Z \times_S S'}(g^*(G, F)) \circ T(g, \gamma)(G, F) = g^* \gamma_Z(G, F)$.

(ii) Let $i_1 : Z_1 \hookrightarrow S$, $i_2 : Z_2 \hookrightarrow Z_1$ be closed embeddings with $S, Z_1, Z_2 \in \text{AnSp}(\mathbb{C})$. Then, for $(G, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S)$,

- there exist a canonical map $T(Z_2/Z_1, \gamma)(G, F) : \Gamma_{Z_2}(G, F) \rightarrow \Gamma_{Z_1}(G, F)$ in $C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S)$ unique up to homotopy such that $\gamma_{Z_1}(G, F) \circ T(Z_2/Z_1, \gamma)(G, F) = \gamma_{Z_2}(G, F)$, together with a distinguish triangle

$$\Gamma_{Z_2}(G, F) \xrightarrow{T(Z_2/Z_1, \gamma)(G, F)} \Gamma_{Z_1}(G, F) \xrightarrow{\text{ad}(j_{2\sharp}^*, j_{2*})(\Gamma_{Z_1}(G, F))} \Gamma_{Z_1 \setminus Z_2}(G, F) \rightarrow \Gamma_{Z_2}(G, F)[1]$$

in $K_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S)$,

- there exist a map $T(Z_2/Z_1, \gamma^\vee)(G, F) : \Gamma_{Z_1}^\vee(G, F) \rightarrow \Gamma_{Z_2}^\vee(G, F)$ in $C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S)$ unique up to homotopy such that $\gamma_{Z_2}^\vee(G, F) = T(Z_2/Z_1, \gamma^\vee)(G, F) \circ \gamma_{Z_1}^\vee(G, F)$, together with a distinguish triangle

$$\Gamma_{Z_1 \setminus Z_2}^\vee(G, F) \xrightarrow{\text{ad}(j_{2\sharp}, j_2^*)(\Gamma_{Z_1}^\vee G)} \Gamma_{Z_1}^\vee(G, F) \xrightarrow{T(Z_2/Z_1, \gamma^\vee)(G, F)} \Gamma_{Z_2}^\vee(G, F) \rightarrow \Gamma_{Z_1 \setminus Z_2}^\vee(G, F)[1]$$

in $K_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S)$.

(iii) Consider a morphism $g : (S', Z) \rightarrow (S, Z)$ with $(S', Z) \rightarrow (S, Z) \in \text{AnSp}(\mathbb{C})$. We denote, for $G \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$ the composite

$$T(D, \gamma^\vee)(G) : g^* \Gamma_Z^\vee G \xrightarrow{\sim} \Gamma_{Z \times_S S'}^\vee g^* G \xrightarrow{T(Z'/Z \times_S S', \gamma^\vee)(G)} \Gamma_{Z'}^\vee g^* G$$

and we have then the factorization $\gamma_{Z'}^\vee(g^* G) : g^* G \xrightarrow{g^* \gamma_Z^\vee(G)} g^* \Gamma_Z^\vee G \xrightarrow{T(D, \gamma^\vee)(G)} \Gamma_{Z'}^\vee g^* G$.

Proof. Similar to definition-proposition 1 or definition-proposition 4. \square

Definition 21. For $S \in \text{AnSp}(\mathbb{C})$, we denote by

$$C_{O_S}(\text{AnSp}(\mathbb{C})^{sm}/S) := C_{e(S)^* O_S}(\text{AnSp}(\mathbb{C})^{sm}/S)$$

the category of complexes of presheaves on $\text{AnSp}(\mathbb{C})^{sm}/S$ endowed with a structure of $e(S)^* O_S$ module, and by

$$C_{O_S \text{ fil}}(\text{AnSp}(\mathbb{C})^{sm}/S) := C_{e(S)^* O_S \text{ fil}}(\text{AnSp}(\mathbb{C})^{sm}/S)$$

the category of filtered complexes of presheaves on $\text{Var}(\mathbb{C})^{sm}/S$ endowed with a structure of $e(S)^* O_S$ module.

Let $S \in \text{AnSp}(\mathbb{C})$. Let $Z \subset S$ a closed subset. Denote by $j : S \setminus Z \hookrightarrow S$ the open complementary embedding,

- For $G \in C_{O_S}(\text{AnSp}(\mathbb{C})^{sm}/S)$, $\Gamma_Z G := \text{Cone}(\text{ad}(j^*, j_*)(G) : F \rightarrow j_* j^* G)[-1]$ has a (unique) structure of $e(S)^* O_S$ module such that $\gamma_Z(G) : \Gamma_Z G \rightarrow G$ is a map in $C_{O_S}(\text{AnSp}(\mathbb{C})^{sm}/S)$. This gives the functor

$$\Gamma_Z : C_{O_S \text{ fil}}(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow C_{\text{fil} O_S}(\text{AnSp}(\mathbb{C})^{sm}/S), (G, F) \mapsto \Gamma_Z(G, F) := (\Gamma_Z G, \Gamma_Z F),$$

together with the canonical map $\gamma_Z((G, F) : \Gamma_Z(G, F) \rightarrow (G, F)$. Let $Z_2 \subset Z$ a closed subset. Then, for $G \in C_{O_S}(\text{AnSp}(\mathbb{C})^{sm}/S)$, $T(Z_2/Z, \gamma)(G) : \Gamma_{Z_2} G \rightarrow \Gamma_Z G$ is a map in $C_{O_S}(\text{AnSp}(\mathbb{C})^{sm}/S)$ (i.e. is $e(S)^* O_S$ linear).

- For $G \in C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S)$, $\Gamma_Z^\vee G := \text{Cone}(\text{ad}(j_\#^*, j^*)(G) : j_\# j^* G \rightarrow G)$ has a unique structure of $e(S)^* O_S$ module, such that $\gamma_Z^\vee(G) : G \rightarrow \Gamma_Z^\vee G$ is a map in $C_{O_S}(\text{AnSp}(\mathbb{C})^{sm}/S)$. This gives the the functor

$$\Gamma_Z^\vee : C_{O_S \text{ fil}}(S) \rightarrow C_{\text{fil} O_S}(S), (G, F) \mapsto \Gamma_Z^\vee(G, F) := (\Gamma_Z^\vee G, \Gamma_Z^\vee F),$$

together with the canonical map $\gamma_Z^\vee((G, F) : (G, F) \rightarrow \Gamma_Z^\vee(G, F)$. Let $Z_2 \subset Z$ a closed subset. Then, for $G \in C_{O_S}(\text{AnSp}(\mathbb{C})^{sm}/S)$, $T(Z_2/Z, \gamma^\vee)(G) : \Gamma_{Z_2}^\vee G \rightarrow \Gamma_Z^\vee G$ is a map in $C_{O_S}(\text{AnSp}(\mathbb{C})^{sm}/S)$ (i.e. is $e(S)^* O_S$ linear).

Definition 22. Let $S \in \text{AnSp}(\mathbb{C})$. Let $Z \subset S$ a closed subset.

(i) We denote by

$$C_Z(\text{AnSp}(\mathbb{C})^{sm}/S) \subset C(\text{AnSp}(\mathbb{C})^{sm}/S),$$

the full subcategory consisting of complexes of presheaves $F^\bullet \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$ such that $a_{usu} H^n(j^* F^\bullet) = 0$ for all $n \in \mathbb{Z}$, where $j : S \setminus Z \hookrightarrow S$ is the complementary open embedding and a_{usu} is the sheafification functor.

(i)' We denote by

$$C_{O_S, Z}(\text{AnSp}(\mathbb{C})^{sm}/S) \subset C_{O_S}(\text{AnSp}(\mathbb{C})^{sm}/S),$$

the full subcategory consisting of complexes of presheaves $F^\bullet \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$ such that $a_{usu} H^n(j^* F^\bullet) = 0$ for all $n \in \mathbb{Z}$, where $j : S \setminus Z \hookrightarrow S$ is the complementary open embedding and a_{usu} is the sheafification functor.

(ii) We denote by

$$C_{fil,Z}(\mathrm{AnSp}(\mathbb{C})^{sm}/S) \subset C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$$

the full subcategory consisting of filtered complexes of presheaves $(F^\bullet, F) \in C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$ such that there exist $r \in \mathbb{N}$ such that $a_{usu} j^* E_r^{p,q}(F^\bullet, F) = 0$ for all $p, q \in \mathbb{Z}$, where $j : S \setminus Z \hookrightarrow S$ is the complementary open embedding and a_{usu} is the sheafification functor. Note that by definition this r does NOT depend on p and q .

(ii)' We denote by

$$C_{O_S fil,Z}(\mathrm{AnSp}(\mathbb{C})^{sm}/S) \subset C_{O_S fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$$

the full subcategory consisting of filtered complexes of presheaves $(F^\bullet, F) \in C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$ such that there exist $r \in \mathbb{N}$ such that $a_{usu} j^* E_r^{p,q}(F^\bullet, F) = 0$ for all $p, q \in \mathbb{Z}$, where $j : S \setminus Z \hookrightarrow S$ is the complementary open embedding and a_{usu} is the sheafification functor. Note that by definition this r does NOT depend on p and q .

Let $S \in \mathrm{AnSp}(\mathbb{C})$ and $Z \subset S$ a closed subset.

- For $(G, F) \in C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$, we have $\Gamma_Z(G, F), \Gamma_Z^\vee(G, F) \in C_{fil,Z}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$.
- For $(G, F) \in C_{O_S fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$, we have $\Gamma_Z(G, F), \Gamma_Z^\vee(G, F) \in C_{O_S fil,Z}(\mathrm{AnSp}(\mathbb{C})^{sm}/S)$.

Let $S \in \mathrm{AnSp}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{AnSp}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote by $\tilde{S}_I = \prod_{i \in I} \tilde{S}_i$. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I$, and for $J \subset I$ the following commutative diagram

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ j_{IJ} \uparrow & & \uparrow p_{IJ} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}$$

where $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ is the projection and $j_{IJ} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. This gives the diagram of analytic spaces $(\tilde{S}_I) \in \mathrm{Fun}(\mathcal{P}(\mathbb{N}), \mathrm{AnSp}(\mathbb{C}))$ which gives the diagram of sites $\mathrm{AnSp}(\mathbb{C})^{sm}/(\tilde{S}_I) \in \mathrm{Fun}(\mathcal{P}(\mathbb{N}), \mathrm{Cat})$. Denote by $m : \tilde{S}_I \setminus (S_I \setminus S_J) \hookrightarrow \tilde{S}_I$ the open embedding.

Definition 23. Let $S \in \mathrm{AnSp}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{AnSp}(\mathbb{C})$. We denote by the full subcategory $C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/(S/(\tilde{S}_I))) \subset C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/(\tilde{S}_I))$ the full subcategory

- whose objects $(G, F) = ((G_I, F)_{I \subset [1, \dots, l]}, u_{IJ})$, with $(G_I, F) \in C_{fil, S_I}(\mathrm{AnSp}(\mathbb{C})^{sm}/\tilde{S}_I)$, and $u_{IJ} : m^*(G_I, F) \rightarrow m^* p_{IJ*}(G_J, F)$ for $I \subset J$, are ∞ -filtered usu local equivalence, satisfying for $I \subset J \subset K$, $p_{IJ*} u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/\tilde{S}_I)$,
- the morphisms $m : ((G, F), u_{IJ}) \rightarrow ((H, F), v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,

$$m = (m_I : (G_I, F) \rightarrow (H_I, F))_{I \subset [1, \dots, l]}$$

such that $v_{IJ} \circ m_I = p_{IJ*} m_J \circ u_{IJ}$ in $C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/\tilde{S}_I)$.

A morphism $m : ((G_I, F), u_{IJ}) \rightarrow ((H_I, F), v_{IJ})$ is said to an r -filtered usu local, equivalence, if all the m_I are r -filtered usu local equivalences.

Denote $L = [1, \dots, l]$ and for $I \subset L$, $p_{0(I)} : S \times \tilde{S}_I \rightarrow S$, $p_{I(0I)} : S \times \tilde{S}_I \rightarrow S_I$ the projections. By definition, we have functors

- $T(S/(\tilde{S}_I)) : C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/S) \rightarrow C_{fil}(\mathrm{AnSp}(\mathbb{C})^{sm}/(S/(\tilde{S}_I))), (G, F) \mapsto (i_{I*} j_I^* F, T(D_{IJ})(j_I^*(G, F))),$

- $T((\tilde{S}_I)/S) : C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/(S/(\tilde{S}_I))) \rightarrow C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S), ((G_I, F), u_{IJ}) \mapsto \text{ho} \lim_{I \subset L} p_{0(0I)*} \Gamma_{S_I}^\vee p_{I(0I)}^*(G_I, F)$.

Note that the functors $T(S/(\tilde{S}_I))$ are NOT embedding, since

$$\text{ad}(i_I^*, i_{I*})(j_I^* F) : i_I^* i_{I*} j_I^* F \rightarrow j_I^* F$$

are usu local equivalence but NOT isomorphism since we are dealing with the morphism of big sites $P(i_I) : \text{AnSp}(\mathbb{C})^{sm}/S_I \rightarrow \text{AnSp}(\mathbb{C})^{sm}/\tilde{S}_I$. However, these functors induces full embeddings

$$T(S/(\tilde{S}_I)) : D_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow D_{fil}(\text{AnSp}(\mathbb{C})^{sm}/(S/(\tilde{S}_I)))$$

since for $F \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$,

$$\text{ho} \lim_{I \subset L} p_{0(0I)*} \Gamma_{S_I} p_{I(0I)}^*(i_I^* j_I^* F) \rightarrow p_{0(0I)*} \Gamma_{S_I} j_I^* F$$

is an equivalence usu local.

Let $f : X \rightarrow S$ a morphism, with $X, S \in \text{AnSp}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ and $X = \cup_{i=1}^l X_i$ be affine open covers and $i_i : S_i \hookrightarrow \tilde{S}_i$, $i'_i : X_i \hookrightarrow \tilde{X}_i$ be closed embeddings. Let $f_i : \tilde{X}_i \rightarrow \tilde{S}_i$ be a lift of the morphism $f_i = f|_{X_i} : X_i \rightarrow S_i$. Then, $f_I = f|_{X_I} : X_I \rightarrow S_I = \cap_{i \in I} S_i \rightarrow \tilde{S}_I = \cap_{i \in I} \tilde{S}_i$ lift to the morphism

$$\tilde{f}_I = \prod_{i \in I} \tilde{f}_i : \tilde{X}_I \rightarrow \tilde{S}_I = \prod_{i \in I} \tilde{S}_i$$

Denote by $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ and $p'_{IJ} : \tilde{X}_J \rightarrow \tilde{X}_I$ the projections. Consider for $J \subset I$ the following commutative diagrams

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ j_{IJ} \uparrow & & \uparrow p_{IJ} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}, \quad D'_{IJ} = \begin{array}{ccc} X_I & \xrightarrow{i'_I} & \tilde{X}_I \\ j'_{IJ} \uparrow & & \uparrow p'_{IJ} \\ X_J & \xrightarrow{i'_J} & \tilde{X}_J \end{array}, \quad D_{fI} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ f_I \uparrow & & \uparrow \tilde{f}_I \\ X_I & \xrightarrow{i'_I} & \tilde{X}_I \end{array}$$

We have then following commutative diagram

$$\begin{array}{ccccc} & & X_I & \xrightarrow{n'_I} & \tilde{X}_I & \longleftarrow & \tilde{X}_I \setminus X_I \\ & \nearrow j'_{IJ} & \uparrow p'_{IJ} & & \uparrow p'_{IJ} & & \uparrow p'_{IJ} \\ i'_J : X_J & \xrightarrow{l_{IJ}} & X_I \times X_I \times \tilde{X}_J \xrightarrow{n'_I \times I} & \tilde{X}_J & \longleftarrow & \tilde{X}_J \setminus X_J \end{array}$$

whose square are cartesian. We then have the pullback functor

$$f^* : C_{(2)fil}(\text{AnSp}(\mathbb{C})^{sm}/S/(\tilde{S}_I)) \rightarrow C_{(2)fil}(\text{AnSp}(\mathbb{C})^{sm}/X/(\tilde{X}_I)), \\ ((G_I, F), u_{IJ}) \mapsto f^*((G_I, F), u_{IJ}) := (\Gamma_{X_I}^\vee \tilde{f}_I^*(G_I, F), \tilde{f}_J^* u_{IJ})$$

with

$$\tilde{f}_J^* u_{IJ} : \Gamma_{X_I}^\vee \tilde{f}_I^*(G_I, F) \xrightarrow{\text{ad}(p_{IJ}^*, p'_{IJ*})(-)} p'_{IJ*} p_{IJ}^* \Gamma_{X_I}^\vee \tilde{f}_I^*(G_I, F) \xrightarrow{T_{\sharp}(p_{IJ}, n'_I)(-)^{-1}} p'_{IJ*} \Gamma_{X_I \times \tilde{X}_J \setminus \tilde{X}_I}^\vee p_{IJ}^* \tilde{f}_I^*(G_I, F) \\ \xrightarrow{p'_{IJ*} \gamma_{\tilde{X}_J}^\vee(-)} p'_{IJ*} \Gamma_{X_J}^\vee p_{IJ}^* \tilde{f}_I^*(G_I, F) = p'_{IJ*} \Gamma_{X_J}^\vee \tilde{f}_J^* p_{IJ}^*(G_I, F) \xrightarrow{\Gamma_{X_J}^\vee \tilde{f}_J^* I(p_{IJ}, p_{IJ*})(-, -)(u_{IJ})} \Gamma_{X_J}^\vee \tilde{f}_J^*(G_J, F)$$

Let $(G, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S)$. Since, $j_I^* i'_I j_I^* f^*(G, F) = 0$, the morphism $T(D_{fI})(j_I^*(G, F)) : \tilde{f}_I^* i_{I*} j_I^*(G, F) \rightarrow i_{I*} j_I^* f^*(G, F)$ factors trough

$$T(D_{fI})(j_I^*(G, F)) : \tilde{f}_I^* i_{I*} j_I^*(G, F) \xrightarrow{\gamma_{\tilde{X}_I}^\vee(-)} \Gamma_{X_I}^\vee \tilde{f}_I^* i_{I*} j_I^*(G, F) \xrightarrow{T^\gamma(D_{fI})(j_I^*(G, F))} i_{I*} j_I^* f^*(G, F)$$

We have then, for $(G, F) \in C_{fil}(S)$, the canonical transformation map

$$\begin{array}{ccc} f^*T(S/(\tilde{S}_I))(G, F) & \xrightarrow{T(f, T(0/I))(G, F)} & T(X/(\tilde{X}_I))(f^*(G, F)) \\ \downarrow = & & \downarrow = \\ (\Gamma_{X_I}^\vee \tilde{f}_I^* i_{I^*} j_I^*(G, F), \tilde{f}_I^* I) & \xrightarrow{T^\gamma(D_{f_I})(j_I^*(G, F))} & (i'_{I^*} j_I^* f^*(G, F), I) \end{array}$$

We have similarly to the algebraic case, we have:

Definition 24. (i) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{AnSp}(\mathbb{C})$. Assume that there exist a factorization $f : X \xrightarrow{i} Y \times S \xrightarrow{p} S$, with $Y \in \text{AnSm}(\mathbb{C})$, $i : X \hookrightarrow Y$ is a closed embedding and p the projection. We then consider

$$Q(X/S) := p_{\sharp} \Gamma_X^\vee \mathbb{Z}_{Y \times S}[d_Y] := \text{Cone}(\mathbb{Z}((Y \times S) \setminus X/S) \rightarrow \mathbb{Z}(Y \times S/S))[d_Y] \in C(\text{AnSp}(\mathbb{C})^{sm}/S).$$

By definition $Q(X/S)$ is projective since it is a complex of two representative presheaves.

(ii) Let $f : X \rightarrow S$ and $g : T \rightarrow S$ two morphism with $X, S, T \in \text{AnSp}(\mathbb{C})$. Assume that there exist a factorization $f : X \xrightarrow{i} Y \times S \xrightarrow{p} S$, with $Y \in \text{AnSm}(\mathbb{C})$, $i : X \hookrightarrow Y$ is a closed embedding and p the projection. We then have the following commutative diagram whose squares are cartesian

$$\begin{array}{ccccc} f : X & \xrightarrow{i} & Y \times S & \xrightarrow{p} & S \\ g' \uparrow & & g'' := (I \times g) \uparrow & & g \uparrow \\ f' : X_T & \xrightarrow{i'} & Y \times T & \xrightarrow{p'} & T \end{array}$$

We then have the canonical isomorphism in $C(\text{AnSp}(\mathbb{C})^{sm}/T)$

$$\begin{aligned} T(f, g, Q) &:= T_{\sharp}(g, p)(-)^{-1} \circ T_{\sharp}(g'', j)(-)^{-1} : \\ g^* Q(X/S) &:= g^* p_{\sharp} \Gamma_X^\vee \mathbb{Z}_{Y \times S}[d_Y] \xrightarrow{\sim} p'_{\sharp} \Gamma_{X_T}^\vee \mathbb{Z}_{Y \times T}[d_Y] =: Q(X_T/T) \end{aligned}$$

with $j : Y \times S \setminus X \hookrightarrow Y \times S$ the closed embedding.

We now define the \mathbb{D}^1 localization property :

Definition 25. Let $S \in \text{Var}(\mathbb{C})$.

(i) A complex $F \in C(\text{AnSp}(\mathbb{C})^{(sm)}/S)$ is said to be \mathbb{D}^1 invariant if for all $U/S \in \text{AnSp}(\mathbb{C})^{(sm)}/S$,

$$F(p_U) : F(U/S) \rightarrow F(U \times \mathbb{D}^1/S)$$

is a quasi-isomorphism, where $p_U : U \times \mathbb{D}^1 \rightarrow U$ is the projection.

(ii) A complex $F \in C(\text{AnSp}(\mathbb{C})^{(sm)}/S)$ is said to be \mathbb{D}^1 local for the usual topology, if for a (hence every) usu local equivalence $k : F \rightarrow G$ with k injective and $G \in C(\text{AnSp}(\mathbb{C})^{(sm)}/S)$ usu fibrant, e.g. $k : F \rightarrow E_\tau(F)$, G is \mathbb{D}^1 invariant for all $n \in \mathbb{Z}$.

(iii) A morphism $m : F \rightarrow G$ with $F, G \in C(\text{AnSp}(\mathbb{C})^{(sm)}/S)$ is said to an (\mathbb{D}^1, usu) local equivalence if for all $H \in C(\text{AnSp}(\mathbb{C})^{(sm)}/S)$ which is \mathbb{A}^1 local for the etale topology

$$\text{Hom}(L(m), E_{usu}(H)) : \text{Hom}(L(G), E_{usu}(H)) \rightarrow \text{Hom}(L(F), E_{usu}(H))$$

is a quasi-isomorphism. Equivalently, $m : F \rightarrow G$ with $F, G \in C(\text{AnSp}(\mathbb{C})^{(sm)}/S)$ is an (\mathbb{D}^1, et) local equivalence if and only if there exists $X_1/S, \dots, X_r/S \in \text{AnSp}(\mathbb{C})^{(sm)}/S$ such that we have in $\text{Ho}_{et}(C(\text{Var}(\mathbb{C})^{(sm)}/S))$

$$\text{Cone}(m) \xrightarrow{\sim} \text{Cone}(\text{Cone}(\mathbb{Z}(X_1 \times \mathbb{D}^1/S) \rightarrow \mathbb{Z}(X_1/S)) \rightarrow \dots \rightarrow \text{Cone}(\mathbb{Z}(X_r \times \mathbb{D}^1/S) \rightarrow \mathbb{Z}(X_r/S)))$$

Proposition 25. *With the weak equivalence the (\mathbb{D}_S^1, usu) equivalence and the fibration the epimorphism with \mathbb{D}_S^1 local and usu fibrant kernels gives a model structure on $C(\text{AnSp}(\mathbb{C})^{sm}/S)$: the left bousfield localization of the projective model structure of $C(\text{AnSp}(\mathbb{C})^{sm}/S)$.*

Proof. See [10]. □

We will consider for the construction of the filtered De Rham realization functor the filtered case :

Definition 26. *Let $S \in \text{AnSp}(\mathbb{C})$.*

- (i) *Let $r \in \mathbb{N}$. A filtered complex $(G, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{(sm)}/S)$ is said to be r -filtered \mathbb{D}^1 invariant if for all $U/S \in \text{AnSp}(\mathbb{C})/S$ and all $p, q \in \mathbb{Z}$,*

$$E_r^{p,q}G(p_U) : E_r^{p,q}(G, F)(U/S) \xrightarrow{\sim} E_r^{p,q}(G, F)(U \times \mathbb{D}^1/S)$$

is an isomorphism of abelian group, where $p_U : U \times \mathbb{D}^1 \rightarrow U$ is the projection. Note that by definition this r does NOT depend on p and q .

- (ii) *A filtered complex $(G, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{(sm)}/S)$ is said to be ∞ -filtered \mathbb{D}^1 invariant if for all $U/S \in \text{AnSp}(\mathbb{C})^{(sm)}/S$, there exist $r \in \mathbb{N}$ such that for all $p, q \in \mathbb{Z}$,*

$$E_r^{p,q}G(p_U) : E_r^{p,q}(G, F)(U/S) \xrightarrow{\sim} E_r^{p,q}(G, F)(U \times \mathbb{D}^1/S)$$

is an isomorphism of abelian group, where $p_U : U \times \mathbb{D}^1 \rightarrow U$ is the projection (note that by definition this r does NOT depend on p and q but may depends on U/S). This implies that, for all $U/S \in \text{AnSp}(\mathbb{C})^{(sm)}/S$,

$$H^n G(p_U) : H^n(G, F)(U/S) \xrightarrow{\sim} H^n(G, F)(U \times \mathbb{D}^1/S)$$

is a filtered isomorphism of filtered abelian groups for all $n \in \mathbb{Z}$.

- (iii) *A filtered complex $(G, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{(sm)}/S)$ is said to be ∞ -filtered \mathbb{D}^1 local for the usual topology if for a (hence every) ∞ -filtered usu local equivalence $k : (G, F) \rightarrow (H, F)$ with k injective and $(H, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{(sm)}/S)$ ∞ -filtered usu fibrant, e.g. $k : (G, F) \rightarrow E_{usu}(G, F)$, (H, F) is ∞ -filtered \mathbb{D}^1 invariant.*

Lemma 3. *Let $S \in \text{Var}(\mathbb{C})$.*

- (i) *Let $(G, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S)$ Then, if $m : F_1 \rightarrow F_2$ with $F_1, F_2 \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$ is an usu local equivalence,*

$$\text{Hom}(m, E_{usu}(G, F)) : \text{Hom}^\bullet(F_2, E_{usu}(G, F)) \rightarrow \text{Hom}^\bullet(F_1, E_{usu}(G, F))$$

is a filtered quasi-isomorphism.

- (ii) *Let $(G, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S)$ be ∞ -filtered \mathbb{D}^1 local for the usual topology. Then, if $m : F_1 \rightarrow F_2$ with $F_1, F_2 \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$ is an (\mathbb{D}^1, et) local equivalence,*

$$\text{Hom}(m, E_{usu}(G, F)) : \text{Hom}^\bullet(F_2, E_{usu}(G, F)) \rightarrow \text{Hom}^\bullet(F_1, E_{usu}(G, F))$$

is an ∞ -filtered quasi-isomorphism.

Proof. Similar to lemma 1. □

Proposition 26. *Let $g : T \rightarrow S$ a morphism with $T, S \in \text{AnSp}(\mathbb{C})$.*

- (i) *The functor $g^* : C(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{sm}/T)$ sends quasi-isomorphism to quasi-isomorphism and equivalence usu local to equivalence usu local, sends (\mathbb{D}^1, usu) local equivalence to (\mathbb{D}^1, usu) local equivalence.*
- (ii) *The functor $g^* : C(\text{AnSp}(\mathbb{C})/S) \rightarrow C(\text{AnSp}(\mathbb{C})/T)$ sends quasi-isomorphism to quasi-isomorphism and equivalence usu local to equivalence usu local, sends (\mathbb{D}^1, usu) local equivalence to (\mathbb{D}^1, usu) local equivalence.*

Proof. Similar to the proof of proposition 19. □

2.9 Presheaves on the big analytical site of pairs

We recall the definition given in subsection 5.1 : For $S \in \text{AnSp}(\mathbb{C})$, $\text{AnSp}(\mathbb{C})^2/S := \text{AnSp}(\mathbb{C})^2/(S, S)$ is by definition (see subsection 2.1) the category whose set of objects is

$$(\text{AnSp}(\mathbb{C})^2/S)^0 := \{((X, Z), h), h : X \rightarrow S, Z \subset X \text{ closed}\} \subset \text{AnSp}(\mathbb{C})/S \times \text{Top}$$

and whose set of morphisms between $(X_1, Z_1)/S = ((X_1, Z_1), h_1), (X_1, Z_1)/S = ((X_2, Z_2), h_2) \in \text{AnSp}(\mathbb{C})^2/S$ is the subset

$$\begin{aligned} & \text{Hom}_{\text{AnSp}(\mathbb{C})^2/S}((X_1, Z_1)/S, (X_2, Z_2)/S) := \\ & \{(f : X_2 \rightarrow X_1), \text{ s.t. } h_1 \circ f = h_2 \text{ and } Z_1 \subset f^{-1}(Z_2)\} \subset \text{Hom}_{\text{AnSp}(\mathbb{C})}(X_1, X_2) \end{aligned}$$

The category $\text{AnSp}(\mathbb{C})^2$ admits fiber products : $(X_1, Z_1) \times_{(S, Z)} (X_2, Z_2) = (X_1 \times_S X_2, Z_1 \times_Z Z_2)$. In particular, for $f : T \rightarrow S$ a morphism with $S, T \in \text{AnSp}(\mathbb{C})$, we have the pullback functor

$$P(f) : \text{AnSp}(\mathbb{C})^2/S \rightarrow \text{AnSp}(\mathbb{C})^2/T, P(f)((X, Z)/S) := (X_T, Z_T)/T, P(f)(g) := (g \times_S f)$$

and we note again $P(f) : \text{AnSp}(\mathbb{C})^2/T \rightarrow \text{AnSp}(\mathbb{C})^2/S$ the corresponding morphism of sites.

We will consider in the construction of the filtered De Rham realization functor the full subcategory $\text{AnSp}(\mathbb{C})^{2,sm}/S \subset \text{AnSp}(\mathbb{C})^2/S$ such that the first factor is a smooth morphism : We will also consider, in order to obtain a complex of D modules in the construction of the filtered De Rham realization functor, the restriction to the full subcategory $\text{AnSp}(\mathbb{C})^{2,pr}/S \subset \text{AnSp}(\mathbb{C})^2/S$ such that the first factor is a projection :

Definition 27. (i) Let $S \in \text{AnSp}(\mathbb{C})$. We denote by

$$\rho_S : \text{AnSp}(\mathbb{C})^{2,sm}/S \hookrightarrow \text{AnSp}(\mathbb{C})^2/S$$

the full subcategory consisting of the objects $(U, Z)/S = ((U, Z), h) \in \text{AnSp}(\mathbb{C})^2/S$ such that the morphism $h : U \rightarrow S$ is smooth. That is, $\text{AnSp}(\mathbb{C})^{2,sm}/S$ is the category

- whose objects are $(U, Z)/S = ((U, Z), h)$, with $U \in \text{AnSp}(\mathbb{C})$, $Z \subset U$ a closed subset, and $h : U \rightarrow S$ a smooth morphism,
- whose morphisms $g : (U, Z)/S = ((U, Z), h_1) \rightarrow (U', Z')/S = ((U', Z'), h_2)$ is a morphism $g : U \rightarrow U'$ of complex algebraic varieties such that $Z \subset g^{-1}(Z')$ and $h_2 \circ g = h_1$.

We denote again $\rho_S : \text{AnSp}(\mathbb{C})^2/S \rightarrow \text{AnSp}(\mathbb{C})^{2,sm}/S$ the associated morphism of site. We have

$$r^s(S) : \text{AnSp}(\mathbb{C})^2 \xrightarrow{r(S) := r(S, S)} \text{AnSp}(\mathbb{C})^2/S \xrightarrow{\rho_S} \text{AnSp}(\mathbb{C})^{2,sm}/S$$

the composite morphism of site.

(ii) Let $S \in \text{AnSp}(\mathbb{C})$. We will consider the full subcategory

$$\mu_S : \text{AnSp}(\mathbb{C})^{2,pr}/S \hookrightarrow \text{AnSp}(\mathbb{C})^2/S$$

whose subset of object consist of those whose morphism is a projection to S :

$$(\text{AnSp}(\mathbb{C})^{2,pr}/S)^0 := \{((Y \times S, X), p), Y \in \text{AnSp}(\mathbb{C}), p : Y \times S \rightarrow S \text{ the projection}\} \subset (\text{AnSp}(\mathbb{C})^2/S)^0.$$

(iii) We will consider the full subcategory

$$\mu_S : (\text{AnSp}(\mathbb{C})^{2,smpr}/S) \hookrightarrow \text{AnSp}(\mathbb{C})^{2,sm}/S$$

whose subset of object consist of those whose morphism is a smooth projection to S :

$$(\text{AnSp}(\mathbb{C})^{2,smpr}/S)^0 := \{((Y \times S, X), p), Y \in \text{SmVar}(\mathbb{C}), p : Y \times S \rightarrow S \text{ the projection}\} \subset (\text{AnSp}(\mathbb{C})^2/S)^0$$

For $f : T \rightarrow S$ a morphism with $T, S \in \text{AnSp}(\mathbb{C})$, we have by definition, the following commutative diagram of sites

$$\begin{array}{ccccc}
\text{AnSp}(\mathbb{C})^2/T & \xrightarrow{\mu_T} & \text{AnSp}(\mathbb{C})^{2,pr}/T & & \\
\downarrow P(f) & \searrow \rho_T & \downarrow H_T & \searrow \rho_T & \\
& & \text{AnSp}(\mathbb{C})^{2,sm}/T & \xrightarrow{P(f)} & \text{AnSp}(\mathbb{C})^{2,smpr}/T \\
& & \downarrow P(f) & & \downarrow P(f) \\
\text{AnSp}(\mathbb{C})^2/S & \xrightarrow{P(f)} & \text{AnSp}(\mathbb{C})^{2,pr}/S & & \text{AnSp}(\mathbb{C})^{2,smpr}/S \\
& \searrow \rho_S & \downarrow P(f) & \searrow \rho_S & \\
& & \text{AnSp}(\mathbb{C})^{2,sm}/S & \xrightarrow{\mu_S} & \text{AnSp}(\mathbb{C})^{2,smpr}/S
\end{array} \quad (36)$$

Recall we have (see subsection 2.1), for $S \in \text{Var}(\mathbb{C})$, the graph functor

$$\begin{aligned}
\text{Gr}_S^{12} : \text{AnSp}(\mathbb{C})/S &\rightarrow \text{AnSp}(\mathbb{C})^{2,pr}/S, \quad X/S \mapsto \text{Gr}_S^{12}(X/S) := (X \times S, X)/S, \\
(g : X/S \rightarrow X'/S) &\mapsto \text{Gr}_S^{12}(g) := (g \times I_S : (X \times S, X) \rightarrow (X' \times S, X'))
\end{aligned}$$

For $f : T \rightarrow S$ a morphism with $T, S \in \text{AnSp}(\mathbb{C})$, we have by definition, the following commutative diagram of sites

$$\begin{array}{ccccc}
\text{AnSp}(\mathbb{C})^{2,pr}/T & \xrightarrow{\text{Gr}_T^{12}} & \text{AnSp}(\mathbb{C})/T & & \\
\downarrow P(f) & \searrow \rho_T & \downarrow \text{Gr}_T^{12} & \searrow \rho_T & \\
& & \text{AnSp}(\mathbb{C})^{2,smpr}/T & \xrightarrow{P(f)} & \text{AnSp}(\mathbb{C})^{sm}/T \\
& & \downarrow P(f) & & \downarrow P(f) \\
\text{AnSp}(\mathbb{C})^{2,pr}/S & \xrightarrow{P(f)} & \text{AnSp}(\mathbb{C})/S & & \text{AnSp}(\mathbb{C})^{sm}/S \\
& \searrow \rho_S & \downarrow P(f) & \searrow \rho_S & \\
& & \text{AnSp}(\mathbb{C})^{2,sm}/S & \xrightarrow{\text{Gr}_S^{12}} & \text{AnSp}(\mathbb{C})^{sm}/S
\end{array} \quad (37)$$

where we recall that $P(f)((X, Z)/S) := ((X_T, Z_T)/T)$, since smooth morphisms are preserved by base change.

As usual, we denote by

$$(f^*, f_*) := (P(f)^*, P(f)_*) : C(\text{AnSp}(\mathbb{C})^{2,sm}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{2,sm}/T)$$

the adjunction induced by $P(f) : \text{AnSp}(\mathbb{C})^{2,sm}/T \rightarrow \text{AnSp}(\mathbb{C})^{2,sm}/S$. Since the colimits involved in the definition of $f^* = P(f)^*$ are filtered, f^* also preserve monomorphism. Hence, we get an adjunction

$$(f^*, f_*) : C_{fil}(\text{AnSp}(\mathbb{C})^{2,sm}/S) \rightleftarrows C_{fil}(\text{AnSp}(\mathbb{C})^{2,sm}/T), \quad f^*(G, F) := (f^*G, f^*F)$$

For $S \in \text{AnSp}(\mathbb{C})$, we denote by $\mathbb{Z}_S := \mathbb{Z}((S, S)/(S, S)) \in \text{PSh}(\text{AnSp}(\mathbb{C})^{2,sm}/S)$ the constant presheaf. By Yoneda lemma, we have for $F \in C(\text{AnSp}(\mathbb{C})^{2,sm}/S)$, $\mathcal{H}om(\mathbb{Z}_S, F) = F$.

For $h : U \rightarrow S$ a smooth morphism with $U, S \in \text{AnSp}(\mathbb{C})$, $P(h) : \text{AnSp}(\mathbb{C})^{2,sm}/S \rightarrow \text{AnSp}(\mathbb{C})^{2,sm}/U$ admits a left adjoint

$$C(h) : \text{AnSp}(\mathbb{C})^{2,sm}/U \rightarrow \text{AnSp}(\mathbb{C})^{2,sm}/S, \quad C(h)((U', Z'), h') = ((U', Z'), h \circ h').$$

Hence $h^* : C(\text{AnSp}(\mathbb{C})^{2,sm}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{2,sm}/U)$ admits a left adjoint

$$h_{\#} : C(\text{AnSp}(\mathbb{C})^{2,sm}/U) \rightarrow C(\text{AnSp}(\mathbb{C})^{2,sm}/S), F \mapsto (h_{\#}F : ((U, Z), h_0) \mapsto \lim_{((U', Z'), h_0 h') \rightarrow ((U, Z), h_0)} F((U', Z')/U))$$

For $F^{\bullet} \in C(\text{AnSp}(\mathbb{C})^{2,sm}/S)$ and $G^{\bullet} \in C(\text{AnSp}(\mathbb{C})^{2,sm}/U)$, we have the adjunction maps

$$\text{ad}(h_{\#}, h^*)(G^{\bullet}) : G^{\bullet} \rightarrow h^* h_{\#} G^{\bullet}, \text{ad}(h_{\#}, h^*)(F^{\bullet}) : h_{\#} h^* F^{\bullet} \rightarrow F^{\bullet}.$$

For a smooth morphism $h : U \rightarrow S$, with $U, S \in \text{AnSp}(\mathbb{C})$, we have the adjunction isomorphism, for $F \in C(\text{AnSp}(\mathbb{C})^{2,sm}/U)$ and $G \in C(\text{AnSp}(\mathbb{C})^{2,sm}/S)$,

$$I(h_{\#}, h^*)(F, G) : \mathcal{H}om^{\bullet}(h_{\#}F, G) \xrightarrow{\sim} h_* \mathcal{H}om^{\bullet}(F, h^*G). \quad (38)$$

For a commutative diagram in $\text{AnSp}(\mathbb{C})$:

$$D = \begin{array}{ccc} V & \xrightarrow{g_2} & U \\ \downarrow h_2 & & \downarrow h_1 \\ T & \xrightarrow{g_1} & S \end{array},$$

where h_1 and h_2 are smooth, we denote by, for $F^{\bullet} \in C(\text{AnSp}(\mathbb{C})^{2,sm}/U)$,

$$T_{\#}(D)(F^{\bullet}) : h_{2\#} g_2^* F^{\bullet} \rightarrow g_1^* h_{1\#} F^{\bullet}$$

the canonical map given by adjunction. If D is cartesian with $h_1 = h$, $g_1 = g$, $f_2 = h' : U_T \rightarrow T$, $g' : U_T \rightarrow U$,

$$T_{\#}(D)(F^{\bullet}) =: T_{\#}(g, h)(F) : h'_* g'^* F^{\bullet} \xrightarrow{\sim} g^* h_{\#} F^{\bullet}$$

is an isomorphism.

We have the support section functors of a closed embedding $i : Z \hookrightarrow S$ for presheaves on the big analytical site of pairs.

Definition 28. Let $i : Z \hookrightarrow S$ be a closed embedding with $S, Z \in \text{AnSp}(\mathbb{C})$ and $j : S \setminus Z \hookrightarrow S$ be the open complementary subset.

(i) We define the functor

$$\Gamma_Z : C(\text{AnSp}(\mathbb{C})^{2,sm}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{2,sm}/S), G^{\bullet} \mapsto \Gamma_Z G^{\bullet} := \text{Cone}(\text{ad}(j^*, j_*)(G^{\bullet}) : G^{\bullet} \rightarrow j_* j^* G^{\bullet})[-1],$$

so that there is then a canonical map $\gamma_Z(G^{\bullet}) : \Gamma_Z G^{\bullet} \rightarrow G^{\bullet}$.

(ii) We have the dual functor of (i) :

$$\Gamma_Z^{\vee} : C(\text{AnSp}(\mathbb{C})^{2,sm}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{2,sm}/S), F \mapsto \Gamma_Z^{\vee}(F^{\bullet}) := \text{Cone}(\text{ad}(j_{\#}, j^*)(G^{\bullet}) : j_{\#} j^* G^{\bullet} \rightarrow G^{\bullet}),$$

together with the canonical map $\gamma_Z^{\vee}(G) : F \rightarrow \Gamma_Z^{\vee}(G)$.

(iii) For $F, G \in C(\text{AnSp}(\mathbb{C})^{2,sm}/S)$, we denote by

$$I(\gamma, \text{hom})(F, G) := (I, I(j_{\#}, j^*)(F, G)^{-1}) : \Gamma_Z \mathcal{H}om(F, G) \xrightarrow{\sim} \mathcal{H}om(\Gamma_Z^{\vee} F, G)$$

the canonical isomorphism given by adjunction.

Note that we have similarly for $i : Z \hookrightarrow S$, $i' : Z' \hookrightarrow Z$ closed embeddings, $g : T \rightarrow S$ a morphism with $T, S, Z \in \text{AnSp}(\mathbb{C})$ and $F \in C(\text{AnSp}(\mathbb{C})^{2,sm}/S)$, the canonical maps in $C(\text{AnSp}(\mathbb{C})^{2,sm}/S)$

- $T(g, \gamma)(F) : g^* \Gamma_Z F \xrightarrow{\sim} \Gamma_{Z \times_S T} g^* F$, $T(g, \gamma^{\vee})(F) : \Gamma_{Z \times_S T}^{\vee} g^* F \xrightarrow{\sim} g^* \Gamma_Z F$

- $T(Z'/Z, \gamma)(F) : \Gamma_Z F \rightarrow \Gamma_Z F$, $T(Z'/Z, \gamma^\vee)(F) : \Gamma_Z^\vee F \rightarrow \Gamma_Z^\vee F$

but we will not use them in this article.

We now define the usual topology on $\text{AnSp}(\mathbb{C})^2/S$.

Definition 29. Let $S \in \text{AnSp}(\mathbb{C})$.

- (i) Denote by τ a topology on $\text{AnSp}(\mathbb{C})$, e.g. the usual topology. The τ covers in $\text{AnSp}(\mathbb{C})^2/S$ of $(X, Z)/S$ are the families of morphisms

$$\{(c_i : (U_i, Z \times_X U_i)/S \rightarrow (X, Z)/S)_{i \in I}, \text{ with } (c_i : U_i \rightarrow X)_{i \in I} \tau \text{ cover of } X \text{ in } \text{AnSp}(\mathbb{C})\}$$

- (ii) Denote by τ the usual or the etale topology on $\text{AnSp}(\mathbb{C})$. The τ covers in $\text{AnSp}(\mathbb{C})^{2,sm}/S$ of $(U, Z)/S$ are the families of morphisms

$$\{(c_i : (U_i, Z \times_U U_i)/S \rightarrow (U, Z)/S)_{i \in I}, \text{ with } (c_i : U_i \rightarrow U)_{i \in I} \tau \text{ cover of } U \text{ in } \text{AnSp}(\mathbb{C})\}$$

- (iii) Denote by τ the usual or the etale topology on $\text{AnSp}(\mathbb{C})$. The τ covers in $\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S$ of $(Y \times S, Z)/S$ are the families of morphisms

$$\{(c_i \times I_S : (U_i \times S, Z \times_{Y \times S} U_i \times S)/S \rightarrow (Y \times S, Z)/S)_{i \in I}, \text{ with } (c_i : U_i \rightarrow Y)_{i \in I} \tau \text{ cover of } Y \text{ in } \text{AnSp}(\mathbb{C})\}$$

Let $S \in \text{AnSp}(\mathbb{C})$. Denote by τ the usual topology on $\text{AnSp}(\mathbb{C})$. In particular, denoting $a_\tau : \text{PSh}(\text{AnSp}(\mathbb{C})^{2,(sm)}/S) \rightarrow \text{Shv}(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$ and $a_\tau : \text{PSh}(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S) \rightarrow \text{Shv}(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ the sheafification functors,

- a morphism $\phi : F \rightarrow G$, with $F, G \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$, is a τ local equivalence if $a_\tau H^n \phi : a_\tau H^n F \rightarrow a_\tau H^n G$ is an isomorphism, a morphism $\phi : F \rightarrow G$, with $F, G \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$, is a τ local equivalence if $a_\tau H^n \phi : a_\tau H^n F \rightarrow a_\tau H^n G$ is an isomorphism,
- a morphism $\phi : (G_1, F) \rightarrow (G_2, F)$, with $(G_1, F), (G_2, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$, is an r -filtered τ local equivalence if for all $p, q \in \mathbb{Z}$, $a_\tau E_r^{p,q} \phi : a_\tau E_r^{p,q}(G_1, F) \rightarrow a_\tau E_r^{p,q}(G_2, F)$ is an isomorphism of sheaves on $\text{AnSp}(\mathbb{C})^{2,(sm)}/S$, a morphism $\phi : (G_1, F) \rightarrow (G_2, F)$, with $(G_1, F), (G_2, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$, is an r -filtered τ local equivalence if for all $p, q \in \mathbb{Z}$ $a_\tau E_r^{p,q} \phi : a_\tau E_r^{p,q}(G_1, F) \rightarrow a_\tau E_r^{p,q}(G_2, F)$ is an isomorphism of sheaves on $\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S$,
- $F^\bullet \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$ is τ fibrant if for all $(U, Z)/S \in \text{AnSp}(\mathbb{C})^{2,sm}/S$ and all τ covers $(c_i : (U_i, Z \times_U U_i)/S \rightarrow (U, Z)/S)_{i \in I}$ of $(U, Z)/S$,

$$F^\bullet(c_i) : F^\bullet((U, Z)/S) \rightarrow \text{Tot}(\oplus_{card I = \bullet} F^\bullet((U_i, Z \times_U U_i)/S))$$

is a quasi-isomorphism of complexes of abelian groups, $F^\bullet \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ is τ fibrant if for all $(Y \times S, Z)/S \in \text{AnSp}(\mathbb{C})^{2,smpr}/S$ and all τ covers $(c_i \times I_S : (U_i \times S, Z \times_{Y \times S} U_i \times S)/S \rightarrow (Y \times S, Z)/S)_{i \in I}$ of $(Y \times S, Z)/S$,

$$F^\bullet(c_i \times I_S) : F^\bullet((Y \times S, Z)/S) \rightarrow \text{Tot}(\oplus_{card I = \bullet} F^\bullet((U_i \times S, Z \times_{Y \times S} U_i \times S)/S))$$

is a quasi-isomorphism of complexes of abelian groups,

- $(F^\bullet, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,sm}/S)$ is r -filtered τ fibrant if for all $(U, Z)/S \in \text{AnSp}(\mathbb{C})^{2,(sm)}/S$ and all τ covers $(c_i : (U_i, Z \times_U U_i)/S \rightarrow (U, Z)/S)_{i \in I}$ of $(U, Z)/S$,

$$E_r^{p,q}(F^\bullet, F)(c_i) : E_r^{p,q}(F^\bullet, F)((U, Z)/S) \rightarrow E_r^{p,q}(\text{Tot}(\oplus_{card I = \bullet} (F^\bullet, F)((U_i, Z \times_U U_i)/S)))$$

is an isomorphism of abelian groups for all $p, q \in \mathbb{Z}$, $(F^\bullet, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ is r -filtered τ fibrant if for all $(Y \times S, Z)/S \in \text{AnSp}(\mathbb{C})^{2,(sm)pr}/S$ and all τ covers $(c_i \times I_S : (U_i \times S, Z \times_{Y \times S} U_i \times S)/S \rightarrow (Y \times S, Z)/S)_{i \in I}$ of $(Y \times S, Z)/S$,

$$E_r^{p,q}(F^\bullet, F)(c_i \times I_S) : E_r^{p,q}(F^\bullet, F)((Y \times S, Z)/S) \rightarrow E_r^{p,q}(\text{Tot}(\oplus_{card I = \bullet} (F^\bullet, F)((U_i \times S, Z \times_{Y \times S} U_i \times S)/S)))$$

is an isomorphism of abelian groups for all $p, q \in \mathbb{Z}$.

Will now define the \mathbb{D}^1 local property on $\text{AnSp}(\mathbb{C})^2/S$.

Definition 30. Let $S \in \text{AnSp}(\mathbb{C})$.

(i) A complex $F \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$, is said to be \mathbb{D}^1 invariant if for all $(X, Z)/S \in \text{AnSp}(\mathbb{C})^{2,(sm)}/S$

$$F(p_X) : F((X, Z)/S) \rightarrow F((X \times \mathbb{D}^1, (Z \times \mathbb{D}^1))/S)$$

is a quasi-isomorphism, where $p_X : (X \times \mathbb{D}^1, (Z \times \mathbb{D}^1)) \rightarrow (X, Z)$ is the projection.

(i)' Similarly, a complex $F \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ is said to be \mathbb{D}^1 invariant if for all $(Y \times S, Z)/S \in \text{AnSp}(\mathbb{C})^{2,(sm)pr}/S$

$$F(p_{Y \times S}) : F((Y \times S, Z)/S) \rightarrow F((Y \times S \times \mathbb{D}^1, (Z \times \mathbb{D}^1))/S)$$

is a quasi-isomorphism

(ii) A complex $F \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$ is said to be \mathbb{D}^1 local for the τ topology induced on $\text{AnSp}(\mathbb{C})^2/S$, if for an (hence every) τ local equivalence $k : F \rightarrow G$ with k injective and $G \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$ τ fibrant, e.g. $k : F \rightarrow E_\tau(F)$, G is \mathbb{D}^1 invariant.

(ii)' Similarly, a complex $F \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ is said to be \mathbb{D}^1 local for the τ topology induced on $\text{AnSp}(\mathbb{C})^2/S$, if for an (hence every) τ local equivalence $k : F \rightarrow G$ with k injective and $G \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ τ fibrant, e.g. $k : F \rightarrow E_\tau(F)$, G is \mathbb{D}^1 invariant.

(iii) A morphism $m : F \rightarrow G$ with $F, G \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$ is said to be an (\mathbb{D}^1, usu) local equivalence if for all $H \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$ which is \mathbb{D}^1 local for the usual topology

$$\text{Hom}(L(m), E_{usu}(H)) : \text{Hom}(L(G), E_{usu}(H)) \rightarrow \text{Hom}(L(F), E_{usu}(H))$$

is a quasi-isomorphism.

(iii)' Similarly, a morphism $m : F \rightarrow G$ with $F, G \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ is said to be an (\mathbb{D}^1, usu) local equivalence if for all $H \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ which is \mathbb{D}^1 local for the usual topology

$$\text{Hom}(L(m), E_{usu}(H)) : \text{Hom}(L(G), E_{usu}(H)) \rightarrow \text{Hom}(L(F), E_{usu}(H))$$

is a quasi-isomorphism.

Equivalently, $m : F \rightarrow G$ with $F, G \in C(\text{AnSp}(\mathbb{C})^{2,sm}/S)$ is an (\mathbb{D}^1, usu) local equivalence if and only if there exists $(X_1, Z_1)/S, \dots, (X_r, Z_r)/S \in \text{AnSp}(\mathbb{C})^{2,sm}/S$ such that we have in $\text{Ho}_{et}(C(\text{AnSp}(\mathbb{C})^{2,sm}/S))$

$$\begin{aligned} \text{Cone}(m) &\xrightarrow{\sim} \text{Cone}(\text{Cone}(\mathbb{Z}((X_1 \times \mathbb{D}^1, Z_1 \times \mathbb{D}^1)/S) \rightarrow \mathbb{Z}((X_1, Z_1)/S)) \\ &\rightarrow \dots \rightarrow \text{Cone}(\mathbb{Z}((X_1 \times \mathbb{D}^1, Z_1 \times \mathbb{D}^1)/S) \rightarrow \mathbb{Z}((X_1, Z_1)/S)) \end{aligned}$$

In the filtered case, we consider :

Definition 31. Let $S \in \text{AnSp}(\mathbb{C})$.

(i) Let $r \in \mathbb{N}$. A filtered complex $(G, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$, is said to be r -filtered \mathbb{D}^1 invariant if for all $(X, Z)/S \in \text{AnSp}(\mathbb{C})^{2,(sm)}/S$

$$E_r^{p,q}(G, F)(p_X) : (G, F)((X, Z)/S) \rightarrow E_r^{p,q}(G, F)((X \times \mathbb{D}^1, (Z \times \mathbb{D}^1))/S)$$

is an isomorphism of abelian groups for all $p, q \in \mathbb{Z}$, where $p_X : (X \times \mathbb{D}^1, (Z \times \mathbb{D}^1)) \rightarrow (X, Z)$ is the projection. Note that by definition this r does NOT depend on p and q .

(i)' Let $r \in \mathbb{N}$. A filtered complex $(G, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$, is said to be r -filtered \mathbb{D}^1 invariant if for all $(Y \times S, Z)/S \in \text{AnSp}(\mathbb{C})^{2,(sm)pr}/S$

$$E_r^{p,q}(G, F)(p_{Y \times S}) : E_r^{p,q}(G, F)((Y \times S, Z)/S) \rightarrow E_r^{p,q}(G, F)((Y \times \mathbb{D}^1 \times S, (Z \times \mathbb{D}^1))/S)$$

is an isomorphism of abelian group for all $p, q \in \mathbb{Z}$. Note that by definition this r does NOT depend on p and q .

(ii) A filtered complex $(G, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,sm}/S)$, is said to be ∞ -filtered \mathbb{D}^1 invariant if for all $(X, Z)/S \in \text{AnSp}(\mathbb{C})^{2,(sm)pr}/S$ there exist $r \in \mathbb{N}$ such that

$$E_r^{p,q}(G, F)(p_X) : E_r^{p,q}(G, F)((X, Z)/S) \rightarrow E_r^{p,q}(G, F)((X \times \mathbb{D}^1, (Z \times \mathbb{D}^1))/S)$$

is an isomorphism of abelian groups for all $p, q \in \mathbb{Z}$, where $p_X : (X \times \mathbb{D}^1, (Z \times \mathbb{D}^1)) \rightarrow (X, Z)$ is the projection (note that by definition this r does NOT depend on p and q but may depends on $(X, Z)/S$). This implies that, for all $(X, Z)/S \in \text{AnSp}(\mathbb{C})^{2,(sm)pr}/S$,

$$H^n(G, F)(p_X) : H^n(G, F)((X, Z)/S) \rightarrow H^n(G, F)((X \times \mathbb{D}^1, (Z \times \mathbb{D}^1))/S)$$

is a filtered isomorphism of filtered abelian groups for all $n \in \mathbb{Z}$.

(ii)' Similarly, a filtered complex $(G, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$, is said to be ∞ -filtered \mathbb{D}^1 invariant if for all $(Y \times S, Z)/S \in \text{AnSp}(\mathbb{C})^{2,(sm)pr}/S$ there exist $r \in \mathbb{N}$ such that

$$E_r^{p,q}(G, F)(p_{Y \times S}) : E_r^{p,q}(G, F)((Y \times S, Z)/S) \rightarrow E_r^{p,q}(G, F)((Y \times \mathbb{A}^1 \times S, (Z \times \mathbb{A}^1))/S)$$

is an isomorphism of abelian group for all $p, q \in \mathbb{Z}$ (note that by definition this r does NOT depend on p and q by may depend on $(Y \times S, Z)/S$). This implies that, for all $(X, Z)/S \in \text{AnSp}(\mathbb{C})^{2,(sm)pr}/S$,

$$H^n(G, F)(p_{Y \times S}) : H^n(G, F)((Y \times S, Z)/S) \rightarrow H^n(G, F)((Y \times S \times \mathbb{D}^1, (Z \times \mathbb{D}^1))/S)$$

is a filtered isomorphism of filtered abelian groups for all $n \in \mathbb{Z}$.

(iii) A filtered complex $(G, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$, is said to be ∞ -filtered \mathbb{D}^1 local for the τ topology induced on $\text{AnSp}(\mathbb{C})^2/S$, if for an (hence every) ∞ -filtered τ local equivalence $k : (G, F) \rightarrow (H, F)$ with k injective and $(H, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$, ∞ -filtered τ fibrant, e.g. $k : (G, F) \rightarrow E_\tau(G, F)$, (H, F) is ∞ -filtered \mathbb{D}^1 invariant.

(iii)' Similarly, a filtered complex $(G, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$, is said to be ∞ -filtered \mathbb{D}^1 local for the τ topology induced on $\text{AnSp}(\mathbb{C})^2/S$, if for an (hence every) ∞ -filtered τ local equivalence $k : (G, F) \rightarrow (H, F)$ with k injective and $(H, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$, ∞ -filtered τ fibrant, e.g. $k : (G, F) \rightarrow E_\tau(G, F)$, (H, F) is ∞ -filtered \mathbb{D}^1 invariant.

Lemma 4. Let $S \in \text{Var}(\mathbb{C})$.

(i) Let $(G, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,smpr}/S)$ Then, if $m : F_1 \rightarrow F_2$ with $F_1, F_2 \in C(\text{AnSp}(\mathbb{C})^{2,smpr}/S)$ is an usu local equivalence,

$$\text{Hom}(m, E_{usu}(G, F)) : \text{Hom}^\bullet(F_2, E_{usu}(G, F)) \rightarrow \text{Hom}^\bullet(F_1, E_{usu}(G, F))$$

is a filtered quasi-isomorphism.

(ii) Let $(G, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,smpr}/S)$ be ∞ -filtered \mathbb{D}^1 local for the etale topology. Then, if $m : F_1 \rightarrow F_2$ with $F_1, F_2 \in C(\text{AnSp}(\mathbb{C})^{2,smpr}/S)$ is an (\mathbb{D}^1, usu) local equivalence,

$$\text{Hom}(m, E_{usu}(G, F)) : \text{Hom}^\bullet(F_2, E_{usu}(G, F)) \rightarrow \text{Hom}^\bullet(F_1, E_{usu}(G, F))$$

is an ∞ -filtered quasi-isomorphism.

Proof. Similar to lemma 2 : follows from lemma 3. \square

We have, similarly to the case of single varieties the following :

Proposition 27. *Let $g : T \rightarrow S$ a morphism with $T, S \in \text{AnSp}(\mathbb{C})$.*

- (i) *The functor $g^* : C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{2,(sm)}/T)$ sends quasi-isomorphism to quasi-isomorphism and equivalence usu local to equivalence usu local, sends (\mathbb{D}^1, usu) local equivalence to (\mathbb{D}^1, usu) local equivalence.*
- (ii) *The functor $g^* : C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/T)$ sends quasi-isomorphism to quasi-isomorphism and equivalence usu local to equivalence usu local, sends (\mathbb{D}^1, usu) local equivalence to (\mathbb{D}^1, usu) local equivalence.*

Proof. Similar to the proof of proposition 21. \square

We also have

Proposition 28. *Let $S \in \text{AnSp}(\mathbb{C})$.*

- (i) *The functor $\text{Gr}_S^{12*} : C(\text{AnSp}(\mathbb{C})/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{2,pr}/S)$ sends usu local equivalence to usu local equivalence.*
- (i)' *The functor $\text{Gr}_S^{12*} : C(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{2,pr}/S)$ sends usu local equivalence to usu local equivalence.*
- (ii) *The functor $\text{Gr}_S^{12*} : C(\text{AnSp}(\mathbb{C})/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{2,pr}/S)$ sends (\mathbb{D}^1, usu) local equivalence to (\mathbb{D}^1, usu) local equivalence.*
- (ii)' *The functor $\text{Gr}_S^{12*} : C(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{2,smpr}/S)$ sends (\mathbb{D}^1, usu) local equivalence to (\mathbb{D}^1, usu) local equivalence.*

Proof. Similar to the proof of proposition 23. \square

We have the following canonical functor :

Definition 32. (i) *For $S \in \text{AnSp}(\mathbb{C})$, we have the functor*

$$\begin{aligned}
(-)^\Gamma &: C(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{2,sm}/S), \\
F &\longmapsto F^\Gamma : (((U, Z)/S) = ((U, Z), h) \mapsto F^\Gamma((U, Z)/S) := (\Gamma_Z^\vee h^* LF)(U/U), \\
&\quad (g : ((U', Z'), h') \rightarrow ((U, Z), h)) \mapsto \\
(F^\Gamma(g) : (\Gamma_Z^\vee h^* LF)(U/U) &\xrightarrow{i_{(\Gamma_Z^\vee h^* LF)(U/U)}} (g^*(\Gamma_Z^\vee h^* LF))(U'/U')) \\
&\quad \xrightarrow{T(g, \gamma^\vee)(h^* LF)(U'/U')} (\Gamma_{Z \times_U U'}^\vee g^* h^* LF)(U'/U') \\
&\quad \xrightarrow{T(Z'/Z \times_U U', \gamma^\vee)(g^* h^* LF)(U'/U')} (\Gamma_{Z'}^\vee g^* h^* LF)(U'/U'))
\end{aligned}$$

where $i_{(\Gamma_Z^\vee h^* LF)(U/U)}$ is the canonical arrow of the inductive limit. Similarly, we have, for $S \in \text{AnSp}(\mathbb{C})$, the functor

$$\begin{aligned}
(-)^\Gamma &: C(\text{AnSp}(\mathbb{C})/S) \rightarrow C(\text{AnSp}(\mathbb{C})^2/S), \\
F &\longmapsto F^\Gamma : (((X, Z)/S) = ((X, Z), h) \mapsto F^\Gamma((X, Z)/S) := (\Gamma_Z^\vee h^* F)(X/X), \\
&\quad (g : ((X', Z'), h') \rightarrow ((X, Z), h)) \mapsto (F^\Gamma(g) : (\Gamma_Z^\vee h^* LF)(X/X) \rightarrow (\Gamma_{Z'}^\vee h'^* LF)(X'/X'))
\end{aligned}$$

Note that for $S \in \text{AnSp}(\mathbb{C})$, $I(S/S) : \mathbb{Z}((S, S)/S) \rightarrow \mathbb{Z}(S/S)^\Gamma$ given by

$$\begin{aligned}
I(S/S)((U, Z), h) &: \mathbb{Z}((S, S)/S)((U, Z), h) \xrightarrow{\gamma_Z^\vee(\mathbb{Z}(U/U))(U/U)} \mathbb{Z}(S/S)^\Gamma((U, Z), h) := (\Gamma_Z^\vee \mathbb{Z}(U/U))(U/U) \\
&\text{is an isomorphism.}
\end{aligned}$$

(ii) Let $f : T \rightarrow S$ a morphism with $T, S \in \text{AnSp}(\mathbb{C})$. For $F \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$, we have the canonical morphism in $C(\text{AnSp}(\mathbb{C})^{2,sm}/T)$

$$\begin{aligned} T(f, \Gamma)(F) &:= T^*(f, \Gamma)(F) : f^*(F^\Gamma) \rightarrow (f^*F)^\Gamma, \\ T(f, \Gamma)(F)((U', Z')/T = ((U', Z'), h')) &:= \\ f^*(F^\Gamma)((U', Z'), h') &:= \lim_{((U', Z'), h') \xrightarrow{l} ((U_T, Z_T), h_T) \xrightarrow{f_U} ((U, Z), h)} (\Gamma_Z^\vee h^* LF)(U/U) \\ &\xrightarrow{F^\Gamma(f_U \circ l)} (\Gamma_Z^\vee l^* f_U^* h^* LF)(U'/U') = (\Gamma_Z^\vee h'^* f^* LF)(U'/U') \\ &\xrightarrow{(\Gamma_Z^\vee h'^* T(f, L)(F))(U'/U')} (\Gamma_Z^\vee h'^* L f^* F)(U'/U') =: (f^*F)^\Gamma((U', Z'), h') \end{aligned}$$

where $f_U : U_T : U \times_S T \rightarrow U$ and $h_T : U_T := U \times_S T \rightarrow T$ are the base change maps, the equality following from the fact that $h \circ f_U \circ l = f \circ h_T \circ l = f \circ h'$. For $F \in C(\text{AnSp}(\mathbb{C})/S)$, we have similarly the canonical morphism in $C(\text{AnSp}(\mathbb{C})^2/T)$

$$T(f, \Gamma)(F) : f^*(F^\Gamma) \rightarrow (f^*F)^\Gamma.$$

(iii) Let $h : U \rightarrow S$ a smooth morphism with $U, S \in \text{AnSp}(\mathbb{C})$. We have, for $F \in C(\text{AnSp}(\mathbb{C})^{sm}/U)$, the canonical morphism in $C(\text{AnSp}(\mathbb{C})^{2,sm}/S)$

$$\begin{aligned} T_\sharp(h, \Gamma)(F) &: h_\sharp(F^\Gamma) \rightarrow (h_\sharp LF)^\Gamma, \\ T_\sharp(h, \Gamma)(F)((U', Z'), h') &: h_\sharp(F^\Gamma)((U', Z'), h') := \lim_{((U', Z'), h') \xrightarrow{l} ((U, U), h)} (\Gamma_Z^\vee l^* LF)(U'/U') \\ &\xrightarrow{(\Gamma_Z^\vee l^* \text{ad}(h_\sharp, h^*)(LF))(U'/U')} (\Gamma_Z^\vee l^* h^* h_\sharp LF)(U'/U') =: (h_\sharp LF)^\Gamma((U', Z')/h') \end{aligned}$$

(iv) Let $i : Z_0 \hookrightarrow S$ a closed embedding with $Z_0, S \in \text{AnSp}(\mathbb{C})$. We have the canonical morphism in $C(\text{AnSp}(\mathbb{C})^{2,sm}/S)$

$$\begin{aligned} T_*(i, \Gamma)(\mathbb{Z}(Z_0/Z_0)) &: i_*(\mathbb{Z}(Z_0/Z_0))^\Gamma \rightarrow (i_*\mathbb{Z}(Z/Z))^\Gamma, \\ T_*(i, \Gamma)(\mathbb{Z}(Z_0/Z_0))((U, Z), h) &: i_*(\mathbb{Z}(Z_0/Z_0))^\Gamma((U, Z), h) := (\Gamma_{Z \times_S Z_0}^\vee \mathbb{Z}(Z_0/Z_0))(U \times_S Z_0) \\ &\xrightarrow{T(i_*, \gamma^\vee)(\mathbb{Z}(Z_0/Z_0))(U \times_S Z_0)} (\Gamma_Z^\vee i_* \mathbb{Z}(Z_0/Z_0))(U \times_S Z_0) =: (i_*\mathbb{Z}(Z/Z))^\Gamma((U, Z), h) \end{aligned}$$

Definition 33. Let $S \in \text{AnSp}(\mathbb{C})$. We have for $F \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$ the canonical map in $C(\text{AnSp}(\mathbb{C})^{sm}/S)$

$$\begin{aligned} \text{Gr}(F) &: \text{Gr}_{S^*}^{12} \mu_{S^*} F^\Gamma \rightarrow F, \\ \text{Gr}(F)(U/S) &: \Gamma_U^\vee p^* F(U \times S/U \times S) \xrightarrow{\text{ad}(l^*, l_*)(p^* F)(U \times S/U \times S)} h^* F(U/U) = F(U/S) \end{aligned}$$

where $h : U \rightarrow S$ is a smooth morphism with $U \in \text{AnSp}(\mathbb{C})$ and $h : U \xrightarrow{l} U \times S \xrightarrow{p} S$ is the graph factorization with l the graph embedding and p the projection.

Proposition 29. Let $S \in \text{AnSp}(\mathbb{C})$.

(i) Then,

- if $m : F \rightarrow G$ with $F, G \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$ is a quasi-isomorphism, $m^\Gamma : F^\Gamma \rightarrow G^\Gamma$ is a quasi-isomorphism in $C(\text{AnSp}(\mathbb{C})^{2,sm}/S)$,
- if $m : F \rightarrow G$ with $F, G \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$ is an usu local equivalence, $m^\Gamma : F^\Gamma \rightarrow G^\Gamma$ is an usu local equivalence in $C(\text{AnSp}(\mathbb{C})^{2,sm}/S)$,
- if $m : F \rightarrow G$ with $F, G \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$ is an $(\mathbb{D}^1, \text{usu})$ local equivalence, $m^\Gamma : F^\Gamma \rightarrow G^\Gamma$ is an $(\mathbb{D}^1, \text{usu})$ local equivalence in $C(\text{AnSp}(\mathbb{C})^{2,sm}/S)$.

(ii) Then,

- if $m : F \rightarrow G$ with $F, G \in C(\text{AnSp}(\mathbb{C})/S)$ is a quasi-isomorphism, $m^\Gamma : F^\Gamma \rightarrow G^\Gamma$ is a quasi-isomorphism in $C(\text{AnSp}(\mathbb{C})^2/S)$,
- if $m : F \rightarrow G$ with $F, G \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$ is an usu local equivalence, $m^\Gamma : F^\Gamma \rightarrow G^\Gamma$ is an usu local equivalence in $C(\text{AnSp}(\mathbb{C})^{2,sm}/S)$,
- if $m : F \rightarrow G$ with $F, G \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$ is an (\mathbb{D}^1, usu) local equivalence, $m^\Gamma : F^\Gamma \rightarrow G^\Gamma$ is an (\mathbb{D}^1, usu) local equivalence in $C(\text{AnSp}(\mathbb{C})^2/S)$.

Proof. Similar to the proof of proposition 24. □

2.10 The analytical functor for presheaves on the big Zariski or etale site and on the big Zariski or etale site of pairs

We have for $f : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$ the following commutative diagram of sites

$$\begin{array}{ccc}
 \text{Dia}(S) := \text{AnSp}(\mathbb{C})/T^{an} & \xrightarrow{\text{An}_T} & \text{Var}(\mathbb{C})/T \\
 \downarrow P(f) & \searrow \rho_T & \downarrow \rho_T \\
 & \text{AnSp}(\mathbb{C})^{sm}/T^{an} & \xrightarrow{\text{An}_T} \text{Var}(\mathbb{C})^{sm}/T \\
 & \downarrow P(f) & \downarrow P(f) \\
 \text{AnSp}(\mathbb{C})/S^{an} & \xrightarrow{\text{An}_S} & \text{Var}(\mathbb{C})/S \\
 \downarrow \rho_S & \downarrow P(f) & \downarrow \rho_S \\
 & \text{AnSp}(\mathbb{C})^{sm}/S^{an} & \xrightarrow{\text{An}_S} \text{Var}(\mathbb{C})^{sm}/S
 \end{array} \tag{39}$$

and

$$\begin{array}{ccc}
 \text{Dia}^{12}(S) := \text{AnSp}(\mathbb{C})^2/T^{an} & \xrightarrow{\text{An}_T} & \text{Var}(\mathbb{C})^2/T \\
 \downarrow P(f) & \searrow \rho_T & \downarrow \rho_T \\
 & \text{AnSp}(\mathbb{C})^{2,sm}/T^{an} & \xrightarrow{\text{An}_T} \text{Var}(\mathbb{C})^{2,sm}/T \\
 & \downarrow P(f) & \downarrow P(f) \\
 \text{AnSp}(\mathbb{C})^2/S^{an} & \xrightarrow{\text{An}_S} & \text{Var}(\mathbb{C})^2/S \\
 \downarrow \rho_S & \downarrow P(f) & \downarrow \rho_S \\
 & \text{AnSp}(\mathbb{C})^{2,sm}/S^{an} & \xrightarrow{\text{An}_S} \text{Var}(\mathbb{C})^{2,sm}/S
 \end{array} \tag{40}$$

For $S \in \text{Var}(\mathbb{C})$ we have the following commutative diagrams of sites

$$\begin{array}{ccccc}
\text{AnSp}(\mathbb{C})^2/S & \xrightarrow{\mu_S} & \text{AnSp}(\mathbb{C})^{2,pr}/S & & \\
\downarrow \text{An}_S & \searrow \rho_S & \downarrow & \searrow \rho_S & \\
& & \text{AnSp}(\mathbb{C})^{2,sm}/S & \xrightarrow{\mu_S} & \text{AnSp}(\mathbb{C})^{2,smpr}/S \\
& & \downarrow & \downarrow \text{An}_S & \downarrow \text{An}_S \\
\text{Var}(\mathbb{C})^2/S & \xrightarrow{\mu_S} & \text{Var}(\mathbb{C})^{2,smpr}/S & & \text{Var}(\mathbb{C})^{2,smpr}/S \\
& \searrow \rho_S & \downarrow & \searrow \rho_S & \\
& & \text{Var}(\mathbb{C})^2/S & \xrightarrow{\mu_S} & \text{Var}(\mathbb{C})^{2,smpr}/S
\end{array}$$

and

$$\begin{array}{ccccc}
\text{AnSp}(\mathbb{C})^{2,pr}/S & \xrightarrow{\text{Gr}_S^{12}} & \text{AnSp}(\mathbb{C})/S & & \\
\downarrow \text{An}_S & \searrow \rho_S & \downarrow & \searrow \rho_S & \\
& & \text{AnSp}(\mathbb{C})^{2,smpr}/S & \xrightarrow{\text{Gr}_S^{12}} & \text{AnSp}(\mathbb{C})^{sm}/S \\
& & \downarrow & \downarrow \text{An}_S & \downarrow \text{An}_S \\
\text{Var}(\mathbb{C})^{2,pr}/S & \xrightarrow{\text{Gr}_S^{12}} & \text{Var}(\mathbb{C})/S & & \text{Var}(\mathbb{C})^{sm}/S \\
& \searrow \rho_S & \downarrow & \searrow \rho_S & \\
& & \text{Var}(\mathbb{C})^{2,sm}/S & \xrightarrow{\text{Gr}_S^{12}} & \text{Var}(\mathbb{C})^{sm}/S
\end{array} \quad (41)$$

For $f : T \rightarrow S$ a morphism in $\text{Var}(\mathbb{C})$ the diagramm $\text{Dia}(S)$ and $\text{Dia}(T)$ commutes with the pullback functors : we have $e(S) \circ P(f) = P(f) \circ e(T)$.

For $S \in \text{Var}(\mathbb{C})$, the analytical functor is

$$(-)^{an} : C_{O_S}(S) \rightarrow C_{O_{S^{an}}}, G \mapsto G^{an} := \text{an}_S^{*mod} G := \text{an}_S^* G \otimes_{\text{an}_S^* O_S} O_{S^{an}}$$

Let $S \in \text{Var}(\mathbb{C})$.

- As $\text{an}_S^* : \text{PSh}(S) \rightarrow \text{PSh}(S^{an})$ preserve monomorphisms (the colimits involved being filtered colimits), we define, for $(G, F) \in C_{(2)fil}(S)$, $\text{an}_S^*(G, F) := (\text{an}_S^* G, \text{an}_S^* F) \in C_{(2)fil}(S^{an})$.
- As $(-)^{an} := \text{an}_S^{*mod} : \text{PSh}_{O_S}(S) \rightarrow \text{PSh}(S^{an})$ preserve monomorphisms (an_S^* preserve monomorphism and $(-) \otimes_{O_S} O_{S^{an}}$ preserve monomorphism since $O_{S^{an}}$ is a flat O_S module), we define, for $(G, F) \in C_{(2)fil}(S)$, $(G, F)^{an} := (G^{an}, \text{an}_S^* F \otimes_{O_S} O_{S^{an}}) \in C_{(2)fil}(S^{an})$.

Let $f : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Then,

- the commutative diagrams of sites $D(\text{An}, f) := (\text{An}_S, f, \text{An}_T, f = f^{an})$ gives, for $G \in C(\text{Var}(\mathbb{C})^{sm}/T)$, the canonical map in $C(\text{AnSp}(\mathbb{C})^{am}/T)$

$$\begin{aligned}
T(\text{An}, f)(G) : \text{An}_S^* f_* G &\xrightarrow{\text{ad}(\text{An}_T^*, \text{An}_{T^*})(G)} \text{An}_S^* f_* \text{An}_{T^*} \text{An}_T^* G = \text{An}_S^* \text{An}_{S^*} f_* \text{An}_T^* G \\
&\xrightarrow{\text{ad}(\text{An}_S^*, \text{An}_{S^*})(f_* \text{An}_T^* G)} f_* \text{An}_T^* G.
\end{aligned}$$

- the commutative diagrams of sites $D(\text{an}, f) := (\text{an}_S, f, \text{an}_T, f)$ gives, for $G \in C(T)$, the canonical map in $C(T^{an})$

$$\begin{aligned}
T(\text{an}, f)(G) : \text{an}_S^* f_* G &\xrightarrow{\text{ad}(\text{an}_T^*, \text{an}_{T^*})(G)} \text{an}_S^* f_* \text{an}_{T^*} \text{an}_T^* G = \text{an}_S^* \text{an}_{S^*} f_* \text{an}_T^* G \\
&\xrightarrow{\text{ad}(\text{an}_S^*, \text{an}_{S^*})(f_* \text{an}_T^* G)} f_* \text{an}_T^* G
\end{aligned}$$

and for $G \in C_{O_T}(T)$, the canonical map in $C_{O_{Tan}}(T^{an})$

$$\begin{aligned} T^{mod}(an, f)(G) : (f_*G)^{an} &:= an_S^{*mod} f_*G \xrightarrow{\text{ad}(an_T^{*mod}, an_{T^*})(G)} \\ an_S^{*mod} f_* an_{T^*} an_T^{*mod} G &= an_S^{*mod} an_{S^*} f_* an_T^{*mod} G \xrightarrow{an_S^{*mod}, an_{S^*}(f_* an_T^* G)} f_* an_T^{*mod} G =: f_* G^{an} \end{aligned}$$

Definition-Proposition 6. Consider a closed embedding $i : Z \hookrightarrow S$ with $S, Z \in \text{Var}(\mathbb{C})$. Then, for $G^\bullet \in C(\text{Var}(\mathbb{C})^{sm}/S)$, there exist a map in $C(\text{AnSp}(\mathbb{C})^{sm}/S)$

$$T(\text{An}, \gamma)(G) : \text{An}_S^* \Gamma_Z G \rightarrow \Gamma_Z \text{An}_S^* G$$

unique up to homotopy, such that $\gamma_Z(\text{An}_S^* G) \circ T(\text{An}, \gamma)(G) = \text{An}_S^* \gamma_Z G$.

Proof. Denote by $j : S \setminus Z \hookrightarrow S$ the open complementary embedding. The map is given by $(I, T(\text{An}, j)(j^*G)) : \text{Cone}(\text{An}_S^* G \rightarrow \text{An}_S^* j_* j^* G) \rightarrow (\text{An}_S^* G \rightarrow j_* j^* \text{An}_S^* G)$. \square

Definition 34. Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume that there exist a factorization $f : X \xrightarrow{i} Y \times S \xrightarrow{p} S$, with $Y \in \text{SmVar}(\mathbb{C})$, $i : X \hookrightarrow Y$ is a closed embedding and p the projection. We then have the canonical isomorphism in $C(\text{AnSp}(\mathbb{C})^{sm}/S^{an})$

$$\begin{aligned} T(f, g, Q) &:= T_{\sharp}(\text{An}, p)(-)^{-1} \circ T_{\sharp}(\text{An}, j)(-)^{-1} : \\ \text{An}_S^* Q(X/S) &:= \text{An}_S^* p_{\sharp} \Gamma_X^{\vee} \mathbb{Z}_{Y \times S}[d_Y] \xrightarrow{\cong} p_{\sharp} \Gamma_{X^{an}}^{\vee} \mathbb{Z}_{Y \times S}[d_Y] =: Q(X^{an}/S^{an}) \end{aligned}$$

with $j : Y \times S \setminus X \hookrightarrow Y \times S$ the closed embedding.

Definition-Proposition 7. Consider a closed embedding $i : Z \hookrightarrow S$ with $S \in \text{Var}(\mathbb{C})$. Then, for $G \in C_{O_S}(S)$, there is a canonical map in $C_{O_{San}}(S^{an})$

$$T^{mod}(an, \gamma)(G) : (\Gamma_Z G)^{an} \rightarrow \Gamma_{Z^{an}} G^{an}$$

unique up to homotopy, such that $\gamma_{Z^{an}}(G^{an}) \circ T^{mod}(an, \gamma)(G) = g^* \gamma_Z G$.

Proof. It is a particular case of definition-proposition 2(i). \square

We recall the first GAGA theorem for coherent sheaf on the projective spaces :

Theorem 11. For $X \in \text{Var}(\mathbb{C})$ and $F \in C_{O_X}(X)$ denote by

$$a(F) : \text{ad}(an_X^{*mod}, an(X)_*)(E(F)) : E(F) \rightarrow an_{X^*}(E(F))^{an} = an_{X^*} E(F^{an}),$$

the canonical morphism.

(i) Let $X \in \text{PVar}(\mathbb{C})$ a proper complex algebraic variety. For $F \in \text{Coh}_{O_X}(X)$ a coherent sheaf, the morphism

$$H^n \Gamma(X, a(F)) : H^n(X, F) = H^n \Gamma(X, E(F)) \rightarrow H^n(X, F^{an}) = H^n \Gamma(X, E(F^{an}))$$

is an isomorphism for all $n \in \mathbb{Z}$.

(ii) Let $f : X \rightarrow S$ a proper morphism with $X, S \in \text{Var}(\mathbb{C})$. For $F \in \text{Coh}_{O_X}(X)$ a coherent sheaf, the morphism

$$H^n f_* a(F) : R^n f_* F = H^n f_*(E(F)) \rightarrow R^n f_* F^{an} = H^n f_* E(F^{an})$$

is an isomorphism for all $n \in \mathbb{Z}$.

Proof. See [28]. (i) reduces to the case where X is projective and (ii) to the case where f is projective. Hence, the theorem reduce to the case of a coherent sheaf $F \in \text{Coh}_{O_{\mathbb{P}^N}}(\mathbb{P}^N)$ on \mathbb{P}^N . \square

2.11 The De Rahm complexes of algebraic varieties and analytical spaces

For $X \in \text{Var}(\mathbb{C})$, we denote by $\iota_X : \mathbb{C}_X \rightarrow \Omega_X^\bullet =: DR(X)$ the canonical inclusion map. More generally, for $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$, we denote by $\iota_{X/S} : f^*O_S \rightarrow \Omega_{X/S}^\bullet =: DR(X/S)$ the canonical inclusion map.

For $X \in \text{AnSp}(\mathbb{C})$, we denote by $\iota_X : \mathbb{C}_X \rightarrow \Omega_X^\bullet =: DR(X)$ the canonical inclusion map. More generally, for $f : X \rightarrow S$ a morphism with $X, S \in \text{AnSp}(\mathbb{C})$, we denote by $\iota_{X/S} : f^*O_S \rightarrow \Omega_{X/S}^\bullet =: DR(X/S)$ the canonical inclusion map.

Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Then, the commutative diagram of site $(an, f) := (f, \text{An}_S, f = f^{an}, an(X))$ gives the transformation map in $C_{O_{S^{an}}}(S^{an})$ (definition 1)

$$T_\omega^O(an, f) : (f_*E(\Omega_{X/S}^\bullet, F_b))^{an} := an_S^{*mod} f_*E(\Omega_{X/S}^\bullet, F_b) \xrightarrow{T(an(X), E)(-) \circ T(an, f)(E(\Omega_{X/S}^\bullet))} (f_*E(an(X)^*(\Omega_{X/S}^\bullet, F_b))) \otimes_{an_S^* O_S} O_{S^{an}} \xrightarrow{m \otimes E(\Omega_{(X^{an}/X)/(S^{an}/S)})} f_*E(\Omega_{X^{an}/S^{an}}^\bullet, F_b)$$

We will give in this paper a relative version for all smooth morphisms of the following theorem of Grothendieck

Theorem 12. *Let $U \in \text{SmVar}(\mathbb{C})$. Denote by $a_U : U \rightarrow \{\text{pt}\}$ the terminal map. Then the map*

$$T_\omega^O(a_U, an) : \Gamma(U, E(\Omega_U^\bullet)) \rightarrow \Gamma(U^{an}, E(\Omega_U^\bullet))$$

is a quasi-isomorphism of complexes.

Proof. Take a compactification (X, D) of U , with $X \in \text{PSmVar}(\mathbb{C})$ and $D = X \setminus U$ a normal crossing divisor. The proof then use proposition 13, the first GAGA theorem (theorem 11 (i)) for the coherent sheaves $\Omega_U^p(nD)$ on X , and the fact (which is specific of the De Rahm complex) that $\Omega_{U^{an}}^\bullet(*D^{an}) \rightarrow j_*E(\Omega_{U^{an}})$ is a quasi-isomorphism. \square

We recall Poincare lemma for smooth morphisms of complex analytic spaces and in particular complex analytic manifold :

Proposition 30. (i) *For $h : U \rightarrow S$ a smooth morphism with $U, S \in \text{AnSp}(\mathbb{C})$, the inclusion map $\iota_{X/S} : h^*O_S \rightarrow \Omega_{U/S}^\bullet =: DR(U/S)$ is a quasi-isomorphism.*

(ii) *For $X \in \text{AnSm}(\mathbb{C})$, the inclusion map $\iota_X : \mathbb{C}_X \rightarrow \Omega_X^\bullet$ is a quasi-isomorphism.*

Proof. Standard. (ii) is a particular case of (i) (the absolute case $S = \{\text{pt}\}$). \square

Remark 4. *We do NOT have poincare lemma in general if $h : U \rightarrow S$ is not a smooth morphism. Already in the absolute case, we can find $X \in \text{Var}(\mathbb{C})$ singular such that the inclusion map $\iota_X : \mathbb{C}_{X^{an}} \rightarrow \Omega_{X^{an}}^\bullet$ is not a quasi-isomorphism. Indeed, we can find exemple of $X \in \text{PVar}(\mathbb{C})$ projective singular where*

$$H^p(c_X) : H^p(X^{an}, \mathbb{C}_{X^{an}}) \xrightarrow{\sim} H^p C_{sing}^\bullet(X^{an})$$

X^{an} being locally contractible since $X^{an} \in \text{CW}$, have not the same dimension then the De Rham cohomology

$$H^p(T_\omega^O(an, a_X)) : \mathbb{H}^p(X, E(\Omega_X^\bullet)) \xrightarrow{\sim} \mathbb{H}^p(X^{an}, E(\Omega_{X^{an}}^\bullet))$$

X being projective, that is are not isomorphic as vector spaces. Hence, in particular, the canonical map

$$H^p \iota_X : H^p(X^{an}, \mathbb{C}_{X^{an}}) \rightarrow \mathbb{H}^p(X^{an}, E(\Omega_{X^{an}}^\bullet))$$

is not an isomorphism.

Consider a commutative diagram

$$D_0 = \begin{array}{ccccc} f : X & \xrightarrow{i} & Y & \xrightarrow{p} & S \\ g' \uparrow & & g'' \uparrow & & g \uparrow \\ f' : X' & \xrightarrow{i'} & Y' & \xrightarrow{p'} & T \end{array}$$

with $X, X', Y, Y', S, T \in \text{Var}(\mathbb{C})$ or $X, X', Y, Y', S, T \in \text{AnSp}(\mathbb{C})$, i, i' being closed embeddings. Denote by D the right square of D_0 . The closed embedding $i' : X' \hookrightarrow Y'$ factors through $i' : X' \xrightarrow{i'_1} X \times_Y Y' \xrightarrow{i'_0} Y'$ where i'_1, i'_0 are closed embeddings. Then, definition-proposition 3 say that

- there is a canonical map,

$$E(\Omega_{((Y'/Y)/(T/S))}) \circ T(g'', E)(-) \circ T(g'', \gamma)(-) : g''^* \Gamma_X E(\Omega_{Y/S}^\bullet, F_b) \rightarrow \Gamma_{X \times_Y Y'} E(\Omega_{Y'/T}^\bullet, F_b)$$

unique up to homotopy such that the following diagram in $C_{g''^* p^* O_S \text{fil}}(Y') = C_{p'^* g^* O_S \text{fil}}(Y')$ commutes

$$\begin{array}{ccc} g''^* \Gamma_X E(\Omega_{Y/S}^\bullet, F_b) & \xrightarrow{E(\Omega_{((Y'/Y)/(T/S))}) \circ T(g'', E)(-) \circ T(g'', \gamma)(-)} & \Gamma_{X \times_Y Y'} E(\Omega_{Y'/T}^\bullet, F_b) , \\ \gamma_X(-) \downarrow & & \downarrow \gamma_{X \times_Y Y'}(-) \\ g''^* E(\Omega_{Y/S}^\bullet, F_b) & \xrightarrow{E(\Omega_{((Y'/Y)/(T/S))}) \circ T(g'', E)(-)} & E(\Omega_{Y'/T}^\bullet, F_b) \end{array}$$

- there is a canonical map,

$$T_\omega^O(D)^\gamma : g^{*mod} LOP_* \Gamma_X E(\Omega_{Y/S}^\bullet, F_b) \rightarrow p'_* \Gamma_{X \times_Y Y'} E(\Omega_{Y'/T}^\bullet, F_b)$$

unique up to homotopy such that the following diagram in $C_{O_T \text{fil}}(T)$ commutes

$$\begin{array}{ccc} g^{*mod} LOP_* \Gamma_X E(\Omega_{Y/S}^\bullet, F_b) & \xrightarrow{T_\omega^O(D)^\gamma} & p'_* \Gamma_{X \times_Y Y'} E(\Omega_{Y'/T}^\bullet, F_b) . \\ \gamma_X(-) \downarrow & & \downarrow \gamma_{X \times_Y Y'}(-) \\ g^{*mod} LOP_* E(\Omega_{Y/S}^\bullet, F_b) & \xrightarrow{T_\omega^O(D)} & p'_* E(\Omega_{Y'/T}^\bullet, F_b) \end{array}$$

- (iii) there is a map in $C_{f'^* O_T \text{fil}}(Y')$

$$T(X'/X \times_Y Y', \gamma)(E(\Omega_{Y'/T}^\bullet, F_b)) : \Gamma_{X'} E(\Omega_{Y'/T}^\bullet, F_b) \rightarrow \Gamma_{X \times_Y Y'} E(\Omega_{Y'/T}^\bullet, F_b)$$

unique up to homotopy such that $\gamma_{X \times_Y Y'}(-) \circ T(X'/X \times_Y Y', \gamma)(-) = \gamma_{X'}(-)$.

Let $h : Y \rightarrow S$ a morphism and $i : X \hookrightarrow Y$ a closed embedding with $S, Y, X \in \text{Var}(\mathbb{C})$. Then, definition-proposition 3 say that

- there is a canonical map

$$E(\Omega_{(Y^{an}/Y)/(S^{an}/S)}) \circ T(an, \gamma)(-) : an(Y)^* \Gamma_X E(\Omega_{Y/S}^\bullet, F_b) \rightarrow \Gamma_{X^{an}} E(\Omega_{Y/S}^\bullet, F_b)$$

unique up to homotopy such that the following diagram in $C_{h^* O_S \text{fil}}(Y^{an})$ commutes

$$\begin{array}{ccc} an(Y)^* \Gamma_X E(\Omega_{Y/S}^\bullet, F_b) & \xrightarrow{E(\Omega_{(Y^{an}/Y)/(S^{an}/S)}) \circ T(an, \gamma)(-)} & \Gamma_{X^{an}} E(\Omega_{Y/S}^\bullet, F_b) \\ \gamma_X(-) \downarrow & & \downarrow \gamma_{X^{an}}(-) \\ an(Y)^* E(\Omega_{Y/S}^\bullet, F_b) & \xrightarrow{E(\Omega_{(Y^{an}/Y)/(S^{an}/S)})} & E(\Omega_{Y/S}^\bullet, F_b) \end{array}$$

- there is a canonical map

$$T_\omega^O(an, h)^\gamma : (h_*\Gamma_X E(\Omega_{Y/S}^\bullet, F_b))^{an} \rightarrow h_*\Gamma_{X^{an}} E(\Omega_{Y/S}^\bullet, F_b)$$

unique up to homotopy such that the following diagram in $C(Y)$ commutes

$$\begin{array}{ccc} (h_*\Gamma_X E(\Omega_{Y/S}^\bullet, F_b))^{an} & \xrightarrow{T_\omega^O(an, h)^\gamma} & h_*\Gamma_{X^{an}} E(\Omega_{Y/S}^\bullet, F_b) \\ \gamma_X(-) \downarrow & & \downarrow \gamma_{X^{an}}(-) \\ (h_*E(\Omega_{Y/S}^\bullet, F_b))^{an} & \xrightarrow{T_\omega^O(an, h)} & h_*E(\Omega_{Y/S}^\bullet, F_b) \end{array}$$

2.12 Resolution of a representable presheaf on $\text{Var}(\mathbb{C})/S$ by Corti-Hanamura presheaves and the functorialities of these resolutions

Definition 35. Let $X_0 \in \text{Var}(\mathbb{C})$ and $Z \subset X_0$ a closed subset. A desingularization of (X_0, Z) is a pair of complex varieties $(X, D) \in \text{Var}^2(\mathbb{C})$, together with a morphism of pair of varieties $\epsilon : (X, D) \rightarrow (X_0, Z)$ such that

- $X \in \text{SmVar}(\mathbb{C})$ and $D := \epsilon^{-1}(Z) \subset X$ is a normal crossing divisor
- $\epsilon : X \rightarrow X_0$ is a proper modification with discriminant Z , that is $\epsilon : X \rightarrow X_0$ is proper and $\epsilon : X \setminus D \xrightarrow{\sim} X \setminus Z$ is an isomorphism.

We have the following well known resolution of singularities of complex algebraic varieties and their functorialities :

Theorem 13. (i) Let $X_0 \in \text{Var}(\mathbb{C})$ and $Z \subset X_0$ a closed subset. There exists a desingularization of (X_0, Z) , that is a pair of complex varieties $(X, D) \in \text{Var}^2(\mathbb{C})$, together with a morphism of pair of varieties $\epsilon : (X, D) \rightarrow (X_0, Z)$ such that

- $X \in \text{SmVar}(\mathbb{C})$ and $D := \epsilon^{-1}(Z) \subset X$ is a normal crossing divisor
- $\epsilon : X \rightarrow X_0$ is a proper modification with discriminant Z , that is $\epsilon : X \rightarrow X_0$ is proper and $\epsilon : X \setminus D \xrightarrow{\sim} X \setminus Z$ is an isomorphism.

(ii) Let $X_0 \in \text{PVar}(\mathbb{C})$ and $Z \subset X_0$ a closed subset. There exists a desingularization of (X_0, Z) , that is a pair of complex varieties $(X, D) \in \text{PVar}^2(\mathbb{C})$, together with a morphism of pair of varieties $\epsilon : (X, D) \rightarrow (X_0, Z)$ such that

- $X \in \text{PSmVar}(\mathbb{C})$ and $D := \epsilon^{-1}(Z) \subset X$ is a normal crossing divisor
- $\epsilon : X \rightarrow X_0$ is a proper modification with discriminant Z , that is $\epsilon : X \rightarrow X_0$ is proper and $\epsilon : X \setminus D \xrightarrow{\sim} X \setminus Z$ is an isomorphism.

Proof. (i):Standard. See [24] for example.

(ii):Follows immediately from (i). □

We use this theorem to construct a resolution of a morphism by Corti-Hanamura morphisms, we will need these resolution in the definition of the filtered De Rham realization functor :

Definition-Proposition 8. (i) Let $h : V \rightarrow S$ a morphism, with $V, S \in \text{Var}(\mathbb{C})$. Let $\bar{S} \in \text{PVar}(\mathbb{C})$ be a compactification of S .

- There exist a compactification $X_0 \in \text{PVar}(\mathbb{C})$ of V such that $h : V \rightarrow S$ extend to a morphism $f_0 = \bar{h}_0 : X_0 \rightarrow \bar{S}$. Denote by $Z = X_0 \setminus V$. We denote by $j : V \hookrightarrow X_0$ the open embedding and by $i_0 : Z \hookrightarrow X_0$ the complementary closed embedding.

- Take, using theorem 13(ii), a desingularization $\epsilon : (X, D) \rightarrow (X_0, Z)$ of the pair (X_0, Z) , with $X \in \text{PSmVar}(\mathbb{C})$ and $D = \cup_{i=1}^s D_i \subset X$ a normal crossing divisor. We denote by $i_\bullet : D_\bullet \hookrightarrow X = X_{c(\bullet)}$ the morphism of simplicial varieties given by the closed embeddings $i_I : D_I = \cap_{i \in I} D_i \hookrightarrow X$. Then the morphisms $f := \bar{h}_0 \circ \epsilon : X \rightarrow \bar{S}$ and $f_{D_\bullet} := f \circ i_\bullet : D_\bullet \rightarrow \bar{S}$ are projective since X and D_I are projective varieties.

(ii) Let $g : V'/S \rightarrow V/S$ a morphism, with $V'/S = (V', h'), V/S = (V, h) \in \text{Var}(\mathbb{C})/S$

- Take (see (i)) a compactification $X_0 \in \text{PVar}(\mathbb{C})$ of V such that $h : V \rightarrow S$ extend to a morphism $f_0 = \bar{h}_0 : X_0 \rightarrow \bar{S}$. Denote by $Z = X_0 \setminus V$. Then, there exist a compactification $X'_0 \in \text{PVar}(\mathbb{C})$ of V' such that $h' : V' \rightarrow S$ extend to a morphism $f'_0 = \bar{h}'_0 : X'_0 \rightarrow \bar{S}$, $g : V' \rightarrow V$ extend to a morphism $\bar{g}_0 : X'_0 \rightarrow X_0$ and $f \circ \bar{g}_0 = f'$ that is \bar{g}_0 is gives a morphism $\bar{g}_0 : X'_0/\bar{S} \rightarrow X_0/\bar{S}$. Denote by $Z' = X'_0 \setminus V'$. We then have the following commutative diagram

$$\begin{array}{ccccc}
 V & \xrightarrow{j} & X_0 & \xleftarrow{i} & Z \\
 \uparrow g & & \uparrow \bar{g}_0 & & \uparrow \bar{g}' \\
 V' & \xrightarrow{j'} & X'_0 & \xleftarrow{i'} & Z' \xleftarrow{i'_{g',0} \bar{g}_0^{-1}(Z)} : i'_{g',0}
 \end{array}$$

- Take using theorem 13 a desingularization $\epsilon : (X, D) \rightarrow (X_0, Z)$ of (X_0, Z) . Then there exist a desingularization $\epsilon'_\bullet : (X', D') \rightarrow (X'_0, Z')$ of (X'_0, Z') and a morphism $\bar{g} : X' \rightarrow X$ such that the following diagram commutes

$$\begin{array}{ccc}
 X'_0 & \xrightarrow{\bar{g}_0} & X_0 \\
 \epsilon'_\bullet \uparrow & & \uparrow \epsilon \\
 X' & \xrightarrow{\bar{g}} & X
 \end{array}$$

We then have the following commutative diagram in $\text{Fun}(\Delta, \text{Var}(\mathbb{C}))$

$$\begin{array}{ccccc}
 V = V_{c(\bullet)} & \xrightarrow{j} & X = X_{c(\bullet)} & \xleftarrow{i_\bullet} & D_{s_g(\bullet)} \\
 \uparrow g & & \uparrow \bar{g} & & \uparrow \bar{g}' \\
 V' = V'_{c(\bullet)} & \xrightarrow{j'} & X' = X'_{c(\bullet)} & \xleftarrow{i'_\bullet} & D'_\bullet \xleftarrow{i'_{g',\bullet} \bar{g}^{-1}(D_{s_g(\bullet)})} : i'_{g',\bullet}
 \end{array}$$

where $i_\bullet : D_\bullet \hookrightarrow X_\bullet$ the morphism of simplicial varieties given by the closed embeddings $i_n : D_n \hookrightarrow X_n$, and $i'_\bullet : D'_\bullet \hookrightarrow X'_\bullet$ the morphism of simplicial varieties given by the closed embeddings $i'_n : D'_n \hookrightarrow X'_n$.

Proof. (i): Let $X_{00} \in \text{PVar}(\mathbb{C})$ be a compactification of V . Let $l_0 : X_0 = \bar{\Gamma}_h \hookrightarrow X_{00} \times \bar{S}$ be the closure of the graph of h and $\bar{f}_0 := p_{\bar{S}} \circ l_0 : X_0 \hookrightarrow X_{00} \times \bar{S} \rightarrow \bar{S}$, $\epsilon_{X_0} := p_{X_{00}} \circ l_0 : X_0 \hookrightarrow X_{00} \times \bar{S} \rightarrow X_{00}$ be the restriction to X_0 of the projections. Then, $X \in \text{PVar}(\mathbb{C})$, $\epsilon_{X_0} : X_0 \rightarrow X_{00}$ is a proper modification which does not affect the open subset $V \subset X_0$, and $f_0 = \bar{h}_0 : X_0 \rightarrow \bar{S}$ is a compactification of h .

(ii): There is two things to prove:

- Let $f_0 : X_0 \rightarrow S$ a compactification of $h : V \rightarrow S$ and $f'_{00} : X'_{00} \rightarrow S$ a compactification of $h' : V' \rightarrow S$ (see (i)). Let $l_0 : X'_0 \hookrightarrow \bar{\Gamma}_g \subset X'_{00} \times_S X_0$ be the closure of the graph of g , $f'_0 := (f'_{00}, f_0) \circ l_0 : X'_0 \hookrightarrow X'_{00} \times_S X_0 \rightarrow S$ and $\bar{g}_0 := p_{X_0} \circ l_0 : X'_0 \hookrightarrow X'_{00} \times_S X_0 \rightarrow X_0$, $\epsilon_{X'_{00}} := p_{X'_{00}} \circ i : X'_0 \hookrightarrow X'_{00} \times_S X_0 \rightarrow X'_{00}$ be the restriction to X' of the projections. Then $\epsilon_{X'_{00}} : X'_0 \rightarrow X'_{00}$ is a proper modification which does not affect the open subset $V' \subset X'_0$, $f'_0 : X'_0 \rightarrow S$ is an other compactification of $h' : V' \rightarrow S$ and $\bar{g}_0 : X'_0 \rightarrow X_0$ is a compactification of g .

- Take, using theorem 13, a desingularization $\epsilon : (X, D) \rightarrow (X_0, Z)$ of the pair (X_0, Z) . Take then, using theorem 13, a desingularization $\epsilon'_1 : (X', D') \rightarrow (X \times_{X_0} X'_0, X \times_{X_0} Z')$ of the pair $(X \times_{X_0} X'_0, X \times_{X_0} Z')$. We consider then following commutative diagram whose square is cartesian :

$$\begin{array}{ccccc}
& & X'_0 & \xrightarrow{\bar{g}_0} & X_0 \\
& & \uparrow \epsilon'_0 & & \uparrow \epsilon \\
& \epsilon' & X \times_{X_0} X'_0 & \xrightarrow{\bar{g}'_0} & X \\
& \uparrow \epsilon'_1 & \uparrow \bar{g} & & \\
X' & & & &
\end{array}$$

and $\epsilon' := \epsilon'_0 \circ \epsilon'_1 : (X', D') \rightarrow (X'_0, Z')$ is a desingularization of the pair $(X \times_{X_0} X'_0, X \times_{X_0} Z')$. □

Let $S \in \text{Var}(\mathbb{C})$. Recall we have the dual functor

$$\mathbb{D}_S : C(\text{Var}(\mathbb{C})/S) \rightarrow C(\text{Var}(\mathbb{C})/S), F \mapsto \mathbb{D}_S(F) := \mathcal{H}om(F, E_{et}(\mathbb{Z}_S))$$

which induces the functor

$$L\mathbb{D}_S : C(\text{Var}(\mathbb{C})/S) \rightarrow C(\text{Var}(\mathbb{C})/S), F \mapsto L\mathbb{D}_S(F) := \mathbb{D}_S(LF) := \mathcal{H}om(LF, E_{et}(\mathbb{Z}_S)).$$

We will use the following resolutions of representable presheaves by Corti-Hanamura presheaves and their the functorialities.

Definition 36. (i) Let $h : U \rightarrow S$ a morphism, with $U, S \in \text{Var}(\mathbb{C})$. Take, see definition-proposition 8, $f_0 = \bar{h}_0 : X_0 \rightarrow S$ a compactification of $h : U \rightarrow S$ and denote by $Z = X_0 \setminus U$. Take, using theorem 13(ii), a desingularization $\epsilon : (X, D) \rightarrow (X_0, Z)$ of the pair (X_0, Z) , with $X \in \text{PSmVar}(\mathbb{C})$ and $D = \cup_{i=1}^s D_i \subset X$ a normal crossing divisor. We denote by $i_\bullet : D_\bullet \hookrightarrow X = X_{c(\bullet)}$ the morphism of simplicial varieties given by the closed embeddings $i_I : D_I = \cap_{i \in I} D_i \hookrightarrow X$. We denote by $j : U \hookrightarrow X$ the open embedding. We then consider the following map in $C(\text{Var}(\mathbb{C})/S)$

$$\begin{aligned}
& r_{(X,D)/S}(\mathbb{Z}(U/S)) : R_{(X,D)/S}(\mathbb{Z}(U/S)) \\
& \xrightarrow{=} \text{Cone}(\text{ad}(i_{\bullet\#}^*, i_\bullet^*)(\mathbb{Z}(X/X)) : f_* E_{et}(i_{\bullet\#} \mathbb{Z}(D_\bullet/D_\bullet), u_{I,J}) \rightarrow f_* E_{et}(\mathbb{Z}(X/X))) \\
& \xrightarrow{(0, \text{ad}(j^*, j_*)(E_{et}(\mathbb{Z}(X/X)))} h_* E_{et}(\mathbb{Z}(U/U))) = \mathbb{D}_S \mathbb{Z}(U/S)
\end{aligned}$$

Note that $r_{(X,D)/S}$ is NOT an equivalence (\mathbb{A}^1, et) local by proposition 20 since $\rho_{S*} i_{\bullet\#} \mathbb{Z}(D_\bullet/D_\bullet) = 0$, whereas $\rho_{S*} \text{ad}(j^*, j_*)(E_{et}(\mathbb{Z}(X/X)))$ is not an equivalence (\mathbb{A}^1, et) local.

- (ii) Let $g : U'/S \rightarrow U/S$ a morphism, with $U'/S = (U', h'), U/S = (U, h) \in \text{Var}(\mathbb{C})/S$. Take, see definition-proposition 8(ii), a compactification $f_0 = \bar{h} : X_0 \rightarrow S$ of $h : U \rightarrow S$ and a compactification $f'_0 = \bar{h}' : X'_0 \rightarrow S$ of $h' : U' \rightarrow S$ such that $g : U'/S \rightarrow U/S$ extend to a morphism $\bar{g}_0 : X'_0/S \rightarrow X_0/S$. Denote $Z = X_0 \setminus U$ and $Z' = X'_0 \setminus U'$. Take, see definition-proposition 8(ii), a desingularization $\epsilon : (X, D) \rightarrow (X_0, Z)$ of (X_0, Z) , a desingularization $\epsilon'_\bullet : (X', D') \rightarrow (X'_0, Z')$ of (X'_0, Z') and a morphism $\bar{g} : X' \rightarrow X$ such that the following diagram commutes

$$\begin{array}{ccc}
X'_0 & \xrightarrow{\bar{g}_0} & X_0 \\
\epsilon'_\bullet \uparrow & & \uparrow \epsilon \\
X' & \xrightarrow{\bar{g}} & X
\end{array}$$

We then have, see definition-proposition 8(ii), the following commutative diagram in $\text{Fun}(\Delta, \text{Var}(\mathbb{C}))$

$$\begin{array}{ccccc}
U = U_{c(\bullet)} & \xrightarrow{j} & X = X_{c(\bullet)} & \xleftarrow{i_\bullet} & D_{s_g(\bullet)} \\
\uparrow g & & \uparrow g & & \uparrow \bar{g}' \\
U' = U'_{c(\bullet)} & \xrightarrow{j'} & X' = X'_{c(\bullet)} & \xleftarrow{i'_\bullet} & D'_\bullet \xleftarrow{i''_{g'_\bullet}} \bar{g}^{-1}(D_{s_g(\bullet)}) : i'_{g'_\bullet}
\end{array} \quad (42)$$

We then consider the following map in $C(\text{Var}(\mathbb{C})/S)$

$$\begin{aligned}
& R_S^{CH}(g) : R_{(X,D)/S}(\mathbb{Z}(U/S)) \\
& \xrightarrow{\cong} \text{Cone}(\text{ad}(i_{\bullet\#}, i_{\bullet}^*)(\mathbb{Z}(X/X)) : f_* E_{et}(i_{\bullet\#}(\mathbb{Z}(D_\bullet/D_\bullet), u_{IJ})) \rightarrow f_* E_{et}(\mathbb{Z}(X/X))) \\
& \xrightarrow{\text{ad}(\bar{g}^*, \bar{g}_*)(-), \text{ad}(\bar{g}^*, \bar{g}_*)(-)} \\
& \text{Cone}(\bar{g}^* \text{ad}(i_{\bullet\#}, i_{\bullet}^*)(\mathbb{Z}(X/X)) : f'_* E_{et}(\bar{g}^*(i_{\bullet\#}(\mathbb{Z}(D_\bullet/D_\bullet), u_{IJ})) \rightarrow f'_* E_{et}(\mathbb{Z}(X'/X'))) \\
& \xrightarrow{(T_\#(\bar{g}, i_\bullet)(-)^{-1}, I)} \\
& \text{Cone}(\text{ad}(i'_{g'_\bullet\#}, i'_{g'_\bullet}{}^*)(\mathbb{Z}(X'/X')) : f'_* E_{et}(i'_{g'_\bullet\#}(\mathbb{Z}(\bar{g}^{-1}(D_\bullet)/\bar{g}^{-1}(D_\bullet)), u_{IJ})) \rightarrow f'_* E_{et}(\mathbb{Z}(X'/X'))) \\
& \xrightarrow{(\text{ad}(i'_{g'_\bullet\#}, i'_{g'_\bullet}{}^*)(\mathbb{Z}(D'_\bullet/D'_\bullet)), I)} \\
& \text{Cone}(\text{ad}(i'_{\bullet\#}, i'_{\bullet}{}^*)(\mathbb{Z}(X'/X')) : f'_* E_{et}(i'_{\bullet\#}(\mathbb{Z}(D'_\bullet/D'_\bullet), u_{IJ})) \rightarrow f'_* E_{et}(\mathbb{Z}(X'/X'))) \\
& \xrightarrow{\cong} R_{(X',D')/S}(\mathbb{Z}(U'/S))
\end{aligned}$$

Then by the diagram (42) and adjonction, the following diagram in $C(\text{Var}(\mathbb{C})/S)$ obviously commutes

$$\begin{array}{ccc}
R_{(X,D)/S}(\mathbb{Z}(U/S)) & \xrightarrow{r_{(X,D)/S}(\mathbb{Z}(U/S))} & \mathbb{D}_S \mathbb{Z}(U/S) = h_* E_{et}(\mathbb{Z}(U/U)) \\
R_S^{CH}(g) \downarrow & & \downarrow T(g, E) \circ \text{ad}(g^*, g_*)(E_{et}(\mathbb{Z}(U/U))) \\
R_{(X',D')/S}(\mathbb{Z}(U'/S)) & \xrightarrow{r_{(X',D')/S}(\mathbb{Z}(U'/S))} & \mathbb{D}_S \mathbb{Z}(U'/S) = h'_* E_{et}(\mathbb{Z}(U'/U'))
\end{array}$$

(iii) Let $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$. We get from (i) and (ii) morphisms in $C(\text{Var}(\mathbb{C})/S)$

$$r_S^{CH}(LF) : R_{(X^*, D^*)/S}(\rho_S^* LF) \rightarrow \mathbb{D}_S \rho_S^* LF$$

- Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Let $h : U \rightarrow S$ a morphism with $U \in \text{Var}(\mathbb{C})$. Consider the cartesian square

$$\begin{array}{ccc}
U_T & \xrightarrow{h'} & T \\
\downarrow g' & & \downarrow g \\
U & \xrightarrow{h} & S
\end{array}$$

Take, see definition-proposition 8(ii), a compactification $f_0 = \bar{h} : X_0 \rightarrow S$ of $h : U \rightarrow S$ and a compactification $f'_0 = g \circ h' : X'_0 \rightarrow S$ of $g \circ h' : U' \rightarrow S$ such that $g' : U_T/S \rightarrow U/S$ extend to a morphism $\bar{g}'_0 : X'_0/S \rightarrow X_0/S$. Denote $Z = X_0 \setminus U$ and $Z' = X'_0 \setminus U_T$. Take, see definition-proposition 8(ii), a desingularization $\epsilon : (X, D) \rightarrow (X_0, Z)$ of (X_0, Z) , a desingularization $\epsilon'_\bullet : (X', D') \rightarrow (X'_0, Z')$ of (X'_0, Z') and a morphism $\bar{g}' : X' \rightarrow X$ such that the following diagram commutes

$$\begin{array}{ccc}
X'_0 & \xrightarrow{\bar{g}'_0} & X_0 \\
\epsilon' \uparrow & & \uparrow \epsilon \\
X' & \xrightarrow{\bar{g}'} & X
\end{array}$$

We then have, see definition-proposition 8(ii), the following commutative diagram in $\text{Fun}(\Delta, \text{Var}(\mathbb{C}))$

$$\begin{array}{ccccc} U = U_{c(\bullet)} & \xrightarrow{j} & X = X_{c(\bullet)} & \xleftarrow{i_\bullet} & D_{s_{g'}(\bullet)} \\ g' \uparrow & & \bar{g}' \uparrow & & (\bar{g}')'_\bullet \uparrow \\ U_T = U_{T,c(\bullet)} & \xrightarrow{j'} & X' = X'_{c(\bullet)} & \xleftarrow{i'_\bullet} & D'_\bullet \xleftarrow{i''_{g'_\bullet}^{-1}} (D_{s_{g'}(\bullet)}) : i'_{g'_\bullet} \end{array}$$

We then consider the following map in $C(\text{Var}(\mathbb{C})/T)$, see definition 36(ii)

$$\begin{aligned} & T(g, R^{CH})(\mathbb{Z}(U/S)) : g^* R_{(X,D)/S}(\mathbb{Z}(U/S)) \\ \xrightarrow{g^* R_S^{CH}(g')} & g^* R_{(X',D')/S}(\mathbb{Z}(U_T/S)) = g^* g_* R_{(X',D')/T}(\mathbb{Z}(U_T/T)) \\ \xrightarrow{\text{ad}(g^*, g_*)(R_{(X',D')/T}(\mathbb{Z}(U_T/T)))} & R_{(X',D')/T}(\mathbb{Z}(U_T/T)) \end{aligned}$$

It gives for $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ the map in $C(\text{Var}(\mathbb{C})/T)$

$$T(g, R^{CH})(LF) : g^* R_{(X^*,D^*)/S}(\rho_S^* LF) \rightarrow R_{(X'^*,D'^*)/T}(\rho_T^* g^* LF)$$

- Let $S_1, S_2 \in \text{Var}(\mathbb{C})$ and $p : S_1 \times S_2 \rightarrow S_1$ the projection. Let $h : U \rightarrow S_1$ a morphism with $U \in \text{Var}(\mathbb{C})$. Consider the cartesian square

$$\begin{array}{ccc} U \times S_2 & \xrightarrow{h \times I} & S_1 \times S_2 \\ \downarrow p' & & \downarrow p \\ U & \xrightarrow{h} & S_1 \end{array}$$

Take, see definition-proposition 8(i), a compactification $f = \bar{h} : X_0 \rightarrow S_1$ of $h : U \rightarrow S_1$. Then $f \times I : X_0 \times S_2 \rightarrow S_1 \times S_2$ is a compactification of $h \times I : U \times S_2 \rightarrow S_1 \times S_2$ and $p' : U \times S_2 \rightarrow U$ extend to $\bar{p}'_0 := p_{X_0} : X_0 \times S_2 \rightarrow X_0$. Denote $Z = X_0 \setminus U$. Take see theorem 13(i), a desingularization $\epsilon : (X, D) \rightarrow (X_0, Z)$ of the pair (X_0, Z) . We then have the following commutative diagram in $\text{Fun}(\Delta, \text{Var}(\mathbb{C}))$ whose squares are cartesian

$$\begin{array}{ccccc} U = U_{c(\bullet)} & \xrightarrow{j} & X & \xleftarrow{i_\bullet} & D_\bullet \\ g \uparrow & & \bar{p}' := p_X \uparrow & & \bar{p}'_\bullet \uparrow \\ U \times S_2 = (U \times S_2)_{c(\bullet)} & \xrightarrow{j \times I} & X \times S_2 & \xleftarrow{i'_\bullet} & D_\bullet \times S_2 \end{array} \quad (43)$$

Then the map in $C(\text{Var}(\mathbb{C})/S_1 \times S_2)$

$$T(p, R^{CH})(\mathbb{Z}(U/S_1)) : p^* R_{(X,D)/S_1}(\mathbb{Z}(U/S_1)) \xrightarrow{\sim} R_{(X \times S_2, D_\bullet \times S_2)/S_1 \times S_2}(\mathbb{Z}(U \times S_2/S_1 \times S_2))$$

is an isomorphism. It gives for $F \in C(\text{Var}(\mathbb{C})^{sm}/S_1)$ the isomorphisms in $C(\text{Var}(\mathbb{C})/S_1 \times S_2)$

$$T(p, R^{CH})(LF) : p^* R_{(X^*,D^*)/S_1}(\rho_{S_1}^* LF) \xrightarrow{\sim} R_{(X^* \times S_2, D^* \times S_2)/S_1 \times S_2}(\rho_{S_1 \times S_2}^* p^* LF)$$

is an isomorphism

- Let $h_1 : U_1 \rightarrow S, h_2 : U_2 \rightarrow S$ two morphisms with $U_1, U_2, S \in \text{Var}(\mathbb{C})$. Denote by $p_1 : U_1 \times_S U_2 \rightarrow U_1$ and $p_2 : U_1 \times_S U_2 \rightarrow U_2$ the projections. Take, see definition-proposition 8(i), a compactification $f_{10} = \bar{h}_1 : X_{10} \rightarrow S$ of $h_1 : U_1 \rightarrow S$ and a compactification $f_{20} = \bar{h}_2 : X_{20} \rightarrow S$ of $h_2 : U_2 \rightarrow S$. Then,

- $f_{10} \times f_{20} : X_{10} \times_S X_{20} \rightarrow S$ is a compactification of $h_1 \times h_2 : U_1 \times_S U_2 \rightarrow S$.
- $\bar{p}_{10} := p_{X_{10}} : X_{10} \times_S X_{20} \rightarrow X_{10}$ is a compactification of $p_1 : U_1 \times_S U_2 \rightarrow U_1$.
- $\bar{p}_{20} := p_{X_{20}} : X_{10} \times_S X_{20} \rightarrow X_{20}$ is a compactification of $p_2 : U_1 \times_S U_2 \rightarrow U_2$.

Denote $Z_1 = X_{10} \setminus U_1$ and $Z_2 = X_{20} \setminus U_2$. Take, see theorem 13(i), a desingularization $\epsilon_1 : (X_1, D) \rightarrow (X_{10}, Z_1)$ of the pair (X_{10}, Z_1) . and a desingularization $\epsilon_2 : (X_2, E) \rightarrow (X_{20}, Z_2)$ of the pair (X_{20}, Z_2) . Take then a desingularization

$$\epsilon_{12} : ((X_1 \times_S X_2)^N, F) \rightarrow (X_1 \times_S X_2, (D \times_S X_2) \cup (X_1 \times_S E))$$

of the pair $(X_1 \times_S X_2, (D \times_S X_2) \cup (X_1 \times_S E))$. We have then the following commutative diagram

$$\begin{array}{ccccc}
 & & X_1 & \xrightarrow{f_1} & S \\
 & & \uparrow \bar{p}_2 & & \uparrow f_2 \\
 & & X_1 \times_S X_2 & \xrightarrow{\bar{p}_1} & X_2 \\
 & \nearrow (\bar{p}_2)^N & \uparrow \epsilon_{12} & \nearrow (\bar{p}_1)^N & \\
 & (X_1 \times_S X_2)^N & & &
 \end{array}$$

and

- $f_1 \times f_2 : X_1 \times_S X_2 \rightarrow S$ is a compactification of $h_1 \times h_2 : U_1 \times_S U_2 \rightarrow S$.
- $(\bar{p}_1)^N := \bar{p}_1 \circ \epsilon_{12} : (X_1 \times_S X_2)^N \rightarrow X_1$ is a compactification of $p_1 : U_1 \times_S U_2 \rightarrow U_1$.
- $(\bar{p}_2)^N := \bar{p}_2 \circ \epsilon_{12} : (X_1 \times_S X_2)^N \rightarrow X_2$ is a compactification of $p_2 : U_1 \times_S U_2 \rightarrow U_2$.

We have then the isomorphism in $C(\text{Var}(\mathbb{C})/S)$

$$\begin{aligned}
 T(\otimes, R_S^{CH})(\mathbb{Z}(U_1/S), \mathbb{Z}(U_2/S)) &:= R_S^{CH}(p_1) \otimes R_S^{CH}(p_2) : \\
 R_{(X_1, D)/S}(\mathbb{Z}(U_1/S)) \otimes R_{(X_2, E)/S}(\mathbb{Z}(U_2/S)) &\xrightarrow{\sim} R_{(X_1 \times_S X_2)^N, F/S}(\mathbb{Z}(U_1 \times_S U_2/S))
 \end{aligned}$$

Hence, for $F_1, F_2 \in C(\text{Var}(\mathbb{C})^{sm}/S)$, we get isomorphisms

$$\begin{aligned}
 T(\otimes, R_S^{CH})(LF_1, LF_2) &: R_{(X_1^*, D^*)/S}(\rho_S^* LF_1) \otimes R_{(X_2^*, E^*)/S}(\rho_S^* LF_2) \\
 &\xrightarrow{\sim} R_{(X_1 \times_S X_2)^N/S}(\rho_S^*(LF_1 \otimes LF_2)).
 \end{aligned}$$

in $C(\text{Var}(\mathbb{C})/S)$.

2.13 The derived categories of filtered complexes of presheaves on a site or of filtered complexes of presheaves of modules on a ringed topos

Definition 37. Let $\mathcal{S} \in \text{Cat}$ a site endowed with topology τ . We denote by $D(\mathcal{S}) := \text{Ho}_{\text{Top}} C(\mathcal{S})$ the localization of the category of complexes of presheaves on \mathcal{S} with respect to top local equivalence and by $D(\tau) : C(\mathcal{S}) \rightarrow D(\mathcal{S})$ the localization functor. We denote for $r = 1, \dots, \infty$, resp. $r = (1, \dots, \infty)^2$,

$$D_{fil, r}(\mathcal{S}) := \text{Ho}_{F_{r\tau}} C_{fil}(\mathcal{S}), \quad D_{2fil, r}(\mathcal{S}) := \text{Ho}_{F_{r\tau}} C_{2fil}(\mathcal{S}),$$

the localizations of the category of filtered complexes of presheaves on \mathcal{S} whose filtration is biregular with respect to r - τ local equivalence. By definition, we have sequences of functors

$$C_{fil}(\mathcal{S}) \rightarrow K_{fil}(\mathcal{S}) \rightarrow D_{fil}(\mathcal{S}) \rightarrow D_{fil, 2}(\mathcal{S}) \rightarrow \dots \rightarrow D_{fil, \infty}(\mathcal{S}).$$

where we recall $K_{fil}(\mathcal{S}) := K(\text{PSh}_{fil}(\mathcal{S}))$.

Definition 38. Let $(\mathcal{S}, O_{\mathcal{S}}) \in \text{RCat}$ where $\mathcal{S} \in \text{Cat}$ is a site endowed with topology τ . We denote by $D_{\mathcal{O}}(\mathcal{S}) := \text{Ho}_{\text{Top}} C_{\mathcal{O}}(\mathcal{S})$ the localization of the category of complexes of presheaves on \mathcal{S} with respect to top local equivalence and by $D(\tau) : C_{\mathcal{O}}(\mathcal{S}) \rightarrow D_{\mathcal{O}}(\mathcal{S})$ the localization functor. We denote for $r = 1, \dots, \infty$, resp. $r = (1, \dots, \infty)^2$,

$$D_{O_{\mathcal{S}}\text{fil},r}(\mathcal{S}) := \text{Ho}_{Frr\tau} C_{O_{\mathcal{S}}\text{fil}}(\mathcal{S}), \quad D_{O_{\mathcal{S}}2\text{fil},r}(\mathcal{S}) := \text{Ho}_{Frr\tau} C_{O_{\mathcal{S}}2\text{fil}}(\mathcal{S}),$$

the localizations of the category of filtered complexes of presheaves on \mathcal{S} whose filtration is biregular with respect to r - τ local equivalence. By definition, we have sequences of functors

$$C_{O_{\mathcal{S}}\text{fil}}(\mathcal{S}) \rightarrow K_{O_{\mathcal{S}}\text{fil}}(\mathcal{S}) \rightarrow D_{O_{\mathcal{S}}\text{fil}}(\mathcal{S}) \rightarrow D_{O_{\mathcal{S}}\text{fil},2}(\mathcal{S}) \rightarrow \dots \rightarrow D_{O_{\mathcal{S}}\text{fil},\infty}(\mathcal{S}).$$

where we recall $K_{O_{\mathcal{S}}\text{fil}}(\mathcal{S}) := K(\text{PSh}_{O_{\mathcal{S}}\text{fil}}(\mathcal{S}))$.

Let $f : \mathcal{T} \rightarrow \mathcal{S}$ a morphism of presite, with $\mathcal{S}, \mathcal{T} \in \text{Cat}$ endowed with a topology τ . If f is a morphism of site, the adjunctions

$$(f^*, f_*) = (f^{-1}, f_*) : C(\mathcal{S}) \rightleftarrows C(\mathcal{T}), \quad (f^*, f_*) = (f^{-1}, f_*) : C_{(2)\text{fil}}(\mathcal{S}) \rightleftarrows C_{(2)\text{fil}}(\mathcal{T}).$$

are Quillen adjunctions. They induces respectively in the derived categories, for $r = (1, \dots, \infty)$, resp. $r = (1, \dots, \infty)$ (note that f^* derive trivially)

$$(f^*, Rf_*) : D(\mathcal{S}) \rightleftarrows D(\mathcal{T}), \quad (f^*, Rf_*) : D_{\text{fil},r}(\mathcal{S}) \rightleftarrows D_{\text{fil},r}(\mathcal{T}).$$

For $F^\bullet \in C(\mathcal{S})$, we have the adjunction maps

$$\text{ad}(f^*, f_*)(F^\bullet) : F^\bullet \rightarrow f_* f^* F^\bullet, \quad \text{ad}(f^*, f_*)(F^\bullet) : f^* f_* F^\bullet \rightarrow F^\bullet,$$

induces in the derived categories, for $(M, F) \in D_{\text{fil}}(\mathcal{S})$ and $(N, F) \in D_{\text{fil}}(\mathcal{T})$, the adjunction maps

$$\text{ad}(f^*, Rf_*)(M) : (M, F) \rightarrow Rf_* f^*(M, F), \quad \text{ad}(f^*, Rf_*)(N, F) : f^* Rf_*(N, F) \rightarrow (N, F).$$

For a commutative diagram of sites :

$$D = \begin{array}{ccc} \mathcal{Y} & \xrightarrow{g_2} & \mathcal{X} \\ \downarrow f_2 & & \downarrow f_1 \\ \mathcal{T} & \xrightarrow{g_1} & \mathcal{S} \end{array},$$

with $\mathcal{Y}, \mathcal{T}, \mathcal{S}, \mathcal{X} \in \text{Cat}$ with topology $\tau_{\mathcal{Y}}, \tau_{\mathcal{T}}, \tau_{\mathcal{S}}, \tau_{\mathcal{X}}$, the maps, for $F \in C(\mathcal{X})$,

$$T(D)(F) : g_1^* f_{1*} F \rightarrow f_{2*} g_2^* F$$

induce in the derived category the maps in $D_{\text{fil},r}(\mathcal{T})$, given by, for $(G, F) \in D_{\text{fil},r}(\mathcal{X})$ with $(G, F) = D(\tau_{\mathcal{X}}, r)((G, F))$,

$$\begin{array}{ccc} g_1^* Rf_{1*}(M, F) & \xrightarrow{T(D)(M, F)} & Rf_{2*} g_2^*(M, F) \\ \uparrow = & & \uparrow = \\ g_1^* f_{1*}(E(G, F)) & \xrightarrow{k \circ T(D)(E(G, F))} & f_{2*} E(g_2^*(E(G, F))) \end{array}.$$

Let $\mathcal{S} \in \text{Cat}$ a site with topology τ . The tensor product of complexes of abelian groups and the internal hom of presheaves on \mathcal{S}

$$((\cdot \otimes \cdot), \mathcal{H}om^\bullet(\cdot, \cdot)) : C(\mathcal{S})^2 \rightarrow C(\mathcal{S}),$$

is a Quillen adjunction which induces in the derived category

$$((\cdot \otimes^L \cdot), R\mathcal{H}om^\bullet(\cdot, \cdot)) : D_{fil,r}(S)^2 \rightarrow D_{fil,r}(S), \quad R\mathcal{H}om^\bullet((M, W), (N, W)) = \mathcal{H}om^\bullet((Q, W), E(G, F)),$$

where, Q is projectively cofibrant such that $M = D(\tau)(Q^\bullet)$ and G such that $N = D(\tau)(G)$.

Let $i : Z \hookrightarrow S$ a closed embedding, with $S, Z \in \text{Top}$. Denote by $j : S \setminus Z \hookrightarrow S$ the open embedding of the complementary subset. The adjunction

$$(i_*, i^!) := (i_*, i^!) : C(Z) \rightarrow C(S), \quad \text{with in this case } i^!F := \ker(F \rightarrow j_*j^*F)$$

is a Quillen adjunction. Since $i^!$ preserve monomorphisms, we have also Quillen adjunctions

$$(i_*, i^!) : C_{(2)fil}(Z) \rightarrow C_{(2)fil}(S), \quad \text{with } i^!(G, F) = (i^!G, F).$$

which induces in the derived category (i_* derive trivially)

$$(i_*, Ri^!) : D_{(2)fil}(Z) \rightarrow D_{(2)fil}(S), \quad \text{with } Ri^!(G, F) = i^!E(G, F).$$

3 Triangulated category of motives

3.1 Definition and the six functor formalism

The category of motives is obtained by inverting the (\mathbb{A}_S^1, et) equivalence. Hence the \mathbb{A}_S^1 local complexes of presheaves play a key role.

Definition 39. *The derived category of motives of complex algebraic varieties over S is the category*

$$DA(S) := \text{Ho}_{\mathbb{A}_S^1, et}(C(\text{Var}(\mathbb{C})^{sm}/S)),$$

which is the localization of the category of complexes of presheaves on $\text{Var}(\mathbb{C})^{sm}/S$ with respect to (\mathbb{A}_S^1, et) local equivalence and we denote by

$$D(\mathbb{A}_S^1, et) := D(\mathbb{A}_S^1) \circ D(et) : C(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow DA(S)$$

the localization functor. We have $DA^-(S) := D(\mathbb{A}_S^1, et)(\text{PSh}(\text{Var}(\mathbb{C})^{sm}/S, C^-(\mathbb{Z}))) \subset DA(S)$ the full subcategory consisting of bounded above complexes.

Definition 40. *The stable derived category of motives of complex algebraic varieties over S is the category*

$$DA_{st}(S) := \text{Ho}_{\mathbb{A}_S^1, et}(C_\Sigma(\text{Var}(\mathbb{C})^{sm}/S)),$$

which is the localization of the category of \mathbb{G}_{mS} -spectra ($\Sigma F^\bullet = F^\bullet \otimes \mathbb{G}_{mS}$) of complexes of presheaves on $\text{Var}(\mathbb{C})^{sm}/S$ with respect to (\mathbb{A}_S^1, et) local equivalence. The functor

$$\Sigma^\infty : C(\text{Var}(\mathbb{C})^{sm}/S) \hookrightarrow C_\Sigma(\text{Var}(\mathbb{C})^{sm}/S)$$

induces the functor $\Sigma^\infty : DA(S) \rightarrow DA_{st}(S)$.

We have all the six functor formalism by [10]. We give a list of the operation we will use :

- For $f : T \rightarrow S$ a morphism with $S, T \in \text{Var}(\mathbb{C})$, the adjunction

$$(f^*, f_*) : C(\text{Var}(\mathbb{C})^{sm}/S) \rightleftarrows C(\text{Var}(\mathbb{C})^{sm}/T)$$

is a Quillen adjunction which induces in the derived categories (f^* derives trivially), $(f^*, Rf_*) : DA(S) \rightleftarrows DA(T)$.

- For $h : V \rightarrow S$ a smooth morphism with $V, S \in \text{Var}(\mathbb{C})$, the adjunction

$$(h_{\sharp}, h^*) : C(\text{Var}(\mathbb{C})^{sm}/V) \rightleftarrows C(\text{Var}(\mathbb{C})^{sm}/S)$$

is a Quillen adjunction which induces in the derived categories (h^* derive trivially) $(Lh_{\sharp}, h^*) =: \text{DA}(V) \rightleftarrows \text{DA}(S)$.

- For $i : Z \hookrightarrow S$ a closed embedding, with $Z, S \in \text{Var}(\mathbb{C})$,

$$(i_*, i^!) := (i_*, i^\perp) : C(\text{Var}(\mathbb{C})^{sm}/Z) \rightleftarrows C(\text{Var}(\mathbb{C})^{sm}/S)$$

is a Quillen adjunction, which induces in the derived categories (i_* derive trivially) $(i_*, Ri^!) : \text{DA}(Z) \rightleftarrows \text{DA}(S)$. The fact that i_* derive trivially (i.e. send (\mathbb{A}^1, et) local equivalence to (\mathbb{A}^1, et) local equivalence is proved in [4].

- For $S \in \text{Var}(\mathbb{C})$, the adjunction given by the tensor product of complexes of abelian groups and the internal hom of presheaves

$$((\cdot \otimes \cdot), \mathcal{H}om^\bullet(\cdot, \cdot)) : C(\text{Var}(\mathbb{C})^{sm}/S)^2 \rightarrow C(\text{Var}(\mathbb{C})^{sm}/S),$$

is a Quillen adjunction, which induces in the derived category

$$, ((\cdot \otimes^L \cdot), R\mathcal{H}om^\bullet(\cdot, \cdot)) : \text{DA}(S)^2 \rightarrow \text{DA}(S),$$

- Let $M, N \in \text{DA}(S)$, Q^\bullet projectively cofibrant such that $M = D(\mathbb{A}^1, et)(Q^\bullet)$, and G^\bullet be \mathbb{A}^1 local for the etale topology such that $N = D(\mathbb{A}^1, et)(G^\bullet)$. Then,

$$R\mathcal{H}om^\bullet(M, N) = \mathcal{H}om^\bullet(Q^\bullet, E(G^\bullet)) \in \text{DA}(S). \quad (44)$$

This is well defined since if $s : Q_1 \rightarrow Q_2$ is a etale local equivalence,

$$\mathcal{H}om(s, E(G)) : \mathcal{H}om(Q_1, E(G)) \rightarrow \mathcal{H}om(Q_2, E(G))$$

is a etale local equivalence for $1 \leq i \leq l$.

- For a commutative diagram in $\text{Var}(\mathbb{C})$:

$$D = \begin{array}{ccc} Y & \xrightarrow{g_2} & X \\ \downarrow f_2 & & \downarrow f_1 \\ T & \xrightarrow{g_1} & S \end{array},$$

and $F \in C(\text{Var}(\mathbb{C})^{sm}/X)$, the transformation map $T(D)(F) : g_1^* f_{1*} F \rightarrow f_{2*} g_2^* F$ induces in the derived category, for $M \in \text{DA}(X)$, $M = D(\mathbb{A}^1, et)(F)$ with F \mathbb{A}^1 local for the etale topology,

$$\begin{array}{ccc} g_1^* Rf_{1*} M & \xrightarrow{T(D)(M)} & Rf_{2*} g_2^* M \\ \uparrow = & & \uparrow = \\ g_1^* f_{1*} E(F) & \xrightarrow{k \circ c \circ T(D)(E(F))} & f_{2*} E(C_*(g_2^* E(F))) \end{array}$$

If D is cartesian with $f_1 = f$, $g_1 = g$, $f_2 = f'$, $g_2 = g'$, we denote

- $T(D)(F) =: T(f, g)(F) : g^* f_* F \rightarrow f'_* g'^* F$,
- $T(D)(M) =: T(f, g)(M) : g^* Rf_* M \rightarrow Rf'_* g'^* M$.

We get from the first point 2 functors :

- The 2-functor $C(\mathrm{Var}(\mathbb{C})^{sm}/\cdot) : \mathrm{Var}(\mathbb{C}) \rightarrow \mathrm{Ab\,Cat}$, given by

$$S \mapsto C(\mathrm{Var}(\mathbb{C})^{sm}/S), (f : T \rightarrow S) \mapsto (f^* : C(\mathrm{Var}(\mathbb{C})^{sm}/S) \rightarrow C(\mathrm{Var}(\mathbb{C})^{sm}/T)).$$

- The 2-functor $\mathrm{DA}(\cdot) : \mathrm{Var}(\mathbb{C}) \rightarrow \mathrm{TriCat}$, given by

$$S \mapsto \mathrm{DA}(S), (f : T \rightarrow S) \mapsto (f^* : \mathrm{DA}(S) \rightarrow \mathrm{DA}(T)).$$

The main theorem is the following :

Theorem 14. [4][10] *The 2-functor $\mathrm{DA}(\cdot) : \mathrm{Var}(\mathbb{C}) \rightarrow \mathrm{TriCat}$, given by*

$$S \mapsto \mathrm{DA}(S), (f : T \rightarrow S) \mapsto (f^* : \mathrm{DA}(S) \rightarrow \mathrm{DA}(T))$$

is a 2-homotopic functor ([4])

From theorem 14, we get in particular

- For $f : T \rightarrow S$ a morphism with $T, S \in \mathrm{Var}(\mathbb{C})$, there by theorem 14 is also a pair of adjoint functor

$$(f_!, f^!) : \mathrm{DA}(S) \rightleftarrows \mathrm{DA}(T)$$

- with $f_! = Rf_*$ if f is proper,
- with $f^! = f^*[d]$ if f is smooth of relative dimension d .

For $h : U \rightarrow S$ a smooth morphism with $U, S \in \mathrm{Var}(\mathbb{C})$ irreducible, have, for $M \in \mathrm{DA}(U)$, an isomorphism

$$Lh_{\sharp}M \rightarrow h_!M[d], \tag{45}$$

in $\mathrm{DA}(S)$.

- For $i : Z \hookrightarrow X$ a closed embedding with $Z, X \in \mathrm{Var}(\mathbb{C})$, denote by $j : S \setminus Z \hookrightarrow S$ the open complementary subset. Then :

- the functor $(i^*, j^*) : \mathrm{DA}(S) \rightarrow \mathrm{DA}(Z) \times \mathrm{DA}(S \setminus Z)$ is conservative,
- on the other hand, it is easy to see that, for $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/Z)$, the adjunction map $\mathrm{ad}(i^*, i_*)(F) : i^*i_*F \rightarrow F$ is an equivalence Zariski local, hence for $M \in \mathrm{DA}(S)$, the induced map in the derived category

$$\mathrm{ad}(i^*, i_*)(M) : i^*i_*M \xrightarrow{\sim} M$$

is an isomorphism.

- For $f : X \rightarrow S$ a proper map, $g : T \rightarrow S$ a morphism, with $T, X, S \in \mathrm{Var}(\mathbb{C})$, and $M \in \mathrm{DA}(X)$,

$$T(f, g)(M) : g^*Rf_*M \rightarrow Rf'_*\tilde{g}^*M$$

is an isomorphism in $\mathrm{DA}(T)$ if f is proper.

Definition 41. *The derived category of extended motives of complex algebraic varieties over S is the category*

$$\underline{\mathrm{DA}}(S) := \mathrm{Ho}_{\mathbb{A}_S^1, et}(C(\mathrm{Var}(\mathbb{C})/S)),$$

which is the localization of the category of complexes of presheaves on $\mathrm{Var}(\mathbb{C})/S$ with respect to (\mathbb{A}_S^1, et) local equivalence and we denote by

$$D(\mathbb{A}_S^1, et) := D(\mathbb{A}_S^1) \circ D(et) : C(\mathrm{Var}(\mathbb{C})/S) \rightarrow \underline{\mathrm{DA}}(S)$$

the localization functor. We have $\underline{\mathrm{DA}}^-(S) := D(\mathbb{A}_S^1, et)(\mathrm{PSh}(\mathrm{Var}(\mathbb{C})/S, C^-(\mathbb{Z}))) \subset \underline{\mathrm{DA}}(S)$ the full subcategory consisting of bounded above complexes.

Remark 5. Let $i : Z \hookrightarrow S$ a closed embedding, with $Z, S \in \text{Var}(\mathbb{C})$.

(i) By theorem 14, for $X/S = (X, h) \in \text{Var}(\mathbb{C})^{sm}/S$,

$$(0, \text{ad}(i^*, i_*)(\mathbb{Z}(X/S))) : \Gamma_Z^\vee \mathbb{Z}(X/S) \rightarrow i_* \mathbb{Z}(X_Z/Z)$$

is an equivalence $(\mathbb{A}^1, \text{et})$ local.

(ii) For $X/S = (X, f) \in \text{Var}(\mathbb{C})/S$,

$$(0, \text{ad}(i^*, i_*)(\mathbb{Z}(X/S))) : \Gamma_Z^\vee \mathbb{Z}(X/S) \rightarrow i_* \mathbb{Z}(X_Z/Z)$$

is NOT an equivalence $(\mathbb{A}^1, \text{et})$ local in general, since for example if $f(X) = Z \subset S$, $\rho_{S*} \mathbb{Z}(X/S) = 0$ but $D(\mathbb{A}^1, \text{et})(\rho_{S*} i_* \mathbb{Z}(X_Z/Z) = i_* \rho_{S*} \mathbb{Z}(X_Z/Z)) \neq 0 \in \underline{\text{DA}}(S)$, hence it is NOT an equivalence $(\mathbb{A}^1, \text{et})$ local in this case by proposition 20. In particular $\underline{\text{DA}}(S)$ does NOT satisfy the localization property.

(ii)' For $X/Z = (X, f) \in \text{Var}(\mathbb{C})/Z$, the inclusion

$$T(i_{\sharp}, i_*) : i_{\sharp} \mathbb{Z}(X/Z) \hookrightarrow i_* \mathbb{Z}(X/Z)$$

is NOT an equivalence $(\mathbb{A}^1, \text{et})$ local by proposition 20 since $\rho_{S*} i_{\sharp} \mathbb{Z}(X/Z) = 0$ but $D(\mathbb{A}^1, \text{et})(\rho_{S*} i_* \mathbb{Z}(X/Z) = i_* \rho_{S*} \mathbb{Z}(X/Z)) \neq 0 \in \underline{\text{DA}}(S)$,

Definition 42. (i) Let $f : X \rightarrow S$ a smooth proper morphism with $X, S \in \text{Var}(\mathbb{C})$ of relative dimension $d = d_X - d_S$. Then, we have the equivalence $(\mathbb{A}^1, \text{et})$ local in $C(\text{Var}(\mathbb{C})^{sm}/S)$

$$T(f_{\sharp}, f_*)(\mathbb{Z}(X/X)) := [\Delta_X] : f_{\sharp} \mathbb{Z}(X/X) \rightarrow f_* E_{\text{et}}(\mathbb{Z}(X/X))(d)[2d]$$

given by the diagonal

$$[\Delta_X] \in \text{CH}^d(X \times_S X) = \text{Hom}_{\underline{\text{DA}}(S)}(f_{\sharp} \mathbb{Z}(X/X), f_* E_{\text{et}}(\mathbb{Z}(X/X))(d)[2d]).$$

(ii) Let $f : X \rightarrow S$ a proper surjective morphism with $X, S \in \text{Var}(\mathbb{C})$ of relative dimension $d = d_X - d_S$. Then, we have the equivalence $(\mathbb{A}^1, \text{et})$ local in $C(\text{Var}(\mathbb{C})/S)$

$$T(f_{\sharp}, f_*)(\mathbb{Z}(X/X)) := [\Delta_X] : f_{\sharp} \mathbb{Z}(X/X) \rightarrow f_* E_{\text{et}}(\mathbb{Z}(X/X))(d)[2d]$$

given by the diagonal

$$[\Delta_X] \in \text{CH}^d(X \times_S X) = \text{Hom}_{\underline{\text{DA}}(S)}(f_{\sharp} \mathbb{Z}(X/X), f_* E_{\text{et}}(\mathbb{Z}(X/X))(d)[2d]).$$

Note that if $f : X \rightarrow S$ is not surjective, $d < 0$ and

$$\text{Hom}_{\underline{\text{DA}}(S)}(f_{\sharp} \mathbb{Z}(X/X), f_* E_{\text{et}}(\mathbb{Z}(X/X))(d)[2d]) = \text{CH}^d(X \times_S X) = 0.$$

(ii)' Let $f : X \rightarrow S$ a proper surjective morphism with $X, S \in \text{Var}(\mathbb{C})$ of relative dimension $d = d_X - d_S$. Then, we have, for $i : Z \hookrightarrow X$ a closed embedding, the equivalence $(\mathbb{A}^1, \text{et})$ local in $C(\text{Var}(\mathbb{C})/S)$

$$T(f_{\sharp}, f_*)(\mathbb{Z}(Z/X)) := [\Delta_Z] : f_{\sharp} \mathbb{Z}(Z/X) \rightarrow f_* E_{\text{et}}(\mathbb{Z}(Z/X))(d)[2d]$$

given by the diagonal (note that $\mathbb{Z}(Z/X)$ is trivially \mathbb{A}^1 invariant since $\mathbb{Z}(Z/X)(S/S) = \mathbb{Z}(Z/X)(\mathbb{A}^1 \times S/S) = 0$, hence $f_* E_{\text{et}}(\mathbb{Z}(Z/X))(d)[2d]$ is \mathbb{A}^1 local for the etale topology)

$$[\Delta_Z] \in \text{CH}^d(Z \times_S Z) = \text{Hom}_{\underline{\text{DA}}(S)}(f_{\sharp} \mathbb{Z}(Z/X), f_* E_{\text{et}}(\mathbb{Z}(Z/X))(d)[2d]).$$

Note that if $f : X \rightarrow S$ is not surjective, $d < 0$ and

$$\text{Hom}_{\underline{\text{DA}}(S)}(f_{\sharp} \mathbb{Z}(Z/X), f_* E_{\text{et}}(\mathbb{Z}(Z/X))(d)[2d]) = \text{CH}^d(Z \times_S Z) = 0.$$

3.2 Constructible motives and resolution of a motive by Corti-Hanamura motives

We now give the definition of the motives of morphisms $f : X \rightarrow S$ which are constructible motives and the definition of the category of Corti-Hanamura motives.

Definition 43. Let $S \in \text{Var}(\mathbb{C})$,

- the homological motive functor is $M(/S) : \text{Var}(\mathbb{C})/S \rightarrow \text{DA}(S)$, $(f : X \rightarrow S) \mapsto M(X/S) := f_! f^! M(S/S)$,
- the cohomological motive functor is $M^\vee(/S) : \text{Var}(\mathbb{C})/S \rightarrow \text{DA}(S)$, $(f : X \rightarrow S) \mapsto M(X/S)^\vee := Rf_* M(X/X) = f_* E(\mathbb{Z}_X)$,
- the Borel-Moore motive functor is $M^{BM}(/S) : \text{Var}(\mathbb{C})/S \rightarrow \text{DA}(S)$, $(f : X \rightarrow S) \mapsto M^{BM}(X/S) := f_! M(X/X)$,
- the (homological) motive with compact support functor is $M_c(/S) : \text{Var}(\mathbb{C})/S \rightarrow \text{DA}(S)$, $(f : X \rightarrow S) \mapsto M_c(X/S) := Rf_* f^! M(S/S)$.

Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume that there exist a factorization $f : X \xrightarrow{i} Y \times S \xrightarrow{p} S$, with $Y \in \text{SmVar}(\mathbb{C})$, $i : X \hookrightarrow Y$ is a closed embedding and p the projection. Then,

$$Q(X/S) := p_! \Gamma_X^\vee \mathbb{Z}_{Y \times S} [d_Y] \in C(\text{Var}(\mathbb{C})^{sm}/S)$$

(see definition 10) is projective, admits transfert, and satisfy $D(\mathbb{A}_S^1, et)(Q^{BM}(X/S)) = M^{BM}(X/S)$.

Definition 44. (i) Let $S \in \text{Var}(\mathbb{C})$. We define the full subcategory $\text{DA}_c(S) \subset \text{DA}(S)$ whose objects are constructible motives to be the thick triangulated category generated by the motives of the form $M(X/S)$, with $f : X \rightarrow S$ a morphism, $X \in \text{Var}(\mathbb{C})$.

(ii) Let $X, S \in \text{Var}(\mathbb{C})$. If $f : X \rightarrow S$ is proper (but not necessary smooth) and X is smooth, $M(X/S)$ is said to be a Corti-Hanamura motive and we have by above in this case $M(X/S) = M^{BM}(X/S)[c] = M(X/S)^\vee[c]$, with $c = \text{codim}(X, X \times S)$ where $f : X \hookrightarrow X \times S \rightarrow S$. We denote by

$$\mathcal{CH}(S) = \{M(X/S)\}_{\{X/S=(X,f), f \text{ pr}, X \text{ sm}\}}^{pa} \subset DM(S)$$

the full subcategory which is the pseudo-abelian completion of the full subcategory whose objects are Corti-Hanamura motives.

(iii) We denote by

$$\mathcal{CH}^0(S) \subset \mathcal{CH}(S)$$

the full subcategory which is the pseudo-abelian completion of the full subcategory whose objects are Corti-Hanamura motives $M(X/S)$ such that the morphism $f : X \rightarrow S$ is projective.

For bounded above motives, we have

Theorem 15. Let $S \in \text{Var}(\mathbb{C})$.

- (i) There exists a unique weight structure ω on $\text{DA}^-(S)$ such that $\text{DA}^-(S)^{\omega=0} = \mathcal{CH}(S)$
- (ii) There exist a well defined functor

$$W(S) : \text{DA}^-(S) \rightarrow K^-(\mathcal{CH}(S)), W(S)(M) = [M^{(\bullet)}]$$

where $M^{(\bullet)} \in C^-(\mathcal{CH}(S))$ is a bounded above weight complex, such that if $m \in \mathbb{Z}$ is the highest weight, we have a generalized distinguish triangle for all $i \leq m$

$$T_i : M^{(i)}[i] \rightarrow M^{(i+1)}[(i+1)] \rightarrow \cdots \rightarrow M^{(m)}[m] \rightarrow M^{w \geq i} \quad (46)$$

Moreover the maps $w(M)^{(\geq i)} : M^{\geq i} \rightarrow M$ induce an isomorphism $w(M) : \text{holim}_i M^{\geq i} \xrightarrow{\sim} M$ in $\text{DA}^-(S)$.

(iii) Denote by $\text{Chow}(S)$ the category of Chow motives, which is the pseudo-abelian completion of the category

- whose set of objects consist of the $X/S = (X, f) \in \text{Var}(\mathbb{C})/S$ such that f is proper and X is smooth,
- whose set of morphisms between X_1/S and X_2/S is $\text{CH}^{d_1}(X_1 \times_S X_2)$, and the composition law is given in [11].

We have then a canonical functor $\text{CH}_S : \text{Chow}(S) \hookrightarrow \text{DA}(S)$, with $\text{CH}_S(X/S) := M(X/S) := \text{Rf}_*\mathbb{Z}(X/X)$, which is a full embedding whose image is the category $\mathcal{CH}(S)$.

Proof. (i): The category $\text{DA}(S)$ is clearly weakly generated by $\mathcal{CH}(S)$. Moreover $\mathcal{CH}(S) \subset \text{DA}(S)$ is negative. Hence, the result follows from [6] theorem 4.3.2 III.

(ii): Follows from (i) by standard fact of weight structure on triangulated categories. See [6] theorem 3.2.2 and theorem 4.3.2 V for example.

(iii): See [12]. □

Theorem 16. *Let $S \in \text{Var}(\mathbb{C})$.*

(i) *There exists a unique weight structure ω on $\text{DA}^-(S)$ such that $\text{DA}^-(S)^{\omega=0} = \mathcal{CH}^0(S)$*

(ii) *There exist a well defined functor*

$$W(S) : \text{DA}^-(S) \rightarrow K^-(\mathcal{CH}^0(S)), \quad W(S)(M) = [M^{(\bullet)}]$$

where $M^{(\bullet)} \in C^-(\mathcal{CH}^0(S))$ is a bounded above weight complex, such that if $m \in \mathbb{Z}$ is the highest weight, we have a generalized distinguish triangle for all $i \leq m$

$$T_i : M^{(i)}[i] \rightarrow M^{(i+1)}[(i+1)] \rightarrow \cdots \rightarrow M^{(m)}[m] \rightarrow M^{w \geq i} \quad (47)$$

Moreover the maps $w(M)^{(\geq i)} : M^{\geq i} \rightarrow M$ induce an isomorphism $w(M) : \text{holim}_i M^{\geq i} \xrightarrow{\sim} M$ in $\text{DA}^-(S)$.

Proof. Similar to the proof of theorem 15. □

Corollary 1. *Let $S \in \text{Var}(\mathbb{C})$. Let $M \in \text{DA}(S)$. Then there exist $(F, W) \in C_{\text{fil}}(\text{Var}(\mathbb{C})^{sm}/S)$ such that $D(\mathbb{A}^1, et)(F) = M$ and $D(\mathbb{A}^1, et)(\text{Gr}_p^W F) \in \mathcal{CH}^0(S)$.*

Proof. By theorem 16, there exist, by induction, for $i \in \mathbb{Z}$, a distinguish triangle in $\text{DA}(S)$

$$T_i : M^{(i)}[i] \xrightarrow{m_i} M^{(i+1)} \xrightarrow{m_{i+1}} \cdots \xrightarrow{m_{m-1}} M^{(m)}[m] \rightarrow M^{w \geq i} \quad (48)$$

with $M^{(j)}[j] \in \mathcal{CH}^0(S)$ and $w(M) : \text{holim}_i M^{\geq i} \xrightarrow{\sim} M$ in $\text{DA}^-(S)$. For $i \in \mathbb{Z}$, take $(F_j)_{j \geq i}, F_{w \geq i} \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $D(\mathbb{A}^1, et)(F_j) = M^{(j)}[j]$, $D(\mathbb{A}^1, et)(F_{w \geq i}) = M^{w \geq i}$ and such that we have in $C(\text{Var}(\mathbb{C})^{sm}/S)$,

$$F_{w \geq i} = \text{Cone}(F_i \xrightarrow{m_i} F_{i+1} \xrightarrow{m_{i+1}} \cdots \xrightarrow{m_{m-1}} F_m) \quad (49)$$

where $m_j : F_j \rightarrow F_{j+1}$ are morphisms in $C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $D(\mathbb{A}^1, et)(m_j) = m_j$. Now set $F = \text{holim}_i F_{w \geq i} \in C(\text{Var}(\mathbb{C})^{sm}/S)$ and $W_i F := F_{w \geq i} \hookrightarrow F$, so that $(F, W) \in C_{\text{fil}}(\text{Var}(\mathbb{C})^{sm}/S)$ satisfy $D(\mathbb{A}^1, et)(\text{Gr}_p^W F) = M^{(p)}[p] \in \mathcal{CH}^0(S)$. □

3.3 The restriction of relative motives to their Zariski sites

Let $S \in \text{Var}(\mathbb{C})$. The adjunction

$$(e(S)^*, e(S)_*) : C(\text{Var}(\mathbb{C})^{sm}/S) \rightleftarrows C(S)$$

is a Quillen adjunction and induces in the derived category

- $(e(S)^*, e(S)_*) : \text{Ho}_{zar}(\text{Var}(\mathbb{C})^{sm}/S) \rightleftarrows D(T) := \text{Ho}_{zar} C(S)$, since $e(S)_*$ sends Zariski local equivalence on the big site $\text{Var}(\mathbb{C})^{sm}/S$ to Zariski local equivalence in the small Zariski site of S ,
- $(e(S)^*, Re(S)_*) : \text{DA}(S) \rightleftarrows D(T) := \text{Ho}_{zar} C(S)$.

We will use in the definition of the De Rahm realization functor on $\text{DA}(S)$ the following proposition concerning the restriction of the derived internal hom functor to the Zariski site :

Proposition 31. *Let $M, N \in \text{DA}(S)$ and $m : M \rightarrow N$ be a morphism. Let $F^\bullet, G^\bullet \in \text{PSh}(\text{Var}(\mathbb{C})^{sm}/S, C(\mathbb{Z}))$ such that $M = D(\mathbb{A}_S^1, et)(F^\bullet)$ and $N = D(\mathbb{A}_S^1, et)(G^\bullet)$. If we take G^\bullet (\mathbb{A}_S^1, et) fibrant and admitting transfert, and F^\bullet cofibrant for the projective model structure, we have*

$$Re(S)_* R\mathcal{H}om^\bullet(M, N) = e(S)_* \mathcal{H}om^\bullet(F^\bullet, G^\bullet)$$

in $D(S)$.

Proof. Since F^\bullet is projectively cofibrant and G^\bullet is (projectively) (\mathbb{A}_S^1, et) fibrant, we have $R\mathcal{H}om^\bullet(M, N) = \mathcal{H}om^\bullet(F^\bullet, G^\bullet)$. Then, $\mathcal{H}om^\bullet(F^\bullet, G^\bullet)$ is \mathbb{A}_S^1 local and admits transfert. On the other hand, we have

$$\mathcal{L}_{\mathbb{A}_S^1} D_{et}(\text{Cor Var}(\mathbb{C})^{sm}/S) = \mathcal{L}_{\mathbb{A}_S^1} D_{zar}(\text{Cor Var}(\mathbb{C})^{sm}/S) \subset D(\text{Var}(\mathbb{C})^{sm}/S)$$

by theorem 10 (ii). This gives the equality of the proposition. \square

We will also have :

Proposition 32. *For $f : T \rightarrow S$ a morphism and $i : Z \hookrightarrow S$ a closed embedding, with $Z, S, T \in \text{Var}(\mathbb{C})$, we have*

$$(i) \quad Re(S)_* Rf_* = Rf_* Re(T)_* \quad \text{and} \quad e(S)^* Rf_* = Rf_* e(T)^*$$

$$(ii) \quad Re(S)_* R\Gamma_Z = R\Gamma_Z Re(S)_*.$$

Proof. (i): Follows from proposition 16 (i) and the fact that f_* preserve (\mathbb{A}^1, et) fibrant complex of presheaves.

(ii): Follows from proposition 16 (ii) and the fact that Γ_Z preserve (\mathbb{A}^1, et) fibrant complex of presheaves. \square

3.4 Motives of complex analytic spaces

The category of motives is obtained by inverting the (\mathbb{D}_S^1, usu) local equivalence. Hence the \mathbb{D}_S^1 local complexes of presheaves plays a key role.

Definition 45. *The derived category of motives of complex algebraic varieties over S is the category*

$$\text{AnDA}(S) := \text{Ho}_{\mathbb{D}_S^1, usu}(C(\text{AnSp}(\mathbb{C})^{sm}/S)),$$

which is the localization of the category of complexes of presheaves on $\text{AnSp}(\mathbb{C})^{sm}/S$ with respect to (\mathbb{D}_S^1, usu) local equivalence and we denote by

$$D(\mathbb{D}_S^1, usu) := D(\mathbb{A}_S^1) \circ D(et) : C(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow \text{AnDA}(S)$$

the localization functor. We have $\text{DA}^-(S) := D(\mathbb{D}_S^1, usu)(\text{PSh}(\text{AnSp}(\mathbb{C})^{sm}/S, C^-(\mathbb{Z}))) \subset \text{DA}(S)$ the full subcategory consisting of bounded above complexes.

Theorem 17. *Let $S \in \text{AnSp}(\mathbb{C})$. The adjunction $(e(S)^*, e(S)_*) : C(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow C(S)$ induces an equivalence of categories*

$$(e(S)^*, Re(S)_*) : \text{AnDM}(S) \xrightarrow{\sim} D(S).$$

In particular, for $F \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$, the adjunction map $\text{ad}(e(S)^, e(S)_*)(F) : e(S)^*e(S)_*F \rightarrow F$ is an equivalence (\mathbb{D}^1, usu) local.*

Proof. See [1]. □

We deduce from this theorem the following :

Proposition 33. *Let $S \in \text{AnSp}(\mathbb{C})$. Let $F, G \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$. If G is \mathbb{D}^1 local, then the canonical map*

$$T(e, \text{hom})(F, G) : e(S)_*\mathcal{H}om(F, G) \rightarrow \mathcal{H}om(e(S)_*F, e(S)_*G)$$

is an equivalence usu local.

Proof. The map $T(e, \text{hom})(F, G)$ is the composite

$$T(e, \text{hom})(F, G) : e(S)_*\mathcal{H}om(F, G) \xrightarrow{\mathcal{H}om(\text{ad}(e(S)^*, e(S)_*)(F), G)} e(S)_*\mathcal{H}om(e(S)^*e(S)_*F, G) \\ \xrightarrow{I(e(S)^*, e(S)_*)(F, G)} \mathcal{H}om(e(S)_*F, e(S)_*G)$$

where the last map is the adjunction isomorphism. The first map is an isomorphism by theorem 17 since G is \mathbb{D}^1 local. □

4 The category of filtered D modules on commutative ringed topos, on commutative ringed spaces, complex algebraic varieties complex analytic spaces and the functorialities

4.1 The The category of filtered D modules on commutative ringed topos, on commutative ringed spaces, and the functorialities

4.1.1 Definitions et functorialities

Let $(\mathcal{S}, O_S) \in \text{RCat}$ with O_S commutative. Recall that $\Omega_{O_S} := \mathbb{D}_S^O(\mathcal{I}_S/\mathcal{I}_S^2) \in \text{PSh}_{O_S}(S)$ is the universal derivation O_S -module together with its derivation map $d : O_S \rightarrow \Omega_{O_S}$, where $\mathcal{I}_S = \ker(s_S : O_S \otimes O_S \rightarrow O_S) \in \text{PSh}_{O_S \times O_S}(S)$ the diagonal ideal.

In the particular case of a ringed space $(S, O_S) \in \text{RTop}$, $s_S : O_S \otimes O_S = \Delta_S^*(p_1^*O_S \otimes p_2^*O_S) \rightarrow O_S$ is the structural morphism of diagonal embedding $\Delta_S : (S, O_S) \hookrightarrow (S \times S, p_1^*O_S \otimes p_2^*O_S)$, $p_1 : S \times S \rightarrow S$ and $p_2 : S \times S \rightarrow S$ being the projections. More generally, for $k \in \mathbb{N}$, $k \geq 1$ we have the sheaf of k -jets $J^k(O_S) := \Delta_S^*\mathcal{I}_S/\mathcal{I}_S^{k+1}$ with in particular $J^1(O_S) = T_S$. We have, for $s \in S$, $J^k(O_S)_s = m_s/m_s^k$ where $m_s \subset O_{S,s}$ is the maximal ideal if $O_{S,s}$ is a local ring.

Definition 46. (i) *Let $(\mathcal{S}, O_S) \in \text{RCat}$ with O_S a commutative sheaf of ring and \mathcal{S} is endowed with a topology τ . We denote by*

$$D(O_S) = \langle O_S, \text{Der}_{O_S}(O_S, O_S) \rangle_{\subset} a_\tau \mathcal{H}om(O_S, O_S)$$

the subsheaf of ring generated by O_S and the subsheaf of derivations $\text{Der}_{O_S}(O_S, O_S) = T_S := \mathbb{D}_S^O \Omega_{O_S}$, $a_\tau : \text{PSh}(\mathcal{S}) \rightarrow \text{Shv}(\mathcal{S})$ being the sheaftification functor.

(ii) *Let $f : \mathcal{X} \rightarrow \mathcal{S}$ be a morphism of site, with $\mathcal{X}, \mathcal{S} \in \text{Cat}$ endowed with topology τ , resp. τ' , and $O_S \in \text{PSh}(\mathcal{S}, \text{cRing})$ a commutative sheaf of ring. We will note in this case by abuse $f^*O_S := a_\tau f^*O_S$ and $f^*D(O_S) := a_\tau f^*D(O_S)$, $a_\tau : \text{PSh}(\mathcal{X}) \rightarrow \text{Shv}(\mathcal{X})$ being the sheaftification functor.*

Let $f : \mathcal{X} \rightarrow \mathcal{S}$ a morphism of site, with $\mathcal{X}, \mathcal{S} \in \text{Cat}$ endowed with topology τ , resp. τ' , and $O_S \in \text{PSh}(\mathcal{S}, \text{cRing})$ a commutative sheaf of ring. Consider the ringed space $(\mathcal{X}, f^*O_S) := (\mathcal{X}, a_\tau f^*O_S) \in \text{RCat}$, $a_\tau : \text{PSh}(\mathcal{X}) \rightarrow \text{Shv}(\mathcal{X})$ being the sheafification functor. Then, the map in $\text{PSh}(\mathcal{X})$

$$T(f, \text{hom})(O_S, O_S) : f^* \mathcal{H}om(O_S, O_S) \rightarrow \mathcal{H}om(f^*O_S, f^*O_S)$$

induces a canonical morphism of sheaf of rings

$$T(f, \text{hom})(O_S, O_S) : a_\tau f^* D(O_S) =: f^* D(O_S) \rightarrow D(a_\tau f^* O_S) =: D(f^* O_S).$$

In the special case of ringed spaces, we have then :

Proposition 34. *Let $f : X \rightarrow S$ is a continuous map, with $X, S \in \text{Top}$ and $O_S \in \text{PSh}(S, \text{cRing})$ a commutative sheaf of ring. Consider the ringed space $(X, f^*O_S) := (X, a_\tau f^*O_S) \in \text{RTop}$, $a_\tau : \text{PSh}(X) \rightarrow \text{Shv}(X)$ being the sheafification functor. Then, the map in $\text{PSh}(X)$*

$$T(f, \text{hom})(O_S, O_S) : f^* \mathcal{H}om(O_S, O_S) \rightarrow \mathcal{H}om(f^*O_S, f^*O_S)$$

induces a canonical isomorphism of sheaf of rings

$$T(f, \text{hom})(O_S, O_S) : f^* D(O_S) =: a_\tau f^* D(O_S) \xrightarrow{\sim} D(a_\tau f^* O_S) =: D(f^* O_S).$$

Proof. For all $x \in X$,

$$T(f, \text{hom})(O_S, O_S)_x : (f^* D(O_S))_x \xrightarrow{\sim} D(O_{S, f(x)}) \xrightarrow{\sim} (D(f^* O_S))_x.$$

Hence, since $a_\tau f^* D(O_S)$ and $D(a_\tau f^* O_S)$ are sheaves,

$$T(f, \text{hom})(O_S, O_S) : f^* D(O_S) =: a_\tau f^* D(O_S) \xrightarrow{\sim} D(a_\tau f^* O_S) =: D(f^* O_S).$$

is an isomorphism □

We will consider presheaves of $D(O_S)$ modules on a ringed topos (\mathcal{S}, O_S) :

Definition 47. *Let $(\mathcal{S}, O_S) \in \text{RCat}$ with O_S a commutative sheaf of ring.*

(i) *We will consider $\text{PSh}_{D(O_S)}(\mathcal{S})$ the category of presheaves of (left) $D(O_S)$ modules on \mathcal{S} and $C_{D(O_S)}(\mathcal{S}) := C(\text{PSh}_{D(O_S)}(\mathcal{S}))$ its category of complexes. We will consider $\text{PSh}_{D(O_S)^{op}}(\mathcal{S})$ the category of presheaves of right $D(O_S)$ modules on \mathcal{S} and $C_{D(O_S)^{op}}(\mathcal{S}) := C(\text{PSh}_{D(O_S)^{op}}(\mathcal{S}))$ its category of complexes. We denote again by abuse*

$$\text{PSh}_{D(O_S) \text{fil}}(\mathcal{S}) = (\text{PSh}_{D(O_S)}(\mathcal{S}), F) := (\text{PSh}_{(D(O_S), F^{ord})}(\mathcal{S}), F)$$

the category of filtered $(D(O_S), F^{ord})$ -module, with, for $-p \leq 0$, $F^{ord, -p} D(O_S) = \{P \in D(O_S), \text{ord}(P) \leq p\}$ and $F^{ord, p} D(O_S) = 0$ for $p > 0$,

- *whose objects are $(M, F) \in (\text{PSh}_{O_S}(\mathcal{S}), F)$ together with a map of filtered presheaves of O_S modules $md : (M, F) \otimes_{O_S} (D(O_S), F^{ord}) \rightarrow (M, F)$, i.e. a map in $(\text{PSh}_{O_S}(\mathcal{S}), F)$,*
- *whose morphism $\phi : (M_1, F) \rightarrow (M_2, F)$ are as usual the morphisms of presheaves $\phi : M_1 \rightarrow M_2$ which are morphism of filtered presheaves (i.e. $\phi(F^p M_1) \subset F^p M_2$) and which are $D(O_S)$ linear (in particular O_S linear).*

Note that this a NOT the category of filtered $D(O_S)$ modules in the usual sense, that is the $(M, F) \in (\text{PSh}_{O_S}(\mathcal{S}), F)$ together with a map $md : (M, F) \otimes_{O_S} D(O_S) \rightarrow (M, F)$ in $(\text{PSh}_{O_S}(\mathcal{S}), F)$, since F^{ord} is NOT the trivial filtration. More precisely the O_S submodules $F^p M \subset M$ are NOT $D(O_S)$ submodules but satisfy Griffithz transversality. We denote by

$$\text{PSh}_{D(O_S)0 \text{fil}}(\mathcal{S}) \subset \text{PSh}_{D(O_S) \text{fil}}(\mathcal{S}), \text{PSh}_{D(O_S)(1,0) \text{fil}}(\mathcal{S}) \subset \text{PSh}_{D(O_S)2 \text{fil}}(\mathcal{S}) := (\text{PSh}_{(D(O_S), F^{ord})}(\mathcal{S}), F, W)$$

the full subcategory consisting of filtered $D(O_S)$ module in the usual sense, resp. the full subcategory such that W is a filtration by $D(O_S)$ submodules.

(ii) We denote again by

$$C_{D(O_S)fil}(\mathcal{S}) \subset C(\text{PSh}_{D(O_S)}(\mathcal{S}), F), C_{D(O_S)2fil}(\mathcal{S}) \subset C(\text{PSh}_{D(O_S)}(\mathcal{S}), F, W)$$

the full subcategory of complexes such that the filtration(s) is (are) regular. We will consider also

$$C_{D(O_S)0fil}(\mathcal{S}) \subset C_{D(O_S)fil}(\mathcal{S}), C_{D(O_S)(1,0)fil}(\mathcal{S}) \subset C_{D(O_S)2fil}(\mathcal{S})$$

the full subcategory consisting of complexes of filtered $D(O_S)$ modules in the usual sense (i.e. by $D(O_S)$ submodule), respectively the full subcategory consisting of complexes of bifiltered $D(O_S)$ modules such that $W^p M \subset M$ are $D(O_S)$ submodules i.e. the filtration W is a filtration in the usual sense, but NOT F wich satisfy only Griffith transversality.

Proposition 35. *Let $(\mathcal{S}, O_S) \in \text{RCat}$ with a O_S commutative sheaf of ring.*

(i) *Let $M \in \text{PSh}_{O_S}(\mathcal{S})$. Then, there is a one to one correspondence between*

- *the $D(O_S)$ module structure on M compatible with the O_S module structure, that is the maps $md : D(O_S) \otimes_{O_S} M \rightarrow M$ in $\text{PSh}_{O_S}(\mathcal{S})$ and*
- *the integrable connexions on M , that is the maps $\nabla : M \rightarrow \Omega_{O_S} \otimes_{O_S} M$ satisfying $\nabla \circ \nabla = 0$ with $\nabla : \Omega_{O_S} \otimes_{O_S} M \rightarrow \Omega_{O_S}^2 \otimes_{O_S} M$ given by $\nabla(\omega \otimes m) = (d\omega) \otimes m + \omega \wedge \nabla(m)$*

(ii) *Let $(M, F) \in \text{PSh}_{O_S fil}(\mathcal{S})$. Then, there is a one to one correspondence between*

- *the $D(O_S)$ module structure on (M, F) compatible with the O_S module structure, that is the maps $md : (D(O_S), F^{ord}) \otimes_{O_S} (M, F) \rightarrow (M, F)$ in $\text{PSh}_{O_S fil}(\mathcal{S})$ and*
- *the integrable connexions on M , that is the maps $\nabla : (M, F) \rightarrow \Omega_{O_S} \otimes_{O_S} (M, F)$ satisfying $\nabla \circ \nabla = 0$ and Griffith transversality (i.e. $\nabla(F^p M) \subset \Omega_{O_S} \otimes_{O_S} F^{p-1} M$).*

Proof. Standard. □

The following proposition tells that the O -tensor product of D modules has a canonical structure of D module.

Definition-Proposition 9. (i) *Let $f : (\mathcal{X}, O_X) \rightarrow (\mathcal{S}, O_S)$ a morphism with $(\mathcal{X}, O_X), (\mathcal{S}, O_S) \in \text{RCat}$ with commutative structural sheaf of ring. For $N \in \text{PSh}_{O_X, D(f^*O_S)}(\mathcal{X})$ and $M \in \text{PSh}_{O_X, D(f^*O_S)}(\mathcal{X})$, $N \otimes_{O_X} M$ has the canonical $D(f^*O_S)$ module structure given by, for $X^\circ \in \mathcal{X}$,*

$$\gamma \in \Gamma(X^\circ, D(f^*O_S)), m \in \Gamma(X^\circ, M), n \in \Gamma(X^\circ, N), \gamma.(n \otimes m) = (\gamma.n) \otimes m + n \otimes (\gamma.m).$$

This gives the functor

$$\begin{aligned} & \text{PSh}_{O_X, D(f^*O_S)fil}(\mathcal{X}) \times \text{PSh}_{O_X, D(f^*O_S)fil}(\mathcal{X}) \rightarrow \text{PSh}_{O_X, D(f^*O_S)fil}(\mathcal{X}), ((M, F), (N, F)) \mapsto \\ & (M, F) \otimes_{O_X} (N, F), F^p(M, F) \otimes_{O_X} (N, F) := \text{Im}(\oplus_{q \in \mathbb{Z}} F^q M \otimes_{O_X} F^{p-q} N \rightarrow M \otimes_{O_X} N) \end{aligned}$$

(ii) *Let $f : (\mathcal{X}, O_X) \rightarrow (\mathcal{S}, O_S)$ a morphism with $(\mathcal{X}, O_X), (\mathcal{S}, O_S) \in \text{RCat}$ with commutative structural sheaf of ring. For $N \in C_{D(O_X), D(f^*O_S)}(\mathcal{X})$ and $M \in C_{D(O_X)op}(\mathcal{X})$, $N \otimes_{D(O_X)} M$ has the canonical $f^*D(O_S)$ module structure given by, for $X^\circ \in \mathcal{X}$,*

$$\gamma \in \Gamma(X^\circ, D(f^*O_S)), m \in \Gamma(X^\circ, M), n \in \Gamma(X^\circ, N), \gamma.(n \otimes m) = (\gamma.n) \otimes m.$$

This gives the functor

$$\begin{aligned} & C_{D(O_X), D(f^*O_S)fil}(\mathcal{X}) \times C_{D(O_X)fil}(\mathcal{X}) \rightarrow C_{D(f^*O_S)fil}(\mathcal{X}), ((M, F), (N, F)) \mapsto \\ & (M, F) \otimes_{D(O_X)} (N, F), F^p(M, F) \otimes_{D(O_X)} (N, F) := \text{Im}(\oplus_{q \in \mathbb{Z}} F^q M \otimes_{D(O_X)} F^{p-q} N \rightarrow M \otimes_{D(O_X)} N) \end{aligned}$$

Note that, by definition, we have for $(M, F) \in (\text{PSh}_{D(O_S)fil}(\mathcal{S}))$, the canonical isomorphism

$$(M, F) \otimes_{D(O_S)} (D(O_S), F^{ord}) \xrightarrow{\sim} (M, F), m \otimes P \mapsto Pm, m \mapsto (m \otimes 1)$$

Proof. Immediate from definition. \square

We now look at the functorialities for morphisms of ringed spaces, using proposition 34. First note that for $f : (X, O_X) \rightarrow (S, O_S)$ a morphism, with $(X, O_X), (S, O_S) \in \text{RTop}$ with structural presheaves commutative sheaves of rings, there is NO canonical morphism between $D(f^*O_S) = f^*D(O_S)$ (see proposition 34) and $D(O_X)$.

We have the pullback functor for (left) D-modules :

Definition-Proposition 10. (i) *Let $f : (X, O_X) \rightarrow (S, O_S)$ a morphism with $(X, O_X), (S, O_S) \in \text{RTop}$ with structural presheaves commutative sheaves of rings. Recall that $f^*D(O_S) = D(f^*O_S)$ in this case. Then for $(M, F) \in \text{PSh}_{D(O_S)\text{fil}}(S)$,*

$$f^{*\text{mod}}(M, F) := (O_X, F_b) \otimes_{f^*O_S} f^*(M, F) \in \text{PSh}_{O_X\text{fil}}(X)$$

has a canonical structure of filtered $D(O_X)$ module given by

$$\text{for } \gamma \in \Gamma(X^\circ, T_{O_X}), n \otimes m \in \Gamma(X^\circ, O_X \otimes_{f^*O_S} f^*M), \gamma \cdot (n \otimes m) := (\gamma \cdot n) \otimes m + n \otimes df(\gamma)(m)$$

$$\text{with } df := \mathbb{D}_S^O \Omega_{O_X/f^*O_S} : T_{O_X} \rightarrow T_{f^*O_S} = f^*T_{O_S} \text{ and } f^*(M, F) \in \text{PSh}_{f^*D(O_S)\text{fil}}(X) = \text{PSh}_{D(f^*O_S)\text{fil}}(X).$$

(ii) *More generally, let $f : (\mathcal{X}, O_X) \rightarrow (\mathcal{S}, O_S)$ a morphism with $(\mathcal{X}, O_X), (\mathcal{S}, O_S) \in \text{RCat}$ with structural presheaves commutative sheaves of rings. Assume that the canonical morphism $T(f, \text{hom})(O_S, O_S) : f^*D(O_S) \rightarrow D(f^*O_S)$ is an isomorphism of sheaves. Then for $(M, F) \in \text{PSh}_{D(O_S)\text{fil}}(\mathcal{S})$,*

$$f^{*\text{mod}}(M, F) := (O_X, F_b) \otimes_{f^*O_S} f^*(M, F) \in \text{PSh}_{O_X\text{fil}}(\mathcal{X})$$

has a canonical structure of filtered $D(O_X)$ module given by

$$\text{for } \gamma \in \Gamma(X^\circ, T_{O_X}), n \otimes m \in \Gamma(X^\circ, O_X \otimes_{f^*O_S} f^*M), \gamma \cdot (n \otimes m) := (\gamma \cdot n) \otimes m + n \otimes df(\gamma)(m)$$

$$\text{with } df := \mathbb{D}_S^O \Omega_{O_X/f^*O_S} : T_{O_X} \rightarrow T_{f^*O_S} = f^*T_{O_S} \text{ and } f^*(M, F) \in \text{PSh}_{f^*D(O_S)\text{fil}}(\mathcal{X}) = \text{PSh}_{D(f^*O_S)\text{fil}}(\mathcal{X}).$$

Proof. Standard. \square

Remark 6. • *Let $f : (X, O_X) \rightarrow (S, O_S)$ a morphism with $(X, O_X), (S, O_S) \in \text{RTop}$ with structural presheaves commutative sheaves of rings. Recall that $f^*D(O_S) = D(f^*O_S)$. Then by definition $f^{*\text{mod}}(O_S, F_b) = (O_X, F_b)$.*

• *More generally, let $f : (\mathcal{X}, O_X) \rightarrow (\mathcal{S}, O_S)$ a morphism with $(\mathcal{X}, O_X), (\mathcal{S}, O_S) \in \text{RCat}$ with structural presheaves commutative sheaves of rings. Assume that the canonical morphism $T(f, \text{hom})(O_S, O_S) : f^*D(O_S) \rightarrow D(f^*O_S)$ is an isomorphism of sheaves. Then by definition $f^{*\text{mod}}(O_S, F_b) = (O_X, F_b)$.*

For the definition of a push-forward functor for a right D module we use the transfert module

Let $f : (X, O_X) \rightarrow (S, O_S)$ be a morphism with $(X, O_X), (S, O_S) \in \text{RTop}$ with structural presheaves commutative sheaves of rings. Then, the transfer module is

$$(D(O_X \rightarrow f^*O_S), F^{\text{ord}}) := f^{*\text{mod}}(D(O_S), F^{\text{ord}}) := f^*(D(O_S), F^{\text{ord}}) \otimes_{f^*O_S} (O_X, F_b)$$

which is a left $D(O_X)$ module and a left and right $f^*D(O_S) = D(f^*O_S)$ module.

Lemma 5. *Let $f_1 : (X, O_X) \rightarrow (Y, O_Y)$, $f_2 : (Y, O_Y) \rightarrow (S, O_S)$ be two morphism with $(X, O_X), (Y, O_Y), (S, O_S) \in \text{RTop}$. We have in $C_{D(O_X), (f_2 \circ f_1)^*D(O_S)\text{fil}}(X)$*

$$(D_{O_X \rightarrow (f_2 \circ f_1)^*O_S}, F^{\text{ord}}) = f_1^*(D_{O_Y \rightarrow f_2^*O_S}, F^{\text{ord}}) \otimes_{f_1^*D(O_Y)} (D_{O_X \rightarrow f_1^*O_Y}, F^{\text{ord}})$$

Proof. Follows immediately from definition. \square

For right D module, we have the direct image functor :

Definition 48. Let $f : (X, O_X) \rightarrow (S, O_S)$ a morphism with $(X, O_X), (S, O_S) \in \text{RTop}$ with structural presheaves commutative sheaves of rings. Then for $(M, F) \in C_{D(O_X)^{op} \text{fil}}(X)$, we define

$$f_{*mod}^{00}(M, F) = f_*(D_{O_X \rightarrow f^*O_S}, F^{ord} \otimes_{D(O_X)} (M, F)) \in C_{D(O_S) \text{fil}}(S)$$

For a closed embedding of topological spaces, there is the V -filtration on the structural sheaf, it will play an important role in this article

Definition 49. (i) Let $(S, O_S) \in \text{RTop}$ a locally ringed space. Let $Z = V(\mathcal{I}_Z) \subset S$ a Zariski closed subset. We set, for $S^\circ \subset S$ an open subset, $p \in \mathbb{Z}$,

- $V_{Z,p}O_S(S^\circ) := O_S(S^\circ)$ if $p > 0$,
- $V_{Z,-q}O_S(S^\circ) := \mathcal{I}_Z^q(S^\circ) \subset O_S(S^\circ)$ for $p = -q \leq 0$.

We immediately check that, by definition, this filtration satisfy Griffithz transversality, that is $(O_S, V_Z) \in \text{PSh}_{D(O_S) \text{fil}}(S)$. For a morphism $g : ((T, O_T), Z') \rightarrow ((S, O_S), Z)$ with $((T, O_T), Z), ((S, O_S), Z) \in \text{RTop}^2$ locally ringed spaces, where Z and Z' are Zariski closed subsets, the structural morphism $a_g : g^*O_S \rightarrow O_T$ is a filtered morphism :

$$a_g : g^*(O_S, V_Z) \rightarrow (O_T, V_{Z'}), h \mapsto a_g(h)$$

(ii) Let $(S, O_S) \in \text{RTop}$. Let $i : Z \hookrightarrow S$ a closed embedding. The V_Z -filtration on O_S (see (i)) gives the filtration, given by for $p \in \mathbb{Z}$,

$$V_{Z,p} \text{Hom}(O_S, O_S) := \left\{ P \in \text{Hom}(O_S, O_S), \text{ s.t. } P\mathcal{I}_Z^k \subset \mathcal{I}_Z^{k-p} \right\}$$

on $\text{Hom}(O_S, O_S)$, which induces the filtration $V_{Z,p}D_S := D_S \cap V_{Z,p} \text{Hom}(O_S, O_S)$ on $D_S \subset \text{Hom}(O_S, O_S)$. We get $(D_S, V_Z) \in \text{PSh}_{\text{fil}}(S, \text{Ring})$ and we call it the V_Z -filtration on D_S .

(iii) Let $(S, O_S) \in \text{RTop}$ a locally ringed space. Let $i : Z = V(\mathcal{I}_Z) \hookrightarrow S$ a Zariski closed embedding and $O_Z := i^*O_S/\mathcal{I}_Z$. We say that $(M, F) \in \text{PSh}_{D(O_S) \text{fil}}(S)$ is specializable on Z if it admits an (increasing) filtration (called a V_Z -filtration) $(M, F, V) \in \text{PSh}_{D(O_S)2\text{fil}}(S)$ compatible with (D_S, V_Z) , that is $V_{Z,p}D_S \cdot V_q M \subset V_{p+q}M$.

(iii)' Under certain hypothesis, a V_Z -filtration on (M, F) if it exist is unique in this case we denote $(M, F, V_Z) \in \text{PSh}_{D(O_S)2\text{fil}}(S)$. If $m : (M_1, F) \rightarrow (M_2, F)$ is a morphism, with $(M_1, F), (M_2, F) \in \text{PSh}_{D(O_S) \text{fil}}(S)$ admitting a unique V_Z filtration, we have $m(V_Z^q F^p M_1) \subset V_Z^q F^p M_2$, that is we get a morphism $m : (M_1, F, V_Z) \rightarrow (M_2, F, V_Z)$. See proposition 89 for the particular case of smooth complex algebraic varieties we will be mainly interested with.

(iv) Let $(S, O_S) \in \text{RTop}$ a locally ringed space. Let $i : Z = V(\mathcal{I}_Z) \hookrightarrow S$ a Zariski closed embedding and $O_Z := i^*O_S/\mathcal{I}_Z$. For $(M, F) \in \text{PSh}_{D(O_S) \text{fil}}(S)$ is specializable on Z , consider for a V_Z filtration $(M, F, V) \in \text{PSh}_{D(O_S)2\text{fil}}(S)$, the quotient map in $\text{PSh}_{O_S \text{fil}}(S)$

$$q_{V0} : (M, F) \rightarrow (M, F)/V_{<0}(M, F) =: \text{Gr}_{V,0}(M, F).$$

Since by definition $\mathcal{I}_Z V_{<0}(M, F) \subset V_{<0}(M, F)$, q_{V0} factors trough in $\text{PSh}_{i^*O_S \text{fil}}(Z)$

$$i^* q_{V0} : i^*(M, F) \xrightarrow{i^* q_Z} i^*_{mod}(M, F) \xrightarrow{\text{spv}(M, F)} i^* \text{Gr}_{V,0}(M, F)$$

where

- $q_Z : (M, F) = (M, F) \otimes_{O_S} O_S \rightarrow (M, F) \otimes_{O_S} i_* O_Z$ is the quotient map, which is given by for $S^\circ \subset S$ an open subset and $m \in \Gamma(S^\circ, M)$, $q_Z(m) := m \otimes 1$,

- $sp_V(M, F) : i^{*mod}(M, F) \rightarrow i^* \text{Gr}_{V,0}(M, F)$ is given by for $S^\circ \subset S$ an open subset, $m \in \Gamma(S^\circ, M)$ and $h \in \Gamma(S^\circ, i_* O_Z)$,

$$sp_V(M, F)(m \otimes h) := hq_{V_0}(m) = q_{V_0}(\tilde{h}m),$$

where $\tilde{h} \in \Gamma(S^\circ, O_S)$ is lift of h , and of course $sp_V(M, F) : i^{*mod}(M, F) \rightarrow i^* \text{Gr}_{V,0}(M, F)$ is unique in $\text{PSh}_{i^* O_S \text{fil}}(Z)$ such that $i^* q_{V_0} = sp_V(M, F) \circ i^* q_Z$ since q_Z is surjective.

The quotient $i^* \text{Gr}_{V,0}(M, F)$ has a structure of D_Z module since for $S^\circ \subset S$ an open subset and $\partial_z \in \Gamma(Z \cap S^\circ, T_Z) \subset \Gamma(S^\circ, T_S)$, we have $\partial_z \in \Gamma(S, V_{Z,0} D_S)$ since for $f = \sum_{i=1}^r t_i h_i \in \Gamma(S^\circ, \mathcal{I}_Z)$, where $(t_i) = \mathcal{I}_Z(S^\circ)$ are generators of the ideal $\mathcal{I}_Z(S^\circ) \subset O_S(S^\circ)$ and $h_i \in \Gamma(S^\circ, O_S)$, we have

$$\partial_z \left(\sum_{i=1}^r t_i h_i \right) = \sum_{i=1}^r (\partial_z(t_i) h_i + t_i \partial_z(h_i)) = \sum_{i=1}^r t_i (\partial_z(h_i)) \in \Gamma(S, \mathcal{I}_Z)$$

as $\partial_z(t_i) = 0$ (only the vector field of T_{O_S} which are transversal to $T_{O_Z} \subset T_{O_S}$ increase the grading), Then, obviously, by definition,

$$sp_V(M, F) : i^{*mod}(M, F) \rightarrow i^*(M, F)/V_{<0}(M, F) =: i^* \text{Gr}_{V,0}(M, F)$$

is $D(O_Z)$ linear, i.e. is a map in $\text{PSh}_{i^* O_S, D(O_Z) \text{fil}}(Z)$ and we call it the specialization map.

Definition-Proposition 11. Let $(S, O_S) \in \text{RTop}$ a locally ringed space. Consider a commutative diagram

$$\begin{array}{ccc} Z_1 = V(\mathcal{I}_1) & \xrightarrow{i_1} & (S, O_S) \\ i'_2 \uparrow & & \uparrow i_2 \\ Z = V(\mathcal{I}) & \xrightarrow{i'_1} & Z_2 = V(\mathcal{I}_2) \end{array}$$

where the maps are Zariski closed embeddings and which is cartesian (i.e. $\mathcal{I} = (\mathcal{I}_1, \mathcal{I}_2)$, in particular $Z = Z_1 \cap Z_2$).

- (i) Let $(M, F) \in \text{PSh}_{D(O_S) \text{fil}}(S)$ admitting a unique V_Z -filtration, a unique V_{Z_1} -filtration and a unique V_{Z_2} -filtration (see definition 49). Let $p, q \in \mathbb{Z}$. Then

- $\text{Gr}_{V_{Z_1}, p}(M, F)$ is a $D(O_{Z_1})$ admitting a unique V_{Z_1} -filtration,
- $\text{Gr}_{V_{Z_2}, q}(M, F)$ is a $D(O_{Z_2})$ admitting a unique V_{Z_2} -filtration,

and the quotient map in $\text{PSh}_{O_{Z_2} \text{fil}}(S)$

$$q_{V_{Z_1} p} : \text{Gr}_{V_{Z_2}, q}(M, F) \rightarrow \text{Gr}_{V_{Z_2}, q} \text{Gr}_{V_{Z_1}, p}(M, F)$$

factors through

$$q_{V_{Z_1} p} : \text{Gr}_{V_{Z_2}, q}(M, F) \xrightarrow{q_{V_{Z_2}, p}} \text{Gr}_{V_{Z_2}, p} \text{Gr}_{V_{Z_2}, q}(M, F) \xrightarrow{G_{Z_1, Z_2}^{p, q}(M, F)} \text{Gr}_{V_{Z_2}, q} \text{Gr}_{V_{Z_1}, p}(M, F)$$

and $G_{Z_1, Z_2}^{p, q}(M, F)$ is an isomorphism in $\text{PSh}_{D(O_Z) \text{fil}}(S)$.

- (ii) If $m : (M_1, F) \rightarrow (M_2, F)$ is a morphism with $(M_1, F), (M_2, F) \in \text{PSh}_{D(O_S) \text{fil}}(S)$ admitting a unique V_Z -filtration, a unique V_{Z_1} -filtration and a unique V_{Z_2} -filtration (see definition 49). Then for all $p, q \in \mathbb{Z}$ the following diagram commutes

$$\begin{array}{ccc} \text{Gr}_{V_{Z_2}, p} \text{Gr}_{V_{Z_2}, q}(M_1, F) & \xrightarrow{G_{Z_1, Z_2}^{p, q}(M_1, F)} & \text{Gr}_{V_{Z_2}, q} \text{Gr}_{V_{Z_1}, p}(M_1, F) \\ \downarrow \text{Gr}_{V_{Z_2}, p} \text{Gr}_{V_{Z_2}, q}(m) & & \downarrow \text{Gr}_{V_{Z_2}, q} \text{Gr}_{V_{Z_1}, p}(m) \\ \text{Gr}_{V_{Z_2}, p} \text{Gr}_{V_{Z_2}, q}(M_2, F) & \xrightarrow{G_{Z_1, Z_2}^{p, q}(M_2, F)} & \text{Gr}_{V_{Z_2}, q} \text{Gr}_{V_{Z_1}, p}(M_2, F) \end{array}$$

Let $(S, O_S) \in \text{RTop}$ a locally ringed space. Consider a commutative diagram

$$\begin{array}{ccccc} Z'_1 = V(\mathcal{I}'_1) & \xrightarrow{l_1} & Z_1 = V(\mathcal{I}_1) & \xrightarrow{i_1} & (S, O_S) \\ i''_2 \uparrow & & i'_2 \uparrow & & i_2 \uparrow \\ Z' = V(\mathcal{I}') & \xrightarrow{l'_1} & Z = V(\mathcal{I}) & \xrightarrow{i'_1} & Z_2 = V(\mathcal{I}_2) \end{array}$$

where the maps are Zariski closed embeddings and whose squares are cartesian (i.e. $\mathcal{I} = (\mathcal{I}_1, \mathcal{I}_2)$ and $\mathcal{I}' = (\mathcal{I}'_1, \mathcal{I})$, in particular $Z = Z_1 \cap Z_2$ and $Z' = Z'_1 \cap Z$). Let $(M, F) \in \text{PSh}_{D(O_S)\text{fil}}(S)$ admitting a unique V_Z -filtration, a unique V_{Z_1} -filtration a unique $V_{Z'}$ -filtration, a unique $V_{Z'_1}$ -filtration, and a unique V_{Z_2} -filtration (see definition 49). Then for all $p, q \in \mathbb{Z}$, $\text{Gr}_{V_{Z'_1}, p} \text{Gr}_{V_{Z_1}, p}(M, F) = \text{Gr}_{V_{Z'_1}, p}(M, F)$ and

$$G_{Z'_1, Z}^{p, q}(G_{Z_1, Z_2}^{p, q}(M, F)) = G_{Z'_1, Z_2}^{p, q}(M_2, F)$$

Proof. Obvious. □

We will also consider the following categories

Definition 50. Let $(\mathcal{X}, O_X) \in \text{RCat}$. We denote by $C_{O_X \text{fil}, D(O_X)}(\mathcal{X})$ the category

- whose objects $(M, F) \in C_{O_X \text{fil}, D(O_X)}(\mathcal{X})$ are filtered complexes of presheaves of O_X modules $(M, F) \in C_{O_X \text{fil}}(\mathcal{X})$ whose cohomology presheaves $H^n(M, F) \in \text{PSh}_{O_X \text{fil}}(\mathcal{X})$ are endowed with a structure of filtered $D(O_X)$ modules for all $n \in \mathbb{Z}$.
- whose set of morphisms $\text{Hom}_{C_{O_X \text{fil}, D(O_X)}(\mathcal{X})}((M, F), (N, F)) \subset \text{Hom}_{C_{O_X \text{fil}}(\mathcal{X})}((M, F), (N, F))$ between $(M, F), (N, F) \in C_{O_X \text{fil}, D(O_X)}(\mathcal{X})$ are the morphisms of filtered complexes of O_X modules $m : (M, F) \rightarrow (N, F)$ such that $H^n m : H^n(M, F) \rightarrow H^n(N, F)$ is $D(O_X)$ linear, i.e. is a morphism of (filtered) $D(O_X)$ modules, for all $n \in \mathbb{Z}$.

4.1.2 The De Rham complex of a (left) filtered D-module and the Spencer complex of a right filtered D-module

Using proposition 35, we define the filtered De Rham complex of a complex of filtered (left) D-modules :

Definition 51. (i) Let $(S, O_S) \in \text{RCat}$ with O_S commutative. Let $(M, F) \in C_{D(O_S)\text{fil}}(S)$. By proposition 35, we have the complex

$$DR(O_S)(M, F) := (\Omega_{O_S}^\bullet, F_b) \otimes_{O_S} (M, F) \in C_{\text{fil}}(\mathcal{S})$$

whose differentials are $d(\omega \otimes m) = (d\omega) \otimes m + \omega \wedge (\nabla m)$.

(ii) More generally, let $f : (\mathcal{X}, O_X) \rightarrow (S, O_S)$ with $(\mathcal{X}, O_X), (S, O_S) \in \text{RCat}$. The quotient map $q : \Omega_{O_X} \rightarrow \Omega_{O_X/f^*O_S}$ induce, for $G \in \text{PSh}_{O_X}(\mathcal{X})$ the quotient map

$$q^p(G) := \wedge^p q \otimes I\Omega_{O_X}^q \otimes_{O_X} G \rightarrow \Omega_{O_X/f^*O_S}^q \otimes_{O_X} G.$$

Let $(M, F) \in C_{D(O_X)\text{fil}}(\mathcal{X})$. By proposition 35, we have the relative De Rham complex

$$DR(O_X/f^*O_S)(M, F) := (\Omega_{X/S}^\bullet, F_b) \otimes_{O_X} (M, F) \in C_{f^*O_S \text{fil}}(\mathcal{X})$$

whose differentials are $d(q^p(M)(\omega \otimes m)) := q^{p+1}(M)((d\omega) \otimes m) + q^{p+1}(M)(\omega \otimes (\nabla m))$.

(iii) Let $(X, O_X)/\mathcal{F} \in \text{FolRTop}$, that is $(X, O_X) \in \text{RTop}$ endowed with a foliation with quotient map $q : \Omega_{O_X} \rightarrow \Omega_{O_X/\mathcal{F}}$. Let $(M, F) \in C_{D(O_X)\text{fil}}(X)$. By proposition 35, we have the foliated De Rham complex

$$DR(O_X/\mathcal{F})(M, F) := (\Omega_{O_X/\mathcal{F}}^\bullet, F_b) \otimes_{O_X} (M, F) \in C_{\text{fil}}(X)$$

whose differentials are $d(q(M)(\omega \otimes m)) := q(M)((d\omega) \otimes m) + q(M)(\omega \otimes (\nabla m))$.

By definition,

- with the notation of (ii) if $\phi : (M_1, F) \rightarrow (M_2, F)$ is a morphism in $C_{D(O_X)fil}(\mathcal{X})$,

$$DR(O_X/f^*O_S)(\phi) := (I \otimes \phi) : (\Omega_{X/S}^\bullet, F) \otimes_{O_X} (M_1, F) \rightarrow (\Omega_{X/S}^\bullet, F) \otimes_{O_X} (M_2, F)$$

is a morphism in $C_{f^*O_S fil}(\mathcal{X})$,

- with the notation of (ii) $DR(O_X)(O_X) = DR(O_X)$ and more generally in the relative case $DR(O_X/f^*O_S)(O_X) = DR(O_X/f^*O_S)$, and with the notation of (iii) $DR(O_X/\mathcal{F})(O_X) = DR(O_X/\mathcal{F})$.

Dually, we have the filtered Spencer complex of a complex of filtered right D-module :

Definition 52. (i) Let $(S, O_S) \in \text{RCat}$ with O_S commutative. Let $(M, F) \in C_{D(O_S)op fil}(\mathcal{S})$. By proposition 35, we have the complex

$$SP(O_S)(M, F) := (T_{O_S}^\bullet, F_b) \otimes_{O_S} (M, F) \in C_{fil}(\mathcal{S})$$

whose differentials are, for $X \in \mathcal{S}$, and $\partial_1 \wedge \cdots \wedge \partial_r \otimes m \in \Gamma(X, T_{O_S}^{r-1} \otimes_{O_S} M)$,

$$d(\partial_1 \wedge \cdots \wedge \partial_r \otimes m) : (\omega \in \Gamma(X, \Omega_{O_S}^{r-1}) \mapsto \sum_i \omega(\partial_1 \wedge \cdots \wedge \partial_i \cdots \partial_r) m - \sum_{i < j} \omega([\partial_i, \partial_j]) m).$$

- (ii) More generally, let $f : (\mathcal{X}, O_X) \rightarrow (\mathcal{S}, O_S)$ with $(\mathcal{X}, O_X), (\mathcal{S}, O_S) \in \text{RCat}$. The quotient map $q : \Omega_{O_X} \rightarrow \Omega_{O_X/f^*O_S}$ induce, for $G \in \text{PSh}_{O_X}(\mathcal{X})$ the injective map

$$q^{\vee, p}(G) := \wedge^p q^\vee \otimes I : T_{O_X/f^*O_S}^q \otimes_{O_X} G \rightarrow T_{O_X}^q \otimes_{O_X} G.$$

Let $(M, F) \in C_{D(O_X)op fil}(\mathcal{X})$. By proposition 35, we have the relative Spencer complex

$$SP(O_X/f^*O_S)(M, F) := (T_{X/S}^\bullet, F_b) \otimes_{O_X} (M, F) \in C_{f^*O_S fil}(\mathcal{X})$$

whose differentials are the one of $SP(O_X)(M, F)$ given in (i) by the embedding $q^\vee : SP(O_X/f^*O_S)(M, F) \hookrightarrow SP(O_X)(M, F)$.

- (iii) Let $(X, O_X)/\mathcal{F} \in \text{FolRTop}$, that is $(X, O_X) \in \text{RTop}$ endowed with a foliation with quotient map $q : \Omega_{O_X} \rightarrow \Omega_{O_X/\mathcal{F}}$. Let $(M, F) \in C_{D(O_X)op fil}(X)$. By proposition 35, we have the foliated Spencer complex

$$SP(O_X/\mathcal{F})(M, F) := (T_{O_X/\mathcal{F}}^\bullet, F_b) \otimes_{O_X} (M, F) \in C_{fil}(X)$$

whose differentials are of $SP(O_X)(M, F)$ given in (i) by the embedding $q^\vee : SP(O_X/\mathcal{F})(M, F) \hookrightarrow SP(O_X)(M, F)$.

By definition, with the notation of (ii) if $\phi : (M_1, F) \rightarrow (M_2, F)$ is a morphism in $C_{D(O_X)op fil}(\mathcal{X})$,

$$SP(O_X/f^*O_S)(\phi) := (I \otimes \phi) : (T_{X/S}^\bullet, F_b) \otimes_{O_X} (M_1, F) \rightarrow (T_{X/S}^\bullet, F_b) \otimes_{O_X} (M_2, F)$$

is a morphism in $C_{f^*O_S fil}(\mathcal{X})$.

Proposition 36. (i) Let $f : (X, O_X) \rightarrow (S, O_S)$ a morphism with $(S, O_S), (X, O_X) \in \text{RTop}$. Assume that the canonical map $T(f, \text{hom})(O_X, O_X) : f^*D(O_X) \rightarrow D(f^*O_X)$ is an isomorphism of sheaves. For $(M, F) \in C_{D(O_X)op, f^*D(O_S)fil}(X)$ and $(M', F), (N, F) \in C_{D(O_X)fil}(X)$, we have canonical isomorphisms in $C_{f^*D(O_S)fil}(X)$:

$$\begin{aligned} (M', F) \otimes_{O_X} (N, F) \otimes_{D(O_X)} (M, F) &= (M', F) \otimes_{D(O_X)} ((M, F) \otimes_{O_X} (N, F)) \\ &= ((M', F) \otimes_{O_X} (M, F)) \otimes_{D(O_X)} (N, F) \end{aligned}$$

- (ii) Let $f : (X, O_X) \rightarrow (S, O_S)$ a morphism with $(S, O_S), (X, O_X) \in \text{RTop}$. For $(M, F) \in C_{D(O_X)fil}(X)$, we have a canonical isomorphisms of filtered f^*O_S modules, i.e. isomorphisms in $C_{f^*O_S fil}(X)$,

$$(\Omega_{O_X/f^*O_S}^\bullet, F_b) \otimes_{O_X} (M, F) = ((\Omega_{O_X/f^*O_S}^\bullet, F_b) \otimes_{O_X} D(O_X)) \otimes_{D(O_X)} (M, F)$$

Proof. These are standard fact of algebra. □

Definition-Proposition 12. Consider a commutative diagram in \mathbf{RCat}

$$D = \begin{array}{ccc} (\mathcal{X}, O_X) & \xrightarrow{f} & (\mathcal{S}, O_S) \\ g' \uparrow & & \uparrow g \\ (\mathcal{X}', O_{X'}) & \xrightarrow{f'} & (\mathcal{T}, O_T) \end{array}$$

with commutative structural sheaf of rings. Assume that the canonical map $T(g', \text{hom})(O_X, O_X) : g'^*D(O_X) \rightarrow D(g'^*O_X)$ is an isomorphism of sheaves.

(i) For $(M, F) \in \text{PSh}_{D(O_X)\text{fil}}(\mathcal{X})$, the graded map in $(\text{PSh}_{g'^*O_X}(\mathbb{N} \times \mathcal{X}'), F)$

$$\begin{aligned} \Omega_{(O_{X'}/g'^*O_X)/(O_T/g^*O_S)}(M, F) &:= m' \circ (\Omega_{(O_{X'}/g'^*O_X)/(O_T/g^*O_S)} \otimes I) : \\ g'^*((\Omega_{O_X/f^*O_S}^\bullet, F_b) \otimes_{O_X} (M, F)) &\rightarrow (\Omega_{O_{X'}/f'^*O_T}^\bullet, F_b) \otimes_{O_{X'}} g'^{\text{mod}}(M, F) \end{aligned}$$

given in degree $p \in \mathbb{N}$ by, for $X'^o \in \mathcal{X}'$ and $X^o \in \mathcal{X}$ such that $g'^*(X^o) \leftarrow X'^o$,

$$\begin{aligned} \Omega_{(O_{X'}/g'^*O_X)/(O_T/g^*O_S)}^p(M)(X'^o) &:= m' \circ (\Omega_{(O_{X'}/g'^*O_X)/(O_T/g^*O_S)}^p \otimes I)(X'^o) : \\ \omega \otimes m \in \Gamma(X^o, \Omega_{O_X}^p \otimes_{O_X} M) &\mapsto \Omega_{O_{X'}/g'^*O_X}(\omega) \otimes (m \otimes 1) \end{aligned}$$

is a map of complexes, that is a map in $C_{(f \circ g')^*O_S\text{fil}}(\mathcal{X}')$.

(ii) For $(M, F) \in C_{D(O_X)\text{fil}}(\mathcal{X})$, we get from (i) by functoriality, the map in $C_{(f \circ g')^*O_S\text{fil}}(\mathcal{X}')$

$$\begin{aligned} \Omega_{(O_{X'}/g'^*O_X)/(O_T/g^*O_S)}(M, F) &:= m' \circ (\Omega_{(O_{X'}/g'^*O_X)/(O_T/g^*O_S)} \otimes I) : \\ g'^*((\Omega_{O_X/f^*O_S}^\bullet, F_b) \otimes_{O_X} (M, F)) &\rightarrow (\Omega_{O_{X'}/f'^*O_T}^\bullet, F_b) \otimes_{O_{X'}} g'^{\text{mod}}(M, F) \end{aligned}$$

(iii) For $(M, F) \in C_{D(O_X)\text{fil}}(\mathcal{X})$, we get from (ii) the canonical transformation map in $C_{O_T\text{fil}}(\mathcal{T})$

$$\begin{aligned} T_\omega^O(D)(M, F) &: g'^{\text{mod}}L_O(f_*E((\Omega_{O_X/f^*O_S}^\bullet, F_b) \otimes_{O_X} (M, F))) \xrightarrow{q} \\ (g^*f_*E((\Omega_{O_X/f^*O_S}^\bullet, F_b) \otimes_{O_X} (M, F))) \otimes_{g^*O_S} O_T &\xrightarrow{T(g', E)(-) \circ T(D)(E(\Omega_{O_X/f^*O_S}^\bullet \otimes_{O_X} (M, F)))} \\ (f'_*E(g'^*((\Omega_{O_X/f^*O_S}^\bullet, F_b) \otimes_{O_X} (M, F)))) \otimes_{g^*O_S} O_T &\xrightarrow{E(\Omega_{(O_{X'}/g'^*O_X)/(O_T/g^*O_S)}(M, F))} \\ (f'_*E((\Omega_{O_{X'}/f'^*O_T}^\bullet, F_b) \otimes_{O_X} (M, F))) \otimes_{g^*O_S} O_T &\xrightarrow{m} f'_*E((\Omega_{O_{X'}/f'^*O_T}^\bullet, F_b) \otimes_{O_{X'}} g'^{\text{mod}}(M, F)) \end{aligned}$$

with $m(n \otimes s) = s.n$.

Proof. (i): We check that the map in $(\text{PSh}_{g'^*O_X}(\mathbb{N} \times \mathcal{X}'), F)$

$$\begin{aligned} \Omega_{(O_{X'}/g'^*O_X)/(O_T/g^*O_S)}(M, F) &:= m' \circ (\Omega_{(O_{X'}/g'^*O_X)/(O_T/g^*O_S)} \otimes I) : \\ g'^*((\Omega_{O_X/f^*O_S}^\bullet, F_b) \otimes_{O_X} (M, F)) &\rightarrow (\Omega_{O_{X'}/f'^*O_T}^\bullet, F_b) \otimes_{O_{X'}} g'^{\text{mod}}(M, F) \end{aligned}$$

is a map in $C_{(f \circ g')^*O_S\text{fil}}(\mathcal{X}')$. But we have, for $X'^o \in \mathcal{X}'$ the following equality in $\Gamma(X'^o, \Omega_{O_{X'}}^{p+1} \otimes_{O_{X'}})$

$g'^{*mod}M)$

$$\begin{aligned}
d(\Omega_{(O_{X'}/g'^*O_X)(O_T/g^*O_S)}^p(M)(\omega \otimes m)) &:= d(\Omega_{O_{X'}/g'^*O_X}^p(\omega) \otimes (m \otimes 1)) \\
&:= d(\Omega_{O_{X'}/g'^*O_X}^p(\omega)) \otimes (m \otimes 1) + \Omega_{O_{X'}/g'^*O_X}^p(\omega) \otimes \nabla(m \otimes 1) \\
&= \Omega_{O_{X'}/g'^*O_X}^{p+1}(d\omega) \otimes (m \otimes 1) + \Omega_{O_{X'}/g'^*O_X}^p(\omega) \otimes \nabla(m) \otimes 1 \\
&= \Omega_{O_{X'}/g'^*O_X}^{p+1}(d\omega) \otimes (m \otimes 1) + \Omega_{O_{X'}/g'^*O_X}^{p+1}(\omega \otimes \nabla(m)) \otimes 1 \\
&=: \Omega_{(O_{X'}/g'^*O_X)(O_T/g^*O_S)}^{p+1}(M)(d(\omega) \otimes m + \omega \otimes \nabla(m)) \\
&=: \Omega_{(O_{X'}/g'^*O_X)(O_T/g^*O_S)}^{p+1}(M)(d(\omega \otimes m))
\end{aligned}$$

since for $\partial' \in T_{O_{X'}}(X'^o)$,

$$\nabla_{\partial'}(m \otimes 1) = \nabla_{dg'(\partial')}(m) \otimes 1 + m \otimes \nabla_{\partial'}1 = \nabla_{dg'(\partial')}(m) \otimes 1 :$$

see in definition-proposition 10 the definition of the $D(O_{X'})$ module structure on the $O_{X'}$ module $g'^{*mod}M := g'^*M \otimes_{g'^*O_X} O_{X'}$.

(ii) and (iii): There is nothing to prove. \square

Remark 7. Consider a commutative diagram in \mathbf{RCat}

$$\begin{array}{ccc}
D = (\mathcal{X}, O_X) & \xrightarrow{f} & (\mathcal{S}, O_S) . \\
g' \uparrow & & \uparrow g \\
(\mathcal{X}', O_{X'}) & \xrightarrow{f'} & (\mathcal{T}, O_T)
\end{array}$$

Assume that the canonical map $T(g', \text{hom})(O_X, O_X) : g'^*D(O_X) \rightarrow D(g'^*O_X)$ is an isomorphism of sheaves. Under the canonical isomorphism $(-)\otimes 1 : (\Omega_{O_X/f^*O_S}^\bullet, F_b) \xrightarrow{\sim} (\Omega_{O_X/f^*O_S}^\bullet, F_b) \otimes_{O_X} (O_X, F_b)$, we have (see definition-proposition 12 and definition 1)

- $\Omega_{(O_{X'}/g'^*O_X)/(O_T/g^*O_S)}(O_X) = \Omega_{(O_{X'}/g'^*O_X)/(O_T/g^*O_S)} : g'^*\Omega_{O_X/f^*O_S}^\bullet \rightarrow \Omega_{O_{X'}/f'^*O_T}^\bullet$
- $T_\omega^O(D)(O_X) = T_\omega^O(D) : g'^{*mod}L_O(f_*E(\Omega_{O_X/f^*O_S}^\bullet, F_b)) \rightarrow f'_*E(\Omega_{O_{X'}/f'^*O_T}^\bullet, F_b)$.

Definition 53. Consider a commutative diagram in \mathbf{RCat}

$$\begin{array}{ccc}
D = (\mathcal{X}, O_X) & \xrightarrow{f} & (\mathcal{S}, O_S) . \\
g' \uparrow & & \uparrow g \\
(\mathcal{X}', O_{X'}) & \xrightarrow{f'} & (\mathcal{T}, O_T)
\end{array}$$

with commutative structural sheaf of rings. Assume that the canonical map $T(g', \text{hom})(O_X, O_X) : g'^*D(O_X) \rightarrow D(g'^*O_X)$ is an isomorphism of sheaves. For $(N, F) \in C_{D(O_{X'}), g'^*D(O_X)}\text{fil}(\mathcal{X}')$, we have by definition-proposition 12 the map in $C_{f^*O_S}\text{fil}(\mathcal{X})$

$$\begin{aligned}
T_\omega^O(g', \otimes)(N, F) &: \Omega_{O_X/f^*O_S}^\bullet \otimes_{O_X} g'_*(N, F) \xrightarrow{\text{ad}(g'^{*mod}, g'_*)(-)} \\
&g'_*(g'^*(\Omega_{O_X/f^*O_S}^\bullet \otimes_{O_X} g'_*N) \otimes_{g'^*O_X} O_{X'}) \xrightarrow{m \circ \Omega_{(O_{X'}/g'^*O_X)/(O_T/g^*O_S)}(g'_*(N, F))} \\
&g'_*(\Omega_{O_X/f^*O_S}^\bullet \otimes_{O_X} g'^{*mod}g'_*(N, F)) \xrightarrow{\text{ad}(g'^{*mod}, g'_*)(N, F)} g'_*(\Omega_{O_{X'}/f'^*O_T}^\bullet \otimes_{O_{X'}}(N, F))
\end{aligned}$$

with $m(n \otimes s) = s.n$ and $g'_*N \in C_{D(O_X)}(\mathcal{X}')$, the structure of $D(O_X)$ module being given by the canonical morphism $\text{ad}(g'^*, g'_*)(D(O_X)) : D(O_X) \rightarrow g'_*g'^*D(O_X)$ applied to $g'_*N \in C_{g'_*g'^*D(O_X)}(\mathcal{X}')$.

We finish this subsection by a proposition for ringed spaces similar to proposition 9

Proposition 37. *Let $f : (X, O_X) \rightarrow (S, O_S)$ a morphism with $(X, O_X), (S, O_S) \in \text{RTop}$ with commutative sheaves of rings. Assume that $\Omega_{O_X/f^*O_S} \in \text{PSh}_{O_X}(X)$ is a locally free O_X module of finite rank.*

(i) *If $\phi : (M, F) \rightarrow (N, F)$ is an r -filtered top local equivalence with $(M, F), (N, F) \in C_{D(O_X)\text{fil}}(X)$, then*

$$DR(O_X/f^*O_S)(\phi) : (\Omega_{O_X/f^*O_S}^\bullet, F_b) \otimes_{O_X} (M, F) \rightarrow (\Omega_{O_X/f^*O_S}^\bullet, F_b) \otimes_{O_T} (N, F)$$

is an r -filtered top local equivalence.

(ii) *Consider a commutative diagram in RTop*

$$D = \begin{array}{ccc} (X, O_X) & \xrightarrow{f} & (S, O_S) \\ g' \uparrow & & \uparrow g \\ (X', O_{X'}) & \xrightarrow{f'} & (T, O_T) \end{array}$$

*with commutative structural sheaf of rings. For $(N, F) \in C_{D(O_{X'})\text{fil}}(X')$, the map in $C_{f^*O_S\text{fil}}(X)$*

$$k \circ T_\omega^O(g', \otimes)(E(N, F)) : (\Omega_{O_X/f^*O_S}^\bullet, F_b) \otimes_{O_X} g'_* E(N, F) \rightarrow g'_* E((\Omega_{O_{X'}/f^*O_T}^\bullet, F_b) \otimes_{O_{X'}} E(N, F))$$

is a filtered top local equivalence (see definition 53).

Proof. (i): Follows from proposition 9 (i) since $\Omega_{O_X/f^*O_S}^\bullet \in C^b(X)$ is then a bounded complex with $\Omega_{O_X/f^*O_S}^n \in \text{PSh}_{O_X}(X)$ a locally free O_X module of finite rank.

(ii): Follows from proposition 9 (ii) since $\Omega_{O_X/f^*O_S}^\bullet \in C^b(X)$ is then a bounded complex with $\Omega_{O_X/f^*O_S}^n \in \text{PSh}_{O_X}(X)$ a locally free O_X module of finite rank. \square

4.1.3 The support section functor for D module on ringed spaces

Let $(S, O_S) \in \text{RTop}$ with O_S commutative. Let $Z \subset S$ a closed subset. Denote by $j : S \setminus Z \hookrightarrow S$ the open complementary embedding,

- For $G \in C_{D(O_S)}(S)$, $\Gamma_Z G := \text{Cone}(\text{ad}(j^*, j_*)(G) : F \rightarrow j_* j^* G)[-1]$ has a (unique) structure of $D(O_S)$ module such that $\gamma_Z(G) : \Gamma_Z G \rightarrow G$ is a map in $C_{D(O_S)}(S)$. This gives the functor

$$\Gamma_Z : C_{D(O_S)\text{fil}}(S) \rightarrow C_{D(O_S)\text{fil}}(S), (G, F) \mapsto \Gamma_Z(G, F)$$

together with the canonical map $\gamma_Z(G, F) : \Gamma_Z(G, F) \rightarrow (G, F)$. Let $Z_2 \subset Z$ a closed subset, then for $G \in C_{D(O_S)}(S)$, $T(Z_2/Z, \gamma)(G) : \Gamma_{Z_2} G \rightarrow \Gamma_Z G$ is a map in $C_{D(O_S)}(S)$.

- For $G \in C_{O_S}(S)$, $\Gamma_Z^\vee G := \text{Cone}(\text{ad}(j_!, j^*)(G) : j_! j^* G \rightarrow G)$ has a unique structure of $D(O_S)$ module, such that $\gamma_Z^\vee(G) : G \rightarrow \Gamma_Z^\vee G$ is a map in $C_{D(O_S)}(S)$. This gives the functor

$$\Gamma_Z^\vee : C_{D(O_S)\text{fil}}(S) \rightarrow C_{D(O_S)\text{fil}}(S), (G, F) \mapsto \Gamma_Z^\vee(G, F),$$

together with the canonical map $\gamma_Z^\vee(G, F) : (G, F) \rightarrow \Gamma_Z^\vee(G, F)$. Let $Z_2 \subset Z$ a closed subset, then for $G \in C_{D(O_S)}(S)$, $T(Z_2/Z, \gamma^\vee)(G) : \Gamma_{Z_2}^\vee G \rightarrow \Gamma_Z^\vee G$ is a map in $C_{D(O_S)}(S)$.

- For $G \in C_{D(O_S)}(S)$,

$$\begin{aligned} \Gamma_Z^{\vee, h} G &:= \mathbb{D}_S^O L_O \Gamma_Z E(\mathbb{D}_S^O G) \\ &:= \text{Cone}(\mathbb{D}_S^O L_O \text{ad}(j_*, j^*)(E(\mathbb{D}_S^O G)) : \mathbb{D}_S^O L_O j_* j^* E(\mathbb{D}_S^O G) \rightarrow \mathbb{D}_S^O L_O E(\mathbb{D}_S^O G)) \end{aligned}$$

has also canonical $D(O_S)$ -module structure, and $\gamma_Z^{\vee,h}(G) : G \rightarrow \Gamma_Z^{\vee,h}G$ is a map in $C_{D(O_S)}$. This gives the functor

$$\Gamma_Z^{\vee,h} : C_{D(O_S)fil}(S) \rightarrow C_{D(O_S)fil}(S), (G, F) \mapsto \Gamma_Z^{\vee,h}(G, F),$$

together with the canonical map $\gamma_Z^{\vee,h}(G, F) : (G, F) \rightarrow \Gamma_Z^{\vee,h}(G, F)$.

- Consider $\mathcal{I}_Z^o \subset O_S$ the ideal of vanishing function on Z and $\mathcal{I}_Z \subset D_S$ the right ideal of D_S generated by \mathcal{I}_Z^o . We have then $\mathcal{I}_Z^D \subset \mathcal{I}_Z$, where $\mathcal{I}_Z^D \subset D_S$ is the left and right ideal consisting of sections which vanish on Z . For $G \in \text{PSh}_{D(O_S)}(S)$, we consider, $S^o \subset S$ being an open subset,

$$\mathcal{I}_Z G(S^o) = \langle \{f.m, m \in G(S^o), f \in \mathcal{I}_Z(S^o)\} \rangle \subset G(S^o)$$

the $D(O_S)$ -submodule generated by the functions which vanish on Z (\mathcal{I}_Z is a right $D(O_S)$ ideal), This gives the functor,

$$\begin{aligned} \Gamma_Z^{\vee,O} &:= \Gamma_Z^{\vee,O,\mathcal{I}_Z} : C_{D(O_S)fil}(S) \rightarrow C_{D(O_S)fil}(S), \\ (G, F) &\mapsto \Gamma_Z^{\vee,O}(G, F) := \text{Cone}(b_Z(G, F) : \mathcal{I}_Z(G, F) \rightarrow (G, F)), b_Z(-) := b_{\mathcal{I}_Z}(-) \end{aligned}$$

together with the canonical map $\gamma_Z^{\vee,O}(G, F) : (G, F) \rightarrow \Gamma_Z^{\vee,O}(G, F)$. which factors through

$$\gamma_Z^{\vee,O}(G) : G \xrightarrow{\gamma_Z^{\vee}(G)} \Gamma_Z^{\vee}G \xrightarrow{b_{S/Z}(G)} \Gamma_Z^{\vee,O}G.$$

with $b_{S/Z}(-) = b_{S/Z}^I$ and we have an homotopy equivalence $c_Z(G) := c_{\mathcal{I}_Z}(G) : \Gamma_Z^{\vee,O}G \rightarrow G/\mathcal{I}_Z G$.

Lemma 6. *Let $(Y, O_Y) \in \text{RTop}$ and $i : X \hookrightarrow Y$ a closed embedding.*

- (i) *For $(M, F) \in C_{D(O_Y)fil}(Y)$ and $(N, F) \in \text{PSh}_{D(O_Y)opfil}(Y)$ such that $a_\tau N$ is a locally free $D(O_Y)$ module of finite rank, the canonical map*

$$\begin{aligned} T(\gamma, \otimes)(E(M, F), (N, F)) &:= (I, T(j, \otimes)(E(M, F), (N, F))) : \\ (\Gamma_X E(M, F)) \otimes_{D(O_Y)} (N, F) &\rightarrow \Gamma_X E((M, F) \otimes_{D(O_Y)} (N, F)) \end{aligned}$$

is an equivalence top local.

- (ii) *For $(M, F) \in C_{D(O_Y)(op)fil}(Y)$ and $(N, F) \in \text{PSh}_{D(O_Y)(op)fil}(Y)$ such that $a_\tau N$ is a locally free O_Y module of finite rank, the canonical map*

$$\begin{aligned} T(\gamma, \otimes)(E(M, F), (N, F)) &:= (I, T(j, \otimes)(E(M, F), (N, F))) : \\ (\Gamma_X E(M, F)) \otimes_{O_Y} (N, F) &\rightarrow \Gamma_X E((M, F) \otimes_{O_Y} (N, F)) \end{aligned}$$

is a filtered top local equivalence.

Proof. Follows from proposition 9. Also note that $T(j, \otimes)(-, -) = T^{mod}(j, \otimes)(-, -)$. □

We now look at the pullback map and the transformation map of De Rahm complexes together with the support section functor. The following is a generalization of definition-proposition 3 :

Definition-Proposition 13. *Consider a commutative diagram in RTop*

$$\begin{array}{ccccc} D_0 = & f : (X, O_X) & \xrightarrow{i} & (Y, O_Y) & \xrightarrow{p} & (S, O_S) \\ & \uparrow g' & & \uparrow g'' & & \uparrow g \\ & f' : (X', O_{X'}) & \xrightarrow{i'} & (Y', O_{Y'}) & \xrightarrow{p'} & (T, O_T) \end{array}$$

with i, i' being closed embeddings. Denote by D the right square of D . We have a factorization $i' : X' \xrightarrow{i'_1} X \times Y Y' \xrightarrow{i'_0} Y'$, where i'_0, i'_1 are closed embedding.

(i) For $(M, F) \in C_{\mathcal{D}(O_Y)fil}(Y)$, the canonical map,

$$E(\Omega_{(O_{Y'}/g''^*O_Y)/(O_T/g^*O_S)}(M, F)) \circ T(g'', E)(-) \circ T(g'', \gamma)(-) : \\ g''^* \Gamma_X E((\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} (M, F)) \rightarrow \Gamma_{X \times_Y Y'} E((\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b) \otimes_{O_{Y'}} g''^{*mod}(M, F))$$

unique up to homotopy such that the following diagram in $C_{g''^*p^*O_S fil}(Y') = C_{p'^*g^*O_S fil}(Y')$ commutes

$$\begin{array}{ccc} g''^* \Gamma_X E((\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} (M, F)) & \xrightarrow{E(\Omega_{(-)/(-)}(M, F) \circ T(g'', E)(-) \circ T(g'', \gamma)(-))} & \Gamma_{X \times_Y Y'} E((\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b) \otimes_{O_{Y'}} (g''^{*mod}(M, F))) \\ \gamma_X(-) \downarrow & & \downarrow \gamma_{X \times_Y Y'}(-) \\ g''^* E((\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} (M, F)) & \xrightarrow{E(\Omega_{(-)/(-)}(M, F) \circ T(g'', E)(-))} & E((\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b) \otimes_{O_{Y'}} g''^{*mod}(M, F)) \end{array}$$

(ii) For $M \in C_{\mathcal{D}}(Y)$, there is a canonical map

$$T_\omega^O(D)(M, F)^\gamma : g^{*mod} \text{Lop}_* \Gamma_X E((\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} (M, F)) \rightarrow \\ p'_* \Gamma_{X \times_Y Y'} E((\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b) \otimes_{O_{Y'}} g''^{*mod}(M, F))$$

unique up to homotopy such that the following diagram in $C_{O_T fil}(T)$ commutes

$$\begin{array}{ccc} g^{*mod} \text{Lop}_* \Gamma_X E((\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} (M, F)) & \xrightarrow{T_\omega^O(D)(M, F)^\gamma} & p'_* \Gamma_{X \times_Y Y'} E((\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b) \otimes_{O_{Y'}} (g''^{*mod}(M, F))) \\ \gamma_X(-) \downarrow & & \downarrow \gamma_{X \times_Y Y'}(-) \\ g^{*mod} \text{Lop}_* E((\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} (M, F)) & \xrightarrow{T_\omega^O(D)(M, F)} & p'_* E((\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b) \otimes_{O_{Y'}} g''^{*mod}(M, F)) \end{array}$$

(iii) For $N \in C_{\mathcal{D}}(Y \times T)$, the canonical map in $C_{h'^*O_T fil}(Y')$

$$T(X'/X \times_Y Y', \gamma)(-) : \Gamma_{X'} E((\Omega_{Y'/T}^\bullet, F_b) \otimes_{O_{Y'}} (N, F)) \rightarrow \Gamma_{X \times_Y Y'} E((\Omega_{O_{Y'}/O_T}^\bullet, F_b) \otimes_{O_{Y'}} (N, F))$$

is unique up to homotopy such that $\gamma_{X \times_Y Y'}(-) \circ T(X'/X \times_Y Y', \gamma)(-) = \gamma_{X'}(-)$.

(iv) For $M = O_Y$, we have $T_\omega^O(D)(O_Y \times_S)^\gamma = T_\omega^O(D)^\gamma$ and $T_\omega^O(X \times_Y Y'/Y')(O_{Y'})^\gamma = T_\omega^O(X \times_Y Y'/Y')^\gamma$ (see definition-proposition 3).

Proof. Immediate from definition. We take for the map of point (ii) the composite

$$\begin{aligned} & T_\omega^O(D)(M, F)^\gamma : g^{*mod} \text{Lop}_* \Gamma_X E((\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} (M, F)) \xrightarrow{q} \\ & g^* p_* \Gamma_X E((\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} (M, F)) \otimes_{g^*O_S} O_T \xrightarrow{T(g'', E)(-) \circ T(g'', \gamma)(-) \circ T(D)(E(\Omega_{O_Y/p^*O_S}^\bullet, F_b))} \\ & (p'_* \Gamma_{X \times_Y Y'} E(g''((\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} (M, F)))) \otimes_{g^*O_S} O_T \xrightarrow{E(\Omega_{(O_{Y'}/g''^*O_Y)/(O_T/g^*O_S)}(M, F))} \\ & p'_* \Gamma_{X \times_Y Y'} E((\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b) \otimes_{O_{Y'}} g''^{*mod}(M, F)) \otimes_{g^*O_S} O_T \xrightarrow{m} \\ & p'_* \Gamma_{X \times_Y Y'} E((\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b) \otimes_{O_{Y'}} g''^{*mod}(M, F)), \end{aligned}$$

with $m(n \otimes s) = s.n$. □

Let $p : (Y, O_Y) \rightarrow (S, O_S)$ a morphism with $(Y, O_Y), (S, O_S) \in \text{RTop}$. Let $i : X \hookrightarrow Y$ a closed embedding. Denote by $j : Y \setminus X \hookrightarrow Y$ the complementary open embedding. Consider, for $(M, F) \in C_{D(O_Y)fil}(Y)$, the map in $C_{p^*O_S fil}(Y)$ (see definition 53):

$$\begin{aligned} k \circ T_w^O(j, \otimes)(E(M, F)) &: (\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} j_* j^* E(M, F) \\ &\xrightarrow{DR(O_Y/p^*O_S)(\text{ad}(j^*, j_*)(-))} \\ j_* j^* ((\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} j_* j^* E(M, F)) &= j_* j^* ((\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} j_* j^* j^* E(M, F)) \\ &\xrightarrow{k \circ DR(O_Y/p^*O_S)(\text{ad}(j^*, j_*)(j^* E(M)))} \\ j_* E(j^* (\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} j^* E(M, F)) &= j_* E(j^* ((\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} E(M, F))) \end{aligned}$$

Definition 54. Let $p : (Y, O_Y) \rightarrow (S, O_S)$ a morphism with $(Y, O_Y), (S, O_S) \in \text{RTop}$. Let $i : X \hookrightarrow Y$ a closed embedding. Denote by $j : Y \setminus X \hookrightarrow Y$ the complementary open embedding. We consider, for $(M, F) \in C_{D(O_Y)fil}(Y)$ the canonical map in $C_{p^*O_S fil}(Y)$

$$\begin{aligned} T_w^O(\gamma, \otimes)(M, F) &:= (I, k \circ T_w^O(j, \otimes)(E(M, F))) : \\ (\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} \Gamma_X E(M, F) &\rightarrow \Gamma_X E((\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} E(M, F)). \end{aligned}$$

Proposition 38. Let $p : (Y, O_Y) \rightarrow (S, O_S)$ a morphism with $(Y, O_Y), (S, O_S) \in \text{RTop}$. Let $i : X \hookrightarrow Y$ a closed embedding. Then, if Ω_{O_Y/p^*O_S} is a locally free O_Y module, for $(M, F) \in C_{D(O_Y)fil}(Y)$

(i) the map

$$T_w^O(\gamma, \otimes)(M, F) : (\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} \Gamma_X E(M, F) \rightarrow \Gamma_X E((\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} E(M, F))$$

is a 1-filtered top local equivalence,

(ii) the map in $D_{p^*O_S fil}(Y)$

$$\begin{aligned} T_w^O(\gamma, \otimes) &:= DR(O_Y/p^*O_S)(k)^{-1} \circ T_w^O(\gamma, \otimes)(M, F) : \\ (\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} \Gamma_X E(M, F) &\rightarrow \Gamma_X E((\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} (M, F)) \end{aligned}$$

is an isomorphism.

Proof. By proposition 37,

- $\text{Gr}_F^p(k \circ T_w^O(j, \otimes)(E(M, F))) : \Omega_{O_Y/p^*O_S}^\bullet \otimes_{O_Y} j_* j^* F^{p-\bullet} E(M) \rightarrow j_* E(j^* (\Omega_{O_Y/p^*O_S}^\bullet \otimes_{O_Y} F^{p-\bullet} E(M)))$ is a top local equivalence and
- $DR(O_Y/p^*O_S)(k) : \Omega_{O_Y/p^*O_S}^\bullet \otimes_{O_Y} (M, F) \rightarrow \Omega_{O_Y/p^*O_S}^\bullet \otimes_{O_Y} E(M, F)$ is a filtered top local equivalence.

□

4.2 The D-modules on smooth complex algebraic varieties and on complex analytic manifold and their functorialities in the filtered case

For convenience, we will work with and state the results for presheaves of D-modules. In this section, it is possible to assume that all the presheaves are sheaves and take the sheafification functor after the pullback functor f^* for a morphism $f : X \rightarrow S$, $X, S \in \text{Var}(\mathbb{C})$ or $X, S \in \text{AnSp}(\mathbb{C})$, and after the internal hom functors and tensor products of presheaves of modules on $S \in \text{Var}(\mathbb{C})$ or $S \in \text{AnSp}(\mathbb{C})$.

For $S = (S, O_S) \in \text{SmVar}(\mathbb{C})$, resp. $S = (S, O_S) \in \text{AnSm}(\mathbb{C})$, we denote by

- $D_S := D(O_S) \subset \mathcal{H}om_{\mathbb{C}_S}(O_S, O_S)$ the subsheaf consisting of differential operators. By a D_S module, we mean a left D_S module.

- we denote by
 - $\text{PSh}_{\mathcal{D}}(S)$ the abelian category of Zariski (resp. usu) presheaves on S with a structure of left D_S module, and by $\mathcal{H}ol_{\mathcal{D}}(S) \subset \mathcal{C}oh_{\mathcal{D}}(S) \subset \text{PSh}_{\mathcal{D}}(S)$ the full subcategories whose objects are coherent, resp. holonomic, sheaves of left D_S modules,
 - $\text{PSh}_{\mathcal{D}^{op}}(S)$ the abelian category of Zariski (resp. usu) presheaves on S with a structure of right D_S module, and by $\mathcal{H}ol_{\mathcal{D}^{op}}(S) \subset \mathcal{C}oh_{\mathcal{D}^{op}}(S) \subset \text{PSh}_{\mathcal{D}^{op}}(S)$ the full subcategories whose objects are coherent, resp. holonomic, sheaves of right D_S modules,

- we denote by

- $C_{\mathcal{D}}(S) = C(\text{PSh}_{\mathcal{D}}(S))$ the category of complexes of Zariski presheaves on S with a structure of D_S module,

$$C_{\mathcal{D},h}(S) \subset C_{\mathcal{D},c}(S) \subset C_{\mathcal{D}}(S)$$

the full subcategories consisting of complexes of presheaves M such that $a_{\tau}H^n(M)$ are coherent (resp. holonomic) sheaves of D_S modules, a_{τ} being the sheafification functor for the Zariski, resp. usual, topology,

- $C_{\mathcal{D}^{op}}(S) = C(\text{PSh}_{\mathcal{D}^{op}}(S))$ the category of complexes of Zariski presheaves on S with a structure of right D_S module,

$$C_{\mathcal{D}^{op},h}(S) \subset C_{\mathcal{D}^{op},c}(S) \subset C_{\mathcal{D}^{op}}(S)$$

the full subcategories consisting of complexes of presheaves M such that $a_{\tau}H^n(M)$ are coherent (resp. holonomic) sheaves of right D_S modules,

- in the filtered case we have

- $C_{\mathcal{D}(2)fil}(S) \subset C(\text{PSh}_{\mathcal{D}}(S), F, W) := C(\text{PSh}_{D(O_S)}(S), F, W)$ the category of (bi)filtered complexes of algebraic (resp. analytic) D_S modules such that the filtration is biregular (see definition 47),

$$C_{\mathcal{D}(2)fil,h}(S) \subset C_{\mathcal{D}(2)fil,c}(S) \subset C_{\mathcal{D}(2)fil}(S),$$

the full subcategories consisting of filtered complexes of presheaves (M, F) such that $a_{\tau}H^n(M)$ are coherent (resp. holonomic) sheaves of D_S modules

- $C_{\mathcal{D}0fil}(S) \subset C_{\mathcal{D}fil}(S)$ the full subcategory such that the filtration is a filtration by D_S submodule (which is stronger than Griffithz transversality), $C_{\mathcal{D}(1,0)fil}(S) \subset C_{\mathcal{D}2fil}(S)$ the full subcategory such that W is a filtration by D_S submodules (see definition 47),

$$C_{\mathcal{D}(1,0)fil,h}(S) = C_{\mathcal{D}2fil,h}(S) \cap C_{\mathcal{D}(1,0)fil}(S) \subset C_{\mathcal{D}2fil,h}(S),$$

the full subcategory consisting of filtered complexes of presheaves (M, F, W) such that $a_{\tau}H^n(M)$ are holonomic sheaves of D_S modules and such that $W^p M \subset M$ are D_S submodules (recall that the O_S submodules $F^p M \subset M$ are NOT D_S submodules but satisfy by definition $md : F^r D_S \otimes F^p M \subset F^{p+r} M$),

- $C_{\mathcal{D}^{op}(2)fil}(S) \subset C(\text{PSh}_{\mathcal{D}^{op}}(S), F, W) := C(\text{PSh}_{D(O_S)^{op}}(S), F, W)$ the category of (bi)filtered complexes of algebraic (resp. analytic) right D_S modules such that the filtration is biregular, as in the left case we consider the subcategories

$$C_{\mathcal{D}^{op}(2)fil,h}(S) \subset C_{\mathcal{D}^{op}(2)fil,c}(S) \subset C_{\mathcal{D}^{op}(2)fil}(S),$$

the full subcategories consisting of filtered complexes of presheaves (M, F) such that $a_{\tau}H^n(M)$ are coherent (resp. holonomic) sheaves of right D_S modules.

For $S = (S, O_S) \in \text{AnSm}(\mathbb{C})$, we have the natural extension $D_S \subset D_S^{\infty} \subset \mathcal{H}om_{\mathbb{C}_S}(O_S, O_S)$ where $D_S^{\infty} \subset \mathcal{H}om_{\mathbb{C}_S}(O_S, O_S)$ is the subsheaf of differential operators of possibly infinite order (see [18]) for the definition of the action of a differential operator of infinite order on O_S) Similarly, we have

- $C_{\mathcal{D}^\infty(2)fil}(S) \subset C(\text{PSh}_{\mathcal{D}^\infty}(S), F, W) := C(\text{PSh}_{D_S^\infty}(S), F, W)$ the category of (bi)filtered complexes of D_S^∞ modules such that the filtration is biregular,

$$C_{\mathcal{D}^\infty(2)fil,h}(S) \subset C_{\mathcal{D}^\infty(2)fil,c}(S) \subset C_{\mathcal{D}^\infty(2)fil}(S),$$

the full subcategories consisting of filtered complexes of presheaves (M, F) such that $a_\tau H^n(M)$ are coherent (resp. holonomic) sheaves of D_S^∞ modules.

- $C_{\mathcal{D}^\infty 0fil}(S) \subset C_{\mathcal{D}^\infty fil}(S)$ the full subcategory such that the filtration is a filtration by D_S^∞ submodule, $C_{\mathcal{D}^\infty(1,0)fil}(S) \subset C_{\mathcal{D}^\infty 2fil}(S)$ the full subcategory such that W is a filtration by D_S^∞ submodules,

$$C_{\mathcal{D}^\infty(1,0)fil,h}(S) = C_{\mathcal{D}^\infty 2fil,h}(S) \cap C_{\mathcal{D}^\infty(1,0)fil}(S) \subset C_{\mathcal{D}^\infty 2fil,h}(S),$$

the full subcategory consisting of filtered complexes of presheaves (M, F, W) such that $a_\tau H^n(M)$ are holonomic sheaves of D_S^∞ modules and such that $W^p M \subset M$ are D_S submodules

- $C_{\mathcal{D}^\infty,op(2)fil}(S) \subset C(\text{PSh}_{\mathcal{D}^\infty,op}(S), F, W) := C(\text{PSh}_{D_S^\infty,op}(S), F, W)$ the category of (bi)filtered complexes of right D_S^∞ modules such that the filtration is biregular,

$$C_{\mathcal{D}^\infty,op(2)fil,h}(S) \subset C_{\mathcal{D}^\infty,op(2)fil,c}(S) \subset C_{\mathcal{D}^\infty,op(2)fil}(S),$$

the full subcategories consisting of filtered complexes of presheaves (M, F) such that $a_\tau H^n(M)$ are coherent (resp. holonomic) sheaves of D_S modules.

For $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$ or with $(X, S) \in \text{AnSm}(\mathbb{C})$,

- we denote by

- $\text{PSh}_{f^*\mathcal{D}}(X)$ the abelian category of Zariski (resp. usu) presheaves on S with a structure of left f^*D_S module, and $C_{f^*\mathcal{D}}(X) = C(\text{PSh}_{f^*\mathcal{D}}(X))$,
- $\text{PSh}_{\mathcal{D},f^*\mathcal{D}}(X)$ the abelian category of Zariski (resp. usu) presheaves on S with a structure of left f^*D_S module and left D_X module, and $C_{\mathcal{D},f^*\mathcal{D}}(X) = C(\text{PSh}_{\mathcal{D},f^*\mathcal{D}}(X))$,
- $\text{PSh}_{\mathcal{D}^{op},f^*\mathcal{D}}(X)$ the abelian category of Zariski (resp. usu) presheaves on S with a structure of left f^*D_S module and right D_X module and $C_{\mathcal{D}^{op},f^*\mathcal{D}}(X) = C(\text{PSh}_{\mathcal{D}^{op},f^*\mathcal{D}}(X))$,

- we denote by

- $C_{f^*\mathcal{D}fil}(X) \subset C(\text{PSh}_{f^*\mathcal{D}}(X), F) := C(\text{PSh}_{f^*D_{(O_S)}}(X), F)$ the category of filtered complexes of algebraic (resp. analytic) f^*D_S modules such that the filtration is biregular,
- $C_{\mathcal{D},f^*\mathcal{D}fil}(X) \subset C(\text{PSh}_{\mathcal{D},f^*\mathcal{D}}(X), F)$ the category of filtered complexes of algebraic (resp. analytic) (f^*D_S, D_X) modules such that the filtration is biregular,
- $C_{\mathcal{D}^{op},f^*\mathcal{D}fil}(X) \subset C(\text{PSh}_{\mathcal{D}^{op},f^*\mathcal{D}}(X), F)$ the category of filtered complexes of algebraic (resp. analytic) (f^*D_S, D_X^{op}) modules such that the filtration is biregular.

For $f : X \rightarrow S$ a morphism with $X, S \in \text{AnSm}(\mathbb{C})$, we denote by

- $C_{f^*\mathcal{D}^\infty fil}(X) \subset C(\text{PSh}_{f^*\mathcal{D}^\infty}(X), F) := C(\text{PSh}_{f^*D_S^\infty}(X), F)$ the category of filtered complexes of $f^*D_S^\infty$ modules such that the filtration is biregular,
- $C_{\mathcal{D}^\infty,f^*\mathcal{D}^\infty fil}(X) \subset C(\text{PSh}_{\mathcal{D}^\infty,f^*\mathcal{D}^\infty}(X), F)$ the category of filtered complexes of $(f^*D_S^\infty, D_X^\infty)$ modules such that the filtration is biregular,
- $C_{\mathcal{D}^\infty,op,f^*\mathcal{D}^\infty fil}(X) \subset C(\text{PSh}_{\mathcal{D}^\infty,op,f^*\mathcal{D}^\infty}(X), F)$ the category of filtered complexes of $(f^*D_S^\infty, D_X^{\infty,op})$ modules such that the filtration is biregular.

For $S \in \text{AnSm}(\mathbb{C})$, we denote by

$$J_S : C_{\mathcal{D}(2)fil}(S) \rightarrow C_{\mathcal{D}^\infty(2)fil}(S), (M, F) \mapsto J_S(M, F) := (M, F) \otimes_{D_S} (D_S^\infty, F^{ord})$$

the natural functor. For $(M, F) \in C_{\mathcal{D}^\infty fil}(S)$, we will consider the map

$$\mathcal{J}_S(M, F) : J_S(M, F) := (M, F) \otimes_{D_S} (D_S^\infty, F^{ord}) \rightarrow (M, F), m \otimes P \mapsto Pm$$

Of course $J_S(C_{\mathcal{D}(1,0)fil}(S)) \subset C_{\mathcal{D}^\infty(1,0)fil}(S)$. More generally, for $f : X \rightarrow S$ a morphism with $X, S \in \text{AnSm}(\mathbb{C})$, we denote by

$$J_{X/S} : C_{f^*\mathcal{D}(2)fil}(X) \rightarrow C_{f^*\mathcal{D}^\infty(2)fil}(X), (M, F) \mapsto J_{X/S}(M, F) := (M, F) \otimes_{f^*(D_S, F)} f^*(D_S^\infty, F)$$

the natural functor, together with, for $(M, F) \in C_{f^*\mathcal{D}^\infty fil}(X)$, the map $\mathcal{J}_S(M, F) : J_S(M, F) \rightarrow (M, F)$.

Definition 55. Let $S \in \text{SmVar}(\mathbb{C})$, resp. $S \in \text{AnSm}(\mathbb{C})$. Let $Z \subset S$ a closed subset and denote by $j : S \setminus Z \hookrightarrow S$ the open embedding.

(i) We denote by

- $\text{PSh}_{\mathcal{D}, Z}(S) \subset \text{PSh}_{\mathcal{D}}(S)$, the full subcategory consisting of presheaves $M \in \text{PSh}_{\mathcal{D}}(S)$, such that $j^*M = 0$,
- $C_{\mathcal{D}, Z}(S) \subset C_{\mathcal{D}}(S)$, the full subcategory consisting of complexes presheaves $M \in C_{\mathcal{D}}(S)$ such that $a_\tau j^* H^n M = 0$ for all $n \in \mathbb{Z}$,
- $C_{\mathcal{D}, Z, h}(S) := C_{\mathcal{D}, Z}(S) \cap C_{\mathcal{D}, h}(S) \subset C_{\mathcal{D}}(S)$ the full subcategory consisting of $M \in C_{\mathcal{D}}(S)$ such that $a_\tau H^n(M)$ are holonomic and $a_\tau j^* H^n M = 0$ for all $n \in \mathbb{Z}$,
- $C_{\mathcal{D}, Z, c}(S) := C_{\mathcal{D}, Z}(S) \cap C_{\mathcal{D}, c}(S) \subset C_{\mathcal{D}}(S)$ the full subcategory consisting of $M \in C_{\mathcal{D}}(S)$ such that $a_\tau H^n(M)$ are coherent and $a_\tau j^* H^n M = 0$ for all $n \in \mathbb{Z}$.

(ii) We denote by

- $C_{\mathcal{D}(2)fil, Z}(S) \subset C_{\mathcal{D}(2)fil}(S)$, the full subcategory consisting of $(M, F) \in C_{\mathcal{D}fil}(S)$ such that there exists $r \in \mathbb{N}$ such that $a_\tau j^* E_r^{p, q}(M, F) = 0$ for all $p, q \in \mathbb{Z}$, note that by definition this r does NOT depend on p and q ,
- $C_{\mathcal{D}(2)fil, Z, h}(S) := C_{\mathcal{D}(2)fil, Z}(S) \cap C_{\mathcal{D}(2)fil, h}(S) \subset C_{\mathcal{D}(2)fil}(S)$ the full subcategory consisting of (M, F) such that $a_\tau H^n(M)$ are holonomic for all $n \in \mathbb{Z}$ and such that there exists $r \in \mathbb{N}$ such that $a_\tau j^* E_r^{p, q}(M, F) = 0$ for all $p, q \in \mathbb{Z}$,
- $C_{\mathcal{D}(2)fil, Z, c}(S) := C_{\mathcal{D}(2)fil, Z}(S) \cap C_{\mathcal{D}(2)fil, c}(S) \subset C_{\mathcal{D}(2)fil}(S)$ the full subcategory consisting of (M, F) such that $a_\tau H^n(M)$ are coherent for all $n \in \mathbb{Z}$ and such that there exists $r \in \mathbb{N}$ such that $a_\tau j^* E_r^{p, q}(M, F) = 0$ for all $p, q \in \mathbb{Z}$.

(iii) We have then the full subcategories

- $C_{\mathcal{D}(1,0)fil, Z}(S) = C_{\mathcal{D}(1,0)fil}(S) \cap C_{\mathcal{D}2fil, Z}(S) \subset C_{\mathcal{D}2fil}(S)$,
- $C_{\mathcal{D}(1,0)fil, Z, h}(S) = C_{\mathcal{D}(1,0)fil}(S) \cap C_{\mathcal{D}2fil, Z, h}(S) \subset C_{\mathcal{D}2fil}(S)$.

Similarly :

Definition 56. Let $S \in \text{AnSm}(\mathbb{C})$. Let $Z \subset S$ a closed subset and denote by $j : S \setminus Z \hookrightarrow S$ the open embedding.

(i) We denote by

- $C_{\mathcal{D}^\infty(2)fil, Z}(S) \subset C_{\mathcal{D}^\infty(2)fil}(S)$. the full subcategory consisting of $(M, F) \in C_{\mathcal{D}^\infty}(S)$ such that j^*M is acyclic

- $C_{\mathcal{D}^\infty(2)fil,Z,h}(S) := C_{\mathcal{D}^\infty(2)fil,Z}(S) \cap C_{\mathcal{D}^\infty(2)fil,h}(S) \subset C_{\mathcal{D}^\infty(2)fil}(S)$ the full subcategory consisting of (M, F) such that $a_\tau H^n(M)$ are holonomic and $a_\tau j^* E_r^{p,q}(M, F)$.

(ii) We have then the full subcategories

- $C_{\mathcal{D}^\infty(1,0)fil,Z}(S) = C_{\mathcal{D}^\infty(1,0)fil}(S) \cap C_{\mathcal{D}^\infty 2fil,Z}(S) \subset C_{\mathcal{D}^\infty 2fil}(S)$,
- $C_{\mathcal{D}^\infty(1,0)fil,Z,h}(S) := C_{\mathcal{D}^\infty(1,0)fil}(S) \cap C_{\mathcal{D}^\infty 2fil,Z,h}(S) \subset C_{\mathcal{D}^\infty(2)fil}(S)$.

Definition 57. (i) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$, or with $X, S \in \text{AnSm}(\mathbb{C})$, we have, for $r = 1, \dots, \infty$, resp. $r = (1, \dots, \infty)^2$, the categories

$$D_{\mathcal{D}, f^* \mathcal{D}(2)fil,r}(S) := \text{Ho}_{Frtop} C_{\mathcal{D}, f^* \mathcal{D}(2)fil}(S), \quad D_{\mathcal{D}^{op}, f^* \mathcal{D}(2)fil,r}(S) := \text{Ho}_{Frtop} C_{\mathcal{D}^{op}, f^* \mathcal{D}(2)fil}(S),$$

the localizations with respect to r -filtered Zariski, resp. usu, local equivalence.

(ii) Let $S \in \text{SmVar}(\mathbb{C})$, or $S \in \text{AnSm}(\mathbb{C})$. We denote by

$$D_{\mathcal{D}(2)fil,\infty,h}(S) \subset D_{\mathcal{D}(2)fil,\infty}(S), \quad D_{\mathcal{D}(1,0)fil,\infty,h}(S) \subset D_{\mathcal{D}2fil,\infty}(S)$$

the full subcategories consisting of the image of $C_{\mathcal{D}(2)fil,h}(S)$, resp. $C_{\mathcal{D}(1,0)fil,h}(S)$, by the localization functor

$$D(\text{top}) : C_{\mathcal{D}(2)fil}(S) \rightarrow D_{\mathcal{D}(2)fil,\infty}(S)$$

that is consisting of $(M, F) \in C_{\mathcal{D}fil}(S)$ such that $a_\tau H^n(M)$ are holonomic, for all $n \in \mathbb{Z}$, resp. consisting of $(M, F, W) \in C_{\mathcal{D}2fil}(S)$ such that $a_\tau H^n(M)$ are holonomic and $W^p M^n \subset M^n$ are D_S submodules for all $n \in \mathbb{Z}$.

(iii) Let $S \in \text{AnSm}(\mathbb{C})$. We denote by

- $D_{\mathcal{D}(2)fil,\infty,rh}(S) \subset D_{\mathcal{D}(2)fil,\infty,h}(S)$, the full subcategory consisting of $(M, F) \in C_{\mathcal{D}fil}(S)$ such that $a_\tau H^n(M)$ are regular holonomic for all $n \in \mathbb{Z}$.
- $D_{\mathcal{D}(1,0)fil,\infty,rh}(S) = D_{\mathcal{D}2fil,\infty,rh}(S) \cap D_{\mathcal{D}(1,0)fil,\infty,h}(S) \subset D_{\mathcal{D}2fil,\infty,h}(S)$, the full subcategory consisting of $(M, F, W) \in C_{\mathcal{D}2fil}(S)$ such that $a_\tau H^n(M)$ are regular holonomic and $W^p M^n \subset M$ are D_S submodules for all $n \in \mathbb{Z}$.

Similarly,

Definition 58. (i) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{AnSm}(\mathbb{C})$, we have, for $r = 1, \dots, \infty$, resp. $r = (1, \dots, \infty)^2$, the categories

$$D_{\mathcal{D}^\infty, f^* \mathcal{D}^\infty(2)fil,r}(S) := \text{Ho}_{Frtop} C_{\mathcal{D}^\infty, f^* \mathcal{D}^\infty(2)fil}(S), \quad D_{\mathcal{D}^\infty, op, f^* \mathcal{D}^\infty(2)fil,r}(S) := \text{Ho}_{Frtop} C_{\mathcal{D}^\infty, op, f^* \mathcal{D}^\infty(2)fil}(S),$$

the localizations with respect to r -filtered usu local equivalence.

(ii) Let $S \in \text{AnSm}(\mathbb{C})$. We denote by

$$D_{\mathcal{D}^\infty(2)fil,\infty,h}(S) \subset D_{\mathcal{D}^\infty(2)fil,\infty}(S), \quad D_{\mathcal{D}^\infty(1,0)fil,\infty,h}(S) \subset D_{\mathcal{D}^\infty 2fil,\infty}(S)$$

the full subcategories consisting of the image of $C_{\mathcal{D}^\infty(2)fil,h}(S)$, resp. $C_{\mathcal{D}^\infty(1,0)fil,h}(S)$, by the localization functor

$$D(\text{top}) : C_{\mathcal{D}^\infty(2)fil}(S) \rightarrow D_{\mathcal{D}^\infty(2)fil,\infty}(S).$$

We begin this subsection by recalling the following well known facts

Proposition 39. Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$.

- (i) The sheaf of differential operators D_S is a locally free sheaf of O_S module. Hence, a coherent D_S module $M \in \text{Coh}_{\mathcal{D}}(S)$ is a quasi-coherent sheaf of O_S modules.

- (ii) A coherent sheaf $M \in \text{Coh}_{O_S}(S)$ of O_S module admits a D_S module structure if and only if it is locally free (of finite rank by coherency) and admits an integrable connexion. In particular if $i : Z \hookrightarrow S$ is a closed embedding for the Zariski topology, then i_*O_Z does NOT admit a D_S module structure since it is a coherent but not locally free O_S module.

Proof. Standard. □

In order to prove a version of the first GAGA theorem for coherent D modules, we will need to following. We start by a definition (cf. [16] definition 1.4.2) :

Definition 59. An $X \in \text{SmVar}(\mathbb{C})$ is said to be *D-affine* if the following two condition hold:

- (i) The global section functor $\Gamma(X, \cdot) : \mathcal{QCoh}_{\mathcal{D}}(X) \rightarrow \text{Mod}(\Gamma(X, D_X))$ is exact.
(ii) If $\Gamma(X, M) = 0$ for $M \in \mathcal{QCoh}_{\mathcal{D}}(X)$, then $M = 0$.

Proposition 40. If $X \in \text{SmVar}(\mathbb{C})$ is *D-affine*, then :

- (i) Any $M \in \mathcal{QCoh}_{\mathcal{D}}(X)$ is generated by its global sections.
(ii) The functor $\Gamma(X, \cdot) : \mathcal{QCoh}_{\mathcal{D}}(X) \rightarrow \text{Mod}(\Gamma(X, D_X))$ is an equivalence of category whose inverse is $L \in \text{Mod}(\Gamma(X, D_X)) \mapsto D_X \otimes_{\Gamma(X, D_X)} L \in \mathcal{QCoh}_{\mathcal{D}}(X)$.
(iii) We have $\Gamma(X, \cdot)(\mathcal{Coh}_{\mathcal{D}}(X)) = \text{Mod}(\Gamma(X, D_X))_f$, that is the global sections of a coherent D_X module is a finite module over the differential operators on X .

Proof. See [16]. □

The following proposition is from Kashiwara.

Proposition 41. Let $S \in \text{AnSm}(\mathbb{C})$.

- (i) For $K \in C_c(S)$ a complex of presheaves with constructible cohomology sheaves, we have $\mathcal{H}om(L(K), E(O_S)) \in C_{\mathcal{D}^\infty, h}(S)$.
(ii) The functor $J_S : C_{\mathcal{D}(2)fil}(S) \rightarrow C_{\mathcal{D}^\infty(2)fil}(S)$ satisfy $J_S(C_{\mathcal{D}(2)fil, rh}(S)) \subset C_{\mathcal{D}^\infty(2)fil, h}(S)$, derive trivially, and induce an equivalence of category

$$J_S : D_{\mathcal{D}(2)fil, \infty, rh}(S) \rightarrow D_{\mathcal{D}^\infty(2)fil, \infty, h}(S).$$

whose inverse satisfy, for $(M, F) \in \mathcal{H}ol_{\mathcal{D}^\infty(2)fil}(S)$ a (filtered) holonomic D_S^∞ module, that $J_S^{-1}(M, F) = (M_{reg}, F) \subset (M, F)$ is the D_S sub-module of M which is the regular part.

- (iii) We have $J_S(C_{\mathcal{D}(1,0)fil, rh}(S)) \subset C_{\mathcal{D}^\infty(1,0)fil, h}(S)$ and $J_S(D_{\mathcal{D}(1,0)fil, \infty, rh}(S)) = D_{\mathcal{D}^\infty(1,0)fil, \infty, h}(S)$.

Proof. Follows from [18]. □

Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$, and let $i : Z \hookrightarrow S$ a closed embedding and denote by $j : S \setminus Z \hookrightarrow S$ the open complementary. For $M \in \text{PSh}_{\mathcal{D}}(S)$, we denote $\mathcal{I}_Z M \subset M$ the (left) D_S submodule given by, for $S^\circ \subset S$ an open subset, $\mathcal{I}_Z M(S^\circ) \subset M(S^\circ)$ is the (left) $D_S(S^\circ)$ submodule

$$\mathcal{I}_Z M(S^\circ) = \langle \{fm, f \in \mathcal{I}_Z(S^\circ), m \in M(S^\circ)\} \rangle \subset M(S^\circ)$$

generated by the elements of the form fm . We denote by $b_Z(M) : \mathcal{I}_Z M \rightarrow M$ the inclusion map and $c_Z(M) : M \rightarrow M/\mathcal{I}_Z M$ the quotient map of (left) D_S modules. For $M \in \text{PSh}_{\mathcal{D}}(S)$, we denote $M\mathcal{I}_Z \subset M$ the right D_S submodule given by, for $S^\circ \subset S$ an open subset, $M\mathcal{I}_Z(S^\circ) \subset M(S^\circ)$ is the right $D_S(S^\circ)$ submodule

$$\mathcal{I}_Z M(S^\circ) = \langle \{mf, f \in \mathcal{I}_Z(S^\circ), m \in M(S^\circ)\} \rangle \subset M(S^\circ)$$

generated by the elements of the form mf . We denote by $b_Z(M) : \mathcal{I}_Z M \rightarrow M$ the inclusion map and $c_Z(M) : M \rightarrow M/\mathcal{I}_Z M$ the quotient map of right D_S modules.

4.2.1 Functorialities

Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{SmVar}(\mathbb{C})$, or let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then, we recall from section 4.1, the transfers modules

- $(D_{X \rightarrow S}, F^{ord}) := f^{*mod}(D_S, F^{ord}) := f^*(D_S, F^{ord}) \otimes_{f^*O_S} (O_X, F_b)$ which is a left D_X module and a left and right f^*D_S module
- $(D_{X \leftarrow S}, F^{ord}) := (K_X, F_b) \otimes_{O_X} (D_{X \rightarrow S}, F^{ord}) \otimes_{f^*O_S} f^*(K_S, F_b)$. which is a right D_X module and a left and right f^*D_S module.

Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then, the transfers modules of infite order are

- $(D_{X \rightarrow S}^\infty, F^{ord}) := f^{*mod}(D_S^\infty, F^{ord}) := f^*(D_S^\infty, F^{ord}) \otimes_{f^*O_S} (O_X, F_b)$ which is a left D_X^∞ module and a left and right $f^*D_S^\infty$ module
- $(D_{X \leftarrow S}^\infty, F^{ord}) := (K_X, F_b) \otimes_{O_X} (D_{X \rightarrow S}^\infty, F^{ord}) \otimes_{f^*O_S} f^*(K_S, F_b)$. which is a right D_X^∞ module and a left and right $f^*D_S^\infty$ module.

We have the following :

Lemma 7. *Let $f_1 : X \rightarrow Y$, $f_2 : Y \rightarrow S$ be two morphism with $X, S, Y \in \text{SmVar}(\mathbb{C})$, or let $f_1 : X \rightarrow Y$, $f_2 : Y \rightarrow S$ be two morphism with $X, S, Y \in \text{AnSm}(\mathbb{C})$.*

- (i) *We have $(D_{X \rightarrow S}, F^{ord}) = f_1^*(D_{Y \rightarrow S}, F^{ord}) \otimes_{f_1^*D_Y} (D_{X \rightarrow Y}, F^{ord})$ in $C_{\mathcal{D}, (f_2 \circ f_1)^* \mathcal{D}fil}(X)$ and*
- $$(D_{X \rightarrow S}, F^{ord}) = f_1^*(D_{Y \rightarrow S}, F^{ord}) \otimes_{f_1^*D_Y} (D_{X \rightarrow Y}, F^{ord}) = f_1^*(D_{Y \rightarrow S}, F^{ord}) \otimes_{f_1^*D_Y}^L (D_{X \rightarrow Y}, F^{ord}).$$
- in $D_{\mathcal{D}, (f_2 \circ f_1)^* \mathcal{D}fil, r}(X)$.*

- (ii) *We have $(D_{X \leftarrow S}, F^{ord}) = f_1^*(D_{Y \rightarrow S}, F^{ord}) \otimes_{f_1^*D_Y} (D_{X \leftarrow Y}, F^{ord})$ in $C_{\mathcal{D}^{op}, (f_2 \circ f_1)^* \mathcal{D}fil}(X)$ and*
- $$(D_{X \leftarrow S}, F^{ord}) = f_1^*(D_{Y \rightarrow S}, F^{ord}) \otimes_{f_1^*D_Y} (D_{X \leftarrow Y}, F^{ord}) = f_1^*(D_{Y \leftarrow S}, F^{ord}) \otimes_{f_1^*D_Y}^L (D_{X \leftarrow Y}, F^{ord}),$$
- in $D_{\mathcal{D}^{op}, (f_2 \circ f_1)^* \mathcal{D}fil, r}(X)$.*

Proof. Follows immediatly from definition. The first assertions of (i) and (ii) are particular cases of lemma 5. See [16] for example. \square

In the analytical case we also have

Lemma 8. *Let $f_1 : X \rightarrow Y$, $f_2 : Y \rightarrow S$ be two morphism with $X, S, Y \in \text{AnSm}(\mathbb{C})$.*

- (i) *We have $(D_{X \rightarrow S}^\infty, F^{ord}) = f_1^*(D_{Y \rightarrow S}^\infty, F^{ord}) \otimes_{f_1^*D_Y^\infty} (D_{X \rightarrow Y}^\infty, F^{ord})$ in $C_{\mathcal{D}^\infty, (f_2 \circ f_1)^* \mathcal{D}^\infty fil}(X)$ and*
- $$(D_{X \rightarrow S}^\infty, F^{ord}) = f_1^*(D_{Y \rightarrow S}^\infty, F^{ord}) \otimes_{f_1^*D_Y^\infty} (D_{X \rightarrow Y}^\infty, F^{ord}) = f_1^*(D_{Y \rightarrow S}^\infty, F^{ord}) \otimes_{f_1^*D_Y^\infty}^L (D_{X \rightarrow Y}^\infty, F^{ord}).$$
- in $D_{\mathcal{D}^\infty, (f_2 \circ f_1)^* \mathcal{D}^\infty fil, r}(X)$.*

- (ii) *We have $(D_{X \leftarrow S}^\infty, F^{ord}) = f_1^*(D_{Y \rightarrow S}^\infty, F^{ord}) \otimes_{f_1^*D_Y^\infty} (D_{X \leftarrow Y}^\infty, F^{ord})$ in $C_{\mathcal{D}^\infty, op, (f_2 \circ f_1)^* \mathcal{D}^\infty fil}(X)$ and*
- $$(D_{X \leftarrow S}^\infty, F^{ord}) = f_1^*(D_{Y \rightarrow S}^\infty, F^{ord}) \otimes_{f_1^*D_Y^\infty} (D_{X \leftarrow Y}^\infty, F^{ord}) = f_1^*(D_{Y \leftarrow S}^\infty, F^{ord}) \otimes_{f_1^*D_Y^\infty}^L (D_{X \leftarrow Y}^\infty, F^{ord}),$$
- in $D_{\mathcal{D}^\infty, op, (f_2 \circ f_1)^* \mathcal{D}^\infty fil, r}(X)$.*

Proof. Similar to the proof of lemma 7 \square

For closed embeddings, we have :

Proposition 42. (i) Let $i : Z \hookrightarrow S$ be a closed embedding with $Z, S \in \text{SmVar}(\mathbb{C})$. Then, $D_{Z \rightarrow S} = i^* D_S / D_S \mathcal{I}_Z$ and it is a locally free (left) D_Z module. Similarly, $D_{Z \leftarrow S} = i^* D_S / \mathcal{I}_Z D_S$ and it is a locally free right D_Z module.

(ii) Let $i : Z \rightarrow S$ be a closed embedding with $Z, S \in \text{AnSm}(\mathbb{C})$. Then, $D_{Z \rightarrow S} = i^* D_S / D_S \mathcal{I}_Z$ and it is a locally free (left) D_Z module. Similarly, $D_{Z \leftarrow S} = i^* D_S / \mathcal{I}_Z D_S$ and it is a locally free right D_Z module.

(iii) Let $i : Z \rightarrow S$ be a closed embedding with $Z, S \in \text{AnSm}(\mathbb{C})$. Then, $D_{Z \rightarrow S}^\infty = i^* D_S^\infty / D_S^\infty \mathcal{I}_Z$ and it is a locally free (left) D_Z^∞ module. Similarly, $D_{Z \leftarrow S}^\infty = i^* D_S^\infty / \mathcal{I}_Z D_S^\infty$ and it is a locally free right D_Z^∞ module.

Proof. (i): See [16].

(ii): See [25].

(iii): Similar to (ii). □

We now enumerate some functorialities we will use, all of them are particular case of the functoriality given in subsection 2.3 for any ringed spaces :

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{Var}(\mathbb{C})$, or let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSp}(\mathbb{C})$. Then, the inverse image functor

$$f^{*mod} : \text{PSh}_{O_S}(S) \rightarrow \text{PSh}_{O_X}(X), \quad M \mapsto f^{*mod} M := O_X \otimes_{f^* O_S} f^* M$$

is a Quillen adjunction which induces in the derived category the functor

$$L f^{*mod} : D_{O_S}(S) \rightarrow D_{O_X}(X), \quad M \mapsto L f^{*mod} M := O_X \otimes_{f^* O_S}^L f^* M = O_X \otimes_{f^* O_S} f^* L_O M,$$

The adjunction $(f^{*mod}, f_*) : \text{PSh}_{O_S}(S) \rightleftharpoons \text{PSh}_{O_X}(X)$ is a Quillen adjunction, the adjunction map are the maps

- for $M \in C_{O_S}(S)$, $\text{ad}(f^{*mod}, f_*)(M) : M \xrightarrow{\text{ad}(f^*, f_*)(M)} f_* f^* M \xrightarrow{f_* m} f_*(f^* M \otimes_{f^* O_S} O_X) = f_* f^{*mod} M$ where $m(m) = m \otimes 1$,
- for $M \in C_{O_X}(X)$, $\text{ad}(f^{*mod}, f_*)(M) : f^{*mod} f_* M = f^* f_* M \otimes_{f^* O_S} O_X \xrightarrow{\text{ad}(f^*, f_*)(M) \otimes_{f^* O_S} O_X} M \otimes_{f^* O_S} O_X \xrightarrow{n} M$, where $n(m \otimes h) = h \cdot m$ is the multiplication map.

- Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$.

- For $M \in C_{\mathcal{D}}(S)$, we have the canonical projective resolution $q : L_{\mathcal{D}}(M) \rightarrow M$ of complexes of D_S modules.
- For $M \in C_{\mathcal{D}}(S)$, there exist a unique structure of D_S module on the flasque presheaves $E^i(M)$ such that $E(M) \in C_{\mathcal{D}}(S)$ (i.e. is a complex of D_S modules) and that the map $k : M \rightarrow E(M)$ is a morphism of complexes of D_S modules.

Let $S \in \text{AnSm}(\mathbb{C})$.

- For $M \in C_{\mathcal{D}^\infty}(S)$, we have the canonical projective resolution $q : L_{\mathcal{D}^\infty}(M) \rightarrow M$ of complexes of D_S^∞ modules.
- For $M \in C_{\mathcal{D}^\infty}(S)$, there exist a unique structure of D_S^∞ module on the flasque presheaves $E^i(M)$ such that $E(M) \in C_{\mathcal{D}^\infty}(S)$ (i.e. is a complex of D_S^∞ modules) and that the map $k : M \rightarrow E(M)$ is a morphism of complexes of D_S^∞ modules.

- Let $S \in \text{SmVar}(\mathbb{C})$ or let $S \in \text{AnSm}(\mathbb{C})$. For $M \in C_{\mathcal{D}^{(op)}}(S)$, $N \in C(S)$, we will consider the induced D module structure (right D_S module in the case one is a left D_S module and the other one is a right one) on the presheaf $M \otimes N := M \otimes_{\mathbb{Z}_S} N$ (see section 2). We get the bifunctor

$$C(S) \times C_{\mathcal{D}}(S) \rightarrow C_{\mathcal{D}}(S), (M, N) \mapsto M \otimes N$$

For $S \in \text{AnSm}(\mathbb{C})$, we also have the bifunctor $C(S) \times C_{\mathcal{D}^\infty}(S) \rightarrow C_{\mathcal{D}^\infty}(S)$, $(M, N) \mapsto M \otimes N$.

- Let $S \in \text{SmVar}(\mathbb{C})$ or let $S \in \text{AnSm}(\mathbb{C})$. For $M, N \in C_{\mathcal{D}^{(op)}}(S)$, $M \otimes_{O_S} N$ (see section 2), has a canonical structure of D_S modules (right D_S module in the case one is a left D_S module and the other one is a right one) given by (in the left case) for $S^o \subset S$ an open subset,

$$m \otimes n \in \Gamma(S^o, M \otimes_{O_S} N), \gamma \in \Gamma(S^o, D_S), \gamma.(m \otimes n) := (\gamma.m) \otimes n - m \otimes \gamma.n$$

This gives the bifunctor

$$C_{\mathcal{D}^{(op)}}(S)^2 \rightarrow C_{\mathcal{D}^{(op)}}(S), (M, N) \mapsto M \otimes_{O_S} N$$

More generally, let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$ or with $X, S \in \text{AnSp}(\mathbb{C})$. Assume S smooth. For $M, N \in C_{f^*\mathcal{D}^{(op)}}(X)$, $M \otimes_{f^*O_S} N$ (see section 2), has a canonical structure of f^*D_S modules (right f^*D_S module in the case one is a left f^*D_S module and the other one is a right one) given by (in the left case) for $X^o \subset X$ an open subset,

$$m \otimes n \in \Gamma(X^o, M \otimes_{f^*O_S} N), \gamma \in \Gamma(X^o, f^*D_S), \gamma.(m \otimes n) := (\gamma.m) \otimes n - m \otimes \gamma.n$$

This gives the bifunctor

$$C_{f^*\mathcal{D}^{(op)}}(X)^2 \rightarrow C_{f^*\mathcal{D}^{(op)}}(X), (M, N) \mapsto M \otimes_{f^*O_S} N$$

For $f : X \rightarrow S$ a morphism with $X, S \in \text{AnSp}(\mathbb{C})$ and S smooth, we also have the bifunctor $C_{f^*\mathcal{D}^\infty, (op)}(X)^2 \rightarrow C_{f^*\mathcal{D}^\infty, (op)}(X)$, $(M, N) \mapsto M \otimes_{f^*O_S} N$.

- Let $S \in \text{SmVar}(\mathbb{C})$ or let $S \in \text{AnSm}(\mathbb{C})$. For $M \in C_{\mathcal{D}^{op}}(S)$ and $N \in C_{\mathcal{D}}(S)$, we have $M \otimes_{D_S} N \in C(S)$ (see section 2). This gives the bifunctor

$$C_{\mathcal{D}^{op}}(S) \times C_{\mathcal{D}}(S) \rightarrow C(S), (M, N) \mapsto M \otimes_{D_S} N$$

For $S \in \text{AnSm}(\mathbb{C})$, we also have the bifunctor $C_{\mathcal{D}^\infty, op}(S) \times C_{\mathcal{D}^\infty}(S) \rightarrow C(S)$, $(M, N) \mapsto M \otimes_{D_S^\infty} N$.

- Let $S \in \text{SmVar}(\mathbb{C})$ or let $S \in \text{AnSm}(\mathbb{C})$. The internal hom bifunctor

$$\mathcal{H}om(\cdot, \cdot) := \mathcal{H}om_{\mathbb{Z}_S}(\cdot, \cdot) : C(S)^2 \rightarrow C(S)$$

induces a bifunctor

$$\mathcal{H}om(\cdot, \cdot) := \mathcal{H}om_{\mathbb{Z}_S}(\cdot, \cdot) : C(S) \times C_{\mathcal{D}}(S) \rightarrow C_{\mathcal{D}}(S)$$

such that, for $F \in C(S)$ and $G \in C_{\mathcal{D}}(S)$, the D_S structure on $\mathcal{H}om^\bullet(F, G)$ is given by

$$\gamma \in \Gamma(S^o, D_S) \mapsto (\phi \in \text{Hom}^p(F|_{S^o}^\bullet, G|_{S^o}) \mapsto (\gamma \cdot \phi : \alpha \in F^\bullet(S^o) \mapsto \gamma \cdot \phi^p(S^o)(\alpha))$$

where $\phi^p(S^o)(\alpha) \in \Gamma(S^o, G)$. For $S \in \text{AnSm}(\mathbb{C})$, it also induce the bifunctor

$$\mathcal{H}om(\cdot, \cdot) := \mathcal{H}om_{\mathbb{Z}_S}(\cdot, \cdot) : C(S) \times C_{\mathcal{D}^\infty}(S) \rightarrow C_{\mathcal{D}^\infty}(S)$$

- Let $S \in \text{SmVar}(\mathbb{C})$ or let $S \in \text{AnSm}(\mathbb{C})$. For $M, N \in C_{\mathcal{D}}(S)$, $\mathcal{H}om_{O_S}(M, N)$, has a canonical structure of D_S modules given by for $S^o \subset S$ an open subset and $\phi \in \Gamma(S^o, \mathcal{H}om(M, O_S))$, $\gamma \in \Gamma(S^o, D_S)$, $(\gamma.\phi)(m) := \gamma.(\phi(m)) - \phi(\gamma.m)$ This gives the bifunctor

$$\text{Hom}_{O_S}^\bullet(-, -) : C_{\mathcal{D}}(S)^2 \rightarrow C_{\mathcal{D}}(S)^{op}, (M, N) \mapsto \mathcal{H}om_{O_S}^\bullet(M, N)$$

In particular, for $M \in C_{\mathcal{D}}(S)$, we get the dual

$$\mathbb{D}_S^O(M) := \mathcal{H}om_{O_S}^\bullet(M, O_S) \in C_{\mathcal{D}}(S)$$

with respect to O_S , together with the canonical map $d(M) : M \rightarrow \mathbb{D}_S^{O,2}(M)$. Let $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$ or with $X, S \in \text{AnSm}(\mathbb{C})$. We have, for $M \in C_{\mathcal{D}}(S)$, the canonical transformation map

$$\begin{aligned} T(f, D^o)(M) : f^{*mod} \mathbb{D}_S^O M &= (f^* \mathcal{H}om_{O_S}(M, O_S)) \otimes_{f^* O_S} O_X \\ \xrightarrow{T^{mod}(f, hom)(M, O_S)} \mathcal{H}om_{O_X}(f^* M \otimes_{f^* O_S} O_X, O_X) &=: \mathbb{D}_X^O(f^{*mod} M). \end{aligned}$$

• Let $S \in \text{SmVar}(\mathbb{C})$ or let $S \in \text{AnSm}(\mathbb{C})$. We have the bifunctors

- $\text{Hom}_{D_S}^\bullet(-, -) : C_{\mathcal{D}}(S)^2 \rightarrow C(S)$, $(M, N) \mapsto \mathcal{H}om_{D_S}^\bullet(M, N)$, and if N is a bimodule (i.e. has a right D_S module structure whose opposite coincide with the left one), $\mathcal{H}om_{D_S}(M, N) \in C_{\mathcal{D}^{op}}(S)$ given by for $S^o \subset S$ an open subset and $\phi \in \Gamma(S^o, \mathcal{H}om(M, N))$, $\gamma \in \Gamma(S^o, D_S)$, $(\phi \cdot \gamma)(m) := (\phi(m)) \cdot \gamma$
- $\text{Hom}_{D_S}(-, -) : C_{\mathcal{D}^{op}}(S)^2 \rightarrow C(S)$, $(M, N) \mapsto \mathcal{H}om_{D_S}(M, N)$ and if N is a bimodule, $\mathcal{H}om_{D_S}(M, N) \in C_{\mathcal{D}}(S)$

For $M \in C_{\mathcal{D}}(S)$, we get in particular the dual with respect \mathbb{D}_S ,

$$\mathbb{D}_S M := \mathcal{H}om_{D_S}(M, D_S) \in C_{\mathcal{D}}(S) ; \mathbb{D}_S^K M := \mathcal{H}om_{D_S}(M, D_S) \otimes_{O_S} \mathbb{D}_S^O w(K_S)[d_S] \in C_{\mathcal{D}}(S)$$

and we have canonical map $d : M \rightarrow \mathbb{D}_S^2 M$. This functor induces in the derived category, for $M \in D_{\mathcal{D}}(S)$,

$$L\mathbb{D}_S M := R\mathcal{H}om_{D_S}(L_D M, D_S) \otimes_{O_S} \mathbb{D}_S^O w(K_S)[d_S] = \mathbb{D}_S^K L_D M \in D_{\mathcal{D}}(S).$$

where $\mathbb{D}_S^O w(K_S) : \mathbb{D}_S^O w(K_S) \rightarrow \mathbb{D}_S^O K_S = K_S^{-1}$ is the dual of the Kozul resolution of the canonical bundle (proposition 62), and the canonical map $d : M \rightarrow L\mathbb{D}_S^2 M$. For $S \in \text{AnSm}(\mathbb{C})$, we also have the bifunctors

- $\text{Hom}_{D_S^\infty}^\bullet(-, -) : C_{\mathcal{D}^\infty}(S)^2 \rightarrow C(S)$, $(M, N) \mapsto \mathcal{H}om_{D_S^\infty}^\bullet(M, N)$, and if N is a bimodule, $\mathcal{H}om_{D_S^\infty}(M, N) \in C_{\mathcal{D}^\infty}(S)$,
- $\text{Hom}_{D_S}(-, -) : C_{\mathcal{D}^\infty, op}(S)^2 \rightarrow C(S)$, $(M, N) \mapsto \mathcal{H}om_{D_S^\infty}(M, N)$ and if N is a bimodule, $\mathcal{H}om_{D_S^\infty}(M, N) \in C_{\mathcal{D}^\infty, op}(S)$

For $M \in C_{\mathcal{D}}(S)$, we get in particular the dual with respect \mathbb{D}_S^∞ ,

$$\mathbb{D}_S^\infty M := \mathcal{H}om_{D_S^\infty}(M, D_S^\infty) \in C_{\mathcal{D}^\infty}(S) , \mathbb{D}_S^{\infty, K} M := \mathcal{H}om_{D_S^\infty}(M, D_S^\infty) \otimes_{O_S} \mathbb{D}_S^O w(K_S)[d_S] \in C_{\mathcal{D}^\infty}(S)$$

and we have canonical maps $d : M \rightarrow \mathbb{D}_S^{\infty, 2} M$, $d : M \rightarrow \mathbb{D}_S^{\infty, K, 2} M$. This functor induces in the derived category, for $M \in D_{\mathcal{D}^\infty}(S)$,

$$L\mathbb{D}_S^\infty M := R\mathcal{H}om_{D_S^\infty}(M, D_S^\infty) \otimes_{O_S} \mathbb{D}_S^O w(K_S)[d_S] = \mathbb{D}_S^{\infty, K} L_{D^\infty} M \in D_{\mathcal{D}^\infty}(S).$$

and the canonical map $d : M \rightarrow L\mathbb{D}_S^{\infty, 2} M$.

• Let $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$ or with $X, S \in \text{AnSm}(\mathbb{C})$. For $N \in C_{\mathcal{D}, f^* \mathcal{D}}(X)$ and $M \in C_{\mathcal{D}}(X)$, $N \otimes_{D_X} M$ has the canonical $f^* D_S$ module structure given by, for $X^o \subset X$ an open subset,

$$\gamma \in \Gamma(X^o, f^* D_S), m \in \Gamma(X^o, M), n \in \Gamma(X^o, N), \gamma \cdot (n \otimes m) = (\gamma \cdot n) \otimes m.$$

This gives the functor

$$C_{\mathcal{D}, f^* \mathcal{D}}(X) \times C_{\mathcal{D}}(X) \rightarrow C_{f^* \mathcal{D}}(X), (M, N) \mapsto M \otimes_{D_X} N$$

- Let $f : X \rightarrow S$ a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. For $N \in C_{\mathcal{D}^\infty, f^* \mathcal{D}^\infty}(X)$ and $M \in C_{\mathcal{D}^\infty}(X)$, $N \otimes_{D_X^\infty} M$ has the canonical $f^* D_S^\infty$ module structure given by, for $X^o \subset X$ an open subset,

$$\gamma \in \Gamma(X^o, f^* D_S), m \in \Gamma(X^o, M), n \in \Gamma(X^o, N), \gamma.(n \otimes m) = (\gamma.n) \otimes m.$$

This gives the functor

$$C_{\mathcal{D}^\infty, f^* \mathcal{D}^\infty}(X) \times C_{\mathcal{D}^\infty}(X) \rightarrow C_{f^* \mathcal{D}^\infty}(X), (M, N) \mapsto M \otimes_{D_X^\infty} N$$

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{SmVar}(\mathbb{C})$, or let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then, for $M \in C_{\mathcal{D}}(S)$, $O_X \otimes_{f^* O_S} f^* M$ has a canonical D_X module structure given by given by, for $X^o \subset X$ an open subset,

$$m \otimes n \in \Gamma(X^o, O_X \otimes_{f^* O_S} f^* M), \gamma \in \Gamma(X^o, D_X), \gamma.(m \otimes n) := (\gamma.m) \otimes n - m \otimes d\gamma.n.$$

This gives the inverse image functor

$$f^{*mod} : \text{PSh}_{\mathcal{D}}(S) \rightarrow \text{PSh}_{\mathcal{D}}(X), M \mapsto f^{*mod} M := O_X \otimes_{f^* O_S} f^* M = D_{X \rightarrow S} \otimes_{f^* D_S} f^* M$$

which induces in the derived category the functor

$$L f^{*mod} : D_{\mathcal{D}}(S) \rightarrow D_{\mathcal{D}}(X), M \mapsto L f^{*mod} M := O_X \otimes_{f^* O_S}^L f^* M = O_X \otimes_{f^* O_S} f^* L_D M,$$

We will also consider the shifted inverse image functor

$$L f^{*mod[-]} := L f^{*mod}[d_S - d_X] : D_{\mathcal{D}}(S) \rightarrow D_{\mathcal{D}}(X).$$

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then, for $M \in C_{\mathcal{D}^\infty}(S)$, $O_X \otimes_{f^* O_S} f^* M$ has a canonical D_X^∞ module structure induced by the finite order case. This gives the inverse image functor

$$f^{*mod} : \text{PSh}_{\mathcal{D}^\infty}(S) \rightarrow \text{PSh}_{\mathcal{D}^\infty}(X), M \mapsto f^{*mod} M := O_X \otimes_{f^* O_S} f^* M = D_{X \rightarrow S} \otimes_{f^* D_S^\infty} f^* M$$

which induces in the derived category the functor

$$L f^{*mod} : D_{\mathcal{D}^\infty}(S) \rightarrow D_{\mathcal{D}^\infty}(X), M \mapsto L f^{*mod} M := O_X \otimes_{f^* O_S}^L f^* M = O_X \otimes_{f^* O_S} f^* L_{D^\infty} M,$$

We will also consider the shifted inverse image functor

$$L f^{*mod[-]} := L f^{*mod}[d_S - d_X] : D_{\mathcal{D}^\infty}(S) \rightarrow D_{\mathcal{D}^\infty}(X).$$

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{SmVar}(\mathbb{C})$, or let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. For $M \in C_{\mathcal{D}}(X)$, $D_{X \leftarrow S} \otimes_{D_X} M$ has the canonical $f^* D_S$ module structure given above. Then, the direct image functor

$$f_{*mod}^0 : \text{PSh}_{\mathcal{D}}(X) \rightarrow \text{PSh}_{\mathcal{D}}(S), M \mapsto f_{*mod} M := f_*(D_{X \leftarrow S} \otimes_{D_X} M)$$

induces in the derived category the functor

$$\int_f = R f_{*mod} : D_{\mathcal{D}}(X) \rightarrow D_{\mathcal{D}}(S), M \mapsto \int_f M = R f_*(D_{X \leftarrow S} \otimes_{D_X}^L M).$$

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. For $M \in C_{\mathcal{D}^\infty}(X)$, $D_{X \leftarrow S} \otimes_{D_X} M$ has the canonical $f^* D_S$ module structure given above. Then, the direct image functor

$$f_{*mod}^{00} : \text{PSh}_{\mathcal{D}^\infty}(X) \rightarrow \text{PSh}_{\mathcal{D}^\infty}(S), M \mapsto f_{*mod} M := f_*(D_{X \leftarrow S}^\infty \otimes_{D_X^\infty} M)$$

induces in the derived category the functor

$$\int_f = R f_{*mod} : D_{\mathcal{D}^\infty}(X) \rightarrow D_{\mathcal{D}^\infty}(S), M \mapsto \int_f M = R f_*(D_{X \leftarrow S}^\infty \otimes_{D_X^\infty}^L M).$$

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. The direct image functor with compact support

$$f_{!mod}^{00} : \text{PSh}_{\mathcal{D}}(X) \rightarrow \text{PSh}_{\mathcal{D}}(S), \quad M \mapsto f_{!mod} M := f_!(D_{S \leftarrow X} \otimes_{D_X} M)$$

induces in the derived category the functor

$$\int_{f!} = Rf_{!mod} : D_{\mathcal{D}}(X) \rightarrow D_{\mathcal{D}}(S), \quad M \mapsto \int_f M = Rf_!(D_{X \leftarrow S} \otimes_{D_X}^L M).$$

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. The direct image functor with compact support

$$f_{!mod}^{00} : \text{PSh}_{\mathcal{D}^\infty}(X) \rightarrow \text{PSh}_{\mathcal{D}^\infty}(S), \quad M \mapsto f_{!mod} M := f_!(D_{S \leftarrow X}^\infty \otimes_{D_X^\infty} M)$$

induces in the derived category the functor

$$\int_{f!} = Rf_{!mod} : D_{\mathcal{D}^\infty}(X) \rightarrow D_{\mathcal{D}^\infty}(S), \quad M \mapsto \int_f M = Rf_!(D_{X \leftarrow S}^\infty \otimes_{D_X^\infty}^L M).$$

- Let $S \in \text{SmVar}(\mathbb{C})$. The analytical functor of a D_S modules has a canonical structure of $D_{S^{an}}$ module:

$$(-)^{an} : C_{\mathcal{D}}(S) \rightarrow C_{\mathcal{D}}(S^{an}), \quad M \mapsto M^{an} := \text{an}_S^{*mod} M := M \otimes_{\text{an}_S^* O_S} O_{S^{an}}$$

which induces in the derived category

$$(-)^{an} : D_{\mathcal{D}}(S) \rightarrow D_{\mathcal{D}}(S^{an}), \quad M \mapsto M^{an} := \text{an}_S^{*mod} M$$

since an_S^{*mod} derive trivially.

The functorialities given above induce :

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{Var}(\mathbb{C})$, or let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSp}(\mathbb{C})$. The adjunction map induces

- for $(M, F) \in C_{O_S \text{fil}}(S)$, the map in $D_{O_S \text{fil}}(S)$

$$\begin{aligned} \text{ad}(Lf^{*mod}, Rf_*)(M, F) : (M, F) &\xrightarrow{\text{koad}(f^*, f_*)(M, F)} f_* E(f^*(M, F)) = Rf_* f^*(M, F) \\ &\xrightarrow{f_* m} Rf_*(f^*(M, F) \otimes_{f^* O_S}^L O_X) = Rf_* f^*(M, F), \end{aligned}$$

where $m(m) = m \otimes 1$,

- for $(M, F) \in C_{O_X \text{fil}}(X)$, the map in $D_{O_X \text{fil}}(X)$

$$\begin{aligned} \text{ad}(Lf^{*mod}, Rf_*)(M, F) : Lf^{*mod} Rf_*(M, F) &= f^* f_* E(M, F) \otimes_{f^* O_S}^L O_X \\ &\xrightarrow{\text{ad}(f^*, f_*)(E(M, F)) \otimes_{f^* O_S}^L O_X} (M, F) \otimes_{f^* O_S}^L O_X \xrightarrow{n} (M, F), \end{aligned}$$

where $n(m \otimes h) = h.m$ is the multiplication map.

- For a commutative diagram in $\text{Var}(\mathbb{C})$ or in $\text{AnSp}(\mathbb{C})$:

$$D = \begin{array}{ccc} Y & \xrightarrow{g_2} & X \\ \downarrow f_2 & & \downarrow f_1 \\ T & \xrightarrow{g_1} & S \end{array},$$

we have, for $(M, F) \in C_{O_X \text{fil}}(X)$, the canonical map in $D_{O_T \text{fil}}(T)$

$$\begin{aligned} T^{\text{mod}}(D)(M, F) : Lg_1^{*\text{mod}} f_{1*}(M, F) &\xrightarrow{\text{ad}(Lf_2^{*\text{mod}}, Rf_{2*})(Lg_1^{*\text{mod}} f_{1*} E(M, F))} \\ Rf_{2*} Lf_2^{*\text{mod}} Lg_1^{*\text{mod}} Rf_{1*}(M, F) &= Rf_{2*} Lg_2^{*\text{mod}} Lf_1^{*\text{mod}} Rf_{1*}(M, F) \\ &\xrightarrow{\text{ad}(Lf_1^{*\text{mod}}, Rf_1)(M, F)} Rf_{2*} Lg_2^{*\text{mod}}(M, F) \end{aligned}$$

the canonical transformation map given by the adjunction maps.

- Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$. For $(M, F) \in C_{\text{fil}}(S)$ and $(N, F) \in C_{\text{fil}}(S)$, recall that (see section 2)

$$F^p((M, F) \otimes (N, F)) := \text{Im}(\oplus_q F^q M \otimes F^{p-q} N \rightarrow M \otimes N)$$

This gives the functor

$$(\cdot, \cdot) : C_{\text{fil}}(S) \times C_{\mathcal{D}\text{fil}}(S) \rightarrow C_{\mathcal{D}\text{fil}}(S), ((M, F), (N, F)) \mapsto (M, F) \otimes (N, F).$$

It induces in the derived categories by taking r-projective resolutions the bifunctors, for $r = 1, \dots, \infty$,

$$(\cdot, \cdot) : D_{\mathcal{D}\text{fil}, r}(S) \times D_{\text{fil}, r}(S) \rightarrow D_{\mathcal{D}\text{fil}, r}(S), ((M, F), (N, F)) \mapsto (M, F) \otimes^L (N, F) = L_D(M, F) \otimes (N, F).$$

For $S \in \text{AnSm}(\mathbb{C})$, it gives the bifunctor

$$(\cdot, \cdot) : C_{\text{fil}}(S) \times C_{\mathcal{D}^\infty \text{fil}}(S) \rightarrow C_{\mathcal{D}^\infty \text{fil}}(S), ((M, F), (N, F)) \mapsto (M, F) \otimes (N, F),$$

and its derived functor.

- Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$ and $O'_S \in \text{PSh}(S, \text{cRing})$ a sheaf of commutative ring. For $(M, F) \in C_{O'_S \text{fil}}(S)$ and $(N, F) \in C_{O'_S \text{fil}}(S)$, recall that (see section 2)

$$F^p((M, F) \otimes_{O'_S} (N, F)) := \text{Im}(\oplus_q F^q M \otimes_{O'_S} F^{p-q} N \rightarrow M \otimes_{O'_S} N)$$

It induces in the derived categories by taking r-projective resolutions the bifunctors, for $r = 1, \dots, \infty$,

$$(\cdot, \cdot) : D_{\mathcal{D}\text{fil}, r}(S)^2 \rightarrow D_{\mathcal{D}\text{fil}, r}(S), ((M, F), (N, F)) \mapsto (M, F) \otimes_{O'_S}^L (N, F).$$

More generally, let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$ or with $X, S \in \text{AnSp}(\mathbb{C})$. Assume S smooth. We have the bifunctors

$$(\cdot, \cdot) : D_{f^* \mathcal{D}\text{fil}, r}(X)^2 \rightarrow D_{f^* \mathcal{D}\text{fil}, r}(X), ((M, F), (N, F)) \mapsto (M, F) \otimes_{f^* O_S}^L (N, F) = (M, F) \otimes_{f^* O_S} Lf^* D(N, F).$$

- Let $S \in \text{SmVar}(\mathbb{C})$ or let $S \in \text{AnSm}(\mathbb{C})$. The hom functor induces the bifunctor

$$\text{Hom}(-, -) : C_{\mathcal{D}\text{fil}}(S) \times C_{\text{fil}}(S) \rightarrow C_{\mathcal{D}(1,0)\text{fil}}(S), ((M, W), (N, F)) \mapsto \mathcal{H}om((M, W), (N, F)).$$

For $S \in \text{AnSm}(\mathbb{C})$, the hom functor also induces the bifunctor

$$\text{Hom}(-, -) : C_{\mathcal{D}^\infty \text{fil}}(S) \times C_{\text{fil}}(S) \rightarrow C_{\mathcal{D}^\infty(1,0)\text{fil}}(S), ((M, W), (N, F)) \mapsto \mathcal{H}om((M, W), (N, F)).$$

Note that the filtration given by W satisfy that the W^p are D_S submodule which is stronger than Griffithz transversality.

- Let $S \in \text{SmVar}(\mathbb{C})$ or let $S \in \text{AnSm}(\mathbb{C})$. The hom functor induces the bifunctor

$$\text{Hom}_{O_S}(-, -) : C_{\mathcal{D}\text{fil}}(S)^2 \rightarrow C_{\mathcal{D}2\text{fil}}(S), ((M, W), (N, F)) \mapsto \mathcal{H}om_{O_S}((M, W), (N, F)).$$

- Let $S \in \text{SmVar}(\mathbb{C})$ or let $S \in \text{AnSm}(\mathbb{C})$. The hom functor induces the bifunctors

- $\text{Hom}_{D_S}(-, -) : C_{\mathcal{D}fil}(S)^2 \rightarrow C_{2fil}(S)$, $((M, W), (N, F)) \mapsto \mathcal{H}om_{D_S}((M, W), (N, F))$,
- $\text{Hom}_{D_S}(-, -) : C_{\mathcal{D}^{op}fil}(S)^2 \rightarrow C_{2fil}(S)$, $((M, W), (N, F)) \mapsto \mathcal{H}om_{D_S}((M, W), (N, F))$.

We get the filtered dual

$$\mathbb{D}_S^K(\cdot) : C_{\mathcal{D}(2)fil}(S) \rightarrow C_{\mathcal{D}(2)fil}(S)^{op}, (M, F) \mapsto \mathbb{D}_S^K(M, F) := \mathcal{H}om_{D_S}((M, F), D_S) \otimes_{O_S} \mathbb{D}_S^O w(K_S)[d_S]$$

together with the canonical map $d(M, F) : (M, F) \rightarrow \mathbb{D}_S^{2,K}(M, F)$. Of course $\mathbb{D}_S^K(\cdot)(C_{\mathcal{D}(1,0)fil}(S)) \subset C_{\mathcal{D}(1,0)fil}(S)$. It induces in the derived categories $D_{\mathcal{D}fil,r}(S)$, for $r = 1, \dots, \infty$, the functors

$$L\mathbb{D}_S(\cdot) : D_{\mathcal{D}(2)fil,r}(S) \rightarrow D_{\mathcal{D}(2)fil,r}(S)^{op}, (M, F) \mapsto L\mathbb{D}_S(M, F) := \mathbb{D}_S^K L_D(M, F).$$

together with the canonical map $d(M, F) : L_D(M, F) \rightarrow \mathbb{D}_S^{2,K} L_D(M, F)$.

- Let $S \in \text{AnSm}(\mathbb{C})$. The hom functor also induces the bifunctors

- $\text{Hom}_{D_S^\infty}(-, -) : C_{\mathcal{D}^\infty fil}(S)^2 \rightarrow C_{2fil}(S)$, $((M, W), (N, F)) \mapsto \mathcal{H}om_{D_S^\infty}((M, W), (N, F))$,
- $\text{Hom}_{D_S}(-, -) : C_{\mathcal{D}^\infty, op fil}(S)^2 \rightarrow C_{2fil}(S)$, $((M, W), (N, F)) \mapsto \mathcal{H}om_{D_S^\infty}((M, W), (N, F))$.

We get the filtered dual

$$\mathbb{D}_S^{\infty,K}(\cdot) : C_{\mathcal{D}^\infty(2)fil}(S) \rightarrow C_{\mathcal{D}^\infty(2)fil}(S)^{op}, (M, F) \mapsto \mathbb{D}_S^{\infty,K}(M, F) := \mathcal{H}om_{D_S^\infty}((M, F), D_S^\infty) \otimes_{O_S} \mathbb{D}_S^O w(K_S)[d_S]$$

together with the canonical map $d(M, F) : (M, F) \rightarrow \mathbb{D}_S^{\infty,K}(M, F)$. Of course $\mathbb{D}_S^{\infty,K}(\cdot)(C_{\mathcal{D}^\infty(1,0)fil}(S)) \subset C_{\mathcal{D}^\infty(1,0)fil}(S)$. It induces in the derived categories $D_{\mathcal{D}^\infty fil,r}(S)$, for $r = 1, \dots, \infty$, the functors

$$L\mathbb{D}_S^\infty(\cdot) : D_{\mathcal{D}^\infty(2)fil,r}(S) \rightarrow D_{\mathcal{D}^\infty(2)fil,r}(S)^{op}, (M, F) \mapsto L\mathbb{D}_S^\infty(M, F) = \mathbb{D}_S^{\infty,K} L_{D^\infty}(M, F).$$

together with the canonical map $d(M, F) : (M, F) \rightarrow L\mathbb{D}_S^{\infty,K}(M, F)$.

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{SmVar}(\mathbb{C})$, or let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then, the inverse image functor

$$f^{*mod} : C_{\mathcal{D}(2)fil}(S) \rightarrow C_{\mathcal{D}(2)fil}(X),$$

$$(M, F) \mapsto f^{*mod}(M, F) := (O_X, F_b) \otimes_{f^* O_S} f^*(M, F) = (D_{X \rightarrow S}, F^{ord}) \otimes_{f^* D_S} f^*(M, F),$$

induces in the derived categories the functors, for $r = 1, \dots, \infty$ (resp. $r \in (1, \dots, \infty)^2$),

$$L f^{*mod} : D_{\mathcal{D}(2)fil,r}(S) \rightarrow D_{\mathcal{D}(2)fil,r}(X),$$

$$(M, F) \mapsto L f^{*mod} M := (O_X, F_b) \otimes_{f^* O_S}^L f^*(M, F) = (O_X, F_b) \otimes_{f^* O_S} f^* L_D(M, F).$$

Of course $f^{*mod}(C_{\mathcal{D}(1,0)fil}(S)) \subset C_{\mathcal{D}(1,0)fil}(X)$. Note that

- If the M is a complex of locally free O_S modules, then $L f^{*mod}(M, F) = f^{*mod}(M, F)$ in $D_{\mathcal{D}(2)fil,\infty}(S)$.
- If the $\text{Gr}_F^p M$ are complexes of locally free O_S modules, then $L f^{*mod}(M, F) = f^{*mod}(M, F)$ in $D_{\mathcal{D}(2)fil}(S)$.

We will consider also the shifted inverse image functors

$$L f^{*mod[-1]} := L f^{*mod}[d_S - d_X] : D_{\mathcal{D}(2)fil,r}(S) \rightarrow D_{\mathcal{D}(2)fil,r}(X).$$

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then, the inverse image functor

$$f^{*mod} : C_{\mathcal{D}^\infty(2)fil}(S) \rightarrow C_{\mathcal{D}^\infty(2)fil}(X), (M, F) \mapsto f^{*mod}(M, F) := (O_X, F_b) \otimes_{f^*O_S} f^*(M, F),$$

induces in the derived categories the functors, for $r = 1, \dots, \infty$ (resp. $r \in (1, \dots, \infty)^2$),

$$\begin{aligned} Lf^{*mod} &: D_{\mathcal{D}^\infty(2)fil,r}(S) \rightarrow D_{\mathcal{D}^\infty(2)fil,r}(X), \\ (M, F) &\mapsto Lf^{*mod}M := (O_X, F_b) \otimes_{f^*O_S}^L f^*(M, F) = (O_X, F_b) \otimes_{f^*O_S} f^*L_{D^\infty}(M, F). \end{aligned}$$

Of course $f^{*mod}(C_{\mathcal{D}^\infty(1,0)fil}(S)) \subset C_{\mathcal{D}^\infty(1,0)fil}(X)$. Note that We will consider also the shifted inverse image functors

$$Lf^{*mod[-1]} := Lf^{*mod}[d_S - d_X] : D_{\mathcal{D}^\infty(2)fil,r}(S) \rightarrow D_{\mathcal{D}^\infty(2)fil,r}(X).$$

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{SmVar}(\mathbb{C})$, or let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then, the direct image functor

$$f_{*mod}^{00} : (\text{PSh}_{\mathcal{D}}(X), F) \rightarrow (\text{PSh}_{\mathcal{D}}(S), F), (M, F) \mapsto f_{*mod}(M, F) := f_*((D_{S \leftarrow X}, F^{ord}) \otimes_{D_X}(M, F))$$

induces in the derived categories by taking r-injective resolutions the functors, for $r = 1, \dots, \infty$,

$$\int_f = Rf_{*mod} : D_{\mathcal{D}(2)fil,r}(X) \rightarrow D_{\mathcal{D}(2)fil,r}(S), (M, F) \mapsto \int_f(M, F) = Rf_*((D_{S \leftarrow X}, F^{ord}) \otimes_{D_X}^L(M, F)).$$

Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphism with $X, Y, S \in \text{SmVar}(\mathbb{C})$ or with $X, Y, S \in \text{AnSm}(\mathbb{C})$. We have, for $(M, F) \in C_{\mathcal{D}fil}(X)$, the canonical transformation map in $D_{\mathcal{D}(2)fil,r}(S)$

$$\begin{aligned} T\left(\int_{f_2} \circ \int_{f_1}, \int_{f_2 \circ f_1}\right)(M, F) : \\ \int_{f_2} \int_{f_1} (M, F) &:= Rf_{2*}((D_{Y \leftarrow S}, F^{ord}) \otimes_{D_Y}^L Rf_{1*}((D_{X \leftarrow Y}, F^{ord}) \otimes_{D_X}^L(M, F))) \\ &\xrightarrow{T(f_1, \otimes)(-, -)} Rf_{2*}Rf_{1*}(f_1^*(D_{Y \leftarrow S}, F^{ord}) \otimes_{D_Y}^L((D_{X \leftarrow Y}, F^{ord}) \otimes_{D_X}^L(M, F))) \\ &\xrightarrow{\sim} Rf_{2*}Rf_{1*}((f_1^*(D_{Y \leftarrow S}, F^{ord}) \otimes_{D_Y}^L(D_{X \leftarrow Y}, F^{ord})) \otimes_{D_X}^L(M, F)) \\ &\xrightarrow{\sim} Rf_{2*}Rf_{1*}((D_{X \leftarrow S}, F^{ord}) \otimes_{D_X}^L(M, F)) := \int_{f_2 \circ f_1}(M, F) \end{aligned}$$

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then, the direct image functor

$$f_{*mod}^{00} : (\text{PSh}_{\mathcal{D}^\infty}(X), F) \rightarrow (\text{PSh}_{\mathcal{D}^\infty}(S), F), (M, F) \mapsto f_{*mod}(M, F) := f_*((D_{S \leftarrow X}^\infty, F^{ord}) \otimes_{D_X^\infty}(M, F))$$

induces in the derived categories by taking r-injective resolutions the functors, for $r = 1, \dots, \infty$,

$$\int_f = Rf_{*mod} : D_{\mathcal{D}^\infty(2)fil,r}(X) \rightarrow D_{\mathcal{D}^\infty(2)fil,r}(S), (M, F) \mapsto \int_f(M, F) = Rf_*((D_{S \leftarrow X}^\infty, F^{ord}) \otimes_{D_X^\infty}^L(M, F)).$$

We have, similarly, for $(M, F) \in C_{\mathcal{D}^\infty fil}(X)$, the canonical transformation map in $D_{\mathcal{D}^\infty(2)fil,r}(S)$

$$T\left(\int_{f_2} \circ \int_{f_1}, \int_{f_2 \circ f_1}\right)(M, F) : \int_{f_2} \int_{f_1} (M, F) \rightarrow \int_{f_2 \circ f_1}(M, F)$$

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then, the direct image functor with compact support

$$f_{!mod}^{00} : (\text{PSh}_{\mathcal{D}}(X), F) \rightarrow (\text{PSh}_{\mathcal{D}}(S), F), (M, F) \mapsto f_{!mod}^{00}(M, F) := f_!((D_{S \leftarrow X}, F^{ord}) \otimes_{D_X}(M, F))$$

induces in the derived categories by taking r-injective resolutions the functors, for $r = 1, \dots, \infty$,

$$\int_{f!} = Rf_{!mod} : D_{\mathcal{D}fil,r}(X) \rightarrow D_{\mathcal{D}fil,r}(S), \quad (M, F) \mapsto \int_f (M, F) = Rf_!((D_{S \leftarrow X}, F^{ord}) \otimes_{D_X}^L (M, F)).$$

We have, similarly, for $(M, F) \in C_{\mathcal{D}fil}(X)$, the canonical transformation map in $D_{\mathcal{D}(2)fil,r}(S)$

$$T(\int_{f_2!} \circ \int_{f_1!}, \int_{(f_2 \circ f_1)!})(M, F) : \int_{f_2!} \int_{f_1!} (M, F) \rightarrow \int_{(f_2 \circ f_1)!} (M, F)$$

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then, the direct image functor with compact support

$$f_{!mod}^{00} : (\text{PSh}_{\mathcal{D}^\infty}(X), F) \rightarrow (\text{PSh}_{\mathcal{D}^\infty}(S), F), \quad (M, F) \mapsto f_{!mod}^{00}(M, F) := f_!((D_{S \leftarrow X}^\infty, F^{ord}) \otimes_{D_X^\infty}^L (M, F))$$

induces in the derived categories by taking r-injective resolutions the functors, for $r = 1, \dots, \infty$,

$$\int_{f!} = Rf_{!mod} : D_{\mathcal{D}^\infty fil,r}(X) \rightarrow D_{\mathcal{D}^\infty fil,r}(S), \quad (M, F) \mapsto \int_f (M, F) = Rf_!((D_{S \leftarrow X}^\infty, F^{ord}) \otimes_{D_X^\infty}^L (M, F)).$$

We have, similarly, for $(M, F) \in C_{\mathcal{D}^\infty fil}(X)$, the canonical transformation map in $D_{\mathcal{D}^\infty(2)fil,r}(S)$

$$T(\int_{f_2!} \circ \int_{f_1!}, \int_{(f_2 \circ f_1)!})(M, F) : \int_{f_2!} \int_{f_1!} (M, F) \rightarrow \int_{(f_2 \circ f_1)!} (M, F)$$

- Let $S \in \text{SmVar}(\mathbb{C})$. The analytical functor for filtered D_S -modules is

$$(\cdot)^{an} : C_{\mathcal{D}(2)fil}(S) \rightarrow C_{\mathcal{D}(2)fil}(S^{an}), \quad (M, F) \mapsto (M, F)^{an} := \text{an}_S^*(M, F) \otimes_{\text{an}_S^* O_S} (O_{S^{an}}, F_b).$$

It induces in the derived categories the functors, for $r = 1, \dots, \infty$,

$$(\cdot)^{an} : D_{\mathcal{D}(2)fil,r}(S) \rightarrow D_{\mathcal{D}(2)fil,r}(S^{an}), \quad (M, F) \mapsto (M, F)^{an} := \text{an}_S^*(M, F) \otimes_{\text{an}_S^* O_S}^L (O_{S^{an}}, F_b).$$

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{SmVar}(\mathbb{C})$, or let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then the functor

$$f^{\hat{*}mod} : C_{\mathcal{D}2fil}(S) \rightarrow C_{\mathcal{D}2fil}(X), \quad (M, F) \mapsto f^{\hat{*}mod}(M, F) := \mathbb{D}_X^K L_D f^{*mod} L_D \mathbb{D}_S^K (M, F)$$

induces in the derived categories the exceptional inverse image functors, for $r = 1, \dots, \infty$ (resp. $r \in (1, \dots, \infty)^2$),

$$L f^{\hat{*}mod} : D_{\mathcal{D}(2)fil,r}(S) \rightarrow D_{\mathcal{D}(2)fil,r}(X), \\ (M, F) \mapsto L f^{\hat{*}mod}(M, F) := L \mathbb{D}_X L f^{*mod} L \mathbb{D}_S (M, F) := f^{\hat{*}mod} L_D (M, F).$$

Of course $f^{\hat{*}mod}(C_{\mathcal{D}(1,0)fil}(S)) \subset C_{\mathcal{D}(1,0)fil}(X)$. We will also consider the shifted exceptional inverse image functors

$$L f^{\hat{*}mod}[-] := L f^{\hat{*}mod}[d_S - d_X] : D_{\mathcal{D}(2)fil,r}(S) \rightarrow D_{\mathcal{D}(2)fil,r}(X).$$

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then the functor

$$f^{\hat{*}mod} : C_{\mathcal{D}^\infty(2)fil}(S) \rightarrow C_{\mathcal{D}^\infty(2)fil}(X), \quad (M, F) \mapsto f^{\hat{*}mod}(M, F) := \mathbb{D}_X^{K,\infty} L_D f^{*mod} L_D \mathbb{D}_S^{K,\infty} (M, F)$$

induces in the derived categories the exceptional inverse image functors, for $r = 1, \dots, \infty$ (resp. $r \in (1, \dots, \infty)^2$),

$$Lf^{\hat{*}mod} : D_{\mathcal{D}^\infty(2)fil,r}(S) \rightarrow D_{\mathcal{D}^\infty(2)fil,r}(X),$$

$$(M, F) \mapsto Lf^{\hat{*}mod}(M, F) := L\mathbb{D}_X^\infty Lf^{\hat{*}mod} L\mathbb{D}_S^\infty(M, F) := f^{\hat{*}mod}(M, F).$$

Of course $f^{\hat{*}mod}(C_{\mathcal{D}^\infty(1,0)fil}(S)) \subset C_{\mathcal{D}^\infty(1,0)fil}(X)$. We will also consider the shifted exceptional inverse image functors

$$Lf^{\hat{*}mod[-]} := Lf^{\hat{*}mod}[d_S - d_X] : D_{\mathcal{D}^\infty(2)fil,r}(S) \rightarrow D_{\mathcal{D}^\infty(2)fil,r}(X).$$

- Let $S_1, S_2 \in \text{SmVar}(\mathbb{C})$ or $S_1, S_2 \in \text{AnSm}(\mathbb{C})$. Consider $p : S_1 \times S_2 \rightarrow S_1$ the projection. Since p is a projection, we have a canonical embedding $p^*D_{S_1} \hookrightarrow D_{S_1 \times S_2}$. For $(M, F) \in C_{\mathcal{D}(2)fil}(S_1 \times S_2)$, (M, F) has a canonical $p^*D_{S_1}$ module structure. Moreover, with this structure, for $(M_1, F) \in C_{\mathcal{D}(2)fil}(S_1)$

$$\text{ad}(p^{*mod}, p)(M_1, F) : (M_1, F) \rightarrow p_*p^{*mod}(M_1, F)$$

is a map of complexes of D_{S_1} modules, and for $(M_{12}, F) \in C_{\mathcal{D}(2)fil}(S_1 \times S_2)$)

$$\text{ad}(p^{*mod}, p)(M_{12}, F) : p^{*mod}p_*(M_{12}, F) \rightarrow (M_{12}, F)$$

is a map of complexes of $D_{S_1 \times S_2}$ modules. Indeed, for the first adjunction map, we note that $p^{*mod}(M_1, F)$ has a structure of $p^*D_{S_1}$ module, hence $p_*p^{*mod}(M_1, F)$ has a structure of $p_*p^*D_{S_1}$ module, hence a structure of D_{S_1} module using the adjunction map $\text{ad}(p^*, p_*)(D_{S_1}) : D_{S_1} \rightarrow p_*p^*D_{S_1}$. For the second adjunction map, we note that (M_{12}, F) has a structure of $p^*D_{S_1}$ module, hence $p_*(M_{12}, F)$ has a structure of $p_*p^*D_{S_1}$, hence a structure of D_{S_1} module using the adjunction map $\text{ad}(p^*, p_*)(D_{S_1}) : D_{S_1} \rightarrow p_*p^*D_{S_1}$.

- Let $S_1, S_2 \in \text{AnSm}(\mathbb{C})$. Consider $p : S_1 \times S_2 \rightarrow S_1$ the projection. Since p is a projection, we have a canonical embedding $p^*D_{S_1}^\infty \hookrightarrow D_{S_1 \times S_2}^\infty$. For $(M, F) \in C_{\mathcal{D}^\infty(2)fil}(S_1 \times S_2)$, (M, F) has a canonical $p^*D_{S_1}^\infty$ module structure. Moreover, with this structure, for $(M_1, F) \in C_{\mathcal{D}^\infty(2)fil}(S_1)$

$$\text{ad}(p^{*mod}, p)(M_1, F) : (M_1, F) \rightarrow p_*p^{*mod}(M_1, F)$$

is a map of complexes of $D_{S_1}^\infty$ modules, and for $(M_{12}, F) \in C_{\mathcal{D}^\infty(2)fil}(S_1 \times S_2)$)

$$\text{ad}(p^{*mod}, p)(M_{12}, F) : p^{*mod}p_*(M_{12}, F) \rightarrow (M_{12}, F)$$

is a map of complexes of $D_{S_1 \times S_2}^\infty$ modules, similarly to the finite order case.

The following proposition concern the commutativity of the inverse images functors and the commutativity of the direct image functors.

Proposition 43. (i) Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphism with $X, Y, S \in \text{SmVar}(\mathbb{C})$.

- Let $(M, F) \in C_{\mathcal{D}(2)fil,r}(S)$. Then $(f_2 \circ f_1)^{*mod}(M, F) = f_1^{*mod}f_2^{*mod}(M, F)$.
- Let $(M, F) \in D_{\mathcal{D}(2)fil,r}(S)$. Then $L(f_2 \circ f_1)^{*mod}(M, F) = Lf_1^{*mod}(Lf_2^{*mod}(M, F))$.

(ii) Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphism with $X, Y, S \in \text{SmVar}(\mathbb{C})$. Let $M \in D_{\mathcal{D}}(X)$. Then,

$$T\left(\int_{f_2} \circ \int_{f_1}, \int_{f_2 \circ f_1}\right)(M) : \int_{f_2} \int_{f_1} (M) \xrightarrow{\sim} \int_{f_2 \circ f_1} (M)$$

is an isomorphism in $D_{\mathcal{D}}(S)$ (i.e. if we forget filtration).

(iii) Let $i_0 : Z_2 \hookrightarrow Z_1$ and $i_1 : Z_1 \hookrightarrow S$ two closed embedding, with $Z_2, Z_1, S \in \text{SmVar}(\mathbb{C})$. Let $(M, F) \in C_{\mathcal{D}(2)\text{fil}}(Z_2)$. Then, $(i_1 \circ i_0)_* \text{mod}(M, F) = i_{1*} \text{mod}(i_{0*} \text{mod}(M, F))$ in $C_{\mathcal{D}(2)\text{fil}}(S)$.

Proof. (i): Obvious : we have

- $f_1^{* \text{mod}} f_2^{* \text{mod}}(M, F) = f_1^*(f_2^*(M, F) \otimes_{f_2^* \mathcal{O}_S} \mathcal{O}_Y) \otimes_{f_1^* \mathcal{O}_Y} \mathcal{O}_X = f_1^* f_2^*(M, F) \otimes_{f_1^* f_2^* \mathcal{O}_S} f_1^* \mathcal{O}_Y \otimes_{f_1^* \mathcal{O}_Y} \mathcal{O}_X = (f_2 \circ f_1)^{* \text{mod}}(M, F)$
- $L f_1^{* \text{mod}} L f_2^{* \text{mod}}(M, F) = f_1^*(f_2^*(M, F) \otimes_{f_2^* \mathcal{O}_S}^L \mathcal{O}_Y) \otimes_{f_1^* \mathcal{O}_Y}^L \mathcal{O}_X = f_1^* f_2^*(M, F) \otimes_{f_1^* f_2^* \mathcal{O}_S}^L f_1^* \mathcal{O}_Y \otimes_{f_1^* \mathcal{O}_Y}^L \mathcal{O}_X = L(f_2 \circ f_1)^{* \text{mod}}(M, F)$

(ii): See [16] : we have by lemma 7

$$\begin{aligned} & \int_{f_2 \circ f_1} M := Rf_{2*} Rf_{1*}(D_{X \leftarrow S} \otimes_{D_X}^L M) \\ \xrightarrow{=} & Rf_{2*} Rf_{1*}((f_1^* D_{Y \leftarrow S} \otimes_{f_1^* D_Y}^L D_{X \leftarrow Y}) \otimes_{D_X}^L M) \xrightarrow{Rf_{2*} T(f_1, \otimes)(D_{Y \leftarrow S}, D_{X \leftarrow Y} \otimes_{D_X}^L M)^{-1}} \\ & Rf_{2*}(D_{Y \leftarrow S} \otimes_{D_Y}^L Rf_{1*}(D_{X \leftarrow Y} \otimes_{D_X}^L M)) =: \int_{f_2} \int_{f_1} M \end{aligned}$$

where, $D_{Y \leftarrow S}$ being a quasi-coherent D_Y module, we used the fact that for $N \in C_{f_1^* \mathcal{D}}(X)$ and $N' \in C_{\mathcal{D}}(Y)$

$$T(f_1, \otimes)(N', N) : N' \otimes_{D_Y}^L Rf_{1*} N \rightarrow Rf_{1*}(f_1^* N' \otimes_{f_1^* D_Y}^L N)$$

is an isomorphism if N' is quasi-coherent, which follows from the fact that f_{1*} commutes with arbitrary (possibly infinite) direct sums (see [16]).

(iii): Denote $i_2 = i_1 \circ i_0 : Z_2 \hookrightarrow S$. We have

$$\begin{aligned} i_{2*} \text{mod}(M, F) &= i_{2*}((M, F) \otimes_{D_{Z_2}} (D_{Z_2 \leftarrow S}, F^{\text{ord}})) \xrightarrow{=} \\ i_{1*} i_{0*}((M, F) \otimes_{D_{Z_2}} (D_{Z_2 \leftarrow Z_1}, F^{\text{ord}}) \otimes_{i_0^* D_{Z_1}}^* i_0^*(D_{Z_1 \leftarrow S}, F^{\text{ord}})) &\xrightarrow{i_{1*} T(i_0, \otimes)(-)^{-1}} i_{1*} \text{mod} i_{0*} \text{mod}((M, F)) \end{aligned}$$

using proposition 7 and proposition 10. \square

Remark 8. Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphism with $X, Y, S \in \text{SmVar}(\mathbb{C})$. Then, for $(M, F) \in D_{\mathcal{D}(2)\text{fil}, \infty}(X)$, $\int_{f_2} \int_{f_1} (M, F)$ is NOT isomorphic to $\int_{f_2 \circ f_1} (M, F)$ in general, the filtrations on the isomorphic cohomology sheaves may be different.

Proposition 44. Let $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$. Then,

- (i) For $(M, F) \in C_{\mathcal{D}(2)\text{fil}, h}(S)$, we have $L f^{* \text{mod}}(M, F) \in D_{\mathcal{D}(2)\text{fil}, \infty, h}(X)$.
- (ii) For $(M, F) \in C_{\mathcal{D}(2)\text{fil}, h}(X)$, we have $\int_f (M, F) \in D_{\mathcal{D}(2)\text{fil}, \infty, h}(S)$.

Proof. Follows immediately from the non filtered case since we look at the complex in the derived category with respect to ∞ -Zariski local equivalence. It says that the pullback and the pushforward of an holonomic D module is still holonomic. See [16] for the non filtered case. \square

The following easy proposition says that the analytical functor commutes with the pullback of D modules and the tensor product. Again it is well known in the non filtered case. Note that for $S \in \text{SmVar}(\mathbb{C})$, $D_S^{\text{an}} = D_{S^{\text{an}}}$.

Proposition 45. (i) Let $f : T \rightarrow S$ a morphism with $T, S \in \text{SmVar}(\mathbb{C})$.

- Let $(M, F) \in C_{\mathcal{D}(2)\text{fil}, r}(S)$. Then $(f^{* \text{mod}}(M, F))^{\text{an}} = f^{* \text{mod}}(M, F)^{\text{an}}$.
- Let $(M, F) \in D_{\mathcal{D}\text{fil}, r}(S)$, for $r = 1, \dots, \infty$. Then, $(L f^{* \text{mod}}(M, F))^{\text{an}} = L f^{* \text{mod}}(M, F)^{\text{an}}$.

(ii) Let $S \in \text{SmVar}(\mathbb{C})$

- Let $(M, F), (N, F) \in C_{\mathcal{D}fil}(S)$. Then, $((M, F) \otimes_{O_S} (N, F))^{an} = (M, F)^{an} \otimes_{O_{S^{an}}} (N, F)^{an}$.
- Let $(M, F), (N, F) \in D_{\mathcal{D}fil,r}(S)$, for $r = 1, \dots, \infty$. Then, $((M, F) \otimes_{O_S}^L (N, F))^{an} = (M, F)^{an} \otimes_{O_{S^{an}}}^L (N, F)^{an}$.

Proof. (i): For $(M, F) \in C_{\mathcal{D},fil}(S)$, we have, since $f^* \text{an}_S^* = \text{an}_X^* f^{an*}$,

$$\begin{aligned} (f^{*mod}(M, F))^{an} &= \text{an}(X)^*(f^*(M, F) \otimes_{f^*O_S} O_X) \otimes_{\text{an}(X)^*O_X} O_{X^{an}} \\ &= f^{an*} \text{an}_S^*(M, F) \otimes_{f^{an*}O_{S^{an}}} O_{X^{an}} =: f^{an*mod}(M^{an}, F) \end{aligned}$$

For $(M, F) \in D_{\mathcal{D},fil,r}(S)$, we take $(M, F) \in C_{\mathcal{D},fil}(S)$ an r -projective f^*O_S module such that $D_{top,r}(M, F) = (M, F)$ so that

$$(L f^{*mod}(M, F))^{an} = (f^{*mod}(M, F))^{an} = f^{an*mod}(M^{an}, F) = L f^{an*mod}(M^{an}, F)$$

(ii): For $(M, F), (N, F) \in C_{\mathcal{D},fil}(S)$, we have

$$\begin{aligned} ((M, F) \otimes_{O_S} (N, F))^{an} &= \text{an}_S^*((M, F) \otimes_{O_S} (N, F)) \otimes_{\text{an}_S^*O_S} O_{S^{an}} \\ &= \text{an}_S^*(M, F) \otimes_{\text{an}_S^*O_S} \text{an}_S^*(N, F) \otimes_{\text{an}_S^*O_S} O_{S^{an}} \\ &= \text{an}_S^*(M, F) \otimes_{\text{an}_S^*O_S} \otimes_{O_{S^{an}}} \otimes_{O_{S^{an}}} \text{an}_S^*(N, F) \otimes_{\text{an}_S^*O_S} O_{S^{an}} \\ &=: (M^{an}, F) \otimes_{O_{S^{an}}} (N^{an}, F) \end{aligned}$$

It implies the isomorphism in the derived category by taking an r -projective resolution of (M, F) (e.g. $(L_D(M), F) = L_D(M, F)$). \square

Proposition 46. (i) Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphism with $X, Y, S \in \text{AnSm}(\mathbb{C})$.

- Let $(M, F) \in C_{\mathcal{D}(2)fil}(S)$ or let $(M, F) \in C_{\mathcal{D}^\infty(2)fil}(S)$. Then $(f_2 \circ f_1)^{*mod}(M, F) = f_1^{*mod} f_2^{*mod}(M, F)$.
- Let $(M, F) \in D_{\mathcal{D}(2)fil,r}(S)$ or let $(M, F) \in D_{\mathcal{D}^\infty(2)fil,r}(S)$. Then $L(f_2 \circ f_1)^{*mod}(M, F) = L f_1^{*mod}(L f_2^{*mod}(M, F))$.

(ii) Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphisms with $X, Y, S \in \text{AnSm}(\mathbb{C})$. Let $M \in D_{\mathcal{D}}(X)$. If f_1 is proper, we have $\int_{f_2 \circ f_1} M = \int_{f_2} (\int_{f_1} M)$.

(ii)' Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphisms with $X, Y, S \in \text{AnSm}(\mathbb{C})$. Let $M \in D_{\mathcal{D}^\infty}(X)$. If f_1 is proper, we have $\int_{f_2 \circ f_1} M = \int_{f_2} (\int_{f_1} M)$.

(iii) Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphisms with $X, Y, S \in \text{AnSm}(\mathbb{C})$. Let $(M, F) \in D_{\mathcal{D}}(X)$. We have $\int_{(f_2 \circ f_1)!} M = \int_{f_2!} (\int_{f_1!} M)$.

(iii)' Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphisms with $X, Y, S \in \text{AnSm}(\mathbb{C})$. Let $M \in D_{\mathcal{D}^\infty}(X)$. We have $\int_{(f_2 \circ f_1)!} M = \int_{f_2!} (\int_{f_1!} M)$.

(iv) Let $i_0 : Z_2 \hookrightarrow Z_1$ and $i_1 : Z_1 \hookrightarrow S$ two closed embedding, with $Z_2, Z_1, S \in \text{AnSm}(\mathbb{C})$. Let $(M, F) \in C_{\mathcal{D}(2)fil}(Z_2)$. Then, $(i_1 \circ i_0)^{*mod}(M, F) = i_1^{*mod}(i_0^{*mod}(M, F))$ in $C_{\mathcal{D}(2)fil}(S)$.

(iv)' Let $i_0 : Z_2 \hookrightarrow Z_1$ and $i_1 : Z_1 \hookrightarrow S$ two closed embedding, with $Z_2, Z_1, S \in \text{AnSm}(\mathbb{C})$. Let $(M, F) \in C_{\mathcal{D}^\infty(2)fil}(Z_2)$. Then, $(i_1 \circ i_0)^{*mod}(M, F) = i_1^{*mod}(i_0^{*mod}(M, F))$ in $C_{\mathcal{D}^\infty(2)fil}(S)$.

Proof. (i): Similar to the proof of proposition 43(i).

(ii): Similar to the proof of proposition 43(ii) : we use lemma 7 and the fact that for $N \in C_{f_1^* \mathcal{D}}(X)$ and $N' \in C_{\mathcal{D}}(Y)$, the canonical morphism

$$T(f_1, \otimes)(N', N) : N' \otimes_{D_Y}^L Rf_{1*} N \rightarrow Rf_{1*}(f_1^* N' \otimes_{f_1^* D_Y}^L N)$$

is an isomorphism if f_1 is proper (in this case $f_{1!} = f_{1*}$).

(ii)': Similar to the proof of proposition 43(ii) : we use lemma 8 and the fact that for $N \in C_{f_1^* \mathcal{D}^\infty}(X)$ and $N' \in C_{\mathcal{D}^\infty}(Y)$, the canonical morphism

$$T(f_1, \otimes)(N', N) : N' \otimes_{D_Y^\infty}^L Rf_{1*} N \rightarrow Rf_{1*}(f_1^* N' \otimes_{f_1^* D_Y^\infty}^L N)$$

is an isomorphism if f_1 is proper (in this case $f_{1!} = f_{1*}$).

(iii)': Similar to the proof of proposition 43(ii) : we use lemma 7 and the fact that for $N \in C_{f_1^* \mathcal{D}}(X)$ and $N' \in C_{\mathcal{D}}(Y)$, the canonical morphism

$$T(f_{1!}, \otimes)(N', N) : N' \otimes_{D_Y}^L Rf_{1!} N \rightarrow Rf_{1!}(f_1^* N' \otimes_{f_1^* D_Y}^L N)$$

is an isomorphism.

(iii)': Similar to the proof of proposition 43(ii) : we use lemma 8 and the fact that for $N \in C_{f_1^* \mathcal{D}^\infty}(X)$ and $N' \in C_{\mathcal{D}^\infty}(Y)$, the canonical morphism

$$T(f_{1!}, \otimes)(N', N) : N' \otimes_{D_Y^\infty}^L Rf_{1!} N \rightarrow Rf_{1!}(f_1^* N' \otimes_{f_1^* D_Y^\infty}^L N)$$

is an isomorphism

(iv)': Similar to the proof of proposition 43(iii) : we have

$$\begin{aligned} i_{2*} \text{mod}(M, F) &= i_{2*}((M, F) \otimes_{D_{Z_2}} (D_{Z_2 \leftarrow S}, F^{\text{ord}})) \xrightarrow{=} \\ i_{1*} i_{0*}((M, F) \otimes_{D_{Z_2}} (D_{Z_2 \leftarrow Z_1}, F^{\text{ord}}) \otimes_{i_0^* D_{Z_1}}^* i_0^*(D_{Z_1 \leftarrow S}, F^{\text{ord}})) &\xrightarrow{i_{1*} T(i_0, \otimes)(-)^{-1}} i_{1*} \text{mod} i_{0*} \text{mod}((M, F)) \end{aligned}$$

using lemma 7 and proposition 10.

(iv)': Similar to (iv): we have

$$\begin{aligned} i_{2*} \text{mod}(M, F) &= i_{2*}((M, F) \otimes_{D_{Z_2}^\infty} (D_{Z_2 \leftarrow S}^\infty, F^{\text{ord}})) \xrightarrow{=} \\ i_{1*} i_{0*}((M, F) \otimes_{D_{Z_2}^\infty} (D_{Z_2 \leftarrow Z_1}^\infty, F^{\text{ord}}) \otimes_{i_0^* D_{Z_1}^\infty}^* i_0^*(D_{Z_1 \leftarrow S}^\infty, F^{\text{ord}})) &\xrightarrow{i_{1*} T(i_0, \otimes)(-)^{-1}} i_{1*} \text{mod} i_{0*} \text{mod}((M, F)) \end{aligned}$$

using lemma 8 and proposition 10. □

Proposition 47. (i) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. For $(M, F) \in C_{\mathcal{D}(2) \text{fil}, h}(S)$, we have $Lf^{* \text{mod}}(M, F) \in D_{\mathcal{D}(2) \text{fil}, \infty, h}(X)$. For $(M, F) \in C_{\mathcal{D}^\infty(2) \text{fil}, h}(S)$, we have $Lf^{* \text{mod}}(M, F) \in D_{\mathcal{D}^\infty(2) \text{fil}, \infty, h}(X)$.

(ii) Let $f : X \rightarrow S$ a proper morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then, for $(M, F) \in C_{\mathcal{D}(2) \text{fil}, h}(X)$, we have $\int_f(M, F) \in D_{\mathcal{D}(2) \text{fil}, \infty, h}(S)$.

(iii) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then, for $(M, F) \in C_{\mathcal{D}^\infty(2) \text{fil}, h}(X)$, we have $\int_f(M, F) \in D_{\mathcal{D}^\infty(2) \text{fil}, \infty, h}(S)$.

Proof. (i) and (ii): Follows immediately from the non filtered case since we look at the complex in the derived category with respect to ∞ -usu local equivalence. It says that the pullback and the proper pushforward of an holonomic D module is still holonomic. See [16] for the non filtered case.

(iii): In the case the morphism is proper, it follows from the finite order case (ii). In the case of an open embedding, it follows from proposition 41(i) : we have for $j : S^\circ \hookrightarrow S$ an open embedding,

$$j_* E(O_{S^\circ}) = j_* \text{Hom}(\mathbb{Z}_{S^\circ}, E(O_{S^\circ})) = \text{Hom}(j! \mathbb{Z}_{S^\circ}, E(O_S)) \in C_{\mathcal{D}^\infty, h}(S).$$

and on the other hand

$$\begin{aligned} T(j, \otimes)(-, -) = T^{\text{mod}}(j, \otimes)(-, -) : \int_j(M, F) &= j_* E(M, F) = j_* E(j^* O_S \otimes_{O_{S^\circ}}(M, F)) \\ &\xrightarrow{\sim} j_* E(O_{S^\circ}) \otimes_{O_S}(M, F) \end{aligned}$$

is an isomorphism by proposition 9. □

For $X, Y \in \text{SmVar}(\mathbb{C})$ or $X, Y \in \text{AnSm}(\mathbb{C})$, we denote by

- $C_{O_X}(X) \times C_{O_Y}(Y) \rightarrow C_{O_{X \times Y}}(X \times Y), (M, N) \mapsto M \cdot N := O_{X \times Y} \otimes_{p_X^* O_X \otimes_{p_Y^* O_Y}} p_X^* M \otimes p_Y^* N,$
- $C_{\mathcal{D}}(X) \times C_{\mathcal{D}}(Y) \rightarrow C_{\mathcal{D}}(X \times Y), (M, N) \mapsto M \cdot N := O_{X \times Y} \otimes_{p_X^* O_X \otimes_{p_Y^* O_Y}} p_X^* M \otimes p_Y^* N$

the natural functors which induces functors in the filtered cases and the derived categories, $p_X : X \times Y \rightarrow X$ and $p_Y : X \times Y \rightarrow Y$ the projections.

We have then the following easy proposition :

Proposition 48. *For $X \in \text{SmVar}(\mathbb{C})$ or $X \in \text{AnSm}(\mathbb{C})$, we have for $(M, F), (N, F) \in C_{O_X, \text{fil}}(X)$ or $(M, F), (N, F) \in C_{\mathcal{D}, \text{fil}}(X)$,*

$$(M, F) \otimes_{O_X} (N, F) = \Delta_X^{* \text{mod}}(M, F) \cdot (N, F)$$

Proof. Standard. □

Definition 60. *Let $f : X \rightarrow S$ a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. We have the canonical map in $C_{f^* \mathcal{D}, \mathcal{D}^\infty}(X)$ modules :*

$$T(f, \infty) : (D_{X \rightarrow S}, F^{\text{ord}}) \otimes_{D_X} (D_X^\infty, F^{\text{ord}}) \rightarrow (D_{X \rightarrow S}^\infty, F^{\text{ord}}), (h_X \otimes P_S) \otimes P_X \mapsto (P_X \cdot h_X \otimes P_S + h_X \otimes df(P_X)) P_S$$

where $h_X \in \Gamma(X^\circ, O_X)$, $P_S \in \Gamma(X^\circ, f^* D_S)$ and $P_X \in \Gamma(X^\circ, D_X^\infty)$. This gives, for $(M, F) \in C_{\mathcal{D}(2) \text{fil}}(S)$, the following transformation map in $C_{\mathcal{D}^\infty(2) \text{fil}}(X)$

$$T(f, \infty)(M, F) : J_X(f^{* \text{mod}}(M, F)) := f^*(M, F) \otimes_{f^* D_S} (D_{X \rightarrow S}, F^{\text{ord}}) \otimes_{D_X} (D_X^\infty, F^{\text{ord}}) \xrightarrow{I \otimes T(f, \infty)} f^*(M, F) \otimes_{f^* D_S} (D_{X \rightarrow S}^\infty, F^{\text{ord}}) = f^*(M, F) \otimes_{f^* D_S} f^* D_S^\infty \otimes_{f^* D_S^\infty} (D_{X \rightarrow S}^\infty, F^{\text{ord}}) =: f^{* \text{mod}} J_S(M, F)$$

where we recall that $J_S(M, F) = (M, F) \otimes_{D_S} (D_S^\infty, F^{\text{ord}})$.

We now look at some properties of the dual functor for D modules : For complex of D module with coherent cohomology we have the following:

Proposition 49. (i) *Let $S \in \text{SmVar}(\mathbb{C})$. For $M \in C_{\mathcal{D}, c}(S)$, the canonical map $d(M) : M \rightarrow \mathbb{D}_S^2 L_D M$ is an equivalence Zariski local.*

(ii) *Let $S \in \text{AnSm}(\mathbb{C})$. For $M \in C_{\mathcal{D}}(S)$, the canonical map $d(M) : M \rightarrow \mathbb{D}_S^2 L_D(M)$ is an equivalence usu local.*

iii) *Let $S \in \text{AnSm}(\mathbb{C})$. For $(M, F) \in C_{\mathcal{D}^\infty}(S)$, the canonical map $d(M) : M \rightarrow \mathbb{D}_S^2 L_{\mathcal{D}^\infty}(M)$ is an equivalence usu local.*

Proof. Standard :follows from the definition of coherent sheaves. See [16] for exemple. □

Let $S_1, S_2 \in \text{SmVar}(\mathbb{C})$ or $S_1, S_2 \in \text{AnSm}(\mathbb{C})$ and $p : S_{12} := S_1 \times S_2 \rightarrow S_1$ the projection. In this case we have a canonical embedding $D_{S_1} \hookrightarrow p_* D_{S_{12}}$. This gives, for $(M, F) \in C_{\mathcal{D} \text{fil}}(S_1 \times S_2)$, the following transformation map in $C_{\mathcal{D} \text{fil}}(S_1)$

$$\begin{aligned} T_*(p, D)(M, F) &: p_* \mathbb{D}_{S_{12}}^K(M, F) := p_* \mathcal{H}om_{D_{S_{12}}}((M, F), D_{S_{12}}) \otimes_{O_{S_{12}}} \mathbb{D}_{S_{12}}^O w(K_{S_{12}})[d_{S_{12}}] \\ &\xrightarrow{T_*(p, \text{hom})(-, -)} \mathcal{H}om_{p_* D_{S_{12}}}(p_*(M, F), p_* D_{S_{12}}) \otimes_{p_* O_{S_{12}}} \mathbb{D}_{p_* S_{12}}^O w(p_* K_{S_{12}})[d_{S_{12}}] \\ &\xrightarrow{\sim} \mathcal{H}om_{D_{S_1}}(p_*(M, F), D_{S_1}) \otimes_{O_{S_1}} \mathbb{D}_{S_1}^O w(K_{S_1})[d_{S_1}] =: \mathbb{D}_{S_1}^K p_*(M, F) \end{aligned}$$

We have the canonical map

$$p(D) : p^{* \text{mod}} D_{S_1} = p^* D_{S_1} \otimes_{p^* O_{S_1}} O_{S_{12}} \rightarrow D_{S_{12}}, \gamma \otimes f \mapsto f \cdot \gamma$$

induced by the embedding $p^*D_{S_1} \hookrightarrow D_{S_{12}}$. This gives, for $(M, F) \in C_{\mathcal{D}fil}(S_1)$, the following transformation map in $C_{\mathcal{D}fil}(S_1 \times S_2)$

$$\begin{aligned}
T(p, D)(M, F) : p^{*mod} \mathbb{D}_{S_1}^K(M, F) &:= p^* \mathcal{H}om_{D_{S_1}}((M, F), D_{S_1}) \otimes_{p^*O_{S_1}} p^{*mod} \mathbb{D}_{S_1}^O w(K_{S_1})[d_{S_1}] \\
&\xrightarrow{T(p, hom)(-, -) \otimes I} \mathcal{H}om_{p^*D_{S_1}}(p^*(M, F), p^*D_{S_1}) \otimes_{p^*O_{S_1}} p^{*mod} \mathbb{D}_{S_1}^O w(K_{S_1})[d_{S_1}] \\
&\xrightarrow{(\phi \mapsto \phi \otimes I_{O_{S_{12}}}) \otimes I} \mathcal{H}om_{D_{S_{12}}}(p^{*mod}(M, F), p^{*mod}D_{S_1}) \otimes_{p^*O_{S_1}} p^{*mod} \mathbb{D}_{S_1}^O w(K_{S_1})[d_{S_1}] \\
&\xrightarrow{I \otimes K^{-1}(S_1/S_{12})} \mathcal{H}om_{D_{S_{12}}}(p^{*mod}(M, F), p^{*mod}D_{S_1}) \otimes_{p^*O_{S_1}} \mathbb{D}_{S_{12}}^O w(K_{S_{12}})[d_{S_{12}}] \\
&\xrightarrow{q(p^*O_{S_1}/O_{S_{12}})} \mathcal{H}om_{D_{S_{12}}}(p^{*mod}(M, F), p^{*mod}D_{S_1}) \otimes_{O_{S_{12}}} \mathbb{D}_{S_{12}}^O w(K_{S_{12}})[d_{S_{12}}] \\
&\xrightarrow{\mathcal{H}om(p^{*mod}(M, F), p(D))} \mathcal{H}om_{D_{S_{12}}}(p^{*mod}(M, F), D_{S_{12}}) \otimes_{O_{S_{12}}} \mathbb{D}_{S_{12}}^O w(K_{S_{12}})[d_{S_{12}}] =: \mathbb{D}_{S_{12}}^K(p^{*mod}(M, F))
\end{aligned}$$

whre $K^{-1}(S_1/S_{12})$ is given by the wedge product with a generator of $\wedge^{d_{S_2}} T_{S_{12}/S_1} \xrightarrow{\sim} K_{S_2}^{-1}$.

In the case $S_1, S_2 \in \text{AnSm}(\mathbb{C})$, we also have the embedding $p^*D_{S_1}^\infty \hookrightarrow D_{S_{12}}^\infty$. This gives in the same way, for $(M, F) \in C_{\mathcal{D}fil}(S_1 \times S_2)$, the following transformation map in $C_{\mathcal{D}fil}(S_1)$

$$T_*(p, D^\infty)(M, F) : p_* \mathbb{D}_{S_{12}}^{\infty, K}(M, F) \rightarrow \mathbb{D}_{S_1}^{\infty, K} p_*(M, F).$$

The map

$$p(D^\infty) : p^{*mod} D_{S_1} = p^* D_{S_1}^\infty \otimes_{p^*O_{S_1}} O_{S_{12}} \rightarrow D_{S_{12}}^\infty, \quad \gamma \otimes f \mapsto f \cdot \gamma$$

induced by the embedding $p^*D_{S_1}^\infty \hookrightarrow D_{S_{12}}^\infty$, gives in the same way, for $(M, F) \in C_{\mathcal{D}^\infty fil}(S_1)$, the transformation map in $C_{\mathcal{D}^\infty fil}(S_1 \times S_2)$

$$\begin{aligned}
T(p, D^\infty)(M, F) : p^{*mod} \mathbb{D}_{S_1}^{\infty, K}(M, F) &:= p^{*mod}(\mathbb{D}_{S_1}^\infty(M, F) \otimes_{O_{S_1}} \mathbb{D}_{S_1}^O w(K_{S_1})[d_{S_1}]) \rightarrow \\
&\mathcal{H}om_{D_{S_{12}}^\infty}(p^{*mod}(M, F), D_{S_{12}}^\infty) \otimes_{O_{S_{12}}} \mathbb{D}_{S_{12}}^O w(K_{S_{12}}[d_{S_{12}}]) =: \mathbb{D}_{S_{12}}^{\infty, K}(p^{*mod}(M, F)),
\end{aligned}$$

given in the same way then $T(p, D)(M, F)$.

Proposition 50. (i) Let $g : T \rightarrow S$ a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. We have, for $M \in D_{\mathcal{D}}(S)$ canonical maps

- $T'(g, D)(M) : L\mathbb{D}_S Lg^{*mod} M \rightarrow Lg^{*mod} L\mathbb{D}_S M$
- $T'(g, D)(M) : Lg^{*mod} L\mathbb{D}_S M \rightarrow L\mathbb{D}_S Lg^{*mod} M$

Moreover, in the case where g is non characteristic with respect to M (e.g if g is smooth), these maps are isomorphism.

(ii) Let $S_1, S_2 \in \text{SmVar}(\mathbb{C})$, $p : S_1 \times S_2 \rightarrow S_1$ the projection. For $M \in D_{\mathcal{D}}(S_1)$, we have $T(p, D)(L_D(M)) = T'(p, D)(M)$ in $D_{\mathcal{D}fil}(S_1 \times S_2)$ (c.f.(i)).

Proof. (i):See [16] for the first map. The second one follows from the first by proposition 49(i) and (iii).
(ii):See the proof of (i) in [16] \square

We have the followings :

Proposition 51. Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphism with $X, Y, S \in \text{SmVar}(\mathbb{C})$. Let $M \in C_{\mathcal{D}, h}(S)$. Then, we have $L(f_2 \circ f_1)^{*mod} M = Lf_1^{*mod}(Lf_2^{*mod} M)$ in $D_{\mathcal{D}, h}(X)$.

Proof. Follows from proposition 43 (i), proposition 44 and proposition 49. \square

Proposition 52. Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphism with $X, Y, S \in \text{AnSm}(\mathbb{C})$.

(i) Let $(M) \in C_{\mathcal{D},h}(S)$. Then, we have $L(f_2 \circ f_1)^{\hat{*}mod}(M) = Lf_1^{\hat{*}mod}(Lf_2^{\hat{*}mod}(M))$ in $D_{\mathcal{D},h}(X)$.

(ii) Let $M \in C_{\mathcal{D}^\infty,h}(S)$. Then, we have $L(f_2 \circ f_1)^{\hat{*}mod}M = Lf_1^{\hat{*}mod}(Lf_2^{\hat{*}mod}M)$ in $D_{\mathcal{D}^\infty,h}(X)$.

Proof. Follows from proposition 46 (i), proposition 47 and proposition 49. \square

In the analytic case, we have the following transformation map which we will use in subsection 5.3:

Definition 61. Let $S \in \text{AnSm}(\mathbb{C})$. We have for $(M, F) \in C_{\mathcal{D}fil}(S)$ the canonical transformation map in $C_{\mathcal{D}^\infty fil}(S)$:

$$\begin{aligned} T(D, \infty)(M, F) : \\ J_S(\mathbb{D}_S^K(M, F)) := \text{Hom}_{D_S}((M, F), D_S) \otimes_{D_S} (D_S^\infty, F^{ord}) \otimes_{O_S} \mathbb{D}_S^O w(K_S)[d_S] \xrightarrow{\text{ev}_{D_S}(\text{hom}, \otimes)(-, -, -) \otimes I} \\ \text{Hom}_{D_S}(L_D(M, F), D_S^\infty) \otimes_{O_S} \mathbb{D}_S^O w(K_S)[d_S] \xrightarrow{I(D_S^\infty/D_S)((M, F), D_S^\infty) \otimes I} \\ \text{Hom}_{D_S^\infty}((M, F) \otimes_{D_S} (D_S^\infty, F^{ord}), D_S^\infty) \otimes_{O_S} \mathbb{D}_S^O w(K_S)[d_S] =: \mathbb{D}_S^{\infty, K} J_S(M, F). \end{aligned}$$

which is an isomorphism.

4.2.2 The (relative) De Rham of a (filtered) complex of a D-module and the filtered De Rham direct image

Recall that for $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$ or with $X, S \in \text{AnSp}(\mathbb{C})$,

$$DR(X/S) := \Omega_{X/S}^\bullet \in C_{f^*O_S}(X)$$

denotes (see section 2) the relative De Rham complex of the morphism of ringed spaces $f : (X, O_X) \rightarrow (S, O_S)$, with $\Omega_{X/S}^p := \wedge^p \Omega_{X/S} \in \text{PSh}_{O_X}(X)$ and $\Omega_{X/S} := \text{coker}(f^* \Omega_S \rightarrow \Omega_X) \in \text{PSh}_{O_X}(X)$. Recall that $\Omega_{X/S}^\bullet \in C_{f^*O_S}(S)$ is a complex of f^*O_S modules, but is NOT a complex of O_X module since the differential is a derivation hence NOT O_X linear. Recall that (see section 4.1), for $(M, F) \in C_{D(O_X)fil}(X)$, we have the relative (filtered) De rham complex of (M, F)

$$DR(X/S)(M, F) := (\Omega_{X/S}^\bullet, F_b) \otimes_{O_X} (M, F) \in C_{f^*O_S fil}(X),$$

and that if $\phi : (M_1, F) \rightarrow (M_2, F)$ a morphism with $(M_1, F), (M_2, F) \in C_{D(O_X)fil}(X)$,

$$(I \otimes \phi) : DR(X/S)(M_1, F) := (\Omega_{X/S}^\bullet, F) \otimes_{O_X} (M_1, F) \rightarrow DR(X/S)(M_2, F) := (\Omega_{X/S}^\bullet, F) \otimes_{O_X} (M_2, F)$$

is by definition a morphism of complexes, that is a morphism in $C_{f^*O_S fil}(X)$. For $(N, F) \in C_{D(O_X)^{op} fil}(X)$, we have the relative (filtered) Spencer complex of (N, F)

$$SP(X/S)(N, F) := (T_{X/S}^\bullet, F_b) \otimes_{O_X} (N, F) \in C_{f^*O_S fil}(X),$$

and that if $\phi : (N_1, F) \rightarrow (N_2, F)$ a morphism with $(N_1, F), (N_2, F) \in C_{D(O_X)^{op} fil}(X)$,

$$(I \otimes \phi) : SP(X/S)(N_1, F) := (T_{X/S}^\bullet, F) \otimes_{O_X} (N_1, F) \rightarrow SP(X/S)(N_2, F) := (T_{X/S}^\bullet, F) \otimes_{O_X} (N_2, F)$$

is by definition a morphism of complexes, that is a morphism in $C_{f^*O_S fil}(X)$.

Proposition 53. Let $f : X \rightarrow S$ a smooth morphism with $X, S \in \text{Var}(\mathbb{C})$ or with $X, S \in \text{AnSp}(\mathbb{C})$, denote $d = d_X - d_S$. The inner product gives, for $(M, F) \in C_{D(O_X)fil}(X)$, an isomorphism in $C_{f^*O_S fil}(X)$ and termwise O_X linear

$$T(DR, SP)(M, F) : T_{X/S}^\bullet \otimes_{O_X} (M, F) \otimes_{O_X} K_{X/S} \xrightarrow{\sim} \Omega_{X/S}^{d-\bullet} \otimes_{O_X} (M, F), \partial \otimes m \otimes \kappa \mapsto \iota(\partial)\kappa \otimes m$$

Proof. Standard. \square

For a commutative diagram in $\text{Var}(\mathbb{C})$ or in $\text{AnSp}(\mathbb{C})$:

$$D = \begin{array}{ccc} X & \xrightarrow{f} & S \\ g' \uparrow & & \uparrow g \\ X' & \xrightarrow{f'} & T \end{array}$$

we have (see section 2) the relative differential map of g' given by the pullback of differential forms:

$$\begin{aligned} \Omega_{(X'/X)/(T/S)} : g'^* \Omega_{X/S} &\rightarrow \Omega_{X'/T}, \text{ given by for } X'^o \subset X', X^o \supset g'(X'^o) \text{ (i.e. } g'^{-1}(X^o) \supset X'^o), \\ \omega \in \Gamma(X^o, \Omega_{X/S}^p) &\mapsto \Omega_{(X'/X)/(T/S)}(X'^o)(\omega) := [g'^* \omega] \in \Gamma(X'^o, \Omega_{X'/T}^p). \end{aligned}$$

Moreover, by definition-proposition 12 (section 4.1), for $(M, F) \in C_{D(O_X)fil}(X)$ the map

$$\Omega_{(X'/X)/(T/S)}(M, F) : g'^*(\Omega_{X/S}^\bullet \otimes_{O_X} (M, F)) \rightarrow \Omega_{X'/T}^\bullet \otimes_{O_{X'}} g'^{*mod}(M, F)$$

given in degree (p, i) by, for $X'^o \subset X'$ an open subset and $X^o \subset X$ an open subset such that $g'^{-1}(X^o) \supset X'^o$ (i.e. $X^o \supset g'(X'^o)$), $\omega \in \Gamma(X^o, \Omega_{X/S}^p)$ and $m \in \Gamma(X^o, M^i)$,

$$\Omega_{(X'/X)/(T/S)}(M, F)(\omega \otimes m) = g'^* \omega \otimes (m \otimes 1)$$

is a map of complexes, that is a map in $C_{f^*O_S fil}(X')$. This give, for $(M, F) \in C_{D(O_X)fil}(X)$, the following transformation map in $C_{O_T fil}(T)$

$$\begin{aligned} T_\omega^O(D)(M, F) : g'^{*mod} L_O(f_* E(\Omega_{X/S}^\bullet \otimes_{O_X} (M, F))) &\xrightarrow{T(g, L_O)(-)} \\ (g^* f_* E(\Omega_{X/S}^\bullet \otimes_{O_X} (M, F))) \otimes_{g^* O_S} O_T &\xrightarrow{T(g', E)(-)\circ T(D)(E(\Omega_{X/S}^\bullet \otimes_{O_X} M))} \\ (f'_* E(g'^*(\Omega_{X/S}^\bullet \otimes_{O_X} (M, F)))) \otimes_{g^* O_S} O_T &\xrightarrow{m \circ E(\Omega_{(X'/X)/(T/S)}(M))} f'_* E(\Omega_{X'/T}^\bullet \otimes_{O_{X'}} g'^{*mod}(M, F)), \end{aligned}$$

with $m'(m) = m \otimes 1$. Under the canonical isomorphism $\Omega_{X/S}^\bullet \xrightarrow{\sim} \Omega_{X/S}^\bullet \otimes_{O_X} O_X$ given by $\omega \mapsto \omega \otimes 1$, we have (see remark 7)

$$T_\omega^O(D)(O_X) = T_\omega^O(D) : g'^{*mod} L_O(f_* E(\Omega_{X/S}^\bullet)) \rightarrow f'_* E(\Omega_{X'/T}^\bullet).$$

Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Again by definition-proposition 12 (section 4.1), for $(M, F) \in C_{D(O_X)fil}(X)$ the map

$$\Omega_{(X^{an}/X)/(S^{an}/S)}(M, F) : an_X^*(\Omega_{X/S}^\bullet \otimes_{O_X} (M, F)) \rightarrow \Omega_{X^{an}/S^{an}}^\bullet \otimes_{O_{X^{an}}} M^{an}$$

given in degree (p, i) by, for $X^o \subset X$ and $X^o \supset X^{oo}$ an open subsets of X for the usual, resp. Zariski topology, $\omega \in \Gamma(X^o, \Omega_{X/S}^p)$ and $m \in \Gamma(X^o, M^i)$,

$$\Omega_{(X^{an}/X)/(S^{an}/S)}(M, F)(\omega \otimes m) = \omega \otimes (m \otimes 1)$$

is a map of complexes, that is a map in $C_{f^*O_{S^{an}} fil}(X^{an})$. This gives, for $(M, F) \in C_{D(O_X)fil}(X)$, we have the following transformation map in $C_{O_{S^{an}} fil}(S^{an})$

$$\begin{aligned} T_\omega^O(an, f)(M, F) : (f_* E(\Omega_{X/S}^\bullet \otimes_{O_X} (M, F)))^{an} &:= an_S^*(f_* E(\Omega_{X/S}^\bullet \otimes_{O_X} (M, F))) \otimes_{an_S^* O_S} O_{S^{an}} \\ \xrightarrow{T(an(X), E)(-)\circ T(an, f)(E(\Omega_{X/S}^\bullet \otimes_{O_X} M))} &(f_* E(an_X^*(\Omega_{X/S}^\bullet \otimes_{O_X} (M, F)))) \otimes_{an_S^* O_S} O_{S^{an}} \\ \xrightarrow{m \circ E(\Omega_{(X^{an}/X)/(S^{an}/S)}(M, F))} &f_* E(\Omega_{X/S}^\bullet \otimes_{O_X^{an}} (M^{an}, F)) \end{aligned}$$

with $m(n \otimes s) = s.n$. Under the canonical isomorphism $\Omega_{X/S}^\bullet \xrightarrow{\sim} \Omega_{X/S}^\bullet \otimes_{O_X} O_X$ given by $\omega \mapsto \omega \otimes 1$, we have (see remark 7)

$$T_\omega^O(an, f)(O_X) = T_\omega^O(an, f) : (f_*E(\Omega_{X/S}^\bullet))^{an} \rightarrow f_*E(\Omega_{X^{an}/S^{an}}^\bullet)$$

Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$ or with $X, S \in \text{AnSp}(\mathbb{C})$. In the case where X is smooth, for $(M, F) = (M^\bullet, F) \in C_{\mathcal{D}fil}(X)$, the differential of the relative De Rham complex of (M, F)

$$DR(X/S)(M, F) := (\Omega_{X/S}^\bullet, F) \otimes_{O_X} (M, F) = \text{Tot}((\Omega_{X/S}^\bullet, F) \otimes_{O_X} (M^\bullet, F)) \in C_{f^*O_S fil}(X)$$

are given by

- $d_{p,p+1} : \Omega_{X/S}^p \otimes_{O_X} M^i \rightarrow \Omega_{X/S}^{p+1} \otimes_{O_X} M^i$, with for $X^o \subset X$ an open affine subset with (x_1, \dots, x_n) local coordinate (since X is smooth, T_X is locally free), $m \in \Gamma(X^o, M^i)$ and $\omega \in \Gamma(X^o, \Omega_{X/S}^p)$,

$$d_{p,p+1}(\omega \otimes m) := (d\omega) \otimes m + \sum_{i=1}^n (dx_i \wedge \omega) \otimes (\partial_i)m$$

- $d_{i,i+1} : \Omega_{X/S}^p \otimes_{O_X} M^i \rightarrow \Omega_{X/S}^p \otimes_{O_X} M^{i+1}$, with for $X^o \subset X$ an open subset, $m \in \Gamma(X^o, M^i)$ and $\omega \in \Gamma(X^o, \Omega_{X/S}^p)$, $d_{i,i+1}(\omega \otimes m) := (\omega \otimes dm)$.

For D_X only, the differential of its De Rham complex $(\Omega_{X/S}^\bullet, F) \otimes_{O_X} D_X$ are right linear, so that

$$(\Omega_{X/S}^\bullet, F_b) \otimes_{O_X} (D_X, F^{ord}) \in C_{\mathcal{D}^{op}, f^*O_S fil}(X)$$

In the particular case of a projection $p : Y \times S \rightarrow S$ with $Y, S \in \text{SmVar}(\mathbb{C})$ or with $Y, S \in \text{AnSm}(\mathbb{C})$ we have :

Proposition 54. *Let $Y, S \in \text{SmVar}(\mathbb{C})$ or $Y, S \in \text{AnSm}(\mathbb{C})$. Let $p : Y \times S \rightarrow S$ the projection. For $(M, F) \in C_{\mathcal{D}fil}(Y \times S)$,*

$$DR(Y \times S/S)(M, F) := (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M, F) \in C_{p^*O_S fil}(Y \times S)$$

is a naturally a complex of filtered p^*D_S modules, that is

$$DR(Y \times S/S)(M, F) := (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M, F) \in C_{p^* \mathcal{D}fil}(Y \times S),$$

where the p^*D_S module structure on $\Omega_{Y \times S/S}^p \otimes_{O_{Y \times S}} M^n$ is given by for $(Y \times S)^o \subset Y \times S$ an open subset,

$$(\gamma \in \Gamma((Y \times S)^o, T_{Y \times S}), \hat{\omega} \otimes m \in \Gamma((Y \times S)^o, \Omega_{Y \times S/S}^p \otimes_{O_{Y \times S}} M^n)) \mapsto \gamma.(\hat{\omega} \otimes m) := (\hat{\omega} \otimes (\gamma.m)).$$

Moreover, if $\phi : (M_1, F) \rightarrow (M_2, F)$ a morphism with $(M_1, F), (M_2, F) \in C_{\mathcal{D}fil}(Y \times S)$,

$$DR(Y \times S/S)(\phi) := (I \otimes \phi) : (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M_1, F) \rightarrow (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M_2, F)$$

is a morphism in $C_{p^* \mathcal{D}fil}(Y \times S)$.

Proof. Standard. □

In the analytic case, we also have

Proposition 55. *Let $Y, S \in \text{AnSm}(\mathbb{C})$. Let $p : Y \times S \rightarrow S$ the projection. For $(M, F) \in C_{\mathcal{D}^\infty fil}(Y \times S)$,*

$$DR(Y \times S/S)(M, F) := (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M, F) \in C_{p^*O_S fil}(Y \times S)$$

is naturally a complex of filtered $p^*D_S^\infty$ modules, that is

$$DR(Y \times S/S)(M, F) := (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M, F) \in C_{p^*\mathcal{D}^\infty \text{fil}}(Y \times S),$$

where the $p^*D_S^\infty$ module structure on $\Omega_{Y \times S/S}^p \otimes_{O_{Y \times S}} M^n$ is given by for $(Y \times S)^\circ \subset Y \times S$ an open subset,

$$(\gamma \in \Gamma((Y \times S)^\circ, T_{Y \times S}), \hat{\omega} \otimes m \in \Gamma((Y \times S)^\circ, \Omega_{Y \times S/S}^p \otimes_{O_{Y \times S}} M^n)) \mapsto \gamma.(\hat{\omega} \otimes m) := (\hat{\omega} \otimes (\gamma.m)).$$

Moreover, if $\phi : (M_1, F) \rightarrow (M_2, F)$ a morphism with $(M_1, F), (M_2, F) \in C_{\mathcal{D}^\infty \text{fil}}(Y \times S)$,

$$DR(Y \times S/S)(\phi) := (I \otimes \phi) : (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M_1, F) \rightarrow (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M_2, F)$$

is a morphism in $C_{p^*\mathcal{D}^\infty \text{fil}}(Y \times S)$.

Proof. Standard : follows from the finite order case (proposition 54). \square

We state on the one hand the commutativity of the tensor product with respect to D_S and with respect to O_S , for $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$ in the filtered case, and on the other hand the commutativity between the tensor product with respect to D_S by D_S and the De Rahm complex :

Proposition 56. (i) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$ or with $X, S \in \text{AnSm}(\mathbb{C})$. For $(M', F) \in C_{\mathcal{D}^\infty \text{op fil}, f^*\mathcal{D}}(X)$ and $(M, F), (N, F) \in C_{\mathcal{D} \text{fil}}(X)$. we have canonical isomorphisms of filtered f^*D_S modules, i.e. isomorphisms in $C_{f^*\mathcal{D}}(X)$,

$$\begin{aligned} (M', F) \otimes_{O_X} (N, F) \otimes_{D_X} (M, F) &= (M', F) \otimes_{D_X} ((M, F) \otimes_{O_X} (N, F)) \\ &= ((M', F) \otimes_{O_X} (M, F)) \otimes_{D_X} (N, F) \end{aligned}$$

(ii) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$ or with $X, S \in \text{AnSp}(\mathbb{C})$. For $(M, F) \in C_{D(O_X) \text{fil}}(X)$, we have a canonical isomorphisms of filtered f^*O_S modules, i.e. isomorphisms in $C_{f^*O_S \text{fil}}(X)$,

$$(\Omega_{X/S}^\bullet, F_b) \otimes_{O_X} (M, F) = ((\Omega_{X/S}^\bullet, F_b) \otimes_{O_X} (D(O_X), F_b)) \otimes_{D(O_X)} (M, F)$$

(iii) Let $p : Y \times S \rightarrow S$ a projection with $Y, S \in \text{SmVar}(\mathbb{C})$ or with $X, S \in \text{AnSm}(\mathbb{C})$. For $(M, F) \in C_{\mathcal{D} \text{fil}}(Y \times S)$, the isomorphisms of filtered p^*O_S modules of (ii)

$$(\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M, F) = ((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (D_{Y \times S}, F_b)) \otimes_{D_{Y \times S}} (M, F)$$

are isomorphisms of filtered p^*D_S modules, that is isomorphism in $C_{p^*\mathcal{D} \text{fil}}(Y \times S)$.

Proof. (i) and (ii) are particular case of proposition 36.

(iii): follows immediately by definition of the p^*D_S module structure. \square

We now look at the functorialities of the relative De Rham complex of a smooth morphisms of smooth complex algebraic varieties :

Proposition 57. Consider a commutative diagram in $\text{SmVar}(\mathbb{C})$ or in $\text{AnSm}(\mathbb{C})$:

$$D = \begin{array}{ccc} Y \times S & \xrightarrow{p} & S \\ g'' = (g_0'' \times g) \uparrow & & \uparrow g \\ Y' \times T & \xrightarrow{p'} & T \end{array}$$

with p and p' the projections. For $(M, F) \in C_{\mathcal{D} \text{fil}}(Y \times S)$ the map in $C_{g''^*p^*O_S \text{fil}}(Y' \times T)$

$$\Omega_{(Y' \times T / Y \times S) / (T / S)}(M, F) : g''^*((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M, F)) \rightarrow (\Omega_{Y' \times T / T}^\bullet, F_b) \otimes_{O_{Y' \times T}} g''^{*\text{mod}}(M, F)$$

given in definition-proposition 12 is a map in $C_{g''_*p^*\mathcal{D}fil}(Y' \times T)$. Hence, for $(M, F) \in C_{\mathcal{D}fil}(Y \times S)$, the map in $C_{O_T fil}(T)$ (with L_D instead of L_O)

$$T_\omega^O(D)(M) : g^{*mod} L_D(p_* E((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M, F))) \rightarrow p'_* E((\Omega_{Y' \times T/T}^\bullet, F_b) \otimes_{O_{Y' \times T}} g''^{*mod} (M, F)),$$

is a map in $C_{\mathcal{D}fil}(T)$.

Proof. Follows immediately by definition. \square

In the analytic case, we also have

Proposition 58. Consider a commutative diagram in $\text{AnSm}(\mathbb{C})$:

$$D = \begin{array}{ccc} Y \times S & \xrightarrow{p} & S \\ g''=(g_0'' \times g) \uparrow & & \uparrow g \\ Y' \times T & \xrightarrow{p'} & T \end{array}$$

with p and p' the projections. For $(M, F) \in C_{\mathcal{D}^\infty fil}(Y \times S)$ the map in $C_{g''_*p^*O_S fil}(Y' \times T)$

$$\Omega_{(Y' \times T / Y \times S) / (T/S)}(M, F) : g''^*((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M, F)) \rightarrow (\Omega_{Y' \times T/T}^\bullet, F_b) \otimes_{O_{Y' \times T}} g''^{*mod} (M, F)$$

is a map in $C_{g''_*p^*\mathcal{D}^\infty fil}(Y' \times T)$. Hence, for $(M, F) \in C_{\mathcal{D}^\infty fil}(Y \times S)$, the map in $C_{O_T fil}(T)$ (with L_D instead of L_O)

$$T_\omega^O(D)(M) : g^{*mod} L_D(p_* E((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M, F))) \rightarrow p'_* E((\Omega_{Y' \times T/T}^\bullet, F_b) \otimes_{O_{Y' \times T}} g''^{*mod} (M, F)),$$

is a map in $C_{\mathcal{D}^\infty fil}(T)$.

Proof. Follows immediately by definition. \square

Similarly, we have

Proposition 59. Let $p : Y \times S \rightarrow S$ a projection with $Y, S \in \text{SmVar}(\mathbb{C})$. For $(M, F) \in C_{\mathcal{D}fil}(Y \times S)$ the map in $C_{p^*O_{S^{an}}}(Y^{an} \times S^{an})$

$$\Omega_{(Y^{an} \times S^{an} / Y \times S) / (S^{an} / S)}(M, F) : \text{an}(Y \times S)^*((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M, F)) \rightarrow (\Omega_{Y^{an} \times S^{an} / S^{an}}^\bullet, F_b) \otimes_{O_{Y^{an} \times S^{an}}} (M^{an}, F)$$

is a map in $C_{p^*\mathcal{D}fil}(Y^{an} \times S^{an})$. For $(M, F) \in C_{\mathcal{D}fil}(Y \times S)$, the map in $C_{O_{S^{an}} fil}(S^{an})$

$$T_\omega^O(\text{an}, h)(M, F) : (p_* E((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M, F)))^{an} \rightarrow p_* E((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y^{an} \times S^{an}}} (M, F)^{an})$$

is a map in $C_{\mathcal{D}fil}(S^{an})$.

Proof. Similar to the proof of proposition 57. \square

Proposition 60. Let $p : Y \times S \rightarrow S$ a projection with $Y, S \in \text{SmVar}(\mathbb{C})$ or with $Y, S \in \text{AnSm}(\mathbb{C})$.

(i) If $\phi : (M, F) \rightarrow (N, F)$ is an r -filtered Zariski, resp. usu, local equivalence with $(M_1, F), (M_2, F) \in C_{\mathcal{D}fil}(Y \times S)$, then

$$DR(Y \times S/S)(\phi) : (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M, F) \rightarrow \Omega_{Y \times S/S}^\bullet \otimes_{O_{Y \times S}} (N, F)$$

is an r -filtered equivalence Zariski, resp. usu, local in $C_{p^*\mathcal{D}fil}(Y \times S)$.

(ii) Consider a commutative diagram in $\text{SmVar}(\mathbb{C})$ or in $\text{AnSm}(\mathbb{C})$

$$D = \begin{array}{ccc} Y \times S & \xrightarrow{p} & S \\ \uparrow l & & \uparrow = \\ V & \xrightarrow{k} & S \end{array}$$

with p the projection. For $(N, F) \in C_{\mathcal{D}, l^* \mathcal{D} \text{fil}}(V)$, the map in $C_{p^* O_S}(Y \times S)$ (see definition 53)

$$\begin{aligned} k \circ T_{\omega}^O(l, \otimes)(E(N, F)) : (\Omega_{Y \times S/S}^{\bullet}, F_b) \otimes_{O_{Y \times S}} l_* E(N, F) &\rightarrow l_* ((\Omega_{V/S}^{\bullet}, F_b) \otimes_{O_V} E(N, F)) \\ &\rightarrow l_* E((\Omega_{V/S}^{\bullet}, F_b) \otimes_{O_V} E(N, F)) \end{aligned}$$

is a filtered equivalence Zariski, resp. usu, local in $C_{p^* \mathcal{D} \text{fil}}(Y \times S)$.

Proof. (i): Follows from proposition 54 that it is a morphism of $p^* D_S$ module. The fact that it is an equivalence Zariski, resp usu, local is a particular case of proposition 37(i).

(ii): Follows from proposition 54 and the first part of proposition 57 that it is a morphism of $h^* D_S$ module. The fact that it is an equivalence Zariski, resp usu, local is a particular case of proposition 37(ii). \square

In the analytical case, we also have

Proposition 61. Let $p : Y \times S \rightarrow S$ a projection with $Y, S \in \text{SmVar}(\mathbb{C})$ or with $Y, S \in \text{AnSm}(\mathbb{C})$.

(i) If $\phi : (M, F) \rightarrow (N, F)$ is an r -filtered usu local equivalence with $(M_1, F), (M_2, F) \in C_{\mathcal{D}^{\infty} \text{fil}}(Y \times S)$, then

$$DR(Y \times S/S)(\phi) : (\Omega_{Y \times S/S}^{\bullet}, F_b) \otimes_{O_{Y \times S}} (M, F) \rightarrow \Omega_{Y \times S/S}^{\bullet} \otimes_{O_{Y \times S}} (N, F)$$

is an r -filtered equivalence usu local in $C_{p^* \mathcal{D}^{\infty} \text{fil}}(Y \times S)$.

(ii) Consider a commutative diagram in $\text{AnSm}(\mathbb{C})$

$$D = \begin{array}{ccc} Y \times S & \xrightarrow{p} & S \\ \uparrow l & & \uparrow = \\ V & \xrightarrow{k} & S \end{array}$$

with p the projection. For $(N, F) \in C_{\mathcal{D}^{\infty}, l^* \mathcal{D}^{\infty} \text{fil}}(V)$, the map in $C_{p^* O_S}(Y \times S)$ (see definition 53)

$$\begin{aligned} k \circ T_{\omega}^O(l, \otimes)(E(N, F)) : (\Omega_{Y \times S/S}^{\bullet}, F_b) \otimes_{O_{Y \times S}} l_* E(N, F) &\rightarrow l_* ((\Omega_{V/S}^{\bullet}, F_b) \otimes_{O_V} E(N, F)) \\ &\rightarrow l_* E((\Omega_{V/S}^{\bullet}, F_b) \otimes_{O_V} E(N, F)) \end{aligned}$$

is a filtered equivalence usu local in $C_{p^* \mathcal{D}^{\infty} \text{fil}}(Y \times S)$.

Proof. Follows from the finite order case : proposition 60. \square

Dually of the De Rahm complex of a D_S module M , we have the Spencer complex of M . In the particular case of D_S , we have the following:

Proposition 62. Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$.

- We have the filtered resolutions of K_S by the following complex of locally free right D_S modules: $\omega(S) : \omega(K_S) := (\Omega_S^{\bullet}, F_b)[d_S] \otimes_{O_S} (D_S, F_b) \rightarrow (K_S, F_b)$ and $\omega(S) : \omega(K_S, F^{\text{ord}}) := (\Omega_S^{\bullet}, F_b)[d_S] \otimes_{O_S} (D_S, F^{\text{ord}}) \rightarrow (K_S, F^{\text{ord}})$
- Dually, we have the filtered resolution of O_S by the following complex of locally free (left) D_S modules: $\omega^{\vee}(S) : \omega(O_S) := (\wedge^{\bullet} T_S, F_b)[d_S] \otimes_{O_S} (D_S, F_b) \rightarrow (O_S, F_b)$ and $\omega^{\vee}(S) : \omega(O_S, F^{\text{ord}}) := (\wedge^{\bullet} T_S, F_b)[d_S] \otimes_{O_S} (D_S, F^{\text{ord}}) \rightarrow (O_S, F^{\text{ord}})$.

Let $S_1, S_2 \in \text{SmVar}(\mathbb{C})$ or $S_1, S_2 \in \text{AnSm}(\mathbb{C})$. Consider the projection $p = p_1 : S_1 \times S_2 \rightarrow S_1$.

- We have the filtered resolution of $D_{S_1 \times S_2 \rightarrow S_1}$ by the following complexes of (left) ($p^*D_{S_1}$ and right $D_{S_1 \times S_2}$) modules :

$$\omega(S_1 \times S_2/S_1) : (\Omega_{S_1 \times S_2/S_1}^\bullet[d_{S_2}], F_b) \otimes_{O_{S_1 \times S_2}} (D_{S_1 \times S_2}, F^{ord}) \rightarrow (D_{S_1 \times S_2 \leftarrow S_1}, F^{ord}).$$

- Dually, we have the filtered resolution of $D_{S_1 \times S_2 \rightarrow S_1}$ by the following complexes of (left) ($p^*D_{S_1}, D_{S_1 \times S_2}$) modules :

$$\omega^\vee(S_1 \times S_2/S_1) : (\wedge^\bullet T_{S_1 \times S_2/S_1}[d_{S_2}], F_b) \otimes_{O_{S_1 \times S_2}} (D_{S_1 \times S_2}, F^{ord}) \rightarrow (D_{S_1 \times S_2 \rightarrow S_1}, F^{ord}),$$

Proof. See [16]. □

In the analytical case, we also have

Proposition 63. *Let $S \in \text{AnSm}(\mathbb{C})$.*

- We have the filtered resolutions of K_S by the following complex of locally free right D_S modules: $\omega(S) : \omega(K_S) := (\Omega_S^\bullet, F_b)[d_S] \otimes_{O_S} (D_S^\infty, F^{ord}) \rightarrow (K_S, F_b)$.
- Dually, we have the filtered resolution of O_S by the following complex of locally free (left) D_S modules: $\omega^\vee(S) : \omega(O_S) := (\wedge^\bullet T_S, F_b)[d_S] \otimes_{O_S} (D_S^\infty, F^{ord}) \rightarrow (O_S, F_b)$.

Let $S_1, S_2 \in \text{AnSm}(\mathbb{C})$. Consider the projection $p = p_1 : S_1 \times S_2 \rightarrow S_1$.

- We have the filtered resolution of $D_{S_1 \times S_2 \rightarrow S_1}^\infty$ by the following complexes of (left) ($p^*D_{S_1}^\infty$ and right $D_{S_1 \times S_2}^\infty$) modules :

$$\omega(S_1 \times S_2/S_1) : (\Omega_{S_1 \times S_2/S_1}^\bullet[d_{S_2}], F_b) \otimes_{O_{S_1 \times S_2}} (D_{S_1 \times S_2}^\infty, F^{ord}) \rightarrow (D_{S_1 \times S_2 \leftarrow S_1}^\infty, F^{ord}).$$

- Dually, we have the filtered resolution of $D_{S_1 \times S_2 \rightarrow S_1}^\infty$ by the following complexes of (left) ($p^*D_{S_1}^\infty, D_{S_1 \times S_2}^\infty$) modules :

$$\omega^\vee(S_1 \times S_2/S_1) : (\wedge^\bullet T_{S_1 \times S_2/S_1}[d_{S_2}], F_b) \otimes_{O_{S_1 \times S_2}} (D_{S_1 \times S_2}^\infty, F^{ord}) \rightarrow (D_{S_1 \times S_2 \rightarrow S_1}^\infty, F^{ord}),$$

Proof. Similar to the finite order case : the first map on the right is a surjection and the kernel are obtained by tensoring D_S^∞ with the kernel of the kozcul resolution of K_S (note that D_S^∞ is a locally free hence flat O_S module). □

Motivated by these resolutions, we make the following definition

Definition 62. (i) Let $i : Z \hookrightarrow S$ be a closed embedding, with $Z, S \in \text{SmVar}(\mathbb{C})$ or with $Z, S \in \text{AnSm}(\mathbb{C})$. Then, for $(M, F) \in \mathcal{C}_{\mathcal{D}fil}(Z)$, we set

$$i_{*mod}(M, F) := i_{*mod}^0(M, F) := i_*((M, F) \otimes_{D_Z} (D_{Z \leftarrow S}, F^{ord})) \in \mathcal{C}_{\mathcal{D}fil}(S)$$

(ii) Let $S_1, S_2 \in \text{SmVar}(\mathbb{C})$ or $S_1, S_2 \in \text{AnSm}(\mathbb{C})$ and $p : S_1 \times S_2 \rightarrow S_1$ be the projection. Then, for $(M, F) \in \mathcal{C}_{\mathcal{D}fil}(S_1 \times S_2)$, we set

$$- p_{*mod}^0(M, F) := p_*(DR(S_1 \times S_2/S_1)(M, F)) := p_*((\Omega_{S_1 \times S_2/S_1}^\bullet, F_b) \otimes_{O_{S_1 \times S_2}} (M, F))[d_{S_2}] \in \mathcal{C}_{\mathcal{D}fil}(S_1),$$

$$- p_{*mod}(M, F) := p_*E(DR(S_1 \times S_2/S_1)(M, F)) := p_*E((\Omega_{S_1 \times S_2/S_1}^\bullet, F_b) \otimes_{O_{S_1 \times S_2}} (M, F))[d_{S_2}] \in \mathcal{C}_{\mathcal{D}fil}(S_1).$$

(iii) Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{SmVar}(\mathbb{C})$ or $X, S \in \text{AnSm}(\mathbb{C})$. Consider the factorization $f : X \xrightarrow{i} X \times S \xrightarrow{p_S} S$, where i is the graph embedding and $p_S : X \times S \rightarrow S$ is the projection. Then, for $(M, F) \in C_{\mathcal{D}fil}(X)$ we set

$$\begin{aligned} - f_{*mod}^{FDR}(M, F) &:= p_{S*mod} i_{*mod}(M, F) \in C_{\mathcal{D}fil}(S), \\ - \int_f^{FDR}(M, F) &:= f_{*mod}^{FDR}(M, F) := p_{S*mod} i_{*mod}(M, F) \in D_{\mathcal{D}fil, \infty}(S). \end{aligned}$$

By proposition 64 below, we have $\int_f^{FDR} M = \int_f M \in D_{\mathcal{D}}(X)$.

(iii) Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{SmVar}(\mathbb{C})$ or $X, S \in \text{AnSm}(\mathbb{C})$. Consider the factorization $f : X \xrightarrow{i} X \times S \xrightarrow{p_S} S$, where i is the graph embedding and $p_S : X \times S \rightarrow S$ is the projection. Then, for $(M, F) \in C_{\mathcal{D}fil}(X)$ we set

$$\begin{aligned} - f_{!mod}^{FDR}(M, F) &:= \mathbb{D}_S^K L_D f_{*mod}^{FDR} \mathbb{D}_S^K L_D(M, F) := \mathbb{D}_S^K L_D p_{S*mod} i_{*mod} \mathbb{D}_{X \times S}^K L_D(M, F) \in C_{\mathcal{D}fil}(S), \\ - \int_{f!}^{FDR}(M, F) &:= f_{!mod}^{FDR}(M, F) := \mathbb{D}_S^K L_D p_{S*mod} i_{*mod} \mathbb{D}_{X \times S}^K L_D(M, F) \in D_{\mathcal{D}fil, \infty}(S). \end{aligned}$$

In the analytical case we also consider :

Definition 63. (i) Let $i : Z \hookrightarrow S$ be a closed embedding with $Z, S \in \text{AnSm}(\mathbb{C})$. Then, for $(M, F) \in C_{\mathcal{D}^\infty fil}(Z)$, we set

$$i_{*mod}(M, F) := i_{*mod}^0(M, F) := i_*((M, F) \otimes_{D_Z^\infty} (D_{Z \leftarrow S}^\infty, F^{ord})) \in C_{\mathcal{D}^\infty fil}(S)$$

(ii) Let $S_1, S_2 \in \text{AnSm}(\mathbb{C})$ and $p : S_1 \times S_2 \rightarrow S_1$ be the projection. For $(M, F) \in C_{\mathcal{D}^\infty fil}(S_1 \times S_2)$, we consider

$$\begin{aligned} - p_{*mod}^0(M, F) &:= p_*(DR(S_1 \times S_2/S_1)(M, F)) := p_*((\Omega_{S_1 \times S_2/S_1}^\bullet, F_b) \otimes_{O_{S_1 \times S_2}} (M, F))[d_{S_2}] \in C_{\mathcal{D}^\infty fil}(S_1), \\ - p_{*mod}(M, F) &:= p_*E(DR(S_1 \times S_2/S_1)(M, F)) := p_*E((\Omega_{S_1 \times S_2/S_1}^\bullet, F_b) \otimes_{O_{S_1 \times S_2}} (M, F))[d_{S_2}] \in C_{\mathcal{D}^\infty fil}(S_1). \end{aligned}$$

(iii) Let $S_1, S_2 \in \text{AnSm}(\mathbb{C})$ and $p : S_1 \times S_2 \rightarrow S_1$ be the projection. For $(M, F) \in C_{\mathcal{D}fil}(S_1 \times S_2)$ or $(M, F) \in C_{\mathcal{D}^\infty fil}(S_1 \times S_2)$, we set

$$\begin{aligned} - p_{!mod}^0(M, F) &:= p_!(DR(S_1 \times S_2/S_1)(M, F)) := p_!((\Omega_{S_1 \times S_2/S_1}^\bullet, F_b) \otimes_{O_{S_1 \times S_2}} (M, F))[d_{S_2}] \in C_{\mathcal{D}fil}(S_1), \\ - p_{!mod}(M, F) &:= p_!E(DR(S_1 \times S_2/S_1)(M, F)) := p_!E((\Omega_{S_1 \times S_2/S_1}^\bullet, F_b) \otimes_{O_{S_1 \times S_2}} (M, F))[d_{S_2}] \in C_{\mathcal{D}fil}(S_1). \end{aligned}$$

(iv) Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{AnSm}(\mathbb{C})$. Consider the factorization $f : X \xrightarrow{i} X \times S \xrightarrow{p_S} S$, where i is the graph embedding and $p_S : X \times S \rightarrow S$ is the projection. Then, for $(M, F) \in C_{\mathcal{D}^\infty fil}(X)$ we set

$$\begin{aligned} - f_{*mod}^{FDR}(M, F) &:= p_{S*mod} i_{*mod}(M, F) \in C_{\mathcal{D}^\infty fil}(S), \\ - \int_f^{FDR}(M, F) &:= f_{*mod}^{FDR}(M, F) := p_{S*mod} i_{*mod}(M, F) \in D_{\mathcal{D}^\infty fil, \infty}(S), \\ - f_{!mod}^{FDR}(M, F) &:= p_{S!mod} i_{*mod}(M, F) \in C_{\mathcal{D}^\infty fil}(S), \\ - \int_{f!}^{FDR}(M, F) &:= f_{!mod}^{FDR}(M, F) := p_{S!mod} i_{*mod}(M, F) \in D_{\mathcal{D}^\infty fil, \infty}(S). \end{aligned}$$

By proposition 65 below, we have $\int_{f!}^{FDR} M = \int_{f!} M \in D_{\mathcal{D}^\infty}(X)$ and $\int_f^{FDR} M = \int_f M \in D_{\mathcal{D}^\infty}(X)$.

(v) Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{AnSm}(\mathbb{C})$. Consider the factorization $f : X \xrightarrow{i} X \times S \xrightarrow{p_S} S$, where i is the graph embedding and $p_S : X \times S \rightarrow S$ is the projection. Then, for $(M, F) \in C_{\mathcal{D}fil}(X)$ we set

$$\begin{aligned} - \int_{!mod}^{FDR}(M, F) &:= p_{S!mod} i_* mod(M, F) \in C_{\mathcal{D}fil}(S), \\ - \int_{f!}^{FDR}(M, F) &:= \int_{!mod}^{FDR}(M, F) := p_{S!mod} i_* mod(M, F) \in D_{\mathcal{D}fil, \infty}(S). \end{aligned}$$

By proposition 65 below, we have $\int_{f!}^{FDR} M = \int_{f!} M \in D_{\mathcal{D}}(X)$.

Proposition 64. (i) Let $i : Z \hookrightarrow S$ a closed embedding with $S, Z \in \text{SmVar}(\mathbb{C})$ or with $S, Z \in \text{AnSm}(\mathbb{C})$. Then for $(M, F) \in C_{\mathcal{D}fil}(Z)$, we have

$$\int_i (M, F) := Ri_*((M, F) \otimes_{D_Z}^L (D_{Z \leftarrow S}, F^{ord})) = i_*((M, F) \otimes_{D_Z} (D_{Z \leftarrow S}, F^{ord})) = i_* mod(M, F).$$

(ii) Let $S_1, S_2 \in \text{SmVar}(\mathbb{C})$ or $S_1, S_2 \in \text{AnSm}(\mathbb{C})$ and $p : S_{12} := S_1 \times S_2 \rightarrow S_1$ be the projection. Then, for $(M, F) \in C_{\mathcal{D}fil}(S_1 \times S_2)$ we have

$$\begin{aligned} \int_p (M, F) : &= Rp_*((M, F) \otimes_{D_{S_1 \times S_2}}^L (D_{S_1 \times S_2 \leftarrow S_1}, F^{ord})) \\ &= p_* E((\Omega_{S_1 \times S_2/S_1}^\bullet, F_b) \otimes_{O_{S_1 \times S_2}} (D_{S_1 \times S_2}, F^{ord}) \otimes_{D_{S_1 \times S_2}} (M, F)[d_{S_2}]) \\ &= p_* E((\Omega_{S_1 \times S_2/S_1}^\bullet, F_b) \otimes_{O_{S_1 \times S_2}} (M, F)[d_{S_2}]) =: p_* mod(M, F). \end{aligned}$$

where the second equality follows from Griffithz transversality (the canonical isomorphism map respect by definition the filtration).

(iii) Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{SmVar}(\mathbb{C})$ or with $X, S \in \text{AnSm}(\mathbb{C})$. Then for $M \in C_{\mathcal{D}}(X)$, we have $\int_f^{FDR} M = \int_f M$.

Proof. (i): Follows from the fact that $D_{Z \leftarrow S}$ is a locally free D_Z module and that i_* is an exact functor. (ii): Since $\Omega_{S_{12}/S_1}^\bullet[d_{S_2}] \otimes_{O_{S_{12}}} D_{S_{12}}$ is a complex of locally free $D_{S_1 \times S_2}$ modules, we have in $D_{fil}(S_1 \times S_2)$, using proposition 62,

$$(D_{S_1 \times S_2 \leftarrow S_1}, F^{ord}) \otimes_{D_{S_1 \times S_2}}^L (M, F) = (\Omega_{S_{12}/S_1}^\bullet[d_{S_2}], F_b) \otimes_{O_{S_{12}}} (D_{S_{12}}, F^{ord}) \otimes_{D_{S_{12}}} (M, F).$$

(iii): Follows from (i) and (ii) by proposition 43 (ii) in the algebraic case and by proposition 46(ii) in the analytic case since a closed embedding is proper. \square

In the analytical case, we also have :

Proposition 65. (i) Let $i : Z \hookrightarrow S$ a closed embedding with $S, Z \in \text{AnSm}(\mathbb{C})$. Then for $(M, F) \in C_{\mathcal{D}^\infty fil}(Z)$, we have $\int_i (M, F) = i_* mod(M, F)$.

(ii) Let $S_1, S_2 \in \text{AnSm}(\mathbb{C})$ and $p : S_{12} := S_1 \times S_2 \rightarrow S_1$ be the projection. Then, for $(M, F) \in C_{\mathcal{D}^\infty fil}(S_1 \times S_2)$ we have

$$\begin{aligned} \int_p (M, F) : &= Rp_*((M, F) \otimes_{D_{S_1 \times S_2}^\infty}^L (D_{S_1 \times S_2 \leftarrow S_1}^\infty, F^{ord})) \\ &= p_* E((\Omega_{S_1 \times S_2/S_1}^\bullet, F_b) \otimes_{O_{S_1 \times S_2}} (D_{S_1 \times S_2}^\infty, F^{ord}) \otimes_{D_{S_1 \times S_2}} (M, F)[d_{S_2}]) \\ &= p_* E((\Omega_{S_1 \times S_2/S_1}^\bullet, F_b) \otimes_{O_{S_1 \times S_2}} (M, F)[d_{S_2}]) =: p_* mod(M, F). \end{aligned}$$

(ii)' Let $S_1, S_2 \in \text{AnSm}(\mathbb{C})$ and $p : S_{12} := S_1 \times S_2 \rightarrow S_1$ be the projection. Then, for $(M, F) \in C_{\mathcal{D}fil}(S_1 \times S_2)$ or $(M, F) \in C_{\mathcal{D}fil}(S_1 \times S_2)$, we have

$$\begin{aligned} \int_{p!} (M, F) : &= R p_!((M, F) \otimes_{D_{S_1 \times S_2}}^L (D_{S_1 \times S_2 \leftarrow S_1}, F^{ord})) \\ &= p_! E((\Omega_{S_1 \times S_2/S_1}^\bullet, F_b) \otimes_{O_{S_1 \times S_2}} (D_{S_1 \times S_2}, F^{ord}) \otimes_{D_{S_1 \times S_2}} (M, F)[d_{S_2}]) \\ &= p_! E((\Omega_{S_1 \times S_2/S_1}^\bullet, F_b) \otimes_{O_{S_1 \times S_2}} (M, F)[d_{S_2}]) =: p_{!mod}(M, F). \end{aligned}$$

(iii) Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. For $M \in C_{\mathcal{D}^\infty}(X)$, we have $\int_f^{FDR} M = \int_f M$ and $\int_{f!}^{FDR} M = \int_{f!} M$. For $M \in C_{\mathcal{D}}(X)$, we have $\int_{f!}^{FDR} M = \int_{f!} M$.

Proof. (i): Follows from the fact that $D_{Z \leftarrow S}^\infty$ is a locally free D_Z^∞ module and that i_* is an exact functor. (ii): Similar to the proof of proposition 64(ii): follows from proposition 63. (ii)': Similar to the proof of proposition 64(ii): follows from proposition 63. (iii): The first assertion follows from (i), (ii) and (ii)' by proposition 46. The second one follows from proposition 64(i) and (ii)' and by proposition 46. □

Proposition 66. Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphism with $X, Y, S \in \text{SmVar}(\mathbb{C})$.

(i) Let $(M, F) \in C_{\mathcal{D}(2)fil}(X)$. Then $\int_{f_2 \circ f_1!}^{FDR} (M, F) = \int_{f_1!}^{FDR} \int_{f_2!}^{FDR} (M, F) \in D_{\mathcal{D}(2)fil, \infty}(S)$.

(ii) Let $(M, F) \in C_{\mathcal{D}(2)fil, h}(X)$. Then $\int_{(f_2 \circ f_1)!}^{FDR} (M, F) = \int_{f_1!}^{FDR} \int_{f_2!}^{FDR} (M, F) \in D_{\mathcal{D}(2)fil, \infty}(S)$.

Proof. See [21]. □

Proposition 67. Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphism with $X, Y, S \in \text{AnSm}(\mathbb{C})$.

(i) Let $(M, F) \in C_{\mathcal{D}^\infty(2)fil}(X)$. Then $\int_{(f_2 \circ f_1)!}^{FDR} (M, F) = \int_{f_1!}^{FDR} \int_{f_2!}^{FDR} (M, F)$.

(ii) Let $(M, F) \in C_{\mathcal{D}^\infty(2)fil, h}(X)$. Then $\int_{f_2 \circ f_1!}^{FDR} (M, F) = \int_{f_1!}^{FDR} \int_{f_2!}^{FDR} (M, F)$.

Proof. Similar to proposition 66. □

Definition 64. (i) Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{SmVar}(\mathbb{C})$ or $X, S \in \text{AnSm}(\mathbb{C})$.

Consider the graph factorization $f : X \xrightarrow{l} X \times S \xrightarrow{p} S$, with l the graph embedding and p the projection. We have the transformation map given by, for $(M, F) \in C_{\mathcal{D}fil}(X)$,

$$T\left(\int_f^{FDR}, \int_f\right)(M, F) : \int_f^{FDR} (M, F) := \int_p \int_l (M, F) \xrightarrow{T(\int_p \circ \int_l, \int_{p \circ l})(M, F)} \int_f (M, F)$$

(ii) Let $j : S^\circ \hookrightarrow S$ an open embedding with $S \in \text{Var}(\mathbb{C})$. Consider the graph factorization $j : S^\circ \xrightarrow{l} S^\circ \times S \xrightarrow{p} S$, with l the graph embedding and p the projection. We have, for $(M, F) \in C_{\mathcal{D}fil}(S^\circ)$, the canonical map in $C_{\mathcal{D}fil}(S)$,

$$\begin{aligned} T(j_{*mod}^{FDR}, j_*)(M, F) : j_{*mod}^{FDR}(M, F) &:= p_* E((\Omega_{S^\circ \times S/S}^\bullet, F_b) \otimes_{O_{S^\circ \times S}} l_{*mod}(M, F)) \xrightarrow{k_{\omega}(S^\circ \times S/S)} \\ &p_* E((D_{S^\circ \times S \leftarrow S}, F^{ord}) \otimes_{D_{S^\circ \times S}} l_*(D_{S^\circ \leftarrow S^\circ \times S} \otimes_{D_{S^\circ}} E(M, F))) \xrightarrow{T(l, \otimes)(-, -)} j_* E(M, F) \end{aligned}$$

We have, for $(M, F) \in C_{\mathcal{D}fil}(S)$, the canonical map in $C_{\mathcal{D}fil}(S)$,

$$\text{ad}(j^*, j_{*mod}^{FDR})(M, F) : (M, F) \xrightarrow{\text{ad}(p^{*mod}, p_*)(M, F)} p_* E((\Omega_{S^\circ \times S/S}^\bullet, F_b) \otimes_{O_{S^\circ \times S}} p^{*mod}(M, F))$$

4.2.3 The support section functors for \mathbf{D} modules and the graph inverse image

Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$. Let $i : Z \hookrightarrow S$ a closed embedding and denote $j : S \setminus Z \hookrightarrow S$ the complementary open embedding. More generally, let $h : Y \rightarrow S$ a morphism with $Y, S \in \text{Var}(\mathbb{C})$ or $Y, S \in \text{AnSp}(\mathbb{C})$, S smooth, and let $i : X \hookrightarrow Y$ a closed embedding and denote by $j : Y \setminus X \hookrightarrow Y$ the open complementary. We then get from section 2 the following functors :

- We get the functor

$$\Gamma_Z : C_{\mathcal{D}(2)\text{fil}}(S) \rightarrow C_{\mathcal{D}(2)\text{fil}}(S),$$

$$(M, F) \mapsto \Gamma_Z(M, F) := \text{Cone}(\text{ad}(j^*, j_*)((M, F))) : (M, F) \rightarrow j_* j^*(M, F)[-1],$$

together we the canonical map $\gamma_Z(M, F) : \Gamma_Z(M, F) \rightarrow (M, F)$, and more generally the functor

$$\Gamma_X : C_{h^*\mathcal{D}(2)\text{fil}}(Y) \rightarrow C_{h^*\mathcal{D}(2)\text{fil}}(Y),$$

$$(M, F) \mapsto \Gamma_X(M, F) := \text{Cone}(\text{ad}(j^*, j_*)((M, F))) : (M, F) \rightarrow j_* j^*(M, F)[-1],$$

together we the canonical map $\gamma_X(M, F) : \Gamma_X(M, F) \rightarrow (M, F)$.

- We get the functor

$$\Gamma_Z^\vee : C_{\mathcal{D}(2)\text{fil}}(S) \rightarrow C_{\mathcal{D}(2)\text{fil}}(S),$$

$$(M, F) \mapsto \Gamma_Z^\vee(M, F) := \text{Cone}(\text{ad}(j_!, j^*)((M, F))) : j_! j^*(M, F) \rightarrow (M, F),$$

together we the canonical map $\gamma_Z^\vee(M, F) : (M, F) \rightarrow \Gamma_Z^\vee(M, F)$, and more generally the functor

$$\Gamma_X^\vee : C_{h^*\mathcal{D}(2)\text{fil}}(Y) \rightarrow C_{h^*\mathcal{D}(2)\text{fil}}(Y),$$

$$(M, F) \mapsto \Gamma_X^\vee(M, F) := \text{Cone}(\text{ad}(j_!, j^*)((M, F))) : j_! j^*(M, F) \rightarrow (M, F),$$

together we the canonical map $\gamma_X^\vee(M, F) : (M, F) \rightarrow \Gamma_X^\vee(M, F)$.

- We get the functor

$$\Gamma_Z^{\vee, h} : C_{\mathcal{D}(2)\text{fil}}(S) \rightarrow C_{\mathcal{D}(2)\text{fil}}(S), (M, F) \mapsto \Gamma_Z^{\vee, h}(M, F) := \mathbb{D}_S^K L_D \Gamma_Z E(\mathbb{D}_S^K(M, F)),$$

together with the factorization

$$\gamma_Z^{\vee, h}(L_D(M, F)) : L_D(M, F) \xrightarrow{\gamma_Z^\vee(L_D(M, F))} \Gamma_Z^\vee L_D(M, F) \xrightarrow{k \circ \mathbb{D}^K I(j_!, j^*)(-) \circ \text{od}(-)} \Gamma_Z^{\vee, h} L_D(M, F),$$

and more generally the functor

$$\Gamma_X^{\vee, h} : C_{h^*\mathcal{D}(2)\text{fil}}(Y) \rightarrow C_{h^*\mathcal{D}(2)\text{fil}}(Y), (M, F) \mapsto \Gamma_X^{\vee, h}(M, F) := \mathbb{D}_Y^{h^*D, K} L_{h^*D} \Gamma_X E(\mathbb{D}_Y^{h^*D, K}(M, F)),$$

together with the factorization

$$\gamma_X^{\vee, h}(L_{h^*D}(M, F)) : L_{h^*D}(M, F) \xrightarrow{\gamma_X^\vee(L_{h^*D}(M, F))} \Gamma_X^\vee L_{h^*D}(M, F)$$

$$\xrightarrow{k \circ \mathbb{D}^{h^*D, K} I(j_!, j^*)(-) \circ \text{od}(-)} \Gamma_X^{\vee, h} L_{h^*D}(M, F).$$

- We get the functor

$$\Gamma_Z^{\vee, O} : C_{\mathcal{D}(2)\text{fil}}(S) \rightarrow C_{\mathcal{D}(2)\text{fil}}(S),$$

$$(M, F) \mapsto \Gamma_Z^{\vee, O}(M, F) := \text{Cone}(b_Z((M, F))) : \mathcal{I}_Z(M, F) \rightarrow (M, F),$$

together we the factorization

$$\gamma_Z^{\vee, O}(M, F) : (M, F) \xrightarrow{\gamma_Z^{\vee, O}(M, F)} \Gamma_Z^{\vee}(M, F) \xrightarrow{b_{S/Z}(M, F)} \Gamma_Z^{\vee, O}(M, F).$$

Since $M \mapsto M/\mathcal{I}_Z M$ is a right exact functor, $M \mapsto \Gamma_Z^{\vee, O} M$ send Zariski, resp. usu, local equivalence between projective complexes of presheaves to Zariski, resp. usu local equivalence, and thus induces in the derived category

$$\begin{aligned} L\Gamma_Z^{\vee, O} : D_{\mathcal{D}fil, \infty}(S) &\rightarrow D_{\mathcal{D}fil, \infty}(S), \\ (M, F) \mapsto \Gamma_Z^{\vee, O} L_D(M, F) &:= \text{Cone}(b_Z(L_D(M, F))) : \mathcal{I}_Z L_D(M, F) \rightarrow L_D(M, F). \end{aligned}$$

- We get the functor

$$\begin{aligned} \Gamma_Z^O : C_{\mathcal{D}(2)fil}(S) &\rightarrow C_{\mathcal{D}(2)fil}(S), \\ (M, F) \mapsto \Gamma_Z^O(M, F) &:= \text{Cone}(b'_Z((M, F))) : (M, F) \rightarrow (M, F) \otimes_{O_S} \mathbb{D}_S^K(\mathcal{I}_Z D_S), \end{aligned}$$

together we the factorization

$$\gamma_Z^O(M, F) : (M, F) \Gamma_Z^O \xrightarrow{b'_{S/Z}(M, F)} \Gamma_Z(M, F) \xrightarrow{\gamma_Z(M, F)} (M, F).$$

- We have, for $(M, F) \in C_{\mathcal{D}fil}(S)$, a canonical isomorphism

$$I(D, \gamma^O)(M, F) : \mathbb{D}_S^K \Gamma_Z^{\vee, O}(M, F) \xrightarrow{\sim} \Gamma_Z^O \mathbb{D}_S^K(M, F)$$

which gives the transformation map in $C_{\mathcal{D}fil}(S)$

$$T(D, \gamma^O)(M, F) : \Gamma_Z^{\vee, O} \mathbb{D}_S^K(M, F) \rightarrow \mathbb{D}_S^K \Gamma_Z^O(M, F)$$

Let $S \in \text{AnSm}(\mathbb{C})$. Let $i : Z \hookrightarrow S$ a closed embedding and denote $j : S \setminus Z \hookrightarrow S$ the complementary open embedding. More generally, let $h : Y \rightarrow S$ a morphism with $Y, S \in \text{AnSp}(\mathbb{C})$, S smooth, and let $i : X \hookrightarrow Y$ a closed embedding and denote by $j : Y \setminus X \hookrightarrow Y$ the open complementary.

- We get the functor

$$\begin{aligned} \Gamma_Z : C_{\mathcal{D}^\infty(2)fil}(S) &\rightarrow C_{\mathcal{D}^\infty(2)fil}(S), \\ (M, F) \mapsto \Gamma_Z(M, F) &:= \text{Cone}(\text{ad}(j^*, j_*))((M, F)) : (M, F) \rightarrow j_* j^*(M, F)[-1], \end{aligned}$$

together we the canonical map $\gamma_Z(M, F) : \Gamma_Z(M, F) \rightarrow (M, F)$, and more generally the functor

$$\begin{aligned} \Gamma_X : C_{h^* \mathcal{D}^\infty(2)fil}(Y) &\rightarrow C_{h^* \mathcal{D}^\infty(2)fil}(Y), \\ (M, F) \mapsto \Gamma_X(M, F) &:= \text{Cone}(\text{ad}(j^*, j_*))((M, F)) : (M, F) \rightarrow j_* j^*(M, F)[-1], \end{aligned}$$

together we the canonical map $\gamma_X(M, F) : \Gamma_X(M, F) \rightarrow (M, F)$.

- We get the functor

$$\begin{aligned} \Gamma_Z^\vee : C_{\mathcal{D}^\infty(2)fil}(S) &\rightarrow C_{\mathcal{D}^\infty(2)fil}(S), \\ (M, F) \mapsto \Gamma_Z^\vee(M, F) &:= \text{Cone}(\text{ad}(j_!, j^*))((M, F)) : j_! j^*(M, F) \rightarrow (M, F), \end{aligned}$$

together we the canonical map $\gamma_Z^\vee(M, F) : (M, F) \rightarrow \Gamma_Z^\vee(M, F)$, and more generally the functor

$$\begin{aligned} \Gamma_X^\vee : C_{h^* \mathcal{D}^\infty(2)fil}(Y) &\rightarrow C_{h^* \mathcal{D}^\infty(2)fil}(Y), \\ (M, F) \mapsto \Gamma_X^\vee(M, F) &:= \text{Cone}(\text{ad}(j_!, j^*))((M, F)) : j_! j^*(M, F) \rightarrow (M, F), \end{aligned}$$

together we the canonical map $\gamma_X^\vee(M, F) : (M, F) \rightarrow \Gamma_X^\vee(M, F)$.

- We get the functor

$$\Gamma_Z^{\vee,h} : C_{\mathcal{D}^\infty(2)fil}(S) \rightarrow C_{\mathcal{D}^\infty(2)fil}(S), (M, F) \mapsto \Gamma_Z^{\vee,h}(M, F) := \mathbb{D}_S^{\infty,K} L_{D^\infty} \Gamma_Z E(\mathbb{D}_S^{\infty,K}(M, F)),$$

together with the factorization

$$\gamma_Z^{\vee,h}(L_{D^\infty}(M, F)) : L_{D^\infty}(M, F) \xrightarrow{\gamma_Z^{\vee}(L_{D^\infty}(M, F))} \Gamma_Z^{\vee} L_{D^\infty}(M, F) \xrightarrow{k \circ \mathbb{D}^\infty I(j!, j^*)(-) \circ d(-)} \Gamma_Z^{\vee,h} L_{D^\infty}(M, F),$$

and more generally the functor

$$\Gamma_X^{\vee,h} : C_{h^* \mathcal{D}^\infty(2)fil}(Y) \rightarrow C_{h^* \mathcal{D}^\infty(2)fil}(Y), (M, F) \mapsto \Gamma_X^{\vee,h}(M, F) := \mathbb{D}_Y^{h^* \infty, K} L_{h^* D^\infty} \Gamma_X E(\mathbb{D}_Y^{h^* \infty, K}(M, F)),$$

together with the factorization

$$\begin{aligned} \gamma_X^{\vee,h}(L_{h^* D^\infty}(M, F)) : L_{h^* D^\infty}(M, F) &\xrightarrow{\gamma_X^{\vee}(L_{h^* D^\infty}(M, F))} \Gamma_X^{\vee} L_{h^* D^\infty}(M, F) \\ &\xrightarrow{k \circ \mathbb{D}^{h^* D^\infty, K} I(j!, j^*)(-) \circ d(-)} \Gamma_X^{\vee,h} L_{h^* D^\infty}(M, F). \end{aligned}$$

- We get the functor

$$\begin{aligned} \Gamma_Z^{\vee,O} : C_{\mathcal{D}^\infty(2)fil}(S) &\rightarrow C_{\mathcal{D}^\infty(2)fil}(S), \\ (M, F) &\mapsto \Gamma_Z^{\vee,O}(M, F) := \text{Cone}(b_Z((M, F)) : \mathcal{I}_Z(M, F) \rightarrow (M, F)), \end{aligned}$$

together we the factorization

$$\gamma_Z^{\vee,O}(M, F) : (M, F) \xrightarrow{\gamma_Z^{\vee}(M, F)} \Gamma_Z^{\vee}(M, F) \xrightarrow{b_{S/Z}(M, F)} \Gamma_Z^{\vee,O}(M, F).$$

- We get the functor

$$\begin{aligned} \Gamma_Z^O : C_{\mathcal{D}^\infty(2)fil}(S) &\rightarrow C_{\mathcal{D}^\infty(2)fil}(S), \\ (M, F) &\mapsto \Gamma_Z^O(M, F) := \text{Cone}(b'_Z((M, F)) : (M, F) \rightarrow (M, F) \otimes_{O_S} \mathbb{D}_S^K(\mathcal{I}_Z D_S)), \end{aligned}$$

together we the factorization

$$\gamma_Z^O(M, F) : (M, F) \Gamma_Z^O \xrightarrow{b'_{S/Z}(M, F)} \Gamma_Z(M, F) \xrightarrow{\gamma_Z(M, F)} (M, F).$$

- We have, for $(M, F) \in C_{\mathcal{D}^\infty fil}(S)$, a canonical isomorphism

$$I(D, \gamma^O)(M, F) : \mathbb{D}_S^{K, \infty} \Gamma_Z^{\vee,O}(M, F) \xrightarrow{\sim} \Gamma_Z^O \mathbb{D}_S^{K, \infty}(M, F)$$

which gives the transformation map in $C_{\mathcal{D}^\infty fil}(S)$

$$T(D, \gamma^O)(M, F) : \Gamma_Z^{\vee,O} \mathbb{D}_S^{\infty, K}(M, F) \rightarrow \mathbb{D}_S^{\infty, K} \Gamma_Z^O(M, F)$$

In the analytic case, we have

Definition 65. Let $S \in \text{AnSm}(\mathbb{C})$. For $(M, F) \in C_{\mathcal{D}^\infty fil}(S)$, we have the map in $C_{\mathcal{D}^\infty fil}(S)$

$$\begin{aligned} T(\infty, \gamma)(M, F) &:= (I, T(j, \otimes)(-, -)) : \\ J_S(\Gamma_Z(M, F)) &:= \Gamma_Z(M, F) \otimes_{D_S} (D_S^\infty, F^{ord}) \rightarrow \Gamma_Z((M, F) \otimes_{D_S} (D_S^\infty, F^{ord})) =: \Gamma_Z J_S(M, F) \end{aligned}$$

Let $i : Z \hookrightarrow S$ a closed embedding, with $Z, S \in \text{SmVar}(\mathbb{C})$ or $Z, S \in \text{AnSm}(\mathbb{C})$. We have the functor

$$i^\sharp : C_{\mathcal{D}^\infty fil}(S) \rightarrow C_{\mathcal{D}^\infty fil}(Z), (M, F) \mapsto i^\sharp(M, F) := \mathcal{H}om_{i^* D_S}((D_{S \leftarrow Z}, F^{ord}), i^*(M, F))$$

where the (left) D_Z module structure on $i^\sharp M$ comes from the right module structure on $D_{S \leftarrow Z}$, resp. O_Z . We denote by

- for $(M, F) \in C_{\mathcal{D}fil}(S)$, the canonical map in $C_{\mathcal{D}fil}(S)$

$$\text{ad}(i_{*mod}, i^\sharp)(M, F) : i_{*mod}i^\sharp(M, F) := i_*(\mathcal{H}om_{i^*D_S}((D_{S \leftarrow Z}, F^{ord}), i^*(M, F)) \otimes_{D_Z} (D_{S \leftarrow Z}, F^{ord})) \rightarrow (M, F), \phi \otimes P \mapsto \phi(P)$$
- for $(N, F) \in C_{\mathcal{D}fil}(Z)$, the canonical map in $C_{\mathcal{D}fil}(Z)$

$$\text{ad}(i_{*mod}, i^\sharp)(N, F) : (N, F) \rightarrow i^\sharp i_{*mod}(N, F) := \mathcal{H}om_{i^*D_S}(D_{S \leftarrow Z}, i^*i_*(N, F) \otimes_{D_Z} (D_{S \leftarrow Z}, F^{ord})) \rightarrow (N, F), n \mapsto (P \mapsto n \otimes P)$$

The functor i^\sharp induces in the derived category the functor :

$$Ri^\sharp : D_{\mathcal{D}(2)fil,r}(S) \rightarrow D_{\mathcal{D}(2)fil,r}(Z),$$

$$(M, F) \mapsto Ri^\sharp(M, F) := R\mathcal{H}om_{i^*D_S}((D_{Z \leftarrow S}, F^{ord}), i^*(M, F)) = \mathcal{H}om_{i^*D_S}((D_{Z \leftarrow S}, F^{ord}), E(i^*(M, F))).$$

Proposition 68. *Let $i : Z \hookrightarrow S$ a closed embedding, with $Z, S \in \text{SmVar}(\mathbb{C})$ or $Z, S \in \text{AnSm}(\mathbb{C})$. The functor $i_{*mod} : C_{\mathcal{D}}(Z) \rightarrow C_{\mathcal{D}}(S)$ admit a right adjoint which is the functor $i^\sharp : C_{\mathcal{D}}(S) \rightarrow C_{\mathcal{D}}(Z)$ and*

$$\text{ad}(i_{*mod}, i^\sharp)(N) : N \rightarrow i^\sharp i_{*mod}N \quad \text{and} \quad \text{ad}(i_{*mod}, i^\sharp)(M) : i_{*mod}i^\sharp M \rightarrow M$$

are the adjunction maps.

Proof. See [16] for the algebraic case. The analytic case is completely analogue. \square

One of the main results in D modules is Kashiwara equivalence :

Theorem 18. *Let $i : Z \hookrightarrow S$ a closed embedding with $Z, S \in \text{SmVar}(\mathbb{C})$, or with $Z, S \in \text{AnSm}(\mathbb{C})$.*

- (i) *The functor $i_{*mod} : \mathcal{Q}Coh_{\mathcal{D}}(Z) \rightarrow \mathcal{Q}Coh_{\mathcal{D}}(S)$ is an equivalence of category whose inverse is $i^\sharp := a_\tau i^\sharp : \mathcal{Q}Coh_{\mathcal{D}}(S) \rightarrow \mathcal{Q}Coh_{\mathcal{D}}(Z)$. That is, for $M \in \mathcal{Q}Coh_{\mathcal{D}}(S)$ and $N \in \mathcal{Q}Coh_{\mathcal{D}}(Z)$, the adjunction maps*

$$\text{ad}(i_{*mod}, i^\sharp)(M) : i_{*mod}i^\sharp M \xrightarrow{\sim} M, \quad \text{ad}(i_{*mod}, i^\sharp)(N) : i^\sharp i_{*mod}N \xrightarrow{\sim} N$$

are isomorphisms.

- (ii) *The functor $\int_i = i_{*mod} : D_{\mathcal{D},c}(Z) \rightarrow D_{\mathcal{D},c}(S)$ is an equivalence of category whose inverse is $Ri^\sharp : D_{\mathcal{D},c}(S) \rightarrow D_{\mathcal{D},c}(Z)$. That is, for $M \in D_{\mathcal{D},c}(S)$ and $N \in D_{\mathcal{D},c}(Z)$, the adjunction maps*

$$\text{ad}(\int_i, Ri^\sharp)(M) : \int_i Ri^\sharp M \xrightarrow{\sim} M, \quad \text{ad}(\int_i, Ri^\sharp)(N) : Ri^\sharp \int_i N \xrightarrow{\sim} N$$

are isomorphisms.

We have a canonical embedding of rings $D_Z \hookrightarrow D_{Z \rightarrow S} := i^*D_S \otimes_{i^*O_S} O_Z$. We denote by $C_{i^*\mathcal{D},Z}(Z)$ the category whose objects are complexes of presheaves M of i^*D_S modules on Z such that the cohomology presheaves $H^n M$ have an induced structure of D_Z modules. We denote by

$$q_K : K_{O_S}(i_*O_Z) \rightarrow i_*O_Z$$

the Kozcul complex which is a resolution of the O_S module i_*O_Z of length $c = \text{codim}(Z, S)$ by locally free sheaves of finite rank. The fact that it is a locally free resolution of finite rank comes from the fact that Z is a locally complete intersection in S since both Z and S are smooth. We denote again

$$q_K = i^*q_K : K_{i^*O_S}(O_Z) := i^*K_{O_S}(i_*O_Z) \rightarrow i^*i_*O_Z = O_Z$$

We denote by $K_{i^*O_S}^\vee(O_Z) := \mathcal{H}om_{i^*O_S}(K_{i^*O_S}(O_Z), i^*O_S)$ its dual, so that we have a canonical map

$$q_K^\vee : K_{i^*O_S}^\vee(O_Z) \rightarrow O_Z[-c].$$

Let $M \in C_{\mathcal{D}}(S)$. The i^*D_S module structure on $\mathcal{H}om_{i^*O_S}(K_{i^*O_S}(O_Z), i^*M)$ and $K_{i^*O_S}(O_Z) \otimes_{i^*O_S} i^*M$ induce a canonical D_Z module structure on the cohomology groups $H^n \mathcal{H}om_{i^*O_S}(K_{i^*O_S}(O_Z), i^*M)$ and $H^n(K_{i^*O_S}(O_Z) \otimes_{i^*O_S} i^*M)$ for all $n \in \mathbb{Z}$.

The projection formula for ringed spaces (proposition 9) implies the following lemma :

Lemma 9. *Let $i : Z \hookrightarrow S$ a closed embedding with $Z, S \in \text{Var}(\mathbb{C})$ or with $Z, S \in \text{AnSp}(\mathbb{C})$. Denote by $j : U := S \setminus Z \hookrightarrow Z$ the open complementary embedding. Then, if i is a locally complete intersection embedding (e.g. if Z, S are smooth), we have for $M \in C_{O_U}(U)$, $Li^{*mod}Rj_*M = 0$.*

Proof. We have

$$\begin{aligned} i_*Li^{*mod}Rj_*M &:= i_*(i^*L_O(j_*E(M)) \otimes_{i^*O_S} O_Z) \xrightarrow{T(i, \otimes)(L_O(j_*E(M)), O_Z)^{-1}} L_O(j_*E(M)) \otimes_{O_S} i_*O_Z \\ &\xrightarrow{q \circ (i_*q_K)^{-1}} (j_*E(M)) \otimes_{O_S} i_*K_{i^*O_S}(O_Z) \xrightarrow{T(j, \otimes)(E(M), K_{O_S}(i_*O_Z))} j_*(E(M) \otimes_{O_U} j^*K_{O_S}(i_*O_Z)), \end{aligned}$$

$T(i, \otimes)(L_O(j_*E(M)), O_Z)$ being an equivalence Zariski, resp. usu, local by proposition 10 and follows from the fact that $j^*K_{O_S}(i_*O_Z)$ is acyclic. But

$$T(j, \otimes)(E(M), K_{O_S}(i_*O_Z)) : (j_*E(M)) \otimes_{O_S} K_{O_S}(i_*O_Z) \rightarrow j_*(E(M) \otimes_{O_U} j^*i_*K_{i^*O_S}(O_Z))$$

is an equivalence Zariski, resp. usu, local by proposition 9 since $K_{O_S}(i_*O_Z)$ is a finite complex of locally free O_S modules of finite rank. \square

Definition-Proposition 14. *Let $i : Z \hookrightarrow S$ a closed embedding with $Z, S \in \text{SmVar}(\mathbb{C})$. Denote by $c = \text{codim}(Z, S)$. Then, for $M \in C_{\mathcal{D}, c}(S)$, we have by Kashiwara equivalence the following map in $C_{\mathcal{D}}(S)$:*

$$\begin{aligned} \mathcal{K}_{Z/S}(M) : \Gamma_Z E(M) &\xrightarrow{\text{ad}(i_{*mod}, i^\sharp)(-)^{-1}} i_{*mod}i^\sharp \Gamma_Z E(M) \\ \xrightarrow{\gamma_Z(-)} i_{*mod}i^\sharp(E(M)) &\xrightarrow{\mathcal{H}om(q_K, E(i^*M)) \circ \mathcal{H}om(O_Z, T(i, E)(M))} i_{*mod}K_{i^*O_S}^\vee(O_Z) \otimes_{i^*O_S} M \end{aligned}$$

which is an equivalence Zariski, resp. usu, local. It gives the isomorphism in $D_{\mathcal{D}}(S)$

$$\mathcal{K}_{Z/S}(M) : R\Gamma_Z M \rightarrow i_{*mod}K_{i^*O_S}^\vee(O_Z) = i_{*mod}Li^{*mod}M[c]$$

Proof. Follows from theorem 18 lemma 9 : see [16] for the algebraic case, the proof in the analytic case is completely similar. \square

Definition 66. *Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{SmVar}(\mathbb{C})$ or $X, S \in \text{AnSm}(\mathbb{C})$. Consider the factorization $f : X \xrightarrow{i} X \times S \xrightarrow{p_S} S$, where i is the graph embedding and $p_S : X \times S \rightarrow S$ is the projection.*

(i) *Then, for $(M, F) \in C_{\mathcal{D}(2)fil}(S)$ we set*

$$f^{*mod[-], \Gamma}(M, F) := \Gamma_X E(p_S^{*mod[-]}(M, F)) \in C_{\mathcal{D}(2)fil, \infty}(X \times S),$$

It induces in the derived category

$$Rf^{*mod[-], \Gamma}(M, F) := f^{*mod[-], \Gamma}(M, F) := \Gamma_X E(p_S^{*mod[-]}(M, F)) \in D_{\mathcal{D}(2)fil, \infty}(X \times S),$$

*By definition-proposition 14, we have in the algebraic case $Li^{*mod}f^{*mod, \Gamma}M = Lf^{*mod}M \in D_{\mathcal{D}}(X)$.*

(ii) *Then, for $(M, F) \in C_{\mathcal{D}(2)fil}(S)$ we set*

$$Lf^{\hat{*}mod[-], \Gamma}(M, F) := \Gamma_X^{\vee, h} L_D p_S^{*mod[-]}(M, F) := \mathbb{D}_S^K L_D \Gamma_X E(\mathbb{D}_S^K L_D p_S^{*mod[-]}(M, F)) \in D_{\mathcal{D}(2)fil, \infty}(X \times S).$$

In the analytical case we also have

Definition 67. *Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{AnSm}(\mathbb{C})$. Consider the factorization $f : X \xrightarrow{i} X \times S \xrightarrow{p_S} S$, where i is the graph embedding and $p_S : X \times S \rightarrow S$ is the projection.*

(i) Then, for $(M, F) \in C_{\mathcal{D}^\infty(2)fil}(S)$ we set

$$f^{*mod[-],\Gamma}(M, F) := \Gamma_X E(p_S^{*mod[-]}(M, F)) \in C_{\mathcal{D}^\infty(2)fil,\infty}(X \times S),$$

It induces in the derived category

$$Rf^{*mod[-],\Gamma}(M, F) := f^{*mod[-],\Gamma}(M, F) := \Gamma_X E(p_S^{*mod[-]}(M, F)) \in D_{\mathcal{D}^\infty(2)fil,\infty}(X \times S),$$

(ii) Then, for $(M, F) \in C_{\mathcal{D}^\infty(2)fil}(S)$ we set

$$Lf^{\hat{*}mod[-],\Gamma}(M, F) := \Gamma_X^{\vee,h} L_D p_S^{*mod[-]}(M, F) := \mathbb{D}_S^K L_D \Gamma_X E(\mathbb{D}_S^K L_D p_S^{*mod[-]}(M, F)) \in D_{\mathcal{D}^\infty(2)fil,\infty}(X \times S).$$

4.2.4 The 2 functors and transformations maps for D modules on the smooth complex algebraic varieties and the complex analytic manifolds

By the definitions and the propositions 43, 44, 66, for the algebraic case, and the propositions 46, 47, 67, for the analytic case,

- we have the 2 functors on $\text{SmVar}(\mathbb{C})$:

- $C_{\mathcal{D}(2)fil}(\cdot) : \text{SmVar}(\mathbb{C}) \rightarrow C_{\mathcal{D}(2)fil}(\cdot)$, $S \mapsto C_{\mathcal{D}(2)fil}(S)$, $(f : T \rightarrow S) \mapsto f^{*mod}$, $(f : T \rightarrow S) \mapsto f^{*mod[-]}$,
- $D_{\mathcal{D}(2)fil,r}(\cdot) : \text{SmVar}(\mathbb{C}) \rightarrow D_{\mathcal{D}(2)fil,r}(\cdot)$, $S \mapsto D_{\mathcal{D}(2)fil,r}(S)$, $(f : T \rightarrow S) \mapsto Lf^{*mod}$, $(f : T \rightarrow S) \mapsto Lf^{*mod[-]}$,
- $D_{\mathcal{D}(2)fil,\infty}(\cdot) : \text{SmVar}(\mathbb{C}) \rightarrow D_{\mathcal{D}(2)fil,\infty}(\cdot)$, $S \mapsto D_{\mathcal{D}(2)fil,\infty}(S)$, $(f : T \rightarrow S) \mapsto \int_f^{FDR}$,

- we have the 2 functors on $\text{AnSm}(\mathbb{C})$:

- $C_{\mathcal{D}(2)fil}(\cdot) : \text{AnSm}(\mathbb{C}) \rightarrow C_{\mathcal{D}(2)fil}(\cdot)$, $S \mapsto C_{\mathcal{D}(2)fil}(S)$, $(f : T \rightarrow S) \mapsto f^{*mod}$, $(f : T \rightarrow S) \mapsto f^{*mod[-]}$,
- $D_{\mathcal{D}(2)fil,r}(\cdot) : \text{AnSm}(\mathbb{C}) \rightarrow D_{\mathcal{D}(2)fil,r}(\cdot)$, $S \mapsto D_{\mathcal{D}(2)fil,r}(S)$, $(f : T \rightarrow S) \mapsto Lf^{*mod}$, $(f : T \rightarrow S) \mapsto Lf^{*mod[-]}$,
- $D_{\mathcal{D}(2)fil,\infty}(\cdot) : \text{AnSm}(\mathbb{C}) \rightarrow D_{\mathcal{D}(2)fil,\infty}(\cdot)$, $S \mapsto D_{\mathcal{D}(2)fil,\infty}(S)$, $(f : T \rightarrow S) \mapsto \int_{f!}^{FDR}$,

- we have also the 2 functors on $\text{AnSm}(\mathbb{C})$:

- $C_{\mathcal{D}^\infty(2)fil}(\cdot) : \text{AnSm}(\mathbb{C}) \rightarrow C_{\mathcal{D}^\infty(2)fil}(\cdot)$, $S \mapsto C_{\mathcal{D}^\infty(2)fil}(S)$, $(f : T \rightarrow S) \mapsto f^{*mod}$, $(f : T \rightarrow S) \mapsto f^{*mod[-]}$,
- $D_{\mathcal{D}^\infty(2)fil,r}(\cdot) : \text{AnSm}(\mathbb{C}) \rightarrow D_{\mathcal{D}^\infty(2)fil,r}(\cdot)$, $S \mapsto D_{\mathcal{D}^\infty(2)fil,r}(S)$, $(f : T \rightarrow S) \mapsto Lf^{*mod}$, $(f : T \rightarrow S) \mapsto Lf^{*mod[-]}$,
- $D_{\mathcal{D}^\infty(2)fil,r}(\cdot) : \text{AnSm}(\mathbb{C}) \rightarrow D_{\mathcal{D}^\infty(2)fil,r}(\cdot)$, $S \mapsto D_{\mathcal{D}^\infty(2)fil,r}(S)$, $(f : T \rightarrow S) \mapsto \int_{f!}^{FDR}$,

inducing the following commutative diagrams of functors :

$$\begin{array}{ccc} \text{SmVar}(\mathbb{C}) & \xrightarrow{f \mapsto f^{*mod}} & C_{\mathcal{D}(2)fil}(\cdot) \\ \downarrow \text{An} & & \downarrow \text{an} \\ \text{AnSm}(\mathbb{C}) & \xrightarrow{f \mapsto f^{*mod}} & C_{\mathcal{D}(2)fil}(\cdot) \xrightarrow{J} C_{\mathcal{D}^\infty(2)fil}(\cdot) \end{array}, \quad \begin{array}{ccc} \text{SmVar}(\mathbb{C}) & \xrightarrow{f \mapsto Lf^{*mod}} & D_{\mathcal{D}(2)fil,r}(\cdot) \\ \downarrow \text{An} & & \downarrow \text{an} \\ \text{AnSm}(\mathbb{C}) & \xrightarrow{f \mapsto Lf^{*mod}} & D_{\mathcal{D}(2)fil,r}(\cdot) \xrightarrow{J} D_{\mathcal{D}^\infty(2)fil,r}(\cdot) \end{array},$$

where, for $S \in \text{AnSm}(\mathbb{C})$,

- $D_{\mathcal{D}(2)fil,\infty,rh}(S) \subset D_{\mathcal{D}(2)fil,\infty,h}(S)$ is the full subcategory consisting of filtered complexes of D_S module whose cohomology sheaves are regular holonomic,
- $J : C_{\mathcal{D}(2)fil}(S) \rightarrow C_{\mathcal{D}^\infty(2)fil}(S)$ is the functor $(M, F) \mapsto J(M, F) := (M, F) \otimes_{D_S} D_S^\infty$, which derive trivially.

We first look at the pullback map and the transformation map of De Rahm complexes (see definition 12 and definition-proposition 13) together with the support section functor :

Proposition 69. *Consider a commutative diagram and a factorization*

$$D_0 = \begin{array}{ccc} X & \xrightarrow{f} & S \\ g' \uparrow & & \uparrow g \\ X' & \xrightarrow{f'} & T \end{array} \quad D_0 = \begin{array}{ccccc} f : X & \xrightarrow{i} & Y \times S & \xrightarrow{p} & S \\ g' \uparrow & & \uparrow g''=I \times g & & \uparrow g \\ f' : X' & \xrightarrow{i'} & Y \times T & \xrightarrow{p'} & T \end{array}$$

with $X, X', Y, S, T \in \text{Var}(\mathbb{C})$ or $X, X', Y, S, T \in \text{AnSp}(\mathbb{C})$, i, i' being closed embeddings, and p, p' the projections. Denote by D the right square of D_0 . We have a factorization $i' : X' \xrightarrow{i'_1} X_T = X \times_{Y \times S} Y \times T \xrightarrow{i'_0} Y \times T$, where i'_0, i'_1 are closed embedding. Assume S, T, Y, Y' are smooth.

(i) For $(M, F) \in C_{\mathcal{D}fil}(Y \times S)$, the canonical map in $C_{p'^*O_Tfil}(Y \times T)$ (c.f. definition-proposition 13),

$$E(\Omega_{((Y' \times T)/(X \times S))/(T/S)}(M, F)) \circ T(g'', E)(-) \circ T(g'', \gamma)(-) : \\ g''^* \Gamma_{XE}((\Omega_{Y \times S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M, F)) \rightarrow \Gamma_{X_T} E((\Omega_{Y \times T/T}^\bullet, F_b) \otimes_{O_{Y \times T}} g''^* \text{mod}(M, F))$$

is a map in $C_{p'^*\mathcal{D}fil}(Y \times T)$.

(ii) For $(M, F) \in C_{\mathcal{D}fil}(Y \times S)$, the canonical map in $C_{O_Tfil}(T)$ (c.f. definition-proposition 13 with L_D instead of L_O)

$$T_\omega^O(D)(M, F)^\gamma : g^* \text{mod} L_{Dp_*} \Gamma_{XE}(\Omega_{Y \times S}^\bullet \otimes_{O_{Y \times S}} (M, F)) \rightarrow p'_* \Gamma_{X_T} E(\Omega_{Y \times T/T}^\bullet \otimes_{O_{Y \times T}} g''^* \text{mod}(M, F))$$

is a map in $C_{\mathcal{D}fil}(T)$.

(iii) For $(N, F) \in C_{\mathcal{D}fil}(Y \times T)$, the canonical map in $C_{p'^*O_Tfil}(Y \times T)$

$$T(X'/X_T, \gamma)(-) : \Gamma_{X'} E((\Omega_{Y \times T/T}^\bullet, F_b) \otimes_{O_{Y \times T}} (N, F)) \rightarrow \Gamma_{X_T} E((\Omega_{Y \times T/T}^\bullet, F_b) \otimes_{O_{Y \times T}} (N, F))$$

is a map in $C_{p'^*\mathcal{D}_0fil}(Y \times T)$.

(iv) For $M = O_Y$, we have $T_\omega^O(D)(O_{Y \times S})^\gamma = T_\omega^O(D)^\gamma$ as complexes of D_T modules and $T_\omega^O(X_T/Y \times T)(O_{Y \times T})^\gamma = T_\omega^O(X_T/Y \times T)^\gamma$ as complexes of p'^*D_T modules.

Proof. Follows by definition from proposition 57. □

In the analytical case, we also have

Proposition 70. *Consider a commutative diagram and a factorization*

$$D_0 = \begin{array}{ccc} X & \xrightarrow{f} & S \\ g' \uparrow & & \uparrow g \\ X' & \xrightarrow{f'} & T \end{array} \quad D_0 = \begin{array}{ccccc} f : X & \xrightarrow{i} & Y \times S & \xrightarrow{p} & S \\ g' \uparrow & & \uparrow g''=I \times g & & \uparrow g \\ f' : X' & \xrightarrow{i'} & Y \times T & \xrightarrow{p'} & T \end{array}$$

with $X, X', Y, S, T \in \text{AnSp}(\mathbb{C})$, i, i' being closed embeddings, and p, p' the projections. Denote by D the right square of D_0 . We have a factorization $i' : X' \xrightarrow{i'_1} X_T = X \times_{Y \times S} Y \times T \xrightarrow{i'_0} Y \times T$, where i'_0, i'_1 are closed embedding. Assume S, T, Y, Y' are smooth.

(i) For $(M, F) \in C_{\mathcal{D}^\infty \text{fil}}(Y \times S)$, the canonical map in $C_{p'^* O_T \text{fil}}(Y \times T)$ (c.f. definition-proposition 13),

$$E(\Omega_{((Y' \times T)/(X \times S))/(T/S)}(M, F)) \circ T(g'', E)(-) \circ T(g'', \gamma)(-) : \\ g''^* \Gamma_X E((\Omega_{Y \times S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M, F)) \rightarrow \Gamma_{X_T} E((\Omega_{Y \times T/T}^\bullet, F_b) \otimes_{O_{Y \times T}} g''^* \text{mod}(M, F))$$

is a map in $C_{p'^* \mathcal{D}^\infty \text{fil}}(Y \times T)$.

(ii) For $(M, F) \in C_{\mathcal{D}^\infty \text{fil}}(Y \times S)$, the canonical map in $C_{O_T \text{fil}}(T)$ (c.f. definition-proposition 13 with $L_{\mathcal{D}^\infty}$ instead of L_O)

$$T_\omega^O(D)(M, F)^\gamma : g^* \text{mod} L_{\mathcal{D}^\infty} p_* \Gamma_X E((\Omega_{Y \times S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M, F)) \rightarrow p'_* \Gamma_{X_T} E((\Omega_{Y \times T/T}^\bullet, F_b) \otimes_{O_{Y \times T}} g''^* \text{mod}(M, F))$$

is a map in $C_{\mathcal{D}^\infty \text{fil}}(T)$.

(iii) For $(N, F) \in C_{\mathcal{D}^\infty \text{fil}}(Y \times T)$, the canonical map in $C_{p'^* O_T \text{fil}}(Y \times T)$

$$T(X'/X_T, \gamma)(-) : \Gamma_{X'} E(\Omega_{Y \times T/T}^\bullet \otimes_{O_{Y \times T}} (N, F)) \rightarrow \Gamma_{X_T} E(\Omega_{Y \times T/T}^\bullet \otimes_{O_{Y \times T}} (N, F))$$

is a map in $C_{p'^* \mathcal{D}^\infty \text{fil}}(Y \times T)$.

(iv) For $M = O_Y$, we have $T_\omega^O(D)(O_{Y \times S})^\gamma = T_\omega^O(D)^\gamma$ as complexes of D_T^∞ modules and $T_\omega^O(X_T/Y \times T)(O_{Y \times T})^\gamma = T_\omega^O(X_T/Y \times T)^\gamma$. as complexes of $p^* D_T^\infty$ modules.

Proof. Follows from proposition 69. \square

Similarly, we have :

Proposition 71. Let $p : Y \times S \rightarrow S$ a projection and $i : X \hookrightarrow Y \times S$ a closed embedding with $S, Y \in \text{SmVar}(\mathbb{C})$.

(i) For $(M, F) \in C_{\mathcal{D} \text{fil}}(Y \times S)$ the canonical map in $C_{p^* O_S \text{fil}}(Y^{an} \times S^{an})$ (see definition-proposition 13)

$$E(\Omega_{(Y^{an} \times S^{an}/Y \times S)/(S^{an}/S)}(M, F)) \circ T(an, \gamma)(-) : \\ (\Gamma_X E((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_Y} (M, F)))^{an} \rightarrow \Gamma_{X^{an}} E((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y^{an} \times S^{an}}} (M, F)^{an})$$

is a map in $C_{h^* \mathcal{D} \text{fil}}(Y^{an} \times S)$.

(ii) For $(M, F) \in C_{\mathcal{D} \text{fil}}(Y \times S)$ the canonical map in $C_{O_S \text{fil}}(S^{an})$ (see definition-proposition 13)

$$T_\omega^O(an, p)(M, F)^\gamma : (p_* \Gamma_X E((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_Y} (M, F)))^{an} \rightarrow p_* \Gamma_{X^{an}} E((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y^{an}}} (M, F)^{an})$$

is a map in $C_{\mathcal{D} \text{fil}}(S^{an})$.

(iii) For $M = O_Y$, we have $T_\omega^O(an, h)(O_Y)^\gamma = T_\omega^O(an, h)^\gamma$ as complexes of D_S modules

Proof. Follows by definition from proposition 59 \square

Let $p : Y \times S \rightarrow S$ a projection with $Y, S \in \text{SmVar}(\mathbb{C})$ or with $Y, S \in \text{AnSm}(\mathbb{C})$. Let $j : V \hookrightarrow Y \times S$ an open embedding. Consider (see proposition 60), for $(M, F) \in C_{\mathcal{D} \text{fil}}(Y \times S)$, the canonical transformation map in $C_{p^* \mathcal{D} \text{fil}}(Y \times S)$

$$k \circ T_\omega^O(j, \otimes)(E(M, F)) : (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_Y} j_* j^* E(M, F) \\ \xrightarrow{DR(Y \times S/S)(\text{ad}(j^*, j_*)(-))} \\ j_* j^* ((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_Y} j_* j^* E(M, F)) = j_* j^* (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_Y} j^* j_* j^* E(M, F) \\ \xrightarrow{k \circ DR(Y \times S/S)(\text{ad}(j^*, j_*)(E(M, F)))} \\ j_* E(j^* (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} j^* E(M, F)) = j_* E(j^* ((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} E(M, F)))$$

We have then :

Proposition 72. *Let $p : Y \times S \rightarrow S$ a projection with $Y, S \in \text{SmVar}(\mathbb{C})$ or with $Y, S \in \text{AnSm}(\mathbb{C})$. Let $i : X \hookrightarrow Y \times S$ a closed embedding. Then, for $(M, F) \in C_{\mathcal{D}fil}(Y \times S)$*

(i) *the canonical map in $C_{p^*\mathcal{D}fil}(Y)$ (definition 54)*

$$T_w^O(\gamma, \otimes)(M, F) := (I, k \circ T_w^O(j, \otimes)(E(M, F))) : \\ (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} \Gamma_X E(M, F) \rightarrow \Gamma_X E((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} E(M, F)),$$

is a (1-)filtered Zariski, resp usu, local equivalence.

(ii) *the map of point (i) gives the following canonical isomorphism in $D_{p^*\mathcal{D}fil}(Y)$*

$$T_w^O(\gamma, \otimes)(M, F) : (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} \Gamma_X E(M, F) \xrightarrow{T_w^O(\gamma, \otimes)(M, F)} \\ \Gamma_X E((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} E(M, F)) \xrightarrow{DR(Y \times S/S)(k)^{-1}} \Gamma_X E((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M, F)).$$

Proof. (i): By proposition 60

- $T_w^O(j, \otimes)(M, F) : (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} j_* j^* E(M, F) \rightarrow j_* E(j^*((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} E(M, F)))$ is a filtered Zariski, resp usu, local equivalence in $C_{p^*\mathcal{D}fil}(Y \times S)$ and
- $DR(Y \times S/S)(k) : (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M, F) \rightarrow (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} E(M, F)$ is a filtered Zariski, resp usu, local equivalence in $C_{p^*\mathcal{D}fil}(Y \times S)$.

(ii): Follows from (i). □

In the analytic case, we also have

Proposition 73. *Let $p : Y \times S \rightarrow S$ a projection with $Y, S \in \text{AnSm}(\mathbb{C})$. Let $i : X \hookrightarrow Y$ a closed embedding. Then, for $(M, F) \in C_{\mathcal{D}^\infty fil}(Y \times S)$*

(i) *the canonical map in $C_{p^*\mathcal{D}^\infty fil}(Y)$*

$$T_w^O(\gamma, \otimes)(M, F) := (I, T_w^O(j, \otimes)(E(M, F))) : \\ (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} \Gamma_X E(M, F) \rightarrow \Gamma_X E((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} E(M, F)).$$

is a map in $C_{p^\mathcal{D}^\infty fil}(Y \times S)$. Proposition 72 says that it is a filtered equivalence usu local,*

(ii) *the map of point (i) gives the following canonical isomorphism in $D_{p^*\mathcal{D}^\infty, fil}(Y \times S)$*

$$T_w^O(\gamma, \otimes)(M, F) : (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} \Gamma_X E(M, F) \xrightarrow{T_w^O(\gamma, \otimes)(M, F)} \\ \Gamma_X E((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} E(M, F)) \xrightarrow{DR(Y \times S/S)(k)^{-1}} \Gamma_X E((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M, F)).$$

Proof. (i): By proposition 61

- $T_w^O(j, \otimes)(M) : \Omega_{Y \times S/S}^\bullet \otimes_{O_{Y \times S}} j_* j^* E(M) \rightarrow j_* E(j^*(\Omega_{Y \times S/S}^\bullet \otimes_{O_{Y \times S}} E(M)))$ is an equivalence usu local in $C_{p^*\mathcal{D}^\infty}(Y \times S)$ and
- $DR(Y \times S/S)(k) : \Omega_{Y \times S/S}^\bullet \otimes_{O_{Y \times S}} M \rightarrow \Omega_{Y \times S/S}^\bullet \otimes_{O_{Y \times S}} E(M)$ is an equivalence usu local in $C_{p^*\mathcal{D}^\infty}(Y \times S)$.

(ii): Follows from (i). □

In the projection case, we consider the following canonical maps : Let $S_1, S_2 \in \text{SmVar}(\mathbb{C})$ or let $S_1, S_2 \in \text{AnSm}(\mathbb{C})$. Denote by $p = p_1 : S_{12} = S_1 \times S_2 \rightarrow S_1$ and $p_2 : S_{12} = S_1 \times S_2 \rightarrow S_2$ the projection. We consider

- $p(M_1, F) : (M_1, F) \rightarrow p_{*mod} p^{*mod[-]}(M_1, F)$ in $C_{\mathcal{D}(2)fil}(S_1)$, for $(M_1, F) \in C_{\mathcal{D}fil(2)}(S_1)$, which is the composite

$$\begin{aligned} p(M_1, F) : (M_1, F) &\xrightarrow{\text{ad}(p^*, p^*)(M_1)} p_* p^*(M_1, F) \xrightarrow{m_1} p_*((\Omega_{S_{12}/S_1}^\bullet, F_b) \otimes_{p^*O_{S_1}} p^*(M_1, F)) \\ &\xrightarrow{=} p_*((\Omega_{S_{12}/S_1}^\bullet, F_b) \otimes_{O_{S_{12}}} p^{*mod}(M_1, F)) = p_{*mod} p^{*mod[-]}(M_1, F) \end{aligned}$$

where $m_1 : p^*M_1 \rightarrow p^*M_1 \otimes_{p^*O_{S_1}} \Omega_{S_{12}/S_1}^\bullet$ is given by $m_1(m) = m \otimes 1$,

- $p(M_{12}, F) : p^{*mod[-]} p_{*mod}(M_{12}, F) \rightarrow (M_{12}, F)$ in $C_{\mathcal{D}fil}(S_1 \times S_2)$, for $(M_{12}, F) \in C_{\mathcal{D}}(S_1 \times S_2)$, which is the composite

$$\begin{aligned} p(M_{12}, F) : p^{*mod[-]} p_{*mod}(M_{12}, F) &= p^* p_*((M_{12}, F) \otimes_{O_{S_{12}}} (\Omega_{S_{12}/S_1}^\bullet, F_b)) \otimes_{p^*O_{S_1}} O_{S_{12}} \\ &\xrightarrow{\text{ad}(p^*, p^*)(-) \otimes_{p^*O_{S_1}} I} (M_{12}, F) \otimes_{O_{S_{12}}} \Omega_{S_{12}/S_1}^\bullet \otimes_{p^*O_{S_1}} O_{S_{12}} = (M_{12}, F) \otimes_{p^*O_{S_1}} \Omega_{S_{12}/S_1}^\bullet \xrightarrow{m_{12}} (M_{12}, F) \end{aligned}$$

where $m_{12} : M_{12} \otimes_{p^*O_{S_1}} \Omega_{S_{12}/S_1}^\bullet \rightarrow M_{12}$ is the multiplication map:

- $m_{12}(M_{12} \otimes_{p^*O_{S_1}} \Omega_{S_{12}/S_1}^p) = 0$ for $p \neq 0$ and
- $m_{12} : M_{12} \otimes_{p^*O_{S_1}} \Omega_{S_{12}/S_1}^0 = M_{12} \otimes_{p^*O_{S_1}} O_{S_{12}} \rightarrow M_{12}$ is given by $m_{12}(m \otimes f) = fm$

We have then $p(p^{*mod[-]}(M_1, F)) \circ p^{*mod[-]} p(M_1, F) = I_{p^{*mod[-]}(M_1, F)}$. It gives the following maps

- $p_!(M_{12}) : (M_{12}, F) \rightarrow p^{*mod[-]} \int_{p_!} (M_{12}, F)$ in $D_{\mathcal{D}(2)fil}(S_1 \times S_2)$, for $(M_{12}, F) \in C_{\mathcal{D}fil, h}(S_1 \times S_2)$, given by

$$\begin{aligned} p_!(M_{12}) : (M_{12}, F) &\xrightarrow{d(M_{12}, F)} L\mathbb{D}_S^2(M_{12}, F) \xrightarrow{L\mathbb{D}_S(p(-) \circ q)} \mathbb{D}_S^K L_D p^{*mod[-]} p_{*mod} E(\mathbb{D}_S^K L_D(M_{12}, F)) \\ &\xrightarrow{T(p, D)(-)^{-1}} p^{*mod} \mathbb{D}_S^K L_D p_{*mod} E(\mathbb{D}_S^K L_D(M_{12}, F)) = p^{*mod[-]} \int_{p_!} (M_{12}, F) \end{aligned}$$

- $p_!(M_1, F) : \int_{p_!} p^{*mod[-]}(M_1, F) \rightarrow (M_1, F)$ in $D_{\mathcal{D}fil}(S_1)$, for $M_1 \in C_{\mathcal{D}fil, h}(S_1)$, given by

$$\begin{aligned} p_!(M_1, F) : \int_{p_!} p^{*mod[-]}(M_1, F) &= \mathbb{D}_S^K L_D p_{*mod} E(\mathbb{D}_S^K p^{*mod[-]} L_D(M_1, F)) \xrightarrow{(\mathbb{D}_S^K k) \circ T(p, D)(-)^{-1}} \\ &\mathbb{D}_S^K p_{*mod} p^{*mod[-]} \mathbb{D}_S^K L_D(M_1, F) \xrightarrow{\mathbb{D}_S^K p(\mathbb{D}_S^K L_D(M_1, F))} \mathbb{D}_S^{K, 2} L_D(M_1, F) \xrightarrow{d(M_1, F)^{-1}} (M_1, F) \end{aligned}$$

so that $p^{*mod[-]}(p_!(M_1, F)) \circ p_!(p^{*mod[-]}(M_1, F)) = I_{p^{*mod[-]}(M_1, F)}$.

Definition 68. (i) Consider a commutative diagram in $\text{SmVar}(\mathbb{C})$ which is cartesian, together with its factorization

$$\begin{array}{ccc} D = (f, q) = X \times T \xrightarrow{f'} T \times S & D = (f, q) = f'' : X \times T \xrightarrow{i''} X \times T \times S \xrightarrow{p''} T \times S, \\ \begin{array}{ccc} g' \downarrow & & \downarrow q \\ X & \xrightarrow{f} & S \end{array} & & \begin{array}{ccc} \downarrow q' & & \downarrow q'' = I \times q \\ f : X & \xrightarrow{i} & X \times S \xrightarrow{p} S \\ & & \downarrow q \end{array} \end{array}$$

where the squares are cartesian, $f = p \circ i$ being the graph factorization and q, q' the projections. We have, for $(M, F) \in C_{\mathcal{D}(2)fil, \infty}(X)$, the following transformation map in $C_{\mathcal{D}(2)fil, \infty}(T \times S)$:

$$\begin{aligned} T^{\mathcal{D}mod}(f, q)(M, F) : q^{*mod} p_* E((\Omega_{X \times S/S}, F_b) \otimes_{O_{X \times S}} i_{*mod}(M, F)) \\ \xrightarrow{T_\omega^O(q, p)(i_{*mod}(M, F))} p''_* E((\Omega_{X \times T \times S/T \times S}, F_b) \otimes_{O_{X \times T \times S}} q''^{*mod} i_{*mod}(M, F)) \\ \xrightarrow{p_* E(T^{\mathcal{D}mod}(i, q'')(M, F) \otimes I)} p''_* E((\Omega_{X \times T \times S/T \times S}, F_b) \otimes_{O_{X \times T \times S}} i''_{*mod} q'^{*mod}(M, F)) \end{aligned}$$

where

$$\begin{aligned}
T^{\mathcal{D}mod}(i, q'')(M, F) : q''^{*mod} i_*^{*mod}(M, F) &:= q''^{*mod} i_*((M, F) \otimes_{D_X} i^{*mod}(D_{X \times S}, F^{ord})) \\
&\xrightarrow{T^{mod}(q'', i)(-)} i_* q'^{*mod}((M, F) \otimes_{D_X} i^{*mod}(D_{X \times S}, F^{ord})) \\
&\xrightarrow{\cong} i_*(q'^{*mod}(M, F) \otimes_{q'^* D_X} q'^* i^{*mod}(D_{X \times S}, F^{ord})) \xrightarrow{\cong} \\
&i_*(q'^{*mod}(M, F) \otimes_{D_{X \times T}} i'^{*mod}(D_{X \times S \times T}, F^{ord})) =: i'^{*mod} q'^{*mod}(M, F)
\end{aligned}$$

(ii) Consider a commutative diagram in $\text{SmVar}(\mathbb{C})$ which is cartesian, together with its factorization

$$\begin{array}{ccc}
D = (f, g) = & X_T \xrightarrow{f'} T & D = (f, g) = & f' : X_T \xrightarrow{i'} X \times T \xrightarrow{p'} T \\
& \downarrow g' & & \downarrow l' & \downarrow l'' = I \times l & \downarrow l \\
& X \xrightarrow{f} S & & f'' : X \times T \xrightarrow{i''} X \times T \times S \xrightarrow{p''} T \times S & & \downarrow q \\
& & & \downarrow q' & \downarrow q'' = I \times q & \downarrow q \\
& & & f : X \xrightarrow{i} X \times S \xrightarrow{p} S & &
\end{array}$$

where the squares are cartesian, $f = p \circ i$, $g = q \circ l$ being the graph factorizations. We have, for $(M, F) \in D_{\mathcal{D}(2)fil, \infty}(X)$, the following transformation map in $D_{\mathcal{D}(2)fil, \infty}(T \times S)$:

$$\begin{aligned}
&T^{\mathcal{D}mod}(f, g)((M, F)) : \\
Rg^{*mod, \Gamma}(M, F) \int_f^{FDR} (M, F) &:= \Gamma_T E(q^{*mod} p_* E((\Omega_{X \times S/S}, F_b) \otimes_{O_{X \times S}} i^{*mod}(M, F))) \\
&\xrightarrow{\Gamma_T E(T^{\mathcal{D}mod}(f, g)(M, F))} \Gamma_T E(p''_* E((\Omega_{X \times T \times S/T \times S}, F_b) \otimes_{O_{X \times T \times S}} i''^{*mod} q'^{*mod}(M, F))) \\
&\xrightarrow{\cong} p''_* \Gamma_{X \times T} E((\Omega_{X \times T \times S/T \times S}, F_b) \otimes_{O_{X \times T \times S}} i''^{*mod} q'^{*mod}(M, F)) \\
&\xrightarrow{T_w^O(\gamma, \otimes)(-)} p''_* E((\Omega_{X \times T \times S/T \times S}, F_b) \otimes_{O_{X \times T \times S}} \Gamma_{X \times T} E(i''^{*mod} q'^{*mod}(M, F))) \\
&\xrightarrow{\cong} p''_* E((\Omega_{X \times T \times S/T \times S}, F_b) \otimes_{O_{X \times T \times S}} (i''^{*mod} q'^{*mod} \Gamma_{X \times T} E(M, F))) =: \int_{f'}^{FDR} Rg'^{*mod, \Gamma}(M, F)
\end{aligned}$$

(ii)' We have, for $M \in D_{\mathcal{D}}(X)$, the following transformation map in $D_{\mathcal{D}}(T)$:

$$\begin{aligned}
&T^{\mathcal{D}mod}(f, g)(M) : \\
g^{*mod}(M, F) \int_f (M) &= l^{*mod} q^{*mod} \int_f M \xrightarrow{q'(M)} l^{*mod} q^{*mod} \int_f q'^{*mod} q'^{*mod} M \\
&\xrightarrow{\cong} l^{*mod} q^{*mod} q'^{*mod} \int_{f''} q'^{*mod} M \xrightarrow{q(-)} l^{*mod} \int_{f''} q'^{*mod} M \\
&\xrightarrow{l^{*mod} \text{ad}(l'^{\sharp}, l'^{*mod})(-)^{-1}} l^{*mod} \int_{f''} l'^{*mod} l'^{\sharp} q'^{*mod} M \xrightarrow{\cong} l^{\sharp} l'^{*mod} \int_{f'} l'^{*mod} q^{*mod} M \\
&\xrightarrow{\text{ad}(l^{\sharp}, l'^{*mod})(-)} \int_{f'} l'^{*mod} q^{*mod} M =: \int_{f'} g'^{*mod}(M)
\end{aligned}$$

where $l'^{*mod} \text{ad}(l'^{\sharp}, l'^{*mod})(-)$ is an isomorphism by lemma 9.

In the analytic case, we have :

Definition 69. Consider a commutative diagram in $\text{AnSm}(\mathbb{C})$ which is cartesian together with a factorization

$$D = (f, g) = \begin{array}{ccc} X_T & \xrightarrow{f'} & T \\ g' \downarrow & & \downarrow g \\ X & \xrightarrow{f} & S \end{array} \quad D = (f, g) = \begin{array}{ccccc} f' : X_T & \xrightarrow{i'} & Y \times T & \xrightarrow{p'} & T \\ \downarrow g' & & \downarrow g'' = I \times g & & \downarrow g \\ f : X & \xrightarrow{i} & Y \times S & \xrightarrow{p} & S \end{array},$$

where $Y \in \text{AnSm}(\mathbb{C})$, i, i' are closed embeddings and p, p' the projections.

(i) We have, for $(M, F) \in D_{\mathcal{D}(2)\text{fil}, \infty, h}(X)$, the following transformation map in $D_{\mathcal{D}(2)\text{fil}, \infty}(T \times S)$

$$T^{\mathcal{D}\text{mod}}(f, g)((M, F)) : Rg^{*\text{mod}, \Gamma} \int_f^{\text{FDR}} (M, F) \rightarrow \int_{f'}^{\text{FDR}} Rg'^{*\text{mod}, \Gamma}(M, F)$$

define in the same way as in definition 68

(ii) For $(M, F) \in D_{\mathcal{D}^\infty(2)\text{fil}, \infty}(X)$, the following transformation map in $D_{\mathcal{D}^\infty(2)\text{fil}, \infty}(T \times S)$

$$T^{\mathcal{D}\text{mod}}(f, g)((M, F)) : Rg^{*\text{mod}, \Gamma} \int_f^{\text{FDR}} (M, F) \rightarrow \int_{f'}^{\text{FDR}} Rg'^{*\text{mod}, \Gamma}(M, F)$$

is defined in the same way as in (ii) : see definition 68.

In the algebraic case, we have the following proposition:

Proposition 74. Consider a cartesian square in $\text{SmVar}(\mathbb{C})$

$$D = \begin{array}{ccc} X_T & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ T & \xrightarrow{g} & S \end{array}$$

(i) For $(M, F) \in D_{\mathcal{D}(2)\text{fil}, \infty, c}(X)$,

$$T^{\mathcal{D}\text{mod}}(f, g)((M, F)) : Rg^{*\text{mod}, \Gamma} \int_f^{\text{FDR}} (M, F) \xrightarrow{\sim} \int_{f'}^{\text{FDR}} Rg'^{*\text{mod}, \Gamma}(M, F)$$

is an isomorphism in $D_{\mathcal{D}(2)\text{fil}, \infty}(T)$.

(ii) For $M \in D_{\mathcal{D}, c}(X)$,

$$T^{\mathcal{D}\text{mod}}(f, g)(M) : g^{*\text{mod}} \int_f M \xrightarrow{\sim} \int_{f'} g'^{*\text{mod}} M$$

is an isomorphism in $D_{\mathcal{D}}(T)$.

Proof. Follows from the projection case and the closed embedding case. □

In the analytic case, we have similarly:

Proposition 75. Consider a cartesian square in $\text{AnSm}(\mathbb{C})$

$$D = \begin{array}{ccc} X_T & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ T & \xrightarrow{g} & S \end{array}$$

(i) Assume that f , hence f' is proper. For $(M, F) \in D_{\mathcal{D}(2)fil, \infty, h}(X)$,

$$T^{\mathcal{D}mod}(f, g)((M, F)) : Rg^{*mod, \Gamma} \int_f^{FDR} (M, F) \xrightarrow{\sim} \int_{f'}^{FDR} Rg'^{*mod, \Gamma}(M, F)$$

is an isomorphism in $D_{\mathcal{D}(2)fil, \infty}(T)$.

(ii) For $(M, F) \in D_{\mathcal{D}^\infty(2)fil, \infty, h}(X)$,

$$T^{\mathcal{D}mod}(f, g)((M, F)) : Rg^{*mod, \Gamma} \int_f^{FDR} (M, F) \xrightarrow{\sim} \int_{f'}^{FDR} Rg'^{*mod, \Gamma}(M, F)$$

is an isomorphism in $D_{\mathcal{D}^\infty(2)fil, \infty}(T)$.

Proof. (i): Similar to the proof of proposition 74.

(ii): Similar to the proof of proposition 74. □

Definition 70. Let $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$.

(i) We have, for $(M, F) \in C_{\mathcal{D}fil}(S)$ and $(N, F) \in C_{\mathcal{D}fil}(X)$, we have the map in $C_{\mathcal{D}fil}(S)$

$$\begin{aligned} & T^{\mathcal{D}mod, 0}(\otimes, f)((M, F), (N, F)) : \\ & (M, F) \otimes_{O_S} f_{*mod}^0(N, F) := (M, F) \otimes_{O_S} f_*((N, F) \otimes_{D_X} (D_{X \leftarrow S}, F^{ord})) \\ & \xrightarrow{T(\otimes, f)(-, -)} f_*(f^*(M, F) \otimes_{f^*O_S} (N, F) \otimes_{D_X} (D_{X \leftarrow S}, F^{ord})) \xrightarrow{=} \\ & f_*(f^{*mod}(M, F) \otimes_{O_X} (N, F) \otimes_{D_X} (D_{X \leftarrow S}, F^{ord})) =: f_{*mod}^0(f^{*mod}(M, F) \otimes_{O_X} (N, F)) \end{aligned}$$

(ii) Consider the cartesian square

$$D = \begin{array}{ccc} X & \xrightarrow{i} & X \times S \\ \downarrow f & & \downarrow f \times I_S \\ S & \xrightarrow{\Delta_S} & S \times S \end{array} ,$$

where $i_f = (f \times I_S) \circ \Delta_X : X \hookrightarrow X \times S$ is the graph embedding. Then, for $(M, F) \in C_{\mathcal{D}(2)fil}(S)$ and $(N, F) \in C_{\mathcal{D}fil}(X)$, we have the map in $D_{\mathcal{D}(2)fil, r}(S)$

$$\begin{aligned} T^{\mathcal{D}mod}(\otimes, f)((M, F), (N, F)) & : \int_f^{FDR} ((N, F) \otimes_{O_X} f_{FDR}^{*mod}(M, F)) = \int_f^{FDR} i_{f, FDR}^{*mod}(p_X^* N \otimes p_S^* M) \\ & \xrightarrow{T^{\mathcal{D}mod}(\Delta_S, f \times I_S)(-)} \Delta_{S, FDR}^{*mod} \int_{(f \times I_S)}^{FDR} (p_X^* N \otimes p_S^* M) = \left(\int_f (N, F) \right) \otimes_{O_S}^L (M, F). \end{aligned}$$

Clearly if $i : Z \hookrightarrow S$ is a closed embedding with $Z, S \in \text{SmVar}(\mathbb{C})$ or with $Z, S \in \text{AnSm}(\mathbb{C})$, then $T^{\mathcal{D}, 0}(\otimes, i)(M, N) = T^{\mathcal{D}}(\otimes, i)(M, N)$ in $D_{\mathcal{D}(2)fil, \infty}(S)$.

We have then the following :

Proposition 76. (i) Let $i : Z \hookrightarrow S$ is a closed embedding with $Z, S \in \text{SmVar}(\mathbb{C})$, then for $(M, F) \in C_{\mathcal{D}fil}(S)$ and $(N, F) \in C_{\mathcal{D}fil}(Z)$

$$T^{\mathcal{D}, 0}(\otimes, i)((M, F), (N, F)) : (M, F) \otimes_{O_S} i_{*mod}(N, F) \xrightarrow{\sim} i_{*mod}(i^{*mod}(M, F) \otimes_{O_Z} (N, F))$$

is an isomorphism in $C_{\mathcal{D}fil}(S)$.

(ii) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$. Then, for $(M, F) \in C_{\mathcal{D}(2)\text{fil}}(X)$ and $(N, F) \in C_{\mathcal{D}(2)\text{fil}}(S)$,

$$T^{\mathcal{D}^{\text{mod}}}(\otimes, f)((M, F), (N, F)) : \int_f^{\text{FDR}} ((M, F) \otimes_{O_X}^L f_{\text{FDR}}^{*\text{mod}}(N, F)) \xrightarrow{\sim} \left(\int_f^{\text{FDR}} (M, F) \right) \otimes_{O_Y}^L (N, F)$$

is an isomorphism in $D_{\mathcal{D}(2)\text{fil}, \infty}(S)$.

Proof. (i): Follows from proposition 10.

(ii): Follows from proposition 74(i). \square

Let $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$. Consider the graph embedding $f : X \xrightarrow{i} X \times S \xrightarrow{p} S$, with $X, Y, S \in \text{SmVar}(\mathbb{C})$. We have, for $(M, F) \in C_{\mathcal{D}\text{fil}}(X)$, the canonical isomorphism in $C_{\mathcal{D}(2)\text{fil}}(S^{\text{an}})$

$$\begin{aligned} \text{an}_X^{*\text{mod}} i_*^{\text{mod}} L_D(p^{*\text{mod}}(M, F) \otimes_{O_{X \times S}} (O_{X \times S}, \mathcal{V}_X)) &\xrightarrow{\cong} \\ i_*^{\text{mod}} L_D p^{*\text{mod}}((M, F)^{\text{an}} \otimes_{O_{X^{\text{an}} \times S^{\text{an}}}} (O_{X^{\text{an}} \times S^{\text{an}}}, \mathcal{V}_{X^{\text{an}}})) &\end{aligned}$$

We then define and study the transformation map between the direct image functor and the analytical functor for D-modules :

Definition 71. Let $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$.

(i) We have for $(M, F) \in C_{\mathcal{D}(2)\text{fil}}(X)$ the canonical map in $C_{\mathcal{D}(2)\text{fil}}(S^{\text{an}})$

$$\begin{aligned} T^{\mathcal{D}^{\text{mod}0}}(\text{an}, f)(M, F) : \text{an}_S^{*\text{mod}}(f_* E((D_{X \leftarrow S}, F^{\text{ord}}) \otimes_{D_X} L_D(M, F))) &\xrightarrow{T^{\text{mod}}(\text{an}, f)(-)} \\ f_*(E((D_{X \leftarrow S}, F^{\text{ord}}) \otimes_{D_X} L_D(M, F)))^{\text{an}} &\xrightarrow{\cong} f_* E(D_{X^{\text{an}} \leftarrow S^{\text{an}}} \otimes_{D_{X^{\text{an}}}} L_D(M^{\text{an}}, F)) \end{aligned}$$

(ii) Consider the graph embedding $f : X \xrightarrow{i} X \times S \xrightarrow{p} S$, with $X, Y, S \in \text{SmVar}(\mathbb{C})$. We have, for $(M, F) \in C_{\mathcal{D}\text{fil}}(X)$, the canonical map in $C_{\mathcal{D}(2)\text{fil}}(S^{\text{an}})$

$$\begin{aligned} T^{\mathcal{D}^{\text{mod}}}(\text{an}, f)(M, F) : \text{an}_S^{*\text{mod}}(p_* E((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} i_*^{\text{mod}}(M, F))) & \\ \xrightarrow{T_\omega^{\mathcal{O}}(\text{an}, p)(i_*^{\text{mod}}(M, F))} p_* E((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y^{\text{an}} \times S^{\text{an}}}} (i_*^{\text{mod}}(M, F))^{\text{an}}) & \\ \xrightarrow{p_*^{\text{mod}} T^{\mathcal{D}^{\text{mod}0}}(\text{an}, i)((M, F))} p_* E((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y^{\text{an}} \times S^{\text{an}}}} i_*^{\text{mod}}((M, F)^{\text{an}})). & \end{aligned}$$

In order to prove that this map gives an isomorphism in the derived category in the non filtered case if f is proper and M coherent, we will need the following (c.f.[16]):

Theorem 19. A product $X \times S$ of a smooth projective variety X and a smooth affine variety S is D-affine.

Proof. See [16] theorem 1.6.5. \square

A main result is that we have the following version of the first GAGA theorem for coherent D-modules :

Theorem 20. Let $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$. Let $M \in D_{\mathcal{D}(2)\text{fil}, c}(X)$, for $r = 1, \dots, \infty$. If f is proper,

$$T^{\mathcal{D}^{\text{mod}}}(\text{an}, f)(M, F) : \left(\int_f M \right)^{\text{an}} \xrightarrow{\sim} \int_{f^{\text{an}}} (M^{\text{an}})$$

is an isomorphism.

Proof. We may assume that f is projective, so that we have a factorization $f : X \xrightarrow{i} \mathbb{P}^N \times S \xrightarrow{p} S$ where i is a closed embedding and p the projection. The question being local on S , we may assume that S is affine. Since $\mathbb{P}^N \times S$ is D -affine by theorem 19, we have by proposition 40(iii) a complex $F \in C_{\mathcal{D}}(\mathbb{P}^N \times S)$ such that $i_* \text{mod} M = F \simeq F \in D_{\mathcal{D},r}(\mathbb{P}^N \times S)$ and each F^n is a direct summand of a free $D_{\mathbb{P}^N \times S}$ module of finite rank. The theorem now follows from the fact that $\int_p D_{\mathbb{P}^N \times S} \simeq D_S[-N]$ and the fact that $(D_S)^{an} = D_S^{an}$. \square

We also have

Definition 72. (i) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$ or with $X, S \in \text{AnSm}(\mathbb{C})$. We have, for $M, N \in C_{\mathcal{D}fil}(X)$, the canonical transformation map in $D_{\mathcal{D}fil,\infty}(S)$

$$\begin{aligned} T^{0,\mathcal{D}}(f, \text{hom})((M, F), (N, F)) &: Rf_* R\mathcal{H}om_{f^* D_S}((M, F), (N, F)) \rightarrow \\ Rf_* R\mathcal{H}om_{D_X}((M, F) \otimes_{D_X} L_D(D_{X \leftarrow S}, F^{ord}), (N, F) \otimes_{D_X} L_D(D_{X \leftarrow S}, F^{ord})) &\xrightarrow{T^0(f, \text{hom})(E(-), E(-))} \\ R\mathcal{H}om_{D_X}(Rf_*((M, F) \otimes_{D_X} L_D(D_{X \leftarrow S}, F^{ord})), Rf_*((N, F) \otimes_{D_X} L_D(D_{X \leftarrow S}, F^{ord}))) &= \\ R\mathcal{H}om_{D_X}(\int_f (M, F), \int_f (N, F)) & \end{aligned}$$

(ii) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. We have, for $(M, F), (N, F) \in C_{\mathcal{D}fil}(X)$, the canonical transformation map in $D_{\mathcal{D}fil,\infty}(S)$

$$\begin{aligned} T^{0,\mathcal{D}}(f_!, \text{hom})((M, F), (N, F)) &: Rf_* \mathcal{H}om_{f^* D_S}((M, F), (N, F)) \rightarrow \\ Rf_* \mathcal{H}om_{D_X}((M, F) \otimes_{D_X} L_D(D_{X \leftarrow S}, F^{ord}), (N, F) \otimes_{D_X} L_D(D_{X \leftarrow S}, F^{ord})) &\xrightarrow{T^0(f_!, \text{hom})(E(-), E(-))} \\ R\mathcal{H}om_{D_X}(Rf_!((M, F) \otimes_{D_X} L_D(D_{X \leftarrow S}, F^{ord})), Rf_!((N, F) \otimes_{D_X} L_D(D_{X \leftarrow S}, F^{ord}))) &= \\ R\mathcal{H}om_{D_X}(\int_{f!} (M, F), \int_{f!} (N, F)) & \end{aligned}$$

Definition 73. Let $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$ or with $X, S \in \text{AnSm}(\mathbb{C})$. We have, for $(M, F), (N, F) \in C_{\mathcal{D}fil}(S)$, the canonical transformation map in $C_{\mathcal{D}fil}(X)$

$$\begin{aligned} T^{\mathcal{D}}(f, \text{hom})((M, F), (N, F)) &: f^* \mathcal{H}om_{D_S}((M, F), (N, F)) \\ \xrightarrow{T(f, \text{hom})((M, F), (N, F))} &\mathcal{H}om_{f^* D_S}(f^*(M, F), f^*(N, F)) \\ \rightarrow \mathcal{H}om_{D_X}(f^*(M, F) \otimes_{f^* D_S} L_{f^* D}(D_{X \rightarrow S}, F^{ord}), f^*(N, F) \otimes_{f^* D_S} L_{f^* D}(D_{X \rightarrow S}, F^{ord})) & \\ \xrightarrow{=} \mathcal{H}om_{D_X}(f^{*mod}(M, F), f^{*mod}(N, F)) & \end{aligned}$$

which is the one given by Kashiwara (see [19]).

In the algebraic case, we have, in the non filtered case, the six functor formalism for holonomic D -modules :

Theorem 21. Let $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$.

(i) We have, for $M \in D_{\mathcal{D},h}(X)$ and $N \in D_{\mathcal{D},h}(S)$ a canonical isomorphism in $D_{\mathcal{D}}(S)$

$$I^{\mathcal{D}mod}(L_{f^* \text{mod}[-1]}, \int_f)(M, N) : Rf_* R\mathcal{H}om_{D_X}(L_{f^* \text{mod}[-1]} N, M) \xrightarrow{\simeq} R\mathcal{H}om_{D_S}(N, \int_f M).$$

(ii) We have, for $M \in D_{\mathcal{D},h}(X)$ and $N \in D_{\mathcal{D},h}(S)$ a canonical isomorphism in $D_{\mathcal{D}}(X)$

$$I^{\mathcal{D}mod}(\int_{f!}, L_{f^* \text{mod}[-1]})(M, N) : R\mathcal{H}om_{D_X}(\int_{f!} M, N) \xrightarrow{\simeq} Rf_* R\mathcal{H}om_{D_S}(M, L_{f^* \text{mod}[-1]} N).$$

Proof. Follows from the projection case and the closed embedding case. □

Corollary 2. *Let $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$. Then,*

- $(L f^{\hat{*}mod[-]}, \int_f) : D_{\mathcal{D},h}(S) \rightarrow D_{\mathcal{D},h}(X)$ is a pair of adjoint functors.
- $(\int_{f!}, L f^{*mod[-]}) : D_{\mathcal{D},h}(S) \rightarrow D_{\mathcal{D},h}(X)$ is a pair of adjoint functors.

Proof. Follows immediately from theorem 21 by taking global sections. □

Consider a commutative diagram in $\text{SmVar}(\mathbb{C})$,

$$D = \begin{array}{ccc} X' & \xrightarrow{f'} & T \\ g' \downarrow & & \downarrow g \\ X & \xrightarrow{f} & S \end{array} .$$

We have, for $M \in C_{\mathcal{D},h}(X)$, the following transformation maps

$$\begin{aligned} T_1^{\mathcal{D}mod}(D)(M) : Lg^{\hat{*}mod[-]} \int_f M &\xrightarrow{\text{ad}(L f'^{\hat{*}mod[-]}, \int_{f'})(-)} \int_{f'} L f'^{\hat{*}mod[-]} Lg^{\hat{*}mod[-]} \int_f M \xrightarrow{\cong} \\ &\int_{f'} Lg_{FDR}'^{\hat{*}mod[-]} L f^{\hat{*}mod[-]} \int_f M \xrightarrow{\text{ad}(L f^{\hat{*}mod[-]}, \int_f)(M)} \int_{f'} Lg_{FDR}'^{\hat{*}mod[-]} M \end{aligned}$$

and

$$\begin{aligned} T_2^{\mathcal{D}mod}(D)(M, F) : \int_{f!} Lg'^{\hat{*}mod[-]} M &\xrightarrow{\text{ad}(\int_{f!}, L f'^{\hat{*}mod[-]})(-)} \int_{f!} Lg'^{\hat{*}mod[-]} L f'^{\hat{*}mod[-]} \int_{f!} M \xrightarrow{\cong} \\ &\int_{f!} L f'^{\hat{*}mod[-]} Lg'^{\hat{*}mod[-]} M \int_{f!} \xrightarrow{\text{ad}(\int_{f!}, L f'^{\hat{*}mod[-]})(-)} Lg'^{\hat{*}mod[-]} \int_{f!}^{FDR} M \end{aligned}$$

Proposition 77. *Consider a cartesian square in $\text{SmVar}(\mathbb{C})$*

$$D = \begin{array}{ccc} X_T & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ T & \xrightarrow{g} & S \end{array}$$

Assume that f (and hence f') is proper. Then, for $(M, F) \in D_{\mathcal{D}(2)fil,\infty,h}(X)$,

- $T_1^{\mathcal{D}mod}(f, g)(M) : Lg^{\hat{*}mod[-]} \int_f M \xrightarrow{\cong} \int_{f'} Lg'^{\hat{*}mod[-]} M$ and
- $T_2^{\mathcal{D}mod}(f, g)(M) : \int_{f!} Lg'^{\hat{*}mod[-]} M \xrightarrow{\cong} Lg^{\hat{*}mod[-]} \int_{f!} M$

are isomorphisms in $D_{\mathcal{D}}(T)$.

Proof. Follows from proposition 74 and the fact that the map $T_1^{\mathcal{D}mod}(f, g)(M)$ is given by the composite

$$\begin{aligned} T_1^{\mathcal{D}mod}(f, g)(M)[d_T - d_S] : Lg^{\hat{*}mod} \int_f (M, F) &= L\mathbb{D}_T Lg^{\hat{*}mod} L\mathbb{D}_S \int_f M \xrightarrow{T(f_*, f!)(-)} \\ &L\mathbb{D}_T Lg^{\hat{*}mod} \int_f L\mathbb{D}_X M \xrightarrow{(L\mathbb{D}_T T^{\mathcal{D}mod}(f, g)(\mathbb{D}_X M))^{-1}} \mathbb{D}_T \int_{f'} Lg'^{\hat{*}mod} \mathbb{D}_X M \\ &\xrightarrow{T(f'_!, f'_*)(-)} \int_{f'} L\mathbb{D}_{X_T} Lg'^{\hat{*}mod} \mathbb{D}_X (M, F) = \int_{f'} Lg'^{\hat{*}mod} M \end{aligned}$$

and the map $T_2^{\mathcal{D}^{mod}}(f, g)(M, F)$ is given by the composite

$$T_2^{\mathcal{D}^{mod}}(f, g)(M)[d_T - d_S] : \int_{f'!} Lg'^{*mod} M = L\mathbb{D}_T \int_{f'} \mathbb{D}_{X_T} Lg'^{*mod} M \xrightarrow{d(-) \circ T(f_*, f_!)(-)} \int_{f'} Lg'^{*mod} M \xrightarrow{T^{\mathcal{D}^{mod}}(f, g)(M)^{-1}} L\mathbb{D}_T Lg'^{*mod} \int_f L\mathbb{D}_X M = Lg'^{*mod}(M, F) \int_{f!} M$$

□

4.3 The D modules on singular algebraic varieties and singular complex analytic spaces

In this subsection by defining the category of complexes of filtered D-modules in the singular case and there functorialities.

4.3.1 Definition

In all this subsection, we fix the notations:

- For $S \in \text{Var}(\mathbb{C})$, we denote by $S = \cup_i S_i$ an open cover such that there exists closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We have then closed embeddings $i_I : S_I := \cap_{i \in I} S_i \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. Then for $I \subset J$, we denote by $j_{IJ} : S_J \hookrightarrow S_I$ the open embedding and $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ the projection, so that $p_{IJ} \circ i_J = i_I \circ j_{IJ}$. This gives the diagram of algebraic varieties $(\tilde{S}_I) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{Var}(\mathbb{C}))$ which gives the diagram of sites $(\tilde{S}_I) := \text{Ouv}(\tilde{S}_I) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{Cat})$. For $I \subset J$, we denote by $m : \tilde{S}_I \setminus (S_I \setminus S_J) \hookrightarrow \tilde{S}_I$ the open embedding.
- For $S \in \text{AnSp}(\mathbb{C})$ we denote by $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. We have then closed embeddings $i_I : S_I = \cap_{i \in I} S_i \hookrightarrow \tilde{S}_I = \prod_{i \in I} \tilde{S}_i$. Then for $I \subset J$, we denote by $j_{IJ} : S_J \hookrightarrow S_I$ the open embedding and $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ the projection, so that $p_{IJ} \circ i_J = i_I \circ j_{IJ}$. This gives the diagram of analytic spaces $(\tilde{S}_I) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{AnSp}(\mathbb{C}))$ which gives the diagram of sites $(\tilde{S}_I) := \text{Ouv}(\tilde{S}_I) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{Cat})$. For $I \subset J$, we denote by $m : \tilde{S}_I \setminus (S_I \setminus S_J) \hookrightarrow \tilde{S}_I$ the open embedding.

The first definition is from [27] remark 2.1.20, where we give a shifted version to have compatibility with perverse sheaves.

Definition 74. *Let $S \in \text{Var}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; or let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then, $\text{PSh}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)) \subset \text{PSh}_{\mathcal{D}(2)fil}((\tilde{S}_I))$ is the full subcategory*

- whose objects are $(M, F) = ((M_I, F)_{I \subset [1, \dots, l]}, s_{IJ})$, with
 - $(M_I, F) \in \text{PSh}_{\mathcal{D}(2)fil}(\tilde{S}_I)$ such that $\mathcal{I}_{S_I} M_I = 0$, in particular $(M_I, F) \in \text{PSh}_{\mathcal{D}(2)fil, S_I}(\tilde{S}_I)$
 - $s_{IJ} : m^*(M_I, F) \xrightarrow{\sim} m^* p_{IJ*}(M_J, F)[d_{\tilde{S}_I} - d_{\tilde{S}_J}]$ for $I \subset J$, are isomorphisms, $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ being the projection, satisfying for $I \subset J \subset K$, $p_{IJ*} s_{JK} \circ s_{IJ} = s_{IK}$;
- the morphisms $m : (M, F) \rightarrow (N, F)$ between $(M, F) = ((M_I, F)_{I \subset [1, \dots, l]}, s_{IJ})$ and $(N, F) = ((N_I, F)_{I \subset [1, \dots, l]}, r_{IJ})$ are by definition a family of morphisms of complexes,

$$m = (m_I : (M_I, F) \rightarrow (N_I, F))_{I \subset [1, \dots, l]}$$

such that $r_{IJ} \circ m_J = p_{IJ*} m_J \circ s_{IJ}$ in $\mathcal{C}_{\mathcal{D}, S_I}(\tilde{S}_I)$.

We denote by

$$\mathcal{H}ol_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)) \subset \mathcal{C}oh_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)) \subset \mathcal{P}Sh_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$$

the full subcategory consisting of $((M_I, F), s_{IJ})$ such that M_I is coherent, resp. holonomic. We have the full subcategories

$$\mathcal{H}ol_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \subset \mathcal{H}ol_{\mathcal{D}2fil}(S/(\tilde{S}_I)), \mathcal{P}Sh_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \subset \mathcal{P}Sh_{\mathcal{D}2fil}(S/(\tilde{S}_I))$$

consisting of $((M_I, F, W), s_{IJ})$ such that $W^p M_I$ are $D_{\tilde{S}_I}$ submodules.

A morphism $m = (m_I) : ((M_I), s_{IJ}) \rightarrow ((N_I), r_{IJ})$ in $C(\mathcal{P}Sh_{\mathcal{D}}(S/(\tilde{S}_I)))$ is a Zariski, resp. usu, local equivalence if all the m_I are Zariski, resp. usu, local equivalences. A morphism $m = (m_I) : ((M_I, F), s_{IJ}) \rightarrow ((N_I, F), r_{IJ})$ in $C(\mathcal{P}Sh_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)))$ is an r-filtered Zariski, resp. usu, local equivalence if all the m_I are r-filtered Zariski, resp. usu, local equivalence.

Let $S \in \text{Var}(\mathbb{C})$ or $S \in \text{AnSp}(\mathbb{C})$.

- If $S \in \text{Var}(\mathbb{C})$, let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$, and let $S = \cup_{i'=1}^{l'} S_{i'}$ an other open cover such that there exist closed embeddings $i_{i'} : S_{i'} \hookrightarrow \tilde{S}_{i'}$ with $\tilde{S}_{i'} \in \text{SmVar}(\mathbb{C})$.
- If $S \in \text{AnSp}(\mathbb{C})$, let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$, and let $S = \cup_{i'=1}^{l'} S_{i'}$ an other open cover such that there exist closed embeddings $i_{i'} : S_{i'} \hookrightarrow \tilde{S}_{i'}$ with $\tilde{S}_{i'} \in \text{AnSm}(\mathbb{C})$.

Denote $L = [1, \dots, l]$, $L' = [1, \dots, l']$ and $L'' := [1, \dots, l] \sqcup [1, \dots, l']$. We have then the refined open cover $S = \cup_{k \in L} S_k$ and we denote for $I \sqcup I' \subset L''$, $S_{I \sqcup I'} := \cap_{k \in I \sqcup I'} S_k$ and $\tilde{S}_{I \sqcup I'} := \prod_{k \in I \sqcup I'} \tilde{S}_k$, so that we have a closed embedding $i_{I \sqcup I'} : S_{I \sqcup I'} \hookrightarrow \tilde{S}_{I \sqcup I'}$. For $I \sqcup I' \subset J \sqcup J'$, denote by $p_{I \sqcup I', J \sqcup J'} : \tilde{S}_{J \sqcup J'} \rightarrow \tilde{S}_{I \sqcup I'}$ the projection. We then have a natural transfer map

$$\begin{aligned} T_S^{L/L'} : \mathcal{P}Sh_{\mathcal{D}fil}(S/(S_I)) &\rightarrow \mathcal{P}Sh_{\mathcal{D}fil}(S/(S_{I'})), \\ ((M_I, F), s_{IJ}) &\mapsto (\text{ho} \lim_{I \in L} p_{I'(I \sqcup I')}^* (p_{I(I \sqcup I')}^{*mod} (M_I, F)) / \mathcal{I}_{S_{I \sqcup I'}}, s_{I'J'}), \end{aligned}$$

with, in the homotopy limit, the natural transition morphisms

$$\begin{aligned} p_{I'(I \sqcup I')}^* \text{ad}(p_{IJ}^{*mod}, p_{IJ*}) (p_{I(I \sqcup I')}^{*mod[-]} (M_I, F)) : \\ p_{I'(J \sqcup J')}^* (p_{J(J \sqcup J')}^{*mod[-]} (M_J, F)) / \mathcal{I}_{S_{J \sqcup J'}} \rightarrow p_{I'(I \sqcup I')}^* (p_{I(I \sqcup I')}^{*mod[-]} (M_I, F)) / \mathcal{I}_{S_{I \sqcup I'}} \end{aligned}$$

for $J \subset I$, and

$$\begin{aligned} s_{I'J'} : \text{holim}_{I \in L} m^* p_{I'(I \sqcup I')}^* (p_{I(I \sqcup I')}^{*mod[-]} (M_I, F)) / \mathcal{I}_{S_{I \sqcup I'}} \rightarrow \\ \text{holim}_{I \in L} p_{I'J'}^* (p_{I'J'}^{*mod[-]} m^* p_{I'(I \sqcup I')}^* (p_{I(I \sqcup I')}^{*mod[-]} ((M_I, F)) / \mathcal{I}_{S_{I \sqcup I'}})) / \mathcal{I}_{S_{J'}} \\ \rightarrow \text{holim}_{I \in L} p_{I'J'}^* p_{J'(I \sqcup J')}^* (p_{I(I \sqcup J')}^{*mod[-]} (M_I, F)) / \mathcal{I}_{S_{I \sqcup J'}} \end{aligned}$$

Definition-Proposition 15. Let $S \in \text{Var}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; or let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then $\mathcal{P}Sh_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$ does not depend on the open covering of S and the closed embeddings and we set

$$\mathcal{P}Sh_{\mathcal{D}(2)fil}(S) := \mathcal{P}Sh_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$$

We denote by $C_{\mathcal{D}(2)fil}^0(S) := C(\mathcal{P}Sh_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)))$ and by $D_{\mathcal{D}(2)fil}^0(S) := \text{Ho}_{\text{Fr}top}(C_{\mathcal{D}(2)fil}^0(S))$ its localization with respect to r-filtered Zariski, resp. usu, local equivalences.

Proof. It is obvious that $T_S^{L/L'} : \text{PSh}_{\mathcal{D}fil}(S/(S_I)) \rightarrow \text{PSh}_{\mathcal{D}}(S/(S_I))$ is an equivalence of category with inverse $T_S^{L'/L} : \text{PSh}_{\mathcal{D}fil}(S/(S_I)) \rightarrow \text{PSh}_{\mathcal{D}}(S/(S_I))$. \square

We now give the definition of our category :

Definition 75. Let $S \in \text{Var}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; or let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then, $C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}(2)fil}(\tilde{S}_I)$ is the full subcategory

- whose objects are $(M, F) = ((M_I, F)_{I \subset [1, \dots, l]}, u_{IJ})$, with
 - $(M_I, F) \in C_{\mathcal{D}(2)fil, S_I}(\tilde{S}_I)$ (see definition 55),
 - $u_{IJ} : m^*(M_I, F) \rightarrow m^*p_{IJ*}(M_J, F)[d_{\tilde{S}_I} - d_{\tilde{S}_J}]$ for $J \subset I$, are morphisms, $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ being the projection, satisfying for $I \subset J \subset K$, $p_{IJ} * u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{\mathcal{D}fil}(\tilde{S}_I)$;
- the morphisms $m : ((M_I, F), u_{IJ}) \rightarrow ((N_I, F), v_{IJ})$ between $(M, F) = ((M_I, F)_{I \subset [1, \dots, l]}, u_{IJ})$ and $(N, F) = ((N_I, F)_{I \subset [1, \dots, l]}, v_{IJ})$ being a family of morphisms of complexes,

$$m = (m_I : (M_I, F) \rightarrow (N_I, F))_{I \subset [1, \dots, l]}$$

such that $v_{IJ} \circ m_I = p_{IJ*}m_J \circ u_{IJ}$ in $C_{\mathcal{D}fil}(\tilde{S}_I)$.

We denote by $C_{\mathcal{D}(2)fil}^{\sim}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$ the full subcategory consisting of objects $((M_I, F), u_{IJ})$ such that the u_{IJ} are ∞ -filtered Zariski, resp. usu, local equivalences.

Let $S \in \text{Var}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; or let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then, We denote by

$$C_{\mathcal{D}(2)fil, h}^{\sim}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}(2)fil, c}^{\sim}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}(2)fil}^{\sim}(S/(\tilde{S}_I))$$

the full subcategories consisting of those $((M_I, F), u_{IJ}) \in C_{\mathcal{D}(2)fil}^{\sim}(S/(\tilde{S}_I))$ such that $(M_I, F) \in C_{\mathcal{D}(2)fil, S_I, c}(\tilde{S}_I)$, that is such that $a_{\tau}H^n M_I$ are coherent for all $n \in \mathbb{Z}$ and all $I \subset [1, \dots, l]$. resp. such that $(M_I, F) \in C_{\mathcal{D}(2)fil, S_I, h}(\tilde{S}_I)$, that is such that $a_{\tau}H^n M_I$ are holonomic for all $n \in \mathbb{Z}$ and all $I \subset [1, \dots, l]$. We denote by

$$C_{\mathcal{D}(1,0)fil}^{\sim}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}2fil}^{\sim}(S/(\tilde{S}_I)), C_{\mathcal{D}(1,0)fil, h}^{\sim}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}2fil, h}^{\sim}(S/(\tilde{S}_I))$$

the full subcategories consisting of those $((M_I, F, W), u_{IJ}) \in C_{\mathcal{D}2fil}^{\sim}(S/(\tilde{S}_I))$ such that $W^p M_I$ are $D_{\tilde{S}_I}$ submodules (resp. and $a_{\tau}H^n M_I$ holonomic).

A morphism $m = (m_I) : ((M_I), u_{IJ}) \rightarrow ((N_I), v_{IJ})$ in $C_{\mathcal{D}}(S/(\tilde{S}_I))$ is a Zariski, resp. usu, local equivalence if all the m_I are Zariski, resp. usu, local equivalence. A morphism $m = (m_I) : ((M_I, F), u_{IJ}) \rightarrow ((N_I, F), v_{IJ})$ in $C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$ is an r -filtered Zariski, resp. usu, local equivalence if all the m_I are r -filtered Zariski, resp. usu, local equivalence.

In the analytic case, we also define in the same way :

Definition 76. Let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then, $C_{\mathcal{D}^{\infty}(2)fil}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}^{\infty}(2)fil}(\tilde{S}_I)$ is the full subcategory

- whose objects are $(M, F) = ((M_I, F)_{I \subset [1, \dots, l]}, u_{IJ})$, with
 - $(M_I, F) \in C_{\mathcal{D}^{\infty}fil, S_I}(\tilde{S}_I)$ (see definition 56),
 - $u_{IJ} : m^*(M_I, F) \rightarrow m^*p_{IJ*}(M_J, F)[d_{\tilde{S}_I} - d_{\tilde{S}_J}]$, for $J \subset I$, are morphisms, $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ being the projection, satisfying for $I \subset J \subset K$, $p_{IJ*}u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{\mathcal{D}^{\infty}fil}(\tilde{S}_I)$;

- the morphisms $m : ((M_I, F), u_{IJ}) \rightarrow ((N_I, F), v_{IJ})$ between $(M, F) = ((M_I, F)_{I \subset [1, \dots, l]}, u_{IJ})$ and $(N, F) = ((N_I, F)_{I \subset [1, \dots, l]}, v_{IJ})$ being a family of morphisms of complexes,

$$m = (m_I : (M_I, F) \rightarrow (N_I, F))_{I \subset [1, \dots, l]}$$

such that $v_{IJ} \circ m_I = p_{IJ*} m_J \circ u_{IJ}$ in $C_{\mathcal{D}^\infty \text{fil}}(\tilde{S}_I)$.

We denote by $C_{\mathcal{D}^\infty(2)\text{fil}}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}^\infty(2)\text{fil}}(S/(\tilde{S}_I))$ the full subcategory consisting of objects $((M_I, F), u_{IJ})$ such that the u_{IJ} are ∞ -filtered usu local equivalence.

Let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. We denote by

$$C_{\mathcal{D}^\infty(2)\text{fil},h}^{\sim}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}^\infty(2)\text{fil},c}^{\sim}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}^\infty(2)\text{fil}}^{\sim}(S/(\tilde{S}_I))$$

the full subcategories consisting of $((M_I, F), u_{IJ}) \in C_{\mathcal{D}^\infty(2)\text{fil}}^{\sim}(S/(\tilde{S}_I))$ such that $(M_I, F) \in C_{\mathcal{D}^\infty(2)\text{fil},S_I,c}(\tilde{S}_I)$, that is such that $a_\tau H^n M_I$ are coherent for all $n \in \mathbb{Z}$ and all $I \subset [1, \dots, l]$, resp. such that $(M_I, F) \in C_{\mathcal{D}^\infty(2)\text{fil},S_I,h}(\tilde{S}_I)$, that is such that $a_\tau H^n M_I$ are holonomic for all $n \in \mathbb{Z}$ and all $I \subset [1, \dots, l]$. We denote by

$$C_{\mathcal{D}^\infty(1,0)\text{fil}}^{\sim}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}^\infty 2\text{fil}}^{\sim}(S/(\tilde{S}_I)), \quad C_{\mathcal{D}^\infty(1,0)\text{fil},h}^{\sim}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}^\infty 2\text{fil},h}^{\sim}(S/(\tilde{S}_I))$$

the full subcategories consisting of those $((M_I, F, W), u_{IJ}) \in C_{\mathcal{D} 2\text{fil}}^{\sim}(S/(\tilde{S}_I))$ such that $W^p M_I$ are $D_{\tilde{S}_I}$ submodules (resp. and $a_\tau H^n M_I$ holonomic).

A morphism $m = (m_I) : ((M_I), u_{IJ}) \rightarrow ((N_I), v_{IJ})$ in $C_{\mathcal{D}^\infty}(S/(\tilde{S}_I))$ is said to an usu local equivalence if all the m_I are usu local equivalences. A morphism $m = (m_I) : ((M_I, F), u_{IJ}) \rightarrow ((N_I, F), v_{IJ})$ in $C_{\mathcal{D}^\infty(2)\text{fil}}(S/(\tilde{S}_I))$ is said to an r -filtered usu local equivalence if all the m_I are r -filtered usu local equivalences.

Definition 77. Let $S \in \text{Var}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; or let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. We denote by

$$D_{\mathcal{D}(2)\text{fil},\infty}(S/(\tilde{S}_I)) := \text{Ho}_{F\infty, \text{top}}(C_{\mathcal{D}(2)\text{fil}}^{\sim}(S/(\tilde{S}_I)))$$

the localizations with respect to ∞ -filtered Zariski, resp. usu, local equivalences. We have

$$D_{\mathcal{D}(1,0)\text{fil},\infty,h}(S/(\tilde{S}_I)) \subset D_{\mathcal{D} 2\text{fil},\infty,h}(S/(\tilde{S}_I)) \subset D_{\mathcal{D} 2\text{fil},\infty}(S/(\tilde{S}_I))$$

the full subcategories which are the image of $C_{\mathcal{D} 2\text{fil},h}(S/(\tilde{S}_I))$, resp. of $C_{\mathcal{D}(1,0)\text{fil},h}(S/(\tilde{S}_I))$, by the localization functor $D(\text{top}) : C_{\mathcal{D}(2)\text{fil}}^{\sim}(S/(\tilde{S}_I)) \rightarrow D_{\mathcal{D}(2)\text{fil},\infty}(S/(\tilde{S}_I))$.

In the analytic case, we also have

Definition 78. Let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. We denote by

$$D_{\mathcal{D}^\infty(2)\text{fil},\infty}(S/(\tilde{S}_I)) := \text{Ho}_{Frtop}(C_{\mathcal{D}^\infty(2)\text{fil}}^{\sim}(S/(\tilde{S}_I)))$$

the localizations with respect to usu local equivalence. We have then

$$D_{\mathcal{D}^\infty(1,0)\text{fil},\infty,h}(S/(\tilde{S}_I)) \subset D_{\mathcal{D}^\infty 2\text{fil},\infty,h}(S/(\tilde{S}_I)) \subset D_{\mathcal{D}^\infty 2\text{fil},\infty}(S/(\tilde{S}_I))$$

the full subcategories which are the image of $C_{\mathcal{D}^\infty 2\text{fil},h}(S/(\tilde{S}_I))$, resp. $C_{\mathcal{D}^\infty(1,0)\text{fil},h}(S/(\tilde{S}_I))$, by the localization functor $D(\text{usu}) : C_{\mathcal{D}^\infty(2)\text{fil}}^{\sim}(S/(\tilde{S}_I)) \rightarrow D_{\mathcal{D}^\infty(2)\text{fil},\infty}(S/(\tilde{S}_I))$.

Definition 79. Let $S \in \text{Var}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Or let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$.

(i) We denote by

$$C_{\mathcal{D}(2)\text{fil}}(S/(\tilde{S}_I))^0 \subset C_{\mathcal{D}(2)\text{fil}}(S/(\tilde{S}_I))$$

the full subcategory consisting of $((M_I, F), u_{IJ}) \in C_{\mathcal{D}(2)\text{fil}}(S/(\tilde{S}_I))$ such that

$$H^n((M_I, F), u_{IJ}) = (H^n(M_I, F), H^n u_{IJ}) \in \text{PSh}_{\mathcal{D}(2)\text{fil}}^0(S/(\tilde{S}_I))$$

that is such that the $H^n u_{IJ}$ are isomorphism. We denote by $D_{\mathcal{D}(2)\text{fil}}(S/(\tilde{S}_I))^0 := D(\text{top})(C_{\mathcal{D}(2)\text{fil}}(S/(\tilde{S}_I))^0)$ its image by the localization functor.

(ii) We have the full embedding functor

$$\begin{aligned} \iota_{S/(\tilde{S}_I)}^0 : C_{\mathcal{D}(2)\text{fil}}^0(S) &:= C_{\mathcal{D}(2)\text{fil}}^0(S/(\tilde{S}_I)) \hookrightarrow C_{\tilde{\mathcal{D}}(2)\text{fil}}^0(S/(\tilde{S}_I)), \\ &((M_I, F), s_{IJ}) \mapsto ((M_I, F), s_{IJ}) \end{aligned}$$

By definition, $\iota_{S/(\tilde{S}_I)}^0(C_{\mathcal{D}(1,0)\text{fil}}^0(S/(\tilde{S}_I))) \subset C_{\tilde{\mathcal{D}}(1,0)\text{fil}}^0(S/(\tilde{S}_I))$. This full embedding induces in the derived category the functor

$$\begin{aligned} \iota_{S/(\tilde{S}_I)}^0 : D_{\mathcal{D}(2)\text{fil},\infty}^0(S) &:= D_{\mathcal{D}(2)\text{fil},\infty}^0(S/(\tilde{S}_I)) \rightarrow D_{\mathcal{D}(2)\text{fil},\infty}(S/(\tilde{S}_I)), \\ &((M_I, F), s_{IJ}) \mapsto ((M_I, F), s_{IJ}). \end{aligned}$$

Proposition 78. Let $S \in \text{Var}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Or let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then,

$$\iota_{S/(\tilde{S}_I)}^0 : D_{\mathcal{D}(2)\text{fil},\infty}^0(S) \rightarrow D_{\mathcal{D}(2)\text{fil},\infty}(S/(\tilde{S}_I))$$

is a full embedding whose image is $D_{\mathcal{D}(2)\text{fil},\infty}(S/(\tilde{S}_I))^0$, that is consists of $((M_I, F), s_{IJ}) \in C_{\mathcal{D}(2)\text{fil}}(S/(\tilde{S}_I))$ such that

$$H^n((M_I, F), s_{IJ}) := (H^n(M_I, F), H^n(s_{IJ})) \in \text{PSh}_{\mathcal{D}}^0(S/(\tilde{S}_I)).$$

and

$$\iota_S^0 := \iota_{S/(\tilde{S}_I)}^0 : D_{\mathcal{D}(2)\text{fil},\infty}^0(S) \xrightarrow{\sim} D_{\mathcal{D}(2)\text{fil},\infty}(S/(\tilde{S}_I))^0$$

the induced equivalence of categories.

Proof. Standard. □

We finish this subsection by the statement a result of kashiwara in the singular case.

Definition 80. Let $S \in \text{AnSp}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$.

(i) We denote by

$$C_{\mathcal{D}(1,0)\text{fil},rh}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}2\text{fil},rh}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}2\text{fil},h}(S/(\tilde{S}_I))$$

the full subcategories consisting of $((M_I, F, W), u_{IJ}) \in C_{\mathcal{D}(2)\text{fil}}(S/(\tilde{S}_I))$ such that $a_{usu} H^n M_I$ are regular holonomic, resp. and $W^p M_I$ are $D_{\tilde{S}_I}$ submodules. We denote by

$$D_{\mathcal{D}(1,0)\text{fil},\infty,rh}(S) \subset D_{\mathcal{D}2\text{fil},\infty,rh}(S) \subset D_{\mathcal{D}2\text{fil},\infty,h}(S)$$

the image of $C_{\mathcal{D}(1,0)\text{fil},rh}(S/(\tilde{S}_I))$, resp. $C_{\mathcal{D}2\text{fil},rh}(S/(\tilde{S}_I))$, by the localization functor $D(\text{usu}) : C_{\tilde{\mathcal{D}}^\infty(2)\text{fil}}^0(S/(\tilde{S}_I)) \rightarrow D_{\mathcal{D}^\infty(2)\text{fil},\infty}(S/(\tilde{S}_I))$.

(ii) We will consider the functor

$$J_S : C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)) \rightarrow C_{\mathcal{D}^\infty(2)fil}(S/(\tilde{S}_I)),$$

$$((M_I, F), u_{IJ}) \mapsto J_S((M_I, F), u_{IJ}) := (J_{\tilde{S}_I}(M_I, F), J(u_{IJ})) := ((M_I \otimes_{D_S} D_S^\infty, F), J(u_{IJ}))$$

with, denoting for short $d_{IJ} := d_{\tilde{S}_J} - d_{\tilde{S}_I}$,

$$J(u_{IJ}) : J(M_I, F) \xrightarrow{J(u_{IJ})} J(p_{IJ*}(M_J, F)[d_{IJ}]) \xrightarrow{T_*(p_{IJ}, J)(-)} p_{IJ*}J(M_J, F)[d_{IJ}].$$

Of course $J_S(C_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))) \subset C_{\mathcal{D}^\infty(1,0)fil}(S/(\tilde{S}_I))$.

Proposition 79. *Let $S \in \text{AnSp}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then the functor*

$$J_S : C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)) \rightarrow C_{\mathcal{D}^\infty(2)fil}(S/(\tilde{S}_I)),$$

satisfy $J_S : C_{\tilde{\mathcal{D}}(2)fil}(S/(\tilde{S}_I)) \subset C_{\tilde{\mathcal{D}}^\infty(2)fil}(S/(\tilde{S}_I))$ and induces an equivalence of category

$$J_S : D_{\mathcal{D}(2)fil, \infty, rh}(S/(\tilde{S}_I)) \rightarrow D_{\mathcal{D}^\infty(2)fil, \infty, h}(S/(\tilde{S}_I)).$$

and $J_S(D_{\mathcal{D}(1,0)fil, \infty, rh}(S/(\tilde{S}_I))) \subset D_{\mathcal{D}^\infty, h(1,0)fil, \infty}(S/(\tilde{S}_I))$.

Proof. Follows immediately from the smooth case (proposition 41). □

4.3.2 Duality in the singular case

The definition of Saito's category comes with a dual functor :

Definition 81. *Let $S \in \text{Var}(\mathbb{C})$ and let $S = \cup S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; or let $S \in \text{AnSp}(\mathbb{C})$ and $S = \cup S_i$ an open cover such that there exist closed embedding $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. We have the dual functor :*

$$\mathbb{D}_S^K : C_{\mathcal{D}fil}^0(S/(\tilde{S}_I)) \rightarrow C_{\mathcal{D}fil}^0(S/(\tilde{S}_I)), ((M_I, F), s_{IJ}) \mapsto (\mathbb{D}_{\tilde{S}_I}^K(M_I, F), s_{IJ}^d),$$

with, denoting for short $d_{IJ} := d_{\tilde{S}_J} - d_{\tilde{S}_I}$,

$$u_{IJ}^q : \mathbb{D}_{\tilde{S}_I}^K(M_I, F) \xrightarrow{\mathbb{D}^K(s_{IJ}^{-1})} \mathbb{D}_{\tilde{S}_I}^K p_{IJ*}(M_J, F)[d_{IJ}] \xrightarrow{T_*(p_{IJ}, D)(-)} p_{IJ*} \mathbb{D}_{\tilde{S}_J}^K(M_J, F)[d_{IJ}]$$

It induces in the derived category the functor

$$L\mathbb{D}_S^K : D_{\mathcal{D}fil}^0(S/(\tilde{S}_I)) \rightarrow D_{\mathcal{D}fil}^0(S/(\tilde{S}_I)), ((M_I, F), s_{IJ}) \mapsto \mathbb{D}_S^K Q((M_I, F), s_{IJ}),$$

with $q : Q((M_I, F), s_{IJ}) \rightarrow ((M_I, F), s_{IJ})$ a projective resolution.

In the analytic case we also define

Definition 82. *Let $S \in \text{AnSp}(\mathbb{C})$ and $S = \cup S_i$ an open cover such that there exist closed embedding $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. We have the dual functor :*

$$\mathbb{D}_S^{K, \infty} : C_{\mathcal{D}^\infty fil}(S/(\tilde{S}_I)) \rightarrow C_{\mathcal{D}^\infty fil}(S/(\tilde{S}_I)), ((M_I, F), u_{IJ}) \mapsto (\mathbb{D}_{\tilde{S}_I}^{K, \infty}(M_I, F), u_{IJ}^d),$$

with u_{IJ}^d defined similarly as in definition 81. It induces in the derived category the functor

$$L\mathbb{D}_S^{K, \infty} : D_{\mathcal{D}^\infty fil}(S/(\tilde{S}_I)) \rightarrow D_{\mathcal{D}^\infty fil}(S/(\tilde{S}_I)), ((M_I, F), u_{IJ}) \mapsto (\mathbb{D}_S^{K, \infty} Q((M_I, F), u_{IJ}^d),$$

with $q : Q((M_I, F), s_{IJ}) \rightarrow ((M_I, F), s_{IJ})$ a projective resolution.

4.3.3 Inverse image in the singular case

We give in this subsection the inverse image functors between our categories.

Let $n : S^o \hookrightarrow S$ be an open embedding with $S \in \text{Var}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; or let $n : S^o \hookrightarrow S$ be an open embedding with $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Denote $S_I^o := n^{-1}(S_I) = S_I \cap S^o$ and $n_I := n|_{S_I^o} : S_I^o \hookrightarrow S^o$ the open embeddings. Consider open embeddings $\tilde{n}_I : \tilde{S}_I^o \hookrightarrow \tilde{S}_I$ such that $\tilde{S}_I^o \cap S_I = S_I^o$, that is which are lift of n_I . We have the functor

$$n^* : C_{\mathcal{D}fil}(S/(\tilde{S}_I)) \rightarrow C_{\mathcal{D}fil}(S^o/(\tilde{S}_I^o)),$$

$$(M, F) = ((M_I, F), u_{IJ}) \mapsto n^*(M, F) := (\tilde{n}_I)^*(M, F) := (\tilde{n}_I^*(M_I, F), n^* u_{IJ})$$

which derive trivially.

Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{Var}(\mathbb{C})$, such that there exist a factorization $f; X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection, and consider $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; or let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{AnSp}(\mathbb{C})$, such that there exist a factorization $f; X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{AnSm}(\mathbb{C})$, l a closed embedding and p_S the projection and consider $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$, with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then, $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. Denote by $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ and $p'_{IJ} : Y \times \tilde{S}_J \rightarrow Y \times \tilde{S}_I$ the projections and by

$$E_{IJ} = \begin{array}{ccc} \tilde{S}_J \setminus S_J & \xrightarrow{m_J} & \tilde{S}_J \\ \downarrow p_{IJ} & & \downarrow p_{IJ} \\ \tilde{S}_I \setminus (S_I \setminus S_J) & \xrightarrow{m_{IJ}} & \tilde{S}_I \end{array}, \quad E'_{IJ} = \begin{array}{ccc} Y \times \tilde{S}_J \setminus X_J & \xrightarrow{m'_J} & Y \times \tilde{S}_J \\ \downarrow p'_{IJ} & & \downarrow p'_{IJ} \\ Y \times \tilde{S}_I \setminus (X_I \setminus X_J) & \xrightarrow{m'_{IJ}} & Y \times \tilde{S}_I \end{array}, \quad E_{fIJ} = \begin{array}{ccc} \tilde{X}_J & \xrightarrow{\tilde{f}_J} & \tilde{S}_J \\ \downarrow p'_{IJ} & & \downarrow p_{IJ} \\ Y \times \tilde{S}_I & \xrightarrow{\tilde{f}_I} & \tilde{S}_I \end{array}$$

the commutative diagrams. The (graph) inverse image functors is :

$$f^{*mod[-], \Gamma} : C_{\mathcal{D}fil}(S/(\tilde{S}_I)) \rightarrow C_{\mathcal{D}fil}(X/(Y \times \tilde{S}_I)),$$

$$(M, F) = ((M_I, F), u_{IJ}) \mapsto f^{*mod[-], \Gamma}(M, F) := (\Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]}(M_I, F)), \tilde{f}_J^{*mod[-]} u_{IJ})$$

with, denoting for short $d_{IJ} := d_{\tilde{S}_I} - d_{\tilde{S}_J}$,

$$\begin{aligned} \tilde{f}_J^{*mod[-]} u_{IJ} : \Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]}(M_I, F)) &\xrightarrow{\Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]}(u_{IJ}))} \Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]} p_{IJ*}(M_J, F)[d_{IJ}]) \\ &\xrightarrow{\Gamma_{X_I} E(T(p_{IJ}^{*mod}, p_{\tilde{S}_I})(-)^{-1})[d_Y + d_{IJ}]} \Gamma_{X_I} E(p'_{IJ*} p_{\tilde{S}_J}^{*mod}(M_J, F)[d_Y + d_{IJ}]) \\ &\xrightarrow{\cong} p'_{IJ*} \Gamma_{X_J} E(p_{\tilde{S}_J}^{*mod[-]}(M_J, F))[d_{IJ}]. \end{aligned}$$

It induces in the derived categories the functor

$$Rf^{*mod[-], \Gamma} : D_{\mathcal{D}(2)fil, \infty}(S/(\tilde{S}_I)) \rightarrow D_{\mathcal{D}(2)fil, \infty}(X/(Y \times \tilde{S}_I)),$$

$$(M, F) = ((M_I, F), u_{IJ}) \mapsto Rf^{*mod[-], \Gamma}(M, F) := (\Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]}(M_I, F)), \tilde{f}_J^{*mod[-]} u_{IJ}).$$

It gives by duality the functor

$$Lf^{\hat{*}mod[-], \Gamma} : D_{\mathcal{D}(2)fil, \infty}(S/(\tilde{S}_I))^0 \rightarrow D_{\mathcal{D}(2)fil, \infty}(X/(Y \times \tilde{S}_I))^0,$$

$$(M, F) = ((M_I, F), u_{IJ}) \mapsto Lf^{\hat{*}mod[-], \Gamma}(M, F) := L\mathbb{D}_S^K Rf^{*mod[-], \Gamma} L\mathbb{D}_S^K \iota_S^{0, -1}(M, F).$$

where $\iota_S^0 : D_{\mathcal{D}(2)fil, \infty}^0(S/(\tilde{S}_I)) \xrightarrow{\cong} D_{\mathcal{D}(2)fil, \infty}(S/(\tilde{S}_I))^0$ is the isomorphism of definition 79.

Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{AnSp}(\mathbb{C})$, such that there exist a factorization $f; X \xrightarrow{l} Y \times S \xrightarrow{ps} S$ with $Y \in \text{AnSm}(\mathbb{C})$, l a closed embedding and p_S the projection and consider $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$, with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then, $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. We have also the functors,

$$f^{*mod[-], \Gamma} : C_{\mathcal{D}^\infty fil}(S/(\tilde{S}_I)) \rightarrow C_{\mathcal{D}^\infty fil}(X/(Y \times \tilde{S}_I)),$$

$$(M, F) = ((M_I, F), u_{IJ}) \mapsto f^{*mod[-], \Gamma}(M, F) := (\Gamma_{X_I}(p_{\tilde{S}_I}^{*mod[-]}(M_I, F)), \tilde{f}_J^{*mod[-]} u_{IJ})$$

with, denoting for short $d_{IJ} := d_{\tilde{S}_J} - d_{\tilde{S}_I}$,

$$\begin{aligned} \tilde{f}_J^{*mod[-]} u_{IJ} : \Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]}(M_I, F)) &\xrightarrow{\Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]})(u_{IJ})} \Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]} p_{IJ*}(M_J, F)[d_{IJ}]) \\ &\xrightarrow{\Gamma_{X_I} E(T(p_{IJ}^{*mod}, p_{\tilde{S}_I})(-)^{-1})[d_Y + d_{IJ}]} \Gamma_{X_I} E(p'_{IJ*} p_{\tilde{S}_J}^{*mod}(M_J, F)[d_Y + d_{IJ}]) \\ &\xrightarrow{=} p'_{IJ*} \Gamma_{X_J} E(p_{\tilde{S}_J}^{*mod[-]}(M_J, F))[d_{IJ}]. \end{aligned}$$

It induces in the derived categories, the functor

$$Rf^{*mod[-], \Gamma} : D_{\mathcal{D}^\infty(2)fil, \infty}(S/(\tilde{S}_I)) \rightarrow D_{\mathcal{D}^\infty(2)fil, \infty}(X/(Y \times \tilde{S}_I)),$$

$$(M, F) = ((M_I, F), u_{IJ}) \mapsto Rf^{*mod[-], \Gamma}(M, F) := (\Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]}(M_I, F)), \tilde{f}_J^{*mod[-]} u_{IJ}).$$

It gives by duality the functor

$$Lf^{\hat{*}mod[-], \Gamma} : D_{\mathcal{D}^\infty(2)fil, \infty}(S/(\tilde{S}_I))^0 \rightarrow D_{\mathcal{D}^\infty(2)fil, \infty}(X/(Y \times \tilde{S}_I))^0,$$

$$(M, F) = ((M_I, F), u_{IJ}) \mapsto Lf^{\hat{*}mod[-], \Gamma}(M, F) := L\mathbb{D}^{K, \infty} Rf^{*mod[-], \Gamma} L\mathbb{D}^{K, \infty} \iota_S^{0, -1}(M, F).$$

where $\iota_S^0 : D_{\mathcal{D}^\infty(2)fil, \infty}^0(S/(\tilde{S}_I)) \xrightarrow{\sim} D_{\mathcal{D}^\infty(2)fil, \infty}(S/(\tilde{S}_I))^0$ is the isomorphism of definition 79.

The following proposition are then easy :

Proposition 80. *Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphism with $X, Y, S \in \text{Var}(\mathbb{C})$. Assume there exist factorizations $f_1 : X \xrightarrow{l_1} Y' \times Y \xrightarrow{p_Y} Y$ and $f_2 : Y \xrightarrow{l_2} Y'' \times S \xrightarrow{p_S} S$ with $Y', Y'' \in \text{SmVar}(\mathbb{C})$, l_1, l_2 closed embeddings and p_S, p_Y the projections. We have then the factorization*

$$f_2 \circ f_1 : X \xrightarrow{(l_2 \circ l_{Y'}) \circ l_1} Y' \times Y'' \times S \xrightarrow{p_S} S.$$

We have, for $(M, F) \in C_{\mathcal{D}(2)fil}^\sim(S/(\tilde{S}_I))$, $R(f_2 \circ f_1)^{*mod[-], \Gamma}(M, F) = Rf_2^{*mod[-], \Gamma} \circ Rf_1^{*mod[-], \Gamma}(M, F)$.

Proof. Follows from the the fact that for $(M, F) = ((M_I, F), u_{IJ}) \in C_{\mathcal{D}(2)fil}^\sim(S/(\tilde{S}_I))$,

$$\begin{aligned} (\Gamma_{X_I} E(\tilde{f}_{1I}^{*mod[-]} \Gamma_{Y_I} E(\tilde{f}_{2I}^{*mod[-]}(M_I, F))), \tilde{f}_{1J}^{*mod[-]}(\tilde{f}_{2J}^{*mod[-]} u_{IJ})) &\xrightarrow{=} \\ (\Gamma_{X_I} E((\tilde{f}_{1I} \circ \tilde{f}_{2I})^{*mod[-]}(M_I, F)), (\tilde{f}_{1J} \circ \tilde{f}_{2J})^{*mod[-]} u_{IJ}^q) & \end{aligned}$$

by proposition 43(i) and the fact that $X_I \subset \tilde{f}_{1I}^{-1}(Y_I)$. □

Proposition 81. *Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphism with $X, Y, S \in \text{Var}(\mathbb{C})$. Assume there exist factorizations $f_1 : X \xrightarrow{l_1} Y' \times Y \xrightarrow{p_Y} Y$ and $f_2 : Y \xrightarrow{l_2} Y'' \times S \xrightarrow{p_S} S$ with $Y', Y'' \in \text{SmVar}(\mathbb{C})$, l_1, l_2 closed embeddings and p_S, p_Y the projections. We have then the factorization*

$$f_2 \circ f_1 : X \xrightarrow{(l_2 \circ l_{Y'}) \circ l_1} Y' \times Y'' \times S \xrightarrow{p_S} S.$$

We have, for $(M, F) \in C_{\mathcal{D}(2)fil}^\sim(S/(\tilde{S}_I))$ or $(M, F) \in C_{\mathcal{D}^\infty(2)fil}^\sim(S/(\tilde{S}_I))$, $R(f_2 \circ f_1)^{*mod[-], \Gamma}(M, F) = Rf_2^{*mod[-], \Gamma} \circ Rf_1^{*mod[-], \Gamma}(M, F)$.

Proof. Similar to the proof of proposition 80. □

4.3.4 Direct image functor in the singular case

We define the direct image functors between our category.

Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{Var}(\mathbb{C})$, and assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S a the projection ; or let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSp}(\mathbb{C})$, and assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{AnSm}(\mathbb{C})$, l a closed embedding and p_S a the projection. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; resp. let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. Denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $X_I = \cap_{i \in I} X_i$. For $I \subset [1, \dots, l]$, denote by $\tilde{S}_I = \prod_{i \in I} \tilde{S}_i$. We then have, for $I \subset [1, \dots, l]$, closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I$ and the following commutative diagrams which are cartesian (we take $Y = \mathbb{P}^{N.o}$ in the algebraic case)

$$\begin{array}{ccccc} f|_{X_I} = f|_{X_I} : X_I & \xrightarrow{l_I} & Y \times S_I & \xrightarrow{p_{S_I}} & S_I & , & Y \times \tilde{S}_I & \xrightarrow{p_{\tilde{S}_I}} & \tilde{S}_I \\ & \searrow & \downarrow i'_I & & \downarrow i_I & & \downarrow p'_{IJ} & & \downarrow p_{IJ} \\ & & Y \times \tilde{S}_I & \xrightarrow{p_{\tilde{S}_I}} & \tilde{S}_I & & Y \times \tilde{S}_I & \xrightarrow{p_{\tilde{S}_I}} & \tilde{S}_I \end{array}$$

with $l_I : l|_{X_I}$, $i'_I = I \times i_I$, p_{S_I} and $p_{\tilde{S}_I}$ are the projections and $p'_{IJ} = I \times p_{IJ}$. Then $\tilde{f}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $f_I = f|_{X_I}$. We define the direct image functor on our category by

$$f_{*mod}^{FDR} : C_{\mathcal{D}(2)fil}(X/(Y \times \tilde{S}_I)) \rightarrow C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)),$$

$$((M_I, F), u_{IJ}) \mapsto (\tilde{f}_{I*mod}^{FDR}(M_I, F), f^k(u_{IJ})) := (p_{\tilde{S}_I*} E((\Omega_{Y \times \tilde{S}_I/\tilde{S}_I}^\bullet, F_b) \otimes_{O_{Y \times \tilde{S}_I}} (M_I, F)[d_Y]), f^k(u_{IJ}))$$

with, denoting for short $d_{IJ} := d_{\tilde{S}_I} - d_{\tilde{S}_I}$,

$$\begin{aligned} & f^k(u_{IJ})[d_Y] : p_{\tilde{S}_I*} E((\Omega_{Y \times \tilde{S}_I/\tilde{S}_I}^\bullet, F_b) \otimes_{O_{Y \times \tilde{S}_I}} (M_I, F)) \\ & \xrightarrow{p_{\tilde{S}_I*} E(DR(Y \times \tilde{S}_I/\tilde{S}_I)(u_{IJ}))} p_{\tilde{S}_I*} E((\Omega_{Y \times \tilde{S}_I/\tilde{S}_I}^\bullet, F_b) \otimes_{O_{Y \times \tilde{S}_I}} p'_{IJ*}(M_J, F)[d_{IJ}]) \\ & \xrightarrow{T_w^O(p_{IJ}, \otimes)(M_I, F)} p_{\tilde{S}_I*} E(p_{IJ*}(\Omega_{Y \times \tilde{S}_J/\tilde{S}_J}^\bullet, F_b) \otimes_{O_{Y \times \tilde{S}_J}} (M_J, F)[d_{IJ}]) \\ & \xrightarrow{=} p_{\tilde{S}_J*} E((\Omega_{Y \times \tilde{S}_J/\tilde{S}_J}^\bullet, F_b) \otimes_{O_{Y \times \tilde{S}_J}} (M_J, F)[d_{IJ}]). \end{aligned}$$

It induces in the derived categories the functor

$$\int_f^{FDR} : D_{\mathcal{D}(2)fil, \infty}(X) \rightarrow D_{\mathcal{D}(2)fil, \infty}(S), ((M_I, F), u_{IJ}) \mapsto (\tilde{f}_{I*mod}^{FDR}(M_I, F), f^k(u_{IJ}))$$

Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSp}(\mathbb{C})$, and assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{AnSm}(\mathbb{C})$, l a closed embedding and p_S a the projection. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. We also have the functors

$$f_{*mod}^{FDR} : C_{\mathcal{D}^\infty(2)fil}(X/(Y \times \tilde{S}_I)) \rightarrow C_{\mathcal{D}^\infty(2)fil}(S/(\tilde{S}_I)),$$

$$((M_I, F), u_{IJ}) \mapsto (\tilde{f}_{I*mod}^{FDR}(M_I, F), f^k(u_{IJ})) := (p_{\tilde{S}_I*} E((\Omega_{Y \times \tilde{S}_I/\tilde{S}_I}^\bullet, F_b) \otimes_{O_{Y \times \tilde{S}_I}} (M_I, F)[d_Y]), f^k(u_{IJ}))$$

where $f^k(u_{IJ})[d_Y]$ is given as above,

$$\int_f^{FDR} : D_{\mathcal{D}^\infty(2)fil, \infty}(X) \rightarrow D_{\mathcal{D}^\infty(2)fil, \infty}(S),$$

$$((M_I, F), u_{IJ}) \mapsto (\tilde{f}_{I*mod}^{FDR}(M_I, F), f^k(u_{IJ})) := (p_{\tilde{S}_I*} E((\Omega_{Y \times \tilde{S}_I/\tilde{S}_I}^\bullet, F_b) \otimes_{O_{Y \times \tilde{S}_I}} (M_I, F)[d_Y]), f^k(u_{IJ}))$$

where $f^k(u_{IJ})[d_Y]$ is given as above.

In the algebraic case, we have the followings:

Proposition 82. *Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphism with $X, Y, S \in \text{QPVar}(\mathbb{C})$ quasi-projective. Then there exist factorizations $f_1 : X \xrightarrow{l_1} Y' \times Y \xrightarrow{p_Y} Y$ and $f_2 : Y \xrightarrow{l_2} Y'' \times S \xrightarrow{p_S} S$ with $Y' = \mathbb{P}^{N, \circ} \subset \mathbb{P}^N, Y'' = \mathbb{P}^{N', \circ} \subset \mathbb{P}^{N'}$ open subsets, l_1, l_2 closed embeddings and p_S, p_Y the projections. We have then the factorization $f_2 \circ f_1 : X \xrightarrow{(l_2 \circ l_{Y'}) \circ l_1} Y' \times Y'' \times S \xrightarrow{p_S} S$. Let $i : S \hookrightarrow \tilde{S}$ a closed embedding with $\tilde{S} = \mathbb{P}^{n, \circ} \subset \mathbb{P}^n$ an open subset.*

(i) *Let $(M, F) \in C_{\mathcal{D}(2)fil}(X/(Y' \times Y'' \times \tilde{S}))$. Then, we have $\int_{f_2 \circ f_1}^{FDR}(M, F) = \int_{f_2}^{FDR}(\int_{f_1}^{FDR}(M, F))$ in $D_{\mathcal{D}(2)fil, \infty}(S/(\tilde{S}_I))$.*

(ii) *Let $(M, F) \in C_{\mathcal{D}(2)fil, h}(X/(Y' \times Y'' \times \tilde{S}))$. Then, we have $\int_{(f_2 \circ f_1)!}^{FDR}(M, F) = \int_{f_2!}^{FDR}(\int_{f_1!}^{FDR}(M, F))$ in $D_{\mathcal{D}(2)fil, \infty, h}(S/(\tilde{S}_I))$.*

Proof. (i): By the smooth case : proposition 66, we have en isomorphism

$$\int_{f_2}^{FDR} \int_{f_1}^{FDR} (M, F) := \int_{p_{\tilde{S}}}^{FDR} \int_{p_{Y' \times \tilde{S}}}^{FDR} (M, F) \xrightarrow{\sim} \int_{p_{\tilde{S}}}^{FDR} (M, F) := \int_{(f_2 \circ f_1)}^{FDR} (M, F).$$

(ii): Follows from (i). □

In the analytic case, we have the followings:

Proposition 83. *Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphism with $X, Y, S \in \text{AnSp}(\mathbb{C})$ quasi-projective. Then there exist factorizations $f_1 : X \xrightarrow{l_1} Y' \times Y \xrightarrow{p_Y} Y$ and $f_2 : Y \xrightarrow{l_2} Y'' \times S \xrightarrow{p_S} S$ with $Y' = \mathbb{P}^{N, \circ} \subset \mathbb{P}^N, Y'' = \mathbb{P}^{N', \circ} \subset \mathbb{P}^{N'}$ open subsets, l_1, l_2 closed embeddings and p_S, p_Y the projections. We have then the factorization $f_2 \circ f_1 : X \xrightarrow{(l_2 \circ l_{Y'}) \circ l_1} Y' \times Y'' \times S \xrightarrow{p_S} S$. Let $i : S \hookrightarrow \tilde{S}$ a closed embedding with $\tilde{S} = \mathbb{P}^{n, \circ} \subset \mathbb{P}^n$ an open subset.*

(i) *Let $(M, F) \in C_{\mathcal{D}^\infty(2)fil, h}(X/(Y' \times Y'' \times \tilde{S}))$. Then, we have $\int_{f_2 \circ f_1}^{FDR}(M, F) = \int_{f_2}^{FDR}(\int_{f_1}^{FDR}(M, F))$ in $D_{\mathcal{D}^\infty(2)fil, \infty}(S/(\tilde{S}_I))$.*

(ii) *Let $(M, F) \in C_{\mathcal{D}^\infty(2)fil, h}(X/(Y' \times Y'' \times \tilde{S}))$. Then, we have $\int_{(f_2 \circ f_1)!}^{FDR}(M, F) = \int_{f_2!}^{FDR}(\int_{f_1!}^{FDR}(M, F))$ in $D_{\mathcal{D}^\infty(2)fil, \infty}(S/(\tilde{S}_I))$.*

Proof. (i): By the smooth case : proposition 67, we have en isomorphism

$$\int_{f_2}^{FDR} \int_{f_1}^{FDR} (M, F) := \int_{p_{\tilde{S}}}^{FDR} \int_{p_{Y' \times \tilde{S}}}^{FDR} (M, F) \xrightarrow{\sim} \int_{p_{\tilde{S}}}^{FDR} (M, F) := \int_{(f_2 \circ f_1)}^{FDR} (M, F).$$

(ii): Follows from (i). □

4.3.5 Tensor product in the singular case

Let $S \in \text{Var}(\mathbb{C})$ and let $S = \cup S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; or let $S \in \text{AnSp}(\mathbb{C})$ and $S = \cup S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. We have the tensor product functors

$$\begin{aligned} (-) \otimes_{\mathcal{O}_S}^{[-]} (-) : C_{\mathcal{D}fil}^2(S/(\tilde{S}_I)) &\rightarrow C_{\mathcal{D}fil}(S/(\tilde{S}_I)), \\ (((M_I, F), u_{IJ}), ((N_I, F), v_{IJ})) &\mapsto ((M_I, F) \otimes_{\mathcal{O}_{\tilde{S}_I}} (N_I, F)[d_{\tilde{S}_I}], u_{IJ} \otimes v_{IJ}), \end{aligned}$$

with, denoting for short $d_{IJ} := d_{\tilde{S}_J} - d_{\tilde{S}_I}$ and $d_I := d_{\tilde{S}_I}$,

$$\begin{aligned} u_{IJ} \otimes v_{IJ} : (M_I, F) \otimes_{O_{\tilde{S}_I}} (N_I, F)[d_I] &\xrightarrow{T(p_{IJ}^{*mod}, p_{IJ})(-)[d_I]} p_{IJ*} p_{IJ}^{*mod}((M_I, F) \otimes_{O_{\tilde{S}_I}} (N_I, F)[d_I]) \\ &\xrightarrow{=} p_{IJ*}(p_{IJ}^{*mod}(M_I, F) \otimes_{O_{\tilde{S}_J}} p_{IJ}^{*mod}(N_I, F)[d_I]) \\ &\xrightarrow{I(p_{IJ}^{*mod}, p_{IJ})(-, -)(u_{IJ}) \otimes I(p_{IJ}^{*mod}, p_{IJ})(-, -)(v_{IJ})[d_I]} p_{IJ*}((M_J, F) \otimes_{O_{\tilde{S}_J}} (N_J, F)[d_J + d_{IJ}]). \end{aligned}$$

Let $S \in \text{AnSp}(\mathbb{C})$ and $S = \cup S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. We have the tensor product functors

$$\begin{aligned} (-) \otimes_{O_S}^{[-]} (-) : C_{\mathcal{D}^\infty \text{fil}}^2(S/(\tilde{S}_I)) &\rightarrow C_{\mathcal{D}^\infty \text{fil}}(S/(\tilde{S}_I)), \\ (((M_I, F), u_{IJ}), ((N_I, F), v_{IJ})) &\mapsto ((M_I, F) \otimes_{O_{\tilde{S}_I}} (N_I, F), u_{IJ} \otimes v_{IJ}), \end{aligned}$$

with $u_{IJ} \otimes v_{IJ}$ as above.

Proposition 84. *Let $S \in \text{Var}(\mathbb{C})$. Denote $\Delta_S : S \hookrightarrow S \times S$ the diagonal embedding. Let $S = \cup S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embedding with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; or let $S \in \text{AnSp}(\mathbb{C})$ and $S = \cup S_i$ an open cover such that there exist closed embedding $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. We have, for $((M_I, F), u_{IJ}), ((N_I, F), v_{IJ}) \in C_{\mathcal{D}^\infty \text{fil}}(S/(\tilde{S}_I))$,*

$$((M_I, F), u_{IJ}) \otimes_{O_{S_I}}^{[-]} ((N_I, F), v_{IJ}) = \Delta_S^{*mod}(((M_I, F), u_{IJ}).((N_I, F), v_{IJ}))$$

Proof. Follows from proposition 48. □

4.3.6 The 2 functors of D modules on the category of complex algebraic varieties and on the category of complex analytic spaces, and the transformation maps

Definition 83. *Consider a commutative diagram in $\text{Var}(\mathbb{C})$ which is cartesian :*

$$D = \begin{array}{ccc} X_T & \xrightarrow{f'} & T \\ \downarrow g' & & \downarrow g \\ X & \xrightarrow{f} & S \end{array} .$$

Assume there exist factorizations $f : X \xrightarrow{l_1} Y_1 \times S \xrightarrow{p_S} S$, $g : T \xrightarrow{l_2} Y_2 \times S \xrightarrow{p_S} S$, with $Y_1, Y_2 \in \text{SmVar}(\mathbb{C})$, l_1, l_2 closed embeddings and p_S, p_S the projections. Then, the above commutative diagram factors through

$$D = \begin{array}{ccccc} f' : X_T & \xrightarrow{l'_1} & Y_1 \times T & \xrightarrow{p_T} & T \\ \downarrow l'_2 & & \downarrow l''_2 = I \times l_2 & & \downarrow l_2 \\ f'' : X \times Y_2 & \xrightarrow{l'_1 = I \times l} & Y_1 \times Y_2 \times S & \xrightarrow{p_{Y_2 \times S}} & Y_2 \times S \\ \downarrow p_X & & \downarrow p_{Y_1 \times S} & & \downarrow p_S \\ f : X & \xrightarrow{l} & Y_1 \times S & \xrightarrow{p_S} & S \end{array} .$$

whose squares are cartesian. Let $S = \cup_i S_i$ be an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then $X = \cup_i X_i$ and $T = \cup_i T_i$ with $X_i := f^{-1}(S_i)$ and $T_i := g^{-1}(S_i)$. Moreover, $f_i = f|_{X_i} : X_i \rightarrow S_i$ lift to $\tilde{f}_i := p_{\tilde{S}_i} : Y_1 \times \tilde{S}_i \rightarrow \tilde{S}_i$ and $g_i = g|_{T_i} : T_i \rightarrow S_i$ lift to

$\tilde{g}_i := p_{\tilde{S}_i} : Y_2 \times \tilde{S}_i \rightarrow \tilde{S}_i$. We then have the following commutative diagram whose squares are cartesian

$$\begin{array}{ccccc}
f' : X_{IT} & \xrightarrow{l'} & Y_1 \times T_I & \xrightarrow{p_T} & T_I \\
\downarrow l'_{2I} & & \downarrow l'_{2I} & & \downarrow l_{2I} \\
X_I \times Y_2 & \xrightarrow{l'_1 = l_1 \times l_1} & Y_1 \times Y_2 \times \tilde{S}_I & \xrightarrow{p_{Y_2 \times \tilde{S}_I}} & Y_2 \times \tilde{S}_I \\
\downarrow p_X & & \downarrow p_{Y_1 \times \tilde{S}_I} & & \downarrow \tilde{g}_I \\
X_I & \xrightarrow{i_I \circ l_I} & Y_1 \times \tilde{S}_I & \xrightarrow{\tilde{f}_I} & \tilde{S}_I
\end{array}$$

We then define, for $(M, F) = ((M_I, F), u_{IJ}) \in C_{\mathcal{D}(2)fil}(X/(Y_1 \times \tilde{S}_I))$, the following canonical transformation map in $D_{\mathcal{D}(2)fil, \infty}(T/(Y_2 \times \tilde{S}_I))$, using proposition 72,

$$\begin{aligned}
& T^{\mathcal{D}mod}(f, g)(M, F) : \\
Rg^{*mod, \Gamma} \int_f^{FDR} (M, F) & := (\Gamma_{T_I} E(\tilde{g}_I^{*mod} p_{\tilde{S}_I^*} E((\Omega_{Y_1 \times \tilde{S}_I / \tilde{S}_I}^\bullet, F_b) \otimes_{O_{Y_1 \times \tilde{S}_I}} (M_I, F))), \tilde{g}_J^{*mod} f^k(u_{IJ})) \\
& \xrightarrow{(T_\omega^O(p_{\tilde{S}_I}, \tilde{g}_I)(M_I, F))} \\
& (\Gamma_{T_I} E(p_{Y_2 \times \tilde{S}_I^*} E((\Omega_{Y_1 \times Y_2 \times \tilde{S}_I / Y_2 \times \tilde{S}_I}^\bullet, F_b) \otimes_{O_{Y_1 \times Y_2 \times \tilde{S}_I}} p_{Y_1 \times \tilde{S}_I}^{*mod} (M_I, F))), f'^k(p_{Y_1 \times \tilde{S}_I}^{*mod} (u_{IJ}))) \\
& \xrightarrow{(T_\omega^O(\gamma, \otimes)(p_{Y_1 \times \tilde{S}_I}^{*mod} (M_I, F)))^{-1}} \\
& (p_{Y_2 \times \tilde{S}_I^*} E((\Omega_{Y_1 \times Y_2 \times \tilde{S}_I / Y_2 \times \tilde{S}_I}^\bullet, F_b) \otimes_{O_{Y_1 \times Y_2 \times \tilde{S}_I}} \Gamma_{Y_1 \times T_I} E(p_{Y_1 \times \tilde{S}_I}^{*mod} ((M_I, F))))), f'^k(\tilde{g}_J''^{*mod} (u_{IJ}^q))) \\
& =: \int_{f'}^{FDR} Rg'^{*mod, \Gamma}(M, F).
\end{aligned}$$

In the analytic case, we have

Definition 84. Consider a commutative diagram in $AnSp(\mathbb{C})$ which is cartesian :

$$D = (f, g) = \begin{array}{ccc} X_T & \xrightarrow{f'} & T \\ \downarrow g' & & \downarrow g \\ X & \xrightarrow{f} & S \end{array} .$$

Assume there exist factorizations $f : X \xrightarrow{l_1} Y_1 \times S \xrightarrow{p_S} S$, $g : T \xrightarrow{l_2} Y_2 \times S \xrightarrow{p_S} S$, with $Y_1, Y_2 \in AnSm(\mathbb{C})$, l_1, l_2 closed embeddings and p_S, p_S the projections.

(i) We have, for $(M, F) \in D_{\mathcal{D}(2)fil, \infty, h}(X/(Y_1 \times \tilde{S}_I))$, the following transformation map in $D_{\mathcal{D}(2)fil, \infty}(T/(Y_2 \times \tilde{S}_I))$

$$T^{\mathcal{D}mod}(f, g)((M, F)) : Rg^{*mod, \Gamma} \int_f^{FDR} (M, F) \rightarrow \int_{f'}^{FDR} Rg'^{*mod, \Gamma}(M, F)$$

define in the same way as in definition 83

(ii) For $(M, F) \in D_{\mathcal{D}^\infty(2)fil, \infty}(X/(Y_1 \times \tilde{S}_I))$, the following transformation map in $D_{\mathcal{D}^\infty(2)fil, \infty}(T/(Y_2 \times \tilde{S}_I))$

$$T^{\mathcal{D}mod}(f, g)((M, F)) : Rg^{*mod, \Gamma} \int_f^{FDR} (M, F) \rightarrow \int_{f'}^{FDR} Rg'^{*mod, \Gamma}(M, F)$$

is defined in the same way as in (ii) : see definition 83.

In the algebraic case, we have the following :

Proposition 85. *Consider a commutative diagram in $\text{Var}(\mathbb{C})$*

$$D = (f, g) = \begin{array}{ccc} X_T & \xrightarrow{f'} & T \\ \downarrow g' & & \downarrow g \\ X & \xrightarrow{f} & S \end{array} .$$

which is cartesian. Assume there exist factorizations $f : X \xrightarrow{l_1} Y_1 \times S \xrightarrow{p_S} S$, $g : T \xrightarrow{l_2} Y_2 \times S \xrightarrow{p_S} S$, with $Y_1, Y_2 \in \text{SmVar}(\mathbb{C})$, l_1, l_2 closed embeddings and p_S, p_S the projections. For $(M, F) = ((M_I, F), u_{IJ}) \in C_{\mathcal{D}(2)\text{fil},c}(X/(Y \times \tilde{S}_I))$,

$$T^{\mathcal{D}\text{mod}}(f, g) : Rg^{*\text{mod},\Gamma} \int_f^{\text{FDR}} (M, F) \rightarrow \int_{f'}^{\text{FDR}} Rg'^{*\text{mod},\Gamma}(M, F)$$

is an isomorphism in $D_{\mathcal{D}(2)\text{fil},\infty}(T/(Y_2 \times \tilde{S}_I))$.

Proof. Similar to the proof of proposition 74: the maps

$$\begin{aligned} T_\omega^O(p_{\tilde{S}_I}, \tilde{g}_I)(M_I, F) : \tilde{g}_I^{*\text{mod}} p_{\tilde{S}_I*} E((\Omega_{Y \times \tilde{S}_I/\tilde{S}_I}^\bullet, F_b) \otimes_{\mathcal{O}_{Y \times \tilde{S}_I}} (M_I, F)) \rightarrow \\ p_{\tilde{T}_I*} E((\Omega_{Y \times \tilde{T}_I/\tilde{T}_I}, F_b) \otimes_{\mathcal{O}_{Y \times \tilde{T}_I}} \tilde{g}_I''^{*\text{mod}}(M_I, F)) \end{aligned}$$

are ∞ -filtered Zariski local equivalences since $\tilde{g}_I : Y_2 \times \tilde{S}_I \rightarrow \tilde{S}_I$ are projections. \square

Proposition 86. *Consider a commutative diagram in $\text{AnSp}(\mathbb{C})$*

$$D = (f, g) = \begin{array}{ccc} X_T & \xrightarrow{f'} & T \\ \downarrow g' & & \downarrow g \\ X & \xrightarrow{f} & S \end{array} .$$

which is cartesian. Assume that f (hence f') is proper and that there exist factorizations $f : X \xrightarrow{l_1} Y_1 \times S \xrightarrow{p_S} S$, $g : T \xrightarrow{l_2} Y_2 \times S \xrightarrow{p_S} S$, with $Y_1, Y_2 \in \text{AnSm}(\mathbb{C})$, l_1, l_2 closed embeddings and p_S, p_S the projections.

(i) For $(M, F) = ((M_I, F), u_{IJ}) \in C_{\mathcal{D}(2)\text{fil},h}(X/(Y_1 \times \tilde{S}_I))$

$$T^{\mathcal{D}\text{mod}}(f, g) : Rg^{*\text{mod},\Gamma} \int_f^{\text{FDR}} (M, F) \rightarrow \int_{f'}^{\text{FDR}} Rg'^{*\text{mod},\Gamma}(M, F)$$

is an isomorphism in $D_{\mathcal{D}(2)\text{fil},\infty}(T/Y_2 \times \tilde{S}_I)$.

(ii) For $(M, F) = ((M_I, F), u_{IJ}) \in C_{\mathcal{D}^\infty(2)\text{fil},h}(X/(Y_1 \times \tilde{S}_I))$

$$T^{\mathcal{D}\text{mod}}(f, g) : Rg^{*\text{mod},\Gamma} \int_f^{\text{FDR}} (M, F) \rightarrow \int_{f'}^{\text{FDR}} Rg'^{*\text{mod},\Gamma}(M, F)$$

is an isomorphism in $D_{\mathcal{D}^\infty(2)\text{fil},\infty}(T/(Y_2 \times \tilde{S}_I))$.

Proof. (i): Similar to the proof of proposition 85.

(ii): Similar to the proof of proposition 85. \square

Definition 85. Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{Var}(\mathbb{C})$, such that there exist a factorization $f; X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection, and consider $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; Then, $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. We have, for $(M, F) = ((M_I, F), u_{IJ}) \in C_{\mathcal{D}(2)\text{fil}}(S/(\tilde{S}_I))$, the canonical transformation map in $D_{\mathcal{D}(2)\text{fil}}(T^{\text{an}}/(\tilde{T}_I^{\text{an}}))$

$$\begin{aligned} & T^{\text{mod}}(\text{an}, \gamma_T)(M, F) : \\ f^{*\text{mod}[-], \Gamma}(M, F)^{\text{an}} & := ((\Gamma_{T_I} E(p_{\tilde{S}_I}^{*\text{mod}[-]}(M_I, F)))^{\text{an}}, (f^{*\text{mod}[-]} u_{IJ})^{\text{an}}) \\ & \xrightarrow{(T^{\text{mod}}(\text{an}, \gamma_{T_I})(-))} (\Gamma_{T_I^{\text{an}}} E((p_{\tilde{S}_I}^{*\text{mod}[-]}(M_I, F))^{\text{an}}), f^{*\text{mod}[-]} u_{IJ}^{\text{an}}) \\ & \xrightarrow{=} (\Gamma_{T_I^{\text{an}}} E(p_{\tilde{S}_I}^{*\text{mod}[-]}(M_I^{\text{an}}, F)), f^{*\text{mod}[-]} u_{IJ}^{\text{an}}) =: f^{*\text{mod}[-], \Gamma}((M, F)^{\text{an}}) \end{aligned}$$

where the equality is obvious (see proposition 45).

Definition 86. Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We have, for $(M, F) = ((M_I, F), u_{IJ}) \in C_{\mathcal{D}\text{fil}}(X/Y \times \tilde{S}_I)$, the following transformation map in $D_{\mathcal{D}\text{fil}}(X^{\text{an}}/(Y \times \tilde{S}_I)^{\text{an}})$

$$\begin{aligned} T^{\mathcal{D}\text{mod}}(\text{an}, f)(M, F) & : \left(\int_f^{\text{FDR}} (M, F) \right)^{\text{an}} = (p_{\tilde{S}_I^*} E((\Omega_{Y \times \tilde{S}_I / \tilde{S}_I}^\bullet, F_b) \otimes_{\mathcal{O}_{Y \times \tilde{S}_I}} L_D(M_I, F)))^{\text{an}}, (f^k(u_{IJ}^q))^{\text{an}}) \\ & \xrightarrow{(T_\omega^{\mathcal{O}}(p_{\tilde{S}_I}, \text{an})(M_I, F))} (p_{\tilde{T}_I^*} E((\Omega_{Y \times \tilde{T}_I / \tilde{T}_I}^\bullet, F_b) \otimes_{\mathcal{O}_{(Y \times \tilde{T}_I)^{\text{an}}} } L_D(M_I, F)^{\text{an}}), f'^k((u_{IJ}^q)^{\text{an}})) =: \int_{f^{\text{an}}}^{\text{FDR}} (M, F)^{\text{an}} \end{aligned}$$

Theorem 22. Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Let $M \in D_{\mathcal{D}\text{fil}, c}(X/Y \times \tilde{S}_I)$. If f is proper,

$$T^{\mathcal{D}}(\text{an}, f)(M) : \left(\int_f M \right)^{\text{an}} \xrightarrow{\sim} \int_{f^{\text{an}}} (M)^{\text{an}}$$

is an isomorphism.

Proof. By theorem 20, $T_\omega^{\mathcal{O}}(p_{\tilde{S}_I}, \text{an})(M_I)$ are usu local equivalences. \square

In the analytic case, we have the following canonical transformation maps

Definition 87. Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{AnSp}(\mathbb{C})$, such that there exist a factorization $f; X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{AnSm}(\mathbb{C})$, l a closed embedding and p_S the projection, and consider $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$, with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$; Then, $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. We have, for $(M, F) = ((M_I, F), u_{IJ}) \in C_{\mathcal{D}(2)\text{fil}}(S/(\tilde{S}_I))$, the canonical transformation map in $D_{\mathcal{D}^\infty\text{fil}}(T/(\tilde{T}_I))$ obtained by the canonical maps given in definition 60 and definition 65 :

$$\begin{aligned} T(f, \infty)(M, F) & : J_T(f^{*\text{mod}[-], \Gamma}(M, F)) := (J_{\tilde{T}_I}(\Gamma_{T_I} E(p_{\tilde{S}_I}^{*\text{mod}[-]}(M_I, F))), J(f^{*\text{mod}[-]} u_{IJ})) \\ & \xrightarrow{(T(\infty, \gamma_{T_I})(-))} (\Gamma_{T_I} E(J_{\tilde{T}_I}(p_{\tilde{S}_I}^{*\text{mod}[-]}(M_I, F))), N_{IJ}) \\ & \xrightarrow{(T(p_{\tilde{S}_I}, \infty)(-))} (\Gamma_{T_I} E(p_{\tilde{S}_I}^{*\text{mod}[-]} J_{\tilde{S}_I}(M_I, F)), f^{*\text{mod}[-]} J(u_{IJ})) =: f^{*\text{mod}[-], \Gamma}(J_S(M, F)) \end{aligned}$$

4.4 The category of complexes of quasi-coherent sheaves whose cohomology sheaves has a structure of D-modules

4.4.1 Definition on a smooth complex algebraic variety or smooth complex analytic space and the functorialities

Let $X \in \text{SmVar}(\mathbb{C})$ or let $X \in \text{AnSm}(\mathbb{C})$. Recall that (see definition 50 section 4.1) $C_{O_X \text{fil}, \mathcal{D}}(X)$ is the category

- whose objects $(M, F) \in C_{O_X \text{fil}, \mathcal{D}}(X)$ are filtered complexes of presheaves of O_X modules $(M, F) \in C_{O_X \text{fil}}(X)$ whose cohomology presheaves $H^n(M, F) \in \text{PSh}_{O_X \text{fil}}(X)$ are endowed with a structure of filtered D_X modules for all $n \in \mathbb{Z}$.
- whose set of morphisms $\text{Hom}_{C_{O_X \text{fil}, \mathcal{D}}(X)}((M, F), (N, F)) \subset \text{Hom}_{C_{O_X \text{fil}}(X)}((M, F), (N, F))$ between $(M, F), (N, F) \in C_{O_X \text{fil}, \mathcal{D}}(X)$ are the morphisms of filtered complexes of O_X modules $m : (M, F) \rightarrow (N, F)$ such that $H^n m : H^n(M, F) \rightarrow H^n(N, F)$ is D_X linear, i.e. is a morphism of (filtered) D_X modules, for all $n \in \mathbb{Z}$.

More generally, let $h : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$ or with $X, S \in \text{AnSm}(\mathbb{C})$. Then, $C_{h^* O_S \text{fil}, h^* \mathcal{D}}(X)$ the category

- whose objects $(M, F) \in C_{h^* O_S \text{fil}, h^* \mathcal{D}}(X)$ are filtered complexes of presheaves of $h^* O_S$ modules $(M, F) \in C_{h^* O_S \text{fil}}(X)$ whose cohomology presheaves $H^n(M, F) \in \text{PSh}_{h^* O_S \text{fil}}(X)$ are endowed with a structure of filtered $h^* D_S$ modules for all $n \in \mathbb{Z}$.
- whose set of morphisms $\text{Hom}_{C_{h^* O_S \text{fil}, h^* \mathcal{D}}(X)}((M, F), (N, F)) \subset \text{Hom}_{C_{h^* O_S \text{fil}}(X)}((M, F), (N, F))$ between $(M, F), (N, F) \in C_{h^* O_S \text{fil}, h^* \mathcal{D}}(X)$ are the morphisms of filtered complexes of $h^* D_S$ modules $m : (M, F) \rightarrow (N, F)$ such that $H^n m : H^n(M, F) \rightarrow H^n(N, F)$ is $h^* D_S$ linear, i.e. is a morphism of (filtered) $h^* D_S$ modules, for all $n \in \mathbb{Z}$.

Definition 88. Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$. Let $Z \subset S$ a closed subset. Denote by $j : S \setminus Z \hookrightarrow S$ the open complementary embedding.

- We denote by $C_{O_S, \mathcal{D}, Z}(S) \subset C_{O_S, \mathcal{D}}(S)$ the full subcategory consisting of $(M, F) \in C_{O_S, \mathcal{D}}(S)$ such that $j^* H^n(M, F) = 0$ for all $n \in \mathbb{Z}$.
- We denote by $C_{O_S \text{fil}, \mathcal{D}, Z}(S) \subset C_{O_S \text{fil}, \mathcal{D}}(S)$ the full subcategory consisting of $(M, F) \in C_{O_S \text{fil}, \mathcal{D}}(S)$ such that there exist $r \in \mathbb{N}$ such that $j^* E_r^{p, q}(M, F) = 0$ for all $p, q \in \mathbb{Z}$, note that by definition r does NOT depend on p and q .

We look at functoriality

- Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$. Let $(M, F) \in C_{O_S \text{fil}, \mathcal{D}}(S)$. Then, the canonical morphism $q : L_O(M, F) \rightarrow (M, F)$ in $C_{O_S \text{fil}}(S)$ being a quasi-isomorphism of O_S modules, we get in a unique way $L_O(M, F) \in C_{O_S \text{fil}, \mathcal{D}}(S)$ such that $q : L_O(M, F) \rightarrow (M, F)$ is a morphism in $C_{O_S \text{fil}, \mathcal{D}}(S)$
- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{SmVar}(\mathbb{C})$, or let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Let $(M, F) \in C_{O_S \text{fil}, \mathcal{D}}(S)$. Then, $f^{* \text{mod}} H^n(M, F) := (O_X, F_b) \otimes_{f^* O_S} f^* H^n(M, F)$ is canonical a filtered D_X module (see section 4.1 or 4.2). Consider the canonical surjective map $q(f) : H^n f^{* \text{mod}}(M, F) \rightarrow f^{* \text{mod}} H^n(M, F)$. Then, $q(f)$ is an isomorphism if f is smooth. Let $h : U \rightarrow S$ be a smooth morphism with $U, S \in \text{SmVar}(\mathbb{C})$, or let $h : U \rightarrow S$ be a smooth morphism with $U, S \in \text{AnSm}(\mathbb{C})$. We get the functor

$$h^{* \text{mod}} : C_{O_S \text{fil}, \mathcal{D}}(S) \rightarrow C_{O_U \text{fil}, \mathcal{D}}(U), (M, F) \mapsto h^{* \text{mod}}(M, F),$$

- Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$, and let $i : Z \hookrightarrow S$ a closed embedding and denote by $j : S \setminus Z \hookrightarrow S$ the open complementary. For $M \in C_{O_S, \mathcal{D}}(S)$, the cohomology presheaves of

$$\Gamma_Z M := \text{Cone}(\text{ad}(j^*, j_*)(M) : M \rightarrow j_* j^* M)[-1]$$

has a canonical D_S -module structure (as $j^* H^n M$ is a $j^* D_S$ module, $H^n j_* j^* M = j_* j^* H^n M$ has an induced structure of D_S module), and $\gamma_Z(M) : \Gamma_Z M \rightarrow M$ is a map in $C_{O_S, \mathcal{D}}(S)$. For $Z_2 \subset Z$ a closed subset and $M \in C_{O_S, \mathcal{D}}(S)$, $T(Z_2/Z, \gamma)(M) : \Gamma_{Z_2} M \rightarrow \Gamma_Z M$ is a map in $C_{O_S, \mathcal{D}}(S)$. We get the functor

$$\begin{aligned} \Gamma_Z : C_{O_S \text{fil}, \mathcal{D}}(S) &\rightarrow C_{O_S \text{fil}, \mathcal{D}}(S), \\ (M, F) \mapsto \Gamma_Z(M, F) &:= \text{Cone}(\text{ad}(j^*, j_*)((M, F)) : (M, F) \rightarrow j_* j^*(M, F))[-1], \end{aligned}$$

together we the canonical map $\gamma_Z(M, F) : \Gamma_Z(M, F) \rightarrow (M, F)$

More generally, let $h : Y \rightarrow S$ a morphism with $Y, S \in \text{Var}(\mathbb{C})$ or $Y, S \in \text{AnSp}(\mathbb{C})$, S smooth, and let $i : X \hookrightarrow Y$ a closed embedding and denote by $j : Y \setminus X \hookrightarrow Y$ the open complementary. For $M \in C_{h^* O_S, h^* \mathcal{D}}(Y)$,

$$\Gamma_X M := \text{Cone}(\text{ad}(j^*, j_*)(M) : M \rightarrow j_* j^* M)[-1]$$

has a canonical $h^* D_S$ -module structure, (as $j^* H^n M$ is a $j^* h^* D_S$ module, $H^n j_* j^* M = j_* j^* H^n M$ has an induced structure of $j^* h^* D_S$ module), and $\gamma_X(M) : \Gamma_X M \rightarrow M$ is a map in $C_{h^* O_S, h^* \mathcal{D}}(Y)$. For $X_2 \subset X$ a closed subset and $M \in C_{h^* O_S, h^* \mathcal{D}}(Y)$, $T(Z_2/Z, \gamma)(M) : \Gamma_{X_2} M \rightarrow \Gamma_X M$ is a map in $C_{h^* O_S, h^* \mathcal{D}}(Y)$. We get the functor

$$\begin{aligned} \Gamma_X : C_{h^* O_S \text{fil}, h^* \mathcal{D}}(Y) &\rightarrow C_{h^* O_S \text{fil}, h^* \mathcal{D}}(Y), \\ (M, F) \mapsto \Gamma_X(M, F) &:= \text{Cone}(\text{ad}(j^*, j_*)((M, F)) : (M, F) \rightarrow j_* j^*(M, F))[-1], \end{aligned}$$

together we the canonical map $\gamma_X(M, F) : \Gamma_X(M, F) \rightarrow (M, F)$

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{SmVar}(\mathbb{C})$, or let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Consider the factorization $f : X \xrightarrow{l} X \times S \xrightarrow{p} S$, where l is the graph embedding and p the projection. We get from the two preceding points the functor

$$f^{*mod, \Gamma} : C_{O_S \text{fil}, \mathcal{D}}(S) \rightarrow C_{O_X \text{fil}, \mathcal{D}}(X \times S), (M, F) \mapsto f^{*mod, \Gamma}(M, F) := \Gamma_X p^{*mod}(M, F),$$

and

$$f^{*mod[-], \Gamma} : C_{O_S \text{fil}, \mathcal{D}}(S) \rightarrow C_{O_X \text{fil}, \mathcal{D}}(X \times S), (M, F) \mapsto f^{*mod[-], \Gamma}(M, F) := \Gamma_X E(p^{*mod}(M, F))[-d_X],$$

which induces in the derived categories the functor

$$Rf^{*mod[-], \Gamma} : D_{O_S \text{fil}, \mathcal{D}}(S) \rightarrow D_{O_X \text{fil}, \mathcal{D}}(X \times S), (M, F) \mapsto Rf^{*mod[-], \Gamma}(M, F) := \Gamma_X E(p^{*mod[-]}(M, F)).$$

For $(M, F) \in C_{O_S \text{fil}, \mathcal{D}}(S)$ or $(M, F) \in C_{O_S \text{fil}}(S)$, the canonical map in $C_{O_X \text{fil}}(X \times S)$

$$\text{ad}(i^{*mod}, i_*)(-) : L_O \Gamma_X E(p^{*mod}(M, F)) \rightarrow i_* i^{*mod} L_O \Gamma_X E(p^{*mod}(M, F))$$

gives in the derived category, the canonical map in $D_{O_X \text{fil}, \infty}(X \times S)$

$$\begin{aligned} I(f^{*mod, \Gamma})(M, F) : Rf^{*mod, \Gamma}(M, F) &= L_O \Gamma_X E(p^{*mod}(M, F)) \xrightarrow{\text{ad}(i^{*mod}, i_*)(-)} \\ i_* i^{*mod} L_O \Gamma_X E(p^{*mod}(M, F)) &\xrightarrow{\sim} i_* i^{*mod} L_O(p^{*mod}(M, F)) = Lf^{*mod}(M, F) \end{aligned}$$

where the isomorphism is given by lemma 9.

- Let $S \in \text{SmVar}(\mathbb{C})$. We have the analytical functor :

$$(-)^{an} : C_{O_S \text{fil}, \mathcal{D}}(S) \rightarrow C_{O_S \text{fil}, \mathcal{D}}(S^{an}), (M, F) \mapsto (M, F)^{an} := \text{an}_S^{*mod}(M, F) := (M, F) \otimes_{\text{an}_S^* O_S} O_{S^{an}}$$

which induces in the derived category

$$(-)^{an} : D_{O_S \text{fil}, \mathcal{D}}(S) \rightarrow D_{O_S \text{fil}, \mathcal{D}}(S^{an}), ((M, F) \mapsto (M, F)^{an} := \text{an}_S^{*mod}(M, F))$$

since an_S^{*mod} is an exact functor.

We have, for $f : T \rightarrow S$ with $T, S \in \text{SmVar}(\mathbb{C})$ or with $T, S \in \text{AnSm}(\mathbb{C})$, the commutative diagrams of functors

$$\begin{array}{ccc} C_{\mathcal{D}\text{fil}}(S) & \xrightarrow{o_S} & C_{O_S \text{fil}, \mathcal{D}}(S) & , & D_{\mathcal{D}\text{fil}, r}(S) & \xrightarrow{o_S} & D_{O_S \text{fil}, \mathcal{D}, r}(S) \\ \downarrow f^{*mod[-], \Gamma} & & \downarrow f^{*mod[-], \Gamma} & & \downarrow Rf^{*mod[-], \Gamma} & & \downarrow Rf^{*mod[-], \Gamma} \\ C_{\mathcal{D}\text{fil}}(T) & \xrightarrow{o_T} & C_{O_S \text{fil}, \mathcal{D}}(T) & & D_{\mathcal{D}\text{fil}, r}(T) & \xrightarrow{o_T} & D_{O_S \text{fil}, \mathcal{D}, r}(T) \end{array}$$

where o_S and o_T are the forgetfull functors.

4.4.2 Definition on a singular complex algebraic variety or singular complex analytic space and the functorialities

Definition 89. Let $S \in \text{Var}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; or let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then, $C_{O_S \text{fil}, \mathcal{D}}(S/(\tilde{S}_I))$ is the category

- whose objects are $(M, F) = ((M_I, F)_{I \subset [1, \dots, l]}, u_{IJ})$, with
 - $(M_I, F) \in C_{O_{\tilde{S}_I} \text{fil}, \mathcal{D}, S_I}(\tilde{S}_I)$,
 - $u_{IJ} : m^*(M_I, F) \rightarrow m^*p_{IJ*}(M_J, F)[d_{\tilde{S}_J} - d_{\tilde{S}_I}]$ for $J \subset I$, are morphisms, $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ being the projection, satisfying for $I \subset J \subset K$, $p_{IJ*}u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{O_{\tilde{S}_I} \text{fil}, \mathcal{D}}(\tilde{S}_I)$;
- whose morphisms $m : ((M_I, F), u_{IJ}) \rightarrow ((N_I, F), v_{IJ})$ between $(M, F) = ((M_I, F)_{I \subset [1, \dots, l]}, u_{IJ})$ and $(N, F) = ((N_I, F)_{I \subset [1, \dots, l]}, v_{IJ})$ are a family of morphisms of complexes,

$$m = (m_I : (M_I, F) \rightarrow (N_I, F))_{I \subset [1, \dots, l]}$$

such that $v_{IJ} \circ m_I = p_{IJ*}m_J \circ u_{IJ}$ in $C_{O_{\tilde{S}_I} \text{fil}, \mathcal{D}}(\tilde{S}_I)$.

We denote by $C_{O_S \text{fil}, \mathcal{D}}^{\sim}(S/(\tilde{S}_I)) \subset C_{O_S \text{fil}, \mathcal{D}}(S/(\tilde{S}_I))$ the full subcategory consisting of objects $((M_I, F), u_{IJ})$ such that the u_{IJ} are ∞ -filtered Zariski, resp. usu, local equivalences, and

$$D_{O_S \text{fil}, \mathcal{D}, \infty}(S/(\tilde{S}_I)) := Ho_{\mathcal{T}, \infty} C_{O_S \text{fil}, \mathcal{D}}^{\sim}(S/(\tilde{S}_I))$$

the derived category.

Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{Var}(\mathbb{C})$, such that there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection, and consider $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; or let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{AnSp}(\mathbb{C})$, such that there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{AnSm}(\mathbb{C})$, l a closed embedding and p_S the projection and consider $S = \cup_{i=1}^l S_i$ an open cover

such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$, with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then, $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. Denote by $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ and $p'_{IJ} : Y \times \tilde{S}_J \rightarrow Y \times \tilde{S}_I$ the projections and by

$$E_{IJ} = \begin{array}{ccc} \tilde{S}_J \setminus S_J & \xrightarrow{m_J} & \tilde{S}_J \\ \downarrow p_{IJ} & & \downarrow p_{IJ} \\ \tilde{S}_I \setminus (S_I \setminus S_J^{m=m_{IJ}}) & \xrightarrow{\quad} & \tilde{S}_I \end{array}, \quad E'_{IJ} = \begin{array}{ccc} Y \times \tilde{S}_J \setminus X_J & \xrightarrow{m'_J} & Y \times \tilde{S}_J \\ \downarrow p'_{IJ} & & \downarrow p'_{IJ} \\ Y \times \tilde{S}_I \setminus (X_I \setminus X_J^{m'=m'_{IJ}}) & \xrightarrow{\quad} & Y \times \tilde{S}_I \end{array}, \quad E_{fIJ} = \begin{array}{ccc} \tilde{X}_J & \xrightarrow{\tilde{f}_J} & \tilde{S}_J \\ \downarrow p'_{IJ} & & \downarrow p_{IJ} \\ Y \times \tilde{S}_I & \xrightarrow{\tilde{f}_I} & \tilde{S}_I \end{array}$$

the commutative diagrams. We then have the filtered De Rham the inverse image functor :

$$f^{*mod[-],\Gamma} : C_{Ofil,\mathcal{D}}(S/(\tilde{S}_I)) \rightarrow C_{Ofil,\mathcal{D}}(X/(Y \times \tilde{S}_I)), \quad (M, F) = ((M_I, F), u_{IJ}) \mapsto \\ f^{*mod[-],\Gamma}(M, F) := (\Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]}(M_I, F))), \tilde{f}_J^{*mod[-]} u_{IJ}$$

with, denoting for short $d_{IJ} := d_{\tilde{S}_J} - d_{\tilde{S}_I}$

$$\tilde{f}_J^{*mod[-]} u_{IJ} : \Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]}(M_I, F)) \xrightarrow{\Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]}(u_{IJ}))} \Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]} p_{IJ*}(M_J, F)[d_{IJ}]) \\ \xrightarrow{\Gamma_{X_I} E(T(p_{IJ}^{*mod}, p_{\tilde{S}_I})(-)^{-1})[d_Y + d_{IJ}]} \Gamma_{X_I} E(p'_{IJ*} p_{\tilde{S}_J}^{*mod}(M_J, F)[d_Y + d_{IJ}]) \\ \xrightarrow{\quad} p'_{IJ*} \Gamma_{X_J} E(p_{\tilde{S}_J}^{*mod[-]}(M_J, F)[d_{IJ}]).$$

It induces in the derived categories, the functor

$$Rf^{*mod[-],\Gamma} : D_{Ofil,\mathcal{D},\infty}(S/(\tilde{S}_I)) \rightarrow D_{Ofil,\mathcal{D},\infty}(X/(Y \times \tilde{S}_I)), \\ (M, F) = ((M_I, F), u_{IJ}) \mapsto \\ Rf^{*mod[-],\Gamma} := f^{*mod[-],\Gamma}(M, F) := (\Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]}(M_I, F)), \tilde{f}_J^{*mod[-]} u_{IJ}).$$

By definition, for $f : T \rightarrow S$ with $T, S \in \text{QPVar}(\mathbb{C})$ or with $T, S \in \text{AnSp}(\mathbb{C})^{QP}$, after considering a factorization $f : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection, the commutative diagrams of functors

$$\begin{array}{ccc} C_{\mathcal{D}fil}(S/(\tilde{S}_I)) & \xrightarrow{o_S} & C_{Ofil,\mathcal{D}}(S/(\tilde{S}_I)) & , & D_{\mathcal{D}fil,r}(S/(\tilde{S}_I)) & \xrightarrow{o_S} & D_{Ofil,\mathcal{D},\infty}(S/(\tilde{S}_I)) \\ \downarrow f^{*mod[-],\Gamma} & & \downarrow f^{*mod[-],\Gamma} & & \downarrow Rf^{*mod[-],\Gamma} & & \downarrow Rf^{*mod[-],\Gamma} \\ C_{\mathcal{D}fil}(T/(Y \times \tilde{S}_I)) & \xrightarrow{o_T} & C_{Ofil,\mathcal{D}}(T/(Y \times \tilde{S}_I)) & & D_{\mathcal{D}fil,\infty}(T/(Y \times \tilde{S}_I)) & \xrightarrow{o_T} & D_{Ofil,\mathcal{D},r}(T/(Y \times \tilde{S}_I)) \end{array}$$

where o_S and o_T are the forgetful functors.

Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{Var}(\mathbb{C})$, such that there exist a factorization $f; X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection, and consider $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; Then, $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. We have, for $(M, F) = ((M_I, F), u_{IJ}) \in C_{Ofil,\mathcal{D}}(S/(\tilde{S}_I))^\vee$, the canonical transformation map in $D_{Ofil,\mathcal{D}}(T^{an}/(\tilde{T}_I^{an}))^\vee$

$$T^{mod}(an, \gamma_T)(M, F) : \\ f^{*mod[-],\Gamma}(M, F)^{an} := ((\Gamma_{T_I} E(p_{\tilde{S}_I}^{*mod[-]}(M_I, F)))^{an}, (f^{*mod[-]} u_{IJ})^{an}) \\ \xrightarrow{(T^{mod}(an, \gamma_{T_I})(-))} (\Gamma_{T_I^{an}} E((p_{\tilde{S}_I}^{*mod[-]}(M_I, F))^{an}), f^{*mod[-]} u_{IJ}^{an}) \\ \xrightarrow{\quad} (\Gamma_{T_I^{an}} E(p_{\tilde{S}_I}^{*mod[-]}(M_I^{an}, F)), f^{*mod[-]} u_{IJ}^{an}) =: f^{*mod[-],\Gamma}((M, F)^{an})$$

where the equality is obvious.

5 The category of mixed Hodge modules on complex algebraic varieties and complex analytic spaces and the functorialities

For $S \in \text{Top}$ a topological space endowed with a stratification $S = \sqcup_{k=1}^d S_k$ by locally closed subsets S_k together with the perversity $p(S_k)$, we denote by $\mathcal{P}(S, W) \subset D_{fil}(S)$ the category of filtered perverse sheaves of abelian groups. For a locally compact (hence Hausdorff) topological space, we denote by $D_c(S) \subset D(S)$ the full subcategory of complexes of presheaves whose cohomology sheaves are constructible.

5.1 The De Rham functor for D modules on a complex analytic space

Let $S \in \text{AnSm}(\mathbb{C})$. Recall we have the dual functor

$$\mathbb{D}_S : C(S) \rightarrow C(S), K \mapsto \mathbb{D}_S(K) := \mathcal{H}om(K, E(\mathbb{Z}_S))$$

which induces the functor

$$L\mathbb{D}_S : D(S) \rightarrow D(S), K \mapsto L\mathbb{D}_S(K) := \mathbb{D}_S(LK) := \mathcal{H}om(LK, E_{et}(\mathbb{Z}_S)).$$

Let $S \in \text{AnSm}(\mathbb{C})$.

- The functor

$$M \in \text{PSh}_{\mathcal{D}}(S) \mapsto DR(S)(M) := \Omega_S^\bullet \otimes_{O_S} M \in C_{\mathbb{C}_S}(S)$$

which sends a D_S module to its De Rham complex (see section 4) induces, after shifting by d_S in order to send holonomic module (degree zero) to perverse sheaves, in the derived category the functor

$$DR(S)^{[-1]} : D_{\mathcal{D}}(S) \rightarrow D_{\mathbb{C}_S}(S), M \mapsto DR(S)^{[-1]}(M) := DR(S)(M)[d_S] := \Omega_S^\bullet \otimes_{O_S} M[d_S] \simeq K_S \otimes_{D_S}^L M \simeq \mathcal{H}om_{D_S}(\mathbb{D}_S L_D M, E(O_S))[d_S]$$

and, by functoriality, the functor

$$DR(S)^{[-1]} : D_{\mathcal{D}0fil, \infty}(S) \rightarrow D_{\mathbb{C}_S fil, \infty}(S), \\ (M, W) \mapsto DR(S)^{[-1]}(M, W) := (\Omega_S^\bullet, F_b) \otimes_{O_S} (M, W)[d_S] = K_S \otimes_{D_S}^L (M, W)$$

- On the other hand, we have the functor

$$C_{\mathbb{C}_S}(S) \rightarrow C_{\mathcal{D}^\infty}(S), K \mapsto \mathcal{H}om_{\mathbb{C}_S}(L_{\mathbb{C}} \mathbb{D}_S(LK), E(O_S))[-d_S]$$

together with, for $K \in C_{\mathbb{C}_S}(S)$, the canonical map

$$s(K) : K \rightarrow DR(S)^{[-1]}(J_S^{-1} \mathcal{H}om_{\mathbb{C}_S}(L_{\mathbb{C}}(\mathbb{D}_S LK), E(O_S))[-d_S]) \\ \xrightarrow{\cong} \mathcal{H}om_{D_S}(\mathbb{D}_S^K L_D J_S^{-1} \mathcal{H}om_{\mathbb{C}_S}(L_{\mathbb{C}}(\mathbb{D}_S LK), E(O_S)), E(O_S)), \\ c \in \Gamma(S^o, L(K)) \mapsto s(K)(c) = (\phi \in \Gamma(S^{oo}, L_{\mathcal{D}} \mathcal{H}om(L_{\mathbb{C}}(K), E(O_S))) \mapsto \phi(c))$$

where $S^{oo} \subset S^o \subset S$ are open subsets.

The main result is Riemann-Hilbert equivalence :

Theorem 23. *Let $S \in \text{AnSm}(\mathbb{C})$.*

- (i) *The functor $J_S : D_{\mathcal{D}, rh}(S) \rightarrow D_{\mathcal{D}^\infty, h}(S)$ is an equivalence of category. Moreover, for $K \in C(S)$, we have $\mathcal{H}om(L(K), E(O_S)) \in C_{\mathcal{D}^\infty, h}(S)$.*

(ii) The restriction of the De Rahm functor to the full subcategory $D_{\mathcal{D},rh}(S) \subset D_{\mathcal{D}}(S)$ is an equivalence of category

$$DR(S)^{[-1]} : D_{\mathcal{D},rh}(S) \xrightarrow{\sim} D_{C_S,c}(S)$$

whose inverse is the functor

$$K \in C_{C_S,c}(S) \mapsto J^{-1}\mathcal{H}om_{C_S}(\mathbb{D}_S L(K), E(O_S))[-d_S],$$

the map $s(K) : K \xrightarrow{\sim} DR(S)^{[-1]}(J^{-1}\mathcal{H}om_{C_S}(L_C \mathbb{D}_S L(K), E(O_S)))$ being an isomorphism.

(iii) The De Rahm functor $DR(S)^{[-1]}$ sends regular holonomic modules to perverse sheaves.

Proof. See [18]. □

Let $S \in \text{AnSp}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$.

- The De Rham functor is in this case

$$DR(S)^{[-1]} : D_{\mathcal{D}0fil,\infty}(S) \rightarrow D_{C_Sfil,\infty}(S), M = ((M_I, W), u_{IJ}) \mapsto DR(S)^{[-1]}(M, W) := (DR(\tilde{S}_I)^{[-1]}(M_I, W), DR^{[-1]}(u_{IJ})) := (\Omega_{\tilde{S}_I}^\bullet \otimes_{O_{\tilde{S}_I}}(M_I, W), DR^{[-1]}(u_{IJ}))$$

with, denoting for short $d_I = d_{\tilde{S}_I}$

$$\begin{aligned} DR^{[-1]}(u_{IJ}) : \Omega_{\tilde{S}_I}^\bullet \otimes_{O_{\tilde{S}_I}}(M_I, W)[d_I] &\xrightarrow{\text{ad}(p_{IJ}, p_{IJ*})(-)} p_{IJ*} p_{IJ}^* \Omega_{\tilde{S}_I}^\bullet \otimes_{O_{\tilde{S}_I}}(M_I, W)[d_I] \\ &\xrightarrow{p_{IJ*} \Omega_{\tilde{S}_J/\tilde{S}_I}[d_I]} p_{IJ*} \Omega_{\tilde{S}_J}^\bullet \otimes_{O_{\tilde{S}_J}} p_{IJ}^{*mod}(M_I, W)[d_I] \\ &\xrightarrow{p_{IJ*} I(p_{IJ}^{*mod}, p_{IJ})(-, -)(u_{IJ})[d_I]} p_{IJ*} \Omega_{\tilde{S}_J}^\bullet \otimes_{O_{\tilde{S}_J}}(M_J, W)[d_J + d_{IJ}] \end{aligned}$$

- Considering the diagrams

$$D_{IJ} = \begin{array}{ccc} \tilde{S}_J & \xrightarrow{p_{IJ}} & \tilde{S}_I \\ i_J \uparrow & & \uparrow i_I \\ S_J & \xrightarrow{j_{IJ}} & S_I \end{array}$$

we get the functor

$$C_{C_Sfil}(S) \xrightarrow{T(S/(S_I))} C_{C_Sfil}(S/(S_I)) \rightarrow C_{\mathcal{D}0fil}(S/(S_I)), \\ (K, W) \mapsto (\mathcal{H}om_{C_{\tilde{S}_I}}(L_C \mathbb{D}_{\tilde{S}_I}(L_{i_I*} j_I^*(K, W)), E(O_{\tilde{S}_I}))[-d_{\tilde{S}_I}], u_{IJ}(K, W))$$

where

$$\begin{aligned} u_{IJ}(K, W) : \mathcal{H}om_{C_{\tilde{S}_I}}(L_C \mathbb{D}_{\tilde{S}_I} L(i_{I*} j_I^*(K, W)), E(O_{\tilde{S}_I}))[-d_{\tilde{S}_I}] \\ \xrightarrow{\text{ad}(p_{IJ}^{*mod}, p_{IJ})(-)} p_{IJ*} p_{IJ}^{*mod}[-1] \mathcal{H}om_{C_{\tilde{S}_I}}(L_C \mathbb{D}_{\tilde{S}_I} L(i_{I*} j_I^*(K, W)), E(O_{\tilde{S}_I}))[-d_{\tilde{S}_J}] \\ \xrightarrow{\mathcal{H}om(-, Eo(p_{IJ})) \circ T(p_{IJ}, \text{hom})(-, -)} p_{IJ*} \mathcal{H}om_{C_{\tilde{S}_I}}(p_{IJ}^* L_C \mathbb{D}_{\tilde{S}_I} L(i_{I*} j_I^*(K, W)), E(O_{\tilde{S}_J}))[-d_{\tilde{S}_J}] \\ \xrightarrow{\mathcal{H}om(T(p_{IJ}, \mathbb{D})(-)^{-1}, -)} p_{IJ*} \mathcal{H}om_{C_{\tilde{S}_I}}(L_C \mathbb{D}_{\tilde{S}_J} p_{IJ}^* L(i_{I*} j_I^*(K, W)), E(O_{\tilde{S}_J}))[-d_{\tilde{S}_J}] \\ \xrightarrow{\mathcal{H}om(L_C \mathbb{D}_{\tilde{S}_J} T^a(D_{IJ})(j_I^*(K, W)), E(O_{\tilde{S}_J}))} p_{IJ*} \mathcal{H}om_{C_{\tilde{S}_J}}(L_C \mathbb{D}_{\tilde{S}_J} L(i_{J*} j_J^*(K, W)), E(O_{\tilde{S}_J}))[-d_{\tilde{S}_J}]. \end{aligned}$$

Moreover, for $(K, W) \in C_{fil}(S)$, we have

$$(\mathcal{H}om_{C_{\tilde{S}_I}}(L_C \mathbb{D}_{\tilde{S}_I} L(i_{I*} j_I^*(K, W)), E(O_{\tilde{S}_I}))[-d_{\tilde{S}_I}], u_{IJ}(K, W)) \in C_{\mathcal{D}\infty fil,h}(S)^0$$

and a canonical map in $D_{fil}(S) = D_{fil}(S/(\tilde{S}_I))$

$$s(K) : T(S/(S_I))(K, W) := (L(i_{I*}j_I^*(K, W)), I) \rightarrow \\ DR(S)^{[-1]}(J_S^{-1}\mathcal{H}om_{\mathbb{C}_{\tilde{S}_I}}(L_{\mathbb{C}}\mathbb{D}_{\tilde{S}_I}L(i_{I*}j_I^*(K, W)), E(O_{\tilde{S}_I}))[-d_{\tilde{S}_I}], u_{IJ}(K, W))$$

Corollary 3. *Let $S \in \text{AnSp}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$. The restriction of the De Rahm functor to the full subcategory $D_{\mathcal{D}, rh}^0(S) \subset D_{\mathcal{D}}^0(S)$ is an equivalence of category*

$$DR(S)^{[-1]} : D_{\mathcal{D}, rh}^0(S) \xrightarrow{\sim} D_{\mathbb{C}S, c}(S)$$

whose inverse is the functor

$$K \mapsto J_S^{-1}(\mathcal{H}om_{\mathbb{C}_{\tilde{S}_I}}(L_{\mathbb{C}}\mathbb{D}_{\tilde{S}_I}L(i_{I*}j_I^*K), E(O_{\tilde{S}_I}))[-d_{\tilde{S}_I}], u_{IJ}(K))$$

the map

$$s(K) : T(S/(S_I))(K, W) := (L(i_{I*}j_I^*(K, W)), I) \rightarrow \\ DR(S)^{[-1]}(J_S^{-1}\mathcal{H}om_{\mathbb{C}_{\tilde{S}_I}}(L_{\mathbb{C}}\mathbb{D}_{\tilde{S}_I}L(i_{I*}j_I^*(K, W)), E(O_{\tilde{S}_I}))[-d_{\tilde{S}_I}], u_{IJ}(K, W))$$

being an isomorphism.

Proof. Follows from theorem 23(ii), see [27]. □

Proposition 87. (i) *Let $S \in \text{AnSm}(\mathbb{C})$. Then, for $M \in C_{\mathcal{D}, c}(S)$, there is a canonical isomorphism*

$$T(D, DR)(M) : \mathbb{D}_S^{\mathbb{C}}DR(S)^{[-1]}(M) \xrightarrow{\sim} DR(S)^{[-1]}(\mathbb{D}_S^K L_D M)$$

(ii) *Let $S \in \text{AnSp}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$. Then, for $M = (M_I, u_{IJ}) \in C_{\mathcal{D}, c}^0(S/(\tilde{S}_I))$, there is a canonical isomorphism*

$$T(D, DR)(M) : \mathbb{D}_S^{\mathbb{C}}DR(S)^{[-1]}(M) \xrightarrow{\sim} DR(S)^{[-1]}(L\mathbb{D}_S^K M)$$

Proof. (i): See [16].

(ii): Follows from (i), see [27]. □

We have the following transformation maps :

- Let $g : T \rightarrow S$ a morphism with $T, S \in \text{AnSm}(\mathbb{C})$. We have, for $(M, W) \in C_{\mathcal{D}0fil, c}(S)$, the canonical transformation map in $D_{fil, c}(T)$:

$$T(g, DR)(M, W) : g^*DR(S)^{[-1]}(M, W) := g^*(\Omega_S^{\bullet} \otimes_{O_S} L_D(M, W))[d_S] \\ \xrightarrow{\Omega_{T/S}(L_D(M, W))[d_S]} \Omega_T^{\bullet} \otimes_{O_T} g^{*mod} L_D(M, W)[d_S] \\ \xrightarrow{=} \Omega_T^{\bullet} \otimes_{O_T} g^{*mod[-1]} L_D(M, W)[d_S] =: DR(T)^{(-)}(Lg^{*mod[-1]}(M, W))$$

Note that this transformation map is NOT an isomorphism in general. It is an isomorphism if g is a smooth morphism. If g is a closed embedding, it is an isomorphism for M non characteristic with respect to g .

- Let $j : S^{\circ} \hookrightarrow S$ an open embedding with $S \in \text{AnSm}(\mathbb{C})$. We have, for $(M, W) \in C_{\mathcal{D}0fil, c}(S^{\circ})$, the canonical transformation map in $D_{fil, c}(S)$:

$$T_*(j, DR)(M, W) : DR(S)^{[-1]}(j_*(M, W)) := \Omega_S^{\bullet} \otimes_{O_S} j_*(M, W)[d_S] \\ \xrightarrow{T_w^{\circ}(j, \otimes)(L_D(M, W))[d_S]} j_*(\Omega_{S^{\circ}}^{\bullet} \otimes_{O_{S^{\circ}}} L_D(M, W))[d_S] =: j_*DR(S)^{[-1]}(M, W)$$

which is an isomorphism (see proposition 72).

- Let $g : T \rightarrow S$ a morphism with $T, S \in \text{AnSp}(\mathbb{C})$. Assume there exist a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{AnSm}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_i S_i$ an open covers such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. We have, for $M = (M_I, u_{IJ}) \in C_{\mathcal{D},c}(S/(\tilde{S}_I))$, the canonical transformation map

$$\begin{aligned}
& T^1(g, DR)(M) : T(T/(Y \times \tilde{S}_I))(g^! DR(S)^{[-1]}(M, W)) \\
& \xrightarrow{=} (\Gamma_{T_I} E(\tilde{g}_I^*(\Omega_{\tilde{S}_I}^\bullet \otimes_{\mathcal{O}_{\tilde{S}_I}} L_D(M_I, W))), \tilde{g}_I^* DR(u_{IJ})) \\
& \xrightarrow{(\Omega_{(Y \times \tilde{S}_I/\tilde{S}_I)}(L_D(M_I, W)))} (\Gamma_{T_I} E(\Omega_{Y \times \tilde{S}_I}^\bullet \otimes_{\mathcal{O}_{Y \times \tilde{S}_I}} \tilde{g}_I^{*mod}(M_I, W)), DR(\tilde{g}_I^{*mod} u_{IJ})) \\
& \xrightarrow{(T_w^{\mathcal{O}}(\gamma, \otimes)(\tilde{g}_I^{*mod} L_D(M, W)))} (\Omega_{Y \times \tilde{S}_I}^\bullet \otimes_{\mathcal{O}_{Y \times \tilde{S}_I}} \Gamma_{T_I} E(\tilde{g}_I^{*mod}(M_I, W)), DR(\tilde{g}_I^{*mod} u_{IJ})) \\
& \xrightarrow{=} DR(T)^{[-1]}(Rg^{*mod[-], \Gamma}(M, W))
\end{aligned}$$

which is an isomorphism.

Proposition 88. *Let $f : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_i S_i$ an open covers such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, for $M = (M_I, u_{IJ}) \in C_{\mathcal{D},h}(S/(\tilde{S}_I))$,*

$$\begin{aligned}
& DR(T)^{[-1]}(T^{mod}(an, \gamma_T)(M)) : \\
& DR(T)^{[-1]}((Rf^{*mod[-], \Gamma} M)^{an}) := DR(T)^{[-1]}(((\Gamma_{T_I} E(\tilde{f}_I^{*mod[-]}(M_I)))^{an}, (f^{*mod[-]} u_{IJ}^q)^{an})) \\
& \rightarrow DR(T)^{[-1]}(Rf^{*mod[-], \Gamma}(M^{an})) := DR(T)^{[-1]}((\Gamma_{T_I^{an}} E(\tilde{f}_I^{*mod[-]}(M_I^{an})), f^{*mod[-]} u_{IJ}^{q, an}))
\end{aligned}$$

is an isomorphism.

Proof. See [16]. □

In the algebraic case, we have by proposition 88 the following canonical isomorphisms:

Definition 90. (i) *Let $f : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have, for $M = (M_I, u_{IJ}) \in C_{\mathcal{D},h}(S/(\tilde{S}_I))^0$, the canonical map*

$$\begin{aligned}
& T^1(f, DR)(M) : f^! DR(S)^{[-1]}(M^{an}) \xrightarrow{T^1(f, DR)(M^{an})} DR(T)^{[-1]}(Rf^{*mod[-], \Gamma}(M^{an})) \\
& \xrightarrow{DR(T)^{[-1]}(T^{mod}(an, \gamma_T)(M))} DR(T)^{[-1]}((Rf^{*mod[-], \Gamma} M)^{an}) =: DR(T)^{[-1]}((Rf^{*mod[-], \Gamma} M)^{an}).
\end{aligned}$$

which is an isomorphism by proposition 88.

(ii) *Let $f : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have, for $M = (M_I, u_{IJ}) \in C_{\mathcal{D},h}(S/(\tilde{S}_I))^0$, the canonical transformation map*

$$\begin{aligned}
& T(f, DR)(M) : DR(T)^{[-1]}((Lf^{\hat{*}mod[-], \Gamma} M)^{an}) := DR(T)^{[-1]}((L\mathbb{D}_T^K Rf^{*mod[-], \Gamma} L\mathbb{D}_S^K M)^{an}) \\
& \xrightarrow{T(D, DR)(-)} L\mathbb{D}_T^K DR(T)^{[-1]}((Rf^{*mod[-], \Gamma} L\mathbb{D}_S^K M)^{an}) \xrightarrow{L\mathbb{D}_T^K T^1(f, DR)(-)} L\mathbb{D}_T^K f^! DR(S)^{[-1]}(L\mathbb{D}_S^K M^{an}) \\
& \xrightarrow{L\mathbb{D}_T^K f^! T(D, DR)(-)^{-1}} L\mathbb{D}_T^K f^! L\mathbb{D}_S^K DR(S)^{[-1]}(M^{an}) = f^* DR(S)^{[-1]}(M^{an})
\end{aligned}$$

which is an isomorphism by (i) and proposition 87.

(iii) Let $f : T \rightarrow S$ a morphism with $T, S \in \text{QPVar}(\mathbb{C})$. Consider a factorization $f : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y = \mathbb{P}^{N,o} \subset \mathbb{P}^N$ an open subset, l a closed embedding, and p_S the projection. We have, for $M \in C_{\mathcal{D},h}(T/Y \times \tilde{S})^0$, the canonical transformation map

$$\begin{aligned} T_*(f, DR)(M) : DR(S)^{[-]} \left(\int_f M^{an} \right) &\xrightarrow{\text{ad}(f^*, Rf_*)(-)} Rf_* f^* DR(S)^{[-]} \left(\int_f M^{an} \right) \xrightarrow{Rf_* T(f, DR)((\int_f M))} \\ &Rf_* DR(T)^{[-]} \left((Lf^{*mod[-], \Gamma} \int_f M)^{an} \right) \xrightarrow{Rf_* DR(T)^{[-]}((\text{ad}(Lf^{*mod[-], \Gamma}, \int_f)(M))^{an})} Rf_* DR(T)^{[-]}(M^{an}) \end{aligned}$$

which is an isomorphism by GAGA in the proper case and by the open embedding case (c.f. proposition 88).

(iv) Let $f : T \rightarrow S$ a morphism with $T, S \in \text{QPVar}(\mathbb{C})$. Consider a factorization $f : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y = \mathbb{P}^{N,o} \subset \mathbb{P}^N$ an open subset, l a closed embedding, and p_S the projection. We have, for $M \in C_{\mathcal{D},h}(T)$, the canonical transformation map

$$\begin{aligned} T_!(f, DR)(M) : Rf_! DR(T)^{[-]}(M^{an}) &\xrightarrow{Rf_! DR(T)^{[-]}(\text{ad}(\int_{f!} Rf^{*mod[-], \Gamma})(M)^{an})} Rf_! DR(T)^{[-]} \left((Rf^{*mod[-], \Gamma} \int_{f!} (M))^{an} \right) \\ &\xrightarrow{T^!(f, DR)(\int_{f!} M)} Rf_! f^! DR(S)^{[-]} \left(\int_{f!} (M) \right) \xrightarrow{\text{ad}(Rf_!, f^!)(-)} DR(S)^{[-]} \left(\int_{f!} M \right) \end{aligned}$$

which is an isomorphism by (iii) and proposition 87.

5.2 The filtered Hodge direct image, the filtered Hodge inverse image, and the hodge support section functors for mixed hodge modules

We consider in the algebraic and analytic case the following categories :

- Let $S \in \text{AnSm}(\mathbb{C})$. The category $C_{\mathcal{D}(1,0)fil,rh}(S) \times_I C_{fil}(S)$ is the category
 - whose set of objects is the set of triples $\{((M, F, W), (K, W), \alpha)\}$ with
$$(M, F, W) \in C_{\mathcal{D}(1,0)fil,rh}(S), (K, W) \in C_{fil}(S), \alpha : (K, W) \otimes \mathbb{C}_S \rightarrow DR(S)^{[-]}((M, W))$$
where $DR(S)^{[-]}$ is the De Rahm functor and α is an isomorphism in $D_{fil}(S)$,
 - and whose set of morphisms are

$$\phi = (\phi_D, \phi_C) : ((M_1, F, W), (K_1, W), \alpha_1) \rightarrow ((M_2, F, W), (K_2, W), \alpha_2)$$

where $\phi_D : (M_1, F, W) \rightarrow (M_2, F, W)$ and $\phi_C : (K_1, W) \rightarrow (K_2, W)$ are morphisms such that $\alpha_2 \circ DR(S)^{[-]}(\phi_D) = \phi_C \circ \alpha_1$ in $D_{fil}(S)$.

We have then the full embedding

$$\text{PSh}_{\mathcal{D}(1,0)fil,rh}(S) \times_I P_{fil}(S) \hookrightarrow C_{\mathcal{D}(1,0)fil,rh}(S) \times_I D_{fil}(S)$$

The category $D_{\mathcal{D}(1,0)fil,\infty,rh}(S) \times_I D_{fil}(S)$ is then the category

- whose set of objects is the set of triples $\{((M, F, W), (K, W), \alpha)\}$ with
$$(M, F, W) \in D_{\mathcal{D}(1,0)fil,\infty,rh}(S), (K, W) \in D_{fil}(S), \alpha : (K, W) \otimes \mathbb{C}_S \rightarrow DR(S)^{[-]}((M, W))$$
where $DR(S)^{[-]}$ is the De Rahm functor and α is an isomorphism in $D_{fil}(S)$,

– and whose set of morphisms are

$$\phi = (\phi_D, \phi_C) : ((M_1, F, W), (K_1, W), \alpha_1) \rightarrow ((M_2, F, W), (K_2, W), \alpha_2)$$

where $\phi_D : (M_1, F, W) \rightarrow (M_2, F, W)$ and $\phi_C : (K_1, W) \rightarrow (K_2, W)$ are morphisms such that $\alpha_2 \circ DR(S)^{[-1]}(\phi_D) = \phi_C \circ \alpha_1$ in $D_{fil}(S)$.

together with the localization functor

$$(D(usu), D(usu)) : C_{\mathcal{D}(1,0)fil,rh}(S) \times_I C_{fil}(S) \rightarrow D_{\mathcal{D}(1,0)fil,\infty,rh}(S) \times_I D_{fil}(S)$$

- Let $S \in \text{AnSp}(\mathbb{C})$. Let $S = \cup_{i \in I} S_i$ an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. The category $C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I C_{fil}(S)$ is the category

– whose set of objects is the set of triples $\{((M_I, F, W), u_{IJ}), (K, W), \alpha\}$ with

$$\begin{aligned} ((M_I, F, W), u_{IJ}) &\in C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)), (K, W) \in C_{fil}(S), \\ \alpha : T(S/(\tilde{S}_I))(K, W) \otimes \mathbb{C}_S &\rightarrow DR(S)^{[-1]}((M_I, W), u_{IJ}) \end{aligned}$$

where $DR(S)^{[-1]}$ is the De Rahm functor and α is an isomorphism in $D_{fil}(S)$,

– and whose set of morphisms are

$$\phi = (\phi_D, \phi_C) : (((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1) \rightarrow (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2)$$

where $\phi_D : (M_1, F, W) \rightarrow (M_2, F, W)$ and $\phi_C : (K_1, W) \rightarrow (K_2, W)$ are morphisms such that $\alpha_2 \circ DR(S)^{[-1]}(\phi_D) = \phi_C \circ \alpha_1$ in $D_{fil}(S)$.

We have then full embeddings

$$\begin{aligned} \text{PSh}_{\mathcal{D}(1,0)fil,rh}^0(S/(\tilde{S}_I)) \times_I P_{fil}(S) &\hookrightarrow C_{\mathcal{D}(1,0)fil,rh}^0(S/(\tilde{S}_I)) \times_I D_{fil}(S) \\ \xrightarrow{\iota_{S/\tilde{S}_I}^0} C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))^0 \times_I D_{fil}(S) &\hookrightarrow C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S) \end{aligned}$$

The category $D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S)$ is then the category

– whose set of objects is the set of triples $\{((M, F, W), u_{IJ}), (K, W), \alpha\}$ with

$$\begin{aligned} ((M, F, W), u_{IJ}) &\in D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I)), (K, W) \in D_{fil}(S), \\ \alpha : (K, W) \otimes \mathbb{C}_S &\rightarrow DR(S)^{[-1]}((M, W), u_{IJ}) \end{aligned}$$

where $DR(S)^{[-1]}$ is the De Rahm functor and α is an isomorphism in $D_{fil}(S)$,

– and whose set of morphisms are

$$\phi = (\phi_D, \phi_C) : (((M_1, F, W), u_{IJ}), (K_1, W), \alpha_1) \rightarrow (((M_2, F, W), u_{IJ}), (K_2, W), \alpha_2)$$

where $\phi_D : ((M_1, F, W), u_{IJ}) \rightarrow ((M_2, F, W), u_{IJ})$ and $\phi_C : (K_1, W) \rightarrow (K_2, W)$ are morphisms such that $\alpha_2 \circ DR(S)^{[-1]}(\phi_D) = \phi_C \circ \alpha_1$.

together with the localization functor

$$(D(usu), D(usu)) : C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I C_{fil}(S) \rightarrow D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S)$$

We have then a full embedding

$$\begin{aligned} D_{\mathcal{D}(1,0)fil,\infty,rh}^0(S/(\tilde{S}_I)) \times_I D_{fil}(S) &\xrightarrow{\iota_{S/\tilde{S}_I}^0} D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I))^0 \times_I D_{fil}(S) \\ &\hookrightarrow D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S) \end{aligned}$$

- Let $S \in \text{SmVar}(\mathbb{C})$. The category $C_{\mathcal{D}(1,0)fil,rh}(S) \times_I C_{fil}(S^{an})$ is the category
 - whose set of objects is the set of triples $\{((M, F, W), (K, W), \alpha)\}$ with

$$(M, F, W) \in C_{\mathcal{D}(1,0)fil,rh}(S), (K, W) \in C_{fil}(S^{an}), \alpha : (K, W) \otimes \mathbb{C}_{S^{an}} \rightarrow DR(S)^{[-1]}((M, W)^{an})$$
 where $DR(S)^{[-1]}$ is the De Rahm functor and α is an isomorphism in $D_{fil}(S^{an})$,
 - and whose set of morphisms are

$$\phi = (\phi_D, \phi_C) : ((M_1, F, W), (K_1, W), \alpha_1) \rightarrow ((M_2, F, W), (K_2, W), \alpha_2)$$
 where $\phi_D : (M_1, F, W) \rightarrow (M_2, F, W)$ and $\phi_C : (K_1, W) \rightarrow (K_2, W)$ are morphisms such that $\alpha_2 \circ DR(S)^{[-1]}(\phi_D) = \phi_C \circ \alpha_1$ in $D_{fil}(S^{an})$.

We have then the full embedding

$$\text{PSh}_{\mathcal{D}(1,0)fil,rh}(S) \times_I P_{fil}(S^{an}) \hookrightarrow C_{\mathcal{D}(1,0)fil,rh}(S) \times_I D_{fil}(S^{an})$$

The category $D_{\mathcal{D}(1,0)fil,\infty,rh}(S) \times_I D_{fil}(S^{an})$ is then the category

- whose set of objects is the set of triples $\{((M, F, W), (K, W), \alpha)\}$

$$(M, F, W) \in D_{\mathcal{D}(1,0)fil,\infty,rh}(S), (K, W) \in D_{fil}(S^{an}), \alpha : (K, W) \otimes \mathbb{C}_S \rightarrow DR(S)^{[-1]}((M, W)^{an})$$
 where $DR(S)^{[-1]}$ is the De Rahm functor and α is an isomorphism in $D_{fil}(S^{an})$,
- and whose set of morphisms are

$$\phi = (\phi_D, \phi_C) : ((M_1, F, W), (K_1, W), \alpha_1) \rightarrow ((M_2, F, W), (K_2, W), \alpha_2)$$

where $\phi_D : (M_1, F, W) \rightarrow (M_2, F, W)$ and $\phi_C : (K_1, W) \rightarrow (K_2, W)$ are morphisms such that $\alpha_2 \circ DR(S)^{[-1]}(\phi_D) = \phi_C \circ \alpha_1$ in $D_{fil}(S^{an})$.

together with the localization functor

$$(D(zar), D(usu)) : C_{\mathcal{D}(1,0)fil,rh}(S) \times_I C_{fil}(S^{an}) \rightarrow D_{\mathcal{D}(1,0)fil,\infty,rh}(S) \times_I D_{fil}(S^{an})$$

- Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i \in I} S_i$ an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_I \in \text{SmVar}(\mathbb{C})$. The category $C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I C_{fil}(S^{an})$ is the category
 - whose set of objects is the set of triples $\{(((M_I, F, W), u_{IJ}), (K, W), \alpha)\}$ with

$$((M_I, F, W), u_{IJ}) \in C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)), (K, W) \in C_{fil}(S^{an}),$$

$$\alpha : T(S/(\tilde{S}_I))(K, W) \otimes \mathbb{C}_S \rightarrow DR(S)^{[-1]}(((M_I, W), u_{IJ})^{an})$$
 where $DR(S)^{[-1]}$ is the De Rahm functor and α is an isomorphism in $D_{fil}(S^{an})$,
 - and whose set of morphisms are

$$\phi = (\phi_D, \phi_C) : (((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1) \rightarrow (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2)$$

where $\phi_D : ((M_1, F, W), u_{IJ}) \rightarrow ((M_2, F, W), u_{IJ})$ and $\phi_C : (K_1, W) \rightarrow (K_2, W)$ are morphisms such that $\alpha_2 \circ DR(S)^{[-1]}(\phi_D) = \phi_C \circ \alpha_1$ in $D_{fil}(S^{an})$.

We have then full embeddings

$$\begin{aligned} \text{PSh}_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I P_{fil}(S^{an}) &\hookrightarrow C_{\mathcal{D}(1,0)fil,rh}^0(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}) \\ \xrightarrow{t_{S/\tilde{S}_I}^0} C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))^0 \times_I D_{fil}(S^{an}) &\hookrightarrow C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}) \end{aligned}$$

The category $D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$ is then the category

– whose set of objects is the set of triples $\{((M_I, F, W), u_{IJ}), (K, W), \alpha\}$ with

$$\begin{aligned} ((M_I, F, W), u_{IJ}) &\in D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I)), (K, W) \in D_{fil}(S^{an}), \\ \alpha &: (K, W) \otimes \mathbb{C}_{S^{an}} \rightarrow DR(S)^{[-1]}(((M_I, W), u_{IJ})^{an}) \end{aligned}$$

where $DR(S)$ is the De Rahm functor and α is an isomorphism in $D_{fil}(S)$,

– and whose set of morphisms are

$$\phi = (\phi_D, \phi_C) : (((M_1, F, W), u_{IJ}), (K_1, W), \alpha_1) \rightarrow (((M_2, F, W), u_{IJ}), (K_2, W), \alpha_2)$$

where $\phi_D : ((M_1, F, W), u_{IJ}) \rightarrow ((M_2, F, W), u_{IJ})$ and $\phi_C : (K_1, W) \rightarrow (K_2, W)$ are morphisms such that $\alpha_2 \circ DR(S)^{[-1]}(\phi_D^{an}) = \phi_C \circ \alpha_1$.

together with the localization functor

$$(D(zar), D(usu)) : C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I C_{fil}(S^{an}) \rightarrow D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$$

We have then a full embedding

$$\begin{aligned} D_{\mathcal{D}(1,0)fil,\infty,rh}^0(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}) &\xrightarrow{i_S^0} D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I))^0 \times_I D_{fil}(S^{an}) \\ &\hookrightarrow D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}) \end{aligned}$$

Proposition 89. *Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$. Let $Z = V(\mathbb{I}) \subset S$ a Zariski closed subset. Consider $(M, F) \in \text{PSh}_{\mathcal{D}(2)fil}(S)$. Then, a V_Z -filtration for (M, F) (see definition 49) if it exists is unique. In particular if $m : (M_1, F) \rightarrow (M_2, F)$ a morphism with $(M_1, F), (M_2, F) \in \text{PSh}_{\mathcal{D}(2)fil}(S)$, we have $m(V_Z^q F^p M_1) \subset V_Z^q F^p M_2$, that is we get $m : (M_1, F, V_Z) \rightarrow (M_2, F, V_Z)$.*

Proof. Standard. See [8] for example. □

The main tool is the nearby and vanishing cycle functors

Definition 91. *Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$. Let $D = V(s) \subset S$ a (Cartier) divisor, where $s \in \Gamma(S, L)$ is a section of the line bundle $L = L_D$ associated to D . We then have the zero section embedding $i : S \hookrightarrow L$. We denote $L_0 = i(S)$ and $j : L^\circ := L \setminus L_0 \hookrightarrow L$ the open complementary subset. Denoting $C_{\mathcal{D}(2)fil}(S)^{sp_i} \subset C_{\mathcal{D}(2)fil}(S)$ the full subcategory consisting of objects for which the V_D filtration exist, which is then unique by proposition 89, we have the nearby cycle functor*

$$\psi_{D1} : C_{\mathcal{D}(2)fil}(S)^{sp} \rightarrow C_{\mathcal{D}(2)fil}(D/(S)), (M, F) \mapsto \psi_{D1}(M, F) := \text{Gr}_{V_S,1} i_{*mod}(M, F)$$

and the vanishing cycle functor

$$\phi_{D1} : C_{\mathcal{D}(2)fil}(S)^{sp} \rightarrow C_{\mathcal{D}(2)fil}(D/(S)), (M, F) \mapsto \phi_{D1}(M, F) := \text{Gr}_{V_S,0} i_{*mod}(M, F).$$

This induces the functors

$$\begin{aligned} \psi_{D1} &: \text{PSh}_{\mathcal{D}(1,0)fil,rh}(S) \times_I P_{fil}(S^{an}) \rightarrow \text{PSh}_{\mathcal{D}2fil,rh,D}(S) \times_I P_{fil,D}(S^{an}), \\ ((M, F, W), (K, W), \alpha) &\mapsto \psi_{D1}((M, F, W), (K, W), \alpha) := (\psi_{D1}(M, F, W), \psi_D(K, W), \psi_D(\alpha)) \end{aligned}$$

and the vanishing cycle functor

$$\begin{aligned} \phi_{D1} &: \text{PSh}_{\mathcal{D}(1,0)fil,rh}(S) \times_I D_{fil}(S^{an}) \rightarrow \text{PSh}_{\mathcal{D}2fil,rh,D}(S) \times_I P_{fil,D}(S^{an}), \\ ((M, F, W), (K, W), \alpha) &\mapsto \phi_{D1}((M, F, W), (K, W), \alpha) := (\phi_{D1}(M, F, W), \phi_D(K, W), \phi_D(\alpha)) \end{aligned}$$

We have the category of mixed Hodge modules over a complex algebraic variety or a complex analytic space S defined by, for S smooth, by induction on dimension of S , and for S singular using embeddings into smooth complex algebraic varieties, resp. smooth complex analytic spaces:

Definition 92. [27]

(i) Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$. The category of Hodge modules over S is the full subcategory

$$\iota_S : HM(S) \hookrightarrow \text{PSh}_{\mathcal{D}fil, rh}(S) \times_I P(S),$$

given inductively by

- for $i_0 : s_0 \hookrightarrow S$ a closed point, $i_{0*}\iota_{\text{pt}} : HM_{s_0}(S) = HS \hookrightarrow \text{PSh}_{\mathcal{D}fil, rh, s_0}(S) \times_I P_{s_0}(S)$ consist of Hodge structures
- $((M, F), K, \alpha) \in \text{PSh}_{\mathcal{D}fil, rh}(S) \times_I P(S)$ belongs to $HM(S)$ if and only if for all proper maps $f : S^\circ \rightarrow \mathbb{A}^1$, $j : S^\circ \hookrightarrow S$ being an open subset, it is specializable along $S^\circ \subset S^\circ \times \mathbb{A}^1$, that is the V_{S° -filtration exists (see proposition 89 for the uniqueness of the V_{S° -filtration), and

$$\phi_{f^{-1}(0), 1}(j^*(M, F), j^*K, j^*\alpha) \in HM_{f^{-1}(0)}(S^\circ) \hookrightarrow \text{PSh}_{\mathcal{D}fil, rh, f^{-1}(0)}(S^\circ) \times_I P_{f^{-1}(0)}(S^\circ)$$

see definition 91.

(ii) Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$. The category of mixed Hodge modules over S is the full subcategory

$$\iota_S : MHM(S) \hookrightarrow MHW(S) \hookrightarrow \text{PSh}_{\mathcal{D}(1,0)fil, rh}(S) \times_I P_{fil}(S),$$

consisting of objects $((M, F, W), (K, W), \alpha) \in \text{PSh}_{\mathcal{D}(1,0)fil, rh}(S) \times_I P_{fil}(S)$ which for all proper maps $f : S^\circ \rightarrow \mathbb{A}^1$, $j : S^\circ \hookrightarrow S$ being an open subset, (M, F, W) is specializable along $S^\circ \subset S^\circ \times \mathbb{A}^1$, that is the V_{S° -filtration exists (see proposition 89 for the uniqueness of the V_{S° -filtration), and satisfy

$$(\text{Gr}_i^W((M, F, W), \text{Gr}_i^W(K, W), \text{Gr}_i^W \alpha) \in HM(S).$$

As usual, for $Z \subset S$ a closed subset and $j : S \setminus Z \hookrightarrow S$ the open complementary subset, we denote $MHM_Z(S) \subset MHM(S)$ the full subcategory consisting of $((M, F, W), (K, W), \alpha) \in MHM(S)$ such that $j^*((M, F, W), (K, W), \alpha) := (j^*(M, F, W), j^*(K, W), j^*\alpha) = 0$.

(iii) Let $S \in \text{Var}(\mathbb{C})$ or $S \in \text{AnSp}(\mathbb{C})$ non smooth. Take an open cover $S = \cup_i S_i$ so that there are closed embedding $S_I \hookrightarrow \tilde{S}_I$, with $S_I \in \text{SmVar}(\mathbb{C})$, resp $S_I \in \text{AnSm}(\mathbb{C})$. The category of mixed Hodge modules over S is the full subcategory

$$\iota_S : MHM(S) \hookrightarrow MHW(S) \hookrightarrow \text{PSh}_{\mathcal{D}(1,0)fil, rh}(S/(\tilde{S}_I)) \times_I P_{fil}(S)$$

consisting of objects

$$((M_I, F, W), u_{IJ}, (K, W), \alpha) \in \text{PSh}_{\mathcal{D}(1,0)fil, rh}^0(S/(\tilde{S}_I)) \times_I P_{fil}(S)$$

such that $((M_I, F, W), T(S/(\tilde{S}_I))(K, W), \alpha) \in (MHM_{S_I}(\tilde{S}_I))$ (see (ii)). The category $MHM(S)$ does NOT depend on the open cover and the closed embedding by proposition 91.

- Let $S \in \text{SmVar}(\mathbb{C})$. We consider the canonical functor

$$\pi_S : C(MHW(S)) \xrightarrow{\iota_S} C_{\mathcal{D}(1,0)fil}(S) \times_I C_{fil}(S^{an}) \xrightarrow{p_S} C_{\mathcal{D}(1,0)fil}(S), ((M, F, W), (K, W), \alpha) \mapsto (M, F, W)$$

where p_S is the projection functor. Then $\pi_S(MHW(S)) \subset \text{PSh}_{\mathcal{D}(1,0)fil}(S)$ is the subcategory consisting of $(M, F, W) \in \text{PSh}_{\mathcal{D}(1,0)fil}(S)$ such that $((M, F, W), (K, W), \alpha) \in MHW(S)$ is a W filtered Hodge module for some $(K, W) \in C_{fil}(S)$. It induces in the derived category the functor

$$\pi_S : D(MHW(S)) \xrightarrow{\iota_S} D_{\mathcal{D}(1,0)fil, \infty}(S) \times_I D_{fil}(S^{an}) \xrightarrow{p_S} D_{\mathcal{D}(1,0)fil, \infty}(S), ((M, F, W), (K, W), \alpha) \mapsto (M, F, W)$$

after localization with respect to ∞ -filtered Zariski and usu local equivalence.

- Let $S \in \text{Var}(\mathbb{C})$ non smooth. Take an open cover $S = \cup_i S_i$ such that there are closed embedding $S_i \hookrightarrow \tilde{S}_i$, with $S_i \in \text{SmVar}(\mathbb{C})$. We consider the canonical functor

$$\begin{aligned} \pi_S : C(MHW(S)) &\hookrightarrow C_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \times_I C_{fil}(S^{an}) \xrightarrow{p_S} C_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)), \\ &(((M_I, F, W), u_{IJ}), (K, W), \alpha) \mapsto ((M_I, F, W), u_{IJ}) \end{aligned}$$

where p_S is the projection functor. Then $\pi_S(MHW(S)) \subset \text{PSh}_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))$ is the subcategory consisting of $((M, F, W), u_{IJ}) \in \text{PSh}_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))$ such that $((M_I, F, W), u_{IJ}), (K, W), \alpha \in MHW(S)$ is a W filtered Hodge module for some $(K, W) \in C_{fil}(S)$. It induces in the derived category the functor

$$\begin{aligned} \pi_S : D(MHW(S)) &\xrightarrow{\iota_S} D_{\mathcal{D}(1,0)fil,\infty}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}) \xrightarrow{p_S} D_{\mathcal{D}(1,0)fil}(S), \\ &(((M_I, F, W), u_{IJ}), (K, W), \alpha) \mapsto ((M, F, W), u_{IJ}) \end{aligned}$$

after localization with respect to ∞ -filtered Zariski and usu local equivalence.

We have from the work of Beilinson the following :

Theorem 24. (i) Let $S \in \text{SmVar}(\mathbb{C})$. Then the full embedding

$$\iota_S : MHM(S) \hookrightarrow \text{PSh}_{\mathcal{D}(1,0)fil,rh}(S) \times_I P_{fil}(S^{an}) \hookrightarrow C_{\mathcal{D}2fil,rh}(S) \times_I D_{fil}(S^{an})$$

induces a full embedding

$$\iota_S : D(MHM(S)) \hookrightarrow D_{\mathcal{D}(1,0)fil,\infty,rh}(S) \times_I D_{fil}(S^{an})$$

whose image consists of $((M, F, W), (K, W), \alpha) \in D_{\mathcal{D}(1,0)fil,\infty,rh}(S) \times_I D_{fil}(S^{an})$ such that

$$(H^n(M, F, W), H^n(K, W), H^n \alpha) \in MHM(S)$$

for all $n \in \mathbb{Z}$.

- (ii) Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i \in I} S_i$ an open cover such that there exists closed embedding $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then the full embedding

$$\iota_S : MHM(S) \hookrightarrow \text{PSh}_{\mathcal{D}(1,0)fil,rh}^0(S/(\tilde{S}_I)) \times_I P_{fil}(S^{an}) \hookrightarrow C_{\mathcal{D}2fil,rh}(S/(\tilde{S}_I)) \times_I C_{fil}(S^{an})$$

induces a full embedding

$$\iota_S : D(MHM(S)) \hookrightarrow D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$$

whose image consists of $((M_I, F, W), u_{IJ}), (K, W), \alpha \in D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$ such that

$$((H^n(M_I, F, W), H^n(u_{IJ})), H^n(K, W), H^n \alpha) \in MHM(S)$$

for all $n \in \mathbb{Z}$.

Proof. (i): We proceed by induction on the dimension of S . For $S = \{\text{pt}\}$, it is the theorem for absolute Hodge complexes, see [9]. We then proceed as in [5] for perverse sheaves.

(ii): Follows from (i). □

We have from [27] the following proposition which shows us how to construct inductively mixed Hodge modules, as we do for perverse sheaves :

Proposition 90. (i) Let $S \in \text{AnSm}(\mathbb{C})$. Let $D = V(s) \subset S$ a (Cartier) divisor, where $s \in \Gamma(S, L)$ is a section of the line bundle $L = L_D$ associated to D . We then have the zero section embedding $i : S \hookrightarrow L$. We denote $L_0 = i(S)$ and $j : L^\circ := L \setminus L_0 \hookrightarrow L$ the open complementary subset. We denote by $MHW(S \setminus D)^{ex} \times_J MHW(D)$ the category whose set of objects consists of

$$\{(\mathcal{M}, \mathcal{N}, a, b), \mathcal{M} \in MHW(S \setminus D)^{ex}, \mathcal{N} \in MHW(D), a : \psi_{D1} \mathcal{M} \rightarrow \mathcal{N}, b : \mathcal{N} \rightarrow \psi_{D1} \mathcal{M}\}$$

where $MHW(S \setminus D)^{ex} \subset MHW(S \setminus D)$ is the full subcategory of extendable objects, i.e. (bi)filtered D_S module which are specializable along D . The functor (see definition 91)

$$(j^*, \phi_{D1}, c, v) : MHW(S) \rightarrow MHW(S \setminus D)^{ex} \times_J MHW(D), \\ ((M, F, W), (K, W), \alpha) \mapsto ((j^*(M, F, W), j^*(K, W), j^* \alpha), \phi_{D1}((M, F, W), (K, W), \alpha), can, var)$$

is an equivalence of category.

(ii) Let $S \in \text{SmVar}(\mathbb{C})$. Let $D = V(s) \subset S$ a (Cartier) divisor, where $s \in \Gamma(S, L)$ is a section of the line bundle $L = L_D$ associated to D . We then have the zero section embedding $i : S \hookrightarrow L$. We denote $L_0 = i(S)$ and $j : L^\circ := L \setminus L_0 \hookrightarrow L$ the open complementary subset. We denote by $MHW(S \setminus D) \times_J MHW(D)$ the category whose set of objects consists of

$$\{(\mathcal{M}, \mathcal{N}, a, b), \mathcal{M} \in MHW(S \setminus D), \mathcal{N} \in MHW(D), a : \psi_{D1} \mathcal{M} \rightarrow \mathcal{N}, b : \mathcal{N} \rightarrow \psi_{D1} \mathcal{M}\}$$

The functor (see definition 91)

$$(j^*, \phi_{D1}, c, v) : MHW(S) \rightarrow MHW(S \setminus D) \times_J MHW(D), \\ ((M, F, W), (K, W), \alpha) \mapsto ((j^*(M, F, W), j^*(K, W), j^* \alpha), \phi_{D1}((M, F, W), (K, W), \alpha), can, var)$$

is an equivalence of category.

Proof. See [27]. □

Let $S \in \text{Var}(\mathbb{C})$ or $S \in \text{AnSp}(\mathbb{C})$.

- If $S \in \text{Var}(\mathbb{C})$, let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$, and let $S = \cup_{i'=1}^{l'} S_{i'}$ an other open cover such that there exist closed embeddings $i_{i'} : S_{i'} \hookrightarrow \tilde{S}_{i'}$ with $\tilde{S}_{i'} \in \text{SmVar}(\mathbb{C})$.
- If $S \in \text{AnSp}(\mathbb{C})$, let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$, and let $S = \cup_{i'=1}^{l'} S_{i'}$ an other open cover such that there exist closed embeddings $i_{i'} : S_{i'} \hookrightarrow \tilde{S}_{i'}$ with $\tilde{S}_{i'} \in \text{AnSm}(\mathbb{C})$.

Denote $L = [1, \dots, l]$, $L' = [1, \dots, l']$ and $L'' := [1, \dots, l] \sqcup [1, \dots, l']$. We have then the refined open cover $S = \cup_{k \in L} S_k$ and we denote for $I \sqcup I' \subset L''$, $S_{I \sqcup I'} := \cap_{k \in I \sqcup I'} S_k$ and $\tilde{S}_{I \sqcup I'} := \prod_{k \in I \sqcup I'} \tilde{S}_k$, so that we have a closed embedding $i_{I \sqcup I'} : S_{I \sqcup I'} \hookrightarrow \tilde{S}_{I \sqcup I'}$. Consider $\pi_S^L(MHM(S)) \subset \text{PSh}_{\mathcal{D}fil}(S/(S_I))$ and $\pi_S^{L'}(MHM(S)) \subset \text{PSh}_{\mathcal{D}fil}(S/(S_{I'}))$. For $I \sqcup I' \subset J \sqcup J'$, denote by $p_{I \sqcup I', J \sqcup J'} : \tilde{S}_{J \sqcup J'} \rightarrow \tilde{S}_{I \sqcup I'}$ the projection. We then have a natural transfer map

$$T_S^{L/L'} : \pi_S^L(MHM(S)) \rightarrow \pi_S^{L'}(MHM(S)), \\ ((M_I, F, W), s_{IJ}) \mapsto (\text{ho} \lim_{I \in L} p_{I'(I \sqcup I')}^* \text{Gr}_{V_{I \sqcup I'}}^{*mod} (M_I, F), s_{I' J'}),$$

with, in the homotopy limit, the natural transition morphisms

$$p_{I'(I \sqcup I')}^* \text{ad}(p_{IJ}^{*mod}, p_{IJ^*}) (p_{I(I \sqcup I')}^{*mod[-]} (M_I, F)) : \\ p_{I'(J \sqcup I')}^* (\text{Gr}_{V_{J \sqcup I'}}^{*mod[-]} (M_J, F)) \rightarrow p_{I'(I \sqcup I')}^* (\text{Gr}_{V_{I \sqcup I'}}^{*mod[-]} (M_I, F))$$

for $J \subset I$, and

$$\begin{aligned} s_{I', J'} &: \text{holim}_{I \in L} m^* p_{I'(I \sqcup I')}^* (\text{Gr}_{V_{I \sqcup I'}} p_{I(I \sqcup I')}^{*mod[-]}(M_I, F) \rightarrow \\ &\text{holim}_{I \in L} p_{I', J'}^* \text{Gr}_{V_{J'}} (p_{I', J'}^{*mod[-]} m^* p_{I'(I \sqcup I')}^* \text{Gr}_{V_{I \sqcup I'}} p_{I(I \sqcup I')}^{*mod[-]}((M_I, F))) \\ &\rightarrow \text{holim}_{I \in L} p_{I', J'}^* p_{J'(I \sqcup J')}^* \text{Gr}_{V_{I \sqcup I'}} p_{I(I \sqcup J')}^{*mod[-]}(M_I, F) \end{aligned}$$

Proposition 91. (i) Let $S \in \text{Var}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then $\pi_S(\text{MHM}(S) \subset \text{PSh}_{\mathcal{D}(2), \text{fil}}(S/(\tilde{S}_I)))$ does not depend on the open covering of S and the closed embeddings. More precisely, let $S = \cup_{i'=1}^{l'} S_{i'}$ an other open cover such that there exist closed embeddings $i_{i'} : S_{i'} \hookrightarrow \tilde{S}_{i'}$ with $\tilde{S}_{i'} \in \text{SmVar}(\mathbb{C})$. Then,

$$T_S^{L/L'} : \pi_S^L(\text{MHM}(S)) \rightarrow \pi_S^{L'}(\text{MHM}(S)),$$

is an equivalence of category with inverse is $T_S^{L'/L} : \pi_S^{L'}(\text{MHM}(S)) \rightarrow \pi_S^L(\text{MHM}(S))$.

(ii) Let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then $\pi_S(\text{MHM}(S) \subset \text{PSh}_{\mathcal{D}(2), \text{fil}}(S/(\tilde{S}_I)))$ does not depend on the open covering of S and the closed embeddings. More precisely, let $S = \cup_{i'=1}^{l'} S_{i'}$ an other open cover such that there exist closed embeddings $i_{i'} : S_{i'} \hookrightarrow \tilde{S}_{i'}$ with $\tilde{S}_{i'} \in \text{AnSm}(\mathbb{C})$. Then,

$$T_S^{L/L'} : \pi_S^L(\text{MHM}(S)) \rightarrow \pi_S^{L'}(\text{MHM}(S)),$$

is an equivalence of category with inverse is $T_S^{L'/L} : \pi_S^{L'}(\text{MHM}(S)) \rightarrow \pi_S^L(\text{MHM}(S))$.

Proof. Follows from proposition 90(see [27]). \square

The main results of Saito, which implies in the algebraic case the six functor formalism on $DMHM(-)$ are the followings

Theorem 25. Let $S \in \text{Var}(\mathbb{C})$. The category of mixed Hodge modules is the full subcategory

$$\iota_S : \text{MHM}(S) \hookrightarrow \text{MHW}(S) \hookrightarrow \text{PSh}_{\mathcal{D}(1,0), \text{fil}, \text{rh}}(S) \times_I \text{P}_{\text{fil}}(S^{an})$$

consisting of objects

$$((M, F, W), (K, W), \alpha) = (((M_I, F, W), u_{IJ}), (K, W), \alpha) \in \text{PSh}_{\mathcal{D}(1,0), \text{fil}, \text{rh}}(S) \times_I \text{P}_{\text{fil}}(S^{an})$$

such that $((M, F, W)^{an}, (K, W), \alpha) := (((M_I^{an}, F, W), u_{IJ}), (K, W), \alpha) \in \text{MHM}(S^{an})$.

Proof. Follows from GAGA and the extendableness in the algebraic case (proposition 90). \square

Definition 93. Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$. We denote by $VMHS(S) \subset \text{PSh}_{\mathcal{D}(1,0), \text{fil}, \text{rh}}(S) \times_I \text{P}_{\text{fil}}(S^{an})$ the full subcategory consisting of variation of mixed Hodge structure, whose objects consist of

$$(((L_S, W) \otimes_{O_S}, F), (L_S, W), \alpha) \in \text{PSh}_{\mathcal{D}(1,0), \text{fil}, \text{rh}}(S) \times_I \text{P}_{\text{fil}}(S^{an})$$

with

- $L_S \in \text{PSh}(S^{an})$ a local system,
- the D_S module structure on $(L_S, W) \otimes_{O_S}$ is given by the flat connection associated to the local system L_S ,
- $F^p(W^q L_S \otimes_{O_S}) \subset (W^q L_S \otimes_{O_S})$ are locally free O_S subbundle satisfying Griffiths transversality for the D_S module structure (i.e. for the flat connection).

- $\alpha : (L_S, W) \rightarrow DR(S)^{[-1]}((L_S, W) \otimes O_S)$ is the isomorphism given by theorem 23.

Theorem 26. Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$.

- (i) A variation of mixed Hodge structure $((L_S, W) \otimes O_S, F), (L_S, W), \alpha \in \text{VMHS}(S)$ (see definition 93) is a mixed module. That is $\text{VMHS}(S) \subset \text{MHM}(S)$.
- (ii) For $((M, F, W), (K, W), \alpha) \in \text{MHM}(S)$ a mixed Hodge module, there exist an open subset $j : S^\circ \hookrightarrow S$, such that $j^*((M, F, W), (K, W), \alpha) := (j^*(M, F, W), j^*(K, W), j^*\alpha) \in \text{VMHS}(S^\circ)$. That is a mixed Hodge module is generically a variation of mixed Hodge structure.

Proof. See [27]. □

Theorem 27. (i) Let $f : X \rightarrow S$ a projective morphism with $X, S \in \text{AnSp}(\mathbb{C})$, where projective means that there exist a factorization $f : X \xrightarrow{l} \mathbb{P}^N \times S \xrightarrow{p_S} S$ with l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^s S_i$ an open cover such that there exists closed embeddings $i_I : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. For $I \subset [1, \dots, s]$, recall that we denote $S_I := \cap_{i \in I} S_i$ and $X_I := f^{-1}(S_I)$. We have then the following commutative diagram

$$\begin{array}{ccccc}
X_I & \xrightarrow{i_I \circ l_I} & \mathbb{P}^N \times \tilde{S}_I & \xrightarrow{p_{\tilde{S}_I}} & \tilde{S}_I \\
\uparrow j'_{IJ} & & \uparrow p'_{IJ} & & \uparrow p_{IJ} \\
X_J & \xrightarrow{i_J \circ l_J} & \mathbb{P}^N \times \tilde{S}_J & \xrightarrow{p_{\tilde{S}_J}} & \tilde{S}_J
\end{array}$$

whose right square is cartesian (see section 5). Then, for

$$((M, F, W), (K, W), \alpha) = (((M_I, F, W), u_{IJ}), (K, W), \alpha) \in \text{MHM}(X),$$

where $((M_I, F, W), u_{IJ}) \in C_{\mathcal{D}2\text{fil}}(X_I/(\mathbb{P}^N \times \tilde{S}_I))$, $(K, W) \in C_{\text{fil}}(X)$, we have for all $n \in \mathbb{Z}$,

$$(H^n \int_f^{\text{FDR}} ((M_I, F, W), u_{IJ}), R^n f_*(K, W), H^n f_*(\alpha)) \in \text{MHM}(S)$$

- (ii) Let $f : X \rightarrow S$ a projective morphism with $X, S \in \text{Var}(\mathbb{C})$, where projective means that there exist a factorization $f : X \xrightarrow{l} \mathbb{P}^N \times S \xrightarrow{p_S} S$ with l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^s S_i$ an open cover such that there exists closed embeddings $i_I : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, s]$, recall that we denote $S_I := \cap_{i \in I} S_i$ and $X_I := f^{-1}(S_I)$. We have then the following commutative diagram

$$\begin{array}{ccccc}
X_I & \xrightarrow{i_I \circ l_I} & \mathbb{P}^N \times \tilde{S}_I & \xrightarrow{p_{\tilde{S}_I}} & \tilde{S}_I \\
\uparrow j'_{IJ} & & \uparrow p'_{IJ} & & \uparrow p_{IJ} \\
X_J & \xrightarrow{i_J \circ l_J} & \mathbb{P}^N \times \tilde{S}_J & \xrightarrow{p_{\tilde{S}_J}} & \tilde{S}_J
\end{array}$$

whose right square is cartesian (see section 5). Then, for

$$((M, F, W), (K, W), \alpha) = (((M_I, F, W), u_{IJ}), (K, W), \alpha) \in D(\text{MHM}(X)),$$

where $((M_I, F, W), u_{IJ}) \in C_{\mathcal{D}2\text{fil}}(X_I/(\mathbb{P}^N \times \tilde{S}_I))$, $(K, W) \in C_{\text{fil}}(X^{an})$, we have

$$H^n \left(\int_f^{\text{FDR}} ((M_I, F, W), u_{IJ}), Rf_*(K, W), f_*(\alpha) \right) \in \text{MHM}(S)$$

for all $n \in \mathbb{Z}$.

Proof. (i): See [27].

(ii): By (i) $(H^n \int_f ((M, F, W)^{an}), R^n f_*(K, W), H^n f_*(\alpha)) \in MHM(S^{an})$ for all $n \in \mathbb{Z}$. On the other hand, $T^{\mathcal{D}}(an, f)(M, F, W) : (\int_f (M, F, W))^{an} \xrightarrow{\sim} \int_f ((M, F, W)^{an})$ is an isomorphism since f is proper by theorem GAGA for mixed hodge modules : see [27]. \square

Theorem 28. (i) Let $S \in \text{AnSp}(\mathbb{C})$. Let $Y \in \text{AnSm}(\mathbb{C})$ and $p_S : Y \times S \rightarrow S$ the projection. Let $S = \cup_{i=1}^s S_i$ an open cover such that there exists closed embeddings $i_I : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. For $I \subset [1, \dots, s]$, recall that we denote $S_I := \cap_{i \in I} S_i$. We have then the following commutative diagram

$$\begin{array}{ccc} Y \times \tilde{S}_I & \xrightarrow{p_{\tilde{S}_I}} & \tilde{S}_I \\ p'_{IJ} \uparrow & & \uparrow p_{IJ} \\ Y \times \tilde{S}_J & \xrightarrow{p_{\tilde{S}_J}} & \tilde{S}_J \end{array}$$

which is cartesian (see section 5). Then, for

$$((M, F, W), (K, W), \alpha) = (((M_I, F, W), u_{IJ}), (K, W), \alpha) \in MHM(S),$$

where $((M_I, F, W), u_{IJ}) \in C_{\mathcal{D}2fil}(S_I/(\tilde{S}_I))$, $(K, W) \in C_{fil}(S)$,

- $(p_S^{*mod[-]}(M, F, W), p_S^*(K, W), p_S^*(\alpha)) := ((p_{\tilde{S}_I}^{*mod[-]}(M_I, F, W), p_{\tilde{S}_J}^{*mod[-]} u_{IJ}), p_S^*(K, W), p_S^*(\alpha)) \in MHM(S)$
- $(p_S^{\hat{*}mod[-]}(M, F, W), p_S^*(K, W), p_S^*(\alpha)) := ((p_{\tilde{S}_I}^{\hat{*}mod[-]}(M_I, F, W), p_{\tilde{S}_J}^{*mod[-]} u_{IJ}), p_S^*(K, W), p_S^*(\alpha)) \in MHM(S)$

(ii) Let $S \in \text{Var}(\mathbb{C})$. Let $Y \in \text{SmVar}(\mathbb{C})$ and $p_S : Y \times S \rightarrow S$ the projection. Let $S = \cup_{i=1}^s S_i$ an open cover such that there exists closed embeddings $i_I : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, s]$, recall that we denote $S_I := \cap_{i \in I} S_i$. We have then the following commutative diagram

$$\begin{array}{ccc} Y \times \tilde{S}_I & \xrightarrow{p_{\tilde{S}_I}} & \tilde{S}_I \\ p'_{IJ} \uparrow & & \uparrow p_{IJ} \\ Y \times \tilde{S}_J & \xrightarrow{p_{\tilde{S}_J}} & \tilde{S}_J \end{array}$$

which is cartesian (see section 5). Then, for

$$((M, F, W), (K, W), \alpha) = (((M_I, F, W), u_{IJ}), (K, W), \alpha) \in D(MHM(S))$$

where $((M_I, F, W), u_{IJ}) \in C_{\mathcal{D}2fil}(S_I/(\tilde{S}_I))$, $(K, W) \in C_{fil}(S^{an})$, we have

- $(p_S^{*mod[-]}, p_S^!)((M, F, W), (K, W), \alpha) := ((p_{\tilde{S}_I}^{*mod[-]}(M_I, F, W), p_{\tilde{S}_J}^{*mod[-]} u_{IJ}), p_S^*(K, W), p_S^*(\alpha)) \in D(MHM(S))$
- $(p_S^{\hat{*}mod[-]}, p_S^*)((M, F, W), (K, W), \alpha) := ((p_{\tilde{S}_I}^{\hat{*}mod[-]}(M_I, F, W), p_{\tilde{S}_J}^{*mod[-]} u_{IJ}), p_S^*(K, W), p_S^*(\alpha)) \in D(MHM(S))$.

Proof. (i): See [27].

(ii): Follows immediately from (i) since $(p_{\tilde{S}_I}^{*mod[-]}(M_I, F, W))^{an} = p_{\tilde{S}_I}^{*mod[-]}((M_I, F, W)^{an})$. \square

We have, by the results of Saito, the following key definition.

Definition 94. (i) Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$. Let $D = V(s) \subset S$ a divisor with $s \in \Gamma(S, L)$ and L a line bundle (S being smooth, D is Cartier). Denote by $j : S^\circ := S \setminus D \hookrightarrow S$ the open complementary embedding. Let $(M, F, W) \in \pi_{S^\circ}(\text{MHW}(S^\circ))$. Assume that (M, F, W) is specializable along D , that is the V_S -filtration exists (see proposition 89 for the uniqueness of a V_S -filtration). Then, by proposition 90,

– there exist

$$j_*^{\text{Hdg}}(M, F, W) := (j^*, \phi_{D1}, c, v)^{-1}((M, F, W), \psi_{D1}(M, F, W)(-1)) \in \pi_S(\text{MHM}(S))$$

unique such that $j^* j_*^{\text{Hdg}}(M, F, W) = (M, F, W)$ and $DR(S)(j_*^{\text{Hdg}}(M, F, W)) = j_* DR(S^\circ)(M, W)$,

– there exist

$$j_!^{\text{Hdg}}(M, F, W) := (j^*, \phi_{D1}, c, v)^{-1}((M, F, W), \psi_{D1}(M, F, W)) \in \pi_S(\text{MHM}(S))$$

unique such that $j^* j_!^{\text{Hdg}}(M, F, W) = (M, F, W)$ and $DR(S)(j_!^{\text{Hdg}}(M, F, W)) = j_! DR(S^\circ)(M, W)$.

Moreover for $(M', F, W) \in \pi_S(\text{MHM}(S))$, by proposition 90

– there is a canonical map $\text{ad}(j^*, j_*^{\text{Hdg}})(M', F, W) : (M', F, W) \rightarrow j_*^{\text{Hdg}} j^*(M', F, W)$ in $\pi_S(\text{MHM}(S))$,

– there is a canonical map $\text{ad}(j_!^{\text{Hdg}}, j^*)(M', F, W) : j_!^{\text{Hdg}} j^*(M', F, W) \rightarrow (M', F, W)$ in $\pi_S(\text{MHM}(S))$.

(ii) Let $S \in \text{SmVar}(\mathbb{C})$. Let $Z = V(\mathcal{I}) \subset S$ an arbitrary closed subset, $\mathcal{I} \subset \mathcal{O}_S$ being an ideal subsheaf. Taking generators $\mathcal{I} = (s_1, \dots, s_r)$, we get $Z = V(s_1, \dots, s_r) = \bigcap_{i=1}^r Z_i \subset S$ with $Z_i = V(s_i) \subset S$, $s_i \in \Gamma(S, \mathcal{L}_i)$ and \mathcal{L}_i a line bundle. Note that Z is an arbitrary closed subset, $d_Z \geq d_X - r$ needing not be a complete intersection. Denote by $j : S^\circ := S \setminus Z \hookrightarrow S$, $j_I : S^{\circ, I} := \bigcap_{i \in I} (S \setminus Z_i) = S \setminus (\bigcup_{i \in I} Z_i) \xrightarrow{j_I^o} S^\circ \xrightarrow{j} S$ the open complementary embeddings, where $I \subset \{1, \dots, r\}$. For $(M, F, W) \in \pi_{S^\circ}(C(\text{MHM}(S^\circ)))$, we define by (i)

– the (bi)-filtered complex of D_S -modules

$$j_*^{\text{Hdg}}(M, F, W) := \varinjlim_{\{(Z_i)_{i \in \{1, \dots, r\}}, Z_i \subset S, \cap Z_i = Z\}, Z_i \subset Z_i} \text{Tot}_{\text{card} I = \bullet} (j_{I*}^{\text{Hdg}} j_I^{o*}(M, F, W)) \in \pi_S(C(\text{MHM}(S))),$$

where the horizontal differential are given by, if $I \subset J$, $d_{IJ} := \text{ad}(j_{IJ*}^{\text{Hdg}}, j_{IJ*}^{\text{Hdg}})(j_I^{o*}(M, F, W))$, $j_{IJ} : S^{\circ, J} \hookrightarrow S^{\circ, I}$ being the open embedding, and $d_{IJ} = 0$ if $I \not\subset J$,

– the (bi)-filtered complex of D_S -modules

$$j_!^{\text{Hdg}}(M, F, W) := \varprojlim_{\{(Z_i)_{i \in \{1, \dots, r\}}, Z_i \subset S, \cap Z_i = Z\}, Z_i \subset Z_i} \text{Tot}_{\text{card} I = -\bullet} (j_{I!}^{\text{Hdg}} j_I^{o*}(M, F, W)) \in \pi_S(C(\text{MHM}(S))),$$

where the horizontal differential are given by, if $I \subset J$, $d_{IJ} := \text{ad}(j_{IJ!}^{\text{Hdg}}, j_{IJ!}^{\text{Hdg}})(j_I^{o*}(M, F, W))$, $j_{IJ} : S^{\circ, J} \hookrightarrow S^{\circ, I}$ being the open embedding, and $d_{IJ} = 0$ if $I \not\subset J$.

By definition, we have for $(M, F, W) \in \pi_{S^\circ}(C(\text{MHM}(S^\circ)))$, $j^* j_*^{\text{Hdg}}(M, F, W) = (M, F, W)$ and $j^* j_!^{\text{Hdg}}(M, F, W) = (M, F, W)$. For $(M', F, W) \in \pi_S(C(\text{MHM}(S)))$, there is, by construction,

– a canonical map $\text{ad}(j^*, j_*^{\text{Hdg}})(M', F, W) : (M', F, W) \rightarrow j_*^{\text{Hdg}} j^*(M', F, W)$,

– a canonical map $\text{ad}(j_!^{\text{Hdg}}, j^*)(M', F, W) : j_!^{\text{Hdg}} j^*(M', F, W) \rightarrow (M', F, W)$.

For $(M, F, W) \in \pi_{S^\circ}(C(\text{MHM}(S^\circ)))$,

– we have the canonical map in $C_{\mathcal{D}(1,0)\text{fil}}(S)$

$$\Gamma(j_*^{\text{Hdg}}, j_*)(M, F, W) := k \circ \text{ad}(j^*, j_*)(j_*^{\text{Hdg}}(M, F, W)) : j_*^{\text{Hdg}}(M, F, W) \rightarrow j_* E(M, F, W),$$

– we have the canonical map in $C_{\mathcal{D}(1,0)fil}(S)$

$$T(j_!, j_!^{Hdg})(M, F, W) := \mathbb{D}_S^K L_D(k \circ \text{ad}(j^*, j_*)(-)) : \\ j_!(M, F, W) := \mathbb{D}_S^K L_D j_* E(\mathbb{D}_S^K(M, F, W)) \rightarrow \mathbb{D}_S^K L_D j_*^{Hdg} \mathbb{D}_S^K(M, F, W) = j_!^{Hdg}(M, F, W)$$

the canonical maps.

Remark 9. Let $j : S^\circ \hookrightarrow S$ an open embedding, with $S \in \text{SmVar}(\mathbb{C})$. Then, for $((M, F, W), (K, W), \alpha) \in \text{MHM}(S^\circ)$,

- the map $T(j_!, j_!^{Hdg})(M, W) : j_!(M, W) \rightarrow j_!^{Hdg}(M, W)$ in $C_{\mathcal{D}0fil}(S)$ is a filtered quasi-isomorphism (apply the functor $DR^{[-1]}(S^\circ)$ and use theorem 23 and theorem 90).
- the map $T(j_*^{Hdg}, j_*)(M, W) : j_*^{Hdg}(M, W) \rightarrow j_* E(M, W)$ in $C_{\mathcal{D}0fil}(S)$ is a filtered quasi-isomorphism (apply the functor $DR^{[-1]}(S^\circ)$ and use theorem 23 and theorem 90).

Hence, for $((M, F, W), (K, W), \alpha) \in \text{MHM}(S^\circ)$,

- we get, for all $p, n \in \mathbb{N}$, monomorphisms

$$F^p H^n T(j_!, j_!^{Hdg})(M, F, W) : F^p H^n j_!(M, F, W) \hookrightarrow F^p H^n j_!^{Hdg}(M, F, W)$$

in $\text{PSh}_{\mathcal{D}}(S)$, but $F^p H^n j_!(M, F, W) \neq F^p H^n j_!^{Hdg}(M, F, W)$ in general (it leads to different F -filtrations).

- we get, for all $p, n \in \mathbb{N}$, monomorphisms

$$T(j_*^{Hdg}, j_*)(M, F, W) : F^p H^n j_*^{Hdg}(M, F, W) \hookrightarrow F^p H^n j_* E(M, F, W)$$

in $\text{PSh}_{\mathcal{D}}(S)$, but $F^p H^n j_*^{Hdg}(M, F, W) \neq F^p H^n j_* E(M, F, W)$ in general (it leads to different F -filtrations).

Definition 95. Let $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$. Consider a compactification $f : X \xrightarrow{j} \bar{X} \xrightarrow{\bar{f}} S$ of f , in particular j is an open embedding and \bar{f} is proper.

(i) For $(M, F, W) \in \pi_X(C(\text{MHM}(X)))$, we define, using definition 94,

$$\int_f^{Hdg} (M, F, W) := \int_{\bar{f}}^{FDR} j_*^{Hdg}(M, F, W) \in D_{\mathcal{D}(1,0)fil,\infty}(S)$$

It does not depends on the choice of the compactification by the unicity of proposition 90. By theorem 27, for $(M, F, W) \in \pi_X(C(\text{MHM}(X)))$, $H^i \int_f^{Hdg}(M, F, W) \in \pi_S(C(\text{MHM}(S)))$ for all $i \in \mathbb{Z}$. Note that $H^i \int_f^{Hdg}(M, F, W) = 0$ for all $i < 0$ if $(M, F, W) \in \pi_X(\text{MHM}(X))$. We then set

- for $(M, F, W) \in \pi_X(\text{MHM}(X))$, $f_*^{Hdg}(M, F, W) := H^0 \int_f^{Hdg}(M, F, W) \in \pi_S(\text{MHM}(S))$,
- $Rf_*^{Hdg}(M, F, W) := f_*^{Hdg} I(M, F, W) \in \pi_S(D(\text{MHM}(S)))$ where $k : (M, F, W) \rightarrow I(M, F, W)$ is the image by π_S of an injective resolution in $\text{MHM}(S)$.

(ii) For $(M, F, W) \in \pi_X(C(\text{MHM}(X)))$, we define, using definition 94,

$$\int_{f!}^{Hdg} (M, F, W) := \int_{\bar{f}}^{FDR} j_!^{Hdg}(M, F, W) \in D_{\mathcal{D}(1,0)fil,\infty}(S)$$

It does not depends on the choice of the compactification by the unicity of proposition 90. By theorem 27, for $(M, F, W) \in \pi_X(C(\text{MHM}(X)))$, $H^i \int_{f!}^{Hdg}(M, F, W) \in \pi_S(C(\text{MHM}(S)))$ for all $i \in \mathbb{Z}$. Note that $H^i \int_{f!}^{Hdg}(M, F, W) = 0$ for all $i < 0$ if $(M, F, W) \in \pi_X(\text{MHM}(X))$. We then set

- for $(M, F, W) \in \pi_X(\text{MHM}(X))$, $f_1^{\text{Hdg}}(M, F, W) := H^0 \int_{f_1!}^{\text{Hdg}}(M, F, W) \in \pi_S(\text{MHM}(S))$,
- $Rf_1^{\text{Hdg}}(M, F, W) := f_1^{\text{Hdg}} I(M, F, W) \in \pi_S(D(\text{MHM}(S)))$ where $k : (M, F, W) \rightarrow I(M, F, W)$ is the image by π_S of an injective resolution in $\text{MHM}(S)$.

Proposition 92. *Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphism with $X, Y, S \in \text{SmVar}(\mathbb{C})$.*

(i) *Let $(M, F, W) \in \pi_X(C(\text{MHM}(X)))$. Then,*

$$R(f_2 \circ f_1)_*^{\text{Hdg}}(M, F) = Rf_{2*}^{\text{Hdg}} Rf_{1*}^{\text{Hdg}}(M, F) \in \pi_S(D(\text{MHM}(S))).$$

(ii) *Let $(M, F, W) \in \pi_S(C(\text{MHM}(S)))$. Then,*

$$R(f_2 \circ f_1)!^{\text{Hdg}}(M, F) = Rf_{2!}^{\text{Hdg}} Rf_{1!}^{\text{Hdg}}(M, F) \in \pi_S(D(\text{MHM}(S)))$$

Proof. (i): Follows from the unicity of the functor j_*^{Hdg} by proposition 90.

(ii): Follows from the unicity of the functor $j_!^{\text{Hdg}}$ by proposition 90. □

We make the following key definition

Definition 96. *Let $S \in \text{SmVar}(\mathbb{C})$. Let $Z \subset S$ a closed subset. Denote by $j : S \setminus Z \hookrightarrow S$ the complementary open embedding.*

(i) *We define using definition 94, the filtered Hodge support section functor*

$$\begin{aligned} & \Gamma_Z^{\text{Hdg}} : \pi_S(C(\text{MHM}(S))) \rightarrow \pi_S(C(\text{MHM}(S))), \\ (M, F, W) & \mapsto \Gamma_Z^{\text{Hdg}}(M, F, W) := \text{Cone}(\text{ad}(j^*, j_*^{\text{Hdg}})(M, F) : (M, F) \rightarrow j_*^{\text{Hdg}} j^*(M, F))[-1], \end{aligned}$$

together we the canonical map $\gamma_Z^{\text{Hdg}}(M, F, W) : \Gamma_Z^{\text{Hdg}}(M, F, W) \rightarrow (M, F, W)$. We then have the canonical map in $C_{\mathcal{D}(2)\text{fil}}(S)$

$$T(\Gamma_Z^{\text{Hdg}}, \Gamma_Z)(M, F, W) := (I, T(j_*^{\text{Hdg}}, j_*)(M, F, W)) : \Gamma_Z^{\text{Hdg}}(M, F, W) \rightarrow \Gamma_Z E(M, F, W)$$

unique up to homotopy such that $\gamma_Z^{\text{Hdg}}(M, F, W) = \gamma_Z(E(M, F, W)) \circ T(\Gamma_Z^{\text{Hdg}}, \Gamma_Z)(M, F, W)$.

(i)' *Since $j_*^{\text{Hdg}} : \pi_{S^\circ}(C(\text{MHM}(S^\circ))) \rightarrow \pi_S(C(\text{MHM}(S)))$ is an exact functor, Γ_Z^{Hdg} induces the functor*

$$\Gamma_Z^{\text{Hdg}} : \pi_S(D(\text{MHM}(S))) \rightarrow \pi_S(D(\text{MHM}(S))), (M, F, W) \mapsto \Gamma_Z^{\text{Hdg}}(M, F, W)$$

(ii) *We define using definition 94, the dual filtered Hodge support section functor*

$$\begin{aligned} & \Gamma_Z^{\vee, \text{Hdg}} : \pi_S(C(\text{MHM}(S))) \rightarrow \pi_S(C(\text{MHM}(S))), \\ (M, F, W) & \mapsto \Gamma_Z^{\vee, \text{Hdg}}(M, F, W) := \text{Cone}(\text{ad}(j_!^{\text{Hdg}}, j^*)(M, F, W) : j_!^{\text{Hdg}}, j^*(M, F, W) \rightarrow (M, F, W)), \end{aligned}$$

together we the canonical map $\gamma_Z^{\vee, \text{Hdg}}(M, F, W) : (M, F, W) \rightarrow \Gamma_Z^{\vee, \text{Hdg}}(M, F, W)$. We then have the canonical map in $C_{\mathcal{D}(2)\text{fil}}(S)$

$$T(\Gamma_Z^{\vee, h}, \Gamma_Z^{\vee, \text{Hdg}})(M, F, W) := (I, T(j_!, j_!^{\text{Hdg}})(M, F, W)) : \Gamma_Z^{\vee, h}(M, F, W) \rightarrow \Gamma_Z^{\vee, \text{Hdg}}(M, F, W)$$

unique up to homotopy such that $\gamma_Z^{\vee, \text{Hdg}}(M, F, W) = T(\Gamma_Z^{\vee, h}, \Gamma_Z^{\vee, \text{Hdg}})(M, F, W) \circ \gamma_Z^{\vee, h}(M, F, W)$.

(ii)' *Since $j_!^{\text{Hdg}} : \pi_{S^\circ}(C(\text{MHM}(S^\circ))) \rightarrow \pi_S(C(\text{MHM}(S)))$ is an exact functor, $\Gamma_Z^{\text{Hdg}, \vee}$ induces the functor*

$$\Gamma_Z^{\vee, \text{Hdg}} : \pi_S(D(\text{MHM}(S))) \rightarrow \pi_S(D(\text{MHM}(S))), (M, F, W) \mapsto \Gamma_Z^{\vee, \text{Hdg}}(M, F, W)$$

We now give the definition of the filtered Hodge inverse image functor :

Definition 97. (i) Let $i : Z \hookrightarrow S$ be a closed embedding, with $Z, S \in \text{SmVar}(\mathbb{C})$. Then, for $(M, F, W) \in \pi_S(C(\text{MHM}(S)))$, we set

$$i_{\text{Hdg}}^{*\text{mod}}(M, F, W) := i^* \mathcal{S}_Z^{-1} \Gamma_Z^{\text{Hdg}}(M, F, W) \in \pi_Z(D(\text{MHM}(Z)))$$

and

$$i_{\text{Hdg}}^{\hat{*}\text{mod}}(M, F, W) := i^* \mathcal{S}_Z^{-1} \Gamma_Z^{\vee, \text{Hdg}}(M, F, W) \in \pi_Z(D(\text{MHM}(Z)))$$

using the fact that $\mathcal{S}_Z : \pi_Z(D(\text{MHM}(Z))) \rightarrow \pi_S(D(\text{MHM}_Z(S)))$ is an equivalence of category since $\mathcal{S}_Z : D(\text{MHM}_Z(S)) \rightarrow D(\text{MHM}_Z(S))$ is an equivalence of category by [27].

(ii) Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{SmVar}(\mathbb{C})$. Consider the factorization $f : X \xrightarrow{i} X \times S \xrightarrow{p_S} S$, where i is the graph embedding and $p_S : X \times S \rightarrow S$ is the projection.

– For $(M, F, W) \in \pi_S(C(\text{MHM}(S)))$ we set

$$f_{\text{Hdg}}^{*\text{mod}}(M, F, W) := i_{\text{Hdg}}^{*\text{mod}} p_S^{*\text{mod}[-]}(M, F, W) \in \pi_X(D(\text{MHM}(X))),$$

– For $(M, F, W) \in \pi_S(C(\text{MHM}(S)))$ we set

$$f_{\text{Hdg}}^{\hat{*}\text{mod}}(M, F, W) := i_{\text{Hdg}}^{\hat{*}\text{mod}} p_S^{*\text{mod}[-]}(M, F, W) \in \pi_X(D(\text{MHM}(X))),$$

If $j : S^\circ \hookrightarrow S$ is a closed embedding, we have (see [27]), for $(M, F, W) \in \pi_S(C(\text{MHM}(S)))$,

$$j_{\text{Hdg}}^{*\text{mod}}(M, F, W) = j_{\text{Hdg}}^{\hat{*}\text{mod}}(M, F, W) = j^*(M, F, W) \in \pi_{S^\circ}(D(\text{MHM}(S^\circ)))$$

(iii) Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{SmVar}(\mathbb{C})$ or $X, S \in \text{AnSm}(\mathbb{C})$. Consider the factorization $f : X \xrightarrow{i} X \times S \xrightarrow{p_S} S$, where i is the graph embedding and $p_S : X \times S \rightarrow S$ is the projection.

– For $(M, F, W) \in \pi_S(C(\text{MHM}(S)))$ we set

$$f_{\text{Hdg}}^{*\text{mod}}(M, F, W) := \Gamma_X^{\text{Hdg}} p_S^{*\text{mod}[-]}(M, F, W) \in \pi_{X \times S}(C(\text{MHM}(X \times S))),$$

We have for $(M, F, W) \in \pi_S(C(\text{MHM}(S)))$, the canonical map in $C_{\mathcal{D}(1,0)\text{fil}}(X \times S)$

$$\begin{aligned} T(f_{\text{Hdg}}^{*\text{mod}}, f^{*\text{mod}, \Gamma})(M, F, W) : f_{\text{Hdg}}^{*\text{mod}, \Gamma}(M, F, W) &:= \Gamma_X^{\text{Hdg}} p_S^{*\text{mod}[-]}(M, F, W) \\ \xrightarrow{T(\Gamma_X^{\text{Hdg}}, \Gamma_X)(-)} \Gamma_X E(p_S^{*\text{mod}[-]}(M, F, W)) &=: f^{*\text{mod}, \Gamma, k}(M, F, W) \end{aligned}$$

– For $(M, F, W) \in \pi_S(C(\text{MHM}(S)))$ we set

$$f_{\text{Hdg}}^{\hat{*}\text{mod}}(M, F, W) := \Gamma_X^{\vee, \text{Hdg}} p_S^{*\text{mod}[-]}(M, F, W) \in \pi_{X \times S}(C(\text{MHM}(X \times S))),$$

We have for $(M, F, W) \in \pi_S(C(\text{MHM}(S)))$, the canonical map in $C_{\mathcal{D}(1,0)\text{fil}}(X \times S)$

$$\begin{aligned} T(f_{\text{Hdg}}^{\hat{*}\text{mod}, \Gamma}, f_{\text{Hdg}}^{\hat{*}\text{mod}})(M, F, W) : f_{\text{Hdg}}^{\hat{*}\text{mod}, \Gamma}(M, F, W) &:= \Gamma_X^{\vee, h} p_S^{*\text{mod}[-]}(M, F, W) \\ \xrightarrow{T(\Gamma_X^{\vee, h}, \Gamma_X^{\vee, \text{Hdg}})(-)} \Gamma_X^{\vee, \text{Hdg}} p_S^{*\text{mod}[-]}(M, F, W) &=: f_{\text{Hdg}}^{\hat{*}\text{mod}}(M, F, W) \end{aligned}$$

Proposition 93. Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphism with $X, Y, S \in \text{SmVar}(\mathbb{C})$.

(i) Let $(M, F, W) \in \pi_S(C(\text{MHM}(S)))$. Then,

$$(f_2 \circ f_1)_{\text{Hdg}}^{*\text{mod}}(M, F) = f_{1\text{Hdg}}^{*\text{mod}} f_{2\text{Hdg}}^{*\text{mod}}(M, F) \in \pi_X(D(\text{MHM}(X))).$$

(ii) Let $(M, F, W) \in \pi_S(C(MHM(S)))$. Then,

$$(f_2 \circ f_1)_{Hdg}^{\hat{*}mod}(M, F) = f_{1Hdg}^{\hat{*}mod} f_{2Hdg}^{\hat{*}mod}(M, F) \in \pi_X(D(MHM(X)))$$

Proof. (i): Follows from the unicity of the functor j_*^{Hdg} .

(ii): Follows from the unicity of the functor j_1^{Hdg} . \square

Definition-Proposition 16. (i) Let $g : S' \rightarrow S$ a morphism with $S', S \in \text{SmVar}(\mathbb{C})$ and $i : Z \hookrightarrow S$ a closed subset. Then, for $(M, F, W) \in \pi_S(C(MHM(S)))$, there is a canonical map in $\pi_S(C(MHM_{S'}(S' \times S)))$

$$T^{Hdg}(g, \gamma)(M, F, W) : g_{Hdg}^{*mod, \Gamma} \Gamma_Z^{Hdg}(M, F, W) \rightarrow \Gamma_{Z \times_S S'}^{Hdg} g_{Hdg}^{*mod, \Gamma}(M, F, W)$$

unique up to homotopy such that

$$\gamma_{Z \times_S S'}^{Hdg}(g_{Hdg}^{*mod, \Gamma}(M, F, W)) \circ T^{Hdg}(g, \gamma)(M, F, W) = g_{Hdg}^{*mod, \Gamma} \gamma_Z^{Hdg}(M, F, W).$$

(i)' Let $g : S' \rightarrow S$ a morphism with $S', S \in \text{SmVar}(\mathbb{C})$ and $i : Z \hookrightarrow S$ a closed subset. Then, for $(M, F, W) \in \pi_S(C(MHM(S)))$, there is a canonical isomorphism in $\pi_S(C(MHM_{S'}(S' \times S)))$

$$T^{Hdg}(g, \gamma^\vee)(M, F, W) : \Gamma_{Z \times_S S'}^{Hdg} \hat{*}mod, \Gamma(M, F, W) \xrightarrow{\sim} g_{Hdg}^{\hat{*}mod, \Gamma} \Gamma_Z^{Hdg}(M, F, W)$$

unique up to homotopy such that

$$\gamma_{Z \times_S S'}^{\vee, Hdg}(g_{Hdg}^{\hat{*}mod, \Gamma}(M, F, W)) \circ g_{Hdg}^{\hat{*}mod, \Gamma} \gamma_Z^{\vee, Hdg}(M, F, W) = T^{Hdg}(g, \gamma)(M, F, W).$$

(ii) Let $S \in \text{SmVar}(\mathbb{C})$ and $i_1 : Z_1 \hookrightarrow S, i_2 : Z_2 \hookrightarrow Z_1$ be closed embeddings. Then, for $(M, F, W) \in \pi_S(C(MHM(S)))$,

– there is a canonical map $T(Z_2/Z_1, \gamma^{Hdg})(M, F, W) : \Gamma_{Z_2}^{Hdg}(M, F, W) \rightarrow \Gamma_{Z_1}^{Hdg}(M, F, W)$ in $\pi_S(C(MHM(S)))$ unique up to homotopy such that

$$\gamma_{Z_1}^{Hdg}(G, F) \circ T(Z_2/Z_1, \gamma^{Hdg})(G, F) = \gamma_{Z_2}^{Hdg}(G, F)$$

together with a distinguish triangle in $K(\pi_S(MHM(S)))$

$$\begin{array}{ccc} \Gamma_{Z_2}^{Hdg}(M, F, W) & \xrightarrow{T(Z_2/Z_1, \gamma^{Hdg})(M, F, W)} & \Gamma_{Z_1}^{Hdg}(M, F, W) \\ \text{ad}(j_2^*, j_{2*}^{Hdg})(\Gamma_{Z_1}^{Hdg}(G, F)) & \xrightarrow{\quad} & \Gamma_{Z_1 \setminus Z_2}^{Hdg}(G, F) \rightarrow \Gamma_{Z_2}^{Hdg}(G, F)[1] \end{array}$$

– there is a canonical map $T(Z_2/Z_1, \gamma^{\vee, Hdg})(M, F, W) : \Gamma_{Z_1}^{\vee, Hdg}(M, F, W) \rightarrow \Gamma_{Z_2}^{\vee, Hdg}(M, F, W)$ in $\pi_S(C(MHM(S)))$ unique up to homotopy such that

$$\gamma_{Z_2}^{\vee, Hdg}(M, F, W) = T(Z_2/Z_1, \gamma^{\vee, Hdg})(M, F, W) \circ \gamma_{Z_1}^{\vee, Hdg}(M, F, W).$$

together with a distinguish triangle in $K(\pi_S((MHM(S))))$

$$\begin{array}{ccc} \Gamma_{Z_1 \setminus Z_2}^{\vee, Hdg}(M, F, W) & \xrightarrow{\text{ad}(j_{2!}^{Hdg}, j_2^*)(M, F, W)} & \Gamma_{Z_1}^{\vee, Hdg}(M, F, W) \\ \xrightarrow{T(Z_2/Z_1, \gamma^{\vee, Hdg})(M, F, W)} & \Gamma_{Z_2}^{\vee, Hdg}(M, F, W) & \rightarrow \Gamma_{Z_2 \setminus Z_1}^{\vee, Hdg}(M, F, W)[1] \end{array}$$

Proof. Follows from the projection case and the closed embedding case using the adjunction maps. \square

We have by proposition 93 and proposition 92 the 2 functors on $\text{SmVar}(\mathbb{C})$:

- $\pi(D(MHM(-))) : \text{SmVar}(\mathbb{C}) \rightarrow \pi(D(MHM(-))), S \mapsto \pi_S(D(MHM(S))), (f : T \rightarrow S) \mapsto Rf_*^{Hdg}$,
- $\pi(D(MHM(-))) : \text{SmVar}(\mathbb{C}) \rightarrow \pi(D(MHM(-))), S \mapsto \pi_S(D(MHM(S))), (f : T \rightarrow S) \mapsto Rj_!^{Hdg}$,
- $\pi(D(MHM(-))) : \text{SmVar}(\mathbb{C}) \rightarrow \pi(D(MHM(-))), S \mapsto \pi_S(D(MHM(S))), (f : T \rightarrow S) \mapsto f_{Hdg}^{*mod}$,
- $\pi(D(MHM(-))) : \text{SmVar}(\mathbb{C}) \rightarrow \pi(D(MHM(-))), S \mapsto \pi_S(D(MHM(S))), (f : T \rightarrow S) \mapsto \tilde{f}_{Hdg}^{*mod}$,

The definitions 96 and 97 immediately extends to the non smooth case :

Definition 98. Let $S \in \text{Var}(\mathbb{C})$. Let $Z \subset S$ a closed subset. Let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Denote $Z_I := Z \cap S_I$. Denote by $j : S \setminus Z \hookrightarrow S$ and $\tilde{j}_I : \tilde{S}_I \setminus Z_I \hookrightarrow \tilde{S}_I$ the complementary open embeddings.

(i) We define using definition 94, the filtered Hodge support section functor

$$\Gamma_Z^{Hdg} : \pi(C(MHM(S))) \rightarrow \pi(C(MHM(S))), ((M_I, F, W), u_{IJ}) \mapsto \Gamma_Z^{Hdg}((M_I, F, W), u_{IJ}) := \text{Cone}(\text{ad}(j^*, j_*^{Hdg})((M_I, F, W), u_{IJ}) : ((M_I, F, W), u_{IJ}) \rightarrow (\tilde{j}_{I*}^{Hdg} \tilde{j}_I^*(M_I, F, W), \tilde{j}_J(u_{IJ}))[-1],$$

together with the canonical map $\gamma_Z^{Hdg}((M_I, F, W), u_{IJ}) : \Gamma_Z^{Hdg}((M_I, F, W), u_{IJ}) \rightarrow ((M_I, F, W), u_{IJ})$. We then have the canonical map in $C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$

$$T(\Gamma_Z^{Hdg}, \Gamma_Z)((M_I, F, W), u_{IJ}) := (I, T(j_*^{Hdg}, j_*)(M, F, W)) : \Gamma_Z^{Hdg}((M_I, F, W), u_{IJ}) \rightarrow (\Gamma_Z E(M_I, F, W), \Gamma(u_{IJ}))$$

unique up to homotopy such that

$$\gamma_Z^{Hdg}((M_I, F, W), u_{IJ}) = (\gamma_{Z_I}(E(M_I, F, W))) \circ T(\Gamma_Z^{Hdg}, \Gamma_Z)((M_I, F, W), u_{IJ}).$$

(i)' Since $\tilde{j}_{I*}^{Hdg} : \pi_{\tilde{S}_I}(C(MHM(\tilde{S}_I \setminus S_I))) \rightarrow \pi_{\tilde{S}_I}(C(MHM(\tilde{S}_I)))$ are exact functors, Γ_Z^{Hdg} induces the functor

$$\Gamma_Z^{Hdg} : \pi_S(D(MHM(S))) \rightarrow \pi_S(D(MHM(S))), ((M_I, F, W), u_{IJ}) \mapsto \Gamma_Z^{Hdg}((M_I, F, W), u_{IJ})$$

(ii) We define using definition 94, the dual filtered Hodge support section functor

$$\Gamma_Z^{\vee, Hdg} : \pi(C(MHM(S))) \rightarrow \pi(C(MHM(S))), ((M_I, F, W), u_{IJ}) \mapsto \Gamma_Z^{\vee, Hdg}((M_I, F, W), u_{IJ}) := \text{Cone}(\text{ad}(j_!^{Hdg}, j^*)((M_I, F, W), u_{IJ}) : \tilde{j}_I^{Hdg} \tilde{j}_I^*((M_I, F, W), u_{IJ}) \rightarrow ((M_I, F, W), u_{IJ})),$$

together with the canonical map $\gamma_Z^{\vee, Hdg}((M_I, F, W), u_{IJ}) : ((M_I, F, W), u_{IJ}) \rightarrow \Gamma_Z^{\vee, Hdg}((M_I, F, W), u_{IJ})$. We then have the canonical map in $C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$

$$T(\Gamma_Z^{\vee, h}, \Gamma_Z^{\vee, Hdg})((M_I, F, W), u_{IJ}) := (I, T(j_!, j_!^{Hdg})((M_I, F, W), u_{IJ})) : (\Gamma_Z^{\vee, h}(M_I, F, W), \Gamma_Z^{\vee, h}(u_{IJ})) \rightarrow \Gamma_Z^{\vee, Hdg}((M_I, F, W), u_{IJ})$$

unique up to homotopy such that

$$\gamma_Z^{\vee, Hdg}((M_I, F, W), u_{IJ}) = T(\Gamma_Z^{\vee, h}, \Gamma_Z^{\vee, Hdg})((M_I, F, W), u_{IJ}) \circ (\gamma_{Z_I}^{\vee, h}(M_I, F, W)).$$

(ii)' Since $\tilde{j}_I^{Hdg} : \pi_{\tilde{S}_I}(C(MHM(\tilde{S}_I \setminus S_I))) \rightarrow \pi_{\tilde{S}_I}(C(MHM(\tilde{S}_I)))$ are exact functors, $\Gamma_Z^{Hdg, \vee}$ induces the functor

$$\Gamma_Z^{\vee, Hdg} : \pi_S(D(MHM(S))) \rightarrow \pi_S(D(MHM(S))), ((M_I, F, W), u_{IJ}) \mapsto \Gamma_Z^{\vee, Hdg}((M_I, F, W), u_{IJ})$$

Definition 99. Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i \in I} S_i$ an open cover such that there exist closed embeddings $i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Denote $X_I := f^{-1}(S_I)$. We have then $X = \cup_{i \in I} X_i$ and the commutative diagrams

$$\begin{array}{ccccc} f : X_I & \xrightarrow{l_I} & Y \times S_I & \xrightarrow{p_{S_I}} & S_I \\ & \searrow & \downarrow i'_I := (I \times i_I) & & \downarrow i_I \\ & & Y \times \tilde{S}_I & \xrightarrow{p_{\tilde{S}_I} := \tilde{f}_I} & \tilde{S}_I \end{array}$$

(i) For $((M_I, F, W), u_{IJ}) \in \pi_S(C(MHM(S)))$ we set (see definition 98 for l)

$$f_{Hdg}^{*mod}((M_I, F, W), u_{IJ}) := \Gamma_X^{Hdg}(p_{\tilde{S}_I}^{*mod[-]}(M_I, F, W), u_{IJ}) \in \pi_X(C(MHM(X))),$$

We have for $((M_I, F, W), u_{IJ}) \in \pi_S(C(MHM(S)))$, the canonical map in $C_{\mathcal{D}(1,0)fil}(X/(Y \times \tilde{S}_I))$

$$\begin{aligned} T(f_{Hdg}^{*mod}, f_{Hdg}^{*mod, \Gamma})((M_I, F, W), u_{IJ}) : f_{Hdg}^{*mod}((M_I, F, W), u_{IJ}) &:= \Gamma_X^{Hdg}(p_{\tilde{S}_I}^{*mod[-]}(M_I, F, W), p_{\tilde{S}_I}^{*mod[-]}u_{IJ}) \\ &\xrightarrow{T(\Gamma_X^{Hdg}, \Gamma_X)(-)} (\Gamma_X E(p_{\tilde{S}_I}^{*mod[-]}(M_I, F, W)), \tilde{f}_I^{*mod[-]}u_{IJ}) =: f^{*mod[-], \Gamma}(M, F, W) \end{aligned}$$

(ii) For $((M_I, F, W), u_{IJ}) \in \pi_S(C(MHM(S)))$ we set (see definition 98 for l)

$$f_{Hdg}^{\hat{*}mod}(M, F, W) := \Gamma_X^{\vee, Hdg}(p_{\tilde{S}_I}^{*mod[-]}(M_I, F, W), p_{\tilde{S}_I}^{*mod[-]}u_{IJ}) \in \pi_X(C(MHM(X))),$$

We have for $(M, F, W) \in \pi_S(C(MHM(S)))$, the canonical map in $C_{\mathcal{D}(1,0)fil}(X/(Y \times \tilde{S}_I))$

$$\begin{aligned} T(f_{Hdg}^{\hat{*}mod, \Gamma}, f_{Hdg}^{\hat{*}mod[-]})((M_I, F, W), u_{IJ}) : f_{Hdg}^{\hat{*}mod[-], \Gamma}(M, F, W) &:= \mathbb{D}_S^K f_{Hdg}^{\hat{*}mod[-], \Gamma} \mathbb{D}_X^K((M_I, F, W), u_{IJ}) \\ &\xrightarrow{\mathbb{D}_S^K T(\Gamma_X^{Hdg}, \Gamma_X)(-)} \Gamma_X^{\vee, Hdg}(p_{\tilde{S}_I}^{*mod[-]}(M_I, F, W), p_{\tilde{S}_I}^{*mod[-]}u_{IJ}) =: f_{Hdg}^{\hat{*}mod[-]}(M, F, W) \end{aligned}$$

From the D-module case on algebraic varieties and the constructible sheaves case on CW complexes, we get :

Definition 100. Let $f : X \rightarrow S$ a morphism with $X, S \in \text{QPVar}(\mathbb{C})$. Then, since X is quasi-projective, there exist a factorization $f : X \xrightarrow{l} \mathbb{P}^{N, o} \times S \xrightarrow{p_S} S$ with $n_0 : \mathbb{P}^{N, o} \hookrightarrow \mathbb{P}^N$ an open subset, l a closed embedding and p_S the projection. Since S is quasi-projective, there exist a closed embedding $i : S \hookrightarrow \tilde{S}$ with $\tilde{S} \in \text{SmVar}(\mathbb{C})$. We have then the commutative diagram

$$\begin{array}{ccccc} f : X & \xrightarrow{l} & \mathbb{P}^{N, o} \times S & \xrightarrow{p_S} & S \\ & \searrow & \downarrow i'_I := (I \times i) & & \downarrow i \\ & & \mathbb{P}^{N, o} \times \tilde{S} & \xrightarrow{p_{\tilde{S}}} & \tilde{S} \\ & & \downarrow n := (n_0 \times I) & & \downarrow = \\ & & \mathbb{P}^N \times \tilde{S} & \xrightarrow{\tilde{p}_{\tilde{S}}} & \tilde{S} \end{array}$$

(i) For $((M, F, W), (K, W), \alpha) \in D(\text{MHM}(X))$, where $(M, F, W) \in C_{\mathcal{D}(1,0)\text{fil}}(X/\mathbb{P}^{N,o} \times \tilde{S})$ and $(K, W) \in C_{\text{fil}}(X^{an})$, we define, using theorem 27(ii) for \bar{p}_S and definition 94 for n ,

$$\begin{aligned} f_{*Hdg}((M, F, W), (K, W), \alpha) &: = (Rf_*^{Hdg}(M, F, W), Rf_*(K, W), f_*(\alpha)) \\ &: = (R\bar{p}_{\tilde{S}^*}^{Hdg} n_*^{Hdg}(M, F, W), Rf_*(K, W), f_*(\alpha)) \in D(\text{MHM}(S)) \end{aligned}$$

with

$$\begin{aligned} f_*(\alpha) : Rf_*(K, W) &\xrightarrow{Rf_*\alpha} Rf_*DR(T)^{[-]}((M, W)^{an}) \\ \xrightarrow{T_*(f, DR)(M, W)} DR(S)^{[-]} &((\int_f (M, W))^{an}) = DR(S)^{[-]}((Rf_*^{Hdg}(M, W))^{an}) \end{aligned}$$

see definition 90 and remark 9.

(ii) For $((M, F, W), (K, W), \alpha) \in D(\text{MHM}(X))$, where $(M, F, W) \in C_{\mathcal{D}(1,0)\text{fil}}(X/\mathbb{P}^{N,o} \times \tilde{S})$ and $(K, W) \in C_{\text{fil}}(X^{an})$, we define, using theorem 27(ii) for \bar{p}_S and definition 94 for n ,

$$\begin{aligned} f_{!Hdg}((M, F, W), (K, W), \alpha) &: = (Rf_{!}^{Hdg}(M, F, W), Rf_{!}(K, W), f_{!}(\alpha)) \\ &: = (R\bar{p}_{\tilde{S}^*}^{Hdg} n_{!}^{Hdg}(M, F, W), Rf_{!}(K, W), f_{!}(\alpha)) \in D(\text{MHM}(S)) \end{aligned}$$

with

$$\begin{aligned} f_{!}(\alpha) : Rf_{!}(K, W) &\xrightarrow{Rf_{!}\alpha} Rf_{!}DR(T)((M, W)^{an}) \\ \xrightarrow{T_{!}(f, DR)(M, W)} DR(S)^{[-]} &((\int_{f!} (M, W))^{an}) = DR(S)^{[-]}((Rf_{!}^{Hdg}(M, W))^{an}) \end{aligned}$$

see definition 90 and remark 9.

(iii) For $((M, F, W), (K, W), \alpha) \in D(\text{MHM}(S))$, where $(M, F, W) \in C_{\mathcal{D}(1,0)\text{fil}}(S/(\tilde{S}))$, $(K, W) \in C_{\text{fil}}(S^{an})$, we define, using definition 99 (see theorem 28(ii) for p_S and definition 96 for $i \circ l$),

$$\begin{aligned} f^{*Hdg}((M, F, W), (K, W), \alpha) &: = (f_{Hdg}^{*mod}(M, F, W), f^*(K, W), f^*\alpha) \\ &: = (\Gamma_X^{\vee, Hdg} p_{\tilde{S}}^{*mod[-]}(M, F, W), \Gamma_X^{\vee} p_S^*(K, W), f^*(\alpha)) \in D(\text{MHM}(X)) \end{aligned}$$

with

$$\begin{aligned} f^*(\alpha) : f^*(K, W) &\xrightarrow{f^*\alpha} f^*DR(S)((M, W)^{an}) \\ \xrightarrow{T(f, DR)((M, W))} DR(T)^{[-]} &((Lf^{*mod[-], \Gamma}(M, W))^{an}) = DR(T)^{[-]}((f_{Hdg}^{*mod}(M, W))^{an}) \end{aligned}$$

see definition 90 and remark 9. For $j : S^o \hookrightarrow S$ an open embedding and $((M, F, W), (K, W), \alpha) \in D(\text{MHM}(S))$, we have (see [27])

$$j^{*Hdg}((M, F, W), (K, W), \alpha) = (j^*(M, F, W), j^*(K, W), j^*\alpha) \in D(\text{MHM}(S^o)).$$

(iv) For $((M, F, W), (K, W), \alpha) \in D(\text{MHM}(S))$, where $(M, F, W) \in C_{\mathcal{D}(1,0)\text{fil}}(S/(\tilde{S}))$, $(K, W) \in C_{\text{fil}}(S^{an})$, we define, using definition 99 (see theorem 28(ii) for p_S and definition 96 for $i \circ l$),

$$\begin{aligned} f^{!Hdg}((M, F, W), (K, W), \alpha) &: = (f_{Hdg}^{*mod}(M, F, W), f^!(K, W), f^!\alpha) \\ &: = (\Gamma_X^{\vee, Hdg} p_{\tilde{S}^!}^{*mod[-]}(M, F, W), R\Gamma_X p_S^*(K, W), f^!(\alpha)) \in D(\text{MHM}(X)) \end{aligned}$$

with

$$\begin{aligned} f^!(\alpha) : f^!(K, W) &\xrightarrow{f^!\alpha} f^!DR(S)((M, W)^{an}) \\ \xrightarrow{T^!(f, DR)((M, W))^{-1}} DR(T)^{[-]} &((Rf^{*mod[-], \Gamma}(M, W))^{an}) = DR(T)^{[-]}((f_{Hdg}^{*mod}(M, W))^{an}) \end{aligned}$$

see definition 90 and remark 9. For $j : S^\circ \hookrightarrow S$ an open embedding and $((M, F, W), (K, W), \alpha) \in D(\text{MHM}(S))$, we have (see [27])

$$j^{!Hdg}((M, F, W), (K, W), \alpha) = (j^*(M, F, W), j^*(K, W), j^*\alpha) \in D(\text{MHM}(S^\circ)).$$

Using the unicity of proposition 90, we see that these definitions does NOT depends on the choice of the factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ of f . Moreover, using the unicity of proposition 90 and proposition 43, we see that they are 2 functors on the category of quasi-projective complex algebraic varieties $(\text{Var}(\mathbb{C}))^{QP}$.

- By definition, we have

$$\iota_S^{-1} \left(\int_{\bar{p}_S}^{FDR} n_*^{Hdg}(M, F, W), Rf_*(K, W), f_*(\alpha) \right) = Rf_{*Hdg}((M, F, W), (K, W), \alpha) \in D(\text{MHM}(S)).$$

and for $j : S^\circ \hookrightarrow S$ an open embedding and $((M, F, W), (K, W), \alpha) \in D(\text{MHM}(S^\circ))$,

$$j_{*Hdg}((M, F, W), (K, W), \alpha) = (j_*^{Hdg}(M, F, W), Rj_*(K, W), j_*\alpha) \in D(\text{MHM}(S)).$$

- By definition, we have

$$\iota_S^{-1} \left(\int_{\bar{p}_S}^{FDR} n_!^{Hdg}(M, F, W), Rf_!(K, W), f_!(\alpha) \right) = Rf_{!Hdg}((M, F, W), (K, W), \alpha) \in D(\text{MHM}(S)).$$

and for $j : S^\circ \hookrightarrow S$ an open embedding and $((M, F, W), (K, W), \alpha) \in D(\text{MHM}(S^\circ))$,

$$j_{!Hdg}((M, F, W), (K, W), \alpha) = (j_!^{Hdg}(M, F, W), j_!(K, W), j_!\alpha) \in D(\text{MHM}(S)).$$

We have then the following

Theorem 29. *Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$, X quasi-projective. Then,*

- (i) $(f^{*Hdg}, f_{*Hdg}) : D(\text{MHM}(S)) \rightarrow D(\text{MHM}(X))$ is a pair of adjoint functors,
- (ii) $(f^{*Hdg}, f_{!Hdg}) : D(\text{MHM}(S)) \rightarrow D(\text{MHM}(X))$ is a pair of adjoint functors.

Proof. For the projection case see section 4. For the open embedding see definition 94. □

Definition 100 gives by proposition 93 and proposition 92 the following 2 functors :

- We have the following 2 functor on the category of complex algebraic varieties

$$\begin{aligned} D(\text{MHW}(\cdot)) : \text{Var}(\mathbb{C}) &\rightarrow \text{TriCat}, S \mapsto D(\text{MHW}(S)), \\ (f : T \rightarrow S) &\mapsto (f^{*Hdg} : ((M, F, W), (K, W), \alpha) \mapsto \\ &f^{*Hdg}((M, F, W), (K, W), \alpha) := (f_{Hdg}^{*mod}(M, F, W), f^*(K, W), f^*(\alpha))). \end{aligned}$$

- We have the following 2 functor on the category of complex quasi-projective algebraic varieties

$$\begin{aligned} D(\text{MHW}(\cdot)) : \text{QPVar}(\mathbb{C}) &\rightarrow \text{TriCat}, S \mapsto D(\text{MHW}(S)), \\ (f : T \rightarrow S) &\mapsto (f_{*Hdg} : ((M, F, W), (K, W), \alpha) \mapsto \\ &f_{*Hdg}((M, F, W), (K, W), \alpha) := (Rf_*^{Hdg}(M, F, W), Rf_*(K, W), f_*(\alpha))). \end{aligned}$$

- We have the following 2 functor on the category of complex quasi-projective algebraic varieties

$$\begin{aligned} D(\text{MHW}(\cdot)) : \text{QPVar}(\mathbb{C}) &\rightarrow \text{TriCat}, S \mapsto D(\text{MHW}(S)), \\ (f : T \rightarrow S) &\mapsto (f_{!Hdg} : ((M, F, W), (K, W), \alpha) \mapsto \\ &f_{!Hdg}((M, F, W), (K, W), \alpha) := (Rf_!^{Hdg}(M, F, W), Rf_!(K, W), f_!(\alpha))). \end{aligned}$$

- We have the following 2 functor on the category of complex algebraic varieties

$$\begin{aligned} D(MHW(\cdot)) : \text{Var}(\mathbb{C}) &\rightarrow \text{TriCat}, S \mapsto D(MHW(S)), \\ (f : T \rightarrow S) &\mapsto (f^{!Hdg} : ((M, F, W), (K, W), \alpha) \mapsto \\ f^{!Hdg}((M, F, W), (K, W), \alpha) &:= (f_{Hdg}^{*mod}(M, F, W), f^!(K, W), f^!(\alpha))). \end{aligned}$$

For a commutative diagram in $\text{Var}(\mathbb{C})$

$$D = \begin{array}{ccc} X & \xrightarrow{f} & S \\ g' \uparrow & & \uparrow g \\ X' & \xrightarrow{f'} & T \end{array}$$

with S, T, X', X quasi-projective, we have, for $((M, F, W), (K, W), \alpha) \in D(MHM(X))$ using theorem 29, the following transformations maps

$$\begin{aligned} &T_1^{Hdg}(D)((M, F, W), (K, W), \alpha) : \\ &g^{*Hdg} f_{*Hdg}((M, F, W), (K, W), \alpha) \xrightarrow{\text{ad}(f'^{*Hdg}, f_{*Hdg})(-)} f'_{*Hdg} f'^{*Hdg} g^{*Hdg} f_{*Hdg}((M, F, W), (K, W), \alpha) \\ \xrightarrow{\cong} f'_{*Hdg} g'^{*Hdg} f^{*Hdg} f_{*Hdg}((M, F, W), (K, W), \alpha) &\xrightarrow{\text{ad}(f'^{*Hdg}, f_{*Hdg})(-)} f'_{*Hdg} g'^{*Hdg}((M, F, W), (K, W), \alpha) \end{aligned}$$

and

$$\begin{aligned} &T_2^{Hdg}(D)((M, F, W), (K, W), \alpha) : \\ &f'_{!Hdg} g'^{!Hdg}((M, F, W), (K, W), \alpha) \xrightarrow{\text{ad}(f'^{!Hdg}, f_{!Hdg})(-)} f'_{!Hdg} g'^{!Hdg} f^{!Hdg} f'_{!Hdg}((M, F, W), (K, W), \alpha) \\ \xrightarrow{\cong} f'_{!Hdg} f'^{!Hdg} g^{!Hdg} f'_{!Hdg}((M, F, W), (K, W), \alpha) &\xrightarrow{\text{ad}(f'^{*Hdg}, f'_{*Hdg})(-)} g^{!Hdg} f'_{!Hdg}((M, F, W), (K, W), \alpha) \end{aligned}$$

One consequence of the unicity of proposition 90 is the following :

Proposition 94. *For a commutative diagram in $\text{Var}(\mathbb{C})$*

$$D = \begin{array}{ccc} X & \xrightarrow{f} & S \\ g' \uparrow & & \uparrow g \\ X_T & \xrightarrow{f'} & T \end{array}$$

which is cartesian, with S, T, X', X quasi-projective and f (hence f' proper), and $((M, F, W), (K, W), \alpha) \in D(MHM(X))$

$$\begin{aligned} &T_1^{Hdg}(f, g) : ((M, F, W), (K, W), \alpha) : \\ &g^{*Hdg} f_{*Hdg}((M, F, W), (K, W), \alpha) \xrightarrow{\cong} f'_{*Hdg} g'^{*Hdg}((M, F, W), (K, W), \alpha) \end{aligned}$$

is an isomorphism.

Proof. See [27]. □

Proposition 95. *(i) Let $S \in \text{AnSp}(\mathbb{C})$. Take an open cover $S = \cup_{i=1}^l S_i$ such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then for $((M_I, F, W), u_{IJ}), ((N_I, F, W), v_{IJ}), (K', W), \alpha')$ $\in MHM(S)$*

$$(((M_I, F, W), u_{IJ}) \otimes_{O_S} ((N_I, F, W), v_{IJ}), (K, W) \otimes (K', W), \alpha \otimes \alpha') \in MHM(S)$$

(ii) Let $S \in \text{Var}(\mathbb{C})$. Take an open cover $S = \cup_{i=1}^l S_i$ such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then for $((M_I, F, W), u_{IJ}, (K, W), \alpha), ((N_I, F, W), v_{IJ}, (K', W), \alpha') \in \text{MHM}(S)$

$$(((M_I, F, W), u_{IJ}) \otimes_{O_S} ((N_I, F, W), v_{IJ}), (K, W) \otimes (K', W), \alpha \otimes \alpha') \in \text{MHM}(S)$$

Proof. See [27]. □

- Let $S \in \text{AnSp}(\mathbb{C})$. Take an open cover $S = \cup_{i=1}^l S_i$ such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. By proposition 95(i), the functor

$$\begin{aligned} ((-) \otimes_{O_S}^{[-]} (-), (-) \otimes (-)) : (C_{\mathcal{D}(1,0)\text{fil}}(S/(\tilde{S}_I)) \times_I C_{\text{fil}}(S))^2 &\rightarrow C_{\mathcal{D}(1,0)\text{fil}}(S/(\tilde{S}_I)) \times_I C(S), \\ (((M_I, F, W), u_{IJ}), (K, W), \alpha), ((N_I, F, W), v_{IJ}), (K', W), \alpha') &\mapsto \\ (((M_I, F, W), u_{IJ}) \otimes_{O_S} ((N_I, F, W), v_{IJ}), (K, W) \otimes (K', W), \alpha \otimes \alpha') & \end{aligned}$$

restricts to a functor $((-) \otimes_{O_S} (-), (-) \otimes (-)) : C(\text{MHM}(S))^2 \rightarrow C(\text{MHM}(S))$.

- Let $S \in \text{Var}(\mathbb{C})$. Take an open cover $S = \cup_{i=1}^l S_i$ such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. By proposition 95(ii), the functor

$$\begin{aligned} (-) \otimes (-) : (C_{\mathcal{D}(1,0)\text{fil}}(S/(\tilde{S}_I)) \times_I C_{\text{fil}}(S^{\text{an}}))^2 &\rightarrow C_{\mathcal{D}(1,0)\text{fil}}(S/(\tilde{S}_I)) \times_I C(S^{\text{an}}), \\ (((M_I, F, W), u_{IJ}), (K, W), \alpha), ((N_I, F, W), v_{IJ}), (K', W), \alpha') &\mapsto \\ (((M_I, F, W), u_{IJ}) \otimes_{O_S} ((N_I, F, W), v_{IJ}), (K, W) \otimes (K', W), \alpha \otimes \alpha') & \end{aligned}$$

restricts to a functor $((-) \otimes_{O_S} (-), (-) \otimes (-)) : C(\text{MHM}(S))^2 \rightarrow C(\text{MHM}(S))$.

For $X \in \text{SmVar}(\mathbb{C})$, we have, by definition

$$\mathbb{Z}_X^{\text{Hdg}} := a_X^* \mathbb{Z}_{\text{pt}}^{\text{Hdg}} := ((O_X, F_b)[d_X], \mathbb{Z}_X, \alpha(X)) \in D(\text{MHM}(X)),$$

with $\alpha(X) : \mathbb{C}_X \hookrightarrow (0 \rightarrow O_X \rightarrow \Omega_X \rightarrow \cdots \rightarrow K_X)$. If $X \in \text{SmVar}(\mathbb{C})$,

$$\mathbb{Z}_X^{\text{Hdg}} := a_X^* \mathbb{Z}_{\text{pt}}^{\text{Hdg}} := ((O_X, F_b)[d_X], \mathbb{Z}_{X^{\text{an}}}, \alpha(X^{\text{an}})) \in D(\text{MHM}(X)).$$

Let $X \in \text{Var}(\mathbb{C})$ non smooth. Take an open cover $X = \cup_{i=1}^l X_i$ such that there exists closed embeddings $i_i : X_i \hookrightarrow \tilde{X}_i$ with $\tilde{X}_i \in \text{SmVar}(\mathbb{C})$. Then, by definition

$$\mathbb{Z}_X^{\text{Hdg}} := a_X^* \mathbb{Z}_{\text{pt}}^{\text{Hdg}} := ((\Gamma_{X_I}^{\vee, \text{Hdg}}(O_{\tilde{X}_I}, F_b)[d_{\tilde{X}_I}], o_{\tilde{S}_J/\tilde{S}_I}), (\mathbb{Z}_{X^{\text{an}}}, W), \alpha(X/\tilde{X}_I)) \in D(\text{MHM}(X)).$$

with

$$\alpha(X/\tilde{X}_I) : (\Gamma_{X_I}^{\vee}, \alpha(\tilde{X}_I)) : T(X/(\tilde{X}_I))(\mathbb{Z}_{X^{\text{an}}}) := (i_{I*} \mathbb{Z}_{X_I^{\text{an}}}, I) \rightarrow DR(X)^{[-]}((\Gamma_{X_I}^{\vee, \text{Hdg}}(O_{\tilde{X}_I})[d_{\tilde{X}_I}], o_{\tilde{S}_J/\tilde{S}_I})$$

We finish this section by the following proposition

Proposition 96. *Let $Y \in \text{PSmVar}(\mathbb{C})$ and $i : Z \hookrightarrow S$ a closed embedding with Z smooth. Denote by $j : U := S \setminus Z \hookrightarrow Y$ the complementary open subset.*

(i) *We have*

$$\begin{aligned} a_{UHdg!} \mathbb{Z}_U^{\text{Hdg}} &:= a_{UHdg!}((O_U, F_b), \mathbb{Z}_{U^{\text{an}}}, \alpha(U)) \xrightarrow{=} \left(\int_{a_Y}^{\text{FDR}} j_!^{\text{Hdg}}(O_U, F_b), (Ra_{U*} \mathbb{Z}_{U^{\text{an}}}, W), a_{U*} \alpha(U) \right) \\ &\xrightarrow{\sim} \left(\int_{a_Y}^{\text{FDR}} \text{Cone}(\mathbb{D}_S^K \text{ad}(i_{*mod}, i^\sharp)(-) : (O_Y, F_b) \rightarrow i_{*mod}(O_D, F_b)), (Ra_{U!} \mathbb{Z}_{U^{\text{an}}}, W), a_{U!} \alpha(U) \right) \\ &\xrightarrow{\sim} (\text{Cone}(E(\Omega_{D/Y})(D) : \Gamma(Y, E(\Omega_Y^\bullet, F_b)) \rightarrow \Gamma(D, E(\Omega_D^\bullet, F_b))), W), (Ra_{U!} \mathbb{Z}_{U^{\text{an}}}, W), a_{U!} \alpha(U) \end{aligned}$$

(ii) We have

$$\begin{aligned}
a_{UHdg*}\mathbb{Z}_U^{Hdg} &:= a_{UHdg*}((O_U, F), \mathbb{Z}_{U^{an}}, \alpha(U)) \xrightarrow{\cong} \left(\int_{a_Y}^{FDR} j_*^{Hdg}(O_U, F, W), (Ra_{U!}\mathbb{Z}_{U^{an}}, W), a_{U!}\alpha(U) \right) \\
&\xrightarrow{\cong} \left(\int_{a_Y}^{FDR} \text{Cone}(\text{ad}(i_{*mod}, i^\sharp)(-) : i_{*mod}(O_D, F_b)[c] \rightarrow (O_Y, F_b)), (Ra_{U*}\mathbb{Z}_{U^{an}}, W), a_{U*}\alpha(U) \right) \\
&\xrightarrow{\cong} (\text{Cone}(i_{D*} : \Gamma(D, E(\Omega_D^\bullet, F_b)))[c] \rightarrow \Gamma(Y, E(\Omega_Y^\bullet, F_b)), (Ra_{U*}\mathbb{Z}_{U^{an}}, W), a_{U*}\alpha(U))
\end{aligned}$$

Proof. See [27]. □

In the case where $D = \cup D_i \subset Y$ is a normal crossing divisor, proposition 96 gives

$$a_{HdgU*}\mathbb{Z}_U^{Hdg} \xrightarrow{\cong} (\Gamma(Y, E(\Omega_Y^\bullet(\log D), F, W)), (Ra_{U*}\mathbb{Z}_{U^{an}}, W), a_{U*}\alpha(U))$$

and

$$a_{HdgU!}\mathbb{Z}_U^{Hdg} := (\Gamma(Y, E(\Omega_Y^\bullet(\text{nul}D), F, W)), (Ra_{U!}\mathbb{Z}_{U^{an}}, W), a_{U!}\alpha(U))$$

6 The algebraic and analytic filtered De Rham realizations for Voevodsky relative motives

6.1 The algebraic filtered De Rham realization functor

6.1.1 The algebraic Gauss-Manin filtered De Rham realization functor and its transformation map with pullbacks

Consider, for $S \in \text{Var}(\mathbb{C})$, the following composition of morphism in RCat (see section 2)

$$\tilde{e}(S) : (\text{Var}(\mathbb{C})/S, O_{\text{Var}(\mathbb{C})/S}) \xrightarrow{\rho_S} (\text{Var}(\mathbb{C})^{sm}/S, O_{\text{Var}(\mathbb{C})^{sm}/S}) \xrightarrow{e(S)} (S, O_S)$$

with, for $X/S = (X, h) \in \text{Var}(\mathbb{C})/S$,

- $O_{\text{Var}(\mathbb{C})/S}(X/S) := O_X(X)$,
- $(\tilde{e}(S)^*O_S(X/S) \rightarrow O_{\text{Var}(\mathbb{C})/S}(X/S)) := (h^*O_S \rightarrow O_X)$.

and $O_{\text{Var}(\mathbb{C})^{sm}/S} := \rho_{S*}O_{\text{Var}(\mathbb{C})/S}$, that is, for $U/S = (U, h) \in \text{Var}(\mathbb{C})^{sm}/S$, $O_{\text{Var}(\mathbb{C})^{sm}/S}(U/S) := O_{\text{Var}(\mathbb{C})/S}(U/S) := O_U(U)$

Definition 101. (i) For $S \in \text{Var}(\mathbb{C})$, we consider the complexes of presheaves

$$\Omega_{/S}^\bullet := \text{coker}(\Omega_{O_{\text{Var}(\mathbb{C})/S}/\tilde{e}(S)^*O_S} : \Omega_{\tilde{e}(S)^*O_S}^\bullet \rightarrow \Omega_{O_{\text{Var}(\mathbb{C})/S}}^\bullet) \in C_{O_S}(\text{Var}(\mathbb{C})/S)$$

which is by definition given by

- for X/S a morphism $\Omega_{/S}^\bullet(X/S) = \Omega_{X/S}^\bullet(X)$
- for $g : X'/S \rightarrow X/S$ a morphism,

$$\begin{aligned}
\Omega_{/S}^\bullet(g) &:= \Omega_{(X'/X)/(S/S)}(X') : \Omega_{X/S}^\bullet(X) \rightarrow g^*\Omega_{X/S}(X') \rightarrow \Omega_{X'/S}^\bullet(X') \\
\omega &\mapsto \Omega_{(X'/X)/(S/S)}(X')(\omega) := g^*(\omega) : (\alpha \in \wedge^k T_{X'}(X') \mapsto \omega(dg(\alpha)))
\end{aligned}$$

(ii) For $S \in \text{Var}(\mathbb{C})$, we consider the complexes of presheaves

$$\Omega_{/S}^\bullet := \rho_{S*}\tilde{\Omega}_{/S}^\bullet = \text{coker}(\Omega_{O_{\text{Var}(\mathbb{C})^{sm}/S}/e(S)^*O_S} : \Omega_{e(S)^*O_S}^\bullet \rightarrow \Omega_{O_{\text{Var}(\mathbb{C})^{sm}/S}}^\bullet) \in C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S)$$

which is by definition given by

- for U/S a smooth morphism $\Omega_{/S}^\bullet(U/S) = \Omega_{U/S}^\bullet(U)$
- for $g : U'/S \rightarrow U/S$ a morphism,

$$\begin{aligned}\Omega_{/S}^\bullet(g) &:= \Omega_{(U'/U)/(S/S)}(U') : \Omega_{U/S}^\bullet(U) \rightarrow g^* \Omega_{U/S}(U') \rightarrow \Omega_{U'/S}^\bullet(U') \\ \omega &\mapsto \Omega_{(U'/U)/(S/S)}(U')(\omega) := g^*(\omega) : (\alpha \in \wedge^k T_{U'}(U') \mapsto \omega(dg(\alpha)))\end{aligned}$$

Remark 10. For $S \in \text{Var}(\mathbb{C})$, $\Omega_{/S}^\bullet \in C(\text{Var}(\mathbb{C})/S)$ is by definition a natural extension of $\Omega_{/S}^\bullet \in C(\text{Var}(\mathbb{C})^{sm}/S)$. However $\Omega_{/S}^\bullet \in C(\text{Var}(\mathbb{C})/S)$ does NOT satisfy *cdh descent*.

For a smooth morphism $h : U \rightarrow S$ with $S, U \in \text{SmVar}(\mathbb{C})$, the cohomology presheaves $H^n \Omega_{U/S}^\bullet$ of the relative De Rham complex

$$DR(U/S) := \Omega_{U/S}^\bullet := \text{coker}(h^* \Omega_S \rightarrow \Omega_U) \in C_{h^* O_S}(U)$$

for all $n \in \mathbb{Z}$, have a canonical structure of a complex of $h^* D_S$ modules given by the Gauss Manin connexion : for $S^o \subset S$ an open subset, $U^o = h^{-1}(S^o)$, $\gamma \in \Gamma(S^o, T_S)$ a vector field and $\hat{\omega} \in \Omega_{U/S}^p(U^o)^c$ a closed form, the action is given by

$$\gamma \cdot [\hat{\omega}] = [\widehat{\iota(\tilde{\gamma})\partial\omega}],$$

$\omega \in \Omega_U^p(U^o)$ being a representative of $\hat{\omega}$ and $\tilde{\gamma} \in \Gamma(U^o, T_U)$ a relevation of γ (h is a smooth morphism), so that

$$DR(U/S) := \Omega_{U/S}^\bullet := \text{coker}(h^* \Omega_S \rightarrow \Omega_U) \in C_{h^* O_S, h^* \mathcal{D}}(U)$$

with this $h^* D_S$ structure. Hence we get $h_* \Omega_{U/S}^\bullet \in C_{O_S, \mathcal{D}}(S)$ considering this structure. Since h is a smooth morphism, $\Omega_{U/S}^p$ are locally free O_U modules.

The point (ii) of the definition 112 above gives the object in $\text{DA}(S)$ which will, for S smooth, represent the algebraic Gauss-Manin De Rham realisation. It is the class of an explicit complex of presheaves on $\text{Var}(\mathbb{C})^{sm}/S$.

Proposition 97. Let $S \in \text{Var}(\mathbb{C})$.

- (i) For $U/S = (U, h) \in \text{Var}(\mathbb{C})^{sm}/S$, we have $e(U)_* h^* \Omega_{/S}^\bullet = \Omega_{U/S}^\bullet$.
- (ii) The complex of presheaves $(\Omega_{/S}^\bullet, F_b) \in C_{O_S \text{ fil}}(\text{Var}(\mathbb{C})^{sm}/S)$ is 2-filtered \mathbb{A}_S^1 local for the étale topology. Note that however, for $p > 0$, the complexes of presheaves $\Omega^{\bullet \geq p}$ are NOT \mathbb{A}_S^1 local. On the other hand, $(\Omega_{/S}^\bullet, F_b)$ admits transfers (recall that means $\text{Tr}(S)_* \text{Tr}(S)^* \Omega_{/S}^p = \Omega_{/S}^p$).
- (iii) If S is smooth, we get $(\Omega_{/S}^\bullet, F_b) \in C_{O_S \text{ fil}, D_S}(\text{Var}(\mathbb{C})^{sm}/S)$ with the structure given by the Gauss Manin connexion. Note that however the D_S structure on the cohomology groups given by Gauss Main connexion does NOT comes from a structure of D_S module structure on the filtered complex of O_S module. The D_S structure on the cohomology groups satisfy a non trivial Griffithz transversality (in the non projection cases), whereas the filtration on the complex is the trivial one.

Proof. (i): Let $h' : V \rightarrow U$ a smooth morphism with $V \in \text{Var}(\mathbb{C})$. We have then

$$h^* \Omega_{/S}^p(V \xrightarrow{h'} U) = \Omega_{/S}^p(V \xrightarrow{h'} U \xrightarrow{h} S).$$

Hence, if $h' : V \hookrightarrow U$ is in particular an open embedding, $h^* \Omega_{/S}^p(V \xrightarrow{h'} U) = \Omega_{U/S}^p(V)$. This proves the equality.

(ii): We prove that $E_{et}(\Omega_{/S}^\bullet, F_b) \in C_{O_S \text{ fil}}(\text{Var}(\mathbb{C})^{sm}/S)$ is 2-filtered \mathbb{A}_S^1 invariant. We follow [20]. Denote by

$$\begin{aligned}P_a &: \text{Var}(\mathbb{C})^{sm}/S \rightarrow \text{Var}(\mathbb{C})^{sm}/S, \\ U/S = (U, h) &\mapsto U \times \mathbb{A}^1/S := (U \times \mathbb{A}^1, h \circ p), \quad g : U/S \rightarrow V/S \mapsto g \times I : U \times \mathbb{A}^1/S \rightarrow V \times \mathbb{A}^1/S\end{aligned}$$

the morphism of site. Consider the map in $C(\text{Var}(\mathbb{C})^{sm}/S)$

$$\phi := \text{ad}(P_a^*, P_{a*})(-) : \Omega_{/S}^\bullet \rightarrow P_{a*}P_a^*\Omega_{/S}^\bullet$$

which is given, for $U/S \in \text{Var}(\mathbb{C})^{sm}/S$ by

$$\text{ad}(P_a^*, P_{a*})(-)(U/S) = \Omega_{(U \times \mathbb{A}^1/U)/(S/S)}(U \times \mathbb{A}^1) : \Omega_{U/S}^\bullet(U) \rightarrow \Omega_{U \times \mathbb{A}^1/S}^\bullet(U \times \mathbb{A}^1), \omega \mapsto p^*\omega$$

where $p : U \times \mathbb{A}^1 \rightarrow U$ is the projection. On the other hand consider the map in $C(\text{Var}(\mathbb{C})^{sm}/S)$

$$\psi := I_0^* : P_{a*}P_a^*\Omega_{/S}^\bullet \rightarrow \Omega_{/S}^\bullet$$

given, for $U/S \in \text{Var}(\mathbb{C})^{sm}/S$ by

$$I_0^*(U/S) : \Omega_{U \times \mathbb{A}^1/S}^\bullet(U \times \mathbb{A}^1) \rightarrow \Omega_{U/S}^\bullet(U), \omega \mapsto i_0^*\omega$$

where $i_0 : U \hookrightarrow U \times \mathbb{A}^1$ is closed embedding given by $i_0(x) := (x, 0)$. Then,

- we have $\phi \circ \psi = I$
- considering the map in $\text{PSh}(\mathbb{N} \times \text{Var}(\mathbb{C})^{sm}/S)$

$$H : P_{a*}P_a^*\Omega_{/S}^\bullet[1] \rightarrow P_{a*}P_a^*\Omega_{/S}^\bullet$$

given for $U/S \in \text{Var}(\mathbb{C})^{sm}/S$ by

$$\begin{aligned} H(U/S)\Omega_{U \times \mathbb{A}^1/S}^p(U \times \mathbb{A}^1) &\rightarrow \Omega_{U \times \mathbb{A}^1/S}^{p-1}(U \times \mathbb{A}^1), \\ H(U/S)(p^*\omega \wedge q^*(f(s)ds)) &= \left(\int_0^t f(s)ds \right) p^*\omega, \quad H(U/S)(p^*\omega \wedge q^*f) = 0, \end{aligned}$$

note that $g(t) = \int_0^t f(s)ds$ is algebraic since $f \in \mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1)$ is a polynomial, we have $\psi \circ \phi - I = \partial H + H\partial$.

This shows that

$$\text{ad}(P_a^*, P_{a*})(-) : \Omega_{/S}^\bullet \rightarrow P_{a*}P_a^*\Omega_{/S}^\bullet$$

is an homotopy equivalence whose inverse is I_0^* . Hence, by proposition 7,

$$\text{ad}(P_a^*, P_{a*})(-) : E_{et}(\Omega_{/S}^\bullet, F_b) \rightarrow E_{et}(P_{a*}P_a^*\Omega_{/S}^\bullet, F_b) = E_{et}P_{a*}(P_a^*\Omega_{/S}^\bullet, F_b)$$

is a 2-filtered quasi-isomorphism, that is $E_{et}(\Omega_{/S}^\bullet, F_b) \in C_{\text{O}_S \text{fil}}(\text{Var}(\mathbb{C})^{sm}/S)$ is 2-filtered \mathbb{A}_S^1 invariant.

(iii): For $h : U \rightarrow S$ a smooth morphism with $U, S \in \text{SmVar}(\mathbb{C})$, recall that the $h^*D_S(U) = D_S(h(U))$ structure on $H^p\Omega_{/S}^\bullet(U/S) := H^p\Omega_{U/S}^\bullet(U)$ is given by, for $\hat{\omega} \in \Omega_{U/S}^p(U)^c$, $\gamma \cdot [\hat{\omega}] = [\iota(\tilde{\gamma})\partial\omega]$, $\omega \in \Omega_U^p(U^o)$ being a representative of $\hat{\omega}$ and $\tilde{\gamma} \in \Gamma(U^o, T_U)$ a relevation of γ (h is a smooth morphism). Now, if $g : V/S \rightarrow U/S$ is a morphism, where $h' : V \rightarrow S$ is a smooth morphism with $V \in \text{SmVar}(\mathbb{C})$, we have

$$g^*(\gamma \cdot \hat{\omega}) = g^*(\iota(\tilde{\gamma})\partial\omega) = \iota(\tilde{\gamma})\partial g^*\omega = \gamma \cdot (g^*\hat{\omega})$$

that is $H^p\Omega_{/S}^\bullet(g) : H^p\Omega^\bullet(U/S) \rightarrow H^p\Omega^\bullet(V/S)$ is a map of $D_S(h(U))$ modules. \square

We have the following canonical transformation map given by the pullback of (relative) differential forms:

Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Consider the following commutative diagram in RCat :

$$\begin{array}{ccc} D(g, e) : (\text{Var}(\mathbb{C})^{sm}/T, \mathcal{O}_{\text{Var}(\mathbb{C})^{sm}/T}) & \xrightarrow{P(g)} & (\text{Var}(\mathbb{C})^{sm}/S, \mathcal{O}_{\text{Var}(\mathbb{C})^{sm}/S}) \\ \downarrow e(T) & & \downarrow e(S) \\ (T, \mathcal{O}_T) & \xrightarrow{P(g)} & (S, \mathcal{O}_S) \end{array}$$

It gives (see section 2) the canonical morphism in $C_{g^*O_S \text{fil}}(\text{Var}(\mathbb{C})^{sm}/T)$

$$\begin{aligned} \Omega_{/(T/S)} &:= \Omega_{(O_{\text{Var}(\mathbb{C})^{sm}/T}/g^*O_{\text{Var}(\mathbb{C})^{sm}/S})/(O_T/g^*O_S)} : \\ g^*(\Omega_{/S}^\bullet, F_b) &= \Omega_{g^*O_{\text{Var}(\mathbb{C})^{sm}/S}/g^*e(S)^*O_S}^\bullet \rightarrow (\Omega_{/T}^\bullet, F_b) = \Omega_{O_{\text{Var}(\mathbb{C})^{sm}/T}/e(T)^*O_T}^\bullet \end{aligned}$$

which is by definition given by the pullback on differential forms : for $(V/T) = (V, h) \in \text{Var}(\mathbb{C})^{sm}/T$,

$$\begin{aligned} \Omega_{/(T/S)}(V/T) : g^*(\Omega_{/S}^\bullet)(V/T) &:= \lim_{(h':U \rightarrow \text{SSm}, g':V \rightarrow U, h, g)} \Omega_{U/S}^\bullet(U) \xrightarrow{\Omega_{(V/U)/(T/S)}(V/T)} \Omega_{V/T}^\bullet(V) =: \Omega_{/T}^\bullet(V/T) \\ \hat{\omega} &\mapsto \Omega_{(V/U)/(T/S)}(V/T)(\omega) := g'^*\hat{\omega}. \end{aligned}$$

If S and T are smooth, $\Omega_{/(T/S)} : g^*(\Omega_{/S}^\bullet, F_b) \rightarrow (\Omega_{/T}^\bullet, F_b)$ is a map in $C_{g^*O_S \text{fil}, g^*D_S}(\text{Var}(\mathbb{C})^{sm}/T)$ It induces the canonical morphisms in $C_{g^*O_S \text{fil}, g^*D_S}(\text{Var}(\mathbb{C})^{sm}/T)$:

$$E\Omega_{/(T/S)} : g^*E_{et}(\Omega_{/S}^\bullet, F_b) \xrightarrow{T(g, E_{et})(\Omega_{/S}^\bullet, F_b)} E_{et}(g^*(\Omega_{/S}^\bullet, F_b)) \xrightarrow{E_{et}(\Omega_{/(T/S)})} E_{et}(\Omega_{/T}^\bullet, F_b).$$

and

$$E\Omega_{/(T/S)} : g^*E_{zar}(\Omega_{/S}^\bullet, F_b) \xrightarrow{T(g, E_{zar})(\Omega_{/S}^\bullet, F_b)} E_{zar}(g^*(\Omega_{/S}^\bullet, F_b)) \xrightarrow{E_{zar}(\Omega_{/(T/S)})} E_{zar}(\Omega_{/T}^\bullet, F_b).$$

Definition 102. (i) Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. We have, for $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$, the canonical transformation in $C_{O_T \text{fil}}(T)$:

$$\begin{aligned} T^O(g, \Omega_{/ \cdot})(F) &: g^{*mod}L_{Oe}(S)_* \mathcal{H}om^\bullet(F, E_{et}(\Omega_{/S}^\bullet, F_b)) \\ &\xrightarrow{=} (g^*L_{Oe}(S)_* \mathcal{H}om^\bullet(F, E_{et}(\Omega_{/S}^\bullet, F_b))) \otimes_{g^*O_S} O_T \\ &\xrightarrow{T(e, g)(-) \circ T(g, L_O)(-)} L_{Oe}(e(T)_* g^* \mathcal{H}om^\bullet(F, E_{et}(\Omega_{/S}^\bullet, F_b))) \otimes_{g^*O_S} O_T \\ &\xrightarrow{T(g, \text{hom})(F, E_{et}(\Omega_{/S}^\bullet)) \otimes I} L_{Oe}(e(T)_* \mathcal{H}om^\bullet(g^*F, g^*E_{et}(\Omega_{/S}^\bullet, F_b))) \otimes_{g^*O_S} O_T \\ &\xrightarrow{ev(\text{hom}, \otimes)(-, -, -)} L_{Oe}(T)_* \mathcal{H}om^\bullet(g^*F, g^*E_{et}(\Omega_{/S}^\bullet, F_b)) \otimes_{g^*e(S)^*O_S} e(T)^*O_T \\ &\xrightarrow{\mathcal{H}om^\bullet(g^*F, E\Omega_{/(T/S)} \otimes I)} L_{Oe}(T)_* \mathcal{H}om^\bullet(g^*F, E_{et}(\Omega_{/T}^\bullet, F_b)) \otimes_{g^*e(S)^*O_S} e(T)^*O_T \\ &\xrightarrow{m} L_{Oe}(T)_* \mathcal{H}om^\bullet(g^*F, E_{et}(\Omega_{/T}^\bullet, F_b)) \end{aligned}$$

where $m(\alpha \otimes h) := h.\alpha$ is the multiplication map.

(ii) Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$, S smooth. Assume there is a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{ps} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and ps the projection. We have, for $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$, the canonical transformation in $C_{O_T \text{fil}}(Y \times S)$:

$$\begin{aligned} T(g, \Omega_{/ \cdot})(F) &: g^{*mod, \Gamma}e(S)_* \mathcal{H}om^\bullet(F, E_{et}(\Omega_{/S}^\bullet, F_b)) \\ &\xrightarrow{=} \Gamma_T E_{zar}(p_S^{*mod}e(S)_* \mathcal{H}om^\bullet(F, E_{et}(\Omega_{/S}^\bullet, F_b))) \\ &\xrightarrow{T^O(p_S, \Omega_{/ \cdot})(F)} \Gamma_T E_{zar}(e(T \times S)_* \mathcal{H}om^\bullet(p_S^*F, E_{et}(\Omega_{/Y \times S}^\bullet, F_b))) \\ &\xrightarrow{=} e(T \times S)_* \Gamma_T (\mathcal{H}om^\bullet(p_S^*F, E_{et}(\Omega_{/Y \times S}^\bullet, F_b))) \\ &\xrightarrow{I(\gamma, \text{hom})(-, -)} e(T \times S)_* \mathcal{H}om^\bullet(\Gamma_T^\vee p_S^*F, E_{et}(\Omega_{/Y \times S}^\bullet, F_b)). \end{aligned}$$

For $Q \in \text{Proj PSh}(\text{Var}(\mathbb{C})^{sm}/S)$,

$$T(g, \Omega_{/ \cdot})(Q) : g^{*mod, \Gamma}e(S)_* \mathcal{H}om^\bullet(Q, E_{et}(\Omega_{/S}^\bullet, F_b)) \rightarrow e(T \times S)_* \mathcal{H}om^\bullet(\Gamma_T^\vee p_S^*Q, E_{et}(\Omega_{/Y \times S}^\bullet, F_b))$$

is a map in $C_{O_T \text{fil}, \mathcal{D}}(Y \times S)$.

The following easy lemma describe these transformation map on representable presheaves :

Lemma 10. *Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$ and $h : U \rightarrow S$ is a smooth morphism with $U \in \text{Var}(\mathbb{C})$. Consider a commutative diagram whose square are cartesian :*

$$\begin{array}{ccccc} g : T & \xrightarrow{l} & S \times Y & \xrightarrow{p_S} & S \\ h' \uparrow & & h'' := h \times I \uparrow & & \uparrow h \\ g' : U_T & \xrightarrow{l'} & U \times Y & \xrightarrow{p_U} & U \end{array}$$

with l, l' the graph embeddings and p_S, p_U the projections. Then $g^*\mathbb{Z}(U/S) = \mathbb{Z}(U_T/T)$ and

(i) we have the following commutative diagram in $C_{O_T \text{fil}}(T)$ (see definition 1 and definition 102(i)) :

$$\begin{array}{ccc} g^{*mod} L_{Oe}(S)_* \mathcal{H}om^\bullet(\mathbb{Z}(U/S), E_{et}(\Omega_{/S}^\bullet, F_b)) & \xrightarrow{T(g, \Omega_{/ \cdot})(\mathbb{Z}(U/S))} & e(T)_* \mathcal{H}om^\bullet(\mathbb{Z}(U_T/T), E_{et}(\Omega_{/T}^\bullet, F_b)) \\ \uparrow k & & \uparrow k \\ g^{*mod} L_{Oe}(S)_* \mathcal{H}om^\bullet(\mathbb{Z}(U/S), E_{zar}(\Omega_{/S}^\bullet, F_b)) & \xrightarrow{T(g, \Omega_{/ \cdot})(\mathbb{Z}(U/S))} & e(T)_* \mathcal{H}om^\bullet(\mathbb{Z}(U_T/T), E_{zar}(\Omega_{/T}^\bullet, F_b)) \\ \downarrow = & & \downarrow = \\ g^{*mod} L_{Oh_*} E_{zar}(\Omega_{U/S}^\bullet, F_b) & \xrightarrow{T_\omega^{mod}(g, h)} & h'_* E_{zar}(\Omega_{U_T/T}^\bullet, F_b) \end{array}$$

(ii) if $Y, S \in \text{SmVar}(\mathbb{C})$, we have the following commutative diagram in $C_{O_T \text{fil}, \mathcal{D}}(Y \times S)$ (see definition 1 and definition 102(ii)) :

$$\begin{array}{ccc} g^{*mod, \Gamma} e(S)_* \mathcal{H}om^\bullet(\mathbb{Z}(U/S), E_{et}(\Omega_{/S}^\bullet, F_b)) & \xrightarrow{T(g, \Omega_{/ \cdot})(\mathbb{Z}(U/S))} & e(Y \times S)_* \mathcal{H}om^\bullet(\Gamma_{U_T}^\vee \mathbb{Z}(U \times Y/S \times Y), E_{et}(\Omega_{/Y \times S}^\bullet, F_b)) \\ \uparrow k & & \uparrow k \\ g^{*mod, \Gamma} L_{Oe}(S)_* \mathcal{H}om^\bullet(\mathbb{Z}(U/S), E_{zar}(\Omega_{/S}^\bullet, F_b)) & \xrightarrow{T(g, \Omega_{/ \cdot})(\mathbb{Z}(U/S))} & e(Y \times S)_* \mathcal{H}om^\bullet(\Gamma_{U_T}^\vee \mathbb{Z}(U \times Y/S \times Y), E_{zar}(\Omega_{/Y \times S}^\bullet, F_b)) \\ \downarrow = & & \downarrow = \\ g^{*mod, \Gamma} h_* E_{zar}(\Omega_{U/S}^\bullet, F_b) & \xrightarrow{T_\omega^O(p_S, h)(-)} & h''_* \Gamma_{U_T} E_{zar}(\Omega_{U \times Y/S \times Y}^\bullet, F_b) \end{array}$$

where $j : T \setminus T \times S \hookrightarrow T \times S$ is the open complementary embedding,

with

$$k : E_{zar}(h^* \Omega_{/S}^\bullet, F_b) \rightarrow E_{et}(E_{zar}(h^* \Omega_{/S}^\bullet, F_b)) = E_{et}(h^* \Omega_{/S}^\bullet, F_b).$$

which is a (1-)filtered Zariski local equivalence.

Proof. The commutative diagram follows from Yoneda lemma and proposition 97(i). On the other hand, $k : E_{zar}(\Omega_{/S}^\bullet, F_b) \rightarrow E_{et}(\Omega_{/S}^\bullet, F_b)$ is a (1-)filtered Zariski local equivalence by theorem 10 and proposition 97(ii) \square

In the projection case, we have the following :

Proposition 98. *Let $p : S_{12} \rightarrow S_1$ is a smooth morphism with $S_1, S_{12} \in \text{AnSp}(\mathbb{C})$. Then if $Q \in C(\text{Var}(\mathbb{C})^{sm}/S_1)$ is projective,*

$$T(p, \Omega_{/ \cdot})(Q) : p^{*mod} e(S_1)_* \mathcal{H}om^\bullet(Q, E_{et}(\Omega_{/S_1}^\bullet, F_b)) \rightarrow e(S_{12})_* \mathcal{H}om^\bullet(p^* Q, E_{et}(\Omega_{/S_{12}}^\bullet, F_b))$$

is an isomorphism.

Proof. Follows from lemma 10 and base change by smooth morphisms of quasi-coherent sheaves. \square

Let $S \in \text{Var}(\mathbb{C})$ and $h : U \rightarrow S$ a morphism with $U \in \text{Var}(\mathbb{C})$. We then have the canonical map given by the wedge product

$$w_{U/S} : \Omega_{U/S}^\bullet \otimes_{O_S} \Omega_{U/S}^\bullet \rightarrow \Omega_{U/S}^\bullet; \alpha \otimes \beta \mapsto \alpha \wedge \beta.$$

Let $S \in \text{Var}(\mathbb{C})$ and $h_1 : U_1 \rightarrow S$, $h_2 : U_2 \rightarrow S$ two morphisms with $U_1, U_2 \in \text{Var}(\mathbb{C})$. Denote $h_{12} : U_{12} := U_1 \times_S U_2 \rightarrow S$ and $p_{112} : U_1 \times_S U_2 \rightarrow U_1$, $p_{212} : U_1 \times_S U_2 \rightarrow U_2$ the projections. We then have the canonical map given by the wedge product

$$w_{(U_1, U_2)/S} : p_{112}^* \Omega_{U_1/S}^\bullet \otimes_{O_S} p_{212}^* \Omega_{U_2/S}^\bullet \rightarrow \Omega_{U_{12}/S}^\bullet; \alpha \otimes \beta \mapsto p_{112}^* \alpha \wedge p_{212}^* \beta$$

which gives the map

$$\begin{aligned} & Ew_{(U_1, U_2)/S} : h_{1*} E_{zar}(\Omega_{U_1/S}^\bullet) \otimes_{O_S} h_{2*} E_{zar}(\Omega_{U_2/S}^\bullet) \\ & \xrightarrow{\text{ad}(p_{112}^*, p_{112*})(-) \otimes \text{ad}(p_{212}^*, p_{212*})(-)} (h_{1*} p_{112*} p_{112}^* E_{zar}(\Omega_{U_1/S}^\bullet)) \otimes_{O_S} (h_{2*} p_{212*} p_{212}^* E_{zar}(\Omega_{U_2/S}^\bullet)) \\ & \xrightarrow{=} h_{12*} (p_{112}^* E_{zar}(\Omega_{U_1/S}^\bullet) \otimes_{h_{12}^* O_S} p_{212}^* E_{zar}(\Omega_{U_2/S}^\bullet)) \\ & \xrightarrow{T(\otimes, E)(-) \circ (T(p_{112}, E)(-) \otimes T(p_{212}, E)(-))} h_{12*} E_{zar}(p_{112}^* \Omega_{U_1/S}^\bullet \otimes_{O_S} p_{212}^* \Omega_{U_2/S}^\bullet) \end{aligned}$$

Let $S \in \text{Var}(\mathbb{C})$. We have the canonical map in $C_{O_S \text{fil}}(\text{Var}(\mathbb{C})^{sm}/S)$

$$w_S : (\Omega_{/S}^\bullet, F_b) \otimes_{O_S} (\Omega_{/S}^\bullet, F_b) \rightarrow (\Omega_{/S}^\bullet, F_b)$$

given by for $h : U \rightarrow S \in \text{Var}(\mathbb{C})^{sm}/S$

$$w_S(U/S) : (\Omega_{U/S}^\bullet, F_b) \otimes_{h^* O_S} (\Omega_{U/S}^\bullet, F_b)(U) \xrightarrow{w_{U/S}(U)} (\Omega_{U/S}^\bullet, F_b)(U)$$

It gives the map

$$Ew_S : E_{et}(\Omega_{/S}^\bullet, F_b) \otimes_{O_S} E_{et}(\Omega_{/S}^\bullet, F_b) \xrightarrow{=} E_{et}((\Omega_{/S}^\bullet, F_b) \otimes_{O_S} (\Omega_{/S}^\bullet, F_b)) \xrightarrow{E_{et}(w_S)} E_{et}(\Omega_{/S}^\bullet, F_b)$$

If $S \in \text{SmVar}(\mathbb{C})$,

$$w_S : (\Omega_{/S}^\bullet, F_b) \otimes_{O_S} (\Omega_{/S}^\bullet, F_b) \rightarrow (\Omega_{/S}^\bullet, F_b)$$

is a map in $C_{O_S \text{fil}, D_S}(\text{Var}(\mathbb{C})^{sm}/S)$.

Definition 103. Let $S \in \text{Var}(\mathbb{C})$. We have, for $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$, the canonical transformation in $C_{O_S \text{fil}}(S)$:

$$\begin{aligned} & T(\otimes, \Omega)(F, G) : e(S)_* \mathcal{H}om(F, E_{et}(\Omega_{/S}^\bullet, F_b)) \otimes_{O_S} e(S)_* \mathcal{H}om(G, E_{et}(\Omega_{/S}^\bullet, F_b)) \\ & \xrightarrow{=} e(S)_* (\mathcal{H}om(F, E_{et}(\Omega_{/S}^\bullet, F_b)) \otimes_{O_S} \mathcal{H}om(G, E_{et}(\Omega_{/S}^\bullet, F_b))) \\ & \xrightarrow{e(S)_* T(\mathcal{H}om, \otimes)(-)} e(S)_* \mathcal{H}om(F \otimes G, E_{et}(\Omega_{/S}^\bullet, F_b) \otimes_{O_S} E_{et}(\Omega_{/S}^\bullet, F_b)) \\ & \xrightarrow{\mathcal{H}om(F \otimes G, Ew_S)} e(S)_* \mathcal{H}om(F \otimes G, E_{et}(\Omega_{/S}^\bullet, F_b)) \end{aligned}$$

If $S \in \text{SmVar}(\mathbb{C})$, $T(\otimes, \Omega)(F, G)$ is a map in $C_{O_S \text{fil}, \mathcal{D}}(S)$.

Lemma 11. Let $S \in \text{Var}(\mathbb{C})$ and $h_1 : U_1 \rightarrow S$, $h_2 : U_2 \rightarrow S$ two smooth morphisms with $U_1, U_2 \in \text{Var}(\mathbb{C})$. Denote $h_{12} : U_{12} := U_1 \times_S U_2 \rightarrow S$ and $p_{112} : U_1 \times_S U_2 \rightarrow U_1$, $p_{212} : U_1 \times_S U_2 \rightarrow U_2$ the projections. We then have the following commutative diagram

$$\begin{array}{ccc} e(S)_* \mathcal{H}om(F, E_{et}(\Omega_{/S}^\bullet, F_b)) \otimes_{O_S} e(S)_* \mathcal{H}om(G, E_{et}(\Omega_{/S}^\bullet, F_b)) & \xrightarrow{T(\otimes, \Omega)(F, G)} & e(S)_* \mathcal{H}om(F \otimes G, E_{et}(\Omega_{/S}^\bullet, F_b)) \\ \uparrow k & & \uparrow k \\ h_{1*} E_{zar}(\Omega_{U_1/S}^\bullet, F_b) \otimes_{O_S} h_{2*} E_{zar}(\Omega_{U_2/S}^\bullet, F_b) & \xrightarrow{Ew_{(U_1, U_2)/S}} & h_{12*} E_{zar}(\Omega_{U_{12}/S}^\bullet, F_b) \end{array}$$

with

$$k : E_{zar}(\Omega_{/S}^\bullet, F_b) \rightarrow E_{et}(E_{zar}(\Omega_{/S}^\bullet, F_b)) = E_{et}(\Omega_{/S}^\bullet, F_b).$$

which is a filtered Zariski local equivalence.

Proof. Follows from Yoneda lemma. \square

Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote by $S_I := \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. Consider, for $I \subset J$, the following commutative diagram

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ j_{IJ} \uparrow & & \uparrow p_{IJ} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}$$

and $j_{IJ} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. Considering the factorization of the diagram D_{IJ} by the fiber product :

$$D_{IJ} = \begin{array}{ccccc} \tilde{S}_J = \tilde{S}_I \times \tilde{S}_{J \setminus I} & \xrightarrow{p_{IJ}} & \tilde{S}_I & & \\ \uparrow & \swarrow i_I \times I & & & \uparrow \\ & S_I \times \tilde{S}_{J \setminus I} & & & \\ \uparrow i_J & \nearrow l_J & & & \uparrow i_I \\ S_J & \xrightarrow{j_{IJ}} & S_I & & \end{array}$$

the square of this factorization being cartesian, we have for $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ the canonical map in $C(\text{Var}(\mathbb{C})^{sm}/\tilde{S}_J)$

$$\begin{aligned} S(D_{IJ})(F) : Li_{J*}j_J^*F &\xrightarrow{q} i_{J*}j_J^*F = (i_I \times I) * l_{J*}j_J^*F \xrightarrow{(i_I \times I)_* \text{ad}(p_{IJ\sharp}^0, p_{IJ\sharp}^0)(-)} \\ &(i_I \times I)_* p_{IJ\sharp}^0 p_{IJ\sharp}^0 l_{J*}j_J^*F \xrightarrow{T(p_{IJ}, i_I)(-)^{-1}} p_{IJ}^* i_{I*} p_{IJ\sharp}^0 l_{J*}j_J^*F = p_{IJ}^* i_{I*} j_I^* F \end{aligned}$$

which factors through

$$S(D_{IJ})(F) : Li_{J*}j_J^*F \xrightarrow{S^q(D_{IJ})(F)} p_{IJ}^* Li_{I*}j_I^*F \xrightarrow{q} p_{IJ}^* i_{I*} j_I^* F$$

Definition 104. (i) Let $S \in \text{SmVar}(\mathbb{C})$. We have the functor

$$C(\text{Var}(\mathbb{C})^{sm}/S)^{op} \rightarrow C_{\text{Fil}, \mathcal{D}}(S), \quad F \mapsto e(S)_* \mathcal{H}om^\bullet(L(i_{I*}j_I^*F), E_{et}(\Omega_{/S}^\bullet, F_b))[-d_S].$$

(ii) Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote by $S_I := \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. We have the functor

$$C(\text{Var}(\mathbb{C})^{sm}/S)^{op} \rightarrow C_{\text{Fil}, \mathcal{D}}(S/(\tilde{S}_I)), \quad F \mapsto (e(\tilde{S}_I)_* \mathcal{H}om^\bullet(L(i_{I*}j_I^*F), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b))[-d_{\tilde{S}_I}], u_{IJ}^q(F))$$

where

$$\begin{aligned} u_{IJ}^q(F)[d_{\tilde{S}_I}] &: e(\tilde{S}_I)_* \mathcal{H}om^\bullet(L(i_{I*}j_I^*F), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)) \\ &\xrightarrow{\text{ad}(p_{IJ}^{*mod}, p_{IJ*})(-)} p_{IJ*} p_{IJ}^{*mod} e(\tilde{S}_I)_* \mathcal{H}om^\bullet(L(i_{I*}j_I^*F), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)) \\ &\xrightarrow{p_{IJ*} T(p_{IJ}, \Omega_{/\tilde{S}_I})(L(i_{I*}j_I^*F))} p_{IJ*} e(\tilde{S}_I)_* \mathcal{H}om^\bullet(p_{IJ}^* L(i_{I*}j_I^*F), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)) \\ &\xrightarrow{p_{IJ*} e(\tilde{S}_I)_* \mathcal{H}om(S^q(D_{IJ})(F), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma}, F_b))} p_{IJ*} e(\tilde{S}_I)_* \mathcal{H}om^\bullet(L(i_{J*}j_J^*F), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)). \end{aligned}$$

For $I \subset J \subset K$, we have obviously $p_{IJ*}u_{JK}(F) \circ u_{IJ}(F) = u_{IK}(F)$.

We will prove in corollary 4 below that $u_{IJ}(F)$ are ∞ -filtered Zariski local equivalence.

We then have the following key proposition

Proposition 99. *Let $S \in \text{Var}(\mathbb{C})$. Let $m : Q_1 \rightarrow Q_2$ be an equivalence (\mathbb{A}^1, et) local in $C(\text{Var}(\mathbb{C})^{sm}/S)$ with Q_1, Q_2 complexes of projective presheaves. Then,*

$$e(S)_* \mathcal{H}om(m, E_{et}(\Omega_{/S}^\bullet, F_b)) : e(S)_* \mathcal{H}om^\bullet(Q_2, E_{et}(\Omega_{/S}^\bullet, F_b)) \rightarrow e(S)_* \mathcal{H}om^\bullet(Q_1, E_{et}(\Omega_{/S}^\bullet, F_b))$$

is an ∞ -filtered quasi-isomorphism (in fact it is a 2-filtered quasi-isomorphism). It is thus an isomorphism in $D_{O_S \text{fil}, \mathcal{D}, \infty}(S)$ if S is smooth.

Proof. By proposition 97(ii), $E_{et}(\Omega_{/S}^\bullet, F_b) \in C_{O_S \text{fil}, \mathcal{D}_S}(\text{Var}(\mathbb{C})^{sm}/S)$ is 2-filtered \mathbb{A}^1 invariant. The result then follows by lemma 1. \square

Definition 105. (i) We define, using definition 104, by proposition 99, the filtered algebraic Gauss-Manin realization functor defined as

$$\mathcal{F}_S^{GM} : \text{DA}_c(S)^{op} \rightarrow D_{O_S \text{fil}, \mathcal{D}, \infty}(S), \quad M \mapsto \mathcal{F}_S^{GM}(M) := e(S)_* \mathcal{H}om^\bullet(L(F), E_{et}(\Omega_{/S}^\bullet, F_b))[-d_S]$$

where $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$,

(ii) Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote by $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. We define, using definition 104 and corollary 4, by proposition 99 the filtered algebraic Gauss-Manin realization functor defined as

$$\begin{aligned} \mathcal{F}_S^{GM} : \text{DA}_c(S)^{op} &\rightarrow D_{O_S \text{fil}, \mathcal{D}, \infty}(S/(\tilde{S}_I)), \quad M \mapsto \\ \mathcal{F}_S^{GM}(M) &:= ((e(\tilde{S}_I)_* \mathcal{H}om^\bullet(L(i_{I*} j_I^* F), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b))[-d_{\tilde{S}_I}], u_{IJ}^q(F)) \end{aligned}$$

where $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$.

Proposition 100. *For $S \in \text{Var}(\mathbb{C})$, the functor \mathcal{F}_S^{GM} is well defined.*

Proof. Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. Let $M \in \text{DA}(S)$. Let $F, F' \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1, et)(F) = D(\mathbb{A}^1, et)(F')$. Then there exist by definition a sequence of morphisms in $C(\text{Var}(\mathbb{C})^{sm}/S)$:

$$F = F_1 \xrightarrow{s_1} F_2 \xleftarrow{s_2} F_3 \xrightarrow{s_3} F_4 \rightarrow \dots \xrightarrow{s_l} F' = F_s$$

where, for $1 \leq k \leq s$, and s_k are (\mathbb{A}^1, et) local equivalence. But if $s : F_1 \rightarrow F_2$ is an equivalence (\mathbb{A}^1, et) local,

$$L(i_{I*} j_I^* s) : L(i_{I*} j_I^* F_1) \rightarrow L(i_{I*} j_I^* F_2)$$

is an equivalence (\mathbb{A}^1, et) local, hence

$$\begin{aligned} \mathcal{H}om(L(i_{I*} j_I^* s), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)) &: (e(\tilde{S}_I)_* \mathcal{H}om(L(i_{I*} j_I^* F_2), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)), u_{IJ}(F_2)) \\ &\rightarrow (e(\tilde{S}_I)_* \mathcal{H}om(L(i_{I*} j_I^* F_1), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)), u_{IJ}(F_1)) \end{aligned}$$

is an ∞ -filtered quasi-isomorphism by proposition 99. \square

Let $f : X \rightarrow S$ a morphism with $S, X \in \text{Var}(\mathbb{C})$. Assume that there is a factorization

$$f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$$

of f , with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We have $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. Denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $X_I = \cap_{i \in I} X_i$. For $I \subset [1, \dots, l]$, denote by $\tilde{S}_I = \prod_{i \in I} \tilde{S}_i$. We then have, for $I \subset [1, \dots, l]$, closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I$ and the following commutative diagrams which are cartesian

$$f_I = f|_{X_I} : X_I \xrightarrow{l_I} Y \times S_I \xrightarrow{p_{S_I}} S_I \quad , \quad Y \times \tilde{S}_J \xrightarrow{p_{\tilde{S}_J}} \tilde{S}_J$$

$$\begin{array}{ccc} & \downarrow i'_I & \downarrow i_I \\ & Y \times \tilde{S}_I \xrightarrow{p_{\tilde{S}_I}} \tilde{S}_I & Y \times \tilde{S}_I \xrightarrow{p_{\tilde{S}_I}} \tilde{S}_I \end{array} \quad , \quad \begin{array}{ccc} & \downarrow p'_{I,J} & \downarrow p_{I,J} \\ & Y \times \tilde{S}_I \xrightarrow{p_{\tilde{S}_I}} \tilde{S}_I & Y \times \tilde{S}_I \xrightarrow{p_{\tilde{S}_I}} \tilde{S}_I \end{array}$$

with $l_I : l|_{X_I}$, $i'_I = I \times i_I$, p_{S_I} and $p_{\tilde{S}_I}$ are the projections and $p'_{I,J} = I \times p_{I,J}$, and we recall that we denote by $j_I : \tilde{S}_I \setminus S_I \hookrightarrow \tilde{S}_I$ and $j'_I : Y \times \tilde{S}_I \setminus X_I \hookrightarrow Y \times \tilde{S}_I$ the open complementary embeddings. We then have the commutative diagrams

$$D_{I,J} = \begin{array}{ccc} S_J & \xrightarrow{i_J} & \tilde{S}_J \\ \downarrow j_{I,J} & & \downarrow p_{I,J} \\ S_I & \xrightarrow{i_I} & \tilde{S}_I \end{array} \quad , \quad D'_{I,J} = \begin{array}{ccc} X_J & \xrightarrow{i'_J \circ l_J} & Y \times \tilde{S}_J \\ \downarrow j'_{I,J} & & \downarrow p'_{I,J} \\ X_I & \xrightarrow{i'_I \circ l_I} & Y \times \tilde{S}_I \end{array}$$

and the factorization of $D'_{I,J}$ by the fiber product:

$$D'_{I,J} = \begin{array}{ccc} X_J & \xrightarrow{i'_I \circ l_I} & Y \times \tilde{S}_J \\ \downarrow j'_{I,J} & & \downarrow p'_{I,J} \\ X_I & \xrightarrow{i'_I \circ l_I} & Y \times \tilde{S}_I \end{array} \quad , \quad D'_{I,J} = \begin{array}{ccc} X_J & \xrightarrow{i'_I \circ l_I} & Y \times \tilde{S}_J \\ \downarrow j'_{I,J} & \searrow \hat{l}_J & \nearrow \hat{i}_I \\ & X_I \times_{Y \times \tilde{S}_I} Y \times \tilde{S}_J = X_I \times \tilde{S}_{J \setminus I} & \\ \downarrow j'_{I,J} & \nearrow \hat{p}_{I,J} & \downarrow p'_{I,J} \\ X_I & \xrightarrow{i'_I \circ l_I} & Y \times \tilde{S}_I \end{array} \quad (50)$$

where $j'_{I,J} : X_J \hookrightarrow X_I$ is the open embedding. Consider

$$F(X/S) := p_{S,\sharp} \Gamma_X^\vee \mathbb{Z}(Y \times S/Y \times S)[d_Y] \in C(\text{Var}(\mathbb{C})^{sm}/S)$$

so that $D(\mathbb{A}^1, et)(F(X/S)) = M^{BM}(X/S)$. Then, by definition,

$$\mathcal{F}_S^{GM}(M^{BM}(X/S)) := (e(\tilde{S}_I)_* \text{Hom}(L(i_{I*} j_I^* F(X/S)), E_{et}(\Omega_{/\tilde{S}_I}, F_b))[-d_{\tilde{S}_I}], u_{\tilde{S}_I}^q(F(X/S)))$$

On the other hand, let

$$Q(X_I/\tilde{S}_I) := p_{\tilde{S}_I,\sharp} \Gamma_{X_I}^\vee \mathbb{Z}(Y \times \tilde{S}_I/Y \times \tilde{S}_I)[d_Y] \in C(\text{Var}(\mathbb{C})^{sm}/\tilde{S}_I),$$

see definition 10. We have then for $I \subset [1, l]$ the following map in $C(\text{Var}(\mathbb{C})^{sm}/\tilde{S}_J)$:

$$N_I(X/S) : Q(X_I/\tilde{S}_I) = p_{\tilde{S}_I,\sharp} \Gamma_{X_I}^\vee \mathbb{Z}(Y \times \tilde{S}_I/Y \times \tilde{S}_I)[d_Y] \xrightarrow{p_{\tilde{S}_I,\sharp} \text{ad}(i'_I, i'_{I*})(-)} \quad (51)$$

$$p_{\tilde{S}_I,\sharp} i'_{I*} \Gamma_{X_I}^\vee \mathbb{Z}(Y \times \tilde{S}_I/Y \times \tilde{S}_I)[d_Y] \xrightarrow{p_{\tilde{S}_I,\sharp} (T(i'_I, \gamma^\vee)(-))^{-1}} p_{\tilde{S}_I,\sharp} i'_{I*} \Gamma_{X_I}^\vee \mathbb{Z}(Y \times S_I/Y \times S_I)[d_Y] \quad (52)$$

$$\xrightarrow{\hat{T}_i(p_{S_I}, i_I)(-)} i_{I*} p_{S_I,\sharp} \Gamma_{X_I}^\vee \mathbb{Z}(Y \times S_I/Y \times S_I)[d_Y] = i_{I*} j_I^* F(X/S) \quad (53)$$

We have then for $I \subset J$ the following commutative diagram in $C(\text{Var}(\mathbb{C})^{sm}/\tilde{S}_J)$:

$$\begin{array}{ccc}
p_{IJ}^* p_{\tilde{S}_I} \Gamma_{X_I}^\vee \mathbb{Z}(Y \times \tilde{S}_I/Y \times \tilde{S}_I)[d_Y] & \xrightarrow{p_{IJ}^* N_I(X/S)} & p_{IJ}^* (i_{I*} j_I^* F(X/S)) \\
\uparrow H_{IJ} & & \uparrow S(D_{IJ})(F(X/S)) \\
p_{\tilde{S}_J} \Gamma_{X_J}^\vee \mathbb{Z}(Y \times \tilde{S}_J/Y \times \tilde{S}_J)[d_Y] & \xrightarrow{N_J(X/S)} & i_{J*} j_J^* F(X/S)
\end{array} \tag{54}$$

with

$$\begin{aligned}
& H_{IJ}[-d_Y] : p_{\tilde{S}_J} \Gamma_{X_J}^\vee \mathbb{Z}(Y \times \tilde{S}_J/Y \times \tilde{S}_J) \\
\cong & p_{\tilde{S}_J} p_{X_J} \Gamma_{X_J}^\vee p_{IJ}^* \mathbb{Z}(Y \times \tilde{S}_I/Y \times \tilde{S}_I) \xrightarrow{\text{Cone}(\text{ad}(p'_{IJ}, p'_{IJ})(-), I)} p_{\tilde{S}_J} \Gamma_{X_I \times \tilde{S}_I}^\vee p_{IJ}^* \mathbb{Z}(Y \times \tilde{S}_I/Y \times \tilde{S}_I) \\
& \xrightarrow{T(p_{IJ}, \gamma^\vee)(-)} p_{\tilde{S}_J} p_{IJ}^* \Gamma_{X_I}^\vee \mathbb{Z}(Y \times \tilde{S}_I/Y \times \tilde{S}_I) \xrightarrow{T_\#(p_{IJ}, p_{\tilde{S}_I})(-)} p_{IJ}^* p_{\tilde{S}_I} \Gamma_{X_I}^\vee \mathbb{Z}(Y \times \tilde{S}_I/Y \times \tilde{S}_I).
\end{aligned}$$

This say that the maps $N_I(X/S)$ induces a map in $C(\text{Var}(\mathbb{C})^{sm}/(S/\tilde{S}_I))$

$$(N_I(X/S)) : (Q(X_I/\tilde{S}_I), H_{IJ}) \rightarrow (i_{I*} j_I^* F(X/S), S(D_{IJ})(F(X/S))).$$

We denote by $v_{IJ}^q(F(X/S))$ the composite

$$\begin{aligned}
& v_{IJ}^q(F(X/S))[d_{\tilde{S}_J}] : e(\tilde{S}_I)_* \mathcal{H}om(Q(X_I/\tilde{S}_I), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)) \\
& \xrightarrow{\text{ad}(p_{IJ}^{*mod}, p_{IJ})(-)} p_{IJ*} p_{IJ}^{*mod} e(\tilde{S}_I)_* \mathcal{H}om(Q(X_I/\tilde{S}_I), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)) \\
& \xrightarrow{p_{IJ*} T(p_{IJ}, \Omega)(Q(X_I/\tilde{S}_I))} p_{IJ*} e(\tilde{S}_I)_* \mathcal{H}om(p_{IJ}^* Q(X_I/\tilde{S}_I), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)) \\
& \xrightarrow{\mathcal{H}om(H_{IJ}, E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b))} p_{IJ*} e(\tilde{S}_I)_* \mathcal{H}om(Q(X_I/\tilde{S}_I), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)).
\end{aligned}$$

On the other hand, we have the following map in $C_{Ofil, \mathcal{D}, S_J}(\tilde{S}_J)$

$$\begin{aligned}
& w_{IJ}(X/S)[d_{\tilde{S}_J}] : p_{\tilde{S}_I} \Gamma_{X_I} E_{zar}(\Omega_{Y \times \tilde{S}_I/\tilde{S}_I}^\bullet, F_b) \xrightarrow{\text{ad}(p_{IJ}^{*mod}, p_{IJ})(-)} p_{IJ*} p_{IJ}^{*mod} p_{\tilde{S}_I} \Gamma_{X_I} E_{zar}(\Omega_{Y \times \tilde{S}_I/\tilde{S}_I}^\bullet, F_b) \\
& \xrightarrow{T_w^\circ(p_{IJ}, p_{\tilde{S}_I})^\gamma} p_{IJ*} p_{\tilde{S}_I} \Gamma_{X_I \times \tilde{S}_I} E_{zar}(\Omega_{Y \times \tilde{S}_I/\tilde{S}_I}^\bullet, F_b) \xrightarrow{\text{Cone}(I, \text{ad}(p'_{IJ}, p_{IJ*})(-))} p_{\tilde{S}_I} \Gamma_{X_I} E_{zar}(\Omega_{Y \times \tilde{S}_I/\tilde{S}_I}^\bullet, F_b).
\end{aligned}$$

Lemma 12. (i) The map in $C(\text{Var}(\mathbb{C})^{sm}/(S/\tilde{S}_I))$

$$(N_I(X/S)) : (Q(X_I/\tilde{S}_I), H_{IJ}) \rightarrow (L(i_{I*} j_I^* F(X/S)), S^q(D_{IJ})(F(X/S))).$$

is an equivalence (\mathbb{A}^1, et) local.

(ii) The maps $(N_I(X/S))$ induces an ∞ -filtered quasi-isomorphism in $C_{Ofil, \mathcal{D}}(S/(\tilde{S}_I))$

$$\begin{aligned}
& (\mathcal{H}om(N_I(X/S), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b))) : \\
& (e(\tilde{S}_I)_* \mathcal{H}om(L(i_{I*} j_I^* F(X/S)), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b))[-d_{\tilde{S}_I}], u_{IJ}^q(F(X/S))) \rightarrow \\
& (e(\tilde{S}_I)_* \mathcal{H}om(Q(X_I/\tilde{S}_I), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b))[-d_{\tilde{S}_I}], v_{IJ}^q(F(X/S)))
\end{aligned}$$

(iii) The maps $(I(\gamma, \text{hom})(-, -))$ and $(k : E_{zar}(p_{\tilde{S}_I}^* \Omega_{/\tilde{S}_I}^\bullet, F_b) \rightarrow E_{et}(p_{\tilde{S}_I}^* \Omega_{/\tilde{S}_I}^\bullet, F_b))$ induce an $(1-)$ filtered Zariski local equivalence in $C_{Ofil, \mathcal{D}}(S/(\tilde{S}_I))$

$$\begin{aligned}
& (k \circ I(\gamma, \text{hom})(-, -)) : (p_{\tilde{S}_I} \Gamma_{X_I} E_{zar}(\Omega_{Y \times \tilde{S}_I/\tilde{S}_I}^\bullet, F_b)[-d_{\tilde{S}_I}], w_{IJ}(X/S)) \\
& \rightarrow (e(\tilde{S}_I)_* \mathcal{H}om(Q(X_I/\tilde{S}_I), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b))[-d_{\tilde{S}_I}], v_{IJ}^q(F(X/S)))
\end{aligned}$$

Proof. (i): Follows from theorem 14.

(ii): These maps induce a morphism in $C_{Ofil, \mathcal{D}}(S/(\tilde{S}_I))$ by construction. The fact that it is an ∞ -filtered quasi-isomorphism follows from (i) and proposition 99.

(iii): These maps induce a morphism in $C_{Ofil, \mathcal{D}}(S/(\tilde{S}_I))$ by construction. \square

Proposition 101. *Let $f : X \rightarrow S$ a morphism with $S, X \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. Denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $X_I = \cap_{i \in I} X_i$. Assume there exist a factorization*

$$f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$$

of f with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We then have, for $I \subset [1, \dots, l]$, the following commutative diagrams which are cartesian

$$\begin{array}{ccc} f_I = f|_{X_I} : X_I & \xrightarrow{l_I} & Y \times S_I \xrightarrow{p_{S_I}} S_I \\ & \searrow & \downarrow i'_I \quad \downarrow i_I \\ & & Y \times \tilde{S}_I \xrightarrow{p_{\tilde{S}_I}} \tilde{S}_I \end{array} \quad , \quad \begin{array}{ccc} Y \times \tilde{S}_J & \xrightarrow{p_{\tilde{S}_J}} & \tilde{S}_J \\ \downarrow p'_{IJ} & & \downarrow p_{IJ} \\ Y \times \tilde{S}_I & \xrightarrow{p_{\tilde{S}_I}} & \tilde{S}_I \end{array}$$

Let $F(X/S) := p_{S, \sharp} \Gamma_X^\vee \mathbb{Z}(Y \times S/Y \times S)[d_Y]$. The transformations maps $(N_I(X/S) : Q(X_I/\tilde{S}_I) \rightarrow i_{I*} j_I^* F(X/S))$ and $(k \circ I(\gamma, \text{hom})(-, -))$, for $I \subset [1, \dots, l]$, induce an isomorphism in $D_{Ofil, \mathcal{D}, \infty}(S/(\tilde{S}_I))$

$$\begin{aligned} & I^{GM}(X/S) : \\ & \mathcal{F}_S^{GM}(M^{BM}(X/S)) := (e(\tilde{S}_I)_* \text{Hom}(L(i_{I*} j_I^* F(X/S)), E_{et}(\Omega_{\tilde{S}_I}^\bullet, F_b))[-d_{\tilde{S}_I}], u_{IJ}^q(F(X/S))) \\ & \xrightarrow{(e(\tilde{S}_I)_* \text{Hom}(LN_I(X/S), E_{et}(\Omega_{\tilde{S}_I}^\bullet, F_b)))} (e(\tilde{S}_I)_* \text{Hom}(Q(X_I/\tilde{S}_I), E_{et}(\Omega_{\tilde{S}_I}^\bullet, F_b))[-d_{\tilde{S}_I}], v_{IJ}^q(F(X/S))) \\ & \xrightarrow{(k \circ I(\gamma, \text{hom})(-, -))^{-1}} (p_{\tilde{S}_I*} \Gamma_{X_I} E_{zar}(\Omega_{Y \times \tilde{S}_I/\tilde{S}_I}^\bullet, F_b)[-d_{\tilde{S}_I}], w_{IJ}(X/S)). \end{aligned}$$

Proof. Follows from lemma 12. \square

Corollary 4. *Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $D(\mathbb{A}^1, et)(F) \in \text{DA}_c(S)$, $u_{IJ}^q(F)$ are ∞ -filtered Zariski local equivalence.*

Proof. Let $f : X \rightarrow S$ a morphism with $X \in \text{Var}(\mathbb{C})$ such that there exist a factorization, $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Then, by lemma 12(ii) and (iii), $u_{IJ}^q(F(X/S))$ are Zariski local equivalences since $w_{IJ}(X/S)$ are isomorphisms. \square

We now define the functorialities of \mathcal{F}_S^{GM} with respect to S which makes \mathcal{F}_{GM}^- a morphism of 2-functor.

Definition 106. *Let $g : T \rightarrow S$ a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Consider the factorization $g : T \xrightarrow{l} T \times S \xrightarrow{p_S} S$ where l is the graph embedding and p_S the projection. Let $M \in \text{DA}_c(S)$ and $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}_S^1, et)(F)$. Then, $D(\mathbb{A}_T^1, et)(g^* F) = g^* M$.*

(i) *We have then the canonical transformation in $D_{Ofil, \mathcal{D}, \infty}(T \times S)$ (see definition 102) :*

$$\begin{aligned} T(g, \mathcal{F}^{GM})(M) : Rg^{*mod[-], \Gamma} \mathcal{F}_S^{GM}(M) & := g^{*mod, \Gamma} e(S)_* \text{Hom}^\bullet(LF, E_{et}(\Omega_{S'}^\bullet, F_b))[d_T] \\ & \xrightarrow{T(g, \Omega_{/ \cdot})(LF)} \\ e(T \times S)_* \text{Hom}^\bullet(\Gamma_T^\vee p_S^* LF, E_{et}(\Omega_{T \times S}^\bullet, F_b))[d_T] & =: \mathcal{F}_{T \times S}^{GM}(l_* g^*(M, W)). \end{aligned}$$

(ii) We have then the canonical transformation in $D_{Ofil,\infty}(T)$ (see definition 102) :

$$T^O(g, \mathcal{F}^{GM})(M, W) : Lg^{*mod[-]} \mathcal{F}_S^{GM}(M) := g^{*mod} e(S)_* \mathcal{H}om^\bullet(LF, E_{et}(\Omega_{/S}^\bullet, F_b)) [d_T] \\ \xrightarrow{T^O(g, \Omega_{/ \cdot})(LF)} \\ e(T)_* \mathcal{H}om^\bullet(g^* LF, E_{et}(\Omega_{/T}^\bullet, F_b)) [d_T] =: \mathcal{F}_T^{GM}(g^* M).$$

We give now the definition in the non smooth case Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$. Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I := g|_{T_I} : T_I \rightarrow S_I$. We recall the commutative diagram :

$$E_{IJg} = (Y \times \tilde{S}_I) \setminus T_I \xrightarrow{m'_I} Y \times \tilde{S}_J, \quad E_{IJ} = \tilde{S}_J \setminus S_J \xrightarrow{m_J} \tilde{S}_J, \quad E'_{IJ} = (Y \times \tilde{S}_J) \setminus T_J \xrightarrow{m'_J} Y \times \tilde{S}_J \\ \downarrow p_{\tilde{S}_I} \quad \downarrow \tilde{g}_I \quad \downarrow p_{I,J} \quad \downarrow p_{I,J} \quad \downarrow p'_{I,J} \quad \downarrow p'_{I,J} \\ \tilde{S}_I \setminus S_I \xrightarrow{m_I} \tilde{S}_I \quad \tilde{S}_I \setminus (S_I \setminus S_J) \xrightarrow{m_{I,J}} \tilde{S}_I \quad (Y \times \tilde{S}_I) \setminus (T_I \setminus T_J) \xrightarrow{m'_{I,J}} Y \times \tilde{S}_I$$

For $I \subset J$, denote by $p_{I,J} : \tilde{S}_J \rightarrow \tilde{S}_I$ and $p'_{I,J} := I_Y \times p_{I,J} : Y \times \tilde{S}_J \rightarrow Y \times \tilde{S}_I$ the projections, so that $\tilde{g}_I \circ p'_{I,J} = p_{I,J} \circ \tilde{g}_J$. Consider, for $I \subset J \subset [1, \dots, l]$, resp. for each $I \subset [1, \dots, l]$, the following commutative diagrams in $\text{Var}(\mathbb{C})$

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ j_{I,J} \uparrow & & \uparrow p_{I,J} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}, \quad D'_{IJ} = \begin{array}{ccc} T_I & \xrightarrow{i'_I} & Y \times \tilde{S}_I \\ j'_{I,J} \uparrow & & \uparrow p'_{I,J} \\ T_J & \xrightarrow{i'_J} & Y \times \tilde{S}_J \end{array}, \quad D_{gI} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ g_I \uparrow & & \uparrow \tilde{g}_I \\ T_I & \xrightarrow{i'_I} & Y \times \tilde{S}_I \end{array},$$

and $j_{I,J} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{I,J} = j_J$. Let $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$. Recall (see section 2) that since $j'_I{}^* i'_I{}^* j'_I{}^* g^* F = 0$, the morphism $T(D_{gI})(j'_I{}^* F) : \tilde{g}_I{}^* i_{I*} j'_I{}^* F \rightarrow i'_I{}^* j'_I{}^* g^* F$ factors through

$$T(D_{gI})(j'_I{}^* F) : \tilde{g}_I{}^* i_{I*} j'_I{}^* F \xrightarrow{\gamma_{X_I}^\vee(-)} \Gamma_{X_I}^\vee \tilde{g}_I{}^* i_{I*} j'_I{}^* F \xrightarrow{T^\gamma(D_{gI})(j'_I{}^* F)} i'_I{}^* j'_I{}^* g^* F$$

We then have, for each $I \subset [1, \dots, l]$, the morphism

$$T^{q,\gamma}(D_{gI})(j'_I{}^* F) : \Gamma_{T_I}^\vee \tilde{g}_I{}^* L(i_{I*} j'_I{}^* F) \xrightarrow{T(\tilde{g}_I, L)(-)} \\ \Gamma_{T_I}^\vee L(\tilde{g}_I{}^* i_{I*} j'_I{}^* F) = L(\Gamma_{T_I}^\vee \tilde{g}_I{}^* i_{I*} j'_I{}^* F) \xrightarrow{L(T^\gamma(D_{gI})(j'_I{}^* F))} L(i'_I{}^* j'_I{}^* g^* F)$$

and the following diagram in $C(\text{Var}(\mathbb{C})^{sm}/Y \times \tilde{S}_I)$ commutes

$$\begin{array}{ccc} \Gamma_{T_I}^\vee \tilde{g}_I{}^* L(i_{I*} j'_I{}^* F) & \xrightarrow{\Gamma_{T_I}^\vee \tilde{g}_I{}^* q_I(F)} & \Gamma_{T_I}^\vee \tilde{g}_I{}^* i_{I*} j'_I{}^* F \\ T^{q,\gamma}(D_{gI})(j'_I{}^* F) \downarrow & & \downarrow T^\gamma(D_{gI})(j'_I{}^* F) \\ L(i'_I{}^* j'_I{}^* g^* F) & \xrightarrow{q_I(g^* F)} & i'_{I*} g_I{}^* j'_I{}^* F = i'_I{}^* j'_I{}^* g^* F \end{array}$$

We have the following commutative diagram in $C(\text{Var}(\mathbb{C})^{sm}/Y \times \tilde{S}_J)$

$$\begin{array}{ccc} p'_{I,J}{}^* \tilde{g}_I{}^* i_{I*} j'_I{}^* F = \tilde{g}_J{}^* p_{I,J}{}^* i_{I*} j'_I{}^* F & \xrightarrow{p'_{I,J}{}^* T(D_{gI})(j'_I{}^* F)} & p'_{I,J}{}^* i'_{I*} g_I{}^* j'_I{}^* F = p'_{I,J}{}^* i'_{I*} j'_I{}^* g^* F \\ \uparrow \tilde{g}_J{}^* S(D_{I,J})(F) & & \uparrow S(D'_{I,J})(g^* F) \\ \tilde{g}_J{}^* i_{J*} j'_{I,J}{}^* j'_I{}^* F = \tilde{g}_J{}^* i_{J*} j'_I{}^* F & \xrightarrow{T(D_{gJ})(j'_I{}^* F)} & i'_{J*} g_J{}^* j'_I{}^* F = i'_{J*} j'_{I,J}{}^* j'_I{}^* g^* F = i'_{J*} j'_I{}^* g^* F \end{array} \quad (55)$$

This gives, after taking the functor L , the following commutative diagram in $C(\text{Var}(\mathbb{C})^{sm}/Y \times \tilde{S}_J)$

$$\begin{array}{ccc} \Gamma_{T_J}^\vee p_{I_J}^{\prime*} \Gamma_{T_I}^\vee \tilde{g}_I^* L(i_{I^*} j_I^* F) = \Gamma_{T_J}^\vee \tilde{g}_J^* p_{I_J}^* L(i_{I^*} j_I^* F) \xrightarrow{p_{I_J}^{\prime*} T^{q,\gamma}(D_{g_I})(j_I^* F)} \Gamma_{T_J}^\vee p_{I_J}^{\prime*} L(i_{I^*} j_I^{\prime*} g^* F) & (56) \\ \uparrow \tilde{g}_J^* S^q(D_{I_J})(F) & & \uparrow S^q(D'_{I_J})(g^* F) \\ \Gamma_{T_J}^\vee \tilde{g}_J^* L(i_{J^*} j_J^* g^* F) \xrightarrow{T^{q,\gamma}(D_{g_J})(j_J^* F)} L(i_{J^*} j_J^{\prime*} g^* F) & & \end{array}$$

The fact that the diagrams (56) commutes says that the maps $T^{q,\gamma}(D_{g_I})(j_I^* F)$ define a morphism in $C(\text{Var}(\mathbb{C})^{sm}/(T/(Y \times \tilde{S}_I)))$

$$(T^{q,\gamma}(D_{g_I})(j_I^* F)) : (\Gamma_{T_I}^\vee \tilde{g}_I^* L(i_{I^*} j_I^* F), \tilde{g}_J^* S^q(D_{I_J})(F)) \rightarrow (L(i_{I^*} j_I^{\prime*} g^* F), S^q(D'_{I_J})(g^* F))$$

Denote for short $d_{YI} := d_{\tilde{S}_I} + d_Y$. We denote by $\tilde{g}_J^* u_{I_J}^q(F)_1$ the composite

$$\begin{aligned} & \tilde{g}_J^* u_{I_J}^q(F)_1 [d_{YJ}] : e(Y \times \tilde{S}_I)_* \Gamma_{T_I} \mathcal{H}om(\tilde{g}_I^* L(i_{I^*} j_I^* F), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b)) \\ & \xrightarrow{\text{ad}(p'_{I_J}, p'_{I_J^*})(-)} p'_{I_J^*} p_{I_J}^{\prime* mod} e(Y \times \tilde{S}_I)_* \Gamma_{T_I} \mathcal{H}om(\tilde{g}_I^* L(i_{I^*} j_I^* F), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b)) \\ & \xrightarrow{T^{mod}(p'_{I_J}, \gamma)(-)} p'_{I_J^*} e(Y \times \tilde{S}_I)_* \Gamma_{T_I \times \tilde{S}_J \setminus I} p_{I_J}^{\prime* mod} \mathcal{H}om(\tilde{g}_J^* L(i_{J^*} j_J^* F), E_{et}(\Omega_{Y \times \tilde{S}_J}^\bullet, F_b)) \\ & \xrightarrow{\text{Cone}(\text{ad}(p'_{I_J}, p'_{I_J^*})(-), I)} p'_{I_J^*} e(Y \times \tilde{S}_I)_* \Gamma_{T_J} p_{I_J}^{\prime* mod} \mathcal{H}om(\tilde{g}_J^* L(i_{J^*} j_J^* F), E_{et}(\Omega_{Y \times \tilde{S}_J}^\bullet, F_b)) \\ & \xrightarrow{T(p'_{I_J}, \Omega_{I^*})(-)} p'_{I_J^*} e(Y \times \tilde{S}_I)_* \Gamma_{T_J} \mathcal{H}om(\tilde{g}_J^* p_{I_J}^* L(i_{I^*} j_I^* F), E_{et}(\Omega_{Y \times \tilde{S}_J}^\bullet, F_b)) \\ & \xrightarrow{\mathcal{H}om(\tilde{g}_J^*(S^q(D_{I_J})(F)), E_{et}(\Omega_{Y \times \tilde{S}_J}^\bullet, F_b))} p'_{I_J^*} e(Y \times \tilde{S}_I)_* \Gamma_{T_J} \mathcal{H}om(\tilde{g}_J^* L(i_{J^*} j_J^* F), E_{et}(\Omega_{Y \times \tilde{S}_J}^\bullet, F_b)) \end{aligned}$$

We denote by $\tilde{g}_J^* u_{I_J}^q(F)_2$ the composite

$$\begin{aligned} & \tilde{g}_J^* u_{I_J}^q(F)_2 [d_{YJ}] : e(\tilde{T}_I)_* \mathcal{H}om(\Gamma_{T_I}^\vee \tilde{g}_I^* L(i_{I^*} j_I^* F), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b)) \\ & \xrightarrow{\text{ad}(p'_{I_J}, p'_{I_J^*})(-)} p'_{I_J^*} p_{I_J}^{\prime* mod} e(\tilde{T}_I)_* \mathcal{H}om(\Gamma_{T_I}^\vee \tilde{g}_I^* L(i_{I^*} j_I^* F), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b)) \\ & \xrightarrow{T(p'_{I_J}, \Omega_{I^*})(-)} p'_{I_J^*} e(\tilde{T}_I)_* \mathcal{H}om(p_{I_J}^{\prime*} \Gamma_{T_I}^\vee \tilde{g}_I^* L(i_{I^*} j_I^* F), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b)) \\ & \xrightarrow{\mathcal{H}om(T(p'_{I_J}, \gamma^\vee)(-), E_{et}(\Omega_{Y \times \tilde{S}_J}^\bullet, F_b))} p'_{I_J^*} e(\tilde{T}_I)_* \mathcal{H}om(\Gamma_{T_I \times \tilde{S}_J \setminus I}^\vee p_{I_J}^{\prime*} \tilde{g}_I^* L(i_{I^*} j_I^* F), E_{et}(\Omega_{Y \times \tilde{S}_J}^\bullet, F_b)) \\ & \xrightarrow{\text{Cone}(\text{ad}(p'_{I_J}, p'_{I_J^*})(-), I)} p'_{I_J^*} e(\tilde{T}_I)_* \mathcal{H}om(\Gamma_{T_J}^\vee p_{I_J}^{\prime*} \tilde{g}_I^* L(i_{I^*} j_I^* F), E_{et}(\Omega_{Y \times \tilde{S}_J}^\bullet, F_b)) \\ & \xrightarrow{\mathcal{H}om(\Gamma_{T_J}^\vee \tilde{g}_J^*(S^q(D_{I_J})(F)), E_{et}(\Omega_{Y \times \tilde{S}_J}^\bullet, F_b))} p'_{I_J^*} e(Y \times \tilde{S}_I)_* \mathcal{H}om(\Gamma_{T_J}^\vee \tilde{g}_J^* L(i_{J^*} j_J^* F), E_{et}(\Omega_{Y \times \tilde{S}_J}^\bullet, F_b)) \end{aligned}$$

We then have then the following lemma :

Lemma 13. (i) *The morphism in $C(\text{Var}(\mathbb{C})^{sm}/(T/(Y \times \tilde{S}_I)))$*

$$(T^{q,\gamma}(D_{g_I})(j_I^* F)) : (\Gamma_{T_I}^\vee L \tilde{g}_I^* i_{I^*} j_I^* F, \tilde{g}_J^* S^q(D_{I_J})(F)) \rightarrow (i_{I^*} j_I^{\prime*} g^* F, S^q(D'_{I_J})(F)(j_I^{\prime*} g^* F))$$

is an equivalence (\mathbb{A}^1, et) local.

(ii) *The maps $\mathcal{H}om((T^{q,\gamma}(D_{g_I})(j_I^* F)), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b))$ induce an ∞ -filtered quasi-isomorphism in $C_{Ofil, \mathcal{D}}(T/(Y \times \tilde{S}_I))$*

$$\begin{aligned} & (\mathcal{H}om(T^{q,\gamma}(D_{g_I})(j_I^* F), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b))) : \\ & (e(Y \times \tilde{T}_I)_* \mathcal{H}om(L(i_{I^*} j_I^{\prime*} g^* F), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b))[-d_{YI}], u_{I_J}^q(g^* F)) \rightarrow \\ & (e(Y \times \tilde{T}_I)_* \mathcal{H}om((\Gamma_{T_I}^\vee L \tilde{g}_I^* i_{I^*} j_I^* F), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b))[-d_{YI}], \tilde{g}_J^* u_{I_J}^q(F)_2) \end{aligned}$$

(iii) The maps $T(\tilde{g}_I, \Omega_\cdot)(L(i_{I*}j_I^*F))$ (see definition 102) induce a morphism in $C_{Ofil, \mathcal{D}}(T/(Y \times \tilde{S}_I))$

$$\begin{aligned} & (T(\tilde{g}_I, \Omega_\cdot)(L(i_{I*}j_I^*F))) : \\ & (\Gamma_{T_I} E_{zar}(\tilde{g}_I^{*mod[-]} e(\tilde{S}_I)_* \mathcal{H}om^\bullet(L(i_{I*}j_I^*F), E_{et}(\Omega_{\tilde{S}_I}^\bullet, F_b)))[-d_{YI}], \tilde{g}_J^{*mod} u_{IJ}^q(F)) \rightarrow \\ & (\Gamma_{T_I}(e(Y \times \tilde{S}_I)_* \mathcal{H}om(\tilde{g}_I^* L(i_{I*}j_I^*F), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b)))[-d_{YI}], \tilde{g}_J^* u_{IJ}^q(F)_1). \end{aligned}$$

Proof. (i): Follows from theorem 14.

(ii): These maps induce a morphism in $C_{Ofil, \mathcal{D}}(T/(Y \times \tilde{S}_I))$ by construction. The fact that this morphism is an ∞ -filtered equivalence Zariski local follows from (i) and proposition 99.

(iii): These maps induce a morphism in $C_{Ofil, \mathcal{D}}(T/(Y \times \tilde{S}_I))$ by construction. \square

Definition 107. Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$. Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I := g|_{T_I} : T_I \rightarrow S_I$. Denote for short $d_{YI} := d_Y + d_{\tilde{S}_I}$. Let $M \in \text{DA}_c(S)$ and $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}_S^1, et)(F)$. Then, $D(\mathbb{A}_T^1, et)(g^*F) = g^*M$. We have, by lemma 13, the canonical transformation in $D_{Ofil, \mathcal{D}, \infty}(T/(Y \times \tilde{S}_I))$

$$\begin{aligned} & T(g, \mathcal{F}^{GM})(M) : Rg^{*mod[-], \Gamma} \mathcal{F}_S^{GM}(M) := \\ & (\Gamma_{T_I} E_{zar}(\tilde{g}_I^{*mod} e(\tilde{S}_I)_* \mathcal{H}om^\bullet(L(i_{I*}j_I^*F), E_{et}(\Omega_{\tilde{S}_I}^\bullet, F_b)))[-d_{YI}], \tilde{g}_J^{*mod} u_{IJ}^q(F)) \\ & \quad \xrightarrow{(\Gamma_{T_I} E(T(\tilde{g}_I, \Omega_\cdot)(L(i_{I*}j_I^*(F, W)))))} \\ & (\Gamma_{T_I} e(Y \times \tilde{S}_I)_* \mathcal{H}om^\bullet(\tilde{g}_I^* L(i_{I*}j_I^*F), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b)))[-d_{YI}], \tilde{g}_J^* u_{IJ}^q(F)_1) \\ & \quad \xrightarrow{(I(\gamma, \text{hom}(-, -)))} \\ & (e(Y \times \tilde{S}_I)_* \mathcal{H}om^\bullet(\Gamma_{T_I}^\vee \tilde{g}_I^* L(i_{I*}j_I^*F), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b)))[-d_{YI}], \tilde{g}_J^* u_{IJ}^q(F)_2) \\ & \quad \xrightarrow{(e(Y \times \tilde{S}_I)_* \mathcal{H}om(T^{q, \gamma}(D_{g_I})(j_I^*F), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b)))^{-1}} \\ & (e(Y \times \tilde{S}_I)_* \mathcal{H}om^\bullet(L(i'_{I*}j_I^* g^*F), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b)))[-d_{YI}], u_{IJ}^q(g^*F)) =: \mathcal{F}_T^{GM}(g^*M). \end{aligned}$$

Proposition 102. (i) Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y_2 \times S \xrightarrow{p_S} S$ with $Y_2 \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y_2 \times \tilde{S}_i$. Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I := g|_{T_I} : T_I \rightarrow S_I$. Let $f : X \rightarrow S$ a morphism with $X \in \text{Var}(\mathbb{C})$. Assume that there is a factorization $f : X \xrightarrow{l} Y_1 \times S \xrightarrow{p_S} S$, with $Y_1 \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have then the following commutative diagram whose squares are cartesian

$$\begin{array}{ccccc} f' : X_T & \longrightarrow & Y_1 \times T & \longrightarrow & T \\ \downarrow & & \downarrow & & \downarrow \\ f'' = f \times I : Y_2 \times X & \longrightarrow & Y_1 \times Y_2 \times S & \longrightarrow & Y_2 \times S \\ \downarrow & & \downarrow & & \downarrow \\ f : X & \longrightarrow & Y_1 \times S & \longrightarrow & S \end{array}$$

Consider $F(X/S) := p_{S,\sharp}\Gamma_X^\vee\mathbb{Z}(Y_1 \times S/Y_1 \times S)[d_{Y_1}]$ and the isomorphism in $C(\text{Var}(\mathbb{C})^{sm}/S)$

$$T(f, g, F(X/S)) : g^*F(X/S) := g^*p_{S,\sharp}\Gamma_X^\vee\mathbb{Z}(Y_1 \times S/Y_1 \times S)[d_{Y_1}] \xrightarrow{\sim} p_{T,\sharp}\Gamma_{X_T}^\vee\mathbb{Z}(Y_1 \times T/Y_1 \times T)[d_{Y_1}] =: F(X_T/T).$$

which gives in $\text{DA}(S)$ the isomorphism $T(f, g, F(X/S)) : g^*M^{BM}(X/S) \xrightarrow{\sim} M^{BM}(X_T/T)$. Then, the following diagram in $D_{\text{O}fil, \mathcal{D}, \infty}(T/(Y_2 \times \tilde{S}_I))$ commutes

$$\begin{array}{ccc} Rg^{*mod, \Gamma} \mathcal{F}_S^{GM}(M^{BM}(X/S)) & \xrightarrow{T(g, \mathcal{F}^{GM})(M^{BM}(X/S))} & \mathcal{F}_T^{GM}(M^{BM}(X_T/T)) \\ \downarrow I^{GM}(X/S) & & \downarrow I^{GM}(X_T/T) \\ g^{*mod[-], \Gamma}(p_{\tilde{S}_I^*} \Gamma_{X_I} E_{zar}(\Omega_{Y_1 \times \tilde{S}_I / \tilde{S}_I}^\bullet, F_b)[-d_{\tilde{S}_I}], w_{IJ}(X/S)) & \xrightarrow{(T(\tilde{g}_I \times I, \gamma)(-) \circ T_w^O(\tilde{g}_I, p_{\tilde{S}_I}))} & (p_{Y_2 \times \tilde{S}_I^*} \Gamma_{X_{T_I}} E_{zar}(\Omega_{Y_2 \times Y_1 \times \tilde{S}_I / Y_2 \times \tilde{S}_I}^\bullet, F_b)[-d_{Y_2 I}], w_{IJ}(X_T/T)) \end{array}$$

(ii) Let $g : T \rightarrow S$ a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Let $f : X \rightarrow S$ a morphism with $X \in \text{Var}(\mathbb{C})$. Assume that there is a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$, with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Consider $F(X/S) := p_{S,\sharp}\Gamma_X^\vee\mathbb{Z}(Y \times S/Y \times S)[d_Y]$ and the isomorphism in $C(\text{Var}(\mathbb{C})^{sm}/S)$

$$T(f, g, F(X/S)) : g^*F(X/S) := g^*p_{S,\sharp}\Gamma_X^\vee\mathbb{Z}(Y \times S/Y \times S)[d_Y] \xrightarrow{\sim} p_{T,\sharp}\Gamma_{X_T}^\vee\mathbb{Z}(Y \times T/Y \times T)[d_Y] =: F(X_T/T).$$

which gives in $\text{DA}(S)$ the isomorphism $T(f, g, F(X/S)) : g^*M^{BM}(X/S) \xrightarrow{\sim} M^{BM}(X_T/T)$. Then, the following diagram in $D_{\text{O}fil, \infty}(T)$ commutes

$$\begin{array}{ccc} Lg^{*mod} \mathcal{F}_S^{GM}(M^{BM}(X/S)) & \xrightarrow{T^O(g, \mathcal{F}^{GM})(M^{BM}(X/S))} & \mathcal{F}_T^{GM}(M^{BM}(X_T/T)) \\ \downarrow I^{GM}(X/S) & & \downarrow I^{GM}(X_T/T) \\ g^{*mod} L_O(p_{S^*} \Gamma_X E_{zar}(\Omega_{Y \times S/S}^\bullet, F_b)[-d_S]) & \xrightarrow{(T(g \times I, \gamma)(-) \circ T_w^O(g, p_S))} & p_{Y \times T^*} \Gamma_{X_T} E_{zar}(\Omega_{Y \times T/T}^\bullet, F_b)[-d_T] \\ \downarrow T_w(\otimes, \gamma)(O_{Y \times S}) & & \downarrow T_w(\otimes, \gamma)(O_{Y \times T}) \\ Lg^{*mod} \int_{p_S}^{FDR} \Gamma_X E(O_{Y \times S}, F_b) & \xrightarrow{T^{\mathcal{D}mod}(g, f)(\Gamma_X E(O_{Y \times S}, F_b))} & \int_{p_T}^{FDR} \Gamma_{X_T} E(O_{Y \times T}, F_b). \end{array}$$

Proof. Follows immediately from definition. \square

We have the following theorem:

Theorem 30. (i) Let $g : T \rightarrow S$ is a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S}$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, for $M \in \text{DA}_c(S)$

$$T(g, \mathcal{F}^{GM})(M) : Rg^{*mod[-], \Gamma} \mathcal{F}_S^{GM}(M) \xrightarrow{\sim} \mathcal{F}_T^{GM}(g^*M)$$

is an isomorphism in $D_{\text{O}Tfil, \mathcal{D}, \infty}(T/(Y \times \tilde{S}_I))$.

(ii) Let $g : T \rightarrow S$ is a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Then, for $M \in \text{DA}_c(S)$

$$T^O(g, \mathcal{F}^{GM})(M) : Lg^{*mod} \mathcal{F}_S^{GM}(M) \xrightarrow{\sim} \mathcal{F}_T^{GM}(g^*M)$$

is an isomorphism in $D_{\text{O}T}(T)$.

(iii) A base change theorem for algebraic De Rham cohomology : Let $g : T \rightarrow S$ is a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Let $h : U \rightarrow S$ a smooth morphism with $U \in \text{Var}(\mathbb{C})$. Then the map (see definition 1)

$$T_w^O(g, h) : Lg^{*mod} Rh_*(\Omega_{U/S}^\bullet, F_b) \xrightarrow{\sim} Rh'_*(\Omega_{U_T/T}^\bullet, F_b)$$

is an isomorphism in $D_{O_T}(T)$.

Proof. (i): Follows from proposition 98.

(ii): Follows from proposition 102(ii) and the base change for algebraic D modules (proposition 74).

(iii): Follows from (ii) and lemma 10. \square

We finish this subsection by a consequence of proposition 97 and theorem 14 :

Theorem 31. Let $X \in \text{PSmVar}(\mathbb{C})$ and $D = \cup D_i \subset X$ a normal crossing divisor. Consider the open embedding $j : U := X \setminus D \hookrightarrow X$. Then, the inclusion

$$j^* : \Gamma(X, E_{zar}(\Omega_X^\bullet(\log D), F_b)) \hookrightarrow \Gamma(U, E_{zar}(\Omega_U^\bullet, F_b))$$

is an ∞ -filtered quasi-isomorphism, that is

$$j^* : F^p H^n(U, \mathbb{C}) := F^p H^n \Gamma(X, E_{zar}(\Omega_X^\bullet(\log D), F_b)) \xrightarrow{\sim} F^p H^n \Gamma(U, E_{zar}(\Omega_U^\bullet, F_b)) =: F_b^p H^n(U, \mathbb{C})$$

is an isomorphism for all $n, p \in \mathbb{Z}$ (note that it is obviously injective since $j^* : H^n \Gamma(X, E_{zar}(\Omega_X^\bullet(\log D))) \xrightarrow{\sim} H^n \Gamma(U, E_{zar}(\Omega_U^\bullet))$ is an isomorphism if we forgot filtrations). Note that however, it is NOT a filtered quasi-isomorphism (for example if U is affine $H^q(U, \Omega_U^p) = 0$ for $q > 0$) that it is not an isomorphism on the E_1 terms of the spectral sequences in general.

Proof. For simplicity, we may assume that $i : D \hookrightarrow X$ is a smooth divisor. Then, by theorem 14, there is a purity isomorphism in $\text{DA}(\mathbb{C})$

$$P(D/X) : \text{Cone}(a_{X\sharp} \text{ad}(j_\sharp, j_*)(\mathbb{Z}(X/X)) : \mathbb{Z}(U) \rightarrow \mathbb{Z}(X)) \xrightarrow{\sim} \mathbb{Z}(D)[1]$$

is an equivalence (\mathbb{A}^1, et) local in $C(\text{SmVar}(\mathbb{C}))$, with $a_X : X \rightarrow \text{pt}$ the terminal map. Hence, by proposition 97, we get an isomorphism in $D_{fil, \infty}(\mathbb{Z})$

$$\begin{aligned} & e(\text{Spec } \mathbb{C})_* \mathcal{H}om(P(D/X), E_{et}(\Omega_{\mathbb{C}}^\bullet, F_b)) : \\ & \text{Cone}(j^* : \Gamma(X, E_{zar}(\Omega_{\mathbb{C}}^\bullet, F_b)) \rightarrow \Gamma(U, E_{zar}(\Omega_{\mathbb{C}}^\bullet, F_b)))[1] \xrightarrow{\sim} \Gamma(D, E_{zar}(\Omega_{\mathbb{C}}^\bullet, F_b))[1] \end{aligned}$$

which gives the isomorphism in $D_{fil, \infty}(\mathbb{Z})$,

$$\begin{aligned} & j^* : \text{Cone}(c(j^*) \circ e(\text{Spec } \mathbb{C})_* \mathcal{H}om(P(D/X), E_{et}(\Omega_{\mathbb{C}}^\bullet, F_b))^{-1} : \\ & \Gamma(D, E_{zar}(\Omega_{\mathbb{C}}^\bullet, F_b))[1] \rightarrow \Gamma(X, E_{zar}(\Omega_{\mathbb{C}}^\bullet, F_b))) \xrightarrow{\sim} \Gamma(U, E_{zar}(\Omega_{\mathbb{C}}^\bullet, F_b)) \end{aligned}$$

where $c(j^*) : \text{Cone}(j^*) \rightarrow \Gamma(X, E_{zar}(\Omega_{\mathbb{C}}^\bullet, F_b))[-1]$ is the canonical map, which induces the isomorphisms

$$H^n j^* : F^p H^n \text{Cone}(\Gamma(D, E_{zar}(\Omega_{\mathbb{C}}^\bullet, F_b))[1] \rightarrow \Gamma(X, E_{zar}(\Omega_{\mathbb{C}}^\bullet, F_b))) \xrightarrow{\sim} F^p H^n \Gamma(U, E_{zar}(\Omega_{\mathbb{C}}^\bullet, F_b)).$$

\square

Definition 108. Let $S \in \text{SmVar}(\mathbb{C})$. We have, for $M, N \in \text{DA}(S)$ and $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ projective such that $M = D(\mathbb{A}^1, et)(F)$ and $N = D(\mathbb{A}^1, et)(G)$, the following transformation map in $D_{O_{fil, \mathcal{D}}}(S)$

$$\begin{aligned} & T(\mathcal{F}_S^{GM}, \otimes)(M, N) : \mathcal{F}_S^{GM}(M) \otimes_{O_S}^{L[-]} \mathcal{F}_S^{GM}(N) := \\ & (e(S)_* \mathcal{H}om(F, E_{et}(\Omega_{\mathbb{C}}^\bullet, F_b))) \otimes_{O_S} (e(S)_* \mathcal{H}om(G, E_{et}(\Omega_{\mathbb{C}}^\bullet, F_b)))[-ds] \\ & \xrightarrow{T(\otimes, \Omega_{\mathbb{C}}^\bullet)(F, G)} e(S)_* \mathcal{H}om(F \otimes G, E_{et}(\Omega_{\mathbb{C}}^\bullet, F_b))[-ds] \\ & \xrightarrow{\cong} e(S)_* \mathcal{H}om(F \otimes G, E_{et}(\Omega_{\mathbb{C}}^\bullet, F_b))[-ds] =: \mathcal{F}_S^{GM}(M \otimes N) \end{aligned}$$

We now give the definition in the non smooth case :

Definition 109. Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open affine covering and denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We have, for $M, N \in \text{DA}(S)$ and $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1, et)(F)$ and $N = D(\mathbb{A}^1, et)(G)$, the following transformation map in $D_{\text{Ofil}, \mathcal{D}}(S/(\tilde{S}_I))$

$$\begin{aligned}
& T(\mathcal{F}_S^{GM}, \otimes)(M, N) : \mathcal{F}_S^{GM}(M) \otimes_{O_S}^{L[-]} \mathcal{F}_S^{GM}(N) := \\
& (e(\tilde{S}_I)_* \mathcal{H}om(L(i_{I*} j_I^* F), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b))[-d_{\tilde{S}_I}], u_{IJ}(F)) \otimes_{O_S}^{[-]} \\
& (e(\tilde{S}_I)_* \mathcal{H}om(L(i_{I*} j_I^* G), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b))[-d_{\tilde{S}_I}], u_{IJ}(G)) \\
& \xrightarrow{=} ((e(\tilde{S}_I)_* \mathcal{H}om(L(i_{I*} j_I^* F), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)) \otimes_{O_{\tilde{S}_I}} \\
& e(\tilde{S}_I)_* \mathcal{H}om(L(i_{I*} j_I^* G), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)))[-d_{\tilde{S}_I}], u_{IJ}(F) \otimes u_{IJ}(G)) \\
& \xrightarrow{(T(\otimes, \Omega_{/\tilde{S}_I})(L(i_{I*} j_I^* F), L(i_{I*} j_I^* G)))} \\
& (e(\tilde{S}_I)_* \mathcal{H}om(L(i_{I*} j_I^* F) \otimes L(i_{I*} j_I^* G), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_{DR}))[-d_{\tilde{S}_I}], v_{IJ}(F \otimes G)) \\
& \xrightarrow{=} (e(\tilde{S}_I)_* \mathcal{H}om(L(i_{I*} j_I^*(F \otimes G), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)))[-d_{\tilde{S}_I}], u_{IJ}(F \otimes G)) =: \mathcal{F}_S^{GM}(M \otimes N)
\end{aligned}$$

Proposition 103. Let $f_1 : X_1 \rightarrow S$, $f_2 : X_2 \rightarrow S$ two morphism with $X_1, X_2, S \in \text{Var}(\mathbb{C})$. Assume that there exist factorizations $f_1 : X_1 \xrightarrow{l_1} Y_1 \times S \xrightarrow{p_S} S$, $f_2 : X_2 \xrightarrow{l_2} Y_2 \times S \xrightarrow{p_S} S$ with $Y_1, Y_2 \in \text{SmVar}(\mathbb{C})$, l_1, l_2 closed embeddings and p_S the projections. We have then the factorization

$$f_1 \times f_2 : X_{12} := X_1 \times_S X_2 \xrightarrow{l_1 \times l_2} Y_1 \times Y_2 \times S \xrightarrow{p_S} S$$

Let $S = \cup_{i=1}^l S_i$ an open affine covering and denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We have, for $M, N \in \text{DA}(S)$ and $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1, et)(F)$ and $N = D(\mathbb{A}^1, et)(G)$, the following commutative diagram in $D_{\text{Ofil}, \mathcal{D}}(S/(\tilde{S}_I))$

$$\begin{array}{ccc}
\mathcal{F}_S^{GM}(M(X_1/S)) \otimes_{O_S}^L \mathcal{F}_S^{GM}(M(X_2/S)) & \xrightarrow{T(\mathcal{F}_S^{GM}, \otimes)(M(X_1/S), M(X_2/S))} & \mathcal{F}_S^{GM}(M(X_1/S) \otimes M(X_2/S)) \\
\downarrow I^{GM}(X_1/S) \otimes I^{GM}(X_2/S) & & \downarrow I^{GM}(X_{12}/S) \\
(p_{\tilde{S}_I^*} \Gamma_{X_{1I}} E_{zar}(\Omega_{Y_1 \times \tilde{S}_I / \tilde{S}_I}^\bullet, F_b)[-d_{\tilde{S}_I}], w_{IJ}(X_1/S)) \otimes_{O_S} & \xrightarrow{(E_{w(Y_1 \times \tilde{S}_I, Y_2 \times \tilde{S}_I)} / \tilde{S}_I)} & (p_{\tilde{S}_I^*} \Gamma_{X_{12I}} E_{zar}(\Omega_{Y_1 \times Y_2 \times \tilde{S}_I / \tilde{S}_I}^\bullet, F_b)[-d_{\tilde{S}_I}], \\
(p_{\tilde{S}_I^*} \Gamma_{X_{2I}} E_{zar}(\Omega_{Y_2 \times \tilde{S}_I / \tilde{S}_I}^\bullet, F_b)[-d_{\tilde{S}_I}], w_{IJ}(X_2/S)) & & w_{IJ}(X_{12}/S))
\end{array}$$

Proof. Immediate from definition. □

6.1.2 The algebraic filtered De Rham realization functor and the commutativity with the six operation

We recall (see section 2), for $f : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$, the commutative diagrams of sites (29) and (30)

$$\begin{array}{ccccc}
 \text{Var}(\mathbb{C})^2/T & \xrightarrow{\mu_T} & \text{Var}(\mathbb{C})^{2,pr}/T & & \\
 \downarrow P(f) & \searrow \rho_T & \downarrow & \searrow \rho_T & \\
 & & \text{Var}(\mathbb{C})^{2,sm}/T & \xrightarrow{P(f)} & \text{Var}(\mathbb{C})^{2,smpr}/T \\
 & & \downarrow & \downarrow P(f) & \downarrow P(f) \\
 \text{Var}(\mathbb{C})^2/S & \xrightarrow{P(f)} & \text{Var}(\mathbb{C})^{2,pr}/S & & \\
 \downarrow & \searrow \rho_S & \downarrow & \searrow \rho_S & \\
 & & \text{Var}(\mathbb{C})^{2,sm}/S & \xrightarrow{\mu_S} & \text{Var}(\mathbb{C})^{2,smpr}/S
 \end{array}$$

and

$$\begin{array}{ccccc}
 \text{Var}(\mathbb{C})^{2,pr}/T & \xrightarrow{\text{Gr}_T^{12}} & \text{Var}(\mathbb{C})/T & & \\
 \downarrow P(f) & \searrow \rho_T & \downarrow & \searrow \rho_T & \\
 & & \text{Var}(\mathbb{C})^{2,smpr}/T & \xrightarrow{P(f)} & \text{Var}(\mathbb{C})^{sm}/T \\
 & & \downarrow & \downarrow P(f) & \downarrow P(f) \\
 \text{Var}(\mathbb{C})^{2,pr}/S & \xrightarrow{P(f)} & \text{Var}(\mathbb{C})/S & & \\
 \downarrow & \searrow \rho_S & \downarrow & \searrow \rho_S & \\
 & & \text{Var}(\mathbb{C})^{2,sm}/S & \xrightarrow{\text{Gr}_S^{12}} & \text{Var}(\mathbb{C})^{sm}/S
 \end{array}$$

Let $S \in \text{Var}(\mathbb{C})$. We have for $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ the canonical map in $C(\text{Var}(\mathbb{C})^{sm}/S)$

$$\begin{aligned}
 \text{Gr}(F) : \text{Gr}_{S^*}^{12} \mu_{S^*} F^\Gamma &\rightarrow F, \\
 \text{Gr}(F)(U/S) : \Gamma_U^\vee p^* F(U \times S/U \times S) &\xrightarrow{\text{ad}(l^*, l_*)(p^* F)(U \times S/U \times S)} h^* F(U/U) = F(U/S)
 \end{aligned}$$

where $h : U \rightarrow S$ is a smooth morphism with $U \in \text{Var}(\mathbb{C})$ and $h : U \xrightarrow{l} U \times S \xrightarrow{p} S$ is the graph factorization with l the graph embedding and p the projection.

Definition 110. (i) For $S \in \text{Var}(\mathbb{C})$, we have the filtered complexes of presheaves

$$(\Omega_{/S}^{\bullet, \Gamma}, F_b) \in C_{O_S \text{ fil}}(\text{Var}(\mathbb{C})^2/S)$$

given by

– for $((X, Z), h) = (X, Z)/S \in \text{Var}(\mathbb{C})^2/S$,

$$(\Omega_{/S}^{\bullet, \Gamma}((X, Z)/S), F_b) := \Gamma_Z^{\vee, h} L_{h^* O}(\Omega_{X/S}^{\bullet, \Gamma}, F_b)(X) := \mathbb{D}_{h^* O_S} L_{h^* O} \Gamma_Z E_{\text{zar}}(\mathbb{D}_{h^* O_S} L_{h^* O}(\Omega_{X/S}^{\bullet, \Gamma}, F_b))(X)$$

– for $g : (X_1, Z_1)/S = ((X_1, Z_1), h_1) \rightarrow (X, Z)/S = ((X, Z), h)$ a morphism in $\text{Var}(\mathbb{C})^2/S$,

$$\begin{aligned} \Omega_{/S}^{\bullet, \Gamma}(g) : \Gamma_Z^{\vee, h} L_{h^*O}(\Omega_{X/S}^{\bullet, \Gamma}, F_b)(X) &\xrightarrow{i_-} g^* \mathbb{D}_{h^*O_S} L_{h^*O} \Gamma_Z E_{zar}(\mathbb{D}_{h^*O_S} L_{h^*O}(\Omega_{X/S}^{\bullet, \Gamma}, F_b))(X_1) \\ &\xrightarrow{=} \mathbb{D}_{h_1^*O_S} L_{h_1^*O} g^* \Gamma_Z E_{zar}(\mathbb{D}_{h^*O_S} L_{h^*O}(\Omega_{X/S}^{\bullet, \Gamma}, F_b))(X_1) \\ \xrightarrow{\mathbb{D}_{h_1^*O_S}((T(g, E)(-) \circ T(g, \gamma)(-))^{-1})(X_1)} &\mathbb{D}_{h_1^*O_S} L_{h_1^*O} \Gamma_{Z \times_X X_1} E_{zar}(g^* \mathbb{D}_{h^*O_S} L_{h^*O}(\Omega_{X/S}^{\bullet, \Gamma}, F_b))(X_1) \\ &\xrightarrow{=} \mathbb{D}_{h_1^*O_S} L_{h_1^*O} \Gamma_{Z \times_X X_1} E_{zar}(\mathbb{D}_{h_1^*O_S} L_{h_1^*O} g^*(\Omega_{X/S}^{\bullet, \Gamma}, F_b))(X_1) \\ \xrightarrow{\mathbb{D}_{h_1^*O_S} T(Z_1/(Z \times_X X_1) \cap Z_1, \gamma)(-)(X_1)} &\mathbb{D}_{h_1^*O_S} L_{h_1^*O} \Gamma_{Z_1} E_{zar}(\mathbb{D}_{h_1^*O_S} L_{h_1^*O} g^*(\Omega_{X/S}^{\bullet, \Gamma}, F_b))(X_1) \\ \xrightarrow{\Gamma_{Z_1}^{\vee, h} L_{h_1^*O}(\Omega_{(X_1/X)/(S/S)})(X_1)} &\mathbb{D}_{h_1^*O_S} L_{h_1^*O} \Gamma_{Z_1} E_{zar}(\mathbb{D}_{h_1^*O_S} L_{h_1^*O}(\Omega_{X_1/S}^{\bullet, \Gamma}, F_b))(X_1) \end{aligned}$$

where i_- is the arrow of the inductive limit.

For $S \in \text{SmVar}(\mathbb{C})$, we get

$$(\Omega_{/S}^{\bullet, \Gamma}, F_b) := \rho_{S*}(\Omega_{/S}^{\bullet, \Gamma}, F_b) \in C_{O_S \text{fil}, D_S}(\text{Var}(\mathbb{C})^{2, \text{sm}}/S)$$

(ii) For $S \in \text{SmVar}(\mathbb{C})$, we have the canonical map $C_{O_S \text{fil}, D_S}(\text{Var}(\mathbb{C})^{\text{sm}}/S)$

$$\text{Gr}^O(\Omega_{/S}) : \text{Gr}_{S^*}^{12} \mu_{S^*}(\Omega_{/S}^{\bullet, \Gamma}, F_b) \rightarrow (\Omega_{/S}^{\bullet, \Gamma}, F_b)$$

given by, for $U/S = (U, h) \in \text{Var}(\mathbb{C})^{\text{sm}}/S$

$$\begin{aligned} \text{Gr}^O(\Omega_{/S})(U/S) : \text{Gr}_{S^*}^{12} \mu_{S^*}(\Omega_{/S}^{\bullet, \Gamma}, F_b)(U/S) &:= \Gamma_U^{\vee, h} L_{h^*O}(\Omega_{U \times S/S}^{\bullet, \Gamma}, F_b)(U \times S) \\ \xrightarrow{\Gamma_U^{\vee, h} L_{h^*O} \text{ad}(i_U^*, i_{U^*})(-)(U \times S)} &\Gamma_U^{\vee, h} L_{h^*O} i_{U^*} i_U^*(\Omega_{U \times S/S}^{\bullet, \Gamma}, F_b)(U \times S) \\ \xrightarrow{\Gamma_U^{\vee, h} L_{h^*O} i_{U^*} \Omega_{(U/U \times S)/(S/S)}(U \times S)} &\Gamma_U^{\vee, h} L_{h^*O} i_{U^*}(\Omega_{U/S}^{\bullet, \Gamma}, F_b)(U \times S) \\ &\xrightarrow{=} (\Omega_{U/S}^{\bullet, \Gamma}, F_b)(U) =: (\Omega_{/S}^{\bullet, \Gamma}, F_b)(U/S) \end{aligned}$$

where $h : U \xrightarrow{i_U} U \times S \xrightarrow{p_S} S$ is the graph factorization with i_U the graph embedding and p_S the projection.

We will use the following map from the property of mixed Hodge module (see section 5) together with the specialization map of a filtered D module for a closed embedding (see definition 49) :

Definition-Proposition 17. (i) Let $l : Z \hookrightarrow S$ a closed embedding with $S, Z \in \text{SmVar}(\mathbb{C})$. Consider an open embedding $j : S^\circ \hookrightarrow S$. We then have the cartesian square

$$\begin{array}{ccc} S^\circ & \xrightarrow{j} & S \\ l' \uparrow & & \uparrow l \\ Z^\circ := Z \times_S S^{\circ j'} & \longrightarrow & Z \end{array}$$

where j' is the open embedding given by base change. Using proposition 89 and theorem 26, the isomorphisms $G_{Z, D}^{p, q}(O_S, F_b)$ for $D \subset S$ a closed subset of definition-proposition 11 induces a canonical isomorphism in $C_{\mathcal{D}(1,0) \text{fil}}(Z)$

$$G(Z, j_!) (O_S, F_b) : l^* \text{Gr}_{V_{Z,0}} j_!^{\text{Hdg}}(O_{S^\circ}, F_b) \xrightarrow{\sim} j_!^{\text{Hdg}}(O_{Z^\circ}, F_b).$$

(ii) Let $l : Z \hookrightarrow S$ and $k : Z' \hookrightarrow Z$ be closed embeddings with $S, Z, Z' \in \text{SmVar}(\mathbb{C})$. Consider an open embedding $j : S^\circ \hookrightarrow S$. We then have the commutative diagram whose squares are cartesian.

$$\begin{array}{ccc} S^\circ & \xrightarrow{j} & S \\ l' \uparrow & & \uparrow l \\ Z^\circ := Z \times_S S^\circ & \xrightarrow{j'} & Z \\ k' \uparrow & & \uparrow k \\ Z'^\circ := Z' \times_S S^\circ & \xrightarrow{j''} & Z' \end{array}$$

where j' is the open embedding given by base change. Then,

$$\begin{aligned} G(Z', j_!)(O_S, F_b) &= G(Z', j'_!)(O_Z, F_b) \circ (k^* \text{Gr}_{V_{Z'}, 0} G(Z, j_!)(O_S, F_b)) : \\ k^* \text{Gr}_{V_{Z'}, 0} l^* \text{Gr}_{V_Z, 0} j_!^{Hdg}(O_{S^\circ}, F_b) &\xrightarrow{k^* \text{Gr}_{V_{Z'}, 0} G(Z, j_!)(O_S, F_b)} k^* \text{Gr}_{V_{Z'}, 0} j_!'^{Hdg}(O_{Z^\circ}, F_b) \\ &\xrightarrow{G(Z', j'_!)(O_Z, F_b)} j_!''^{Hdg}(O_{Z'^\circ}, F_b). \end{aligned}$$

in $C_{\mathcal{D}(1,0)fil}(Z')$.

(iii) Consider a commutative diagram whose squares are cartesian

$$\begin{array}{ccccc} S^{\circ\circ} & \xrightarrow{j_2} & S^\circ & \xrightarrow{j_1} & S \\ l'' \uparrow & & \uparrow l' & & \uparrow l \\ Z^{\circ\circ} := Z \times_S S^{\circ\circ} & \xrightarrow{j_2'} & Z^\circ := Z \times_S S^\circ & \xrightarrow{j_1'} & Z \end{array}$$

where j_1, j_2 , and hence j_1', j_2' are open embeddings. We have then the following commutative diagram

$$\begin{array}{ccc} l^* \text{Gr}_{V_Z, 0} j_1!^{Hdg}(O_{S^\circ}, F_b) & \xrightarrow{\text{ad}(j_2!^{Hdg}, j_2'^*)(O_{S^\circ}, F_b)} & l^* \text{Gr}_{V_Z, 0} (j_1 \circ j_2)!^{Hdg}(O_{S^{\circ\circ}}, F_b) \\ \downarrow G(Z, j_!)(O_S, F_b) & & \downarrow G(Z, (j_1 \circ j_2)!)(O_S, F_b) \\ j_1!'^{Hdg}(O_{Z^\circ}, F_b) & \xrightarrow{\text{ad}(j_2!'^{Hdg}, j_2'^*)(O_{Z^\circ}, F_b)} & (j_1' \circ j_2')!^{Hdg}(O_{Z^{\circ\circ}}, F_b) \end{array}$$

Proof. (i): By definition of $j_1!^{Hdg} : \pi_{S^\circ}(MHM(S^\circ)) \rightarrow \pi_S(C(MHM(S)))$, we have to construct the isomorphism for each complement of a (Cartier) divisor $j = j_D : S^\circ = S \setminus D \hookrightarrow S$. In this case, we have the closed embedding $i : S \hookrightarrow L$ given by the zero section of the line bundle $L = L_D$ associated to D . We have then, using definition-proposition 11, the canonical isomorphism in $\pi_Z(MHW(Z^\circ) \times_J MHW(D \cap Z))$

$$\begin{aligned} G_0(Z, j_!)(O_S, F_b) &: (j'^*, \phi_{(D \cap Z)_1}, c, v)(l^* \text{Gr}_{V_Z, 0} j_!^{Hdg}(O_{S^\circ}, F_b)) \xrightarrow{=} \\ (j'^* l^* \text{Gr}_{V_Z, 0} j_!^{Hdg}(O_{S^\circ}, F_b), \phi_{(D \cap Z)_1}(l^* \text{Gr}_{V_Z, 0} j_!^{Hdg}(O_{S^\circ}, F_b)), \text{can}, \text{var}) &\xrightarrow{(I, G_{Z, S}^{p, q}(j_!^{Hdg}(O_{S^\circ}, F_b)))} \\ (l^* \text{Gr}_{V_{Z^\circ}, 0} j'^* j_!^{Hdg}(O_{S^\circ}, F_b), l^* \text{Gr}_{V_{D \cap Z}, 0} \phi_{D1} j_!^{Hdg}(O_{S^\circ}, F_b), \text{can}, \text{var}) &\xrightarrow{=} \\ (l^* \text{Gr}_{V_{Z^\circ}, 0}(O_{S^\circ}, F_b), l^* \text{Gr}_{V_{D \cap Z}, 0} \psi_{D1}(O_{S^\circ}, F_b), N, I) &\xrightarrow{(I, G_{Z, S}^{p, q}(i_{*mod}(O_{S^\circ}, F_b)))} \\ ((O_{Z^\circ}, F_b), l^* \psi_{(D \cap Z)_1}(O_{S^\circ}, F_b), N, I) &=: (j'^*, \phi_{(D \cap Z)_1}, c, v)(j_1!'^{Hdg}(O_{Z^\circ}, F_b)) \end{aligned}$$

Hence, using proposition 90

$$\begin{aligned} G(Z, j_!)(O_S, F_b) &:= (j'^*, \phi_{(D \cap Z)_1}, c, v)^{-1}(G_0(Z, j_!)(i_{*mod}(O_S, F_b))) : \\ l^* \text{Gr}_{V_Z, 0} j_!^{Hdg}(O_{S^\circ}, F_b) &\xrightarrow{\sim} j_1!'^{Hdg}(O_{Z^\circ}, F_b) \end{aligned}$$

is an isomorphism in $\pi_Z(MHM(Z))$. Now for $j : S^\circ = S \setminus R \hookrightarrow S$ an arbitrary open embedding, we set

$$G(Z, j_!)(O_S, F_b) := \varprojlim_{(D_i), R \subset D_i \subset S} (G(Z, j_{D_i!})(j_{D_i}^*(O_S, F_b)) : l^* \text{Gr}_{V_Z, 0} j_!^{Hdg}(O_{S^\circ}, F_b) \xrightarrow{\sim} j_!^{Hdg}(O_{Z^\circ}, F_b)$$

(ii): Follows from definition-proposition 11.

(iii): Follows from definition-proposition 11. \square

Using definition-proposition 16 in the projection case, and the specialization map given in definition 49 and the isomorphism of definition-proposition 17, in the closed embedding case, we have the following canonical map :

Definition 111. Consider a commutative diagram in $\text{SmVar}(\mathbb{C})$ whose square are cartesian

$$\begin{array}{ccccc} Z_T & \xrightarrow{i'} & T & \xleftarrow{j'} & T \setminus Z_T \\ \downarrow g' & \searrow l' & \downarrow & \searrow l & \downarrow l \\ & & T \times_S Z & \xrightarrow{I \times i} & T \times_S S \\ & \swarrow p_Z & \downarrow & \swarrow p_S & \downarrow p_S \\ Z & \xrightarrow{i} & S & \xleftarrow{j} & S \setminus Z \end{array}$$

where i and hence $I \times i$ and i' , are closed embeddings, $j, I \times j, j'$ are the complementary open embeddings and $g : T \xrightarrow{l} T \times S \xrightarrow{p_S} S$ is the graph factorization, where l is the graph embedding and p_S the projection. We have, by definition-proposition 16, definition 49, proposition 89 and theorem 26, the canonical map in $\mathcal{C}_{\mathcal{D}(1,0)fil}(T)$

$$\begin{aligned} a(g, Z)(O_S, F_b) : g^{*mod} \Gamma_Z^{\vee, Hdg}(O_S, F_b) &= l^{*mod} p_S^{*mod} \Gamma_Z^{\vee, Hdg}(O_S, F_b) \\ &\xrightarrow{l^{*mod} T^{Hdg}(p, \gamma^\vee)(O_S, F_b)^{-1}} l^{*mod} \Gamma_{T \times Z}^{\vee, Hdg}(O_{T \times S}, F_b) \\ &\xrightarrow{sp_{V_Z}(\Gamma_{T \times Z}^{\vee, Hdg}(O_{T \times S}, F_b))} l^* \text{Gr}_{V_T, 0}(\Gamma_{T \times Z}^{\vee, Hdg}(O_{T \times S}, F_b)) \xrightarrow{G(T, (I \times j)_!)(O_{T \times S}, F_b)} \Gamma_{Z_T}^{\vee, Hdg}(O_T, F_b) \end{aligned}$$

where the last map is the isomorphism given in definition-proposition 17.

Lemma 14. (i) For $g : T \rightarrow S$ and $g' : T' \rightarrow T$ two morphism with $S, T, T' \in \text{SmVar}(\mathbb{C})$, considering the commutative diagram whose squares are cartesian

$$\begin{array}{ccccc} Z_{T'} & \xrightarrow{i''} & T' & \xleftarrow{j''} & T' \setminus Z_{T'} \\ \downarrow g' & & \downarrow g' & & \downarrow g' \\ Z_T & \xrightarrow{i'} & T & \xleftarrow{j'} & T \setminus Z_T \\ \downarrow g & & \downarrow g & & \downarrow g \\ Z & \xrightarrow{i} & S & \xleftarrow{j} & S \setminus Z \end{array}$$

we have then

$$\begin{aligned} a(g \circ g', Z)(O_S, F_b) &= a(g', Z_T)(O_T, F_b) \circ (g'^{*mod} a(g, Z)(O_S, F_b)) : \\ (g \circ g')^{*mod} \Gamma_Z^{\vee, Hdg}(O_S, F_b) &= g'^{*mod} g^{*mod} \Gamma_Z^{\vee, Hdg}(O_S, F_b) \xrightarrow{g'^{*mod} a(g, Z)(O_S, F_b)} g'^{*mod} \Gamma_{Z_T}^{\vee, Hdg}(O_T, F_b) \\ &\xrightarrow{a(g', Z_T)(O_T, F_b)} \Gamma_{Z_{T'}}^{\vee, Hdg}(O_{T'}, F_b). \end{aligned}$$

(ii) For $g : T \rightarrow S$ a morphism with $S, T \in \text{SmVar}(\mathbb{C})$, considering the commutative diagram whose squares are cartesian

$$\begin{array}{ccccc} Z'_T & \xrightarrow{k'} & Z_T & \xrightarrow{i'} & T \\ \downarrow g & & \downarrow g & & \downarrow g \\ Z' & \xrightarrow{k} & Z & \xrightarrow{i} & S \end{array}$$

we have then the following commutative diagram

$$\begin{array}{ccc} g^{*mod} \Gamma_Z^{\vee, Hdg}(O_S, F_b) & \xrightarrow{g^{*mod} T(Z'/Z, \gamma^{\vee, Hdg})(O_S, F_b)} & g^{*mod} \Gamma_{Z'}^{\vee, Hdg}(O_S, F_b) \\ \downarrow a(g, Z)(O_S, F_b) & & \downarrow a(g, Z')(O_S, F_b) \\ \Gamma_{Z_T}^{\vee, Hdg}(O_T, F_b) & \xrightarrow{T(Z'_T/Z_T, \gamma^{\vee, Hdg})(O_T, F_b)} & \Gamma_{Z'_T}^{\vee, Hdg}(O_T, F_b) \end{array}$$

Proof. (i): Follows from definition-proposition 17 (ii)

(ii): Follows from definition-proposition 17 (iii) □

We can now define the main object :

Definition 112. (i) For $S \in \text{SmVar}(\mathbb{C})$, we consider the filtered complexes of presheaves

$$(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR}) \in C_{D_S fil}(\text{Var}(\mathbb{C})^{2, smpr} / S)$$

given by,

– for $(Y \times S, Z)/S = ((Y \times S, Z), p) \in \text{Var}(\mathbb{C})^{2, smpr} / S$,

$$(\Omega_{/S}^{\bullet, \Gamma, pr}((Y \times S, Z)/S), F_{DR}) := ((\Omega_{Y \times S/S}^{\bullet}, F_b) \otimes_{O_{Y \times S}} \Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b))(Y \times S)$$

with the structure of p^*D_S module given by proposition 54,

– for $g : (Y_1 \times S, Z_1)/S = ((Y_1 \times S, Z_1), p_1) \rightarrow (Y \times S, Z)/S = ((Y \times S, Z), p)$ a morphism in $\text{Var}(\mathbb{C})^{2, smpr} / S$, denoting for short $\hat{Z} := Z \times_{Y \times S} (Y_1 \times S)$,

$$\begin{aligned} \Omega_{/S}^{\bullet, \Gamma, pr}(g) &: ((\Omega_{Y \times S/S}^{\bullet}, F_b) \otimes_{O_{Y \times S}} \Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b))(Y \times S) \\ &\xrightarrow{i_-} g^*((\Omega_{Y \times S/S}^{\bullet}, F_b) \otimes_{O_{Y \times S}} \Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b))(Y_1 \times S) \\ &\xrightarrow{\Omega_{(Y_1 \times S/Y \times S)/(S/S)}(\Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b))(Y_1 \times S)} (\Omega_{Y_1 \times S/S}^{\bullet}, F_b) \otimes_{O_{Y_1 \times S}} g^{*mod} \Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b)(Y_1 \times S) \\ &\xrightarrow{DR(Y_1 \times S/S)(a(g, Z)(O_{Y \times S}, F_b))(Y_1 \times S)} (\Omega_{Y_1 \times S/S}^{\bullet}, F_b) \otimes_{O_{Y_1 \times S}} \Gamma_{\hat{Z}}^{\vee, Hdg}(O_{Y_1 \times S}, F_b)(Y_1 \times S) \\ &\xrightarrow{DR(Y_1 \times S/S)(T(Z_1/\hat{Z}, \gamma^{\vee, Hdg})(O_{Y_1 \times S}, F_b))(Y_1 \times S)} (\Omega_{Y_1 \times S/S}^{\bullet}, F_b) \otimes_{O_{Y_1 \times S}} \Gamma_{Z_1}^{\vee, Hdg}(O_{Y_1 \times S}, F_b)(Y_1 \times S), \end{aligned}$$

where

* i_- is the arrow of the inductive limit,

* we recall that

$$\begin{aligned} \Omega_{(Y_1 \times S/Y \times S)/(S/S)}(\Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b)) &: g^*((\Omega_{Y \times S/S}^{\bullet}, F_b) \otimes_{O_{Y \times S}} \Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b)) \\ &\rightarrow (\Omega_{Y_1 \times S/S}^{\bullet}, F_b) \otimes_{O_{Y_1 \times S}} g^{*mod} \Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b) \end{aligned}$$

is the map given in definition-proposition 12, which is $p_1^*D_S$ linear by proposition 57,

* the map

$$a(g, Z)(O_{Y \times S}, F_b) : g^{*mod} \Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b) \rightarrow \Gamma_{\hat{Z}}^{\vee, Hdg}(O_{Y_1 \times S}, F_b)$$

is the map given in definition 111

* the map

$$T(Z_1/\hat{Z}, \gamma^{\vee, Hdg})(O_{Y_1 \times S}, F_b) : \Gamma_{\hat{Z}}^{\vee, Hdg}(O_{Y_1 \times S}, F_b) \rightarrow \Gamma_{Z_1}^{\vee, Hdg}(O_{Y_1 \times S}, F_b)$$

is given in definition-proposition 16.

For $g : ((Y_1 \times S, Z_1), p_1) \rightarrow ((Y \times S, Z), p)$ and $g' : ((Y'_1 \times S, Z'_1), p_1) \rightarrow ((Y_1 \times S, Z_1), p)$ two morphisms in $\text{Var}(\mathbb{C})^{2, smpr}/S$, we have

$$\begin{aligned} \Omega_{/S}^{\bullet, \Gamma, pr}(g \circ g') &= \Omega_{/S}^{\bullet, \Gamma, pr}(g') \circ \Omega_{/S}^{\bullet, \Gamma, pr}(g) : ((\Omega_{Y \times S/S}^{\bullet, \Gamma, pr}, F_b) \otimes_{O_{Y \times S}} \Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b))(Y \times S) \\ &\xrightarrow{\Omega_{/S}^{\bullet, \Gamma, pr}(g)} (\Omega_{Y_1 \times S/S}^{\bullet, \Gamma, pr}, F_b) \otimes_{O_{Y_1 \times S}} \Gamma_{Z_1}^{\vee, Hdg}(O_{Y_1 \times S}, F_b)(Y_1 \times S) \\ &\xrightarrow{\Omega_{/S}^{\bullet, \Gamma, pr}(g')} (\Omega_{Y'_1 \times S/S}^{\bullet, \Gamma, pr}, F_b) \otimes_{O_{Y'_1 \times S}} \Gamma_{Z'_1}^{\vee, Hdg}(O_{Y'_1 \times S}, F_b)(Y'_1 \times S), \end{aligned}$$

since, denoting for short $\hat{Z} := Z \times_{Y \times S} (Y_1 \times S)$ and $\hat{Z}' := Z \times_{Y \times S} (Y'_1 \times S)$

– we have by lemma 14(i)

$$a(g \circ g', \hat{Z}')(O_{Y \times S}, F_b) = a(g', \hat{Z})(O_{Y_1 \times S}, F_b) \circ g'^{*mod} a(g, Z)(O_{Y \times S}, F_b),$$

– we have by lemma 14(ii)

$$\begin{aligned} &T(Z'_1/\hat{Z}', \gamma^{\vee, Hdg})(O_{Y'_1 \times S}, F_b) \circ a(g', \hat{Z})(O_{Y_1 \times S}, F_b) \\ &= a(g', Z_1)(O_{Y_1 \times S}, F_b) \circ g'^{*mod} T(Z_1/\hat{Z}, \gamma^{\vee, Hdg})(O_{Y_1 \times S}, F_b). \end{aligned}$$

(ii) For $S \in \text{SmVar}(\mathbb{C})$, we have the canonical map $C_{O_S \text{fil}, D_S}(\text{Var}(\mathbb{C})^{sm}/S)$

$$\text{Gr}(\Omega_{/S}) : \text{Gr}_{S*}^{12}(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR}) \rightarrow (\Omega_{/S}^{\bullet, \Gamma}, F_b)$$

given by, for $U/S = (U, h) \in \text{Var}(\mathbb{C})^{sm}/S$

$$\begin{aligned} \text{Gr}(\Omega_{/S})(U/S) : \text{Gr}_{S*}^{12}(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR})(U/S) &:= ((\Omega_{U \times S/S}^{\bullet, \Gamma, pr}, F_b) \otimes_{O_{U \times S}} \Gamma_U^{\vee, Hdg}(O_{U \times S}, F_b))(U \times S) \\ &\xrightarrow{\text{ad}(i_U^*, i_{U*})(-)(U \times S)} i^*((\Omega_{U \times S/S}^{\bullet, \Gamma, pr}, F_b) \otimes_{O_{U \times S}} \Gamma_U^{\vee, Hdg}(O_{U \times S}, F_b))(U) \\ &\xrightarrow{\Omega_{(U/U \times S)/(S/S)}(-)(U)} ((\Omega_{U/S}^{\bullet, \Gamma}, F_b) \otimes_{O_U} i_U^{*mod} \Gamma_U^{\vee, Hdg}(O_{U \times S}, F_b))(U) \\ &\xrightarrow{DR(U/S)(a(i_U, U))(U)} (\Omega_{U/S}^{\bullet, \Gamma}, F_b)(U) =: (\Omega_{/S}^{\bullet, \Gamma}, F_b)(U/S) \end{aligned}$$

where $h : U \xrightarrow{i_U} U \times S \xrightarrow{ps} S$ is the graph factorization with i_U the graph embedding and ps the projection, note that $a(i_U, U)$ is an isomorphism since for $j_U : U \times S \setminus U \hookrightarrow U \times S$ the open complementary $i_U^{*mod} j_U^{Hdg}(M, F, W) = 0$.

Definition 113. For $S \in \text{SmVar}(\mathbb{C})$, we have the canonical map $C_{O_S \text{fil}, D_S}(\text{Var}(\mathbb{C})^{2, smpr}/S)$

$$T(\Omega_{/S}^{\Gamma}) : \mu_{S*}(\Omega_{/S}^{\bullet, \Gamma}, F_b) \rightarrow (\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR})$$

given by, for $(Y \times S, X)/S = ((Y \times S, Z), p) \in \text{Var}(\mathbb{C})^{2, \text{smpr}}/S$

$$\begin{aligned} & T(\Omega_{/S}^\Gamma)((Y \times S, Z)/S) : \\ & (\Omega_{/S}^{\bullet, \Gamma}, F_b)((Y \times S, Z)/S) := \mathbb{D}_{p^*O_S} L_{p^*O} \Gamma_Z E_{\text{Zar}}(\mathbb{D}_{p^*O_S} L_{p^*O}(\Omega_{Y \times S/S}^{\bullet, \Gamma}, F_b))(Y \times S) \\ & \xrightarrow{\text{DR}(Y \times S/S)(\gamma_Z^{\vee, \text{Hdg}}(O_{Y \times S})(Y \times S))} \\ & \mathbb{D}_{p^*O_S} L_{p^*O} \Gamma_Z E_{\text{Zar}}(\mathbb{D}_{p^*O_S} L_{p^*O}((\Omega_{Y \times S/S}^{\bullet, \Gamma}, F_b) \otimes_{O_{Y \times S}} \Gamma_Z^{\vee, \text{Hdg}}(O_{Y \times S}, F_b))(Y \times S) \xrightarrow{\cong} \\ & ((\Omega_{Y \times S/S}^{\bullet, \Gamma}, F_b) \otimes_{O_{Y \times S}} \Gamma_Z^{\vee, \text{Hdg}}(O_{Y \times S}, F_b))(Y \times S) =: (\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{\text{DR}})((Y \times S, Z)/S). \end{aligned}$$

By definition $\text{Gr}(\Omega_{/S}) \circ \text{Gr}_{S^*} T(\Omega_{/S}^\Gamma) = \text{Gr}^O(\Omega_{/S})$.

Remark 11. (i) Let $S \in \text{Var}(\mathbb{C})$. We have by definition $\circ_{12^*}(\Omega_{/S}^{\bullet, \Gamma}, F_b) = (\Omega_{/S}^{\bullet, \Gamma}, F_b) \in C_{O_S \text{fil}}(\text{Var}(\mathbb{C})^{\text{sm}}/S)$.
Moreover, if $S \in \text{SmVar}(\mathbb{C})$, $\circ_{12^*}(\Omega_{/S}^{\bullet, \Gamma}, F_b) = (\Omega_{/S}^{\bullet, \Gamma}, F_b) \in C_{O_S \text{fil}, D_S}(\text{Var}(\mathbb{C})^{\text{sm}}/S)$.

(ii) Let $S \in \text{Var}(\mathbb{C})$. Then, $(\Omega_{/S}^{\bullet, \Gamma}, F_b) \in C_{O_S \text{fil}}(\text{Var}(\mathbb{C})^2/S)$ is a natural extension of

$$(\Omega_{/S}^{\bullet, \Gamma}, F_b) := \rho_{S^*}(\Omega_{/S}^{\bullet, \Gamma}, F_b) \in C_{O_S \text{fil}}(\text{Var}(\mathbb{C})^{2, \text{sm}}/S),$$

but does NOT satisfy cdh descent.

The point definition 112 above gives the object in $\text{DA}(S)$ which will, for S smooth, represent the algebraic De Rham realisation in D_S modules. It is the class of an explicit complex of presheaves on $\text{Var}(\mathbb{C})^{\text{sm}}/S$.

Proposition 104. Let $S \in \text{Var}(\mathbb{C})$.

(i) The complex of presheaves $(\Omega_{/S}^{\bullet, \Gamma}, F_b) \in C_{O_S \text{fil}}(\text{Var}(\mathbb{C})^{2, \text{sm}}/S)$ is 2-filtered \mathbb{A}_S^1 local for the Zariski or etale topology (see definition 16) and admits transferts (i.e. $\text{Tr}(S)_* \text{Tr}(S)^* \Omega_{/S}^{\bullet, \Gamma} = \Omega_{/S}^{\bullet, \Gamma}$).

(ii) The complex of presheaves $(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{\text{DR}}) \in C_{D_S \text{fil}}(\text{Var}(\mathbb{C})^{2, \text{smpr}}/S)$ is 2-filtered \mathbb{A}_S^1 local for the Zariski or etale topology (see definition 16) and admits transferts (i.e. $\text{Tr}(S)_* \text{Tr}(S)^* \Omega_{/S}^{\bullet, \Gamma, \text{pr}} = \Omega_{/S}^{\bullet, \Gamma, \text{pr}}$).

Proof. (i): Follows from proposition 97.

(ii): Let $(Y \times S, Z)/S = ((Y \times S, Z), p) \in \text{Var}^{2, \text{smpr}}(\mathbb{C})$ and $p_a : (Y \times \mathbb{A}^1 \times S, Z \times \mathbb{A}^1) \rightarrow (Y \times S, Z)$ the projection, $i_0 : (Y \times S, Z) \rightarrow (Y \times \mathbb{A}^1 \times S, Z \times \mathbb{A}^1)$ closed embedding. Then,

$$a(p_a, Z) : p_a^{* \text{mod}} \Gamma_Z^{\vee, \text{Hdg}}(O_{Y \times S}, F_b) \rightarrow \Gamma_{Z \times \mathbb{A}^1}^{\vee, \text{Hdg}}(O_{Y \times \mathbb{A}^1 \times S}, F_b).$$

a quasi-isomorphism in $\pi_{Y \times \mathbb{A}^1 \times S}(C(\text{MHM}(Y \times \mathbb{A}^1 \times S)))$. Since a morphism of mixed Hodge module is strict for the F-filtration,

$$a(p_a, Z) : p_a^{* \text{mod}} \Gamma_Z^{\vee, \text{Hdg}}(O_{Y \times S}, F_b) \rightarrow \Gamma_{Z \times \mathbb{A}^1}^{\vee, \text{Hdg}}(O_{Y \times \mathbb{A}^1 \times S}, F_b).$$

is a (1)-filtered quasi-isomorphism in $C_{\text{Dfil}}(Y \times \mathbb{A}^1 \times S)$. On the other hand, as, for all $p \in \mathbb{Z}$,

$$\begin{aligned} & I(p_a^*, p_{a^*})(-, -)(\Omega_{(Y \times \mathbb{A}^1 \times S/Y \times S)(S/S)}(\Gamma_Z^{\vee, \text{Hdg}}(O_{Y \times S}, F_b))) : \\ & \Omega_{Y \times S/S}^{\bullet} \otimes_{O_{Y \times S}} F^{p-\bullet} \Gamma_Z^{\vee, \text{Hdg}}(O_{Y \times S}, F_b) \\ & \rightarrow p_{a^*}(\Omega_{Y \times \mathbb{A}^1 \times S/S}^{\bullet} \otimes_{O_{Y \times \mathbb{A}^1 \times S}} p_a^{* \text{mod}} F^{p-\bullet} \Gamma_Z^{\vee, \text{Hdg}}(O_{Y \times S}, F_b)) \end{aligned}$$

is an homotopy equivalence whose inverse is

$$\begin{aligned}
& p_{a*}I(i_0^*, i_{0*})(-, -)(\Omega_{(Y \times S/Y \times \mathbb{A}^1 \times S)(S/S)}(p_a^{*mod}(\Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b)))) : \\
& \quad p_{a*}(\Omega_{Y \times \mathbb{A}^1 \times S/S}^\bullet \otimes_{O_{Y \times \mathbb{A}^1 \times S}} p_a^{*mod} F^{p-\bullet} \Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b)) \\
& \rightarrow p_{a*}i_{0*}(\Omega_{Y \times S/S}^\bullet \otimes_{O_{Y \times S}} (i_0^{*mod} p_a^{*mod} F^{p-\bullet} \Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b))) \\
& \quad \xrightarrow{=} \Omega_{Y \times S/S}^\bullet \otimes_{O_{Y \times S}} F^{p-\bullet} \Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b)
\end{aligned}$$

(see the proof of proposition 97), for all $p \in \mathbb{Z}$, the map

$$\begin{aligned}
& E_{et}(I(p_a^*, p_{a*})(-, -)(\Omega_{(Y \times \mathbb{A}^1 \times S/Y \times S)(S/S)}(\Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b)))) : \\
& \quad E_{et}(\Omega_{Y \times S/S}^\bullet \otimes_{O_{Y \times S}} F^{p-\bullet} \Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b)) \\
& \rightarrow E_{et}(p_{a*}(\Omega_{Y \times \mathbb{A}^1 \times S/S}^\bullet \otimes_{O_{Y \times \mathbb{A}^1 \times S}} p_a^{*mod} F^{p-\bullet} \Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b)))
\end{aligned}$$

is a 2-filtered quasi-isomorphism by proposition 7. Hence,

$$\begin{aligned}
& E_{et}(\Omega_{/S}^{\bullet, \Gamma, pr}(p_a)) : E_{et}((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} \Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b)) \\
& \rightarrow E_{et}((\Omega_{Y \times \mathbb{A}^1 \times S/S}^\bullet, F_b) \otimes_{O_{Y \times \mathbb{A}^1 \times S}} \Gamma_{Z \times \mathbb{A}^1}^{\vee, Hdg}(O_{Y \times \mathbb{A}^1 \times S}, F_b))
\end{aligned}$$

is a 2-filtered isomorphism. \square

We have the following canonical transformation map given by the pullback of (relative) differential forms:

- Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. We have the canonical morphism in $C_{g^*O_S \text{fil}, g^*D_S}(\text{Var}(\mathbb{C})^{2, sm}/T)$

$$\Omega_{/(T/S)}^\Gamma : g^*(\Omega_{/S}^{\bullet, \Gamma}, F_b) \rightarrow (\Omega_{/T}^{\bullet, \Gamma}, F_b)$$

induced by the pullback of differential forms : for $((V, Z_1)/T) = ((V, Z_1), h) \in \text{Var}(\mathbb{C})^{2, sm}/T$,

$$\begin{aligned}
& \Omega_{/(T/S)}^\Gamma((V, Z_1)/T) : \\
& g^*\Omega_{/S}^{\bullet, \Gamma}((V, Z_1)/T) := \lim_{(h:(U, Z) \rightarrow \text{Ssm}, g_1:(V, Z_1) \rightarrow (U_T, Z_T), h, g)} \Omega_{/S}^{\bullet, \Gamma}((U, Z)/S) \\
& \quad \xrightarrow{\Omega_{/S}^{\bullet, \Gamma}(g' \circ g_1)} \Omega_{/S}^{\bullet, \Gamma}((V, Z_1)/S) \xrightarrow{\Gamma_{Z_1}^{\vee, h} q(Y_1 \times T)} \Omega_{/T}^{\bullet, \Gamma}((V, Z_1)/T),
\end{aligned}$$

where $g' : U_T := U \times_S T \rightarrow U$ is the base change map and $q : \Omega_{Y_1 \times T/S}^\bullet \rightarrow \Omega_{Y_1 \times T/T}^\bullet$ is the quotient map. If $T, S \in \text{SmVar}(\mathbb{C})$,

$$\Omega_{/(T/S)}^\Gamma : g^*(\Omega_{/S}^{\bullet, \Gamma}, F_b) \rightarrow (\Omega_{/T}^{\bullet, \Gamma}, F_b)$$

is a morphism in $C_{g^*O_S \text{fil}, g^*D_S}(\text{Var}(\mathbb{C})^{2, sm}/T)$ It induces the canonical morphisms in $C_{g^*O_S \text{fil}, g^*D_S}(\text{Var}(\mathbb{C})^{2, sm}/T)$:

$$E\Omega_{/(T/S)}^\Gamma : g^*E_{et}(\Omega_{/S}^{\bullet, \Gamma}, F_b) \xrightarrow{T(g, E_{et})(\Omega_{/S}^{\bullet, \Gamma}, F_b)} E_{et}(g^*(\Omega_{/S}^{\bullet, \Gamma}, F_b)) \xrightarrow{E_{et}(\Omega_{/(T/S)}^\Gamma)} E_{et}(\Omega_{/T}^{\bullet, \Gamma}, F_b)$$

and

$$E\Omega_{/(T/S)}^\Gamma : g^*E_{zar}(\Omega_{/S}^{\bullet, \Gamma}, F_b) \xrightarrow{T(g, E_{zar})(\Omega_{/S}^{\bullet, \Gamma}, F_b)} E_{zar}(g^*(\Omega_{/S}^{\bullet, \Gamma}, F_b)) \xrightarrow{E_{zar}(\Omega_{/(T/S)}^\Gamma)} E_{zar}(\Omega_{/T}^{\bullet, \Gamma}, F_b).$$

- Let $g : T \rightarrow S$ a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. We have the canonical morphism in $C_{g^* D_S \text{fil}}(\text{Var}(\mathbb{C})^{2, \text{smpr}}/T)$

$$\Omega_{/(T/S)}^{\Gamma, \text{pr}} : g^*(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}) \rightarrow (\Omega_{/T}^{\bullet, \Gamma, \text{pr}}, F_{DR})$$

induced by the pullback of differential forms : for $((Y_1 \times T, Z_1)/T) = ((Y_1 \times T, Z_1), p) \in \text{Var}(\mathbb{C})^{2, \text{smpr}}/T$,

$$\begin{aligned} & \Omega_{/(T/S)}^{\Gamma, \text{pr}}((Y_1 \times T, Z_1)/T) : \\ g^* \Omega_{/S}^{\bullet, \Gamma, \text{pr}}((Y_1 \times T, Z_1)/T) & := \lim_{(h: (Y \times S, Z) \rightarrow S, g_1: (Y_1 \times T, Z_1) \rightarrow (Y \times T, Z_T), h, g)} \Omega_{/S}^{\bullet, \Gamma, \text{pr}}((Y \times T, Z)/S) \\ & \xrightarrow{\Omega_{/S}^{\bullet, \Gamma, \text{pr}}(g' \circ g_1)} \Omega_{/S}^{\bullet, \Gamma, \text{pr}}((Y_1 \times T, Z_1)/S) \xrightarrow{q(-)(Y_1 \times T)} \Omega_{/T}^{\bullet, \Gamma, \text{pr}}((Y_1 \times T, Z_1)/T), \end{aligned}$$

where $g' = (I_Y \times g) : Y \times T \rightarrow Y \times S$ is the base change map and $q(M) : \Omega_{Y_1 \times T/S} \otimes_{\mathcal{O}_{Y_1 \times T}}(M, F) \rightarrow \Omega_{Y_1 \times T/T} \otimes_{\mathcal{O}_{Y_1 \times T}}(M, F)$ is the quotient map. It induces the canonical morphisms in $C_{g^* D_S \text{fil}}(\text{Var}(\mathbb{C})^{2, \text{smpr}}/T)$:

$$E\Omega_{/(T/S)}^{\Gamma, \text{pr}} : g^* E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}) \xrightarrow{T(g, E)(-)} E_{et}(g^*(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})) \xrightarrow{E_{et}(\Omega_{/(T/S)}^{\Gamma, \text{pr}})} E_{et}(\Omega_{/T}^{\bullet, \Gamma, \text{pr}}, F_{DR})$$

and

$$E\Omega_{/(T/S)}^{\Gamma, \text{pr}} : g^* E_{zar}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}) \xrightarrow{T(g, E)(-)} E_{zar}(g^*(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})) \xrightarrow{E_{zar}(\Omega_{/(T/S)}^{\Gamma, \text{pr}})} E_{zar}(\Omega_{/T}^{\bullet, \Gamma, \text{pr}}, F_{DR}).$$

Definition 114. Let $g : T \rightarrow S$ a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. We have, for $F \in C(\text{Var}(\mathbb{C})^{2, \text{smpr}}/S)$, the canonical transformation in $C_{\mathcal{D}\text{fil}}(T)$:

$$\begin{aligned} T(g, \Omega_{/S}^{\Gamma, \text{pr}})(F) & : g^{* \text{mod}} L_D e(S)_* \text{Gr}_{S^*}^{12} \mathcal{H}om^{\bullet}(F, E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})) \\ & \xrightarrow{=} (g^* L_D e(S)_* \mathcal{H}om^{\bullet}(F, E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}))) \otimes_{g^* \mathcal{O}_S} \mathcal{O}_T \\ & \xrightarrow{T(g, \text{Gr}^{12})(-) \circ T(e, g)(-) \circ q} e(T)_* \text{Gr}_{T^*}^{12} g^* \mathcal{H}om^{\bullet}(F, E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})) \otimes_{g^* \mathcal{O}_S} \mathcal{O}_T \\ & \xrightarrow{(T(g, \text{hom})(-, -) \otimes I)} e(T)_* \text{Gr}_{T^*}^{12} \mathcal{H}om^{\bullet}(g^* F, g^* E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})) \otimes_{g^* \mathcal{O}_S} \mathcal{O}_T \\ & \xrightarrow{ev(\text{hom}, \otimes)(-, -, -)} e(T)_* \text{Gr}_{T^*}^{12} \mathcal{H}om^{\bullet}(g^* F, g^* E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})) \otimes_{g^* e(S)^* \mathcal{O}_S} e(T)^* \mathcal{O}_T \\ & \xrightarrow{\mathcal{H}om^{\bullet}(g^* F, (E\Omega_{/(T/S)}^{\Gamma, \text{pr}} \otimes m))} e(T)_* \text{Gr}_{T^*}^{12} \mathcal{H}om^{\bullet}(g^* F, E_{et}(\Omega_{/T}^{\bullet, \Gamma, \text{pr}}, F_{DR})) \end{aligned}$$

Let $S \in \text{Var}(\mathbb{C})$. Recall that for and $h : U \rightarrow S$ a morphism with $U \in \text{Var}(\mathbb{C})$, we have the canonical map given by the wedge product

$$w_{U/S} : \Omega_{U/S}^{\bullet} \otimes_{\mathcal{O}_S} \Omega_{U/S}^{\bullet} \rightarrow \Omega_{U/S}^{\bullet}; \alpha \otimes \beta \mapsto \alpha \wedge \beta.$$

Let $S \in \text{Var}(\mathbb{C})$ and $h_1 : U_1 \rightarrow S, h_2 : U_2 \rightarrow S$ two morphisms with $U_1, U_2 \in \text{Var}(\mathbb{C})$. Denote $h_{12} : U_{12} := U_1 \times_S U_2 \rightarrow S$ and $p_{112} : U_1 \times_S U_2 \rightarrow U_1, p_{212} : U_1 \times_S U_2 \rightarrow U_2$ the projections. Recall we have the canonical map given by the wedge product

$$w_{(U_1, U_2)/S} : p_{112}^* \Omega_{U_1/S}^{\bullet} \otimes_{\mathcal{O}_S} p_{212}^* \Omega_{U_2/S}^{\bullet} \rightarrow \Omega_{U_{12}/S}^{\bullet}; \alpha \otimes \beta \mapsto p_{112}^* \alpha \wedge p_{212}^* \beta$$

which gives the map

$$Ew_{(U_1, U_2)/S} : h_{1*} E_{zar}(\Omega_{U_1/S}^{\bullet}) \otimes_{\mathcal{O}_S} h_{2*} E_{zar}(\Omega_{U_2/S}^{\bullet}) \rightarrow h_{12*} E_{zar}(p_{112}^* \Omega_{U_1/S}^{\bullet} \otimes_{\mathcal{O}_S} p_{212}^* \Omega_{U_2/S}^{\bullet})$$

Let $S \in \text{SmVar}(\mathbb{C})$.

- The complex of presheaves $(\Omega_{/S}^{\bullet, \Gamma}, F_b) \in C_{O_S \text{ fil}, D_S}(\text{Var}(\mathbb{C})^{2, sm}/S)$ have a monoidal structure given by the wedge product of differential forms: for $h : (U, Z) \rightarrow S \in \text{Var}(\mathbb{C})^2/S$, the map

$$DR(-)(\gamma_Z^{\vee, h}(-)) \circ w_{U/S} : (\Omega_{U/S}^{\bullet, \Gamma}, F_b) \otimes_{p^*O_S} (\Omega_{U/S}^{\bullet, \Gamma}, F_b) \rightarrow \Gamma_Z^{\vee, h} L_{h^*O_S}(\Omega_{U/S}^{\bullet, \Gamma}, F_b)$$

factors through

$$\begin{aligned} & DR(-)(\gamma_Z^{\vee, Hdg}(-)) \circ w_{U/S} : (\Omega_{U/S}^{\bullet, \Gamma}, F_b) \otimes_{p^*O_S} (\Omega_{U/S}^{\bullet, \Gamma}, F_b) \\ & \xrightarrow{DR(-)(\gamma_Z^{\vee, h}(-)) \otimes DR(-)(\gamma_Z^{\vee, h}(-))} \Gamma_Z^{\vee, h} L_{h^*O_S}(\Omega_{U/S}^{\bullet, \Gamma}, F_b) \otimes_{p^*O_S} \Gamma_Z^{\vee, h} L_{h^*O_S}(\Omega_{U/S}^{\bullet, \Gamma}, F_b) \\ & \xrightarrow{(DR(-)(\gamma_Z^{\vee, h}(-)) \circ w_{U/S})^\gamma} \Gamma_Z^{\vee, h} L_{h^*O_S}(\Omega_{U/S}^{\bullet, \Gamma}, F_b) \end{aligned}$$

unique up to homotopy, giving the map in $C_{O_S \text{ fil}, D_S}(\text{Var}(\mathbb{C})^{2, smpr}/S)$:

$$w_S : (\Omega_{/S}^{\bullet, \Gamma}, F_b) \otimes_{O_S} (\Omega_{/S}^{\bullet, \Gamma}, F_b) \rightarrow (\Omega_{/S}^{\bullet, \Gamma}, F_b)$$

given by for $h : (U, Z) \rightarrow S \in \text{Var}(\mathbb{C})^{2, sm}/S$,

$$\begin{aligned} w_S((U, Z)/S) : & (\Gamma_Z^{\vee, h} L_{h^*O_S}(\Omega_{U/S}^{\bullet, \Gamma}, F_b) \otimes_{p^*O_S} \Gamma_Z^{\vee, h} L_{h^*O_S}(\Omega_{U/S}^{\bullet, \Gamma}, F_b))(U) \\ & \xrightarrow{(DR(-)(\gamma_Z^{\vee, h}(-)) \circ w_{U/S})^\gamma(U)} \Gamma_Z^{\vee, h} L_{h^*O_S}(\Omega_{U/S}^{\bullet, \Gamma}, F_b)(U) \end{aligned}$$

which induces the map in $C_{O_S \text{ fil}, D_S}(\text{Var}(\mathbb{C})^{2, sm}/S)$

$$Ew_S : E_{et}(\Omega_{/S}^{\bullet, \Gamma}, F_b) \otimes_{O_S} E_{et}(\Omega_{/S}^{\bullet, \Gamma}, F_b) \xrightarrow{\cong} E_{et}((\Omega_{/S}^{\bullet, \Gamma}, F_b) \otimes_{O_S} (\Omega_{/S}^{\bullet, \Gamma}, F_b)) \xrightarrow{E_{et}(w_S)} E_{et}(\Omega_{/S}^{\bullet, \Gamma}, F_b)$$

given by the functoriality of the Godement resolution (see section 2).

- The complex of presheaves $(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR}) \in C_{D_S \text{ fil}}(\text{Var}(\mathbb{C})^{2, smpr}/S)$ have a monoidal structure given by the wedge product of differential forms: for $p : (Y \times S, Z) \rightarrow S \in \text{Var}(\mathbb{C})^{2, smpr}/S$, the map

$$\begin{aligned} DR(-)(\gamma_Z^{\vee, Hdg}(-)) \circ w_{Y \times S/S} : & (\Omega_{Y \times S/S}^{\bullet, \Gamma, pr} \otimes_{O_{Y \times S}} (O_{Y \times S}, F_b)) \otimes_{p^*O_S} (\Omega_{Y \times S/S}^{\bullet, \Gamma, pr} \otimes_{O_{Y \times S}} (O_{Y \times S}, F_b)) \\ & \rightarrow \Omega_{Y \times S/S}^{\bullet, \Gamma, pr} \otimes_{O_{Y \times S}} \Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b) \end{aligned}$$

factors through

$$\begin{aligned} & DR(-)(\gamma_Z^{\vee, Hdg}(-)) \circ w_{Y \times S/S} : \\ & (\Omega_{Y \times S/S}^{\bullet, \Gamma, pr} \otimes_{O_{Y \times S}} (O_{Y \times S}, F_b)) \otimes_{p^*O_S} (\Omega_{Y \times S/S}^{\bullet, \Gamma, pr} \otimes_{O_{Y \times S}} (O_{Y \times S}, F_b)) \\ & \xrightarrow{DR(-)(\gamma_Z^{\vee, Hdg}(-)) \otimes DR(-)(\gamma_Z^{\vee, Hdg}(-))} \\ & (\Omega_{Y \times S/S}^{\bullet, \Gamma, pr} \otimes_{O_{Y \times S}} \Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b)) \otimes_{p^*O_S} \Omega_{Y \times S/S}^{\bullet, \Gamma, pr} \otimes_{O_{Y \times S}} \Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b) \\ & \xrightarrow{(DR(-)(\gamma_Z^{\vee, Hdg}(-)) \circ w_{Y \times S/S})^\gamma} \Omega_{Y \times S/S}^{\bullet, \Gamma, pr} \otimes_{O_{Y \times S}} \Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b) \end{aligned}$$

unique up to homotopy, giving the map in $C_{D_S \text{ fil}}(\text{Var}(\mathbb{C})^{2, smpr}/S)$:

$$w_S : (\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR}) \otimes_{O_S} (\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR}) \rightarrow (\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR})$$

given by for $p : (Y \times S, Z) \rightarrow S \in \text{Var}(\mathbb{C})^{2, smpr}/S$,

$$\begin{aligned} & w_S((Y \times S, Z)/S) : \\ & (((\Omega_{Y \times S/S}^{\bullet, \Gamma, pr} \otimes_{O_{Y \times S}} \Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b)) \otimes_{p^*O_S} (\Omega_{Y \times S/S}^{\bullet, \Gamma, pr} \otimes_{O_{Y \times S}} \Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b)))(Y \times S) \\ & \xrightarrow{(DR(-)(\gamma_Z^{\vee, Hdg}(-)) \circ w_{Y \times S/S})^\gamma(Y \times S)} (\Omega_{Y \times S/S}^{\bullet, \Gamma, pr} \otimes_{O_{Y \times S}} \Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b))(Y \times S) \end{aligned}$$

which induces the map in $C_{D_S \text{fil}}(\text{Var}(\mathbb{C})^{2, \text{smpr}}/S)$

$$\begin{aligned} Ew_S : E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}) \otimes_{O_S} E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}) &\xrightarrow{\cong} \\ E_{et}((\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}) \otimes_{O_S} (\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})) &\xrightarrow{E_{et}(w_S)} E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}) \end{aligned}$$

by the functoriality of the Godement resolution (see section 2).

Definition 115. *Let $S \in \text{SmVar}(\mathbb{C})$. We have, for $F, G \in C(\text{Var}(\mathbb{C})^{2, \text{smpr}}/S)$, the canonical transformation in $C_{D\text{fil}}(S)$:*

$$\begin{aligned} &T(\otimes, \Omega)(F, G) : \\ e(S)_* \text{Gr}_{S^*}^{12} \mathcal{H}om(F, E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})) \otimes_{O_S} e(S)_* \text{Gr}_{S^*}^{12} \mathcal{H}om(G, E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})) \\ &\xrightarrow{\cong} e(S)_* \text{Gr}_{S^*}^{12} (\mathcal{H}om(F, E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})) \otimes_{O_S} \mathcal{H}om(G, E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}))) \\ &\xrightarrow{T(\mathcal{H}om, \otimes)(-)} e(S)_* \text{Gr}_{S^*}^{12} \mathcal{H}om(F \otimes G, E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}) \otimes_{O_S} E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})) \\ &\xrightarrow{\cong} e(S)_* \text{Gr}_{S^*}^{12} \mathcal{H}om(F \otimes G, (E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}) \otimes_{O_S} E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}))) \\ &\xrightarrow{\mathcal{H}om(F \otimes G, Ew_S)} e(S)_* \text{Gr}_{S^*}^{12} \mathcal{H}om(F \otimes G, E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})). \end{aligned}$$

We now define the filtered De Rahm realization functor.

Definition 116. (i) *Let $S \in \text{SmVar}(\mathbb{C})$. We have, using definition 112 and definition 36, the functor*

$$\begin{aligned} &C(\text{Var}(\mathbb{C})^{\text{sm}}/S) \rightarrow C_{D\text{fil}}(S), F \mapsto \\ \varinjlim_{r(X^*, D^*)/S(LF)} e(S)_* \text{Gr}_{S^*}^{12} \mathcal{H}om^\bullet(\rho_{S^*} \text{Gr}_{S^*}^{12*} R_{(X^*, D^*)/S}(\rho_S^* L(F)), E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}))[-d_S] \end{aligned}$$

(ii) *Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote by $S_I := \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. Consider, for $I \subset J$, the following commutative diagram*

$$D_{I,J} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ j_{I,J} \uparrow & & \uparrow p_{I,J} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}$$

and $j_{I,J} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{I,J} = j_J$. We have, using definition 112 and definition 36, the functor

$$\begin{aligned} &C(\text{Var}(\mathbb{C})^{\text{sm}}/S) \rightarrow C_{D\text{fil}}(S/(\tilde{S}_I)), F \mapsto \\ (\varinjlim_{r(-)} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\rho_{\tilde{S}_I^*} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, \text{pr}}, F_{DR}))[-d_{\tilde{S}_I}], u_{I,J}^q(F)) \end{aligned}$$

where we have denoted for short $e'(\tilde{S}_I) = e(\tilde{S}_I) \circ \text{Gr}_{\tilde{S}_I}^{12}$, the limits run over the Corti-Hanamura resolutions

$$r_{(X^*, D^*)/\tilde{S}_I}(L(i_{I*} j_I^* F)) : R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)) \rightarrow \mathbb{D}_{\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F))$$

and

$$\begin{aligned}
& u_{IJ}^q(F)[d_{\tilde{S}_J}] : e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\rho_{\tilde{S}_I} \mathrm{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})) \\
& \quad \xrightarrow{\mathrm{ad}(p_{IJ}^{*mod}, p_{IJ})(-)} \\
& p_{IJ*} p_{IJ}^{*mod} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\rho_{\tilde{S}_I} \mathrm{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})) \\
& \quad \xrightarrow{p_{IJ*} T(p_{IJ}, \Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr})(-)} \\
& p_{IJ*} e'(\tilde{S}_J)_* \mathcal{H}om^\bullet(\rho_{\tilde{S}_J} \mathrm{Gr}_{\tilde{S}_J}^{12*} p_{IJ}^* R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{et}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR})) \\
& \quad \xrightarrow{\mathcal{H}om(\mathrm{Gr}_{\tilde{S}_J}^{12*} T(p_{IJ}, R^{CH})(Li_{I*} j_I^* F)^{-1}, E_{et}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR}))} \\
& p_{IJ*} e'(\tilde{S}_J)_* \mathcal{H}om^\bullet(\rho_{\tilde{S}_J} \mathrm{Gr}_{\tilde{S}_J}^{12*} R_{(X^* \times \tilde{S}_{J \setminus I}, D^* \times \tilde{S}_{J \setminus I})/\tilde{S}_J}(\rho_{\tilde{S}_J}^* p_{IJ}^* L(i_{I*} j_I^* F)), E_{et}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR})) \\
& \quad \xrightarrow{\mathcal{H}om(\mathrm{Gr}_{\tilde{S}_J}^{12*} R_{\tilde{S}_J}^{CH}(T^q(D_{IJ})(j_I^* F)), E_{et}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR}))} \\
& p_{IJ*} e'(\tilde{S}_J)_* \mathcal{H}om^\bullet(\rho_{\tilde{S}_J} \mathrm{Gr}_{\tilde{S}_J}^{12*} R_{(X^* \times \tilde{S}_{J \setminus I}, D^* \times \tilde{S}_{J \setminus I})/\tilde{S}_J}(\rho_{\tilde{S}_J}^* L(i_{J*} j_J^* F)), E_{et}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR})).
\end{aligned}$$

For $I \subset J \subset K$, we have obviously $p_{IJ*} u_{JK}(F) \circ u_{IJ}(F) = u_{IK}(F)$. We will prove in corollary 5 below that $u_{IJ}(F)$ are ∞ -filtered Zariski local equivalence.

We have the following key proposition :

Proposition 105. *Let $S \in \mathrm{SmVar}(\mathbb{C})$.*

(i) *Let $m : Q_1 \rightarrow Q_2$ be an etale local equivalence local with $Q_1, Q_2 \in C(\mathrm{Proj} \mathrm{PSh}(\mathrm{Var}(\mathbb{C})^{sm}/S))$ complexes of representable presheaves. Then,*

$$\begin{aligned}
& e(S)_* \mathrm{Gr}_{S^*}^{12} \mathcal{H}om^\bullet(\rho_{S^*} \mathrm{Gr}_{S^*}^{12*} R^{CH}(m), E_{et}(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR}))[-d_S] : \\
& \quad \varinjlim_{r(X^*, D^*)/S(Q_1)} e(S)_* \mathrm{Gr}_{S^*}^{12} \mathcal{H}om^\bullet(\rho_{S^*} \mathrm{Gr}_{S^*}^{12*} R_{(X^*, D^*)/S}(\rho_{S^*}^* Q_1), E_{et}(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR}))[-d_S] \\
& \rightarrow \quad \varinjlim_{r(X^*, D^*)/S(Q_2)} e(S)_* \mathrm{Gr}_{S^*}^{12} \mathcal{H}om^\bullet(\rho_{S^*} \mathrm{Gr}_{S^*}^{12*} R_{(X^*, D^*)/S}(\rho_{S^*}^* Q_2), E_{et}(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR}))[-d_S]
\end{aligned}$$

is a 2-filtered quasi-isomorphism. It is thus an isomorphism in $D_{\mathcal{D}fil, \infty}(S)$.

(ii) *Let $m : Q_1 \rightarrow Q_2$ be an equivalence (\mathbb{A}^1, et) local with $Q_1, Q_2 \in C(\mathrm{Proj} \mathrm{PSh}(\mathrm{Var}(\mathbb{C})^{sm}/S))$ complexes of representable presheaves. Then,*

$$\begin{aligned}
& e(S)_* \mathrm{Gr}_{S^*}^{12} \mathcal{H}om^\bullet(\rho_{S^*} \mathrm{Gr}_{S^*}^{12*} R^{CH}(m), E_{et}(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR}))[-d_S] : \\
& \quad \varinjlim_{r(X^*, D^*)/S(Q_1)} e(S)_* \mathrm{Gr}_{S^*}^{12} \mathcal{H}om^\bullet(\rho_{S^*} \mathrm{Gr}_{S^*}^{12*} R_{(X^*, D^*)/S}(\rho_{S^*}^* Q_1), E_{et}(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR}))[-d_S] \\
& \rightarrow \quad \varinjlim_{r(X^*, D^*)/S(Q_2)} e(S)_* \mathrm{Gr}_{S^*}^{12} \mathcal{H}om^\bullet(\rho_{S^*} \mathrm{Gr}_{S^*}^{12*} R_{(X^*, D^*)/S}(\rho_{S^*}^* Q_2), E_{et}(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR}))[-d_S]
\end{aligned}$$

is an 2-filtered quasi-isomorphism. It is thus an isomorphism in $D_{\mathcal{D}fil, \infty}(S)$.

Proof. (i): Let $h : U \rightarrow S$ a smooth morphism with $U \in \mathrm{Var}(\mathbb{C})$. Note that U is smooth since S is smooth. Let $(r_i : U_i \rightarrow U)_{i \in L=[1, \dots, s]}$ an etale cover of U . Consider the Check cover

$$\begin{aligned}
& (U_\bullet/S) := ((U_L, h_L) := (U_1 \times_U U_2 \times \dots \times_U U_s, h \circ (r_1 \times \dots \times r_s)) \\
& \quad \xrightarrow{r_{L, I}} \dots \xrightarrow{i_{I, i}} \sqcup_{i \in L} (U_i, h_i) := \sqcup_{i \in L} (U_i, h \circ r_i) \xrightarrow{r = \sqcup_{i \in L} r_i} (U, h)
\end{aligned}$$

Take (see definition-proposition 8) a compactification of $r : (U_\bullet/S) \rightarrow U/S$

$$(X_{\bullet 0}/S := ((X_{L0}, f_{L0}) \xrightarrow{\bar{r}_{L,I0}} \cdots \xrightarrow{\bar{r}_{Ii0}} \sqcup_{i \in L} (X_i, f_{i0})) \xrightarrow{\bar{r}_0} (X_0, f_0))$$

where $f_0 : X_0 \rightarrow S$ is a compactification of $h : U \rightarrow S$ and $f_{I0} : X_{I0} \rightarrow S$ are compactification of $h_I : U_I := U_{i_1} \times U \times \cdots \times U_{i_l} \rightarrow S$. Denote $Z := X_0 \setminus U$ and $Z_I := X_{I0} \setminus U_I$. Take (see again definition-proposition 8) a resolution $\epsilon : (X, D) \rightarrow (X_0, Z)$ of the pair (X_0, Z) and resolutions $\epsilon_I : (X_I, D_I) \rightarrow (X_{I0}, Z_I)$ of the pairs (X_{I0}, Z_I) , fitting in an other compactification of $r : (U_\bullet/S) \rightarrow U/S$

$$(X_\bullet/S : (X_L, f_L) \xrightarrow{\bar{r}_{L,I}} \cdots \xrightarrow{\bar{r}_{Ii}} \sqcup_{i \in L} (X_i, f_i)) \xrightarrow{\bar{r}} (X, f)$$

Denote by $j : U \hookrightarrow X$ and $j_\bullet : U_\bullet \hookrightarrow X_\bullet$ the open embeddings. We have the factorization

$$j_\bullet : U_\bullet \xrightarrow{j_\bullet^{10}} \bar{r}^{-1}(U) \xrightarrow{j_\bullet^0} X^\bullet$$

where j^{10} and j^0 are open embeddings. Consider the graph embeddings $\bar{r} : X_\bullet \xrightarrow{l_\bullet} X_\bullet \times X \xrightarrow{p_{I0}} X$. Denote for short $j^s := j \times I : U \times S \hookrightarrow X \times S$, $j_\bullet^s := j_\bullet \times I : U_\bullet \times S \hookrightarrow X_\bullet \times S$ and $\bar{r}^s := \bar{r} \times I : X_\bullet \times S \rightarrow X \times S$. We have the factorization

$$j_\bullet^s : U_\bullet \times S \xrightarrow{j_\bullet^{s10} := (j_\bullet^{10} \times I)} \bar{r}^{s,-1}(U \times S) \xrightarrow{j_\bullet^{s0} := (j_\bullet^0 \times I)} X^\bullet \times S$$

We have then by definition the following commutative diagram, with for short $e = e(S) \circ \text{Gr}_S^{12}$,

$$\begin{array}{ccc} e_* \mathcal{H}om^\bullet(\rho_{S*} \text{Gr}_S^{12*} R_{(X,D)/S}(\mathbb{Z}(U/S)), E_{et}(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR})) & \xrightarrow{\mathcal{H}om(T(f_\#, f_*)(-), -)} & p_{S*} E_{et}((\Omega_{X \times S/S}^\bullet, F_b) \otimes_{O_{X \times S}} j_!^{s, Hdg}(O_{U \times S}, F_b)) [2d_X] \\ \uparrow & & \downarrow \\ \mathcal{H}om^\bullet(\rho_{S*} \text{Gr}_S^{12*} R^{CH}(r), -) & & 'p_{X*} E_{et}(\Omega_{(X_\bullet \times S/X \times S)/(S/S)}(j_!^{s, Hdg}(O_{U \times S}, F_b)) \circ \text{ad}(\bar{r}^{s*}, \bar{r}_*^s)(-)) \downarrow \\ & & p_{X_\bullet*} E_{et}((\Omega_{X_\bullet \times S/S}^\bullet, F_b) \otimes_{O_{X_\bullet \times S}} \bar{r}^{s* \text{mod}} j_!^{s, Hdg}(O_{U \times S}, F_b)) \downarrow \\ & & 'DR(X_\bullet \times S/S)(G(U_\bullet \times S, j_\#^s)(O_{U \times S}, F_b) \circ T^{\vee, Hdg}(p_{I0} \times I, j_s)(-)) \downarrow \\ & & p_{S*} E_{et}((\Omega_{X_\bullet \times S/S}^\bullet, F_b) \otimes_{O_{X_\bullet \times S}} j_\bullet^{s0, Hdg}(O_{\bar{r}_s^{-1}(U \times S)}, F_b)) [2d_X] \\ & & \uparrow \\ e_* \mathcal{H}om^\bullet(\rho_{S*} \text{Gr}_S^{12*} R_{(X_\bullet, D_\bullet)/S}(\mathbb{Z}(U_\bullet/S)), E_{et}(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR})) & \xrightarrow{\mathcal{H}om(T(f_{\bullet\#}, f_{\bullet*})(-), -)} & p_{S*} E_{et}((\Omega_{X_\bullet \times S/S}^\bullet, F_b) \otimes_{O_{X_\bullet \times S}} j_\bullet^{s, Hdg}(O_{U_\bullet \times S}, F_b)) [2d_X] \\ & & \uparrow \\ & & DR(X_\bullet \times S/S)(j_\bullet^{s0, Hdg} \text{ad}(j_\bullet^{s10, Hdg}, j_\bullet^{s10*})(O_{\bar{r}_s^{-1}(U \times S)}, F_b)) \uparrow \end{array}$$

where,

- since $f : X \rightarrow S$ is surjective proper morphism, we have the equivalences (\mathbb{A}^1, et) local in $C(\text{Var}(\mathbb{C})/S)$

$$T(f_\#, f_*)(\mathbb{Z}(X/X)) := [\Delta_X] : f_\#(\mathbb{Z}(X/X)) \rightarrow f_* E_{et}(\mathbb{Z}(X/X))(d_X - d_S)[2d_X - 2d_S]$$

and

$$T(f_\#, f_*)(\mathbb{Z}(D_I/X)) := [\Delta_{D_I}] : f_\#(\mathbb{Z}(D_I/X)) \rightarrow f_* E_{et}(\mathbb{Z}(D_I/X))(d_X - d_S)[2d_X - 2d_S]$$

given in definition 42,

- since $f_\bullet : X_\bullet \rightarrow S$ is given by surjective proper morphisms, we have the equivalences (\mathbb{A}^1, et) local in $C(\text{Var}(\mathbb{C})/S)$

$$T(f_{\bullet\#}, f_{\bullet*})(\mathbb{Z}(X_\bullet/X_\bullet)) := ([\Delta_{X_\bullet}]) : f_{\bullet\#}(\mathbb{Z}(X_\bullet/X_\bullet)) \rightarrow f_{\bullet*} E_{et}(\mathbb{Z}(X_\bullet/X_\bullet))(d_X - d_S)[2d_X - 2d_S]$$

and

$$T(f_{\bullet\#}, f_{\bullet*})(\mathbb{Z}(D_{\bullet I}/X_\bullet)) := ([\Delta_{D_{\bullet I}}]) : f_{\bullet\#}(\mathbb{Z}(D_{\bullet I}/X_\bullet)) \rightarrow f_{\bullet*} E_{et}(\mathbb{Z}(D_{\bullet I}/X_\bullet))(d_X - d_S)[2d_X - 2d_S]$$

given in definition 42.

Now, the two horizontal arrows are 2-filtered quasi-isomorphism by proposition 104 and lemma 2. The arrow given by the composition of the two maps of the right column is a filtered quasi-isomorphism by proposition 94. The upper arrow of the right column is a filtered quasi-isomorphism by proposition 92. Hence the arrow of the left column is a 2-filtered quasi-isomorphism which proves (i).

(ii): Let $h : U \rightarrow S$ a smooth morphism with $U \in \text{Var}(\mathbb{C})$. Denote $p : U \times \mathbb{A}^1 \rightarrow U$ the projection. Let $f_0 : X_0 \rightarrow S$ is a compactification of $h : U \rightarrow S$ (see definition-proposition 8) Denote $Z := X_0 \setminus U$ and $Z_I := X_{I_0} \setminus U_I$. Then $\bar{p}_0 : X_0 \times \mathbb{P}^1 \rightarrow X_0$ is a compactification of $p : U \times \mathbb{A}^1 \rightarrow U$. Take (see again definition-proposition 8) a resolution $\epsilon : (X, D) \rightarrow (X_0, Z)$ of the pair (X_0, Z) . Then

$$\epsilon' : (X \times \mathbb{P}^1, (D \times \mathbb{P}^1) \cup (X \times \{\infty\})) \rightarrow (X_0 \times \mathbb{P}^1, (Z \times \mathbb{P}^1) \cup (X_0 \times \{\infty\}))$$

is a resolution of the pair $(X_0 \times \mathbb{P}^1, (Z \times \mathbb{P}^1) \cup (X_0 \times \{\infty\}))$. Denote by $j : U \hookrightarrow X$ the open embedding. Denote for short $j^s := j \times I : U \times S \hookrightarrow X \times S$, $j^{sa} := j \times j_a \times I : U \times \mathbb{A}^1 \times S \hookrightarrow X \times \mathbb{P}^1 \times S$ $\bar{p}^s := \bar{p} \times I : X \times \mathbb{P}^1 \times S \rightarrow X \times S$. We have then by definition the following commutative diagram, with for short $e = e(S) \circ \text{Gr}_S^{12*}$,

$$\begin{array}{ccc} e_* \text{Hom}^\bullet(\rho_{S*} \text{Gr}_S^{12*} R_{(X,D)/S}(\mathbb{Z}(U/S)), E_{et}(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR})) & \xrightarrow{\mathcal{H}om(T(f_\sharp, f_*)(-), -)} & p_{S*} E_{et}((\Omega_{X \times S/S}^\bullet, F_b) \otimes_O j_!^{s, Hdg}(O_{U \times S}, F_b)) \\ \uparrow & & \downarrow \\ \text{Hom}^\bullet(\rho_{S*} \text{Gr}_S^{12*} R^{CH}(p), -) & & p_{S*} E_{et}((\Omega_{X \times \mathbb{P}^1 \times S/S}^\bullet, F_b) \otimes_O \bar{p}^{s* mod} j_!^{s, Hdg}(O_{U \times S}, \\ & & \downarrow \\ e_* \text{Hom}^\bullet(\rho_{S*} \text{Gr}_S^{12*} R_{(X \times \mathbb{P}^1, -)/S}(\mathbb{Z}(U \times \mathbb{A}^1/S)), E_{et}(\Omega_{/S}^{\bullet, \Gamma, pr} \mathcal{H}om(T((f \circ p_X)_\sharp, (f \circ p_X)_*)(-), -), F_{DR})) & \xrightarrow{\quad} & p_{S*} E_{et}((\Omega_{X \times \mathbb{P}^1 \times S/S}^\bullet, F_b) \otimes_O j_!^{sa, Hdg}(O_{U \times \mathbb{A}^1 \times S}, F_b)) \end{array}$$

where,

- since $f : X \rightarrow S$ is surjective proper morphism, we have the equivalences (\mathbb{A}^1, et) local in $C(\text{Var}(\mathbb{C})/S)$

$$T(f_\sharp, f_*)(\mathbb{Z}(X/X)) := [\Delta_X] : f_\sharp(\mathbb{Z}(X/X)) \rightarrow f_* E_{et}(\mathbb{Z}(X/X))(d_X - d_S)[2d_X - 2d_S]$$

and

$$T(f_\sharp, f_*)(\mathbb{Z}(D_I/X)) := [\Delta_{D_I}] : f_\sharp(\mathbb{Z}(D_I/X)) \rightarrow f_* E_{et}(\mathbb{Z}(D_I/X))(d_X - d_S)[2d_X - 2d_S]$$

given in definition 42,

- since $f \circ p_X : X \times \mathbb{P}^1 \rightarrow S$ is a surjective proper morphism, we have the equivalences (\mathbb{A}^1, et) local in $C(\text{Var}(\mathbb{C})/S)$

$$\begin{aligned} T((f \circ p)_\sharp, (f \circ p)_*)(\mathbb{Z}(X \times \mathbb{P}^1/X \times \mathbb{P}^1)) &:= [\Delta_{X \times \mathbb{P}^1}] : (f \circ p)_\sharp(\mathbb{Z}(X \times \mathbb{P}^1/X \times \mathbb{P}^1)) \\ &\rightarrow (f \circ p)_* E_{et}(\mathbb{Z}(X \times \mathbb{P}^1/X \times \mathbb{P}^1))(d_X + 1 - d_S)[2d_X + 1 - 2d_S] \end{aligned}$$

and

$$\begin{aligned} T((f \circ p)_\sharp, (f \circ p)_*)(\mathbb{Z}(D_I \times \mathbb{P}^1/X \times \mathbb{P}^1)) &:= [\Delta_{D_I \times \mathbb{P}^1}] : (f \circ p)_\sharp(\mathbb{Z}(D_I \times \mathbb{P}^1/X \times \mathbb{P}^1)) \\ &\rightarrow (f \circ p)_* E_{et}(\mathbb{Z}(D_I \times \mathbb{P}^1/X \times \mathbb{P}^1))(d_X + 1 - d_S)[2d_X + 1 - 2d_S] \end{aligned}$$

given in definition 42.

Now, by proposition 104 and lemma 2, the two horizontal arrows of the right column are 2-filtered quasi-isomorphism. The arrow given by the composition of the two maps of the right column is a filtered quasi-isomorphism by proposition 94. Hence the arrow of the left column is a 2-filtered quasi-isomorphism which proves (ii). \square

Definition 117. (i) Let $S \in \text{SmVar}(\mathbb{C})$. We define using definition 116(i) and proposition 105(ii) the filtered algebraic De Rahm realization functor defined as

$$\begin{aligned} \mathcal{F}_S^{FDR} : \text{DA}_c(S) &\rightarrow D_{\mathcal{D}fil, \infty}(S), M \mapsto \mathcal{F}_S^{FDR}(M) := \\ &\varinjlim_{r_{(X^*, D^*)/S}(L(F))} e(S)_* \text{Gr}_{S^*}^{12} \mathcal{H}om^\bullet(\rho_{S^*} \text{Gr}_{S^*}^{12*} R_{(X^*, D^*)/S}(\rho_S^* L(F)), E_{et}(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR}))[-d_S] \\ &\xrightarrow{\sim} e(S)_* \text{Gr}_{S^*}^{12} \mathcal{H}om^\bullet(\rho_{S^*} \text{Gr}_{S^*}^{12*} R_{(X^*, D^*)/S}(\rho_S^* L(F)), E_{et}(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR}))[-d_S] \end{aligned}$$

where $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$.

(i)' For the Corti-Hanamura weight structure W on $\text{DA}_c(S)^-$, we define using definition 116(i) and proposition 105(ii)

$$\begin{aligned} \mathcal{F}_S^{FDR} : \text{DA}_c^-(S) &\rightarrow D_{\mathcal{D}^-(1,0)fil, \infty}(S), M \mapsto \mathcal{F}_S^{FDR}((M, W)) := \\ &\varinjlim_{r_{(X^*, D^*)/S}(L(F, W))} e(S)_* \text{Gr}_{S^*}^{12} \mathcal{H}om^\bullet(\rho_{S^*} \text{Gr}_{S^*}^{12*} R_{(X^*, D^*)/S}(\rho_S^* L(F, W)), E_{et}(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR}))[-d_S] \\ &\xrightarrow{\sim} e(S)_* \text{Gr}_{S^*}^{12} \mathcal{H}om^\bullet(\rho_{S^*} \text{Gr}_{S^*}^{12*} R_{(X^*, D^*)/S}(\rho_S^* L(F, W)), E_{et}(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR}))[-d_S] \end{aligned}$$

where $(F, W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)((F, W))$ using corollary 1. Note that the filtration induced by W is a filtration by sub D_S module, which is a stronger property than Griffithz transversality. Of course, the filtration induced by F satisfy only Griffithz transversality in general.

(ii) Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote by $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. We define, using definition 116(ii), proposition 105(ii) and corollary 5, the filtered algebraic De Rahm realization functor defined as

$$\begin{aligned} \mathcal{F}_S^{FDR} : \text{DA}_c(S) &\rightarrow D_{\mathcal{D}fil, \infty}(S/(\tilde{S}_I)), M \mapsto \mathcal{F}_S^{FDR}(M) := \\ &(\varinjlim_{r_-(\tilde{S}_I)} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\rho_{\tilde{S}_I^*} \text{Gr}_{\tilde{S}_I^*}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F)) \end{aligned}$$

where $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$, see definition 116.

(ii)' For the Corti-Hanamura weight structure W on $\text{DA}_c^-(S)$, using definition 116(ii), proposition 105(ii) and corollary 5,

$$\begin{aligned} \mathcal{F}_S^{FDR} : \text{DA}_c^-(S) &\rightarrow D_{\mathcal{D}^-(1,0)fil, \infty}(S/(\tilde{S}_I)), M \mapsto \mathcal{F}_S^{FDR}((M, W)) := \\ &(\varinjlim_{r_-(\tilde{S}_I)} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\rho_{\tilde{S}_I^*} \text{Gr}_{\tilde{S}_I^*}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* (F, W))), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F, W)) \end{aligned}$$

where $(F, W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $(M, W) = D(\mathbb{A}^1, et)(F, W)$ using corollary 1. Note that the filtration induced by W is a filtration by sub $D_{\tilde{S}_I}$ -modules, which is a stronger property than Griffithz transversality. Of course, the filtration induced by F satisfy only Griffithz transversality in general.

Proposition 106. For $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$, the functor \mathcal{F}_S^{FDR} is well defined.

Proof. Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. Let $M \in \text{DA}(S)$. Let $F, F' \in C(\text{Var}(\mathbb{C})^{sm}/S)$

such that $M = D(\mathbb{A}_1, et)(F) = D(\mathbb{A}_1, et)(F')$. Then there exist by definition a sequence of morphisms in $C(\text{Var}(\mathbb{C})^{sm}/S)$:

$$F = F_1 \xrightarrow{s_1} F_2 \xleftarrow{s_2} F_3 \xrightarrow{s_3} F_4 \rightarrow \dots \xrightarrow{s_l} F' = F_s$$

where, for $1 \leq k \leq s$, and s_k are (\mathbb{A}^1, et) local equivalence. But if $s : F_1 \rightarrow F_2$ is an equivalence (\mathbb{A}^1, et) local,

$$L(i_{I*}j_I^*s) : L(i_{I*}j_I^*F_1) \rightarrow L(i_{I*}j_I^*F_2)$$

is an equivalence (\mathbb{A}^1, et) local, hence

$$\begin{aligned} & \text{Hom}(\text{Gr}_{\tilde{S}_I}^{12*} R_{\tilde{S}_I}^{CH}(L(i_{I*}j_I^*s)), E_{et}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})) : \\ & (\varinjlim_{r_-(\tilde{S}_I)} e'(\tilde{S}_I)_* \text{Hom}(\rho_{\tilde{S}_I^*} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I^*}^* L(i_{I*}j_I^*F_1)), E_{et}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})), u_{IJ}^q(F_1)) \\ & \rightarrow (\varinjlim_{r_-(\tilde{S}_I)} e'(\tilde{S}_I)_* \text{Hom}(\rho_{\tilde{S}_I^*} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I^*}^* L(i_{I*}j_I^*F_2)), E_{et}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})), u_{IJ}^q(F_2)) \end{aligned}$$

is an ∞ -filtered quasi-isomorphism by proposition 105. \square

Let $f : X \rightarrow S$ a morphism with $S, X \in \text{Var}(\mathbb{C})$. Assume there exists a factorization

$$f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$$

of f , with l a closed embedding, $Y \in \text{SmVar}(\mathbb{C})$ and p_S the projection. Let $\bar{Y} \in \text{PSmVar}(\mathbb{C})$ a compactification of Y with $\bar{Y} \setminus Y = D$ a normal crossing divisor, denote $k : D \hookrightarrow \bar{Y}$ the closed embedding and $n : Y \hookrightarrow \bar{Y}$ the open embedding. Denote $\bar{X} \subset \bar{Y} \times S$ the closure of $X \subset Y \times S$. We have then the following commutative diagram in $\text{Var}(\mathbb{C})$

$$\begin{array}{ccccc} X & \xrightarrow{l} & Y \times S & & \\ \downarrow & & \downarrow & \searrow^{ps} & \\ \bar{X} & \xrightarrow{l} & \bar{Y} \times S & \xrightarrow{\bar{p}_S} & S \\ \uparrow & \nearrow^{l_Z} & \uparrow & \nearrow^{(k \times I)} & \\ Z := \bar{X} \setminus X & \longrightarrow & D \times S & & \end{array}$$

Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We have $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. Denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $X_I = \cap_{i \in I} X_i$. For $I \subset [1, \dots, l]$, denote by $\tilde{S}_I = \prod_{i \in I} \tilde{S}_i$. We then have, for $I \subset [1, \dots, l]$, closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I$ and for $I \subset J$, the following commutative diagrams which are cartesian

$$\begin{array}{ccccc} f_I = f|_{X_I} : X_I & \xrightarrow{l_I} & Y \times S_I & \xrightarrow{p_{S_I}} & S_I & , & Y \times \tilde{S}_J & \xrightarrow{p_{\tilde{S}_J}} & \tilde{S}_J \\ & & \downarrow i'_I & & \downarrow i_I & & \downarrow p'_{IJ} & & \downarrow p_{IJ} \\ & & Y \times \tilde{S}_I & \xrightarrow{p_{\tilde{S}_I}} & \tilde{S}_I & & Y \times \tilde{S}_I & \xrightarrow{p_{\tilde{S}_I}} & \tilde{S}_I \end{array}$$

with $l_I : l|_{X_I}$, $i'_I = I \times i_I$, p_{S_I} and $p_{\tilde{S}_I}$ are the projections and $p'_{IJ} = I \times p_{IJ}$, and we recall that we denote by $j_I : \tilde{S}_I \setminus S_I \hookrightarrow \tilde{S}_I$ and $j'_I : Y \times \tilde{S}_I \setminus X_I \hookrightarrow Y \times S_I$ the open complementary embeddings. We then have the commutative diagrams

$$\begin{array}{ccc} D_{IJ} = S_J & \xrightarrow{i_J} & \tilde{S}_J \\ \downarrow j_{IJ} & & \downarrow p_{IJ} \\ S_I & \xrightarrow{i_I} & \tilde{S}_I \end{array} \quad , \quad \begin{array}{ccc} D'_{IJ} = X_J & \xrightarrow{i'_J \circ l_J} & Y \times \tilde{S}_J \\ \downarrow j'_{IJ} & & \downarrow p'_{IJ} \\ X_I & \xrightarrow{i'_I \circ l_I} & Y \times \tilde{S}_I \end{array}$$

and the factorization of $D'_{I,J}$ by the fiber product:

$$\begin{array}{ccccc}
D'_{I,J} = X_J & \xrightarrow{i'_I \circ l_I} & Y \times \tilde{S}_J & , & D'_{I,J} = X_J & \xrightarrow{i'_I \circ l_I} & Y \times \tilde{S}_J \\
\downarrow j'_{I,J} & & \downarrow p'_{I,J} & & \downarrow j'_{I,J} & & \downarrow p'_{I,J} \\
X_I & \xrightarrow{i'_I \circ l_I} & Y \times \tilde{S}_I & & X_I & \xrightarrow{i'_I \circ l_I} & Y \times \tilde{S}_I \\
& & & & \downarrow j'_{I,J} & & \downarrow p'_{I,J} \\
& & & & X_I & \xrightarrow{i'_I \circ l_I} & Y \times \tilde{S}_I \\
& & & & \uparrow \hat{p}_{I,J} & & \uparrow \hat{i}_I \\
& & & & X_I \times_{Y \times \tilde{S}_I} Y \times \tilde{S}_J = X_I \times \tilde{S}_{J \setminus I} & & Y \times \tilde{S}_J \\
& & & & \downarrow \hat{i}_J & & \downarrow \hat{i}_I \\
& & & & X_I & \xrightarrow{i'_I \circ l_I} & Y \times \tilde{S}_I
\end{array} \tag{57}$$

where $j'_{I,J} : X_J \hookrightarrow X_I$ is the open embedding. Consider

$$F(X/S) := p_{S,\#} \Gamma_X^\vee \mathbb{Z}(Y \times S/Y \times S) \in C(\text{Var}(\mathbb{C})^{sm}/S)$$

so that $D(\mathbb{A}^1, et)(F(X/S)) = M(X/S)$ since Y is smooth. Then, by definition,

$$\begin{aligned}
\mathcal{F}_S^{FDR}(M(X/S)) &:= \left(\lim_{r-(-)} e'(\tilde{S}_I)_* \mathcal{H}om(\rho_{\tilde{S}_I^*} \text{Gr}_{\tilde{S}_I}^{12*} R_{(\tilde{Y} \times \tilde{S}_I)^*, E^*} / \tilde{S}_I}(\rho_{\tilde{S}_I^*}^* L(i_{I*} j_I^* F(X/S))), \right. \\
&\quad \left. E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, pr}, F_{DR})[-d_{\tilde{S}_I}], u_{I,J}^q(F(X/S)) \right)
\end{aligned}$$

On the other hand, let

$$Q(X_I/\tilde{S}_I) := p_{\tilde{S}_I,\#} \Gamma_{X_I}^\vee \mathbb{Z}(Y \times \tilde{S}_I/Y \times \tilde{S}_I) \in C(\text{Var}(\mathbb{C})^{sm}/\tilde{S}_I),$$

We have then for $I \subset [1, l]$ the map (51) in $C(\text{Var}(\mathbb{C})^{sm}/\tilde{S}_J)$:

$$\begin{aligned}
N_I(X/S) : Q(X_I/\tilde{S}_I) &= p_{\tilde{S}_I,\#} \Gamma_{X_I}^\vee \mathbb{Z}(Y \times \tilde{S}_I/Y \times \tilde{S}_I) \xrightarrow{\text{ad}(i'_I, i'_{I*})(-)} \\
&\xrightarrow{p_{\tilde{S}_I,\#} i'_{I*} i'_I \Gamma_{X_I}^\vee \mathbb{Z}(Y \times S_I/Y \times S_I)} \xrightarrow{T(i'_I, \gamma^\vee)(-)} p_{\tilde{S}_I,\#} i'_{I*} \Gamma_{X_I}^\vee \mathbb{Z}(Y \times S_I/Y \times S_I) \\
&\xrightarrow{\hat{T}_i(p_{S_I}, i_I)(-)} i_{I*} p_{S_I,\#} \Gamma_{X_I}^\vee \mathbb{Z}(Y \times S_I/Y \times S_I) = i_{I*} j_I^* F(X/S).
\end{aligned}$$

We then have the commutative diagram in $C(\text{Var}(\mathbb{C})^{sm}/\tilde{S}_J)$

$$\begin{array}{ccc}
p_{I,J}^* Q(X_I/\tilde{S}_I) & \xrightarrow{p_{I,J}^* N_I(X/S)} & p_{I,J}^* L i_{I*} j_I^* F(X/S) \\
\downarrow H_{I,J} & & \downarrow T^q(D_{I,J})(j_I^* F(X/S)) \\
Q(X_J/\tilde{S}_J) & \xrightarrow{N_J(X/S)} & L i_{J*} j_J^* F(X/S)
\end{array} \tag{58}$$

with

$$\begin{aligned}
H_{I,J} : p_{I,J}^* p_{\tilde{S}_I,\#} \Gamma_{X_I}^\vee \mathbb{Z}(Y \times \tilde{S}_I/Y \times \tilde{S}_I) &\xrightarrow{T_i(p_{I,J}, p_{\tilde{S}_I})(-)^{-1}} p_{\tilde{S}_J,\#} p_{I,J}^* \Gamma_{X_I}^\vee \mathbb{Z}(Y \times \tilde{S}_I/Y \times \tilde{S}_I) \\
&\xrightarrow{p_{\tilde{S}_J,\#} T(p_{I,J}, \gamma^\vee)(-)} p_{\tilde{S}_J,\#} \Gamma_{X_I \times \tilde{S}_{J \setminus I}}^\vee \mathbb{Z}(Y \times \tilde{S}_J/Y \times \tilde{S}_J) \xrightarrow{p_{\tilde{S}_J,\#} T(X_J/X_I \times \tilde{S}_{J \setminus I}, \gamma^\vee)(-)} p_{\tilde{S}_J,\#} \Gamma_{X_J}^\vee \mathbb{Z}(Y \times \tilde{S}_J/Y \times \tilde{S}_J).
\end{aligned}$$

The diagram 58 say that the maps $N_I(X/S)$ induces a map in $C(\text{Var}(\mathbb{C})^{sm}/(S/\tilde{S}_I))$

$$\begin{aligned}
(N_I(X/S)) : (Q(X_I/\tilde{S}_I), I(p_{I,J}^*, p_{I,J*})(-, -)(H_{I,J})) \\
\rightarrow (L i_{J*} j_J^* F(X/S), I(p_{I,J}^*, p_{I,J*})(-, -)(T^q(D_{I,J})(j_I^* F(X/S)))).
\end{aligned}$$

Denote $\bar{X}_I := \bar{X} \cap (\bar{Y} \times S_I) \subset \bar{Y} \times \tilde{S}_I$ the closure of $X_I \subset \bar{Y} \times \tilde{S}_I$, and $Z_I := Z \cap (\bar{Y} \times S_I) = \bar{X}_I \setminus X_I$. Consider for $I \subset [1, \dots, l]$ and $I \subset J$ the following commutative diagrams in $\text{Var}(\mathbb{C})$

$$\begin{array}{ccc}
X_I & \xrightarrow{l_I} & Y \times \tilde{S}_I \\
\downarrow & & \downarrow (n \times I) \\
\bar{X}_I & \xrightarrow{l_I} & \bar{Y} \times \tilde{S}_I \\
\uparrow & \nearrow l_{Z_I} & \uparrow (k \times I) \\
Z_I = \bar{X}_I \setminus X_I & \longrightarrow & D \times \tilde{S}_I
\end{array}
, \quad
\begin{array}{ccc}
\bar{Y} \times \tilde{S}_J & \xrightarrow{p_{\tilde{S}_J}} & \tilde{S}_J \\
p'_{IJ} \downarrow & & \downarrow p_{IJ} \\
\bar{Y} \times \tilde{S}_I & \xrightarrow{p_{\tilde{S}_I}} & \tilde{S}_I
\end{array}$$

Let $\epsilon_1 : ((\bar{Y} \times \tilde{S}_I)_1, E_1) \rightarrow (\bar{Y} \times \tilde{S}_I, Z_I)$ a desingularization of the pair $(\bar{Y} \times \tilde{S}_I, Z_I)$, $\epsilon_2 : ((\bar{Y} \times \tilde{S}_I)_2, E_2) \rightarrow (\bar{Y} \times \tilde{S}_I, \bar{X}_I)$ a desingularization of the pair $(\bar{Y} \times \tilde{S}_I, \bar{X}_I)$ and a morphism $\epsilon_{12} : (\bar{Y} \times \tilde{S}_I)_2 \rightarrow (\bar{Y} \times \tilde{S}_I)_1$ such that the following diagram commutes (see definition-proposition 8) :

$$\begin{array}{ccc}
(\bar{Y} \times \tilde{S}_I)_2 & \xrightarrow{\epsilon_{12}} & (\bar{Y} \times \tilde{S}_I)_1 \\
\downarrow \epsilon_2 & & \downarrow \epsilon_1 \\
\bar{Y} \times \tilde{S}_I & \xrightarrow{=} & \bar{Y} \times \tilde{S}_I
\end{array}$$

We have then the two canonical maps in $C(\text{Var}(\mathbb{C})^{2,smpr}/(\tilde{S}_I))$

$$\begin{aligned}
& \rho_{\tilde{S}_I^*} R_{((\bar{Y} \times \tilde{S}_I)^*, E^*)/\tilde{S}_I}(\rho_{\tilde{S}_I^*}^* Q(X_I/\tilde{S}_I)) \\
:= & \rho_{\tilde{S}_I^*} \text{Gr}_{\tilde{S}_I}^{12*}(\text{Cone}(\text{Cone}(p_{\tilde{S}_I^*} \epsilon_{1*} E_{et}(l_{Z_I \bullet}(\mathbb{Z}(E_{1\bullet}/E_{1\bullet})), u_{IJ})) \rightarrow p_{\tilde{S}_I^*} \epsilon_{1*} E_{et}(\mathbb{Z}((\bar{Y} \times \tilde{S}_I)_1/(\bar{Y} \times \tilde{S}_I)_1))) \\
& \rightarrow \text{Cone}(\bar{p}_{\tilde{S}_I^*} \epsilon_{2*} E_{et}(l_{I \bullet}(\mathbb{Z}(E_{2\bullet}/E_{2\bullet})), u_{IJ})) \rightarrow p_{\tilde{S}_I^*} \epsilon_{2*} E_{et}(\mathbb{Z}((\bar{Y} \times \tilde{S}_I)_2/(\bar{Y} \times \tilde{S}_I)_2))) \\
& \xleftarrow{(T((\bar{p}_{\tilde{S}_I} \circ \epsilon_1)_\#, (\bar{p}_{\tilde{S}_I} \circ \epsilon_1)_*)(-), T((p_{\tilde{S}_I} \circ \epsilon_1)_\#, (p_{\tilde{S}_I} \circ \epsilon_1)_*)(-))} \\
& \rho_{\tilde{S}_I^*} \text{Gr}_{\tilde{S}_I}^{12*}(\text{Cone}(\text{Cone}(p_{\tilde{S}_I \#} \epsilon_{1\#} l_{Z_I \bullet}(\mathbb{Z}(E_{1\bullet}/E_{1\bullet})), u_{IJ}) \rightarrow p_{\tilde{S}_I \#} \epsilon_{1\#} \mathbb{Z}((\bar{Y} \times \tilde{S}_I)_1/(\bar{Y} \times \tilde{S}_I)_1))) \\
& \rightarrow \text{Cone}(p_{\tilde{S}_I \#} \epsilon_{2\#} l_{I \bullet}(\mathbb{Z}(E_{2\bullet}/E_{2\bullet})), u_{IJ}) \rightarrow p_{\tilde{S}_I \#} \epsilon_{2\#} \mathbb{Z}((\bar{Y} \times \tilde{S}_I)_2/(\bar{Y} \times \tilde{S}_I)_2)))[-2d_Y] \\
& \xrightarrow{=} \text{Cone}(\mathbb{Z}((E_{1\bullet} \times \tilde{S}_I, E_{1\bullet})/\tilde{S}_I), u_{IJ}) \rightarrow \mathbb{Z}((E_{2\bullet} \times \tilde{S}_I, E_{2\bullet})/\tilde{S}_I), u_{IJ}) \\
& \xrightarrow{I_\delta((\bar{X}_I, Z_I)/\tilde{S}_I) := (\mathbb{Z}(l_{Z_I} \circ \epsilon_1 \times I), \mathbb{Z}(l_I \circ \epsilon_2 \times I))} \\
& \text{Cone}(\mathbb{Z}((\bar{Y} \times \tilde{S}_I, Z_I)/\tilde{S}_I) \rightarrow \mathbb{Z}((\bar{Y} \times \tilde{S}_I, \bar{X}_I)/\tilde{S}_I)))[-2d_Y]
\end{aligned}$$

where,

- since $p_{\tilde{S}_I} \circ \epsilon_1 : (\bar{Y} \times \tilde{S}_I)_1 \rightarrow \tilde{S}_I$ is a surjective proper morphism, we have the equivalences (\mathbb{A}^1, et) local in $C(\text{Var}(\mathbb{C})/\tilde{S}_I)$

$$T((p_{\tilde{S}_I} \circ \epsilon_1)_\#, (p_{\tilde{S}_I} \circ \epsilon_1)_*)(\mathbb{Z}((\bar{Y} \times \tilde{S}_I)_1/(\bar{Y} \times \tilde{S}_I)_1)) : (p_{\tilde{S}_I} \circ \epsilon_1)_\# \mathbb{Z}((\bar{Y} \times \tilde{S}_I)_1/(\bar{Y} \times \tilde{S}_I)_1) \rightarrow (p_{\tilde{S}_I} \circ \epsilon_1)_* E_{et}(\mathbb{Z}((\bar{Y} \times \tilde{S}_I)_1/(\bar{Y} \times \tilde{S}_I)_1))$$

and

$$T((p_{\tilde{S}_I} \circ \epsilon_1)_\#, (p_{\tilde{S}_I} \circ \epsilon_1)_*)(\mathbb{Z}(E_{1\bullet}/(\bar{Y} \times \tilde{S}_I)_1)) : (p_{\tilde{S}_I} \circ \epsilon_1)_\# \mathbb{Z}(E_{1\bullet}/(\bar{Y} \times \tilde{S}_I)_1) \rightarrow (p_{\tilde{S}_I} \circ \epsilon_1)_* E_{et}(\mathbb{Z}(E_{1\bullet}/(\bar{Y} \times \tilde{S}_I)_1))(d_Y)[2d_Y]$$

given in definition 42,

- since $p_{\tilde{S}_I} \circ \epsilon_2 : (\bar{Y} \times \tilde{S}_I)_2 \rightarrow \tilde{S}_I$ is a surjective proper morphism, we have the equivalences (\mathbb{A}^1, et) local in $C(\text{Var}(\mathbb{C})/\tilde{S}_I)$

$$T((p_{\tilde{S}_I} \circ \epsilon_2)_\#, (p_{\tilde{S}_I} \circ \epsilon_2)_*)(\mathbb{Z}((\bar{Y} \times \tilde{S}_I)_2/(\bar{Y} \times \tilde{S}_I)_1)) : (p_{\tilde{S}_I} \circ \epsilon_2)_\# \mathbb{Z}((\bar{Y} \times \tilde{S}_I)_2/(\bar{Y} \times \tilde{S}_I)_2) \rightarrow (p_{\tilde{S}_I} \circ \epsilon_2)_* E_{et}(\mathbb{Z}((\bar{Y} \times \tilde{S}_I)_2/(\bar{Y} \times \tilde{S}_I)_2))$$

and

$$T((p_{\tilde{S}_I} \circ \epsilon_2)_\# , (p_{\tilde{S}_I} \circ \epsilon_2)_*) (\mathbb{Z}(E_{2\bullet} / (\bar{Y} \times \tilde{S}_I)_2)) : (p_{\tilde{S}_I} \circ \epsilon_2)_\# \mathbb{Z}(E_{2\bullet} / (\bar{Y} \times \tilde{S}_I)_2) \rightarrow (p_{\tilde{S}_I} \circ \epsilon_2)_* E_{et}(\mathbb{Z}(E_{2\bullet} / (\bar{Y} \times \tilde{S}_I)_2))(d_Y)[2]$$

given in definition 42.

We denote by $v_{IJ}^q(F(X/S))$ the composite

$$\begin{aligned} v_{IJ}^q(F(X/S))[d_{\tilde{S}_J}] : e'(\tilde{S}_I)_* \mathcal{H}om(\rho_{\tilde{S}_I}^* \text{Gr}_{\tilde{S}_I}^{12*} R_{((\bar{Y} \times \tilde{S}_I)^*, E^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* Q(X_I/\tilde{S}_I)), E_{et}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})) \\ \xrightarrow{p_{IJ*} T(p_{IJ}, \Omega_{\tilde{S}_I}^{\Gamma, pr})(-) \circ \text{ad}(p_{IJ}^{*mod}, p_{IJ})(-)} \\ e'(\tilde{S}_J)_* \mathcal{H}om(\rho_{\tilde{S}_J}^* \text{Gr}_{\tilde{S}_J}^{12*} p_{IJ}^* R_{((\bar{Y} \times \tilde{S}_I)^*, E^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* Q(X_I/\tilde{S}_I)), E_{et}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})) \\ \xrightarrow{\mathcal{H}om(T(p_{IJ}, R^{CH})(Q(X_I/\tilde{S}_I))^{-1}, E_{et}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))} \\ p_{IJ*} e'(\tilde{S}_J)_* \mathcal{H}om(\rho_{\tilde{S}_J}^* \text{Gr}_{\tilde{S}_J}^{12*} R_{((\bar{Y} \times \tilde{S}_I)^* \times \tilde{S}_{J \setminus I}, E^* \times \tilde{S}_{J \setminus I})/\tilde{S}_J}(\rho_{\tilde{S}_J}^* p_{IJ}^* Q(X_I/\tilde{S}_I)), E_{et}(\Omega_{\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR})) \\ \xrightarrow{\mathcal{H}om(\rho_{\tilde{S}_J}^* \text{Gr}_{\tilde{S}_J}^{12*} R_{\tilde{S}_J}^{CH}(H_{IJ}), E_{et}(\Omega_{\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR}))} \\ p_{IJ*} e'(\tilde{S}_J)_* \mathcal{H}om(\text{Gr}_{\tilde{S}_J}^{12*} R_{((\bar{Y} \times \tilde{S}_I)^*, E^*)/\tilde{S}_J}(\rho_{\tilde{S}_J}^* Q(X_J/\tilde{S}_J)), E_{et}(\Omega_{\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR})). \end{aligned}$$

On the other hand, we have in $\pi_{\bar{X}}(C(MHM(\bar{X}))) \subset C_{\mathcal{D}fil}(\bar{X}/(\bar{Y} \times \tilde{S}_I))$

$$\begin{aligned} (\text{Cone}(T(Z_I/\bar{X}_I, \gamma^{\vee, Hdg})(-))) : (\Gamma_{\bar{X}_I}^{\vee, Hdg}(O_{\bar{Y} \times \tilde{S}_I}, F_b), x_{IJ}(\bar{X}/S)) \rightarrow (\Gamma_{Z_I}^{\vee, Hdg}(O_{\bar{Y} \times \tilde{S}_I}, F_b), x_{IJ}(Z/S)) \\ \xrightarrow{\cong} (n \times I)!^{Hdg}(\Gamma_{X_I}^{\vee, Hdg}(O_{Y \times \tilde{S}_I}, F_b), x_{IJ}(X/S)) \end{aligned}$$

with

- for the closed embedding $\bar{X} \subset \bar{Y} \times S$ we consider the map in $\pi_{\bar{Y} \times \tilde{S}_J}(C(MHM(\bar{Y} \times \tilde{S}_J)))$

$$\begin{aligned} x_{IJ}(\bar{X}/S) : \Gamma_{\bar{X}_I}^{\vee, Hdg}(O_{\bar{Y} \times \tilde{S}_I}, F_b) \xrightarrow{\text{ad}(p_{IJ}^{\prime *mod}, p_{IJ}^{\prime *})(-)} p_{IJ*} p_{IJ}^{\prime *mod} \Gamma_{\bar{X}_I}^{\vee, Hdg}(O_{\bar{Y} \times \tilde{S}_I}, F_b) \\ \xrightarrow{T(\bar{X}_J/p_{IJ}^{\prime -1}(\bar{X}_I), \gamma^{\vee})(-) \circ T(p_{IJ}^{\prime}, \gamma^{\vee})(-)} \Gamma_{\bar{X}_J}^{\vee, Hdg}(O_{\bar{Y} \times \tilde{S}_J}, F_b), \end{aligned}$$

- for the closed embedding $Z \subset \bar{Y} \times S$ we consider the map in $\pi_{\bar{Y} \times \tilde{S}_J}(C(MHM(\bar{Y} \times \tilde{S}_J)))$

$$\begin{aligned} x_{IJ}(Z/S) : \Gamma_{Z_I}^{\vee, Hdg}(O_{\bar{Y} \times \tilde{S}_I}, F_b) \xrightarrow{\text{ad}(p_{IJ}^{\prime *mod}, p_{IJ}^{\prime *})(-)} p_{IJ*} p_{IJ}^{\prime *mod} \Gamma_{Z_I}^{\vee, Hdg}(O_{\bar{Y} \times \tilde{S}_I}, F_b) \\ \xrightarrow{T(Z_J/p_{IJ}^{\prime -1}(Z_I), \gamma^{\vee})(-) \circ T(p_{IJ}^{\prime}, \gamma^{\vee})(-)} \Gamma_{Z_J}^{\vee, Hdg}(O_{\bar{Y} \times \tilde{S}_J}, F_b), \end{aligned}$$

- for the closed embedding $X \subset Y \times S$ we consider the map in $\pi_{Y \times \tilde{S}_J}(C(MHM(Y \times \tilde{S}_J)))$

$$\begin{aligned} x_{IJ}(X/S) : \Gamma_{X_I}^{\vee, Hdg}(O_{Y \times \tilde{S}_I}, F_b) \xrightarrow{\text{ad}(p_{IJ}^{\prime *mod}, p_{IJ}^{\prime *})(-)} p_{IJ*} p_{IJ}^{\prime *mod} \Gamma_{X_I}^{\vee, Hdg}(O_{Y \times \tilde{S}_I}, F_b) \\ \xrightarrow{T(X_J/X_I \times \tilde{S}_{J \setminus I}, \gamma^{\vee, Hdg})(-) \circ T(p_{IJ}^{\prime}, \gamma^{\vee})(-)} \Gamma_{X_J}^{\vee, Hdg}(O_{Y \times \tilde{S}_J}, F_b). \end{aligned}$$

The maps $x_{IJ}(X/S)$ gives the following maps in $C_{\mathcal{D}fil, S_J}(\tilde{S}_J)$

$$\begin{aligned} w_{IJ}(X/S)[2d_Y] : p_{\tilde{S}_I*} E_{zar}((\Omega_{\bar{Y} \times \tilde{S}_I/\tilde{S}_I}^{\bullet}, F_b) \otimes_{O_{\bar{Y} \times \tilde{S}_I}} (n \times I)!^{Hdg} \Gamma_{X_I}^{\vee, Hdg}(O_{Y \times \tilde{S}_I}, F_b)) \\ \xrightarrow{\text{ad}(p_{IJ}^{\prime *mod}, p_{IJ})(-)} p_{IJ*} p_{IJ}^{\prime *mod} p_{\tilde{S}_I*} E_{zar}((\Omega_{\bar{Y} \times \tilde{S}_I/\tilde{S}_I}^{\bullet}, F_b) \otimes_{O_{Y \times \tilde{S}_I}} (n \times I)!^{Hdg} \Gamma_{X_I}^{\vee, Hdg}(O_{Y \times \tilde{S}_I}, F_b)) \\ \xrightarrow{p_{IJ*} T_w^O(p_{IJ}, p_{\tilde{S}_I})(-)} p_{\tilde{S}_J*} E_{zar}((\Omega_{\bar{Y} \times \tilde{S}_J/\tilde{S}_J}^{\bullet}, F_b) \otimes_{O_{\bar{Y} \times \tilde{S}_J}} p_{IJ}^{\prime *mod} (n \times I)!^{Hdg} \Gamma_{X_I}^{\vee, Hdg}(O_{Y \times \tilde{S}_I}, F_b)) \\ \xrightarrow{(p_{\tilde{S}_J*} E(DR(-)(x_{IJ}(X/S)))} \rightarrow p_{\tilde{S}_J*} E_{zar}((\Omega_{\bar{Y} \times \tilde{S}_J/\tilde{S}_J}^{\bullet}, F_b) \otimes_{O_{\bar{Y} \times \tilde{S}_J}} (n \times I)!^{Hdg} \Gamma_{X_J}^{\vee, Hdg}(O_{\bar{Y} \times \tilde{S}_J}, F_b)). \end{aligned}$$

We have then the following lemma

Lemma 15. (i) The map in $C(\text{Var}(\mathbb{C})^{sm}/(S/\tilde{S}_I))$

$$(N_I(X/S)) : (Q(X_I/\tilde{S}_I), H_{IJ}) \rightarrow (L(i_{I*}j_I^*F(X/S)), T^q(D_{IJ})(F(X/S))).$$

is an equivalence (\mathbb{A}^1, et) local.

(ii) The maps $(N_I(X/S))$ induces an ∞ -filtered quasi-isomorphism in $C_{\mathcal{D}fil}(S/(\tilde{S}_I))$

$$\begin{aligned} & (\text{Hom}(\text{Gr}_{\tilde{S}_I}^{12*} R_{\tilde{S}_I}^{CH}(N_I(X/S)), E_{et}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr}, F_b))) : \\ & (\varinjlim_{r-(-)} e'(\tilde{S}_I)_* \mathcal{H}om(\rho_{\tilde{S}_I} \text{Gr}_{\tilde{S}_I}^{12*} R_{(-,-)/-}(\rho_{\tilde{S}_I}^* L(i_{I*}j_I^*F(X/S))), E_{et}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F(X/S))) \\ & \rightarrow (\varinjlim_{r-(-)} e'(\tilde{S}_I)_* \mathcal{H}om(\rho_{\tilde{S}_I} \text{Gr}_{\tilde{S}_I}^{12*} R_{(-,-)/-}(\rho_{\tilde{S}_I}^* Q(X_I/\tilde{S}_I)), E_{et}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], v_{IJ}^q(F(X/S))) \end{aligned}$$

(iii) The maps $(I_\delta((\bar{X}_I, Z_I)/\tilde{S}_I))$ induce an ∞ -filtered Zariski local equivalence in $C_{\mathcal{D}fil}(S/(\tilde{S}_I))$

$$\begin{aligned} & (\text{Hom}(\rho_{\tilde{S}_I} I_\delta((\bar{X}_I, Z_I)/\tilde{S}_I), k) : \\ & (p_{\tilde{S}_I} E_{zar}((\Omega_{\bar{Y} \times \tilde{S}_I/\tilde{S}_I}^{\bullet}, F_b) \otimes_{O_{\bar{Y} \times \tilde{S}_I}} (n \times I)_!^{Hdg}(\Gamma_{\bar{X}_I}^{\vee, Hdg}(O_{Y \times \tilde{S}_I}, F_b))[-2d_Y - d_{\tilde{S}_I}], w_{IJ}(X/S)) \\ & \rightarrow (e'(\tilde{S}_I)_* \mathcal{H}om(\rho_{\tilde{S}_I} \text{Gr}_{\tilde{S}_I}^{12*} R_{((\bar{Y} \times \tilde{S}_I)^*, E^*)/\tilde{S}_J}(\rho_{\tilde{S}_I}^* Q(X_I/\tilde{S}_I)), E_{et}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], v_{IJ}^q(F(X/S))) \end{aligned}$$

Proof. (i): See lemma 12(i)

(ii): These maps induce a morphism in $C_{\mathcal{D}}(S/(\tilde{S}_I))$ by construction. It is an ∞ -filtered quasi-isomorphism by (i) and proposition 105.

(iii): The fact that these maps define a morphism in $C_{\mathcal{D}}(S/(\tilde{S}_I))$ follows from the commutative diagrams in $C(\text{Var}(\mathbb{C})^{2, smpr}/\tilde{S}_J)$ for $I \subset J$

$$\begin{array}{ccc} p_{IJ}^* \rho_{\tilde{S}_I} \text{Gr}_{\tilde{S}_I}^{12*} R_{((\bar{Y} \times \tilde{S}_I)^*, E^*)/\tilde{S}_J}(\rho_{\tilde{S}_I}^* Q(X_I/\tilde{S}_I)) & \xrightarrow{p_{IJ}^* I_\delta((\bar{X}_I, Z_I)/\tilde{S}_I)} & p_{IJ}^* \text{Cone}(\mathbb{Z}((\bar{Y} \times \tilde{S}_I, Z_I)/\tilde{S}_I) \rightarrow \mathbb{Z}((\bar{Y} \times \tilde{S}_I, \bar{X}_I)/\tilde{S}_I))[d_Y] \\ \downarrow T(p_{IJ}, R^{CH})(-) & & \downarrow = \\ \rho_{\tilde{S}_J} \text{Gr}_{\tilde{S}_J}^{12*} R_{((\bar{Y} \times \tilde{S}_J)^*, E^*)/\tilde{S}_J}(\rho_{\tilde{S}_J}^* p_{IJ}^* Q(X_I/\tilde{S}_I)) & \xrightarrow{I_\delta((p_{IJ}^{\prime -1}(\bar{X}_I), p_{IJ}^{\prime -1}(Z_I))/\tilde{S}_I)} & \text{Cone}(\mathbb{Z}((\bar{Y} \times \tilde{S}_J, p_{IJ}^{\prime -1}(Z_I))/\tilde{S}_J) \rightarrow \mathbb{Z}((\bar{Y} \times \tilde{S}_J, p_{IJ}^{\prime -1}(\bar{X}_I)/\tilde{S}_J))[d_Y] \\ \uparrow R^{CH}(H_{IJ}) & & \uparrow (\mathbb{Z}(\bar{Y} \times \tilde{S}_J), \mathbb{Z}(\bar{Y} \times \tilde{S}_J)) \\ \rho_{\tilde{S}_J} \text{Gr}_{\tilde{S}_J}^{12*} R_{((\bar{Y} \times \tilde{S}_J)^*, E^*)/\tilde{S}_J}(\rho_{\tilde{S}_J}^* Q(X_J/\tilde{S}_J)) & \xrightarrow{I_\delta((\bar{X}_J, Z_J)/\tilde{S}_J)} & \text{Cone}(\mathbb{Z}((\bar{Y} \times \tilde{S}_J, Z_J)/\tilde{S}_J) \rightarrow \mathbb{Z}((\bar{Y} \times \tilde{S}_J, \bar{X}_J)/\tilde{S}_J))[d_Y] \end{array}$$

On the other hand, it is an ∞ -filtered quasi-isomorphism by proposition 93 since we have by Yoneda lemma the following commutative diagram

$$\begin{array}{ccc} p_{\tilde{S}_I} E_{zar}((\Omega_{\bar{Y} \times \tilde{S}_I/\tilde{S}_I}^{\bullet}, F_b) \otimes_{O_{\bar{Y} \times \tilde{S}_I}} (n \times I)_!^{Hdg}(\Gamma_{\bar{X}_I}^{\vee, Hdg}(O_{\bar{Y} \times \tilde{S}_I}, F_b))[-2d_Y]) & \xrightarrow{\sim} & \text{Cone}(p_{\tilde{S}_I} E_{zar}(\Omega_{(E_{2\bullet} \times \tilde{S}_I/\tilde{S}_I)}^{\bullet}, F_b) \otimes_{E_{2\bullet} \times \tilde{S}_I} (\Gamma_{E_{2\bullet}}^{\vee, Hdg}(O_{E_{2\bullet} \times \tilde{S}_I}, F_b))) \rightarrow \\ & & p_{\tilde{S}_I} E_{zar}(\Omega_{(E_{1\bullet} \times \tilde{S}_I/\tilde{S}_I)}^{\bullet}, F_b) \otimes_{E_{1\bullet} \times \tilde{S}_I} (\Gamma_{E_{2\bullet}}^{\vee, Hdg}(O_{E_{1\bullet} \times \tilde{S}_I}, F_b)))[-2d_Y] \\ & \searrow \text{Hom}(\rho_{\tilde{S}_I} I_\delta((\bar{X}_I, Z_I)/\tilde{S}_I), k) & \downarrow \\ & & (e'(\tilde{S}_I)_* \mathcal{H}om(\rho_{\tilde{S}_I} \text{Gr}_{\tilde{S}_I}^{12*} R_{((\bar{Y} \times \tilde{S}_I)^*, E^*)/\tilde{S}_J}(\rho_{\tilde{S}_I}^* Q(X_I/\tilde{S}_I)), E_{et}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))) \end{array}$$

and on the other hand by proposition 104 and lemma 2 since

$$(T((p_{\tilde{S}_I} \circ \epsilon_1)_\# , (p_{\tilde{S}_I} \circ \epsilon_1)_*)(-), T((p_{\tilde{S}_I} \circ \epsilon_2)_\# , (p_{\tilde{S}_I} \circ \epsilon_2)_*)(-))$$

is an equivalence (\mathbb{A}^1, et) local. \square

Proposition 107. *Let $f : X \rightarrow S$ a morphism with $S, X \in \text{Var}(\mathbb{C})$. Assume there exist a factorization*

$$f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$$

of f with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $\tilde{Y} \in \text{PSmVar}(\mathbb{C})$ a compactification of Y with $\tilde{Y} \setminus Y = D$ a normal crossing divisor, denote $k : D \hookrightarrow \tilde{Y}$ the closed embedding and $n : Y \hookrightarrow \tilde{Y}$ the open embedding. Denote $\tilde{X} \subset \tilde{Y} \times S$ the closure of $X \subset \tilde{Y} \times S$. We have then the following commutative diagram in $\text{Var}(\mathbb{C})$

$$\begin{array}{ccccc} X & \xrightarrow{l} & Y \times S & & \\ \downarrow & & \downarrow & \searrow^{p_S} & \\ \tilde{X} & \xrightarrow{l} & \tilde{Y} \times S & \xrightarrow{\tilde{p}_S} & S \\ \uparrow & \nearrow^{l_Z} & \uparrow & \nearrow^{(k \times I)} & \\ Z := \tilde{X} \setminus X & \longrightarrow & D \times S & & \end{array} .$$

Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. Denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $X_I = \cap_{i \in I} X_i$. Denote $\tilde{X}_I := \tilde{X} \cap (\tilde{Y} \times S_I) \subset \tilde{Y} \times \tilde{S}_I$ the closure of $X_I \subset \tilde{Y} \times \tilde{S}_I$, and $Z_I := Z \cap (\tilde{Y} \times S_I) = \tilde{X}_I \setminus X_I \subset \tilde{Y} \times \tilde{S}_I$. We have then for $I \subset [1, \dots, l]$, the following commutative diagram in $\text{Var}(\mathbb{C})$

$$\begin{array}{ccccc} X_I & \xrightarrow{l_I} & Y \times \tilde{S}_I & & \\ \downarrow & & \downarrow & \searrow^{p_{\tilde{S}_I}} & \\ \tilde{X}_I & \xrightarrow{l_I} & \tilde{Y} \times \tilde{S}_I & \xrightarrow{\tilde{p}_{\tilde{S}_I}} & \tilde{S}_I \\ \uparrow & \nearrow^{l_{Z_I}} & \uparrow & \nearrow^{(k \times I)} & \\ Z_I = \tilde{X}_I \setminus X_I & \longrightarrow & D \times \tilde{S}_I & & \end{array} .$$

Let $F(X/S) := p_{S,\#} \Gamma_X^\vee \mathbb{Z}(Y \times S/Y \times S) \in C(\text{Var}(\mathbb{C})^{sm}/S)$. We have then the following isomorphism in $D_{\mathcal{D}fil, \infty}(S/(\tilde{S}_I))$

$$\begin{aligned} I(X/S) : \mathcal{F}_S^{FDR}(M(X/S)) &\xrightarrow{=} \\ (\varinjlim_{r-(-)} e'(\tilde{S}_I)_* \text{Hom}(\rho_{\tilde{S}_I^*} \text{Gr}_{\tilde{S}_I}^{12*} R_{((\tilde{Y} \times \tilde{S}_I)^*, E^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} J_I^* F(X/S))), & \\ E_{et}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], u_{IJ}^q(F(X/S))) &\xrightarrow{(\text{Hom}(\text{Gr}_{\tilde{S}_I}^{12*} R_{\tilde{S}_I}^{QH}(N_I(X/S)), E_{et}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr}, F_b)))} \\ (\varinjlim_{r-(-)} e'(\tilde{S}_I)_* \text{Hom}(\rho_{\tilde{S}_I^*} \text{Gr}_{\tilde{S}_I}^{12*} R_{((\tilde{Y} \times \tilde{S}_I)^*, E^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* Q(X_I/\tilde{S}_I))), & \\ E_{et}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], v_{IJ}^q(F(X/S))) &\xrightarrow{(\text{Hom}(\rho_{\tilde{S}_I^*} I_\delta((\tilde{X}_I, Z_I)/\tilde{S}_I), k)[-d_{\tilde{S}_I}])^{-1}} \\ (p_{\tilde{S}_I^*} E_{zar}((\Omega_{\tilde{Y} \times \tilde{S}_I/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_b) \otimes_{O_{Y \times \tilde{S}_I}} (n \times I)! \text{Hdg} \Gamma_{X_I}^{\vee, \text{Hdg}}(O_{Y \times \tilde{S}_I}, F_b))[-2d_Y - d_{\tilde{S}_I}], w_{IJ}(X/S)) & \\ \xrightarrow{=} \iota_S R f_{I*} \text{Hdg}(\Gamma_{X_I}^{\vee, \text{Hdg}}(O_{Y \times \tilde{S}_I}, F_b), x_{IJ}(X/S)). & \end{aligned}$$

Proof. Follows from lemma 15. \square

Corollary 5. *Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $D(\mathbb{A}^1, et)(F) \in \text{DA}_c(S)$, $u_{IJ}^q(F)$ are ∞ -filtered Zariski local equivalence.*

Proof. Follows from the resolution of a constructible motive by Corti-Hanamura motives and proposition 107. By theorem 15, M is isomorphic in $DA(S)$ to a generalized distinguish triangle of Corti-Hanamura motives, i.e.

$$M \xrightarrow{\sim} \text{Cone}(M(X_0/S)[d_0] \rightarrow \cdots \rightarrow M(X_n/S)[d_n])$$

where $f_0 : X_0 \rightarrow S, \dots, f_n : X_n \rightarrow S$ are projective morphism with $X_0, \dots, X_n \in \text{SmVar}(\mathbb{C})$. By lemma 15(ii) and (iii), $u_{IJ}^q(F(X_r/S))$ are ∞ -filtered Zariski local equivalence since $w_{IJ}(X_r/S)$ are ∞ -filtered Zariski local equivalences. \square

Corollary 6. *Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, for $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1, et)(F) \in \text{DA}_c(S)$,*

$$\begin{aligned} H^i \mathcal{F}_S^{FDR}(M, W) &:= ((e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\rho_{\tilde{S}_I^*} \text{Gr}_{\tilde{S}_I}^{12*} R_{((\tilde{Y} \times \tilde{S}_I)^*, E^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I^*} j_I^*(F, W))))), \\ &E_{et}(\Omega_{\tilde{S}_I}^{\bullet, pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F, W)) \in \pi_S(\text{MHM}(S)) \end{aligned}$$

for all $i \in \mathbb{Z}$.

Proof. Follows from the resolution of a constructible motive by Corti-Hanamura motives and proposition 107. By theorem 15, M is isomorphic in $DA(S)$ to a generalized distinguish triangle of Corti-Hanamura motives, i.e.

$$M \xrightarrow{\sim} \text{Cone}(M(X_0/S)[d_0] \rightarrow \cdots \rightarrow M(X_n/S)[d_n])$$

where $f_0 : X_0 \rightarrow S, \dots, f_n : X_n \rightarrow S$ are projective morphism with $X_0, \dots, X_n \in \text{SmVar}(\mathbb{C})$. For $0 \leq r \leq n$, consider a factorization $f_r : X_r \xrightarrow{l_r} Y \times S \xrightarrow{p_S} S$, where l_r is a closed embedding, $Y \in \mathbb{P}^N$ and p_S the projection. Then,

$$\begin{aligned} \mathcal{F}_S^{FDR}(M(X_r/S)) &= (p_{\tilde{S}_I^*}((\Omega_{Y \times \tilde{S}_I/\tilde{S}_I}^\bullet, F_b) \otimes_{O_{Y \times \tilde{S}_I}} \Gamma_{X_r}^{\vee, Hdg}(O_{Y \times \tilde{S}_I}, F_b))[-2d_Y - d_{\tilde{S}_I}], w_{IJ}(X_r/S)) \\ &= \iota_S R f_{r^*}^{Hdg}(O_{X_r}, F_b) \in D_{\mathcal{D}fil, \infty}(S) \end{aligned}$$

by proposition 107. We conclude by theorem 27 since f_r are projective. \square

Proposition 108. *For $S \in \text{Var}(\mathbb{C})$ not smooth, the functor (see corollary 6)*

$$\iota_S^{-1} \mathcal{F}_S^{FDR} : \text{DA}_c^-(S)^{op} \rightarrow \pi_S(D(\text{MHM}(S)))$$

does not depend on the choice of the open cover $S = \cup_i S_i$ and the closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$.

Proof. Let $S = \cup_{i=l+1}^{i=l'} S_i$ is an other open cover together with closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with

$\tilde{S}_i \in \text{SmVar}(\mathbb{C})$ for $l+1 \leq i \leq l'$. Then, for $J' \subset I' \subset [l+1, \dots, l'] = L'$ and $J \subset I \subset L = [1, \dots, l]$,

$$\begin{aligned}
& T_S^{L/L'} (\iota_S^{-1} (\varinjlim_{r_-(\cdot)} e'(\tilde{S}_I) * \mathcal{H}om(\rho_{\tilde{S}_I} * \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}(F))) \\
& := ((\text{ho} \lim_{I \in L} p_{I'(I \sqcup I')} * \Gamma_{\tilde{S}_{I \sqcup I'}}^{\vee, Hdg} p_{I(I \sqcup I')}^{*mod[-]}) (\iota_S^{-1} (\varinjlim_{r_-(\cdot)} e'(\tilde{S}_I) * \mathcal{H}om(\rho_{\tilde{S}_I} * \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), \\
& \quad E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], T_S^{L/L'}(u_{IJ}(F))))_I \\
& \quad \xrightarrow{(\text{ho} \lim_{I \in L} u_{I(I \sqcup I')}(F))} \\
& (\text{ho} \lim_{I \in L} p_{I'(I \sqcup I')} * \Gamma_{\tilde{S}_{I \sqcup I'}}^{\vee, Hdg} p_{I(I \sqcup I')}^{*mod[-]} p_{I(I \sqcup I')} \varinjlim_{r_-(\cdot)} e'(\tilde{S}_{(I \sqcup I')}) * \mathcal{H}om(\rho_{\tilde{S}_I} * \text{Gr}_{\tilde{S}_{I \sqcup I'}}^{12*} \\
& R_{(X^*, D^*)/\tilde{S}_{I \sqcup I'}}(\rho_{\tilde{S}_I}^* L(i_{(I \sqcup I')*} j_{(I \sqcup I')}^* F)), E_{et}(\Omega_{/\tilde{S}_{I \sqcup I'}}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_{I \sqcup I'}}], u_{(I \sqcup I')(I \sqcup I')}(F)) \\
& \quad \xrightarrow{\text{ad}(p_{I(I \sqcup I')}, p_{I(I \sqcup I')})(-) \circ \gamma_{\tilde{S}_{I \sqcup I'}}^{\vee, Hdg}(-)} \\
& (\text{ho} \lim_{I \in L} p_{I'(I \sqcup I')} \varinjlim_{r_-(\cdot)} e'(\tilde{S}_{(I \sqcup I')}) * \mathcal{H}om(\rho_{\tilde{S}_I} * \text{Gr}_{\tilde{S}_{I \sqcup I'}}^{12*} R_{(X^*, D^*)/\tilde{S}_{I \sqcup I'}}(\rho_{\tilde{S}_I}^* L(i_{(I \sqcup I')*} j_{(I \sqcup I')}^* F)), \\
& \quad E_{et}(\Omega_{/\tilde{S}_{I \sqcup I'}}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_{I \sqcup I'}}], u_{(I \sqcup I')(I \sqcup I')}(F)) \xleftarrow{(\text{ho} \lim_{I \in L} u_{I(I \sqcup I')}(F))} \\
& (\varinjlim_{r_-(\cdot)} e'(\tilde{S}_{I'}) * \mathcal{H}om(\rho_{\tilde{S}_{I'}} * \text{Gr}_{\tilde{S}_{I'}}^{12*} R_{(X^*, D^*)/\tilde{S}_{I'}}(\rho_{\tilde{S}_{I'}}^* L(i_{I'*} j_{I'}^* F)), E_{et}(\Omega_{/\tilde{S}_{I'}}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_{I'}}], u_{I'J'}(F))
\end{aligned}$$

is an ∞ -filtered Zariski local equivalence, since all the morphisms are ∞ -filtered Zariski local equivalences by corollary 5 and proposition 91. \square

We have the canonical transformation map between the filtered De Rham realization functor and the Gauss-Manin realization functor :

Definition 118. Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Let $M \in \text{DA}_c(S)$ and $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1, et)(F)$. We have, using definition 112(ii), the canonical map in $D_{O_S \text{fil}, \mathcal{D}, \infty}(S/(\tilde{S}_I))$

$$\begin{aligned}
& T(\mathcal{F}_S^{GM}, \mathcal{F}_S^{FDR})(M) : \\
& \mathcal{F}_S^{GM}(\text{LD}_S M) := (e(\tilde{S}_I) * \mathcal{H}om^\bullet(L(i_{I*} j_I^* \mathbb{D}_S L F), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)), u_{IJ}^q(F)) \\
& \quad \xrightarrow{\sim} (e(\tilde{S}_I) * \mathcal{H}om^\bullet(\text{LD}_{\tilde{S}_I} L(i_{I*} j_I^* F), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)), u_{IJ}^{q,d}(F)) \\
& \quad \xrightarrow{\mathcal{H}om(-, \text{Gr}(\Omega_{\tilde{S}_I}))^{-1}} (e(\tilde{S}_I) * \mathcal{H}om^\bullet(\text{LD}_{\tilde{S}_I} L(i_{I*} j_I^* F), \text{Gr}_{\tilde{S}_I}^{12} E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})), u_{IJ}^{q,d}(F)) \\
& \quad \xrightarrow{I(\text{Gr}_{\tilde{S}_I}^{12*}, \text{Gr}_{\tilde{S}_I}^{12})(-, -)} (e'(\tilde{S}_I) * \mathcal{H}om^\bullet(\text{Gr}_{\tilde{S}_I}^{12*} \text{LD}_{\tilde{S}_I} L(i_{I*} j_I^* F), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})), u_{IJ}^{q,d}(F)) \\
& \quad \xrightarrow{(\mathcal{H}om^\bullet(\rho_{\tilde{S}_I} * \text{Gr}_{\tilde{S}_I}^{12*} r_{(X^*, D^*)/\tilde{S}_I}^{CH}(L(i_{I*} j_I^* F)), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})))} \\
& (\varinjlim_{r_-(\cdot)} e'(\tilde{S}_I) * \mathcal{H}om(\rho_{\tilde{S}_I} * \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F)) \\
& \quad =: \mathcal{F}_S^{FDR}(M)
\end{aligned}$$

We now define the functorialities of \mathcal{F}_S^{FDR} with respect to S which makes \mathcal{F}_{FDR}^- a morphism of 2 functor.

Definition 119. Let $S \in \text{Var}(\mathbb{C})$. Let $Z \subset S$ a closed subset. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Denote $Z_I := Z \cap S_I$. We then have closed embeddings $Z_I \hookrightarrow S_I \hookrightarrow \tilde{S}_I$.

(i) For $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$, we will consider the following canonical map in $\pi_S(D(\text{MHM}(S))) \subset D_{\mathcal{D}(1,0)\text{fil}}(S/(\tilde{S}_I))$

$$\begin{aligned}
& T(\Gamma_Z^{\vee, Hdg}, \Omega_{/S}^{\Gamma, pr})(F, W) : \\
& \Gamma_Z^{\vee, Hdg} \iota_S^{-1} \left(\varinjlim_{r-(-)} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\rho_{\tilde{S}_I^*} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I^*}^* L(i_{I^*} j_I^*(F, W))), \right. \\
& \quad \left. E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], u_{IJ}^q(F, W) \right) \\
& \quad \xrightarrow{\mathcal{H}om^\bullet(\rho_{\tilde{S}_I^*} \text{Gr}_{\tilde{S}_I}^{12*} r^{CH}(\gamma^{\vee, Z_I}(L(i_{I^*} j_I^*(F, W))), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))} \\
& \Gamma_Z^{\vee, Hdg} \iota_S^{-1} \left(\varinjlim_{r-(-)} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\rho_{\tilde{S}_I^*} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^{*,*}, D^{*,*})/\tilde{S}_I}(\rho_{\tilde{S}_I^*}^* \Gamma_{Z_I}^{\vee} L(i_{I^*} j_I^*(F, W))), \right. \\
& \quad \left. E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], u_{IJ}^{q, Z}(F, W) \right) \\
& \xrightarrow{=} \iota_S^{-1} \left(\varinjlim_{r-(-)} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\rho_{\tilde{S}_I^*} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^{*,*}, D^{*,*})/\tilde{S}_I}(\rho_{\tilde{S}_I^*}^* \Gamma_{Z_I}^{\vee} L(i_{I^*} j_I^*(F, W))), \right. \\
& \quad \left. E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], u_{IJ}^{q, Z}(F, W) \right).
\end{aligned}$$

with

$$\begin{aligned}
& u_{IJ}^{q, Z}(F)[d_{\tilde{S}_I}] : e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\rho_{\tilde{S}_I^*} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I^*}^* \Gamma_{Z_I}^{\vee} L(i_{I^*} j_I^*(F))), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})) \\
& \quad \xrightarrow{p_{IJ*} T(p_{IJ}, \Omega^{\gamma, pr})(-) \circ \text{ad}(p_{IJ}^{*mod}, p_{IJ})(-)} \\
& \quad p_{IJ*} e'(\tilde{S}_J)_* \mathcal{H}om^\bullet(\text{Gr}_{\tilde{S}_J}^{12*} p_{IJ}^* R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I^*}^* \Gamma_{Z_I}^{\vee} L(i_{I^*} j_I^*(F))), E_{et}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR})) \\
& \quad \xrightarrow{\mathcal{H}om(\text{Gr}_{\tilde{S}_J}^{12*} T(p_{IJ}, R^{CH})(L(i_{I^*} j_I^*(F))^{-1}), E_{et}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR}))} \\
& p_{IJ*} e'(\tilde{S}_J)_* \mathcal{H}om^\bullet(\rho_{\tilde{S}_J^*} \text{Gr}_{\tilde{S}_J}^{12*} R_{(X^* \times \tilde{S}_{J \setminus I}, D^* \times \tilde{S}_{J \setminus I})/\tilde{S}_J}(\rho_{\tilde{S}_J^*}^* p_{IJ}^* \Gamma_{Z_I}^{\vee} L(i_{I^*} j_I^*(F))), E_{et}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR})) \\
& \quad \xrightarrow{\mathcal{H}om(\rho_{\tilde{S}_J^*} \text{Gr}_{\tilde{S}_J}^{12*} R_{\tilde{S}_J}^{CH}(T^q(D_{IJ})(j_I^* F) \circ T(Z_J/Z_I \times \tilde{S}_{J \setminus I}, \gamma^{\vee})(-) \circ T(p_{IJ}, \gamma)(-)), E_{et}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR}))} \\
& p_{IJ*} e'(\tilde{S}_J)_* \mathcal{H}om^\bullet(\rho_{\tilde{S}_J^*} \text{Gr}_{\tilde{S}_J}^{12*} R_{(X^* \times \tilde{S}_{J \setminus I}, D^* \times \tilde{S}_{J \setminus I})/\tilde{S}_J}(\rho_{\tilde{S}_J^*}^* \Gamma_{Z_J}^{\vee} L(i_{J^*} j_J^*(F))), E_{et}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR})).
\end{aligned}$$

(ii) For $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$, we have also the following canonical map in $\pi_S(D(\text{MHM}(S))) \subset D_{\mathcal{D}(1,0)\text{fil}}(S/(\tilde{S}_I))$

$$\begin{aligned}
& T(\Gamma_Z^{Hdg}, \Omega_{/S}^{\Gamma, pr})(F, W) : \\
& \iota_S^{-1} \left(\varinjlim_{r-(-)} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\rho_{\tilde{S}_I^*} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^{*,*}, D^{*,*})/\tilde{S}_I}(\rho_{\tilde{S}_I^*}^* L\Gamma_{Z_I} E(i_{I^*} j_I^* \mathbb{D}_S(F, W))), \right. \\
& \quad \left. E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], u_{IJ}^{q, Z, d}(F, W) \right) \\
& \xrightarrow{=} \Gamma_Z^{Hdg} \iota_S^{-1} \left(\varinjlim_{r-(-)} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\rho_{\tilde{S}_I^*} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^{*,*}, D^{*,*})/\tilde{S}_I}(\rho_{\tilde{S}_I^*}^* L\Gamma_{Z_I} E(i_{I^*} j_I^* \mathbb{D}_S(F, W))), \right. \\
& \quad \left. E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], u_{IJ}^{q, Z, d}(F, W) \right) \\
& \quad \xrightarrow{\mathcal{H}om^\bullet(\rho_{\tilde{S}_I^*} \text{Gr}_{\tilde{S}_I}^{12*} r^{CH}(\gamma^{Z_I}(-)), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))} \\
& \Gamma_Z^{Hdg} \iota_S^{-1} \left(\varinjlim_{r-(-)} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\rho_{\tilde{S}_I^*} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I^*}^* L(i_{I^*} j_I^* \mathbb{D}_S(F, W))), \right. \\
& \quad \left. E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], u_{IJ}^q(F, W) \right)
\end{aligned}$$

with

$$\begin{aligned}
& u_{I,J}^{q,Z}(F)[d_{\tilde{S}_I}] : e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\rho_{\tilde{S}_I}^* \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L\Gamma_{Z_I} E(i_{I*} j_I^* \mathbb{D}_S L F)), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})) \\
& \xrightarrow{p_{IJ*} T(p_{IJ}, \Omega^{\gamma, pr})(-) \circ \text{ad}(p_{IJ}^{*mod}, p_{IJ})(-)} \\
& p_{IJ*} e'(\tilde{S}_J)_* \mathcal{H}om^\bullet(\text{Gr}_{\tilde{S}_J}^{12*} p_{IJ}^* R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L\Gamma_{Z_I} E(i_{I*} j_I^* \mathbb{D}_S L F)), E_{et}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR})) \\
& \xrightarrow{\mathcal{H}om(\text{Gr}_{\tilde{S}_J}^{12*} T(p_{IJ}, R^{CH})(Li_{I*} j_I^* F)^{-1}, E_{et}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR}))} \\
& p_{IJ*} e'(\tilde{S}_J)_* \mathcal{H}om^\bullet(\rho_{\tilde{S}_J}^* \text{Gr}_{\tilde{S}_J}^{12*} R_{(X^* \times \tilde{S}_{J \setminus I}, D^* \times \tilde{S}_{J \setminus I})/\tilde{S}_J}(\rho_{\tilde{S}_J}^* L p_{IJ}^* \Gamma_{Z_I} E(i_{I*} j_I^* \mathbb{D}_S L F)), E_{et}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR})) \\
& \xrightarrow{\mathcal{H}om(\rho_{\tilde{S}_J}^* \text{Gr}_{\tilde{S}_J}^{12*} R_{\tilde{S}_J}^{CH}(\mathbb{D}_{\tilde{S}_J} S^q(D_{I,J})(\mathbb{D}_S L F)), E_{et}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR}))} \\
& p_{IJ*} e'(\tilde{S}_J)_* \mathcal{H}om^\bullet(\rho_{\tilde{S}_J}^* \text{Gr}_{\tilde{S}_J}^{12*} R_{(X^* \times \tilde{S}_{J \setminus I}, D^* \times \tilde{S}_{J \setminus I})/\tilde{S}_J}(\rho_{\tilde{S}_J}^* \Gamma_{Z_J} E(i_{J*} j_J^* \mathbb{D}_S L F)), E_{et}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR})).
\end{aligned}$$

This transformation map will, with the projection case, gives the transformation between the pullback functor :

Definition 120. Let $g : T \rightarrow S$ a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Consider the factorization $g : T \xrightarrow{l} T \times S \xrightarrow{p_S} S$ where l is the graph embedding and p_S the projection. Let $M \in \text{DA}_{\mathbb{C}}(S)^-$ and $(F, W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ such that $(M, W) = D(\mathbb{A}_S^1, et)(F, W)$. Then, $D(\mathbb{A}_T^1, et)(g^* F) = g^* M$ and there exist $(F', W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/T \times S)$ and an equivalence (\mathbb{A}^1, et) local $e : \Gamma^\vee p_S^*(F, W) \rightarrow (F', W)$ such that $D(\mathbb{A}_{T \times S}^1, et)(F', W) = (\Gamma^\vee p_S^* M, W)$. We have then the canonical transformation in $\pi_T(D(MHM(T)))$ using definition 114 and definition 119(i) :

$$\begin{aligned}
& T(g, \mathcal{F}^{FDR})(M) : g^{\hat{*}mod, Hdg} \mathcal{F}_S^{FDR}(M) := \\
& \Gamma_T^{\vee, Hdg} \iota_T^{-1} (p_S^{*mod[-]}) \left(\varinjlim_{r(X^*, D^*)/S(-)} e(S)_* \text{Gr}_{S^*}^{12*} \mathcal{H}om^\bullet(\rho_{S^*} \text{Gr}_S^{12*} R_{(X^*, D^*)/S}(\rho_S^* L(F, W)), \right. \\
& \quad \left. E_{et}(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR}))[-d_S] \right) \xrightarrow{T(p_S, \Omega_{/S}^{\Gamma, pr})(-)} \\
& \Gamma_T^{\vee, Hdg} \left(\left(\varinjlim_{r(X^*, D^*)/S(-)} e'(T \times S)_* \mathcal{H}om^\bullet(\rho_{T \times S^*} \text{Gr}_{T \times S}^{12*} p_S^* R_{(X^*, D^*)/S}(\rho_S^* L(F)), \right. \right. \\
& \quad \left. \left. E_{et}(\Omega_{/T \times S}^{\bullet, \Gamma, pr}, F_{DR}))[-d_S] \right) \xrightarrow{\mathcal{H}om(T(p_S, R^{CH})(L(F, W))^{-1}, E_{et}(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR}))[-d_S]} \\
& \Gamma_T^{\vee, Hdg} \iota_T^{-1} \left(\varinjlim_{r(X^*, D^*)/S(-)} e'(T \times S)_* \mathcal{H}om^\bullet(\rho_{T \times S^*} \text{Gr}_{T \times S}^{12*} R_{(T \times X^*, T \times D^*)/T \times S}(\rho_{T \times S}^* p_S^* L(F, W)), \right. \\
& \quad \left. E_{et}(\Omega_{/T \times S}^{\bullet, \Gamma, pr}, F_{DR}))[-d_S] \right) \xrightarrow{=} \\
& \Gamma_T^{\vee, Hdg} \iota_T^{-1} \left(\varinjlim_{r(X'^*, D'^*)/T \times S(-)} e'(T \times S)_* \mathcal{H}om^\bullet(\rho_{T \times S^*} \text{Gr}_{T \times S}^{12*} R_{(X'^*, D'^*)/T \times S}(\rho_{T \times S}^* p_S^* L(F, W)), \right. \\
& \quad \left. E_{et}(\Omega_{/T \times S}^{\bullet, \Gamma, pr}, F_{DR}))[-d_S] \right) \xrightarrow{T(\Gamma_T^{\vee, Hdg}, \Omega_{/T \times S}^{\Gamma, pr})(F, W)} \\
& \left(\varinjlim_{r(X'^*, D'^*)/T \times S(-)} e'(T \times S)_* \mathcal{H}om^\bullet(\rho_{T \times S^*} \text{Gr}_{T \times S}^{12*} R_{(X'^*, D'^*)/T \times S}(\rho_{T \times S}^* \Gamma_T^\vee p_S^* L(F, W)), \right. \\
& \quad \left. E_{et}(\Omega_{/T \times S}^{\bullet, \Gamma, pr}, F_{DR}))[-d_S] \right) \xrightarrow{\mathcal{H}om(\mathcal{R}_{T \times S}^{CH}(e), -)} \\
& \left(\varinjlim_{r(X'^*, D'^*)/T \times S(-)} e'(T \times S)_* \mathcal{H}om^\bullet(\rho_{T \times S^*} \text{Gr}_{T \times S}^{12*} R_{(X'^*, D'^*)/T \times S}(\rho_{T \times S}^* L(F', W)), \right. \\
& \quad \left. E_{et}(\Omega_{/T \times S}^{\bullet, \Gamma, pr}, F_{DR}))[-d_S] \right) =: \mathcal{F}_{T \times S}^{FDR}(l_* g^* M) = \mathcal{F}_T^{FDR}(g^* M)
\end{aligned}$$

where the last equality follows from proposition 108.

We give now the definition in the non smooth case Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$, Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I := g|_{T_I} : T_I \rightarrow S_I$. We recall the commutative diagram :

$$E_{IJg} = (Y \times \tilde{S}_I) \setminus T_I \xrightarrow{m'_I} Y \times \tilde{S}_J, E_{IJ} = \begin{array}{ccc} \tilde{S}_J \setminus S_J & \xrightarrow{m_J} & \tilde{S}_J \\ \downarrow p_{IJ} & & \downarrow p_{IJ} \\ \tilde{S}_I \setminus (S_I \setminus S'_J) & \xrightarrow{m_{IJ}} & \tilde{S}_I \end{array} \quad E'_{IJ} = \begin{array}{ccc} (Y \times \tilde{S}_J) \setminus T_J & \xrightarrow{m'_J} & Y \times \tilde{S}_I \\ \downarrow p'_{IJ} & & \downarrow p'_{IJ} \\ (Y \times \tilde{S}_I) \setminus (T_I \setminus T'_J) & \xrightarrow{m'_{IJ}} & Y \times \tilde{S}_I \end{array}$$

For $I \subset J$, denote by $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ and $p'_{IJ} := I_Y \times p_{IJ} : Y \times \tilde{S}_J \rightarrow Y \times \tilde{S}_I$ the projections, so that $\tilde{g}_I \circ p'_{IJ} = p_{IJ} \circ \tilde{g}_J$. Consider, for $I \subset J \subset [1, \dots, l]$, resp. for each $I \subset [1, \dots, l]$, the following commutative diagrams in $\text{Var}(\mathbb{C})$

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ j_{IJ} \uparrow & & \uparrow p_{IJ} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}, D'_{IJ} = \begin{array}{ccc} T_I & \xrightarrow{i'_I} & Y \times \tilde{S}_I \\ j'_{IJ} \uparrow & & \uparrow p'_{IJ} \\ T_J & \xrightarrow{i'_J} & Y \times \tilde{S}_J \end{array} \quad D_{gI} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ g_I \uparrow & & \uparrow \tilde{g}_I \\ T_I & \xrightarrow{i'_I} & Y \times \tilde{S}_I \end{array},$$

and $j_{IJ} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. Let $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$. Recall (see section 2) that since $j'_I{}^* i'_I{}^* j'_I{}^* g^* F = 0$, the morphism $T(D_{gI})(j'_I{}^* F) : \tilde{g}_I^* i_{I*} j'_I{}^* F \rightarrow i'_{I*} j'_I{}^* g^* F$ factors through

$$T(D_{gI})(j'_I{}^* F) : \tilde{g}_I^* i_{I*} j'_I{}^* F \xrightarrow{\gamma_{X_I}^\vee(-)} \Gamma_{X_I}^\vee \tilde{g}_I^* i_{I*} j'_I{}^* F \xrightarrow{T^\gamma(D_{gI})(j'_I{}^* F)} i'_{I*} j'_I{}^* g^* F.$$

and that the fact that the diagrams (56) commutes says that the maps $T^{q,\gamma}(D_{gI})(j'_I{}^* F)$ define a morphism in $C(\text{Var}(\mathbb{C})^{sm}/(T/(Y \times \tilde{S}_I)))$

$$(T^{q,\gamma}(D_{gI})(j'_I{}^* F)) : (\Gamma_{T_I}^\vee \tilde{g}_I^* L(i_{I*} j'_I{}^* F), T^q(D_{IJ})(j'_I{}^* F) \circ T(T_I/T_I \times \tilde{S}_J \setminus I, \gamma^\vee)(-) \circ T(p'_{IJ}, \gamma^\vee)(-)) \\ \rightarrow (L(i'_{I*} j'_I{}^* g^* F), T^q(D'_{IJ})(j'_I{}^* g^* F))$$

Denote for short $d_{YI} := -d_Y - d_{\tilde{S}_I}$. We denote by $\tilde{g}_J^* u_{IJ}^q(F)$ the composite

$$\begin{aligned} & \tilde{g}_J^* u_{IJ}^q(F)[-d_{YJ}] : \\ & e'(Y \times \tilde{S}_I)_* \mathcal{H}om(\rho_{Y \times \tilde{S}_I} \text{Gr}_{Y \times \tilde{S}_I}^{12*} R_{(X'^*, E^*)/Y \times \tilde{S}_I}(\rho_{Y \times \tilde{S}_I}^* \tilde{g}_I^* L(i_{I*} j'_I{}^* F)), E_{et}(\Omega_{Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})) \\ & \xrightarrow{p'_{IJ*} T(p'_{IJ}, \Omega^{\Gamma, pr})(-) \circ \text{ad}(p'_{IJ*} \text{mod}, p'_{IJ*})(-)} \\ & p'_{IJ*} e'(Y \times \tilde{S}_I)_* \mathcal{H}om(\rho_{Y \times \tilde{S}_J} \text{Gr}_{Y \times \tilde{S}_J}^{12*} p'_{IJ*} R_{(X'^*, E^*)/Y \times \tilde{S}_I}(\rho_{Y \times \tilde{S}_I}^* \tilde{g}_I^* L(i_{I*} j'_I{}^* F)), E_{et}(\Omega_{Y \times \tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR})) \\ & \xrightarrow{\mathcal{H}om(\rho_{Y \times \tilde{S}_J} \text{Gr}_{Y \times \tilde{S}_J}^{12*} T(p'_{IJ}, R^{CH})(-)^{-1}, E_{et}(\Omega_{Y \times \tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR}))} \\ & p'_{IJ*} e'(Y \times \tilde{S}_I)_* \mathcal{H}om(\rho_{Y \times \tilde{S}_J} \text{Gr}_{Y \times \tilde{S}_J}^{12*} R_{(-, -)/Y \times \tilde{S}_J}(\rho_{Y \times \tilde{S}_J}^* p'_{IJ*} \tilde{g}_I^* L(i_{I*} j'_I{}^* F)), E_{et}(\Omega_{Y \times \tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR})) \\ & \xrightarrow{\mathcal{H}om(\rho_{Y \times \tilde{S}_J} \text{Gr}_{Y \times \tilde{S}_J}^{12*} R_{Y \times \tilde{S}_J}^{CH}(T^q(D_{IJ})(j'_I{}^* F)), E_{et}(\Omega_{Y \times \tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR}))} \\ & e'(Y \times \tilde{S}_J)_* \mathcal{H}om(\rho_{Y \times \tilde{S}_J} \text{Gr}_{Y \times \tilde{S}_J}^{12*} R_{(X'^*, E^*)/Y \times \tilde{S}_I}(\rho_{Y \times \tilde{S}_J}^* \tilde{g}_J^* L(i_{J*} j'_J{}^* F)), E_{et}(\Omega_{Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})) \end{aligned}$$

We denote by $\tilde{g}_J^{*,\gamma} u_{IJ}^q(F)$ the composite

$$\begin{aligned}
& \tilde{g}_J^{*,\gamma} u_{IJ}^q(F)[-d_{YJ}] : \\
& e'(Y \times \tilde{S}_I)_* \mathcal{H}om(\rho_{Y \times \tilde{S}_I} \text{Gr}_{Y \times \tilde{S}_I}^{12*} R_{(X'^*, E^*)/Y \times \tilde{S}_I}(\rho_{Y \times \tilde{S}_I}^* \Gamma_{T_I}^\vee \tilde{g}_I^* L(i_{I*} j_I^* F)), E_{et}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})) \\
& \xrightarrow{p'_{IJ*} T(p'_{IJ}, \Omega^{\Gamma, pr})(-) \circ \text{ad}(p'_{IJ*} \text{mod}, p'_{IJ*})(-)} \\
& p'_{IJ*} e'(Y \times \tilde{S}_I)_* \mathcal{H}om(\rho_{Y \times \tilde{S}_J} \text{Gr}_{Y \times \tilde{S}_J}^{12*} p'_{IJ*} R_{(X'^*, E^*)/Y \times \tilde{S}_I}(\rho_{Y \times \tilde{S}_I}^* \Gamma_{T_I}^\vee \tilde{g}_I^* L(i_{I*} j_I^* F)), E_{et}(\Omega_{/Y \times \tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR})) \\
& \xrightarrow{\mathcal{H}om(\rho_{Y \times \tilde{S}_J} \text{Gr}_{Y \times \tilde{S}_J}^{12*} R_{Y \times \tilde{S}_J}^{CH} (T^q(D_{IJ})(j_I^* F) \circ T(T_I/T_I \times \tilde{S}_J \setminus \gamma^\vee)(-) \circ T(p'_{IJ}, \gamma^\vee)(-)), E_{et}(\Omega_{/Y \times \tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR}))} \\
& p'_{IJ*} e'(Y \times \tilde{S}_I)_* \mathcal{H}om(\rho_{Y \times \tilde{S}_J} \text{Gr}_{Y \times \tilde{S}_J}^{12*} R_{(-, -)/Y \times \tilde{S}_J}(\rho_{Y \times \tilde{S}_J}^* p'_{IJ*} \Gamma_{T_I}^\vee \tilde{g}_I^* L(i_{I*} j_I^* F)), E_{et}(\Omega_{/Y \times \tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR})) \\
& \xrightarrow{\mathcal{H}om(\rho_{Y \times \tilde{S}_J} \text{Gr}_{Y \times \tilde{S}_J}^{12*} R_{Y \times \tilde{S}_J}^{CH} (T^q(D_{IJ})(j_I^* F) \circ T(T_I/T_I \times \tilde{S}_J \setminus \gamma^\vee)(-) \circ T(p'_{IJ}, \gamma^\vee)(-)), E_{et}(\Omega_{/Y \times \tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR}))} \\
& e'(Y \times \tilde{S}_J)_* \mathcal{H}om(\rho_{Y \times \tilde{S}_J} \text{Gr}_{Y \times \tilde{S}_J}^{12*} R_{(X'^*, E^*)/Y \times \tilde{S}_J}(\rho_{Y \times \tilde{S}_J}^* \Gamma_{T_J}^\vee \tilde{g}_J^* L(i_{J*} j_J^* F)), E_{et}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))
\end{aligned}$$

We then have then the following lemma :

Lemma 16. (i) *The morphism in $C(\text{Var}(\mathbb{C})^{sm}/(T/(Y \times \tilde{S}_I)))$*

$$\begin{aligned}
& (T^{q,\gamma}(D_{gI})(j_I^* F)) : (\Gamma_{T_I}^\vee L \tilde{g}_I^* i_{I*} j_I^* F, T^q(D_{IJ})(j_I^* F) \circ T(T_I/T_I \times \tilde{S}_J \setminus \gamma^\vee)(-) \circ T(p'_{IJ}, \gamma^\vee)(-)) \\
& \rightarrow (L i_{I*} j_I^* g^* F, T^q(D'_{IJ})(j_I^* g^* F))
\end{aligned}$$

is an equivalence (\mathbb{A}^1, et) local.

(ii) *The maps $\mathcal{H}om((T^{q,\gamma}(D_{gI})(j_I^* F)), E_{et}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))$ induce an ∞ -filtered quasi-isomorphism in $C_{\mathcal{D}fil}(T/(Y \times \tilde{S}_I))$*

$$\begin{aligned}
& (\mathcal{H}om(\rho_{Y \times \tilde{S}_I} \text{Gr}_{Y \times \tilde{S}_I}^{12*} R_{Y \times \tilde{S}_I}^{CH} (T^{q,\gamma}(D_{gI})(j_I^* F)), E_{et}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[d_{YI}] : \\
& (e'(-)_* \mathcal{H}om(\rho_{Y \times \tilde{S}_I} \text{Gr}_{Y \times \tilde{S}_I}^{12*} R_{(X^*, D^*)/-}(\rho_{Y \times \tilde{S}_I}^* \Gamma_{T_I}^\vee \tilde{g}_I^* L(i_{I*} j_I^* F)), E_{et}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[d_{YI}], \tilde{g}_J^{*,\gamma} u_{IJ}^q(F)) \\
& \rightarrow (e'(-)_* \mathcal{H}om(\rho_{Y \times \tilde{S}_I} \text{Gr}_{Y \times \tilde{S}_I}^{12*} R_{(X^*, D^*)/-}(\rho_{Y \times \tilde{S}_I}^* L(i'_{I*} j_I^* g^* F)), E_{et}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[d_{YI}], u_{IJ}^q(g^* F))
\end{aligned}$$

(iii) *The maps $T(\tilde{g}_I, \Omega_{/Y \times \tilde{S}_I}^{\Gamma, pr})(-)$ (see definition 114), induce a morphism in $C_{\mathcal{D}fil}(T/(Y \times \tilde{S}_I))$*

$$\begin{aligned}
& T(\tilde{g}_I, \Omega_{/Y \times \tilde{S}_I}^{\Gamma, pr})(\rho_{\tilde{S}_I} \text{Gr}_{\tilde{S}_I}^{12*} L(i_{I*} j_I^* F))[d_{YI}] : \\
& (\tilde{g}_I^{*mod} e'(\tilde{S}_I)_* \mathcal{H}om(\rho_{\tilde{S}_I} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[d_{YI}], \tilde{g}_J^{*mod} u_{IJ}^q(F)) \\
& \rightarrow (e'(-)_* \mathcal{H}om(\rho_{Y \times \tilde{S}_I} \text{Gr}_{Y \times \tilde{S}_I}^{12*} \tilde{g}_I^* R_{(X^*, D^*)/Y \times \tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{et}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[d_{YI}], \tilde{g}_J^* u_{IJ}^q(F)).
\end{aligned}$$

Proof. (i): Follows from theorem 14.

(ii): These morphism induce a morphism in $C_{\mathcal{D}fil}(T/(Y \times \tilde{S}_I))$ by construction. The fact that this morphism is an ∞ -filtered equivalence Zariski local follows from (i) and proposition 105.

(iii): These morphism induce a morphism in $C_{\mathcal{D}fil}(T/(Y \times \tilde{S}_I))$ by construction. \square

Definition 121. *Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{ps} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and ps the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$. Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I := g_{T_I} : T_I \rightarrow S_I$. Let $M \in \text{DA}_c(S)^-$ and $(F, W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ such that $(M, W) = D(\mathbb{A}_S^1, et)(F, W)$. Then, $D(\mathbb{A}_T^1, et)(g^* F) = g^* M$ and there exist $(F', W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$*

and an equivalence (\mathbb{A}^1, et) local $e : g^*(F, W) \rightarrow (F', W)$ such that $D(\mathbb{A}_T^1, et)(F', W) = (g^*M, W)$. We have, using definition 114 and definition 119(i), by lemma 16, the canonical map in $\pi_T(D(MHM(T))) \subset D_{\mathcal{D}(1,0)fil,\infty}(T/(Y \times \tilde{S}_I))$

$$\begin{aligned}
& T(g, \mathcal{F}^{FDR})(M) : g_{Hdg}^{*mod} \iota_S^{-1} \mathcal{F}_S^{FDR}(M) := \\
& (\Gamma_T^{\vee, Hdg} \iota_T^{-1} (\tilde{g}_I^{*mod} (\varinjlim_{r-(-)} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\rho_{\tilde{S}_I} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^*(L(i_{I^*} j_I^*(F, W)))) \\
& E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{YI}], \tilde{g}_J^{*mod} u_{IJ}^q(F, W)) \xrightarrow{(T(\tilde{g}_I, \Omega_{/\tilde{S}_I}^{\Gamma, pr})(\text{Gr}_{\tilde{S}_I}^{12*} L(i_{I^*} j_I^*(F, W))))} \\
& \Gamma_T^{\vee, Hdg} \iota_T^{-1} (\varinjlim_{r-(-)} e'(Y \times \tilde{S}_I)_* \mathcal{H}om(\rho_{Y \times \tilde{S}_I} \text{Gr}_{Y \times \tilde{S}_I}^{12*} \tilde{g}_I^* R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I^*} j_I^*(F, W)))) \\
& E_{et}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}), \tilde{g}_J^* u_{IJ}^q(F, W)) \xrightarrow{\mathcal{H}om(T(\tilde{g}_I, R^{CH})(-)^{-1}, -)} \\
& \Gamma_T^{\vee, Hdg} \iota_T^{-1} (\varinjlim_{r-(-)} e'(Y \times \tilde{S}_I)_* \mathcal{H}om(\rho_{Y \times \tilde{S}_I} \text{Gr}_{Y \times \tilde{S}_I}^{12*} R_{(Y \times X^*, Y \times D^*)/Y \times \tilde{S}_I}(\rho_{Y \times \tilde{S}_I}^* \tilde{g}_I^* L(i_{I^*} j_I^*(F, W)))) \\
& E_{et}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}), \tilde{g}_J^* u_{IJ}^q(F, W)) \\
& \xrightarrow{=} \Gamma_T^{\vee, Hdg} \iota_T^{-1} (\varinjlim_{r-(-)} e'(Y \times \tilde{S}_I)_* \mathcal{H}om(\rho_{Y \times \tilde{S}_I} \text{Gr}_{Y \times \tilde{S}_I}^{12*} R_{(X'^*, E'^*)/Y \times \tilde{S}_I}(\rho_{Y \times \tilde{S}_I}^* \tilde{g}_I^* L(i_{I^*} j_I^*(F, W)))) \\
& E_{et}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}), \tilde{g}_J^* u_{IJ}^q(F, W)) \xrightarrow{T(\Gamma_T^{\vee, Hdg}, \Omega_{/\tilde{S}_I}^{\Gamma, pr})(F, W)} \\
& \iota_T^{-1} (\varinjlim_{r-(-)} e'(Y \times \tilde{S}_I)_* \mathcal{H}om(\rho_{Y \times \tilde{S}_I} \text{Gr}_{Y \times \tilde{S}_I}^{12*} R_{(X'^*, E^*)/Y \times \tilde{S}_I}(\rho_{Y \times \tilde{S}_I}^* \Gamma_{T_I}^{\vee} \tilde{g}_I^* L(i_{I^*} j_I^*(F, W)))) \\
& E_{et}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}), \tilde{g}_J^{*, \gamma} u_{IJ}^q(F, W)) \\
& \xrightarrow{(\mathcal{H}om(\rho_{Y \times \tilde{S}_I} \text{Gr}_{Y \times \tilde{S}_I}^{12*} R_{Y \times \tilde{S}_I}^{CH}(T^{q, \gamma}(D_{gI})(j_I^*(F, W))), E_{et}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[d_{YI}])} \\
& \iota_T^{-1} (\varinjlim_{r-(-)} e'(Y \times \tilde{S}_I)_* \mathcal{H}om(\rho_{Y \times \tilde{S}_I} \text{Gr}_{Y \times \tilde{S}_I}^{12*} R_{(X'^*, E^*)/Y \times \tilde{S}_I}(\rho_{Y \times \tilde{S}_I}^* L(i'_{I^*} j'_I{}^* g^*(F, W)))) \\
& E_{et}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}), u_{IJ}^q(g^*(F, W)) \xrightarrow{\mathcal{H}om(R_{Y \times \tilde{S}_I}^{CH}(Li'_{I^*} j'_I{}^*(e)), -)} \\
& \iota_T^{-1} (\varinjlim_{r-(-)} e'(Y \times \tilde{S}_I)_* \mathcal{H}om(\rho_{Y \times \tilde{S}_I} \text{Gr}_{Y \times \tilde{S}_I}^{12*} R_{(X'^*, E^*)/Y \times \tilde{S}_I}(\rho_{Y \times \tilde{S}_I}^* L(i'_{I^*} j'_I{}^*(F', W)))) \\
& E_{et}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}), u_{IJ}^q(F', W)) \xrightarrow{=} \mathcal{F}_T^{FDR}(g^*M)
\end{aligned}$$

Proposition 109. Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y_2 \times S \xrightarrow{ps} S$ with $Y_2 \in \text{SmVar}(\mathbb{C})$, l a closed embedding and ps the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y_2 \times \tilde{S}_i$. Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I := g|_{T_I} : T_I \rightarrow S_I$. Let $f : X \rightarrow S$ a morphism with $X \in \text{Var}(\mathbb{C})$ such that there exists a factorization $f : X \xrightarrow{l} Y_1 \times S \xrightarrow{ps} S$, with $Y_1 \in \text{SmVar}(\mathbb{C})$, l a closed embedding

and p_S the projection. We have then the following commutative diagram whose squares are cartesian

$$\begin{array}{ccccccc}
f' : X_T & \longrightarrow & Y_1 \times T & \longrightarrow & T & & \\
\downarrow g' & \searrow & \downarrow & \searrow & \downarrow & \searrow & \\
& & Y_1 \times X & \longrightarrow & Y_1 \times Y & \times S & \longrightarrow & Y_2 \times S \\
& \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \\
f : X & \longrightarrow & Y_1 \times S & \longrightarrow & S & &
\end{array}$$

Take a smooth compactification $\bar{Y}_1 \in \text{PSmVar}(\mathbb{C})$ of Y_1 , denote $\bar{X}_I \subset \bar{Y}_1 \times \tilde{S}_I$ the closure of X_I , and $Z_I := \bar{X}_I \setminus X_I$. Consider $F(X/S) := p_{S,\#} \Gamma_X^\vee \mathbb{Z}(Y_1 \times S / Y_1 \times S)[d_{Y_1}]$ and the isomorphism in $C(\text{Var}(\mathbb{C})^{sm}/T)$

$$\begin{aligned}
T(f, g, F(X/S)) : g^* F(X/S) &:= g^* p_{S,\#} \Gamma_X^\vee \mathbb{Z}(Y_1 \times S / Y_1 \times S) \xrightarrow{\sim} \\
& p_{T,\#} \Gamma_{X_T}^\vee \mathbb{Z}(Y_1 \times T / Y_1 \times T) =: F(X_T/T).
\end{aligned}$$

which gives in $\text{DA}(T)$ the isomorphism $T(f, g, F(X/S)) : g^* M(X/S) \xrightarrow{\sim} (X_T/T)$. Then the following diagram in $\pi_T(D(\text{MHM}(T))) \subset D_{\mathcal{D}(1,0)fil,\infty}(T/(Y_2 \times \tilde{S}_I))$, where the horizontal maps are given by proposition 107, commutes

$$\begin{array}{ccc}
g_{Hdg}^{\hat{*}mod} \iota_S^{-1} \mathcal{F}_S^{FDR}(M(X/S)) & \xrightarrow{g_{Hdg}^{\hat{*}mod} I(X/S)} & g_{Hdg}^{\hat{*}mod} Rf_!^{Hdg}(\Gamma_{X_I}^{\vee, Hdg}(O_{Y_1 \times \tilde{S}_I}, F_b), x_{IJ}(X/S)) \\
\downarrow {}'T(g, \mathcal{F}^{FDR})(M(X/S)) & & \downarrow T(p_{\tilde{S}_I}, \gamma^{\vee, Hdg})(-) \\
& & Rf_!'^{Hdg} g_{Hdg}^{\hat{*}mod}(\Gamma_{X_I}^{\vee, Hdg}(O_{Y_1 \times \tilde{S}_I}, F_b), x_{IJ}(X/S)) \\
& & \downarrow = \\
\iota_T^{-1} \mathcal{F}_T^{FDR}(M(X_T/T)) & \xrightarrow{I(X_T/T)} & Rf_!'^{Hdg}(\Gamma_{X_{T_I}}^{\vee, Hdg}(O_{Y_2 \times Y_1 \times \tilde{S}_I}, F_b), x_{IJ}(X_T/T)).
\end{array}$$

Proof. Follows immediately from definition. \square

Theorem 32. Let $g : T \rightarrow S$ a morphism, with $S, T \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $M \in \text{DA}_c(S)$. Then map in $\pi_T(D(\text{MHM}(T)))$

$$T(g, \mathcal{F}^{FDR})(M) : g_{Hdg}^{\hat{*}mod} \mathcal{F}_S^{FDR}(M) \xrightarrow{\sim} \mathcal{F}_T^{FDR}(g^* M)$$

given in definition 121 is an isomorphism.

Proof. Follows from proposition 109 and proposition 107. \square

Definition 122. • Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have, for $M \in \text{DA}_c(X)$, the following transformation map in $\pi_S(D(\text{MHM}(S)))$

$$\begin{aligned}
T_*(f, \mathcal{F}^{FDR})(M) : \mathcal{F}_S^{FDR}(Rf_* M) &\xrightarrow{\text{ad}(f_{Hdg}^{\hat{*}mod}, Rf_*^{Hdg})(-)} Rf_*^{Hdg} f_{Hdg}^{\hat{*}mod} \mathcal{F}_S^{FDR}(Rf_* M) \\
\underline{T(f, \mathcal{F}^{FDR})(Rf_* M)} &\xrightarrow{Rf_*^{Hdg} \mathcal{F}_X^{FDR}(f^* Rf_* M)} \xrightarrow{\mathcal{F}_X^{FDR}(\text{ad}(f^*, Rf_*)(M))} Rf_*^{Hdg} \mathcal{F}_X^{FDR}(M)
\end{aligned}$$

Clearly, for $p : Y \times S \rightarrow S$ a projection with $Y \in \text{PSmVar}(\mathbb{C})$, we have, for $M \in \text{DA}_c(Y \times S)$, $T_*(p, \mathcal{F}^{FDR})(M) = T_!(p, \mathcal{F}^{FDR})(M)[d_Y]$

- Let $S \in \text{Var}(\mathbb{C})$. Let $Y \in \text{SmVar}(\mathbb{C})$ and $p : Y \times S \rightarrow S$ the projection. We have then, for $M \in \text{DA}(Y \times S)$ the following transformation map in $\pi_S(D(\text{MHM}(S)))$

$$\begin{aligned} T_!(p, \mathcal{F}^{FDR})(M) &: p_!^{Hdg} \mathcal{F}_{Y \times S}^{FDR}(M) \xrightarrow{\mathcal{F}_{Y \times S}^{FDR}(\text{ad}(Lp_*, p^*)(M))} R p_!^{Hdg} \mathcal{F}_{Y \times S}^{FDR}(p^* L p_! M) \\ &\xrightarrow{T(p, \mathcal{F}^{FDR})(Lp_!(M, W))} R p_!^{Hdg} p^{\hat{*}mod[-]} \mathcal{F}_S^{FDR}(L p_! M) \xrightarrow{T(p^{\hat{*}mod}, p^{\hat{*}mod})(-)} p_!^{Hdg} p^{\hat{*}mod[-]} \\ &\quad \mathcal{F}_S^{FDR}(L p_! M) \xrightarrow{\text{ad}(R p_!^{Hdg}, p^{\hat{*}mod[-]})(\mathcal{F}_S^{FDR}(L p_! M))} \mathcal{F}_S^{FDR}(L p_! M) \end{aligned}$$

- Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have then, using the second point, for $M \in \text{DA}(X)$ the following transformation map in $\pi_S(D(\text{MHM}(S)))$

$$\begin{aligned} T_!(f, \mathcal{F}^{FDR})(M) &: R p_!^{Hdg} \mathcal{F}_X^{FDR}(M, W) := R p_!^{Hdg} \mathcal{F}_{Y \times S}^{FDR}(l_* M) \\ &\xrightarrow{T_!(p, \mathcal{F}^{FDR})(l_* M)} \mathcal{F}_S^{FDR}(L p_! l_* M) \xrightarrow{=} \mathcal{F}_S^{FDR}(R f_! M) \end{aligned}$$

- Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have, using the third point, for $M \in \text{DA}(S)$, the following transformation map in $\pi_X(D(\text{MHM}(X)))$

$$\begin{aligned} T^!(f, \mathcal{F}^{FDR})(M) &: \mathcal{F}_X^{FDR}(f^! M) \xrightarrow{\text{ad}(R f_!^{Hdg}, f_{Hdg}^{\hat{*}mod})(\mathcal{F}_X^{FDR}(f^! M))} f_{Hdg}^{\hat{*}mod} R f_!^{Hdg} \mathcal{F}_X^{FDR}(f^! M) \\ &\xrightarrow{T_!(p_S, \mathcal{F}^{FDR})(\mathcal{F}^{FDR}(f^! M))} f_{Hdg}^{\hat{*}mod} \mathcal{F}_S^{FDR}(R f_! f^! M) \xrightarrow{\mathcal{F}_S^{FDR}(\text{ad}(R f_!, f^!)(M))} f_{Hdg}^{\hat{*}mod} \mathcal{F}_S^{FDR}(M) \end{aligned}$$

Proposition 110. Let $S \in \text{Var}(\mathbb{C})$. Let $Y \in \text{SmVar}(\mathbb{C})$ and $p : Y \times S \rightarrow S$ the projection. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i^o : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, l]$, we denote by $S_I = \cap_{i \in I} S_i$, $j_I^o : S_I \hookrightarrow S$ and $j_I : Y \times S_I \hookrightarrow Y \times S$ the open embeddings. We then have closed embeddings $i_I : Y \times S_I \hookrightarrow Y \times \tilde{S}_I$. and we denote by $p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ the projections. Let $f' : X' \rightarrow Y \times S$ a morphism, with $X' \in \text{Var}(\mathbb{C})$ such that there exists a factorization $f' : X' \xrightarrow{l'} Y' \times Y \times S \xrightarrow{p'} Y \times S$ with $Y' \in \text{SmVar}(\mathbb{C})$, l' a closed embedding and p' the projection. Denoting $X'_I := f'^{-1}(Y \times S_I)$, we have closed embeddings $i'_I : X'_I \hookrightarrow Y' \times Y \times \tilde{S}_I$. Consider

$$F(X'/Y \times S) := p_{Y \times S, \#} \Gamma_{X'}^\vee \mathbb{Z}(Y' \times Y \times S/Y' \times Y \times S)[d_{Y'}] \in C(\text{Var}(\mathbb{C})^{sm}/Y \times S)$$

and $F(X'/S) := p_{\#} F(X'/Y \times S) \in C(\text{Var}(\mathbb{C})^{sm}/S)$, so that $L p_{\#} M(X'/Y \times S)[-2d_{Y'}] =: M(X'/S)$. Then, the following diagram in $\pi_S(D(\text{MHM}(S))) \subset D_{\mathcal{D}(1,0)fil,\infty}(S/(Y \times \tilde{S}_I))$, where the vertical maps are given by proposition 107, commutes

$$\begin{array}{ccc} R p^{Hdg} \mathcal{F}_{Y \times S}^{FDR}(M(X'/Y \times S)) & \xrightarrow{T_!(p, \mathcal{F}^{FDR})(M(X'/Y \times S))} & \mathcal{F}_S^{FDR}(M(X'/S)) \\ \uparrow R p^{Hdg} \mathcal{F}_{Y \times S}^{FDR}(M(X'/Y \times S)) & & \uparrow I(X'/S) \\ R p^{Hdg} R f_!^{Hdg} (\Gamma_{X'_I}^\vee, \text{Hdg}(O_{Y' \times Y \times \tilde{S}_I}, F_b), x_{IJ}(X'/Y \times S)) & \xrightarrow{=} & R f_!^{Hdg} (\Gamma_{X'_I}^\vee, \text{Hdg}(O_{Y' \times Y \times \tilde{S}_I}, F_b), x_{IJ}(X'/S)) \end{array}$$

Proof. Immediate from definition. \square

Theorem 33. (i) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Then, for $M \in \text{DA}_c(X)$, the map given in definition 122

$$T_!(f, \mathcal{F}^{FDR})(M) : R f_!^{Hdg} \mathcal{F}_X^{FDR}(M) \xrightarrow{\simeq} \mathcal{F}_S^{FDR}(R f_! M)$$

is an isomorphism in $\pi_S(D(\text{MHM}(S)))$.

(ii) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$, S quasi-projective. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have, for $M \in \text{DA}_c(X)$, the map given in definition 122

$$T_*(f, \mathcal{F}^{FDR})(M) : \mathcal{F}_S^{FDR}(Rf_*M) \xrightarrow{\sim} Rf_*^{Hdg} \mathcal{F}_X^{FDR}(M)$$

is an isomorphism in $\pi_S(D(\text{MHM}(S)))$.

(iii) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$, S quasi-projective. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Then, for $M \in \text{DA}_c(S)$, the map given in definition 122

$$T^!(f, \mathcal{F}^{FDR})(M) : \mathcal{F}_X^{FDR}(f^!M) \xrightarrow{\sim} f_{Hdg}^{*mod} \mathcal{F}_S^{FDR}(M)$$

is an isomorphism in $\pi_X(D(\text{MHM}(X)))$.

Proof. (i): By proposition 110 and proposition 107, for $S \in \text{Var}(\mathbb{C})$, $Y \in \text{SmVar}(\mathbb{C})$, $p : Y \times S \rightarrow S$ the projection and $M \in \text{DA}_c(Y \times S)$,

$$T_!(p, \mathcal{F}^{FDR})(M) : Rp_!^{Hdg} \mathcal{F}_{Y \times S}^{FDR}(M) \rightarrow \mathcal{F}_S^{FDR}(Rp_!M)$$

is an isomorphism.

(ii): Consider first an open embedding $n : S^o \hookrightarrow S$ with $S \in \text{Var}(\mathbb{C})$ so that there exist a closed embedding $i : S \hookrightarrow \tilde{S}$ with $\tilde{S} \in \text{SmVar}(\mathbb{C})$. Then, since

$$n^* : C(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{sm}/S^o)$$

is surjective, $n^* : \text{DA}(S) \rightarrow \text{DA}(S^o)$ is surjective. Denote by $i : Z = S \setminus S^o \hookrightarrow S$ the complementary closed embedding. By [1], $\text{DA}_c(S)$ is generated by motives of the form

$$\begin{aligned} \text{DA}_c(S) = & \langle M(X'/S) = f'_*E(\mathbb{Z}_{X'}), f' : X' \rightarrow S \text{ proper with } X' \in \text{SmVar}(\mathbb{C}), \\ & \text{s.t. } f'^{-1}(Z) = X' \text{ or } f'^{-1}(Z) = \cup D_i = D \subset X' \rangle. \end{aligned}$$

If $f'^{-1}(Z) = X'$, $n^*M(X'/S) = 0$. So let consider the case $f'^{-1}(Z) = \cup_{i=1}^l D_i = D \subset X'$ is a normal crossing divisor. Denote $f'_D : f'_{|D} : D \rightarrow Z$, $D_I = \cap_{i \in I} D_i$ and $i'_I : D_I \hookrightarrow X'$, $n' : X'^o := X' \setminus D \hookrightarrow X'$ the complementary open embedding and $f'^o : f'_{|X'^o} : X'^o \rightarrow S^o$. Denote $L = [1, \dots, l]$. We have then a generalized distinguish triangle in $\text{DA}(X')$

$$\begin{aligned} a(n', i') : n'_* n'^* E_{et}(\mathbb{Z}_{X'}) & \xrightarrow{\sim} \text{Cone}(\gamma_D(-) : \Gamma_D E_{et}(\mathbb{Z}_{X'}) \rightarrow E_{et}(\mathbb{Z}_{X'})) \\ & \xrightarrow{\sim} \text{Cone}(\Gamma_{D_L} E_{et}(\mathbb{Z}_{X'}) \rightarrow \cdots \rightarrow \bigoplus_{i=1}^l E_{et}(\mathbb{Z}_{X'})) \\ & \xrightarrow{\sim} \text{Cone}(i'_{L*} i'_L{}^! E_{et}(\mathbb{Z}_{X'}) \rightarrow \cdots \rightarrow \bigoplus_{i=1}^l i'_{i*} i'_i{}^! E_{et}(\mathbb{Z}_{X'}) \xrightarrow{\text{ad}(i'_{i*}, i'_i{}^!)(E_{et}(\mathbb{Z}_{X'}))} E_{et}(\mathbb{Z}_{X'})) \\ & \xrightarrow{\sim} \text{Cone}(i'_{L*} \mathbb{Z}_{D_L}[-l] \rightarrow \cdots \rightarrow \bigoplus_{i=1}^l i'_{i*} \mathbb{Z}_{D_i}[-1] \rightarrow \mathbb{Z}_{X'}) \end{aligned}$$

where the first isomorphism is the image of an homotopy equivalence by definition, the second one is the image of an homotopy equivalence by definition-proposition 4(ii), the third one follows by the localization property (see section 3, theorem 14) and the last one follows from purity since the D_I and X' are smooth

(see section 3, theorem 14). Similarly, we have a generalized distinguish triangle in $D(MHM(X'))$

$$\begin{aligned}
a^{mod}(n', i') : n_*'^{Hdg} n'^*(O_{X'}, F_b) &\xrightarrow{\sim} \text{Cone}(\gamma_D^{Hdg}(O_{X'}, F_b) : \Gamma_D^{Hdg}(O_{X'}, F_b) \rightarrow (O_{X'}, F_b)) \\
&\xrightarrow{\sim} \text{Cone}(\Gamma_{D_L}^{Hdg}(O_{X'}, F_b) \rightarrow \cdots \rightarrow \bigoplus_{i=1}^l \Gamma_{D_i}^{Hdg}(O_{X'}, F_b) \xrightarrow{\oplus_i \gamma_{D_i}^{Hdg}(-)} (O_{X'}, F_b)) \\
&\xrightarrow{\sim} \text{Cone}(i'_{L*mod}(O_{D_L}, F_b)[-l] \rightarrow \cdots \rightarrow \bigoplus_{i=1}^l i'_{i*mod}(O_{D_i}, F_b)[-1] \\
&\quad \xrightarrow{\text{ad}(i'_{i*mod}, i_i^\sharp)(O_{X'}, F_b)} \cdots \rightarrow (O_{X'}, F_b))
\end{aligned}$$

where the first isomorphism is the image of an homotopy equivalence by definition, the second one is the image of an homotopy equivalence by definition-proposition 16, and the third one follows by the localization property of mixed Hodge modules (see section 5). Consider $n^*M(X'/S) = M(X'^o/S^o)$. We have then the factorization

$$\begin{aligned}
&T_*(n, \mathcal{F}^{FDR})(n^*M(X'/S)) : \mathcal{F}_S^{FDR}(Rn_*n^*M(X'/S)) = \mathcal{F}_S^{FDR}(n_*n^*f'_*E_{et}(\mathbb{Z}_{X'})) \\
&\xrightarrow{\mathcal{F}_S^{FDR}(Rn_*T(n, f')(E_{et}(\mathbb{Z}_{X'})))} \mathcal{F}_S^{FDR}(n_*f'_o n'^*E_{et}(\mathbb{Z}_{X'})) = \mathcal{F}_S^{FDR}(f'_*n'_*n'^*E_{et}(\mathbb{Z}_{X'})) \\
&\quad \xrightarrow{T_*(f', \mathcal{F}^{FDR})(n'_*n'^*E_{et}(\mathbb{Z}_{X'}))} Rf'_*Hdg \mathcal{F}_{X'}^{FDR}(n'_*n'^*E_{et}(\mathbb{Z}_{X'})) \\
&\quad \xrightarrow{\mathcal{F}_{X'}^{FDR}(a(n', i'))} Rf'_*Hdg \mathcal{F}_{X'}^{FDR}(\text{Cone}(i'_{L*}\mathbb{Z}_{D_L}[-l] \rightarrow \cdots \rightarrow \bigoplus_{i=1}^l i'_{i*}\mathbb{Z}_{D_i}[-1] \rightarrow \mathbb{Z}_{X'})) \\
&\xrightarrow{\cong} \text{Cone}(Rf'_*Hdg \mathcal{F}_{X'}^{FDR}(i'_{L*}i'_L\mathbb{Z}_{X'}[-l]) \rightarrow \cdots \rightarrow \bigoplus_{i=1}^l Rf'_*Hdg \mathcal{F}_{X'}^{FDR}(i'_{i*}i'_i\mathbb{Z}_{X'}[-1]) \rightarrow Rf'_*Hdg \mathcal{F}_{X'}^{FDR}(\mathbb{Z}_{X'})) \\
&\quad \xrightarrow{(T_*(i'_L, \mathcal{F}^{FDR})(-) \circ T(i'_L, \mathcal{F}^{FDR})(-))} \text{Cone}(Rf'_*Hdg \int_{i'_L} i'_{L, Hdg} \mathcal{F}_{X'}^{FDR}(\mathbb{Z}_{X'})[-l] \rightarrow \\
&\quad \cdots \rightarrow \bigoplus_{i=1}^l Rf'_*Hdg i'_{L*mod} i'_{L, Hdg} \mathcal{F}_{X'}^{FDR}(\mathbb{Z}_{X'}) \rightarrow Rf'_*Hdg \mathcal{F}_{X'}^{FDR}(\mathbb{Z}_{X'})) \\
&\xrightarrow{\cong} \text{Cone}(Rf'_*Hdg i'_{L*mod} i'_{L, Hdg} \mathcal{F}_{X'}^{FDR}(O_{X'}, F_b)[-l] \rightarrow \cdots \rightarrow \bigoplus_{i=1}^l Rf'_*Hdg i'_{i*mod} i'_{i, Hdg} \mathcal{F}_{X'}^{FDR}(O_{X'}, F_b)[-1] \int_{f'} (O_{X'}, F_b)) \\
&\quad \xrightarrow{\cong} Rf'_*Hdg \text{Cone}(i'_{L*mod}(O_{D_L}, F_b)[-l] \rightarrow \cdots \rightarrow \bigoplus_{i=1}^l i'_{i*mod}(O_{D_i}, F_b)) \\
&\quad \xrightarrow{a^{mod}(n', i')} Rf'_*Hdg n'^*Hdg n'^*(O_{X'}, F) = n_*^{Hdg} Rf'_o(O_{X'^o}, F_b) \xrightarrow{I(X/\tilde{S}^o)} n_*^{Hdg} \mathcal{F}_S^{FDR}(n^*M(X'/S))
\end{aligned}$$

Since all the morphism involved are isomorphisms, $T_*(n, \mathcal{F}^{FDR})(n^*M(X'/S))$ is an isomorphism. Hence, $T_*(n, \mathcal{F}^{FDR})(M)$ is an isomorphism for all $M \in \text{DA}(S^o)$. Consider now the case of a general morphism $f : X \rightarrow S$, $X, S \in \text{Var}(\mathbb{C})$, S quasi-projective, which factors through $f : X \xrightarrow{l} Y \times S \xrightarrow{ps} S$ with some $Y \in \text{SmVar}(\mathbb{C})$. By definition, for $M \in \text{DA}_c(X)$

$$\begin{aligned}
&T_*(f, \mathcal{F}^{FDR})(M) : \mathcal{F}_S^{FDR}(Rf_*M) = \mathcal{F}_S^{FDR}(Rp_{S*}l_*M) \\
&\xrightarrow{T_*(ps, \mathcal{F}^{FDR})(l_*M)} Rp_{S*}^{Hdg} \mathcal{F}_{Y \times S}^{FDR}(l_*M) =: Rf_*^{Hdg} \mathcal{F}_X^{FDR}(M).
\end{aligned}$$

Hence, we have to show that for $S \in \text{Var}(\mathbb{C})$, $Y \in \text{SmVar}(\mathbb{C})$, $p : Y \times S \rightarrow S$ the projection, and $M \in \text{DA}_c(Y \times S)$,

$$T_*(p, \mathcal{F}^{FDR})(M) : \mathcal{F}_S^{FDR}(Rp_*M) \rightarrow Rp_*^{Hdg} \mathcal{F}_{Y \times S}^{FDR}(M)$$

is an isomorphism. Take a smooth compactification $\bar{Y} \in \text{PSmVar}(\mathbb{C})$ of Y . Denote by $n_0 : Y \hookrightarrow \bar{Y}$ and $n := n_0 \times I_S : Y \times S \hookrightarrow \bar{Y} \times S$ the open embeddings and by $\bar{p} : \bar{Y} \times S \rightarrow S$ the projection. We have $p = \bar{p} \circ n : Y \times S \rightarrow S$, which gives the factorization

$$\begin{aligned} T_*(p, \mathcal{F}^{FDR})(M) : \mathcal{F}_S^{FDR}(Rp_*M) &= \mathcal{F}_S^{FDR}(R\bar{p}_*Rn_*M) \xrightarrow{T_*(\bar{p}, \mathcal{F}^{FDR})(Rn_*M)} R\bar{p}_*^{Hdg} \mathcal{F}_{\bar{Y} \times S}^{FDR}(Rn_*M) \\ &\xrightarrow{T_*(n, \mathcal{F}^{FDR})(M)} R\bar{p}_*^{Hdg} n_*^{Hdg} \mathcal{F}_{Y \times S}^{FDR}(M) = Rp_*^{Hdg} \mathcal{F}_{Y \times S}^{FDR}(M). \end{aligned}$$

By the open embedding case $T_*(n, \mathcal{F}^{FDR})(M)$ is an isomorphism. On the other hand, since \bar{p} is proper, $T_*(\bar{p}, \mathcal{F}^{FDR})(Rn_*M) = T_!(\bar{p}, \mathcal{F}^{FDR})(Rn_*M)$ is an isomorphism by (i).

(iii): Denote by $n : Y \times S \setminus X \hookrightarrow Y \times S$ the complementary open embedding. We have, for $M \in \text{DA}_c(S)$, the factorization

$$\begin{aligned} T^!(f, \mathcal{F}^{FDR})(M) : \mathcal{F}_X^{FDR}(f^!M) &= \mathcal{F}_{Y \times S}^{FDR}(l_*l^!p_S^!M) \xrightarrow{\mathcal{F}_{Y \times S}^{FDR}(a(n,l))} \mathcal{F}_{Y \times S}^{FDR}(\text{Cone}(p_S^!M \rightarrow Rn_*n^*p_S^!M)[-1]) \\ &\xrightarrow{=} \text{Cone}(\mathcal{F}_{Y \times S}^{FDR}(p_S^!M) \rightarrow \mathcal{F}_{Y \times S}^{FDR}(Rn_*n^*p_S^!M))[-1] \\ &\xrightarrow{(T(n, \mathcal{F}^{FDR})(p_S^!M) \circ T^!(p_S, \mathcal{F}^{FDR})(M)), T^!(p_S, \mathcal{F}^{FDR})(M))} \\ &\text{Cone}(p_S^{*mod[-]} \mathcal{F}_S^{FDR}(M) \rightarrow n_*^{Hdg} n^* p_S^{*mod[-]} \mathcal{F}_S^{FDR}(M))[-1] \xrightarrow{\sim} f_{Hdg}^{*mod} \mathcal{F}_S^{FDR}(M). \end{aligned}$$

By (ii), $T(n, \mathcal{F}^{FDR})(p_S^!M)$ is an isomorphism. On the other hand, since p_S is a smooth morphism, $T^!(p_S, \mathcal{F}^{FDR})(M) = T(p_S, \mathbb{D}\mathcal{F}^{FDR}(M))[d_Y]$; hence, $T^!(p_S, \mathcal{F}^{FDR})(M)$ is an isomorphism by theorem 32. \square

Lemma 17. *Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$. Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I := g|_{T_I} : T_I \rightarrow S_I$. Let $M \in \text{DA}_c(S)^-$ and $(F, W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ such that $(M, W) = D(\mathbb{A}_S^1, et)(F, W)$. Then, $g^!M = L\mathbb{D}_S g^* L\mathbb{D}_S M$, $D(\mathbb{A}_T^1, et)(g^* \mathbb{D}_S L F) = g^* L\mathbb{D}_S M$ and there exist $(F', W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ and an equivalence (\mathbb{A}^1, et) local $e : (F', W) \rightarrow (g^* \mathbb{D}_S L(F, W))$ such that $D(\mathbb{A}_T^1, et)(F', W) = g^* L\mathbb{D}_S(M, W)$ and, using definition 114 and definition 119(ii) and lemma 16, the map in $\pi_T(D(MHM(T))) \subset D_{\mathcal{D}(1,0)fil, \infty}(T/(Y \times \tilde{S}_I))$*

$$T^!(g, \mathcal{F}^{FDR})(M) : \mathcal{F}_T^{FDR}(g^!M) \rightarrow g_{Hdg}^{*mod} \mathcal{F}_S^{FDR}(M)$$

given in definition 122 is the inverse of the following map

$$\begin{aligned}
& T^!, -1(g, \mathcal{F}^{FDR})(M) : g_{Hdg}^{*mod} \iota_S^{-1} \mathcal{F}_S^{FDR}(M) \\
\rightleftharpoons & (\Gamma_T^{Hdg} \iota_T^{-1} (\tilde{g}_I^{*mod} (\varinjlim_{r_-(\tilde{S}_I)} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\rho_{\tilde{S}_I}^* \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* \mathbb{D}_S L(F, W))), \\
& E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{YI}], \tilde{g}_I^{*mod} u_{IJ}^q(\mathbb{D}_S L(F, W))) \xrightarrow{(T(\tilde{g}_I, \Omega_{/\tilde{S}_I}^{\Gamma, pr})(-))} \\
& \Gamma_T^{Hdg} \iota_T^{-1} (\varinjlim_{r_-(\tilde{S}_I)} e'(Y \times \tilde{S}_I)_* \mathcal{H}om(\rho_{Y \times \tilde{S}_I} \text{Gr}_{Y \times \tilde{S}_I}^{12*} \tilde{g}_I^* R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* \mathbb{D}_S L(F, W))), \\
& E_{et}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}), \tilde{g}_I^* u_{IJ}^q(\mathbb{D}_S L(F, W))) \xrightarrow{\mathcal{H}om(T(\tilde{g}_I, R^{CH})(-)^{-1}, -)} \\
\Gamma_T^{Hdg} \iota_T^{-1} (\varinjlim_{r_-(\tilde{S}_I)} e'(Y \times \tilde{S}_I)_* \mathcal{H}om(\rho_{Y \times \tilde{S}_I} \text{Gr}_{Y \times \tilde{S}_I}^{12*} R_{(Y \times X^*, Y \times D^*)/Y \times \tilde{S}_I}(\rho_{Y \times \tilde{S}_I}^* \tilde{g}_I^* L(i_{I*} j_I^* \mathbb{D}_S L(F, W))), \\
& E_{et}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}), \tilde{g}_I^* u_{IJ}^q(\mathbb{D}_S L(F, W))) \\
\rightleftharpoons & \Gamma_T^{Hdg} \iota_T^{-1} (\varinjlim_{r_-(\tilde{S}_I)} e'(Y \times \tilde{S}_I)_* \mathcal{H}om(\rho_{Y \times \tilde{S}_I} \text{Gr}_{Y \times \tilde{S}_I}^{12*} R_{(X'^*, E^*)/Y \times \tilde{S}_I}(\rho_{Y \times \tilde{S}_I}^* \tilde{g}_I^* L(i_{I*} j_I^* \mathbb{D}_S L(F, W))), \\
& E_{et}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[d_{YI}], \tilde{g}_I^* u_{IJ}^q(\mathbb{D}_S L(F, W))) \xrightarrow{T(\Gamma_T^{Hdg}, \Omega_{/\tilde{S}_I}^{\Gamma, pr})(F, W)^{-1}} \\
& \iota_T^{-1} (\varinjlim_{r_-(\tilde{S}_I)} e'(Y \times \tilde{S}_I)_* \mathcal{H}om(\rho_{Y \times \tilde{S}_I} \text{Gr}_{Y \times \tilde{S}_I}^{12*} R_{(X'^*, E^*)/Y \times \tilde{S}_I}(\rho_{Y \times \tilde{S}_I}^* L\Gamma_{T_I} E(\tilde{g}_I^* L(i_{I*} j_I^* \mathbb{D}_S L(F, W))), \\
& E_{et}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[d_{YI}], \tilde{g}_I^{*, \gamma, d} u_{IJ}^q(F, W)) \\
& \xrightarrow{(\mathcal{H}om(\rho_{Y \times \tilde{S}_I} \text{Gr}_{Y \times \tilde{S}_I}^{12*} R_{Y \times \tilde{S}_I}^{CH}(\mathbb{D}_{Y \times \tilde{S}_I} T^{q, \gamma}(D_{g_I})(j_I^* \mathbb{D}_S L(F, W))), E_{et}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[d_{YI}])^{-1}} \\
& \iota_T^{-1} (\varinjlim_{r_-(\tilde{S}_I)} e'(Y \times \tilde{S}_I)_* \mathcal{H}om(\rho_{Y \times \tilde{S}_I} \text{Gr}_{Y \times \tilde{S}_I}^{12*} R_{(X'^*, E^*)/Y \times \tilde{S}_I}(\rho_{Y \times \tilde{S}_I}^* L\mathbb{D}_{Y \times \tilde{S}_I} L(i'_{I*} j'_I{}^* g^* \mathbb{D}_S L(F, W))), \\
& E_{et}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[d_{YI}], u_{IJ}^{q, d}(g^* \mathbb{D}_S L(F, W))) \xrightarrow{\mathcal{H}om(R_{Y \times \tilde{S}_I}^{CH}(\mathbb{D}_{Y \times \tilde{S}_I} Li'_{I*} j'_I{}^*(e), -))} \\
& \iota_T^{-1} (\varinjlim_{r_-(\tilde{S}_I)} e'(Y \times \tilde{S}_I)_* \mathcal{H}om(\rho_{Y \times \tilde{S}_I} \text{Gr}_{Y \times \tilde{S}_I}^{12*} R_{(X'^*, E^*)/Y \times \tilde{S}_I}(\rho_{Y \times \tilde{S}_I}^* L\mathbb{D}_{Y \times \tilde{S}_I} L(i'_{I*} j'_I{}^* g^* \mathbb{D}_S L(F, W))), \\
& E_{et}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[d_{YI}], u_{IJ}^{q, d}(L(F', W))) \rightleftharpoons \mathcal{F}_T^{FDR}(g^! M)
\end{aligned}$$

Proposition 111. *Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{ps} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and ps the projection. Let $S = \bigcup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, $T = \bigcup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$. Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I := g|_{T_i} : T_i \rightarrow S_I$. Let $M \in \text{DA}_c(S)$ and $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}_S^1, et)(F)$. Then, $D(\mathbb{A}_T^1, et)(g^* F) = g^* M$. Then the following diagram in $D_{Ofil, \mathcal{D}, \infty}(T/(Y \times \tilde{S}_I))$ commutes*

$$\begin{array}{ccc}
Rg^{*mod[-], \Gamma} \mathcal{F}_S^{GM}(\mathbb{L}\mathbb{D}_S M) & \xrightarrow{Rg^{*mod[-], \Gamma} T(\mathcal{F}_S^{GM}, \mathcal{F}_S^{FDR})(M)} & Rg^{*mod[-], \Gamma} \mathcal{F}_S^{FDR}(M) \xleftarrow{T(g^{*mod}, Rg^{*mod[-], \Gamma})(\mathcal{F}_S^{FDR}(M))} g_{Hdg}^{*mod} \mathcal{F}_S^{FDR}(M) \\
\downarrow T(g, \mathcal{F}^{GM})(\mathbb{L}\mathbb{D}_S M) & & \downarrow T^!(g, \mathcal{F}^{FDR})(M)^{-1} \\
\mathcal{F}_T^{GM}(g^* \mathbb{L}\mathbb{D}_S M = \mathbb{L}\mathbb{D}_T g^! M) & \xrightarrow{T(\mathcal{F}_T^{GM}, \mathcal{F}_T^{FDR})(g^* M)} & \mathcal{F}_T^{FDR}(g^! M)
\end{array}$$

Proof. Follows from lemma 17. □

Definition 123. Let $S \in \text{SmVar}(\mathbb{C})$. We have, for $M, N \in \text{DA}(S)$ and $(F, W), (G, W) \in C_{\text{fil}}(\text{Var}(\mathbb{C})^{\text{sm}}/S)$ projective such that $(M, W) = D(\mathbb{A}^1, \text{et})(F, W)$ and $(N, W) = D(\mathbb{A}^1, \text{et})(G, W)$, the following transformation map in $\pi_S(D(\text{MHM}(S)))$

$$\begin{aligned}
& T(\mathcal{F}_S^{FDR}, \otimes)(M, N) : \mathcal{F}_S^{FDR}(M) \otimes_{O_S}^L \mathcal{F}_S^{FDR}(N) \\
& \xrightarrow{=} \left(\varinjlim_{r-(-)} e'(S)_* \text{Hom}(\rho_{S*} \text{Gr}_S^{12*} R_{(X^*, D^*)/S}(\rho_S^*(F, W)), E_{\text{et}}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}))[-d_S] \right) \otimes_{O_S}^{[-1]} \\
& \quad \left(\varinjlim_{r-(-)} e'(S)_* \text{Hom}(\rho_{S*} \text{Gr}_S^{12*} R_{(X^*, D^*)/S}(\rho_S^*(G, W)), E_{\text{et}}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}))[-d_S] \right) \\
& \xrightarrow{=} \varinjlim_{r-(-), r-(-)} (e'(S)_* \text{Hom}(\rho_{S*} \text{Gr}_S^{12*} R_{(X^*, D^*)/S}(\rho_S^*(F, W)), E_{\text{et}}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})) \otimes_{O_S} \\
& \quad e'(S)_* \text{Hom}(\rho_{S*} \text{Gr}_S^{12*} R_{(X^*, D^*)/S}(\rho_S^*(G, W)), E_{\text{et}}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}))[-d_S] \\
& \quad \xrightarrow{T(\otimes, \Omega_{/S}^{\Gamma, \text{pr}})(-, -)} \\
& \left(\varinjlim_{r-(-), r-(-)} e'(S)_* \text{Hom}(\rho_{S*} \text{Gr}_S^{12*} R_{(X^*, D^*)/S}(\rho_S^*(F, W)) \otimes \rho_{S*} \text{Gr}_S^{12*} R_{(X^*, D^*)/S}(\rho_S^*(G, W)), \right. \\
& \quad \left. E_{\text{et}}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}))[-d_S] \right) \xrightarrow{\text{Hom}(T(\otimes, R_S^{CH})(-, -)^{-1}, -)} \\
& e'(S)_* \text{Hom}(\rho_{S*} \text{Gr}_S^{12*} R_{(X^*, D^*)/S}(\rho_S^*((F, W) \otimes (G, W))), E_{\text{et}}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}))[-d_S] \xrightarrow{=} \mathcal{F}_S^{FDR}(M \otimes N)
\end{aligned}$$

We now give the definition in the non smooth case :

Definition 124. Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open affine covering and denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We have, for $M, N \in \text{DA}(S)$ and $(F, W), (G, W) \in C_{\text{fil}}(\text{Var}(\mathbb{C})^{\text{sm}}/S)$ such that $(M, W) = D(\mathbb{A}^1, \text{et})(F, W)$ and $(N, W) = D(\mathbb{A}^1, \text{et})(G, W)$, the following transformation map in

$$\pi_S(D(MHM(S))) \subset D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))$$

$$\begin{aligned}
& T(\mathcal{F}_S^{FDR}, \otimes)(M, N) : \mathcal{F}_S^{FDR}(M) \otimes_{O_S}^{L[-]} \mathcal{F}_S^{FDR}(N) \\
& \xrightarrow{=} (\varinjlim_{r-(-)} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\rho_{\tilde{S}_I^*} \mathrm{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I^*}^* L(i_{I^*} j_I^*(F, W))), \\
& \quad E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], u_{IJ}(F, W)) \otimes_{O_S}^{[-]} \\
& (\varinjlim_{r-(-)} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\rho_{\tilde{S}_I^*} \mathrm{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I^*}^* L(i_{I^*} j_I^*(G, W))), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], u_{IJ}(G, W)) \\
& \quad \xrightarrow{=} (\varinjlim_{r-(-), r-(-)} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\rho_{\tilde{S}_I^*} \mathrm{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I^*}^* L(i_{I^*} j_I^*(F, W))), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})) \otimes_{O_{\tilde{S}_I}} \\
& \quad e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\rho_{\tilde{S}_I^*} \mathrm{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I^*}^* L(i_{I^*} j_I^*(F, W))), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], u_{IJ}(F) \otimes u_{IJ}(G)) \\
& \quad \xrightarrow{(T(\otimes, \Omega_{/\tilde{S}_I}^{\Gamma, pr})(-, -))} \\
& (\varinjlim_{r-(-), r-(-)} e'(\tilde{S}_I)_* \mathcal{H}om(\rho_{\tilde{S}_I^*} \mathrm{Gr}_{\tilde{S}_I}^{12*}, R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I^*}^* L(i_{I^*} j_I^*(F, W))) \otimes R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I^*}^* L(i_{I^*} j_I^*(F, W))), \\
& \quad E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], v_{IJ}(F \otimes G)) \xrightarrow{\mathcal{H}om(T(\otimes, R_{\tilde{S}_I}^{CH})(-, -)^{-1}, -)} \\
& (\varinjlim_{r-(-)} e'(\tilde{S}_I)_* \mathcal{H}om(\rho_{\tilde{S}_I^*} \mathrm{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I^*}^* L(i_{I^*} j_I^*(F, W))) \otimes L(i_{I^*} j_I^*(F, W))), \\
& \quad E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], u_{IJ}(F \otimes G)) \\
& \quad \xrightarrow{\mathcal{H}om(R_{(-, -)/-}(T(\otimes, L)(-, -), -)} \\
& (\varinjlim_{r-(-)} e'(\tilde{S}_I)_* \mathcal{H}om(\rho_{\tilde{S}_I^*} \mathrm{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I^*}^* (L(i_{I^*} j_I^*((F, W) \otimes (G, W))))), \\
& \quad E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], u_{IJ}(F \otimes G)) \xrightarrow{=} \mathcal{F}_S^{FDR}(M \otimes N)
\end{aligned}$$

Proposition 112. Let $f_1 : X_1 \rightarrow S$, $f_2 : X_2 \rightarrow S$ two morphism with $X_1, X_2, S \in \mathrm{Var}(\mathbb{C})$. Assume that there exist factorizations $f_1 : X_1 \xrightarrow{l_1} Y_1 \times S \xrightarrow{p_S} S$, $f_2 : X_2 \xrightarrow{l_2} Y_2 \times S \xrightarrow{p_S} S$ with $Y_1, Y_2 \in \mathrm{SmVar}(\mathbb{C})$, l_1, l_2 closed embeddings and p_S the projections. We have then the factorization

$$f_{12} := f_1 \times f_2 : X_{12} := X_1 \times_S X_2 \xrightarrow{l_1 \times l_2} Y_1 \times Y_2 \times S \xrightarrow{p_S} S$$

Let $S = \cup_{i=1}^l S_i$ an open affine covering and denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. We have then the following commutative diagram in $\pi_S(DMHM(S)) \subset D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))$ where the vertical maps are given by proposition 107

$$\begin{array}{ccc}
\mathcal{F}_S^{FDR}(M(X_1/S)) \otimes_{O_S}^{L[-]} \mathcal{F}_S^{FDR}(M(X_2/S)) & \xrightarrow{I(X_1/S) \otimes I(X_2/S)} & Rf_{1!}^{Hdg}(\Gamma_{X_{1I}}^{\vee, Hdg}(O_{Y_1 \times \tilde{S}_I}, F_b), x_{IJ}(X_1/S)) \otimes_{O_S} \\
\downarrow T(\mathcal{F}_S^{FDR}, \otimes)(M(X_1/S), M(X_2/S)) & & \downarrow (Ew_{(Y_1 \times \tilde{S}_I, Y_2 \times \tilde{S}_I)/\tilde{S}_I}) \\
\mathcal{F}_S^{FDR}(M(X_1/S) \otimes M(X_2/S) = M(X_1 \times_S X_2/S)) & \xrightarrow{I(X_{12}/S)} & Rf_{12!}^{Hdg}(\Gamma_{X_{1I} \times_S X_{2I}}^{\vee, Hdg}(O_{Y_1 \times Y_2 \times \tilde{S}_I}, F_b), x_{IJ}(X_1/S)).
\end{array}$$

Proof. Immediate from definition. \square

Theorem 34. Let $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open affine covering and denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$.

Then, for $M, N \in \text{DA}_e(S)$, the map in $\pi_S(D(\text{MHM}(S)))$

$$T(\mathcal{F}_S^{FDR}, \otimes)(M, N) : \mathcal{F}_S^{FDR}(M) \otimes_{O_S}^L \mathcal{F}_S^{FDR}(N) \xrightarrow{\sim} \mathcal{F}_S^{FDR}(M \otimes N)$$

given in definition 124 is an isomorphism.

Proof. Follows from proposition 112. □

We have the following easy proposition

Proposition 113. *Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open affine covering and denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We have, for $M, N \in \text{DA}(S)$ and $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1, et)(F)$ and $N = D(\mathbb{A}^1, et)(G)$, the following commutative diagram in $D_{O_S \text{ fil}, \mathcal{D}, \infty}(S/(\tilde{S}_I))$*

$$\begin{array}{ccc} \mathcal{F}_S^{GM}(\text{LD}_S M) \otimes_{O_S}^L \mathcal{F}_S^{GM}(\text{LD}_S N) & \xrightarrow{T(\mathcal{F}_S^{GM}, \mathcal{F}_S^{FDR})(M) \otimes T(\mathcal{F}_S^{GM}, \mathcal{F}_S^{FDR})(N)} & \mathcal{F}_S^{FDR}(M) \otimes_{O_S}^L \mathcal{F}_S^{FDR}(N) \\ \downarrow T(\mathcal{F}_S^{GM}, \otimes)(\text{LD}_S M, \text{LD}_S N) & & \downarrow T(\mathcal{F}_S^{FDR}, \otimes)(M, N) \\ \mathcal{F}_S^{GM}(\text{LD}_S(M \otimes N)) & \xrightarrow{T(\mathcal{F}_S^{GM}, \mathcal{F}_S^{FDR})(M \otimes N)} & \mathcal{F}_S^{FDR}(M \otimes N) \end{array}$$

Proof. Immediate from definition. □

6.2 The analytic filtered De Rham realization functor

6.2.1 The analytic Gauss-Manin filtered De Rham realization functor and its transformation map with pullbacks

Consider, for $S \in \text{AnSp}(\mathbb{C})$, the following composition of morphism in RCat (see section 2)

$$\tilde{e}(S) : (\text{AnSp}(\mathbb{C})/S, O_{\text{AnSp}(\mathbb{C})/S}) \xrightarrow{\rho_S} (\text{AnSp}(\mathbb{C})^{sm}/S, O_{\text{AnSp}(\mathbb{C})^{sm}/S}) \xrightarrow{e(S)} (S, O_S)$$

with, for $X/S = (X, h) \in \text{AnSp}(\mathbb{C})/S$,

- $O_{\text{AnSp}(\mathbb{C})/S}(X/S) := O_X(X)$,
- $(\tilde{e}(S)^* O_S(X/S) \rightarrow O_{\text{AnSp}(\mathbb{C})/S}(X/S)) := (h^* O_S \rightarrow O_X)$.

and $O_{\text{AnSp}(\mathbb{C})^{sm}/S} := \rho_{S*} O_{\text{AnSp}(\mathbb{C})/S}$, that is, for $U/S = (U, h) \in \text{AnSp}(\mathbb{C})^{sm}/S$, $O_{\text{AnSp}(\mathbb{C})^{sm}/S}(U/S) := O_{\text{AnSp}(\mathbb{C})/S}(U/S) := O_U(U)$

Definition 125. (i) For $S \in \text{Var}(\mathbb{C})$, we consider the complexes of presheaves

$$\Omega_{/S}^\bullet := \text{coker}(\Omega_{O_{\text{AnSp}(\mathbb{C})/S}/\tilde{e}(S)^* O_S} : \Omega_{\tilde{e}(S)^* O_S}^\bullet \rightarrow \Omega_{O_{\text{AnSp}(\mathbb{C})/S}}^\bullet) \in C_{O_S}(\text{AnSp}(\mathbb{C})/S)$$

which is by definition given by

- for X/S a morphism $\Omega_{/S}^\bullet(X/S) = \Omega_{X/S}^\bullet(X)$
- for $g : X'/S \rightarrow X/S$ a morphism,

$$\begin{aligned} \Omega_{/S}^\bullet(g) &:= \Omega_{(X'/X)/(S/S)}(X') : \Omega_{X/S}^\bullet(X) \rightarrow g^* \Omega_{X/S}(X') \rightarrow \Omega_{X'/S}^\bullet(X') \\ \omega &\mapsto \Omega_{(X'/X)/(S/S)}(X')(\omega) := g^*(\omega) : (\alpha \in \wedge^k T_{X'}(X') \mapsto \omega(dg(\alpha))) \end{aligned}$$

(ii) For $S \in \text{AnSp}(\mathbb{C})$, we consider the complexes of presheaves

$$\tilde{\Omega}_{/S}^\bullet := \rho_{S*} \tilde{\Omega}_{/S}^\bullet = \text{coker}(\Omega_{O_{\text{AnSp}(\mathbb{C})^{sm}/S}/e(S)^* O_S} : \Omega_{e(S)^* O_S}^\bullet \rightarrow \Omega_{O_{\text{AnSp}(\mathbb{C})^{sm}/S}}^\bullet) \in C_{O_S}(\text{AnSp}(\mathbb{C})^{sm}/S)$$

which is by definition given by

- for U/S a smooth morphism $\Omega_{/S}^\bullet(U/S) = \Omega_{U/S}^\bullet(U)$
- for $g : U'/S \rightarrow U/S$ a morphism,

$$\begin{aligned}\Omega_{/S}^\bullet(g) &:= \Omega_{(U'/U)/(S/S)}(U') : \Omega_{U/S}^\bullet(U) \rightarrow g^* \Omega_{U/S}(U') \rightarrow \Omega_{U'/S}^\bullet(U') \\ \omega &\mapsto \Omega_{(U'/U)/(S/S)}(U')(\omega) := g^*(\omega) : (\alpha \in \wedge^k T_{U'}(U') \mapsto \omega(dg(\alpha)))\end{aligned}$$

Remark 12. For $S \in \text{AnSp}(\mathbb{C})$, $\Omega_{/S}^\bullet \in C(\text{AnSp}(\mathbb{C})/S)$ is by definition a natural extension of $\Omega_{/S}^\bullet \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$. However $\Omega_{/S}^\bullet \in C(\text{AnSp}(\mathbb{C})/S)$ does NOT satisfy cdh descent.

For a smooth morphism $h : U \rightarrow S$ with $S, U \in \text{AnSm}(\mathbb{C})$, the cohomology presheaves $H^n \Omega_{U/S}^\bullet$ of the relative De Rham complex

$$DR(U/S) := \Omega_{U/S}^\bullet := \text{coker}(h^* \Omega_S \rightarrow \Omega_U) \in C_{h^* O_S}(U)$$

for all $n \in \mathbb{Z}$, have a canonical structure of a complex of $h^* D_S^\infty$ modules given by the Gauss Manin connexion : for $S^o \subset S$ an open subset, $U^o = h^{-1}(S^o)$, $\gamma \in \Gamma(S^o, T_S)$ a vector field and $\hat{\omega} \in \Omega_{U/S}^p(U^o)^c$ a closed form, the action is given by

$$\gamma \cdot [\hat{\omega}] = [\widehat{(\tilde{\gamma})\partial\omega}],$$

$\omega \in \Omega_U^p(U^o)$ being a representative of $\hat{\omega}$ and $\tilde{\gamma} \in \Gamma(U^o, T_U)$ a relevement of γ (h is a smooth morphism), so that

$$DR(U/S) := \Omega_{U/S}^\bullet := \text{coker}(h^* \Omega_S \rightarrow \Omega_U) \in C_{h^* O_S, h^* D^\infty}(U)$$

with this $h^* D_S^\infty$ structure. Hence we get $h_* \Omega_{U/S}^\bullet \in C_{O_S, D^\infty}(S)$ considering this structure. Since h is a smooth morphism, $\Omega_{U/S}^p$ are locally free O_U modules.

The point (ii) of the definition 134 above gives the object in $DA(S)$ which will, for S smooth, represent the analytic Gauss-Manin De Rham realisation. It is the class of an explicit complex of presheaves on $\text{AnSp}(\mathbb{C})^{sm}/S$.

Proposition 114. Let $S \in \text{Var}(\mathbb{C})$.

(i) For $U/S = (U, h) \in \text{AnSp}(\mathbb{C})^{sm}/S$, we have $e(U)_* h^* \Omega_{/S}^\bullet = \Omega_{U/S}^\bullet$.

(ii) The complex of presheaves $(\Omega_{/S}^\bullet, F_b) \in C_{O_S \text{fil}}(\text{AnSp}(\mathbb{C})^{sm}/S)$ is 2-filtered \mathbb{D}_S^1 invariant. Note that however, for $p > 0$, the complexes of presheaves $\Omega_{/S}^{\bullet \geq p}$ are NOT \mathbb{D}_S^1 local. On the other hand, $(\Omega_{/S}^\bullet, F_b)$ admits transferts (recall that means $\text{Tr}(S)_* \text{Tr}(S)^* \Omega_{/S}^p = \Omega_{/S}^p$).

(iii) If S is smooth, we get $(\Omega_{/S}^\bullet, F_b) \in C_{O_S \text{fil}, D_S^\infty}(\text{Var}(\mathbb{C})^{sm}/S)$ with the structure given by the Gauss Manin connexion. Note that however the D_S^∞ structure on the cohomology groups given by Gauss Main connexion does NOT comes from a structure of D_S^∞ module structure on the filtered complex of O_S module. The D_S structure on the cohomology groups satisfy a non trivial Griffithz transversality (in the non projection cases), whereas the filtration on the complex is the trivial one.

Proof. Similar to the proof of proposition 97. □

We have the following canonical transformation map given by the pullback of (relative) differential forms:

Let $g : T \rightarrow S$ a morphism with $T, S \in \text{AnSp}(\mathbb{C})$. Consider the following commutative diagram in RCat :

$$\begin{array}{ccc} D(g, e) : (\text{AnSp}(\mathbb{C})^{sm}/T, O_{\text{AnSp}(\mathbb{C})^{sm}/T}) & \xrightarrow{P(g)} & (\text{AnSp}(\mathbb{C})^{sm}/S, O_{\text{AnSp}(\mathbb{C})^{sm}/S}) \\ \downarrow e(T) & & \downarrow e(S) \\ (T, O_T) & \xrightarrow{P(g)} & (S, O_S) \end{array}$$

It gives (see section 2) the canonical morphism in $C_{g^*O_S \text{fil}}(\text{Var}(\mathbb{C})^{sm}/T)$

$$\begin{aligned} \Omega_{/T/S} &:= \Omega_{(O_{\text{AnSp}(\mathbb{C})^{sm}/T}/g^*O_{\text{AnSp}(\mathbb{C})^{sm}/S})/(O_T/g^*O_S)} : \\ g^*(\Omega_{/S}^\bullet, F_b) &= \Omega_{g^*O_{\text{AnSp}(\mathbb{C})^{sm}/S}/g^*e(S)^*O_S} \rightarrow (\Omega_{/T}^\bullet, F_b) = \Omega_{O_{\text{AnSp}(\mathbb{C})^{sm}/T}/e(T)^*O_T} \end{aligned}$$

which is by definition given by the pullback on differential forms : for $(V/T) = (V, h) \in \text{Var}(\mathbb{C})^{sm}/T$,

$$\begin{aligned} \Omega_{/T/S}(V/T) : g^*(\Omega_{/S}^\bullet)(V/T) &:= \lim_{(h':U \rightarrow \text{SSm}, g':V \rightarrow U, h, g)} \Omega_{U/S}^\bullet(U) \xrightarrow{\Omega_{(V/U)/(T/S)}(V/T)} \Omega_{V/T}^\bullet(V) =: \Omega_{/T}^\bullet(V/T) \\ \hat{\omega} &\mapsto \Omega_{(V/U)/(T/S)}(V/T)(\omega) := g'^*\hat{\omega}. \end{aligned}$$

If S and T are smooth, $\Omega_{/T/S} : g^*(\Omega_{/S}^\bullet, F_b) \rightarrow (\Omega_{/T}^\bullet, F_b)$ is a map in $C_{g^*O_S \text{fil}, g^*D_S^\infty}(\text{AnSp}(\mathbb{C})^{sm}/T)$ It induces the canonical morphism in $C_{g^*O_S \text{fil}, g^*D_S^\infty}(\text{AnSp}(\mathbb{C})^{sm}/T)$:

$$E\Omega_{/T/S} : g^*E_{usu}(\Omega_{/S}^\bullet, F_b) \xrightarrow{T(g, E_{usu})(\Omega_{/S}^\bullet, F_b)} E_{usu}(g^*(\Omega_{/S}^\bullet, F_b)) \xrightarrow{E_{usu}(\Omega_{/T/S})} E_{usu}(\Omega_{/T}^\bullet, F_b).$$

Definition 126. (i) Let $g : T \rightarrow S$ a morphism with $T, S \in \text{AnSp}(\mathbb{C})$. We have, for $F \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$, the canonical transformation in $C_{O_T \text{fil}}(T)$:

$$\begin{aligned} T^O(g, \Omega_{/ \cdot})(F) &: g^{*mod}L_{Oe}(S)_*\mathcal{H}om^\bullet(F, E_{usu}(\Omega_{/S}^\bullet, F_b)) \\ &\xrightarrow{=} (g^*L_{Oe}(S)_*\mathcal{H}om^\bullet(F, E_{et}(\Omega_{/S}^\bullet, F_b))) \otimes_{g^*O_S} O_T \\ &\xrightarrow{T(e, g)(-) \circ T(g, L_O)(-)} L_{Oe}(e(T)_*g^*\mathcal{H}om^\bullet(F, E_{usu}(\Omega_{/S}^\bullet, F_b))) \otimes_{g^*O_S} O_T \\ &\xrightarrow{T(g, \text{hom})(F, E_{et}(\Omega_{/S}^\bullet)) \otimes I} L_{Oe}(e(T)_*\mathcal{H}om^\bullet(g^*F, g^*E_{usu}(\Omega_{/S}^\bullet, F_b))) \otimes_{g^*O_S} O_T \\ &\xrightarrow{ev(\text{hom}, \otimes)(-, -, -)} L_{Oe}(T)_*\mathcal{H}om^\bullet(g^*F, g^*E_{usu}(\Omega_{/S}^\bullet, F_b)) \otimes_{g^*e(S)^*O_S} e(T)^*O_T \\ &\xrightarrow{\mathcal{H}om^\bullet(g^*F, E\Omega_{/T/S}) \otimes I} L_{Oe}(T)_*\mathcal{H}om^\bullet(g^*F, E_{usu}(\Omega_{/T}^\bullet, F_b)) \otimes_{g^*e(S)^*O_S} e(T)^*O_T \\ &\xrightarrow{m} L_{Oe}(T)_*\mathcal{H}om^\bullet(g^*F, E_{usu}(\Omega_{/T}^\bullet, F_b)) \end{aligned}$$

where $m(\alpha \otimes h) := h \cdot \alpha$ is the multiplication map.

(ii) Let $g : T \rightarrow S$ a morphism with $T, S \in \text{AnSp}(\mathbb{C})$, S smooth. Assume there is a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with l a closed embedding, $Y \in \text{AnSm}(\mathbb{C})$ and p_S the projection. We have, for $F \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$, the canonical transformation in $C_{O_T \text{fil}, \mathcal{D}^\infty}(Y \times S)$:

$$\begin{aligned} T^O(g, \Omega_{/ \cdot})(F) &: g^{*mod, \Gamma}e(S)_*\mathcal{H}om^\bullet(F, E_{usu}(\Omega_{/S}^\bullet, F_b)) \\ &\xrightarrow{=} \Gamma_T E_{usu}(p_S^{*mod}e(S)_*\mathcal{H}om^\bullet(F, E_{usu}(\Omega_{/S}^\bullet, F_b))) \\ &\xrightarrow{T^O(p_S, \Omega_{/ \cdot})(F)} \Gamma_T E_{usu}(e(T \times S)_*\mathcal{H}om^\bullet(p_S^*F, E_{usu}(\Omega_{/T \times S}^\bullet, F_b))) \\ &\xrightarrow{=} e(T \times S)_*\Gamma_T(\mathcal{H}om^\bullet(p_S^*F, E_{usu}(\Omega_{/T \times S}^\bullet, F_b))) \\ &\xrightarrow{I(\gamma, \text{hom})(-, -)} e(T \times S)_*\mathcal{H}om^\bullet(\Gamma_T^V p_S^*F, E_{usu}(\Omega_{/T \times S}^\bullet, F_b)). \end{aligned}$$

For $Q \in \text{Proj PSh}(\text{AnSp}(\mathbb{C})^{sm}/S)$,

$$T(g, \Omega_{/ \cdot})(Q) : g^{*mod, \Gamma}e(S)_*\mathcal{H}om^\bullet(Q, E_{usu}(\Omega_{/S}^\bullet, F_b)) \rightarrow e(T \times S)_*\mathcal{H}om^\bullet(\Gamma_T^V p_S^*Q, E_{usu}(\Omega_{/Y \times S}^\bullet, F_b))$$

is a map in $C_{O_T \text{fil}, \mathcal{D}^\infty}(Y \times S)$.

The following easy lemma describe these transformation map on representable presheaves :

Lemma 18. Let $g : T \rightarrow S$ a morphism with $T, S \in \text{AnSp}(\mathbb{C})$ and $h : U \rightarrow S$ is a smooth morphism with $U \in \text{AnSp}(\mathbb{C})$. Consider a commutative diagram whose square are cartesian :

$$\begin{array}{ccccc} g : T & \xrightarrow{l} & S \times Y & \xrightarrow{ps} & S \\ h' \uparrow & & h'' := h \times I \uparrow & & \uparrow h \\ g' : U_T & \xrightarrow{l'} & U \times Y & \xrightarrow{p_U} & U \end{array}$$

with l, l' the graph embeddings and ps, p_U the projections. Then $g^*\mathbb{Z}(U/S) = \mathbb{Z}(U_T/T)$ and

(i) we have the following commutative diagram in $C_{O_T \text{fil}}(T)$ (see definition 1 and definition 126(i)) :

$$\begin{array}{ccc} g^{*mod} L_{Oe}(S)_* \mathcal{H}om^\bullet(\mathbb{Z}(U/S), E_{usu}(\Omega_{/S}^\bullet, F_b)) & \xrightarrow{T(g, \Omega_{/ \cdot})(\mathbb{Z}(U/S))} & e(T)_* \mathcal{H}om^\bullet(\mathbb{Z}(U_T/T), E_{usu}(\Omega_{/T}^\bullet, F_b)) \\ \downarrow = & & \downarrow = \\ g^{*mod} L_{Oh_*} E_{usu}(\Omega_{U/S}^\bullet, F_b) & \xrightarrow{T_\omega^{mod}(g, h)} & h'_* E_{usu}(\Omega_{U_T/T}^\bullet, F_b) \end{array}$$

(ii) if $Y, S \in \text{AnSm}(\mathbb{C})$, we have the following commutative diagram in $C_{O_T \text{fil}, \mathcal{D}^\infty}(T)$ (see definition 1 and definition 126(ii)) :

$$\begin{array}{ccc} g^{*mod, \Gamma} e(S)_* \mathcal{H}om^\bullet(\mathbb{Z}(U/S), E_{usu}(\Omega_{/S}^\bullet, F_b)) & \xrightarrow{T(g, \Omega_{/ \cdot})(\mathbb{Z}(U/S))} & e(T)_* \mathcal{H}om^\bullet(\mathbb{Z}(U_T/T), E_{usu}(\Omega_{/T}^\bullet, F_b)) \\ \downarrow = & & \downarrow = \\ g^{*mod, \Gamma} h_* E_{usu}(\Omega_{U/S}^\bullet, F_b) & \xrightarrow{T_\omega^O(\gamma, \otimes)(-) \circ T_\omega^O(p_S, h)(-)} & h'_* E_{usu}(\Omega_{U_T/T}^\bullet, F_b) \end{array}$$

where $j : T \setminus T \times S \hookrightarrow T \times S$ is the open complementary embedding,

Proof. Obvious. □

Proposition 115. Let $p : S_{12} \rightarrow S_1$ is a smooth morphism with $S_1, S_{12} \in \text{AnSp}(\mathbb{C})$. Then if $Q \in C(\text{AnSp}(\mathbb{C})^{sm}/S_1)$ is projective,

$$T(p, \Omega_{/ \cdot})(Q) : p^{*mod} e(S_1)_* \mathcal{H}om^\bullet(Q, E_{usu}(\Omega_{/S_1}^\bullet, F_b)) \rightarrow e(S_{12})_* \mathcal{H}om^\bullet(p^*Q, E_{usu}(\Omega_{/S_{12}}^\bullet, F_b))$$

is an isomorphism.

Proof. Similar to the proof of proposition 98. □

Let $S \in \text{AnSp}(\mathbb{C})$ and $h : U \rightarrow S$ a morphism with $U \in \text{AnSp}(\mathbb{C})$. We then have the canonical map given by the wedge product

$$w_{U/S} : \Omega_{U/S}^\bullet \otimes_{O_S} \Omega_{U/S}^\bullet \rightarrow \Omega_{U/S}^\bullet; \alpha \otimes \beta \mapsto \alpha \wedge \beta.$$

Let $S \in \text{Var}(\mathbb{C})$ and $h_1 : U_1 \rightarrow S, h_2 : U_2 \rightarrow S$ two morphisms with $U_1, U_2 \in \text{AnSp}(\mathbb{C})$. Denote $h_{12} : U_{12} := U_1 \times_S U_2 \rightarrow S$ and $p_{112} : U_1 \times_S U_2 \rightarrow U_1, p_{212} : U_1 \times_S U_2 \rightarrow U_2$ the projections. We then have the canonical map given by the wedge product

$$w_{(U_1, U_2)/S} : p_{112}^* \Omega_{U_1/S}^\bullet \otimes_{O_S} p_{212}^* \Omega_{U_2/S}^\bullet \rightarrow \Omega_{U_{12}/S}^\bullet; \alpha \otimes \beta \mapsto p_{112}^* \alpha \wedge p_{212}^* \beta$$

which gives the map

$$\begin{aligned} & Ew_{(U_1, U_2)/S} : h_{1*} E_{usu}(\Omega_{U_1/S}^\bullet) \otimes_{O_S} h_{2*} E_{usu}(\Omega_{U_2/S}^\bullet) \\ & \xrightarrow{\text{ad}(p_{112}^*, p_{212}^*)(-) \otimes \text{ad}(p_{112}^*, p_{212}^*)(-)} (h_{1*} p_{112}^* p_{112}^* E_{usu}(\Omega_{U_1/S}^\bullet)) \otimes_{O_S} (h_{2*} p_{212}^* p_{212}^* E_{usu}(\Omega_{U_2/S}^\bullet)) \\ & \xrightarrow{=} h_{12*} (p_{112}^* E_{usu}(\Omega_{U_1/S}^\bullet) \otimes_{h_{12}^* O_S} p_{212}^* E_{usu}(\Omega_{U_2/S}^\bullet)) \\ & \xrightarrow{T(\otimes, E)(-) \circ (T(p_{112}, E)(-) \otimes T(p_{212}, E)(-))} h_{12*} E_{zar}(p_{112}^* \Omega_{U_1/S}^\bullet \otimes_{O_S} p_{212}^* \Omega_{U_2/S}^\bullet) \end{aligned}$$

Let $S \in \text{AnSp}(\mathbb{C})$. We have the canonical map in $C_{O_S \text{fil}}(\text{AnSp}(\mathbb{C})^{sm}/S)$

$$w_S : (\Omega_{/S}^\bullet, F_b) \otimes_{O_S} (\Omega_{/S}^\bullet, F_b) \rightarrow (\Omega_{/S}^\bullet, F_b)$$

given by for $h : U \rightarrow S \in \text{AnSp}(\mathbb{C})^{sm}/S$

$$w_S(U/S) : (\Omega_{U/S}^\bullet, F_b) \otimes_{h^*O_S} (\Omega_{U/S}^\bullet, F_b)(U) \xrightarrow{w_{U/S}(U)} (\Omega_{U/S}^\bullet, F_b)(U)$$

It gives the map

$$Ew_S : E_{usu}(\Omega_{/S}^\bullet, F_b) \otimes_{O_S} E_{usu}(\Omega_{/S}^\bullet, F_b) \xrightarrow{=} E_{usu}((\Omega_{/S}^\bullet, F_b) \otimes_{O_S} (\Omega_{/S}^\bullet, F_b)) \xrightarrow{E_{usu}(w_S)} E_{usu}(\Omega_{/S}^\bullet, F_b)$$

If $S \in \text{AnSm}(\mathbb{C})$,

$$w_S : (\Omega_{/S}^\bullet, F_b) \otimes_{O_S} (\Omega_{/S}^\bullet, F_b) \rightarrow (\Omega_{/S}^\bullet, F_b)$$

is a map in $C_{O_S \text{fil}, D_\infty}(\text{Var}(\mathbb{C})^{sm}/S)$.

Definition 127. Let $S \in \text{AnSp}(\mathbb{C})$. We have, for $F, G \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$, the canonical transformation in $C_{O_S \text{fil}}(S)$:

$$\begin{aligned} T(\otimes, \Omega)(F, G) &: e(S)_* \mathcal{H}om(F, E_{usu}(\Omega_{/S}^\bullet, F_b)) \otimes_{O_S} e(S)_* \mathcal{H}om(G, E_{usu}(\Omega_{/S}^\bullet, F_b)) \\ &\xrightarrow{=} e(S)_*(\mathcal{H}om(F, E_{usu}(\Omega_{/S}^\bullet, F_b)) \otimes_{O_S} \mathcal{H}om(G, E_{usu}(\Omega_{/S}^\bullet, F_b))) \\ &\xrightarrow{e(S)_* T(\mathcal{H}om, \otimes)(-)} e(S)_* \mathcal{H}om(F \otimes G, E_{usu}(\Omega_{/S}^\bullet, F_b)) \otimes_{O_S} E_{usu}(\Omega_{/S}^\bullet, F_b) \\ &\xrightarrow{\mathcal{H}om(F \otimes G, E_{ws})} e(S)_* \mathcal{H}om(F \otimes G, E_{usu}(\Omega_{/S}^\bullet, F_b)) \end{aligned}$$

If $S \in \text{AnSm}(\mathbb{C})$, $T(\otimes, \Omega)(F, G)$ is a map in $C_{O_S \text{fil}, D_\infty}(S)$.

Lemma 19. Let $S \in \text{AnSp}(\mathbb{C})$ and $h_1 : U_1 \rightarrow S$, $h_2 : U_2 \rightarrow S$ two smooth morphisms with $U_1, U_2 \in \text{AnSp}(\mathbb{C})$. Denote $h_{12} : U_{12} := U_1 \times_S U_2 \rightarrow S$ and $p_{112} : U_1 \times_S U_2 \rightarrow U_1$, $p_{212} : U_1 \times_S U_2 \rightarrow U_2$ the projections. We then have the following commutative diagram

$$\begin{array}{ccc} e(S)_* \mathcal{H}om(F, E_{usu}(\Omega_{/S}^\bullet, F_b)) \otimes_{O_S} e(S)_* \mathcal{H}om(G, E_{usu}(\Omega_{/S}^\bullet, F_b)) \xrightarrow{T(\otimes, \Omega)(F, G)} e(S)_* \mathcal{H}om(F \otimes G, E_{usu}(\Omega_{/S}^\bullet, F_b)) & & \\ \downarrow = & & \downarrow = \\ h_{1*} E_{usu}(\Omega_{U_1/S}^\bullet, F_b) \otimes_{O_S} h_{2*} E_{usu}(\Omega_{U_2/S}^\bullet, F_b) \xrightarrow{Ew_{(U_1, U_2)/S}} h_{12*} E_{zar}(\Omega_{U_{12}/S}^\bullet, F_b) & & \end{array}$$

Proof. Follows from Yoneda lemma. \square

We now define the analytic Gauss Manin De Rahm realization functor.

Let $S \in \text{AnSp}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$ an affine space. For $I \subset [1, \dots, l]$, denote by $S_I := \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. Consider, for $I \subset J$, the following commutative diagram

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ j_{IJ} \uparrow & & \uparrow p_{IJ} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}$$

and $j_{IJ} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. Considering the factorization of the diagram D_{IJ} by the fiber product :

$$\begin{array}{ccc}
 D_{IJ} = \tilde{S}_J = \tilde{S}_I \times \tilde{S}_{J \setminus I} & \xrightarrow{p_{IJ}} & \tilde{S}_I \\
 \uparrow i_J & \swarrow i_I \times I & \uparrow i_I \\
 & S_I \times \tilde{S}_{J \setminus I} & \\
 \uparrow l_J & \searrow p_{IJ}^0 & \\
 S_J & \xrightarrow{j_{IJ}} & S_I
 \end{array}$$

the square of this factorization being cartesian, we have for $F \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$ the canonical map in $C(\text{AnSp}(\mathbb{C})^{sm}/\tilde{S}_J)$

$$\begin{aligned}
 S(D_{IJ})(F) : Li_{J*}j_J^*F &\xrightarrow{q} i_{J*}j_J^*F = (i_I \times I) * l_{J*}j_J^*F \xrightarrow{(i_I \times I)_* \text{ad}(p_{IJ\sharp}^0, p_{IJ\sharp}^0)(-)} \\
 &(i_I \times I)_* p_{IJ\sharp}^0 p_{IJ\sharp}^0 l_{J*}j_J^*F \xrightarrow{T(p_{IJ}, i_I)(-)^{-1}} p_{IJ}^* i_{I*} p_{IJ\sharp}^0 l_{J*}j_J^*F = p_{IJ}^* i_{I*} j_I^* F
 \end{aligned}$$

which factors through

$$S(D_{IJ})(F) : Li_{J*}j_J^*F \xrightarrow{S^q(D_{IJ})(F)} p_{IJ}^* Li_{I*}j_I^*F \xrightarrow{q} p_{IJ}^* i_{I*} j_I^* F$$

Definition 128. (i) Let $S \in \text{AnSm}(\mathbb{C})$. We have the functor

$$\text{Hom}^\bullet(\cdot, E_{usu}(\Omega_{/S}^\bullet, F_b)) : C(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow C_{O_{S\text{fil}}, D_S^\infty}(S), F \mapsto e(S)_* \text{Hom}^\bullet(LF, E_{usu}(\Omega_{/S}^\bullet, F_b)).$$

(ii) Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote by $S_I := \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. We have the functor

$$C(\text{Var}(\mathbb{C})^{sm}/S)^{op} \rightarrow C_{O_{\text{fil}}, \mathcal{D}^\infty}(S/(\tilde{S}_I)), F \mapsto (e(\tilde{S}_I)_* \text{Hom}^\bullet(\text{An}_{\tilde{S}_I}^* L(i_{I*}j_I^*F), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b)))[-d_{\tilde{S}_I}], u_{IJ}^q(F))$$

where

$$\begin{aligned}
 u_{IJ}^q(F)[d_{\tilde{S}_I}] &: e(\tilde{S}_I)_* \text{Hom}^\bullet(\text{An}_{\tilde{S}_I}^* L(i_{I*}j_I^*F), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b)) \\
 &\xrightarrow{\text{ad}(p_{IJ}^{*mod}, p_{IJ*})(-)} p_{IJ*} p_{IJ}^{*mod} e(\tilde{S}_I)_* \text{Hom}^\bullet(\text{An}_{\tilde{S}_I}^* L(i_{I*}j_I^*F), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b)) \\
 &\xrightarrow{p_{IJ*} T(p_{IJ}, \Omega_{/S})(L(i_{I*}j_I^*F))} p_{IJ*} e(\tilde{S}_I)_* \text{Hom}^\bullet(\text{An}_{\tilde{S}_I}^* p_{IJ}^* L(i_{I*}j_I^*F), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b)) \\
 &\xrightarrow{p_{IJ*} e(\tilde{S}_I)_* \text{Hom}(\text{An}_{\tilde{S}_I}^* S^q(D_{IJ})(F), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b))} p_{IJ*} e(\tilde{S}_I)_* \text{Hom}^\bullet(\text{An}_{\tilde{S}_I}^* L(i_{I*}j_I^*F), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b)).
 \end{aligned}$$

For $I \subset J \subset K$, we have obviously $p_{IJ*} u_{JK}(F) \circ u_{IJ}(F) = u_{IK}(F)$.

We will prove in corollary 7 below that $u_{IJ}(F)$ are ∞ -filtered usu local equivalence.

Proposition 116. Let $S \in \text{AnSp}(\mathbb{C})$. Let $m : Q_1 \rightarrow Q_2$ be an equivalence (\mathbb{D}^1 , et) local in $C(\text{AnSp}(\mathbb{C})^{sm}/S)$ with Q_1, Q_2 complexes of projective presheaves. Then,

$$e(S)_* \text{Hom}(m, E_{usu}(\Omega_{/S}^\bullet, F_b)) : e(S)_* \text{Hom}^\bullet(Q_2, E_{usu}(\Omega_{/S}^\bullet, F_b)) \rightarrow e(S)_* \text{Hom}^\bullet(Q_1, E_{usu}(\Omega_{/S}^\bullet, F_b))$$

is a quasi-isomorphism. It is thus an isomorphism in $D_{O_{S\text{fil}}, \mathcal{D}^\infty, \infty}(S)$ if S is smooth.

Proof. Similar to the proof of proposition 99. \square

Definition 129. (i) We define, according to proposition 116, the filtered analytic Gauss-Manin realization functor defined as

$$\begin{aligned} \mathcal{F}_{S,an}^{GM} : \text{DA}_c(S)^{op} &\rightarrow D_{O_S \text{fil}, \mathcal{D}^\infty, \infty}(S), \quad M \mapsto \\ \mathcal{F}_{S,an}^{GM}(M) &:= e(S)_* \mathcal{H}om^\bullet(\text{An}_S^* L(F), E_{usu}(\Omega_{/S}^\bullet, F_b))[-d_S] \\ &= e(S)_* \mathcal{H}om^\bullet(L(F), \text{An}_{S^*} E_{usu}(\Omega_{/S}^\bullet, F_b))[-d_S] \end{aligned}$$

where $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$,

(ii) Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote by $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. We define the filtered analytic Gauss-Manin realization functor defined as

$$\begin{aligned} \mathcal{F}_{S,an}^{GM} : \text{DA}_c(S)^{op} &\rightarrow D_{O_S \text{fil}, \mathcal{D}^\infty, \infty}(S/(\tilde{S}_I)), \quad M \mapsto \\ \mathcal{F}_{S,an}^{GM}(M) &:= ((e(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^* F), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b))[-d_{\tilde{S}_I}], u_{IJ}^q(F)) \\ &= ((e(\tilde{S}_I)_* \mathcal{H}om^\bullet(L(i_{I*} j_I^* F), \text{An}_{\tilde{S}_I^*} E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b))[-d_{\tilde{S}_I}], u_{IJ}^q(F)) \end{aligned}$$

where $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$, see definition 128 and corollary 7.

Proposition 117. For $S \in \text{Var}(\mathbb{C})$, the functor \mathcal{F}_S^{GM} is well defined.

Proof. Similar to the proof of proposition 100. \square

Proposition 118. Let $f : X \rightarrow S$ a morphism with $S, X \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. Denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $X_I = \cap_{i \in I} X_i$. Assume there exist a factorization

$$f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$$

of f with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We then have, for $I \subset [1, \dots, l]$, the following commutative diagrams which are cartesian

$$\begin{array}{ccc} f_I = f|_{X_I} : X_I & \xrightarrow{l_I} & Y \times S_I \xrightarrow{p_{S_I}} S_I \\ & \searrow & \downarrow i'_I \quad \downarrow i_I \\ & & Y \times \tilde{S}_I \xrightarrow{p_{\tilde{S}_I}} \tilde{S}_I \end{array} \quad , \quad \begin{array}{ccc} Y \times \tilde{S}_J & \xrightarrow{p_{\tilde{S}_J}} & \tilde{S}_J \\ \downarrow p'_{IJ} & & \downarrow p_{IJ} \\ Y \times \tilde{S}_I & \xrightarrow{p_{\tilde{S}_I}} & \tilde{S}_I \end{array}$$

Let $F(X/S) := p_{S,*} \Gamma_X^\vee \mathbb{Z}(Y \times S/Y \times S)[d_Y]$. The transformations maps $(N_I(X/S) : Q(X_I/\tilde{S}_I) \rightarrow i_{I*} j_I^* F(X/S))$ and $(k \circ I(\gamma, \text{hom})(-, -))$, for $I \subset [1, \dots, l]$, induce an isomorphism in $D_{O_S \text{fil}, \mathcal{D}^\infty, \infty}(S/(\tilde{S}_I))$

$$\begin{aligned} & I^{GM}(X/S) : \\ & \mathcal{F}_{S,an}^{GM}(M^{BM}(X/S)) := (e(\tilde{S}_I)_* \mathcal{H}om(\text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^* F(X/S)), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b))[-d_{\tilde{S}_I}], u_{IJ}^q(F(X/S))) \\ & \xrightarrow{(e(\tilde{S}_I)_* \mathcal{H}om(\text{An}_{\tilde{S}_I}^* N_I(X/S), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b)))} (e(\tilde{S}_I)_* \mathcal{H}om(\text{An}_{\tilde{S}_I}^* Q(X_I/\tilde{S}_I), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b))[-d_{\tilde{S}_I}], v_{IJ}^q(F(X/S))) \\ & \xrightarrow{(e(\tilde{S}_I)_* \mathcal{H}om(T(\text{An}, \gamma^\vee)(-)^{-1}, E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b)))} (e(\tilde{S}_I)_* \mathcal{H}om(Q(X_I^{an}/\tilde{S}_I^{an}), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b))[-d_{\tilde{S}_I}], v_{IJ}^q(F(X/S))) \\ & \xrightarrow{(I(\gamma, \text{hom})(-, -))^{-1}} (p_{\tilde{S}_I,*} \Gamma_{X_I} E_{usu}(\Omega_{Y \times \tilde{S}_I/\tilde{S}_I}^\bullet, F_b)[-d_{\tilde{S}_I}], w_{IJ}(X/S)). \end{aligned}$$

Proof. Similar to the proof of proposition 101. \square

Corollary 7. *Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $D(\mathbb{A}^1, et)(F) \in \text{DA}_c(S)$, $u_{I,J}^q(F)$ are ∞ -filtered usu local equivalence.*

Proof. Similar to corollary 4. \square

We now define the functorialities of \mathcal{F}_S^{GM} with respect to S which makes \mathcal{F}_{GM}^- a morphism of 2-functor.

Definition 130. *Let $g : T \rightarrow S$ a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Consider the factorization $g : T \xrightarrow{l} T \times S \xrightarrow{p_S} S$ where l is the graph embedding and p_S the projection. Let $M \in \text{DA}_c(S)$ and $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}_S^1, et)(F)$. Then, $D(\mathbb{A}_T^1, et)(g^*F) = g^*M$.*

(i) *We have then the canonical transformation in $D_{\mathcal{D}^\infty \text{fil}, \infty}(T \times S)$ (see definition 126) :*

$$\begin{aligned} T(g, \mathcal{F}^{GM})(M) : Rg^{*mod[-], \Gamma} \mathcal{F}_{S, an}^{GM}(M) &:= g^{*mod, \Gamma} e(S)_* \mathcal{H}om^\bullet(\text{An}_S^* L(F), E_{usu}(\Omega_{/S}^\bullet, F_b)) [d_T] \\ &\xrightarrow{T(g, \Omega_{/ \cdot})(\text{An}_S^* L(F))} e(T \times S)_* \mathcal{H}om^\bullet(\Gamma_T^\vee p_S^* \text{An}_S^* L(F), E_{usu}(\Omega_{/Y \times S}^\bullet, F_b)) [d_T] \\ &\xrightarrow{\mathcal{H}om(T(\text{An}, \gamma^\vee)(p_S^* L F)^{-1}, -)} \mathcal{F}_{T \times S, an}^{GM}(l_* g^* M). \end{aligned}$$

where the last isomorphism in the derived category comes from proposition 117.

(ii) *We have then the canonical transformation in $D_{\text{Ofil}, \infty}(T)$ (see definition 126) :*

$$\begin{aligned} T^O(g, \mathcal{F}^{GM})(M) : Lg^{*mod} \mathcal{F}_{S, an}^{GM}(M) &:= g^{*mod} L_O e(S)_* \mathcal{H}om^\bullet(\text{An}_S^* L(F), E_{usu}(\Omega_{/S}^\bullet, F_b)) \\ &\xrightarrow{T(g, \Omega_{/ \cdot})(\text{An}_S^* L(F))} e(T \times S)_* \mathcal{H}om^\bullet(g^* \text{An}_S^* L(F), E_{usu}(\Omega_{/Y \times S}^\bullet, F_b)) =: \mathcal{F}_{T, an}^{GM}(g^* M). \end{aligned}$$

We give now the definition in the non smooth case Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$, Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I := g|_{T_I} : T_I \rightarrow S_I$. We recall the commutative diagram :

$$\begin{array}{ccccc} E_{IJg} = (Y \times \tilde{S}_I) \setminus T_I \xrightarrow{m'_I} Y \times \tilde{S}_J, & E_{IJ} = \tilde{S}_J \setminus S_J \xrightarrow{m_J} \tilde{S}_J & E'_{IJ} = (Y \times \tilde{S}_J) \setminus T_J \xrightarrow{m'_J} Y \times \tilde{S}_J \\ \downarrow p_{S_I} & \downarrow p_{IJ} & \downarrow p'_{IJ} \\ \tilde{S}_I \setminus S_I \xrightarrow{m_I} \tilde{S}_I & \tilde{S}_I \setminus (S_I \setminus S_J) \xrightarrow{m_{IJ}} \tilde{S}_I & (Y \times \tilde{S}_I) \setminus (T_I \setminus T_J) \xrightarrow{m'_{IJ}} Y \times \tilde{S}_I \\ & \downarrow p_{IJ} & \downarrow p'_{IJ} \end{array}$$

For $I \subset J$, denote by $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ and $p'_{IJ} := I_Y \times p_{IJ} : Y \times \tilde{S}_J \rightarrow Y \times \tilde{S}_I$ the projections, so that $\tilde{g}_I \circ p'_{IJ} = p_{IJ} \circ \tilde{g}_J$. Consider, for $I \subset J \subset [1, \dots, l]$, resp. for each $I \subset [1, \dots, l]$, the following commutative diagrams in $\text{Var}(\mathbb{C})$

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ j_{IJ} \uparrow & & \uparrow p_{IJ} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}, \quad D'_{IJ} = \begin{array}{ccc} T_I & \xrightarrow{i'_I} & Y \times \tilde{S}_I \\ j'_{IJ} \uparrow & & \uparrow p'_{IJ} \\ T_J & \xrightarrow{i'_J} & Y \times \tilde{S}_J \end{array}, \quad D_{gI} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ g_I \uparrow & & \uparrow \tilde{g}_I \\ T_I & \xrightarrow{i'_I} & Y \times \tilde{S}_I \end{array},$$

and $j_{IJ} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. Let $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$. The fact that the diagrams (56) commutes says that the maps $T^{q, \gamma}(D_{gI})(j_I^* F)$ define a morphism in $C(\text{Var}(\mathbb{C})^{sm}/(T/(Y \times \tilde{S}_I)))$

$$(T^{q, \gamma}(D_{gI})(j_I^* F)) : (\Gamma_{T_I}^\vee \tilde{g}_I^* L(i_{I*} j_I^* F), \tilde{g}_I^* S^q(D_{IJ})(F)) \rightarrow (L(i'_{I*} j'^*_I g^* F), S^q(D'_{IJ})(g^* F))$$

We then have then the following lemma :

Lemma 20. (i) *The morphism in $C(\text{Var}(\mathbb{C})^{\text{sm}}/(T/(Y \times \tilde{S}_I)))$*

$$(T^{q,\gamma}(D_{g_I})(j_I^*F)) : (\Gamma_{T_I}^\vee L\tilde{g}_I^*i_{I*}j_I^*F, \tilde{g}_J^*S^q(D_{I,J})(F)) \rightarrow (i'_{I*}j'_I{}^*g^*F, S^q(D'_{I,J})(g^*F))$$

is an equivalence (\mathbb{A}^1, et) local.

(ii) *The maps $\mathcal{H}om(\text{An}_{Y \times \tilde{S}_I}^*(T^{q,\gamma}(D_{g_I})(j_I^*F)), E_{usu}(\Omega_{/Y \times \tilde{S}_I}^\bullet, F_b))$ induce an ∞ -filtered quasi-isomorphism in $C_{\text{Ofil}, \mathcal{D}^\infty}(T/(Y \times \tilde{S}_I))$*

$$\begin{aligned} & (\mathcal{H}om(\text{An}_{Y \times \tilde{S}_I}^* T^{q,\gamma}(D_{g_I})(j_I^*F), E_{usu}(\Omega_{/Y \times \tilde{S}_I}^\bullet, F_b))) : \\ & (e(Y \times \tilde{T}_I)_* \mathcal{H}om(\text{An}_{Y \times \tilde{S}_I}^* L(i'_{I*}j'_I{}^*g^*F), E_{usu}(\Omega_{/Y \times \tilde{S}_I}^\bullet, F_b))[-d_{YI}], u_{I,J}^q(g^*F)) \rightarrow \\ & (e(Y \times \tilde{T}_I)_* \mathcal{H}om(\text{An}_{Y \times \tilde{S}_I}^* \Gamma_{T_I}^\vee L(\tilde{g}_I^*i_{I*}j_I^*F), E_{usu}(\Omega_{/Y \times \tilde{S}_I}^\bullet, F_b))[-d_{YI}], \tilde{g}_J^*u_{I,J}^q(F)_2) \end{aligned}$$

(iii) *The maps $T(\tilde{g}_I, \Omega_\cdot)(L(i_{I*}j_I^*F))$ (see definition 126) induce a morphism in $C_{\text{Ofil}, \mathcal{D}^\infty}(T/(Y \times \tilde{S}_I))$*

$$\begin{aligned} & (T(\tilde{g}_I, \Omega_\cdot)(L(i_{I*}j_I^*F))) : \\ & (\Gamma_{T_I} E_{zar}(\tilde{g}_I^{*mod[-]} e(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* L(i_{I*}j_I^*F), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b)))[-d_{YI}], \tilde{g}_J^{*mod} u_{I,J}^q(F)) \rightarrow \\ & (\Gamma_{T_I} (e(Y \times \tilde{S}_I)_* \mathcal{H}om(\text{An}_{Y \times \tilde{S}_I}^* \tilde{g}_I^* L(i_{I*}j_I^*F), E_{usu}(\Omega_{/Y \times \tilde{S}_I}^\bullet, F_b)))[-d_{YI}], \tilde{g}_J^* u_{I,J}^q(F)_1). \end{aligned}$$

Proof. (i): Follows from theorem 14

(ii): Similar to lemma 13(ii).

(iii): Similar to lemma 13(iii). □

Definition 131. *Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$. Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I := g|_{T_I} : T_I \rightarrow S_I$. Denote for short $d_{YI} := d_Y + d_{\tilde{S}_I}$. Let $M \in \text{DA}_c(S)$ and $F \in C(\text{Var}(\mathbb{C})^{\text{sm}}/S)$ such that $M = D(\mathbb{A}_S^1, et)(F)$. Then, $D(\mathbb{A}_T^1, et)(g^*F) = g^*M$. We have, by lemma 13, the canonical transformation in $D_{\text{Ofil}, \mathcal{D}^\infty, \infty}(T/(Y \times \tilde{S}_I))$*

$$\begin{aligned} & T(g, \mathcal{F}^{GM})(M) : Rg^{*mod[-], \Gamma} \mathcal{F}_{S, an}^{GM}(M) := \\ & (\Gamma_{T_I} E_{zar}(\tilde{g}_I^{*mod} e(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* L(i_{I*}j_I^*F), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b)))[-d_{YI}], \tilde{g}_J^{*mod} u_{I,J}^q(F)) \\ & \quad \xrightarrow{(\Gamma_{T_I} E(T(\tilde{g}_I, \Omega_\cdot)(\text{An}_{\tilde{S}_I}^* L(i_{I*}j_I^*F))))} \\ & (\Gamma_{T_I} e(Y \times \tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{Y \times \tilde{S}_I}^* \tilde{g}_I^* L(i_{I*}j_I^*F), E_{usu}(\Omega_{/Y \times \tilde{S}_I}^\bullet, F_b)))[-d_{YI}], \tilde{g}_J^* u_{I,J}^q(F)_1) \\ & \quad \xrightarrow{(I(\gamma, \text{hom}(-, -)))} \\ & (e(Y \times \tilde{S}_I)_* \mathcal{H}om^\bullet(\Gamma_{T_I}^\vee \text{An}_{Y \times \tilde{S}_I}^* \tilde{g}_I^* L(i_{I*}j_I^*F), E_{usu}(\Omega_{/Y \times \tilde{S}_I}^\bullet, F_b)))[-d_{YI}], \tilde{g}_J^* u_{I,J}^q(F)_2) \\ & \quad \xrightarrow{(e(Y \times \tilde{S}_I)_* \mathcal{H}om^\bullet(T(\text{An}_{\tilde{S}_I}^\vee)(\tilde{g}_I^* L(i_{I*}j_I^*F))^{-1}, E_{usu}(\Omega_{/Y \times \tilde{S}_I}^\bullet, F_b)))} \\ & (e(Y \times \tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{Y \times \tilde{S}_I}^* \Gamma_{T_I}^\vee \tilde{g}_I^* L(i_{I*}j_I^*F), E_{usu}(\Omega_{/Y \times \tilde{S}_I}^\bullet, F_b)))[-d_{YI}], \tilde{g}_J^* u_{I,J}^q(F)_2) \\ & \quad \xrightarrow{(e(Y \times \tilde{S}_I)_* \mathcal{H}om(\text{An}_{Y \times \tilde{S}_I}^* T^{q,\gamma}(D_{g_I})(j_I^*F), E_{usu}(\Omega_{/Y \times \tilde{S}_I}^\bullet, F_b)))^{-1}} \\ & (e(Y \times \tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{Y \times \tilde{S}_I}^* L(i'_{I*}j'_I{}^*g^*F), E_{usu}(\Omega_{/Y \times \tilde{S}_I}^\bullet, F_b)))[-d_{YI}], u_{I,J}^q(g^*F)) =: \mathcal{F}_{T, an}^{GM}(g^*M). \end{aligned}$$

Proposition 119. (i) Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y_2 \times S \xrightarrow{p_S} S$ with $Y_2 \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y_2 \times \tilde{S}_i$. Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I := g|_{T_I} : T_I \rightarrow S_I$. Let $f : X \rightarrow S$ a morphism with $X \in \text{Var}(\mathbb{C})$. Assume that there is a factorization $f : X \xrightarrow{l} Y_1 \times S \xrightarrow{p_S} S$, with $Y_1 \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have then the following commutative diagram whose squares are cartesian

$$\begin{array}{ccccc} f' : X_T & \longrightarrow & Y_1 \times T & \longrightarrow & T \\ \downarrow & & \downarrow & & \downarrow \\ f'' = f \times I : Y_2 \times X & \longrightarrow & Y_1 \times Y_2 \times S & \longrightarrow & Y_2 \times S \\ \downarrow & & \downarrow & & \downarrow \\ f : X & \longrightarrow & Y_1 \times S & \longrightarrow & S \end{array}$$

Consider $F(X/S) := p_{S,\#} \Gamma_X^\vee \mathbb{Z}(Y_1 \times S/Y_1 \times S)[d_{Y_1}]$ and the isomorphism in $C(\text{Var}(\mathbb{C})^{sm}/S)$

$$T(f, g, F(X/S)) : g^* F(X/S) := g^* p_{S,\#} \Gamma_X^\vee \mathbb{Z}(Y_1 \times S/Y_1 \times S)[d_{Y_1}] \xrightarrow{\sim} p_{T,\#} \Gamma_{X_T}^\vee \mathbb{Z}(Y_1 \times T/Y_1 \times T)[d_{Y_1}] =: F(X_T/T).$$

which gives in $\text{DA}(S)$ the isomorphism $T(f, g, F(X/S)) : g^* M^{BM}(X/S) \xrightarrow{\sim} M^{BM}(X_T/T)$. Then, the following diagram in $D_{\text{Ofil}, \mathcal{D}^{\infty, \infty}}(T/(Y_2 \times \tilde{S}_I))$ commutes

$$\begin{array}{ccc} Rg^{*mod, \Gamma} \mathcal{F}_{S, an}^{GM}(M^{BM}(X/S)) & \xrightarrow{T(g, \mathcal{F}^{GM})(M^{BM}(X/S))} & \mathcal{F}_{T, an}^{GM}(M^{BM}(X_T/T)) \\ \downarrow I^{GM}(X/S) & & \downarrow I^{GM}(X_T/T) \\ g^{*mod[-], \Gamma}(p_{\tilde{S}_I^*} \Gamma_{X_I} E_{usu}(\Omega_{Y_1 \times \tilde{S}_I / \tilde{S}_I}^\bullet, F_b)[-d_{\tilde{S}_I}], w_{IJ}(X/S)) & \xrightarrow{(T(\tilde{g}_I \times I, \gamma)(-) \circ T_w^O(\tilde{g}_I, p_{\tilde{S}_I}))} & (p_{Y_2 \times \tilde{S}_I^*} \Gamma_{X_{T_I}} E_{usu}(\Omega_{Y_2 \times Y_1 \times \tilde{S}_I / Y_2 \times \tilde{S}_I}^\bullet, F_b)[-d_{Y_2 I}], w_{IJ}(X_T/T)) \end{array}$$

(ii) Let $g : T \rightarrow S$ a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Let $f : X \rightarrow S$ a morphism with $X \in \text{Var}(\mathbb{C})$. Assume that there is a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$, with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Consider $F(X/S) := p_{S,\#} \Gamma_X^\vee \mathbb{Z}(Y \times S/Y \times S)[d_Y]$ and the isomorphism in $C(\text{Var}(\mathbb{C})^{sm}/S)$

$$T(f, g, F(X/S)) : g^* F(X/S) := g^* p_{S,\#} \Gamma_X^\vee \mathbb{Z}(Y \times S/Y \times S)[d_Y] \xrightarrow{\sim} p_{T,\#} \Gamma_{X_T}^\vee \mathbb{Z}(Y \times T/Y \times T)[d_Y] =: F(X_T/T).$$

which gives in $\text{DA}(S)$ the isomorphism $T(f, g, F(X/S)) : g^* M^{BM}(X/S) \xrightarrow{\sim} M^{BM}(X_T/T)$. Then, the following diagram in $D_{\text{Ofil}, \infty}(T)$ commutes

$$\begin{array}{ccc} Lg^{*mod} \mathcal{F}_{S, an}^{GM}(M^{BM}(X/S)) & \xrightarrow{T^O(g, \mathcal{F}^{GM})(M^{BM}(X/S))} & \mathcal{F}_{T, an}^{GM}(M^{BM}(X_T/T)) \\ \downarrow I^{GM}(X/S) & & \downarrow I^{GM}(X_T/T) \\ g^{*mod} L_O(p_{S^*} \Gamma_X E_{usu}(\Omega_{Y \times S / S}^\bullet, F_b)[-d_S]) & \xrightarrow{(T(g \times I, \gamma)(-) \circ T_w^O(g, p_S))} & p_{Y \times T^*} \Gamma_{X_T} E_{usu}(\Omega_{Y \times T / T}^\bullet, F_b)[-d_T] \\ \downarrow T_w(\otimes, \gamma)(O_{Y \times S}) & & \downarrow T_w(\otimes, \gamma)(O_{Y \times T}) \\ Lg^{*mod} \int_{p_S}^{FDR} (\Gamma_X E_{usu}(O_{Y \times S}, F_b)) & \xrightarrow{T^{\mathcal{D}^{mod}}(p_S, f)(-)} & \int_{p_T}^{FDR} (\Gamma_{X_T} E_{usu}(O_{Y \times T}, F_b)). \end{array}$$

Proof. Follows immediately from definition. \square

We have the following theorem:

Theorem 35. (i) Let $g : T \rightarrow S$ is a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S}$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \bigcup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, for $M \in \text{DA}_c(S)$

$$T(g, \mathcal{F}_{an}^{GM})(M) : Rg^{*mod[-], \Gamma} \mathcal{F}_{S, an}^{GM}(M) \rightarrow \mathcal{F}_{T, an}^{GM}(g^*M)$$

is an isomorphism in $D_{O_T \text{fil}, \mathcal{D}^\infty, \infty}(T/(Y \times \tilde{S}_I))$.

(ii) Let $g : T \rightarrow S$ is a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Then, for $M \in \text{DA}_c(S)$

$$T(g, \mathcal{F}_{an}^{GM})(M) : Lg^{*mod} \mathcal{F}_{S, an}^{GM}(M) \rightarrow \mathcal{F}_{T, an}^{GM}(g^*M)$$

is an isomorphism in $D_{O_T}(T)$.

Proof. (i): Follows from proposition 115.

(ii): : **First proof** : Follows from proposition 119, proposition 124 and proposition 86.

: **Second proof** : In the analytic case only, we can give a direct proof of this proposition : Indeed, let $g : T \rightarrow S$ is a morphism with $T, S \in \text{AnSp}(\mathbb{C})$ and let $h : U \rightarrow S$ a smooth morphism with $U \in \text{AnSp}(\mathbb{C})$, then,

$$T_\omega^O(g, h) : g^{*mod} L_{D^\infty} h_* E(\Omega_{U/S}^\bullet, F) \rightarrow h'_* E(\Omega_{U_T/T}^\bullet, F)$$

is an equivalence usu local : consider the following commutative diagram

$$\begin{array}{ccc} g^{*mod} L_O(h_* E(\mathbb{Z}_U) \otimes O_S) & \xrightarrow{T(g, h)(E(\mathbb{Z}_U))} & h'_* E(\mathbb{Z}_{U_T}) \otimes O_T \\ \downarrow g^{*mod} L_O T(h, \otimes)(-, -) & & \downarrow T(h', \otimes)(-, -) \\ g^{*mod} L_O h_* E(h^* O_S) & \xrightarrow{T^{mod}(g, h)(h^* O_S)} & h'_* E(h'^* O_T) \\ \downarrow g^{*mod} L_O h_* E(\iota_{U/S}) & & \downarrow h'_* E(\iota_{U_T/T}) \\ g^{*mod} L_O h_* E(\Omega_{U/S}^\bullet) & \xrightarrow{T_\omega^O(g, h)} & h'_* E(\Omega_{U_T/T}^\bullet) \end{array}$$

then,

- the maps $T(h', \otimes)(-, -)$ and $T(h, \otimes)(-, -)$ are usu local equivalence by proposition 9,
- since $h : U \rightarrow S$ is a smooth morphism, the inclusion $\iota_{U/S} : h^* O_S \rightarrow \Omega_{U/S}^\bullet$ is a quasi-isomorphism,
- since $h' : U_T \rightarrow T$ is a smooth morphism, the inclusion $\iota_{U_T/T} : h'^* O_T \rightarrow \Omega_{U_T/T}^\bullet$ is a quasi-isomorphism,
- since U, U_T, S, T are paracompact topological spaces (in particular Hausdorff), $T(g, h)(E(\mathbb{Z}_U)) : g^* h_* E(\mathbb{Z}_U) \rightarrow h'_* E(\mathbb{Z}_{U_T})$ is a quasi-isomorphism.

This fact, together with lemma 18, proves the proposition. \square

We finish this subsection by a consequence of proposition 114 and theorem 14 :

Theorem 36. Let $X \in \text{PSmVar}(\mathbb{C})$ and $D = \cup D_i \subset X$ a normal crossing divisor. Consider the open embedding $j : U := X \setminus D \hookrightarrow X$. Then, the inclusion

$$j^* : \Gamma(X, E_{usu}(\Omega_X^\bullet(\log D), F_b)) \hookrightarrow \Gamma(U, E_{usu}(\Omega_U^\bullet, F_b))$$

is an ∞ -filtered quasi-isomorphism, that is

$$j^* : F^p H^n(U, \mathbb{C}) := F^p H^n \Gamma(X, E_{usu}(\Omega_X^\bullet(\log D), F_b)) \xrightarrow{\sim} F^p H^n \Gamma(U, E_{usu}(\Omega_U^\bullet, F_b)) =: F_b^p H^n(U, \mathbb{C})$$

is an isomorphism for all $n, p \in \mathbb{Z}$ (note that it is obviously injective since $j^* : H^n \Gamma(X, E_{usu}(\Omega_X^\bullet(\log D))) \xrightarrow{\sim} H^n \Gamma(U, E_{usu}(\Omega_U^\bullet))$ is an isomorphism if we forgot filtrations). Note that however, it is NOT a filtered quasi-isomorphism (for example if U is affine $H^q(U, \Omega_U^p) = 0$ for $q > 0$) that it is not an isomorphism on the E_1 terms of the spectral sequences in general.

Proof. Similar to the proof of theorem 31. \square

Definition 132. Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open affine covering and denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We have, for $M, N \in \text{DA}(S)$ and $F, G \in C(\text{Var}(\mathbb{C})^{\text{sm}}/S)$ such that $M = D(\mathbb{A}^1, \text{et})(F)$ and $N = D(\mathbb{A}^1, \text{et})(G)$, the following transformation map in $D_{\text{Ofil}, \mathcal{D}^\infty}(S/(\tilde{S}_I))$

$$\begin{aligned} & T(\mathcal{F}_{S, \text{an}}^{GM}, \otimes)(M, N) : \\ & \mathcal{F}_{S, \text{an}}^{GM}(M) \otimes_{O_S}^L \mathcal{F}_{S, \text{an}}^{GM}(N) := (e(\tilde{S}_I)_* \mathcal{H}om(\text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^* F), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b)), u_{IJ}(F)) \otimes_{O_S} \\ & \quad (e(\tilde{S}_I)_* \mathcal{H}om(\text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^* G), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b)), u_{IJ}(G)) \\ & \quad \xrightarrow{=} ((e(\tilde{S}_I)_* \mathcal{H}om(\text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^* F), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b)) \otimes_{O_{\tilde{S}_I}} \\ & \quad e(\tilde{S}_I)_* \mathcal{H}om(\text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^* G), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b))), u_{IJ}(F) \otimes u_{IJ}(G)) \\ & \quad \xrightarrow{(T(\otimes, \Omega_{/\tilde{S}_I}^\bullet)(L(i_{I*} j_I^* F), L(i_{I*} j_I^* G)))} \\ & \quad (e(\tilde{S}_I)_* \mathcal{H}om(\text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^* F) \otimes \text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^* G), E_{\text{et}}(\Omega_{/\tilde{S}_I}^\bullet, F_b)), v_{IJ}(F \otimes G)) \\ & \quad \xrightarrow{=} (e(\tilde{S}_I)_* \mathcal{H}om(\text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^*(F \otimes G), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b))), u_{IJ}(F \otimes G)) =: \mathcal{F}_{S, \text{an}}^{GM}(M \otimes N) \end{aligned}$$

We have in the analytical case the following :

Proposition 120. Let $S \in \text{Var}(\mathbb{C})$. Then, for $M, N \in \text{DA}_c(S)$

$$T(\otimes, \mathcal{F}_{S, \text{an}}^{GM})(M, N) : \mathcal{F}_{S, \text{an}}^{GM}(M \otimes N) \xrightarrow{\sim} \mathcal{F}_{S, \text{an}}^{GM}(M) \otimes_{O_S}^L \mathcal{F}_{S, \text{an}}^{GM}(N)$$

is an isomorphism.

Proof. Assume first that S is smooth. Let $h_1 : U_1 \rightarrow S$ and $h_2 : U_2 \rightarrow S$ smooth morphisms with $U_1, U_2 \in \text{Var}(\mathbb{C})$ and consider $h_{12} : U_1 \times_S U_2 \rightarrow S$. We then have by lemma 19 the following commutative diagram

$$\begin{array}{ccc} e(S)_* \mathcal{H}om(\mathbb{Z}(U_1/S) \otimes \mathbb{Z}(U_2/S), E(\Omega_{/S}^\bullet, F_b)) \xrightarrow{T(\otimes, \mathcal{F}_{S, \text{an}}^{GM}(M^{BM}(U_1/S), M^{BM}(U_2/S)))} e(S)_* \mathcal{H}om(\mathbb{Z}(U_1/S), E(\Omega_{/S}^\bullet, F_b)) \otimes_{O_S} e(S)_* \mathcal{H}om(\mathbb{Z}(U_2/S), E(\Omega_{/S}^\bullet, F_b)) \\ \downarrow = \\ h_{12*} E(\Omega_{U_1 \times_S U_2/S}, F) \xrightarrow{Ew_{(U_1, U_2)/S}} h_{1*} E(\omega_{U_1/S}, F_b) \otimes_{O_S} h_{2*} E(\Omega_{U_2/S}, F_b) \\ \downarrow \iota \otimes \iota \\ h_{12*} E(h_{12}^* O_S) \xrightarrow{Ew_{(U_1, U_2)/S}} h_{1*} E(h_1^* O_S) \otimes_{O_S} h_{2*} E(h_2^* O_S) \\ \downarrow T(h_{12}, \otimes)(O_S, \mathbb{Z}_{U_{12}}) \\ h_{12*} E(\mathbb{Z}_{U_{12}}) \otimes_{O_S} \xrightarrow{Ew_{(U_1, U_2)/S}} (h_{1*} E(\mathbb{Z}_{U_1}) \otimes_{O_S} h_{2*} E(\mathbb{Z}_{U_2})) \end{array}$$

Since $U_1, U_2 \in \text{AnSp}(\mathbb{C})$ are locally contractible topological spaces, the lower row is an equivalence usu local by Kunnetth formula for topological spaces (see section 2). This proves the proposition in the case S is smooth. Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. By definition, for $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1, et)(F)$ and $N = D(\mathbb{A}^1, et)(G)$,

$$\begin{aligned} & T(\otimes, \mathcal{F}_{S,an}^{GM}(M, N)) : \\ & e(\tilde{S}_I)_* \mathcal{H}om(\text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^*(F \otimes G), E_{usu}(\Omega_{\tilde{S}_I}^\bullet, F_b)), u_{IJ}(F \otimes G)) \\ & \xrightarrow{(T(\otimes, \Omega_{\tilde{S}_I})(\text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^* F), \text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^* G)))} \\ & (e(\tilde{S}_I)_* \mathcal{H}om(\text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^* F), E_{usu}(\Omega_{\tilde{S}_I}^\bullet, F_b)), u_{IJ}(F)) \otimes_{O_S} \\ & (e(\tilde{S}_I)_* \mathcal{H}om(\text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^* G), E_{usu}(\Omega_{\tilde{S}_I}^\bullet, F_b)), u_{IJ}(G)) \end{aligned}$$

Since $L(i_{I*} j_I^* F), L(i_{I*} j_I^* G) \in \text{DA}_c(\tilde{S}_I)$, by the smooth case applied to \tilde{S}_I for each I , $T(\otimes, \mathcal{F}_{S,an}^{FDR}(M, N))$ is an equivalence usu local. \square

6.2.2 The analytic filtered De Rham realization functor

Recall from section 2 that, for $S \in \text{Var}(\mathbb{C})$ we have the following commutative diagrams of sites

$$\begin{array}{ccccc} \text{AnSp}(\mathbb{C})^2/S & \xrightarrow{\mu_S} & \text{AnSp}(\mathbb{C})^{2,pr}/S & & \\ \downarrow \text{Ans} & \searrow \rho_S & \downarrow \rho_S & \searrow \rho_S & \\ \text{AnSp}(\mathbb{C})^{2,sm}/S & \xrightarrow{\mu_S} & \text{AnSp}(\mathbb{C})^{2,smpr}/S & & \\ \downarrow \text{Ans} & \searrow \rho_S & \downarrow \text{Ans} & \searrow \rho_S & \\ \text{Var}(\mathbb{C})^2/S & \xrightarrow{\mu_S} & \text{Var}(\mathbb{C})^{2,smpr}/S & & \\ \downarrow \text{Ans} & \searrow \rho_S & \downarrow \text{Ans} & \searrow \rho_S & \\ \text{Var}(\mathbb{C})^2/S & \xrightarrow{\mu_S} & \text{Var}(\mathbb{C})^{2,smpr}/S & & \end{array}$$

and

$$\begin{array}{ccccc} \text{AnSp}(\mathbb{C})^{2,pr}/S & \xrightarrow{\text{Gr}_S^{12}} & \text{AnSp}(\mathbb{C})/S & & \\ \downarrow \text{Ans} & \searrow \rho_S & \downarrow \text{Gr}_S^{12} & \searrow \rho_S & \\ \text{AnSp}(\mathbb{C})^{2,smpr}/S & \xrightarrow{\text{Gr}_S^{12}} & \text{AnSp}(\mathbb{C})^{sm}/S & & \\ \downarrow \text{Ans} & \searrow \rho_S & \downarrow \text{Ans} & \searrow \rho_S & \\ \text{Var}(\mathbb{C})^{2,pr}/S & \xrightarrow{\text{Gr}_S^{12}} & \text{Var}(\mathbb{C})/S & & \\ \downarrow \text{Ans} & \searrow \rho_S & \downarrow \text{Ans} & \searrow \rho_S & \\ \text{Var}(\mathbb{C})^{2,sm}/S & \xrightarrow{\text{Gr}_S^{12}} & \text{Var}(\mathbb{C})^{sm}/S & & \end{array}, \quad (59)$$

and that for $f : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$ we have the following commutative diagrams of site,

$$\begin{array}{ccccc}
\text{AnSp}(\mathbb{C})^2/T^{an} & \xrightarrow{\text{An}_T} & \text{Var}(\mathbb{C})^2/T & & \\
\downarrow P(f) & \searrow \rho_T & \downarrow P(f) & \searrow \rho_T & \\
& \text{AnSp}(\mathbb{C})^{2,sm}/T^{an} & \xrightarrow{\text{An}_T} & \text{Var}(\mathbb{C})^{2,sm}/T & \\
& \downarrow P(f) & & \downarrow P(f) & \\
\text{AnSp}(\mathbb{C})^2/S^{an} & \xrightarrow{\text{An}_S} & \text{Var}(\mathbb{C})^2/S & & \\
& \searrow \rho_S & \downarrow P(f) & \searrow \rho_S & \\
& \text{AnSp}(\mathbb{C})^{2,sm}/S^{an} & \xrightarrow{\text{An}_S} & \text{Var}(\mathbb{C})^{2,sm}/S & \\
& & & \downarrow P(f) &
\end{array}$$

Definition 133. (i) For $S \in \text{AnSp}(\mathbb{C})$, we consider the filtered complexes of presheaves

$$(\Omega_{/S}^{\bullet,\Gamma}, F_b) \in C_{O_S \text{fil}}(\text{AnSp}(\mathbb{C})^2/S)$$

given by

– for $((X, Z), h) = (X, Z)/S \in \text{AnSp}(\mathbb{C})^2/S$,

$$\begin{aligned}
(\Omega_{/S}^{\bullet,\Gamma}((X, Z)/S), F_b) &: = \Gamma_Z^{\vee, h} L_{h^*O}(\Omega_{X/S}^{\bullet,\Gamma}, F_b)(X) \\
&: = \mathbb{D}_{h^*O_S} L_{h^*O} \Gamma_Z E_{usu}(\mathbb{D}_{h^*O_S} L_{h^*O}(\Omega_{X/S}^{\bullet,\Gamma}, F_b))(X)
\end{aligned}$$

– for $g : (X_1, Z_1)/S = ((X_1, Z_1), h_1) \rightarrow (X, Z)/S = ((X, Z), h)$ a morphism in $\text{AnSp}(\mathbb{C})^2/S$,

$$\begin{aligned}
\Omega_{/S}^{\bullet,\Gamma}(g) &: \mathbb{D}_{h^*O_S} L_{h^*O} \Gamma_Z E_{usu}(\mathbb{D}_{h^*O_S} L_{h^*O}(\Omega_{X/S}^{\bullet,\Gamma}, F_b))(X) \\
&\quad \mathbb{D}_{h_1^*O_S} L_{h_1^*O} \Gamma_{Z_1} E_{usu}(\mathbb{D}_{h_1^*O_S} L_{h_1^*O}(\Omega_{X_1/S}^{\bullet,\Gamma}, F_b))(X_1)
\end{aligned}$$

is given as in definition 110(i). For $S \in \text{AnSm}(\mathbb{C})$, we consider the complexes of presheaves

$$(\Omega_{/S}^{\bullet,\Gamma}, F_b) := \rho_{S^*}(\Omega_{/S}^{\bullet,\Gamma}, F_b) \in C_{O_S \text{fil}, D_S^\infty}(\text{AnSp}(\mathbb{C})^{2,sm}/S)$$

(ii) For $S \in \text{AnSm}(\mathbb{C})$, we have the canonical map $C_{O_S \text{fil}, D_S^\infty}(\text{AnSp}(\mathbb{C})^{sm}/S)$

$$\text{Gr}^O(\Omega_{/S}) : \text{Gr}_{S^*}^{12} \mu_{S^*}(\Omega_{/S}^{\bullet,\Gamma}, F_b) \rightarrow (\Omega_{/S}^{\bullet,\Gamma}, F_b)$$

given as in definition 110(ii).

Definition 134. (i) For $S \in \text{SmVar}(\mathbb{C})$, we consider, using definition 112(i), the filtered complexes of presheaves

$$(\Omega_{/S^{an}}^{\bullet,\Gamma,pr}, F_{DR}) \in C_{D_S^\infty \text{fil}}(\text{Var}(\mathbb{C})^{2,smpr}/S)$$

given by,

– for $(Y \times S, Z)/S = ((Y \times S, Z), p) \in \text{Var}(\mathbb{C})^{2,smpr}/S$,

$$(\Omega_{/S^{an}}^{\bullet,\Gamma,pr}((Y \times S, Z)/S), F_{DR}) := ((\Omega_{(Y \times S)^{an}}^{\bullet,\Gamma,pr}, F_b) \otimes_{O_{(Y \times S)^{an}}} (\Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b))^{an})(Y \times S)^{an}$$

with the structure of p^*D_S module given by proposition 54.

– for $g : (Y_1 \times S, Z_1)/S = ((Y_1 \times S, Z_1), p_1) \rightarrow (Y \times S, Z)/S = ((Y \times S, Z), p)$ a morphism in $\text{Var}(\mathbb{C})^{2, \text{smpr}}/S$,

$$\Omega_{/S^{an}}^{\bullet, \Gamma, pr}(g) := (\Omega_{/S}^{\bullet, \Gamma, pr}(g))^{an} : ((\Omega_{(Y \times S)^{an}/S^{an}}^{\bullet, \Gamma, pr}, F_b) \otimes_{O_{(Y \times S)^{an}}} (\Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b))^{an})((Y \times S)^{an}) \rightarrow ((\Omega_{(Y_1 \times S)^{an}/S^{an}}^{\bullet, \Gamma, pr}, F_b) \otimes_{O_{(Y_1 \times S)^{an}}} (\Gamma_{Z_1}^{\vee, Hdg}(O_{Y_1 \times S}, F_b))^{an})((Y_1 \times S)^{an}).$$

For $S \in \text{SmVar}(\mathbb{C})$, we get the filtered complexes of presheaves

$$(\Omega_{/S^{an}}^{\bullet, \Gamma, pr, an}, F_{DR}) := \text{An}_S^{*mod}(\Omega_{/S^{an}}^{\bullet, \Gamma, pr}, F_{DR}) := \text{An}_S^*(\Omega_{/S^{an}}^{\bullet, \Gamma, pr}, F_{DR}) \otimes_{O_S} O_{S^{an}} \in C_{D_S^\infty \text{fil}}(\text{AnSp}(\mathbb{C})^{2, \text{smpr}}/S).$$

(ii) For $S \in \text{SmVar}(\mathbb{C})$, we have the canonical map $C_{O_S \text{fil}, D_S^\infty}(\text{Var}(\mathbb{C})^{sm}/S)$

$$\text{Gr}(\Omega_{/S^{an}}) : \text{Gr}_{S^*}^{12}(\Omega_{/S^{an}}^{\bullet, \Gamma, pr}, F_b) \rightarrow \text{An}_{S^*}(\Omega_{/S}^{\bullet, \Gamma}, F_b)$$

given by

$$\begin{aligned} \text{Gr}(\Omega_{/S^{an}})(U/S) &:= (\text{Gr}(\Omega_{/S})(U/S))^{an} \otimes m : \\ J_S((\Omega_{(U \times S)^{an}/S^{an}}^{\bullet, \Gamma, pr}, F_b) \otimes_{O_{(U \times S)^{an}}} (\Gamma_U^{\vee, Hdg}(O_{U \times S}, F_b))^{an})((U \times S)^{an}) &\rightarrow (\Omega_{U^{an}/S^{an}}^{\bullet, \Gamma}, F_b), \end{aligned}$$

where $\text{Gr}(\Omega_{/S^{an}})(U/S)(\omega \otimes m \otimes P) := P(\text{Gr}(\Omega_{/S})(U/S)(\omega \otimes m))$ with $P \in \Gamma(S, D_S^\infty)$, see definition 112(ii), which gives by adjonction

$$\text{Gr}(\Omega_{/S^{an}}) := I(\text{An}_S^{*mod}, \text{An}_S)(\text{Gr}(\Omega_{/S^{an}})) : J_S(\text{Gr}_{S^*}^{12}(\Omega_{/S^{an}}^{\bullet, \Gamma, pr, an}, F_b)) \rightarrow (\Omega_{/S}^{\bullet, \Gamma}, F_b)$$

in $C_{O_S \text{fil}, D_S^\infty}(\text{AnSp}(\mathbb{C})^{sm}/S)$.

Definition 135. For $S \in \text{SmVar}(\mathbb{C})$, we have the canonical map in $C_{O_S \text{fil}, D_S^\infty}(\text{Var}(\mathbb{C})^{2, \text{smpr}}/S)$

$$T(\Omega_{/S^{an}}^\Gamma) : \text{An}_{S^*} \mu_{S^*}(\Omega_{/S^{an}}^{\bullet, \Gamma}, F_b) \rightarrow (\Omega_{/S^{an}}^{\bullet, \Gamma, pr}, F_{DR})$$

given by, for $(Y \times S, X)/S = ((Y \times S, Z), p) \in \text{Var}(\mathbb{C})^{2, \text{smpr}}/S$

$$\begin{aligned} T(\Omega_{/S^{an}}^\Gamma)((Y \times S, Z)/S) &:= (T(\Omega_{/S}^\Gamma)((Y \times S, Z)/S))^{an} : \\ (\Omega_{/S}^{\bullet, \Gamma}, F_b)((Y \times S)^{an}, Z^{an})/S &:= \mathbb{D}_{p^* O_S} L_{p^* O} \Gamma_Z E_{usu}(\mathbb{D}_{p^* O_S} L_{p^* O}(\Omega_{(Y \times S)^{an}/S^{an}}^{\bullet, \Gamma}, F_b))((Y \times S)^{an}) \rightarrow \\ ((\Omega_{(Y \times S)^{an}/S^{an}}^{\bullet, \Gamma}, F_b) \otimes_{O_{(Y \times S)^{an}}} (\Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b))^{an}) &((Y \times S)^{an}) =: (\Omega_{/S^{an}}^{\bullet, \Gamma, pr}, F_{DR})((Y \times S, Z)/S), \end{aligned}$$

see definition 113. By definition we have $\text{Gr}^O(\Omega_{/S^{an}}) = \text{Gr}(\Omega_{/S^{an}}) \circ T(\Omega_{/S^{an}}^\Gamma)$.

Proposition 121. (i) Let $S \in \text{AnSp}(\mathbb{C})$. The complex of presheaves $(\Omega_{/S}^{\bullet, \Gamma}, F_b) \in C_{O_S \text{fil}}(\text{AnSp}(\mathbb{C})^{2, sm}/S)$ is ∞ -filtered \mathbb{D}_S^1 invariant (see definition 16) and admits transferts (i.e. $\text{Tr}(S)_* \text{Tr}(S)^* \Omega_{/S}^{\bullet, \Gamma} = \Omega_{/S}^{\bullet, \Gamma}$).

(ii) Let $S \in \text{SmVar}(\mathbb{C})$. The complex of presheaves $(\Omega_{/S^{an}}^{\bullet, \Gamma, pr, an}, F_{DR}) \in C_{D_S^\infty \text{fil}}(\text{AnSp}(\mathbb{C})^{2, \text{smpr}}/S)$ is ∞ -filtered \mathbb{D}_S^1 local for the usual or etale topology (see definition 16) and admits transferts.

Proof. Similar to proposition 104. □

We have the following canonical transformation map given by the pullback of (relative) differential forms:

Let $g : T \rightarrow S$ a morphism with $T, S \in \text{AnSm}(\mathbb{C})$.

- We have the canonical morphism in $C_{g^*O_S \text{fil}, g^*D_S^\infty}(\text{AnSp}(\mathbb{C})^{2,sm}/T)$

$$\Omega_{/(T/S)}^\Gamma : g^*(\Omega_{/S}^{\bullet,\Gamma}, F_b) \rightarrow (\Omega_{/T}^{\bullet,\Gamma}, F_b)$$

induced by the pullback of differential forms : for $((V, Z_1)/T) = ((V, Z_1), h) \in \text{AnSp}(\mathbb{C})^{2,sm}/T$,

$$\begin{aligned} & \Omega_{/(T/S)}^\Gamma((V, Z_1)/T) : \\ g^*\Omega_{/S}^{\bullet,\Gamma}((V, Z_1)/T) & := \lim_{(h:(U,Z) \rightarrow \text{SSm}, g_1:(V,Z_1) \rightarrow (U_T, Z_T), h, g)} \Omega_{/S}^{\bullet,\Gamma}((U, Z)/S) \\ & \xrightarrow{\Omega_{/S}^{\bullet,\Gamma}(g' \circ g_1)} \Omega_{/S}^{\bullet,\Gamma}((V, Z_1)/S) \xrightarrow{\Gamma_{Z_1}^{\vee, h} q(Y_1 \times T)} \Omega_{/T}^{\bullet,\Gamma}((V, Z_1)/T), \end{aligned}$$

where $g' : U_T := U \times_S T \rightarrow U$ is the base change map and $q : \Omega_{Y_1 \times T/S}^{\bullet} \rightarrow \Omega_{Y_1 \times T/T}^{\bullet}$ is the quotient map. It induces the canonical morphisms in $C_{g^*O_S \text{fil}, g^*D_S^\infty}(\text{AnSp}(\mathbb{C})^{2,sm}/T)$:

$$E\Omega_{/(T/S)}^\Gamma : g^*E_{et}(\Omega_{/S}^{\bullet,\Gamma}, F_b) \xrightarrow{T(g, E_{et})(\Omega_{/S}^{\bullet,\Gamma}, F_b)} E_{et}(g^*(\Omega_{/S}^{\bullet,\Gamma}, F_b)) \xrightarrow{E_{et}(\Omega_{/(T/S)}^\Gamma)} E_{et}(\Omega_{/T}^{\bullet,\Gamma}, F_b)$$

- We have the canonical morphism in $C_{g^*D_S^\infty \text{fil}}(\text{Var}(\mathbb{C})^{2,smpr}/T)$

$$\Omega_{/(T/S)an}^{\Gamma, pr} : g^*(\Omega_{/San}^{\bullet,\Gamma, pr}, F_{DR}) \rightarrow (\Omega_{/Tan}^{\bullet,\Gamma, pr}, F_{DR})$$

induced by the pullback of differential forms : for $((Y_1 \times T, Z_1)/T) = ((Y_1 \times T, Z_1), p) \in \text{Var}(\mathbb{C})^{2,smpr}/T$,

$$\begin{aligned} & \Omega_{/(T/S)an}^{\Gamma, pr}((Y_1 \times T, Z_1)/T) : \\ g^*\Omega_{/San}^{\bullet,\Gamma, pr}((Y_1 \times T, Z_1)/T) & := \lim_{(h:(Y \times S, Z) \rightarrow S, g_1:(Y_1 \times T, Z_1) \rightarrow (Y \times T, Z_T), h, g)} \Omega_{/San}^{\bullet,\Gamma, pr}((Y \times T, Z)/S) \\ & \xrightarrow{\Omega_{/San}^{\bullet,\Gamma, pr}(g' \circ g_1)} \Omega_{/San}^{\bullet,\Gamma, pr}((Y_1 \times T, Z_1)/S) \xrightarrow{q(-)((Y_1 \times T)^{an})} \Omega_{/Tan}^{\bullet,\Gamma, pr}((Y_1 \times T, Z_1)/T), \end{aligned}$$

where $g' = (I_Y \times g) : Y \times T \rightarrow Y \times S$ is the base change map and

$$q(M) : \Omega_{(Y_1 \times T)^{an}/San} \otimes_{O_{(Y_1 \times T)^{an}}} (M, F) \rightarrow \Omega_{(Y_1 \times T)^{an}/Tan} \otimes_{O_{(Y_1 \times T)^{an}}} (M, F)$$

is the quotient map. It induces the canonical morphisms in $C_{g^*D_S \text{fil}}(\text{Var}(\mathbb{C})^{2,smpr}/T)$:

$$E\Omega_{/(T/S)an}^{\Gamma, pr} : g^*E_{et}(\Omega_{/San}^{\bullet,\Gamma, pr}, F_{DR}) \xrightarrow{T(g, E)(-)} E_{et}(g^*(\Omega_{/San}^{\bullet,\Gamma, pr}, F_{DR})) \xrightarrow{E_{et}(\Omega_{/(T/S)an}^{\Gamma, pr})} E_{et}(\Omega_{/Tan}^{\bullet,\Gamma, pr}, F_{DR})$$

and

$$E\Omega_{/(T/S)an}^{\Gamma, pr} : g^*E_{zar}(\Omega_{/San}^{\bullet,\Gamma, pr}, F_{DR}) \xrightarrow{T(g, E)(-)} E_{zar}(g^*(\Omega_{/San}^{\bullet,\Gamma, pr}, F_{DR})) \xrightarrow{E_{zar}(\Omega_{/(T/S)an}^{\Gamma, pr})} E_{zar}(\Omega_{/Tan}^{\bullet,\Gamma, pr}, F_{DR}).$$

Definition 136. Let $g : T \rightarrow S$ a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. We have, for $F \in C(\text{Var}(\mathbb{C})^{2,smpr}/S)$, the canonical transformation in $C_{\mathcal{D}^\infty \text{fil}}(T)$:

$$\begin{aligned} & T(g, \Omega_{/S}^{\Gamma, pr})(F) : g^{*mod} L_D e(S)_* \text{Gr}_{S^*}^{12} \mathcal{H}om^\bullet(\text{An}_S^* F, E_{et}(\Omega_{/San}^{\bullet,\Gamma, pr, an}, F_{DR})) \\ & \xrightarrow{=} (g^* L_D e(S)_* \mathcal{H}om^\bullet(F, E_{et}(\Omega_{/San}^{\bullet,\Gamma, pr, an}, F_{DR}))) \otimes_{g^*O_S} O_T \\ & \xrightarrow{T(g, \text{Gr}^{12})(-) \circ T(e, g)(-) \circ q} e(T)_* \text{Gr}_{T^*}^{12} g^* \mathcal{H}om^\bullet(\text{An}_S^* F, E_{et}(\Omega_{/San}^{\bullet,\Gamma, pr}, F_{DR})) \otimes_{g^*O_S} O_T \\ & \xrightarrow{(T(g, \text{hom})(-, -) \otimes I)} e(T)_* \text{Gr}_{T^*}^{12} \mathcal{H}om^\bullet(\text{An}_T^* g^* F, g^* E_{et}(\Omega_{/San}^{\bullet,\Gamma, pr, an}, F_{DR})) \otimes_{g^*O_S} O_T \\ & \xrightarrow{ev(\text{hom}, \otimes)(-, -, -)} e(T)_* \text{Gr}_{T^*}^{12} \mathcal{H}om^\bullet(\text{An}_T^* g^* F, g^* E_{et}(\Omega_{/San}^{\bullet,\Gamma, pr, an}, F_{DR})) \otimes_{g^*e(S)^*O_S} e(T)^* O_T \\ & \xrightarrow{\mathcal{H}om^\bullet(\text{An}_T^* g^* F, (E\Omega_{/(T/S)}^{\Gamma, pr} \otimes m))} e(T)_* \text{Gr}_{T^*}^{12} \mathcal{H}om^\bullet(\text{An}_T^* g^* F, E_{et}(\Omega_{/Tan}^{\bullet,\Gamma, pr, an}, F_{DR})) \end{aligned}$$

- Let $S \in \text{AnSm}(\mathbb{C})$. We have the map in $C_{O_S \text{fil}, D_S}(\text{Var}(\mathbb{C})^{2, \text{smpr}}/S)$:

$$w_S : (\Omega_{/S}^{\bullet, \Gamma}, F_b) \otimes_{O_S} (\Omega_{/S}^{\bullet, \Gamma}, F_b) \rightarrow (\Omega_{/S}^{\bullet, \Gamma}, F_b) :$$

given by for $h : (U, Z) \rightarrow S \in \text{Var}(\mathbb{C})^{2, \text{sm}}/S$,

$$\begin{aligned} w_S((U, Z)/S) : (\Gamma_Z^{\vee, h} L_{h^* O_S}(\Omega_{U/S}^{\bullet, \Gamma}, F_b) \otimes_{p^* O_S} \Gamma_Z^{\vee, h} L_{h^* O_S}(\Omega_{U/S}^{\bullet, \Gamma}, F_b))(U) \\ \xrightarrow{(DR(-)(\gamma_Z^{\vee, h}(-)) \circ w_{U/S})^\gamma(U)} \Gamma_Z^{\vee, h} L_{h^* O_S}(\Omega_{U/S}^{\bullet, \Gamma}, F_b)(U) \end{aligned}$$

which induces the map in $C_{O_S \text{fil}, D_S}(\text{Var}(\mathbb{C})^{2, \text{sm}}/S)$

$$Ew_S : E_{et}(\Omega_{/S}^{\bullet, \Gamma}, F_b) \otimes_{O_S} E_{et}(\Omega_{/S}^{\bullet, \Gamma}, F_b) \xrightarrow{\cong} E_{et}((\Omega_{/S}^{\bullet, \Gamma}, F_b) \otimes_{O_S} (\Omega_{/S}^{\bullet, \Gamma}, F_b)) \xrightarrow{E_{et}(w_S)} E_{et}(\Omega_{/S}^{\bullet, \Gamma}, F_b).$$

- Let $S \in \text{SmVar}(\mathbb{C})$. We have the map in $C_{D_S^\infty \text{fil}}(\text{Var}(\mathbb{C})^{2, \text{smpr}}/S)$:

$$w_S : (\Omega_{/S^{\text{an}}}^{\bullet, \Gamma, \text{pr}}, F_{DR}) \otimes_{O_S} (\Omega_{/S^{\text{an}}}^{\bullet, \Gamma, \text{pr}}, F_{DR}) \rightarrow (\Omega_{/S^{\text{an}}}^{\bullet, \Gamma, \text{pr}}, F_{DR})$$

given by for $p : (Y \times S, Z) \rightarrow S \in \text{Var}(\mathbb{C})^{2, \text{smpr}}/S$,

$$\begin{aligned} w_S((Y \times S, Z)/S) : \\ (((\Omega_{Y \times S/S}^{\bullet, \Gamma, \text{pr}} \otimes_{O_{Y \times S}} \Gamma_Z^{\vee, \text{Hdg}}(O_{Y \times S}, F_b)) \otimes_{p^* O_S} (\Omega_{Y \times S/S}^{\bullet, \Gamma, \text{pr}} \otimes_{O_{Y \times S}} \Gamma_Z^{\vee, \text{Hdg}}(O_{Y \times S}, F_b)))(Y \times S) \\ \xrightarrow{(DR(-)(\gamma_Z^{\vee, \text{Hdg}}(-)) \circ w_{Y \times S/S})^\gamma(Y \times S)} (\Omega_{Y \times S/S}^{\bullet, \Gamma, \text{pr}} \otimes_{O_{Y \times S}} \Gamma_Z^{\vee, \text{Hdg}}(O_{Y \times S}, F_b))(Y \times S) \end{aligned}$$

which induces the map in $C_{D_S^\infty \text{fil}}(\text{Var}(\mathbb{C})^{2, \text{smpr}}/S)$

$$\begin{aligned} Ew_S : E_{et}(\Omega_{/S^{\text{an}}}^{\bullet, \Gamma, \text{pr}}, F_{DR}) \otimes_{O_S} E_{et}(\Omega_{/S^{\text{an}}}^{\bullet, \Gamma, \text{pr}}, F_{DR}) \xrightarrow{\cong} \\ E_{et}((\Omega_{/S^{\text{an}}}^{\bullet, \Gamma, \text{pr}}, F_{DR}) \otimes_{O_S} (\Omega_{/S^{\text{an}}}^{\bullet, \Gamma, \text{pr}}, F_{DR})) \xrightarrow{E_{et}(w_S)} E_{et}(\Omega_{/S^{\text{an}}}^{\bullet, \Gamma, \text{pr}}, F_{DR}) \end{aligned}$$

by the functoriality of the Godement resolution (see section 2).

Definition 137. Let $S \in \text{SmVar}(\mathbb{C})$. We have, for $F, G \in C(\text{Var}(\mathbb{C})^{2, \text{smpr}}/S)$, the canonical transformation in $C_{D^\infty \text{fil}}(S^{\text{an}})$:

$$\begin{aligned} T(\otimes, \Omega)(F, G) : \\ e(S)_* \text{Gr}_{S^*}^{12} \mathcal{H}om(\text{An}_S^* F, E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}, \text{an}}, F_{DR})) \otimes_{O_S} e(S)_* \text{Gr}_{S^*}^{12} \mathcal{H}om(\text{An}_S^* G, E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}, \text{an}}, F_{DR})) \\ \xrightarrow{\cong} e(S)_* \text{Gr}_{S^*}^{12} (\mathcal{H}om(\text{An}_S^* F, E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}, \text{an}}, F_{DR})) \otimes_{O_S} \mathcal{H}om(\text{An}_S^* G, E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}, \text{an}}, F_{DR}))) \\ \xrightarrow{T(\mathcal{H}om, \otimes)(-)} e(S)_* \text{Gr}_{S^*}^{12} \mathcal{H}om(\text{An}_S^* F \otimes \text{An}_S^* G, E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})) \otimes_{O_S} E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})) \\ \xrightarrow{\cong} e(S)_* \text{Gr}_{S^*}^{12} \mathcal{H}om(\text{An}_S^*(F \otimes G), E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})) \otimes_{O_S} E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})) \\ \xrightarrow{\mathcal{H}om(F \otimes G, \text{An}_S^{* \text{mod}} Ew_S)} e(S)_* \text{Gr}_{S^*}^{12} \mathcal{H}om(F \otimes G, E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}, \text{an}}, F_{DR})). \end{aligned}$$

We now define the filtered analytic De Rahm realization functor.

Definition 138. (i) Let $S \in \text{SmVar}(\mathbb{C})$. We have, using definition 134 and definition 36, the functor

$$\begin{aligned} C(\text{Var}(\mathbb{C})^{\text{sm}}/S) \rightarrow C_{D^\infty \text{fil}}(S^{\text{an}}), F \mapsto \\ \varinjlim_{r_{(X^*, D^*)/S}(LF)} e'(S)_* \mathcal{H}om^\bullet(\text{An}_S^* \rho_{S^*} \text{Gr}_S^{12*} R_{(X^*, D^*)/S}(\rho_S^* L(F)), E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}, \text{an}}, F_{DR}))[-d_S] \\ = e'(S)_* \mathcal{H}om^\bullet(\rho_{S^*} \text{Gr}_S^{12*} R_{(X^*, D^*)/S}(\rho_S^* L(F)), \text{An}_S^* E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}, \text{an}}, F_{DR}))[-d_S] \end{aligned}$$

denoting for short $e'(S) = e(S) \circ \text{Gr}_S^{12}$.

(ii) Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote by $S_I := \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. Consider, for $I \subset J$, the following commutative diagram

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ j_{IJ} \uparrow & & \uparrow p_{IJ} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}$$

and $j_{IJ} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. We have, using definition 134 and definition 36, the functor

$$\begin{aligned} & C(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C_{\mathcal{D}^\infty \text{fil}}(S^{an}/(\tilde{S}_I^{an})), F \mapsto \\ & (\varinjlim_{r-(-)} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \rho_{\tilde{S}_I}^* \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{et}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F)) \\ & = (\varinjlim_{r-(-)} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\rho_{\tilde{S}_I}^* \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), \text{An}_{\tilde{S}_I}^* E_{et}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F)) \end{aligned}$$

where we have denoted for short $e'(\tilde{S}_I) = e(\tilde{S}_I) \circ \text{Gr}_{\tilde{S}_I}^{12}$, the limits run over the Corti-Hanamura resolutions

$$r_{(X^*, D^*)/\tilde{S}_I}(L(i_{I*} j_I^* F)) : R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)) \rightarrow \mathbb{D}_{\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F))$$

and

$$\begin{aligned} & u_{IJ}^q(F)[d_{\tilde{S}_J}] : e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \rho_{\tilde{S}_I}^* \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{et}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})) \\ & \xrightarrow{\text{ad}(p_{IJ}^{*mod}, p_{IJ})(-)} \\ & p_{IJ*} p_{IJ}^{*mod} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \rho_{\tilde{S}_I}^* \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{et}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})) \\ & \xrightarrow{p_{IJ*} T(p_{IJ}, \Omega_{\tilde{S}_I}^{\bullet, pr})(-)} \\ & p_{IJ*} e'(\tilde{S}_J)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \rho_{\tilde{S}_J}^* \text{Gr}_{\tilde{S}_I}^{12*} p_{IJ}^* R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{et}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})) \\ & \xrightarrow{\mathcal{H}om(\text{Gr}_{\tilde{S}_J}^{12*} T(p_{IJ}, R^{CH})(Li_{I*} j_I^* F)^{-1}, E_{et}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR}))} \\ & p_{IJ*} e'(\tilde{S}_J)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \rho_{\tilde{S}_J}^* \text{Gr}_{\tilde{S}_J}^{12*} R_{(X^* \times \tilde{S}_{J \setminus I}, D^* \times \tilde{S}_{J \setminus I})/\tilde{S}_I}(\rho_{\tilde{S}_I}^* p_{IJ}^* L(i_{I*} j_I^* F)), E_{et}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})) \\ & \xrightarrow{\mathcal{H}om(\text{Gr}_{\tilde{S}_J}^{12*} R_{\tilde{S}_J}^{CH}(T^q(D_{IJ})(j_I^* F)), E_{et}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR}))} \\ & p_{IJ*} e'(\tilde{S}_J)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \rho_{\tilde{S}_J}^* \text{Gr}_{\tilde{S}_J}^{12*} R_{(X^* \times \tilde{S}_{J \setminus I}, D^* \times \tilde{S}_{J \setminus I})/\tilde{S}_J}(\rho_{\tilde{S}_J}^* L(i_{J*} j_J^* F)), E_{et}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})). \end{aligned}$$

For $I \subset J \subset K$, we have obviously $p_{IJ*} u_{JK}(F) \circ u_{IJ}(F) = u_{IK}(F)$. We will prove in corollary 8 below that $u_{IJ}(F)$ are ∞ -filtered Zariski local equivalence.

We have the following key proposition :

Proposition 122. *Let $S \in \text{SmVar}(\mathbb{C})$.*

(i) *Let $m : Q_1 \rightarrow Q_2$ be an etale local equivalence local with $Q_1, Q_2 \in C(\text{Proj PSh}(\text{Var}(\mathbb{C})^{sm}/S))$ complexes of representable presheaves. Then,*

$$\begin{aligned} & e'(S)_* \mathcal{H}om^\bullet(\text{An}_S^* \rho_{S^*} \text{Gr}_S^{12*} R^{CH}(\rho_S^*(m)), E_{et}(\Omega_S^{\bullet, \Gamma, pr, an}, F_{DR}))[-d_S] : \\ & \varinjlim_{r_{(X^*, D^*)/S}(Q_1)} e'(S)_* \mathcal{H}om^\bullet(\text{An}_S^* \rho_{S^*} \text{Gr}_S^{12*} R_{(X^*, D^*)/S}(\rho_S^* Q_1), E_{et}(\Omega_S^{\bullet, \Gamma, pr, an}, F_{DR}))[-d_S] \\ & \rightarrow \varinjlim_{r_{(X^*, D^*)/S}(Q_2)} e'(S)_* \mathcal{H}om^\bullet(\text{An}_S^* \rho_{S^*} \text{Gr}_S^{12*} R_{(X^*, D^*)/S}(\rho_S^* Q_2), E_{et}(\Omega_S^{\bullet, \Gamma, pr, an}, F_{DR}))[-d_S] \end{aligned}$$

is an ∞ -filtered quasi-isomorphism. It is thus an isomorphism in $D_{\mathcal{D}fil,\infty}(S)$.

(ii) Let $m : Q_1 \rightarrow Q_2$ be an equivalence (\mathbb{A}^1, et) local with $Q_1, Q_2 \in C(\text{Proj PSh}(\text{Var}(\mathbb{C})^{sm}/S))$ complexes of representable presheaves. Then,

$$\begin{aligned} & e'(S)_* \mathcal{H}om^\bullet(\text{An}_S^* \rho_{S*} \text{Gr}_S^{12*} R^{CH}(\rho_S^*(m)), E_{et}(\Omega_{/S}^{\bullet,\Gamma,pr,an}, F_{DR}))[-d_S] : \\ & \varinjlim_{r_{(X^*, D^*)/S}(Q_1)} e'(S)_* \mathcal{H}om^\bullet(\text{An}_S^* \rho_{S*} \text{Gr}_S^{12*} R_{(X^*, D^*)/S}(\rho_S^* Q_1), E_{et}(\Omega_{/S}^{\bullet,\Gamma,pr,an}, F_{DR}))[-d_S] \\ & \rightarrow \varinjlim_{r_{(X^*, D^*)/S}(Q_2)} e'(S)_* \mathcal{H}om^\bullet(\text{An}_S^* \rho_{S*} \text{Gr}_S^{12*} R_{(X^*, D^*)/S}(\rho_S^* Q_2), E_{et}(\Omega_{/S}^{\bullet,\Gamma,pr,an}, F_{DR}))[-d_S] \end{aligned}$$

is an ∞ -filtered quasi-isomorphism. It is thus an isomorphism in $D_{\mathcal{D}fil,\infty}(S)$.

Proof. Similar to the proof of proposition 105. \square

Definition 139. (i) Let $S \in \text{SmVar}(\mathbb{C})$. We define using definition 138(i) and proposition 122(ii) the filtered algebraic De Rahm realization functor defined as

$$\begin{aligned} & \mathcal{F}_{S,an}^{FDR} : \text{DA}_c(S) \rightarrow D_{\mathcal{D}^\infty fil,\infty}(S^{an}), M \mapsto \mathcal{F}_S^{FDR}(M) := \\ & \varinjlim_{r_{(X^*, D^*)/S}(L(F))} e'(S)_* \mathcal{H}om^\bullet(\text{An}_S^* \rho_{S*} \text{Gr}_S^{12*} R_{(X^*, D^*)/S}(\rho_S^* L(F)), E_{et}(\Omega_{/S}^{\bullet,\Gamma,pr,an}, F_{DR}))[-d_S] \\ & \xrightarrow{\sim} e'(S)_* \mathcal{H}om^\bullet(\text{An}_S^* \rho_{S*} \text{Gr}_S^{12*} R_{(X^*, D^*)/S}(\rho_S^* L(F)), E_{et}(\Omega_{/S}^{\bullet,\Gamma,pr,an}, F_{DR}))[-d_S] \end{aligned}$$

where $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$.

(i)' For the Corti-Hanamura weight structure W on $\text{DA}_c(S)^-$, we define using definition 138(i) and proposition 122(ii)

$$\begin{aligned} & \mathcal{F}_{S,an}^{FDR} : \text{DA}_c^-(S) \rightarrow D_{\mathcal{D}^\infty(1,0)fil,\infty}^-(S^{an}), M \mapsto \mathcal{F}_S^{FDR}((M, W)) := \\ & \varinjlim_{r_{(X^*, D^*)/S}(L(F, W))} e'(S)_* \mathcal{H}om^\bullet(\text{An}_S^* \rho_{S*} \text{Gr}_S^{12*} R_{(X^*, D^*)/S}(\rho_S^* L(F, W)), E_{et}(\Omega_{/S}^{\bullet,\Gamma,pr,an}, F_{DR}))[-d_S] \\ & \xrightarrow{\sim} e'(S)_* \mathcal{H}om^\bullet(\text{An}_S^* \rho_{S*} \text{Gr}_S^{12*} R_{(X^*, D^*)/S}(\rho_S^* L(F, W)), E_{et}(\Omega_{/S}^{\bullet,\Gamma,pr,an}, F_{DR}))[-d_S] \end{aligned}$$

where $(F, W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)((F, W))$ using corollary 1. Note that the filtration induced by W is a filtration by sub D_S module, which is a stronger property than Griffithz transversality. Of course, the filtration induced by F satisfy only Griffithz transversality in general.

(ii) Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote by $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. We define, using definition 138(ii), proposition 122(ii) and corollary 8, the filtered algebraic De Rahm realization functor defined as

$$\begin{aligned} & \mathcal{F}_{S,an}^{FDR} : \text{DA}_c(S) \rightarrow D_{\mathcal{D}^\infty fil,\infty}(S^{an}/(\tilde{S}_I^{an})), M \mapsto \mathcal{F}_S^{FDR}(M) := \\ & (\varinjlim_{r_{(-)}} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \rho_{\tilde{S}_I*} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet,\Gamma,pr,an}, F_{DR}))[-d_{\tilde{S}_I}], u_{I,J}^q(F)) \end{aligned}$$

where $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$, see definition 116.

(ii)' For the Corti-Hanamura weight structure W on $\text{DA}_c^-(S)$, using definition 116(ii), proposition 105(ii) and corollary 5,

$$\begin{aligned} & \mathcal{F}_{S,an}^{FDR} : \text{DA}_c^-(S) \rightarrow D_{\mathcal{D}^\infty(1,0)fil,\infty}^-(S^{an}/(\tilde{S}_I^{an})), M \mapsto \mathcal{F}_S^{FDR}((M, W)) := \\ & (\varinjlim_{r_{(-)}} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \rho_{\tilde{S}_I*} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^*(F, W))), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet,\Gamma,pr,an}, F_{DR}))[-d_{\tilde{S}_I}], u_{I,J}^q(F, W)) \end{aligned}$$

where $(F, W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $(M, W) = D(\mathbb{A}^1, et)(F, W)$ using corollary 1. Note that the filtration induced by W is a filtration by sub $D_{\tilde{S}_I}$ -modules, which is a stronger property than Griffithz transversality. Of course, the filtration induced by F satisfy only Griffithz transversality in general.

Proposition 123. For $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$, the functor $\mathcal{F}_{S, an}^{F, DR}$ is well defined.

Proof. Similar to the proof of proposition 106. □

Proposition 124. Let $f : X \rightarrow S$ a morphism with $S, X \in \text{Var}(\mathbb{C})$. Assume there exist a factorization

$$f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$$

of f with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $\bar{Y} \in \text{PSmVar}(\mathbb{C})$ a compactification of Y with $\bar{Y} \setminus Y = D$ a normal crossing divisor, denote $k : D \hookrightarrow \bar{Y}$ the closed embedding and $n : Y \hookrightarrow \bar{Y}$ the open embedding. Denote $\bar{X} \subset \bar{Y} \times S$ the closure of $X \subset Y \times S$. We have then the following commutative diagram in $\text{Var}(\mathbb{C})$

$$\begin{array}{ccc} X & \xrightarrow{l} & Y \times S \\ \downarrow & & \downarrow (n \times I) \\ \bar{X} & \xrightarrow{l} & \bar{Y} \times S \\ \uparrow & \nearrow l_Z & \uparrow (k \times I) \\ Z := \bar{X} \setminus X & \longrightarrow & D \times S \end{array} \begin{array}{c} \\ \\ \\ \\ \end{array} \begin{array}{c} \\ \\ \xrightarrow{p_S} \\ \xrightarrow{\bar{p}_S} \\ \nearrow \end{array} \begin{array}{c} \\ \\ S \\ S \end{array}$$

Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. Denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $X_I = \cap_{i \in I} X_i$. Denote $\bar{X}_I := \bar{X} \cap (\bar{Y} \times S_I) \subset \bar{Y} \times \tilde{S}_I$ the closure of $X_I \subset \bar{Y} \times \tilde{S}_I$, and $Z_I := Z \cap (\bar{Y} \times S_I) = \bar{X}_I \setminus X_I \subset \bar{Y} \times \tilde{S}_I$. We have then for $I \subset [1, \dots, l]$, the following commutative diagram in $\text{Var}(\mathbb{C})$

$$\begin{array}{ccc} X_I & \xrightarrow{l_I} & Y \times \tilde{S}_I \\ \downarrow & & \downarrow (n \times I) \\ \bar{X}_I & \xrightarrow{l_I} & \bar{Y} \times \tilde{S}_I \\ \uparrow & \nearrow l_{Z_I} & \uparrow (k \times I) \\ Z_I = \bar{X}_I \setminus X_I & \longrightarrow & D \times \tilde{S}_I \end{array} \begin{array}{c} \\ \\ \\ \\ \end{array} \begin{array}{c} \\ \\ \xrightarrow{p_{S_I}} \\ \xrightarrow{\bar{p}_{S_I}} \\ \nearrow \end{array} \begin{array}{c} \\ \\ \tilde{S}_I \\ \tilde{S}_I \end{array}$$

Let $F(X/S) := p_{S,\#}\Gamma_X^\vee \mathbb{Z}(X \times S/X \times S)$. We have then the following isomorphism in $D_{\mathcal{D}fil,\infty}(S/(\tilde{S}_I))$

$$\begin{aligned}
I(X/S) &: \mathcal{F}_{S,an}^{FDR}(M(X/S)) \xrightarrow{\cong} \\
& \left(\varinjlim_{r-(-)} e'(\tilde{S}_I)_* \mathcal{H}om(\text{An}_{\tilde{S}_I}^* \rho_{\tilde{S}_I^*} \text{Gr}_{\tilde{S}_I}^{12*} R_{((\tilde{Y} \times \tilde{S}_I)^*, E^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F(X/S))), \right. \\
& E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})[-d_{\tilde{S}_I}], u_{IJ}^q(F(X/S))) \xrightarrow{(\mathcal{H}om(\text{An}_{\tilde{S}_I}^* \text{Gr}_{\tilde{S}_I}^{12*} R_{\tilde{S}_I}^{CH}(N_I(X/S)), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_b)))} \\
& \left(\varinjlim_{r-(-)} e'(\tilde{S}_I)_* \mathcal{H}om(\text{An}_{\tilde{S}_I}^* \rho_{\tilde{S}_I^*} \text{Gr}_{\tilde{S}_I}^{12*} R_{((\tilde{Y} \times \tilde{S}_I)^*, E^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* Q(X_I/\tilde{S}_I)), \right. \\
& E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})[-d_{\tilde{S}_I}], v_{IJ}^q(F(X/S))) \xrightarrow{(\mathcal{H}om(\text{An}_{\tilde{S}_I}^* \rho_{\tilde{S}_I^*} I_\delta((\tilde{X}_I, Z_I)/\tilde{S}_I), k)[-d_{\tilde{S}_I}])^{-1}} \\
& (\bar{p}_{\tilde{S}_I^*} E_{usu}((\Omega_{\tilde{Y} \times \tilde{S}_I/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_b) \otimes_{O_{\tilde{Y} \times \tilde{S}_I}} (n \times I)_!^{Hdg} \Gamma_{X_I}^{\vee, Hdg}(O_{(Y \times \tilde{S}_I)^{an}}, F_b))[-2d_Y - d_{\tilde{S}_I}], w_{IJ}(X/S)) \\
& \xrightarrow{\cong} \iota_S Rf_!^{Hdg}(\Gamma_{X_I}^{\vee, Hdg}(O_{(Y \times \tilde{S}_I)^{an}}, F_b), x_{IJ}(X/S)).
\end{aligned}$$

Proof. Similar to the proof of proposition 107. \square

Corollary 8. Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $D(\mathbb{A}^1, et)(F) \in \text{DA}_c(S)$, $u_{IJ}^q(F)$ are ∞ -filtered usu local equivalence.

Proof. Similar to the proof of corollary 5. \square

Corollary 9. Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, for $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1, et)(F) \in \text{DA}_c(S)$,

$$\begin{aligned}
H^i \mathcal{F}_{S,an}^{FDR}(M, W) &:= ((e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \rho_{\tilde{S}_I^*} \text{Gr}_{\tilde{S}_I}^{12*} R_{((\tilde{Y} \times \tilde{S}_I)^*, E^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^*(F, W))), \\
& E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], u_{IJ}^q(F, W)) \in \pi_S(MHM(S^{an}))
\end{aligned}$$

for all $i \in \mathbb{Z}$.

Proof. Similar to the proof of corollary 6. \square

Proposition 125. For $S \in \text{Var}(\mathbb{C})$ not smooth, the functor (see corollary 6)

$$\iota_S^{-1} \mathcal{F}_{S,an}^{FDR} : \text{DA}_c^-(S)^{op} \rightarrow \pi_S(D(MHM(S^{an})))$$

does not depend on the choice of the open cover $S = \cup_i S_i$ and the closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$.

Proof. Similar to the proof of proposition 108. \square

We have the canonical transformation map between the filtered analytic De Rham realization functor and the analytic Gauss-Manin realization functor :

Definition 140. Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Let $M \in \text{DA}_c(S)$ and $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that

$M = D(\mathbb{A}^1, et)(F)$. We have, using definition 134(ii), the canonical map in $D_{O_S \text{fil}, D^\infty, \infty}(S^{an}/(\tilde{S}^{an}))$

$$\begin{aligned}
& T(\mathcal{F}_{S,an}^{GM}, \mathcal{F}_{S,an}^{FDR})(M) : \\
& \mathcal{F}_{S,an}^{GM}(L\mathbb{D}_S M) := (e(\tilde{S}_I)_* \text{Hom}^\bullet(\text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^* \mathbb{D}_S L F), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)), u_{IJ}^q(F)) \\
& \quad \xrightarrow{\sim} (e(\tilde{S}_I)_* \text{Hom}^\bullet(\text{An}_{\tilde{S}_I}^* L\mathbb{D}_{\tilde{S}_I} L(i_{I*} j_I^* F), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)), u_{IJ}^{q,d}(F)) \\
& \xrightarrow{\mathcal{H}om(-, \text{Gr}(\Omega_{\tilde{S}_I}^{q^n}))^{-1}} J_S(e(\tilde{S}_I)_* \text{Hom}^\bullet(\text{An}_{\tilde{S}_I}^* L\mathbb{D}_{\tilde{S}_I} L(i_{I*} j_I^* F), \text{Gr}_{\tilde{S}_I}^{12} E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})), u_{IJ}^{q,d}(F)) \\
& \xrightarrow{I(\text{Gr}_{\tilde{S}_I}^{12*}, \text{Gr}_{\tilde{S}_I}^{12*})(-, -)} J_S(e'(\tilde{S}_I)_* \text{Hom}^\bullet(\text{An}_{\tilde{S}_I}^* \text{Gr}_{\tilde{S}_I}^{12*} L\mathbb{D}_{\tilde{S}_I} L(i_{I*} j_I^* F), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})), u_{IJ}^{q,d}(F)) \\
& \quad \xrightarrow{(\text{Hom}^\bullet(\text{An}_{\tilde{S}_I}^* \rho_{\tilde{S}_I} \text{Gr}_{\tilde{S}_I}^{12*} r_{(X^*, D^*)/\tilde{S}_I}^{CH}(L(i_{I*} j_I^* F)), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})))} \\
& J_S(\varinjlim_{r-(-)} e'(\tilde{S}_I)_* \text{Hom}(\text{An}_{\tilde{S}_I}^* \rho_{\tilde{S}_I} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F)) \\
& \quad \quad \quad =: J_S(\mathcal{F}_{S,an}^{FDR}(M))
\end{aligned}$$

We now define the functorialities of \mathcal{F}_S^{FDR} with respect to S which makes \mathcal{F}_{FDR}^- a morphism of 2 functor.

Definition 141. Let $S \in \text{Var}(\mathbb{C})$. Let $Z \subset S$ a closed subset. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Denote $Z_I := Z \cap S_I$. We then have closed embeddings $Z_I \hookrightarrow S_I \hookrightarrow \tilde{S}_I$.

(i) For $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$, we will consider the following canonical map in $\pi_S(D(\text{MHM}(S^{an}))) \subset D_{\mathcal{D}(1,0)\text{fil}}(S^{an}/(\tilde{S}^{an}))$

$$\begin{aligned}
& T(\Gamma_Z^{\vee, Hdg}, \Omega_{/S}^{\Gamma, pr, an})(F, W) : \\
& \Gamma_Z^{\vee, Hdg} \iota_S^{-1}(\varinjlim_{r-(-)} e'(\tilde{S}_I)_* \text{Hom}^\bullet(\text{An}_{\tilde{S}_I}^* \rho_{\tilde{S}_I} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^*(F, W))), \\
& \quad \quad \quad E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F, W)) \\
& \quad \quad \quad \xrightarrow{\mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \rho_{\tilde{S}_I} \text{Gr}_{\tilde{S}_I}^{12*} r_{(X^*, D^*)/\tilde{S}_I}^{CH}(\gamma^{\vee, Z_I}(L(i_{I*} j_I^*(F, W))), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})))} \\
& \Gamma_Z^{\vee, Hdg} \iota_S^{-1}(\varinjlim_{r-(-)} e'(\tilde{S}_I)_* \text{Hom}^\bullet(\text{An}_{\tilde{S}_I}^* \rho_{\tilde{S}_I} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, *, D^*, *)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* \Gamma_{Z_I}^\vee L(i_{I*} j_I^*(F, W))), \\
& \quad \quad \quad E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^{q,Z}(F, W)) \\
& \quad \quad \quad \xrightarrow{\cong} \iota_S^{-1}(\varinjlim_{r-(-)} e'(\tilde{S}_I)_* \text{Hom}^\bullet(\text{An}_{\tilde{S}_I}^* \rho_{\tilde{S}_I} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, *, D^*, *)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* \Gamma_{Z_I}^\vee L(i_{I*} j_I^*(F, W))), \\
& \quad \quad \quad E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^{q,Z}(F, W)),
\end{aligned}$$

with $u_{IJ}^{q,Z}(F)$ given as in definition 119(i).

(ii) For $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$, we have also the following canonical map in $\pi_S(D(\text{MHM}(S^{an}))) \subset$

$$D_{\mathcal{D}(1,0)fil}(S^{an}/(\tilde{S}^{an}))$$

$$\begin{aligned}
& T(\Gamma_Z^{Hdg}, \Omega_{/S}^{\Gamma, pr})(F, W) : \\
& \iota_S^{-1}(\varinjlim_{r-(-)} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \rho_{\tilde{S}_I} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^{*,*}, D^{*,*})/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L\Gamma_{Z_I} E(i_{I*} j_I^* \mathbb{D}_S(F, W))), \\
& \quad E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], u_{IJ}^{q, Z, d}(F, W)) \\
& \xrightarrow{=} \Gamma_Z^{Hdg} \iota_S^{-1}(\varinjlim_{r-(-)} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \rho_{\tilde{S}_I} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^{*,*}, D^{*,*})/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L\Gamma_{Z_I} E(i_{I*} j_I^* \mathbb{D}_S(F, W))), \\
& \quad E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], u_{IJ}^{q, Z, d}(F, W)) \\
& \quad \xrightarrow{\mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \rho_{\tilde{S}_I} \text{Gr}_{\tilde{S}_I}^{12*} r^{CH}(\gamma^{Z_I}(-)), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))} \\
& \Gamma_Z^{Hdg} \iota_S^{-1}(\varinjlim_{r-(-)} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \rho_{\tilde{S}_I} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* \mathbb{D}_S(F, W))), \\
& \quad E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], u_{IJ}^q(F, W))
\end{aligned}$$

with $u_{IJ}^{q, Z}(F)$ given as in definition 119(ii).

Definition 142. Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$. Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I := g|_{T_I} : T_I \rightarrow S_I$. Let $M \in \text{DA}_c(S)$ and $(F, W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ such that $(M, W) = D(\mathbb{A}_S^1, et)(F, W)$. Then, $D(\mathbb{A}_T^1, et)(g^*F) = g^*M$ and there exist $(F', W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ and an equivalence (\mathbb{A}^1, et) local $e : g^*(F, W) \rightarrow (F', W)$ such that $D(\mathbb{A}_T^1, et)(F', W) = (g^*M, W)$. We have, using definition 136 and definition 141(i), the canonical

map in $\pi_T(D(MHM(T^{an}))) \subset D_{\mathcal{D}(1,0)fil,\infty}(T^{an}/(Y^{an} \times \tilde{S}_I^{an}))$

$$\begin{aligned}
& T(g, \mathcal{F}_{an}^{FDR})(M) : g_{Hdg}^{\ast mod} \iota_S^{-1} \mathcal{F}_{S,an}^{FDR}(M) := \\
& (\Gamma_T^{\vee, Hdg} \iota_T^{-1} (\tilde{g}_I^{\ast mod} (\varinjlim_{r_-(\cdot)} e'(\tilde{S}_I) \ast \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^\ast, \rho_{\tilde{S}_I} \ast \text{Gr}_{\tilde{S}_I}^{12^\ast} R_{(X^\ast, D^\ast)/\tilde{S}_I}(\rho_{\tilde{S}_I}^\ast(L(i_{I^\ast} j_I^\ast(F, W))))), \\
& E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR}))[-d_{Y_I}], \tilde{g}_J^{\ast mod} u_{I,J}^q(F, W)) \xrightarrow{(T(\tilde{g}_I, \Omega_{/\tilde{S}_I}^{\Gamma, pr, an})(\text{An}_{\tilde{S}_I}^\ast, \text{Gr}_{\tilde{S}_I}^{12^\ast} L(i_{I^\ast} j_I^\ast(F, W))))} \\
& \Gamma_T^{\vee, Hdg} \iota_T^{-1} (\varinjlim_{r_-(\cdot)} e'(-) \ast \mathcal{H}om(\text{An}_{Y \times \tilde{S}_I}^\ast, \rho_{Y \times \tilde{S}_I} \ast \text{Gr}_{Y \times \tilde{S}_I}^{12^\ast} \tilde{g}_I^\ast R_{(X^\ast, D^\ast)/\tilde{S}_I}(\rho_{\tilde{S}_I}^\ast L(i_{I^\ast} j_I^\ast(F, W))))), \\
& E_{et}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR}), \tilde{g}_J^\ast u_{I,J}^q(F, W)) \xrightarrow{\mathcal{H}om(T(\tilde{g}_I, R^{CH})(-)^{-1}, -)} \\
& \Gamma_T^{\vee, Hdg} \iota_T^{-1} (\varinjlim_{r_-(\cdot)} e'(-) \ast \mathcal{H}om(\text{An}_{Y \times \tilde{S}_I}^\ast, \rho_{Y \times \tilde{S}_I} \ast \text{Gr}_{Y \times \tilde{S}_I}^{12^\ast} R_{(Y \times X^\ast, Y \times D^\ast)/Y \times \tilde{S}_I}(\rho_{Y \times \tilde{S}_I}^\ast \tilde{g}_I^\ast L(i_{I^\ast} j_I^\ast(F, W))))), \\
& E_{et}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR}), \tilde{g}_J^\ast u_{I,J}^q(F, W)) \\
& \xrightarrow{=} \Gamma_T^{\vee, Hdg} \iota_T^{-1} (\varinjlim_{r_-(\cdot)} e'(-) \ast \mathcal{H}om(\text{An}_{Y \times \tilde{S}_I}^\ast, \rho_{Y \times \tilde{S}_I} \ast \text{Gr}_{Y \times \tilde{S}_I}^{12^\ast} R_{(X'^\ast, E'^\ast)/Y \times \tilde{S}_I}(\rho_{Y \times \tilde{S}_I}^\ast \tilde{g}_I^\ast L(i_{I^\ast} j_I^\ast(F, W))))), \\
& E_{et}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR}), \tilde{g}_J^\ast u_{I,J}^q(F, W)) \\
& \xrightarrow{T(\Gamma_T^{\vee, Hdg}, \Omega_{/\tilde{S}_I}^{\Gamma, pr, an})(F, W)} \\
& E_{et}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})[d_{Y_I}], \tilde{g}_J^\ast u_{I,J}^q(F, W)) \xrightarrow{T(\Gamma_T^{\vee, Hdg}, \Omega_{/\tilde{S}_I}^{\Gamma, pr, an})(F, W)} \\
& \iota_T^{-1} (\varinjlim_{r_-(\cdot)} e'(-) \ast \mathcal{H}om(\text{An}_{Y \times \tilde{S}_I}^\ast, \rho_{Y \times \tilde{S}_I} \ast \text{Gr}_{Y \times \tilde{S}_I}^{12^\ast} R_{(X'^\ast, E'^\ast)/Y \times \tilde{S}_I}(\rho_{Y \times \tilde{S}_I}^\ast \Gamma_{T_I}^\vee \tilde{g}_I^\ast L(i_{I^\ast} j_I^\ast(F, W))))), \\
& E_{et}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})[d_{Y_I}], \tilde{g}_J^{\ast, \gamma} u_{I,J}^q(F, W)) \\
& \xrightarrow{(\mathcal{H}om(\text{An}_{Y \times \tilde{S}_I}^\ast, \rho_{Y \times \tilde{S}_I} \ast \text{Gr}_{Y \times \tilde{S}_I}^{12^\ast} R_{Y \times \tilde{S}_I}^{CH}(T^{q, \gamma}(D_{g_I})(j_I^\ast(F, W))), E_{et}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})[d_{Y_I}])} \\
& \iota_T^{-1} (\varinjlim_{r_-(\cdot)} e'(-) \ast \mathcal{H}om(\text{An}_{Y \times \tilde{S}_I}^\ast, \rho_{Y \times \tilde{S}_I} \ast \text{Gr}_{Y \times \tilde{S}_I}^{12^\ast} R_{(X'^\ast, E'^\ast)/Y \times \tilde{S}_I}(\rho_{Y \times \tilde{S}_I}^\ast L(i_{I^\ast} j_I^\ast g^\ast(F, W))))), \\
& E_{et}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})[d_{Y_I}], u_{I,J}^q(g^\ast(F, W)) \xrightarrow{\mathcal{H}om(R_{Y \times \tilde{S}_I}^{CH}(L_{i_{I^\ast} j_I^\ast}(e)),)} \\
& \iota_T^{-1} (\varinjlim_{r_-(\cdot)} e'(-) \ast \mathcal{H}om(\text{An}_{Y \times \tilde{S}_I}^\ast, \rho_{Y \times \tilde{S}_I} \ast \text{Gr}_{Y \times \tilde{S}_I}^{12^\ast} R_{(X'^\ast, E'^\ast)/Y \times \tilde{S}_I}(\rho_{Y \times \tilde{S}_I}^\ast L(i_{I^\ast} j_I^\ast(F', W))))), \\
& E_{et}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})[d_{Y_I}], u_{I,J}^q(F', W)) \xrightarrow{=} \mathcal{F}_{T,an}^{FDR}(g^\ast M)
\end{aligned}$$

Proposition 126. Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y_2 \times S \xrightarrow{p_S} S$ with $Y_2 \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y_2 \times \tilde{S}_i$. Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I := g|_{T_I} : T_I \rightarrow S_I$. Let $f : X \rightarrow S$ a morphism with $X \in \text{Var}(\mathbb{C})$ such that there exists a factorization $f : X \xrightarrow{l} Y_1 \times S \xrightarrow{p_S} S$, with $Y_1 \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have then the following commutative diagram whose squares are cartesian

$$\begin{array}{ccccccc}
f' : X_T & \longrightarrow & Y_1 \times T & \longrightarrow & T & & \\
\downarrow g' & \searrow & \downarrow & \searrow & \downarrow & \searrow & \\
& & Y_1 \times X & \longrightarrow & Y_1 \times Y_2 \times S & \longrightarrow & Y_2 \times S \\
& \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \\
f : X & \longrightarrow & Y_1 \times S & \longrightarrow & S & &
\end{array}$$

Take a smooth compactification $\bar{Y}_1 \in \text{PSmVar}(\mathbb{C})$ of Y_1 , denote $\bar{X}_I \subset \bar{Y}_1 \times \tilde{S}_I$ the closure of X_I , and $Z_I := \bar{X}_I \setminus X_I$. Consider $F(X/S) := p_{S,\sharp} \Gamma_{\bar{X}}^\vee \mathbb{Z}(Y_1 \times S/Y_1 \times S)$ and the isomorphism in $C(\text{Var}(\mathbb{C})^{sm}/T)$

$$T(f, g, F(X/S)) : g^* F(X/S) := g^* p_{S,\sharp} \Gamma_{\bar{X}}^\vee \mathbb{Z}(Y_1 \times S/Y_1 \times S) \xrightarrow{\sim} p_{T,\sharp} \Gamma_{\bar{X}_T}^\vee \mathbb{Z}(Y_1 \times T/Y_1 \times T) =: F(X_T/T).$$

which gives in $\text{DA}(T)$ the isomorphism $T(f, g, F(X/S)) : g^* M(X/S) \xrightarrow{\sim} M(X_T/T)$. Then the following diagram in $\pi_T(D(\text{MHM}(T))) \subset D_{\mathcal{D}(1,0)fil,\infty}(T/(Y_2 \times \tilde{S}_I))$, where the horizontal maps are given by proposition 124, commutes

$$\begin{array}{ccc} g_{Hdg}^{\hat{*}mod} l_S^{-1} \mathcal{F}_{S,an}^{FDR}(M(X/S)) & \xrightarrow{g_{Hdg}^{\hat{*}mod} I(X/S)} & g_{Hdg}^{\hat{*}mod} Rf_!^{Hdg}(\Gamma_{\bar{X}_I}^{\vee, Hdg}(O_{(Y_1 \times \tilde{S}_I)^{an}}, F_b), x_{IJ}(X/S)) \\ \downarrow T(g, \mathcal{F}^{FDR})(M^{BM}(X/S)) & & \downarrow T(p_{\tilde{S}_I}, \gamma^{\vee, Hdg})(-) \\ & & Rf_!^{Hdg} g_{Hdg}^{\hat{*}mod}(\Gamma_{\bar{X}_I}^{\vee, Hdg}(O_{(Y_1 \times \tilde{S}_I)^{an}}, F_b), x_{IJ}(X/S)) \\ & & \downarrow = \\ l_T^{-1} \mathcal{F}_{T,an}^{FDR}(M(X_T/T)) & \xrightarrow{I(X_T/T)} & Rf_!^{Hdg}(\Gamma_{\bar{X}_{T_I}}^{\vee, Hdg}(O_{(Y_2 \times Y_1 \times \tilde{S}_I)^{an}}, F_b), x_{IJ}(X_T/T)). \end{array}$$

Proof. Follows immediately from definition. \square

Theorem 37. Let $g : T \rightarrow S$ a morphism, with $S, T \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $M \in \text{DA}_c(S)$. Then map in $\pi_T(D(\text{MHM}(T^{an})))$

$$T(g, \mathcal{F}_{an}^{FDR})(M) : g_{Hdg}^{\hat{*}mod} \mathcal{F}_{S,an}^{FDR}(M) \xrightarrow{\sim} \mathcal{F}_{T,an}^{FDR}(g^* M)$$

given in definition 142 is an isomorphism.

Proof. Follows from proposition 126 and proposition 124. \square

Definition 143. • Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have, for $M \in \text{DA}_c(X)$, the following transformation map in $\pi_S(D(\text{MHM}(S^{an})))$

$$\begin{array}{ccc} T_*(f, \mathcal{F}_{an}^{FDR})(M) : \mathcal{F}_{S,an}^{FDR}(Rf_* M) & \xrightarrow{\text{ad}(f_{Hdg}^{\hat{*}mod}, Rf_*^{Hdg})(-)} & Rf_*^{Hdg} f_{Hdg}^{\hat{*}mod} \mathcal{F}_{S,an}^{FDR}(Rf_* M) \\ \xrightarrow{T(f, \mathcal{F}_{an}^{FDR})(Rf_* M)} & Rf_*^{Hdg} \mathcal{F}_{X,an}^{FDR}(f^* Rf_* M) & \xrightarrow{\mathcal{F}_X^{FDR}(\text{ad}(f^*, Rf_*)(M))} Rf_*^{Hdg} \mathcal{F}_{X,an}^{FDR}(M) \end{array}$$

Clearly, for $p : Y \times S \rightarrow S$ a projection with $Y \in \text{PSmVar}(\mathbb{C})$, we have, for $M \in \text{DA}_c(Y \times S)$, $T_*(p, \mathcal{F}^{FDR})(M) = T_1(p, \mathcal{F}^{FDR})(M)[2d_Y]$

- Let $S \in \text{Var}(\mathbb{C})$. Let $Y \in \text{SmVar}(\mathbb{C})$ and $p : Y \times S \rightarrow S$ the projection. We have then, for $M \in \text{DA}(Y \times S)$ the following transformation map in $\pi_S(D(\text{MHM}(S^{an})))$

$$\begin{array}{ccc} T_1(p, \mathcal{F}_{an}^{FDR})(M) : p_!^{Hdg} \mathcal{F}_{Y \times S, an}^{FDR}(M) & \xrightarrow{\mathcal{F}_{Y \times S, an}^{FDR}(\text{ad}(Lp_\sharp, p^*)(M))} & Rp_!^{Hdg} \mathcal{F}_{Y \times S, an}^{FDR}(p^* Lp_\sharp(M)) \\ \xrightarrow{T(p, \mathcal{F}^{FDR})(Lp_\sharp(M, W))} & Rp_!^{Hdg} p^{\hat{*}mod[-]} \mathcal{F}_{S,an}^{FDR}(Lp_\sharp M) & \xrightarrow{T(p^{\hat{*}mod}, p^{\hat{*}mod})(-)} p_!^{Hdg} p^{\hat{*}mod[-]} \\ & \mathcal{F}_{S,an}^{FDR}(Lp_\sharp M) & \xrightarrow{\text{ad}(Rp_!^{Hdg}, p^{\hat{*}mod[-]})(\mathcal{F}_{S,an}^{FDR}(Lp_\sharp M))} \mathcal{F}_{S,an}^{FDR}(Lp_\sharp M) \end{array}$$

- Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have then, using the second point, for $M \in \text{DA}(X)$ the following transformation map in $\pi_S(D(\text{MHM}(S^{an})))$

$$T_l(f, \mathcal{F}_{an}^{FDR})(M) : R p_!^{Hdg} \mathcal{F}_X^{FDR}(M) := R p_!^{Hdg} \mathcal{F}_{Y \times S, an}^{FDR}(l_* M) \\ \xrightarrow{T_l(p_S, \mathcal{F}_{an}^{FDR})(l_* M)} \mathcal{F}_{S, an}^{FDR}(L p_{\sharp} l_* M) \xrightarrow{\cong} \mathcal{F}_{S, an}^{FDR}(R f_! M)$$

- Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have, using the third point, for $M \in \text{DA}(S)$, the following transformation map in $\pi_X(D(\text{MHM}(X^{an})))$

$$T^l(f, \mathcal{F}_{an}^{FDR})(M) : \mathcal{F}_{X, an}^{FDR}(f^!(M, W)) \xrightarrow{\text{ad}(R f_!^{Hdg}, f_{Hdg}^{*mod})(\mathcal{F}_{X, an}^{FDR}(f^! M))} f_{Hdg}^{*mod} R f_!^{Hdg} \mathcal{F}_{X, an}^{FDR}(f^! M) \\ \xrightarrow{T_l(p_S, \mathcal{F}_{an}^{FDR})(\mathcal{F}_{an}^{FDR}(f^! M))} f_{Hdg}^{*mod} \mathcal{F}_{S, an}^{FDR}(R f_! f^!(M, W)) \xrightarrow{\mathcal{F}_{S, an}^{FDR}(\text{ad}(R f_!, f^!)(M))} f_{Hdg}^{*mod} \mathcal{F}_{S, an}^{FDR}(M)$$

Proposition 127. Let $S \in \text{Var}(\mathbb{C})$. Let $Y \in \text{SmVar}(\mathbb{C})$ and $p : Y \times S \rightarrow S$ the projection. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i^o : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, l]$, we denote by $S_I = \cap_{i \in I} S_i$, $j_I^o : S_I \hookrightarrow S$ and $j_I : Y \times S_I \hookrightarrow Y \times S$ the open embeddings. We then have closed embeddings $i_I : Y \times S_I \hookrightarrow Y \times \tilde{S}_I$. and we denote by $p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ the projections. Let $f' : X' \rightarrow Y \times S$ a morphism, with $X' \in \text{Var}(\mathbb{C})$ such that there exists a factorization $f' : X' \xrightarrow{l'} Y' \times Y \times S \xrightarrow{p'} Y \times S$ with $Y' \in \text{SmVar}(\mathbb{C})$, l' a closed embedding and p' the projection. Denoting $X'_I := f'^{-1}(Y \times S_I)$, we have closed embeddings $i'_I : X'_I \hookrightarrow Y' \times Y \times \tilde{S}_I$. Consider

$$F(X'/Y \times S) := p_{Y \times S, \sharp} \Gamma_{X'}^{\vee} \mathbb{Z}(Y' \times Y \times S/Y' \times Y \times S) \in C(\text{Var}(\mathbb{C})^{sm}/Y \times S)$$

and $F(X'/S) := p_{\sharp} F(X'/Y \times S) \in C(\text{Var}(\mathbb{C})^{sm}/S)$, so that $L p_{\sharp} M(X'/Y \times S)[-2d_Y] =: M(X'/S)$. Then, the following diagram in $\pi_S(D(\text{MHM}(S^{an}))) \subset D_{\mathcal{D}(1,0)fil, \infty}(S^{an}/(Y^{an} \times \tilde{S}_I^{an}))$, where the vertical maps are given by proposition 124, commutes

$$\begin{array}{ccc} R p^{Hdg}! \mathcal{F}_{Y \times S, an}^{FDR}(M(X'/Y \times S)) & \xrightarrow{T_l(p_S, \mathcal{F}_{an}^{FDR})(M(X'/Y \times S))} & \mathcal{F}_{S, an}^{FDR}(M(X'/S)) \\ \uparrow R p^{Hdg}!(I(X'/Y \times S)) & & \uparrow I(X'/S) \\ R p^{Hdg}! R f_!^{Hdg}(\Gamma_{X'_I}^{\vee, Hdg}(O_{(Y' \times Y \times \tilde{S}_I)^{an}}, F_b), x_{IJ}(X'/Y \times S)) & \xrightarrow{=} & R f_!^{Hdg}(\Gamma_{X'_I}^{\vee, Hdg}(O_{(Y' \times Y \times \tilde{S}_I)^{an}}, F_b), x_{IJ}(X'/S)) \end{array}$$

Proof. Immediate from definition. \square

Theorem 38. (i) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Then, for $M \in \text{DA}_c(X)$,

$$T_l(f, \mathcal{F}_{an}^{FDR})(M) : R f_!^{Hdg} \mathcal{F}_{X, an}^{FDR}(M) \xrightarrow{\sim} \mathcal{F}_{S, an}^{FDR}(R f_! M)$$

is an isomorphism in $\pi_S(D(\text{MHM}(S^{an})))$

(ii) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$, S quasi-projective. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have, for $M \in \text{DA}_c(X)$,

$$T_*(f, \mathcal{F}_{an}^{FDR})(M) : \mathcal{F}_{S, an}^{FDR}(R f_* M) \xrightarrow{\sim} R f_*^{Hdg} \mathcal{F}_{X, an}^{FDR}(M)$$

is an isomorphism in $\pi_S(D(\text{MHM}(S^{an})))$.

(iii) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$, S quasi-projective. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Then, for $M \in \text{DA}_c(S)$

$$T^!(f, \mathcal{F}_{an}^{FDR})(M) : \mathcal{F}_{X,an}^{FDR}(f^!M) \xrightarrow{\sim} f^{*mod} \mathcal{F}_{S,an}^{FDR}(M)$$

is an isomorphism in $\pi_X(D(\text{MHM}(X^{an})))$.

Proof. Similar to the proof of theorem 33. □

Proposition 128. Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$. Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I := g|_{T_I} : T_I \rightarrow S_I$. Let $M \in \text{DA}_c(S)$ and $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}_S^1, et)(F)$. Then, $D(\mathbb{A}_T^1, et)(g^*F) = g^*M$. Then the following diagram in $D_{\text{Ful}, \mathcal{D}^{\infty, \infty}}(T^{an}/(Y^{an} \times \tilde{S}_I^{an}))$ commutes

$$\begin{array}{ccc} Rg^{*mod[-], \Gamma} \mathcal{F}_{S,an}^{GM}(L\mathbb{D}_S M) & \xrightarrow{Rg^{*mod[-], \Gamma} T(\mathcal{F}_{S,an}^{GM}, \mathcal{F}_{S,an}^{FDR})(M)} Rg^{*mod[-], \Gamma} \mathcal{F}_{S,an}^{FDR}(M) & \xleftarrow{g_{Hdg}^{*mod[-], \Gamma} T(\mathcal{F}_{S,an}^{FDR}, Rg^{*mod[-], \Gamma} \mathcal{F}_{S,an}^{GM})(M)} \mathcal{F}_{S,an}^{FDR}(M) \\ \downarrow T(g, \mathcal{F}_{an}^{GM})(L\mathbb{D}_S M) & & \downarrow T^!(g, \mathcal{F}_{an}^{FDR})(M)^{-1} \\ \mathcal{F}_{T,an}^{GM}(g^*L\mathbb{D}_S M = L\mathbb{D}_T g^!M) & \xrightarrow{T(\mathcal{F}_{T,an}^{GM}, \mathcal{F}_{T,an}^{FDR})(g^*M)} & \mathcal{F}_{T,an}^{FDR}(g^!M) \end{array}$$

Proof. Similar to the proof of proposition 111. □

Definition 144. Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open affine covering and denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We have, for $M, N \in \text{DA}(S)$ and $(F, W), (G, W) \in C_{\text{ful}}(\text{Var}(\mathbb{C})^{sm}/S)$ such that $(M, W) = D(\mathbb{A}^1, et)(F, W)$ and $(N, W) = D(\mathbb{A}^1, et)(G, W)$, the following transformation map in

$$\pi_S(D(MHM(S^{an}))) \subset D_{\mathcal{D}(1,0)fil}(S^{an}/(\tilde{S}_I^{an}))$$

$$\begin{aligned}
& T(\mathcal{F}_{S,an}^{FDR}, \otimes)(M, N) : \mathcal{F}_{S,an}^{FDR}(M) \otimes_{O_S}^{L[-]} \mathcal{F}_{S,an}^{FDR}(N) \\
& \xrightarrow{=} (\varinjlim_{r-(-)} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \rho_{\tilde{S}_I} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^*(F, W))), \\
& \quad E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})[-d_{\tilde{S}_I}], u_{IJ}(F, W)) \otimes_{O_S}^{[-]} \\
& (\varinjlim_{r-(-)} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \rho_{\tilde{S}_I} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^*(G, W))), \\
& \quad E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})[-d_{\tilde{S}_I}], u_{IJ}(G, W)) \\
& \xrightarrow{=} (\varinjlim_{r-(-), r-(-)} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \rho_{\tilde{S}_I} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^*(F, W))), \\
& \quad E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})) \otimes_{O_{\tilde{S}_I}} \\
& \quad e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \rho_{\tilde{S}_I} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^*(F, W))), \\
& \quad E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})[-d_{\tilde{S}_I}], u_{IJ}(F) \otimes u_{IJ}(G)) \xrightarrow{(T(\otimes, \Omega_{/\tilde{S}_I}^{\Gamma, pr, an})(-, -), -)} \\
& (\varinjlim_{r-(-), r-(-)} e'(\tilde{S}_I)_* \mathcal{H}om(\text{An}_{\tilde{S}_I}^* \rho_{\tilde{S}_I} \text{Gr}_{\tilde{S}_I}^{12*} (R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^*(F, W))) \otimes \\
& \quad R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^*(F, W))), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})[-d_{\tilde{S}_I}], v_{IJ}(F \otimes G)) \\
& \quad \xrightarrow{\mathcal{H}om(T(\otimes, R_{\tilde{S}_I}^{GH})(-, -)^{-1}, -)} \\
& (\varinjlim_{r-(-)} e'(\tilde{S}_I)_* \mathcal{H}om(\text{An}_{\tilde{S}_I}^* \rho_{\tilde{S}_I} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* (L(i_{I*} j_I^*(F, W)) \otimes L(i_{I*} j_I^*(F, W))), \\
& \quad E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})[-d_{\tilde{S}_I}], u_{IJ}(F \otimes G)) \\
& \quad \xrightarrow{\mathcal{H}om(R_{(-, -)/-}(T(\otimes, L)(-, -), -)} \\
& (\varinjlim_{r-(-)} e'(\tilde{S}_I)_* \mathcal{H}om(\text{An}_{\tilde{S}_I}^* \rho_{\tilde{S}_I} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* (L(i_{I*} j_I^*((F, W) \otimes (G, W))))), \\
& \quad E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})[-d_{\tilde{S}_I}], u_{IJ}(F \otimes G)) \xrightarrow{=} \mathcal{F}_{S,an}^{FDR}(M \otimes N)
\end{aligned}$$

Proposition 129. Let $f_1 : X_1 \rightarrow S$, $f_2 : X_2 \rightarrow S$ two morphism with $X_1, X_2, S \in \text{Var}(\mathbb{C})$. Assume that there exist factorizations $f_1 : X_1 \xrightarrow{l_1} Y_1 \times S \xrightarrow{ps} S$, $f_2 : X_2 \xrightarrow{l_2} Y_2 \times S \xrightarrow{ps} S$ with $Y_1, Y_2 \in \text{SmVar}(\mathbb{C})$, l_1, l_2 closed embeddings and ps the projections. We have then the factorization

$$f_{12} := f_1 \times f_2 : X_{12} := X_1 \times_S X_2 \xrightarrow{l_1 \times l_2} Y_1 \times Y_2 \times S \xrightarrow{ps} S$$

Let $S = \cup_{i=1}^l S_i$ an open affine covering and denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We have then the following commutative diagram in $\pi_S(DMHM(S^{an})) \subset D_{\mathcal{D}(1,0)fil}(S^{an}/(\tilde{S}_I^{an}))$ where the vertical maps are given by proposition 124

$$\begin{array}{ccc}
& & Rf_{1!}^{Hdg}(\Gamma_{X_{1I}}^{\vee, Hdg}(O_{(Y_1 \times \tilde{S}_I)^{an}}, F_b), x_{IJ}(X_1/S)) \otimes_{O_S} \\
\mathcal{F}_{S,an}^{FDR}(M(X_1/S)) \otimes_{O_S}^{L[-]} \mathcal{F}_{S,an}^{FDR}(M(X_2/S)) & \xrightarrow{I(X_1/S) \otimes I(X_2/S)} & Rf_{2!}^{Hdg}(\Gamma_{X_{2I}}^{\vee, Hdg}(O_{(Y_2 \times \tilde{S}_I)^{an}}, F_b), x_{IJ}(X_2/S)) \\
\downarrow T(\mathcal{F}_{S,an}^{FDR}, \otimes)(M(X_1/S), M(X_2/S)) & & \downarrow (Ew_{(Y_1 \times \tilde{S}_I, Y_2 \times \tilde{S}_I)/\tilde{S}_I}) \\
\mathcal{F}_{S,an}^{FDR}(M(X_1/S) \otimes M(X_2/S) = M(X_1 \times_S X_2/S)) & \xrightarrow{I(X_{12}/S)} & Rf_{12!}^{Hdg}(\Gamma_{X_{1I} \times_S X_{2I}}^{\vee, Hdg}(O_{(Y_1 \times Y_2 \times \tilde{S}_I)^{an}}, F_b), x_{IJ}(X_1/S)).
\end{array}$$

Proof. Immediate from definition. \square

Theorem 39. Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open affine covering and denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, for $M, N \in \text{DA}_c(S)$, the map in $\pi_S(D(\text{MHM}(S^{\text{an}})))$

$$T(\mathcal{F}_{S,\text{an}}^{\text{FDR}}, \otimes)(M, N) : \mathcal{F}_{S,\text{an}}^{\text{FDR}}(M) \otimes_{O_S}^L \mathcal{F}_{S,\text{an}}^{\text{FDR}}(N) \xrightarrow{\sim} \mathcal{F}_{S,\text{an}}^{\text{FDR}}(M \otimes N)$$

given in definition 144 is an isomorphism.

Proof. Follows from proposition 129. \square

We have the following easy proposition

Proposition 130. Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open affine covering and denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We have, for $M, N \in \text{DA}(S)$ and $F, G \in C(\text{Var}(\mathbb{C})^{\text{sm}}/S)$ such that $M = D(\mathbb{A}^1, \text{et})(F)$ and $N = D(\mathbb{A}^1, \text{et})(G)$, the following commutative diagram in $D_{O_S \text{ fil}, \mathcal{D}^\infty, \infty}(S^{\text{an}}/(\tilde{S}_I^{\text{an}}))$

$$\begin{array}{ccc} \mathcal{F}_{S,\text{an}}^{\text{GM}}(\mathbb{L}\mathbb{D}_S M) \otimes_{O_S}^L \mathcal{F}_{S,\text{an}}^{\text{GM}}(\mathbb{L}\mathbb{D}_S N) & \xrightarrow{T(\mathcal{F}_{S,\text{an}}^{\text{GM}}, \mathcal{F}_{S,\text{an}}^{\text{FDR}})(M) \otimes T(\mathcal{F}_{S,\text{an}}^{\text{GM}}, \mathcal{F}_{S,\text{an}}^{\text{FDR}})(N)} & \mathcal{F}_{S,\text{an}}^{\text{FDR}}(M) \otimes_{O_S}^L \mathcal{F}_{S,\text{an}}^{\text{FDR}}(N) \\ \downarrow T(\mathcal{F}_{S,\text{an}}^{\text{GM}}, \otimes)(\mathbb{L}\mathbb{D}_S M, \mathbb{L}\mathbb{D}_S N) & & \downarrow T(\mathcal{F}_{S,\text{an}}^{\text{FDR}}, \otimes)(M, N) \\ \mathcal{F}_{S,\text{an}}^{\text{GM}}(\mathbb{D}_S L(M \otimes N)) & \xrightarrow{T(\mathcal{F}_{S,\text{an}}^{\text{GM}}, \mathcal{F}_{S,\text{an}}^{\text{FDR}})(M \otimes N)} & \mathcal{F}_{S,\text{an}}^{\text{FDR}}(M \otimes N) \end{array}$$

Proof. Immediate from definition. \square

6.3 The transformation map between the analytic De Rahm functor and the analytification of the algebraic De Rahm functor

6.3.1 The transformation map between the analytic Gauss Manin realization functor and the analytification of the algebraic Gauss Manin realization functor

Recall from section 2 that, for $f : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$, we have the following commutative diagram of sites (39)

$$\begin{array}{ccc} \text{AnSp}(\mathbb{C})/T^{\text{an}} & \xrightarrow{\text{An}_T} & \text{Var}(\mathbb{C})/T \\ \downarrow P(f) & \searrow \rho_T & \downarrow \rho_T \\ \text{AnSp}(\mathbb{C})^{\text{sm}}/T^{\text{an}} & \xrightarrow{\text{An}_T} & \text{Var}(\mathbb{C})^{\text{sm}}/T \\ \downarrow P(f) & \searrow \rho_T & \downarrow P(f) \\ \text{AnSp}(\mathbb{C})/S^{\text{an}} & \xrightarrow{\text{An}_S} & \text{Var}(\mathbb{C})/S \\ \downarrow \rho_S & \searrow \rho_S & \downarrow \rho_S \\ \text{AnSp}(\mathbb{C})^{\text{sm}}/S^{\text{an}} & \xrightarrow{\text{An}_S} & \text{Var}(\mathbb{C})^{\text{sm}}/S \end{array}$$

We have the following canonical transformation map given by the pullback of (relative) differential forms: Let $S \in \text{Var}(\mathbb{C})$. Consider the following commutative diagram in RCat :

$$\begin{array}{ccc} D(\text{an}, e) : (\text{AnSp}(\mathbb{C})^{\text{sm}}/S^{\text{an}}, O_{\text{AnSp}(\mathbb{C})^{\text{sm}}/T}) & \xrightarrow{\text{An}_S} & (\text{Var}(\mathbb{C})^{\text{sm}}/S, O_{\text{Var}(\mathbb{C})^{\text{sm}}/S}) \\ \downarrow e(T) & & \downarrow e(S) \\ (S^{\text{an}}, O_{S^{\text{an}}}) & \xrightarrow{\text{an}_S} & (S, O_S) \end{array}$$

It gives (see section 2) the canonical morphism in $C_{\text{an}_S^* O_S}(\widehat{\text{AnSp}(\mathbb{C})^{sm}/S^{an}})$

$$\begin{aligned} \Omega_{/(S^{an}/S)} &:= \Omega_{(O_{\text{AnSp}(\mathbb{C})^{sm}/S^{an}}/\text{An}_S^* O_{\text{Var}(\mathbb{C})^{sm}/S})/(O_{S^{an}}/\text{an}_S^* O_S)} : \\ \text{An}_S^*(\Omega_{/S}^\bullet, F_b) &= (\Omega_{\text{An}_S^* O_{\text{Var}(\mathbb{C})^{sm}/S}/\text{An}_S^* e(S)^* O_S}^\bullet, F_b) \rightarrow (\Omega_{/S^{an}}^\bullet, F_b) = (\Omega_{O_{\text{AnSp}(\mathbb{C})^{sm}/S^{an}}/e(S^{an})^* O_{S^{an}}}^\bullet, F_b) \end{aligned}$$

which is by definition given by the analytification on differential forms : for $(V/S^{an}) = (V, h) \in \text{AnSp}(\mathbb{C})^{sm}/S^{an}$,

$$\begin{aligned} \Omega_{/(S^{an}/S)}(V/S^{an}) : \hat{\omega} \in \text{An}_S^*(\Omega_{/S}^r)(V/S^{an}) &:= \lim_{(h':U \rightarrow \text{SSM}, g':V \rightarrow U^{an}, h, g)} \Omega_{/S}^r(U/S) \\ \mapsto \Omega_{(V/U)/(S^{an}/S)}(V/S^{an})(\omega) &:= \widehat{\text{an}_S^*(\omega)} \in \Omega_{S^{an}}^r(V/S^{an}); \end{aligned}$$

with $\omega \in \Gamma(U, \Omega_U^r)$ is such that $q(\omega) = \hat{\omega}$. If $S \in \text{SmVar}(\mathbb{C})$, the map $\Omega_{/(T/S)} : \text{An}_S^* \Omega_{/S}^\bullet \rightarrow \Omega_{/S^{an}}^\bullet$ is a map in $C_{O_S \text{fil}, \mathcal{D}}(\text{AnSp}(\mathbb{C})^{sm}/S^{an})$. It induces the canonical morphism in $C_{O_S \text{fil}, \mathcal{D}}(\text{AnSp}(\mathbb{C})^{sm}/S^{an})$:

$$E\Omega_{/(S^{an}/S)} : \text{An}_S^* E_{et}(\Omega_{/S}^\bullet, F_b) \xrightarrow{T(\text{An}_S, E)(\Omega_{/S}^\bullet, F_b)} E_{et}(\text{An}_S^*(\Omega_{/S}^\bullet, F_b)) \xrightarrow{E(\Omega_{/(S^{an}/S)})} E_{et}(\Omega_{/S^{an}}^\bullet, F_b)$$

We have the following canonical transformation map given by the analytical functor:

Definition 145. Let $S \in \text{SmVar}(\mathbb{C})$.

(i) For $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$, we have the canonical transformation map in $C_{O \text{fil}, \mathcal{D}}(S^{an})$

$$\begin{aligned} &T(\text{an}, \Omega_{/ \cdot})(F) : \\ &((e(S)_* \mathcal{H}om^\bullet(F, E_{et}(\Omega_{/S}^\bullet, F_b)))^{an}) := O_{S^{an}} \otimes_{\text{an}_S^* O_S} \text{an}_S^*(e(S)_* \mathcal{H}om^\bullet(F, E_{et}(\Omega_{/S}^\bullet, F_b))) \\ &\xrightarrow{T(\text{an}, e)(-)} O_{S^{an}} \otimes_{\text{an}_S^* O_S} (e(S^{an})_* \text{An}_S^* \mathcal{H}om^\bullet(F, E_{et}(\Omega_{/S}^\bullet, F_b))) \\ &\xrightarrow{T(\text{An}, \text{hom})(F, E_{et}(\Omega_{/S}, F_b))} O_{S^{an}} \otimes_{\text{an}_S^* O_S} (e(S^{an})_* \mathcal{H}om^\bullet(\text{An}_S^* F, \text{An}_S^* E_{et}(\Omega_{/S}^\bullet, F_b))) \\ &\xrightarrow{\mathcal{H}om(\text{An}_S^* F, E\Omega_{/(S^{an}/S)} \otimes m)} e(S^{an})_* \mathcal{H}om^\bullet(\text{An}_S^* F, E_{et}(\Omega_{/S^{an}}^\bullet, F_b)) \end{aligned}$$

(ii) We get from (i), for $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$, the canonical transformation map in $\text{PSh}_{\mathcal{D}^\infty}(S^{an})$

$$\begin{aligned} &T^n(\text{an}, \Omega_{/ \cdot})(F) : J_S H^n((e(S)_* \mathcal{H}om^\bullet(F, E_{et}(\Omega_{/S}^\bullet, F_b)))^{an}) \\ &\xrightarrow{J_S(H^n T(\text{an}, \Omega_{/ \cdot})(F))} J_S(e(S^{an})_* \mathcal{H}om^\bullet(\text{An}_S^* F, E_{et}(\Omega_{/S^{an}}^\bullet, F_b))) \\ &\xrightarrow{J_S(-)} e(S^{an})_* \mathcal{H}om^\bullet(\text{An}_S^* F, E_{et}(\Omega_{/S^{an}}^\bullet, F_b)) \end{aligned}$$

Lemma 21. Let $S \in \text{SmVar}(\mathbb{C})$.

(i) For $h : U \rightarrow S$ a smooth morphism with $U \in \text{SmVar}(\mathbb{C})$, the following diagram commutes

$$\begin{array}{ccc} e(S)_* \mathcal{H}om^\bullet(\mathbb{Z}(U/S), E_{et}(\Omega_{/S}^\bullet, F_b))^{an} & \xrightarrow{T(\Omega_{/ \cdot}, \text{an})(\mathbb{Z}(U/S))} & e(S^{an})_* \mathcal{H}om^\bullet(\mathbb{Z}(U^{an}/S^{an}), E_{et}(\Omega_{/S^{an}}^\bullet, F_b)) \cdot \\ \uparrow k & & \uparrow = \\ (h_* E_{zar}(\Omega_{U/S}, F_b))^{an} & \xrightarrow{T_\omega^O(\text{an}, h)} & h_{an*} E_{usu}(\Omega_{U^{an}/S^{an}}, F_b) \end{array}$$

(ii) For $h : U \rightarrow S$ a smooth morphism with $U \in \text{SmVar}(\mathbb{C})$, the following diagram commutes

$$\begin{array}{ccc} J_S H^n((e(S)_* \mathcal{H}om^\bullet(\mathbb{Z}(U/S), E_{et}(\Omega_{/S}^\bullet, F_b)))^{an}) & \xrightarrow{T^n(\Omega_{/ \cdot}, \text{an})(\mathbb{Z}(U/S))} & e(S^{an})_* \mathcal{H}om^\bullet(\mathbb{Z}(U^{an}/S^{an}), E_{et}(\Omega_{/S^{an}}^\bullet, F_b)) \cdot \\ \uparrow k & & \uparrow = \\ J_S((h_* E_{zar}(\Omega_{U/S}, F_b))^{an}) & \xrightarrow{J_S(-) \circ J_S T_\omega^O(\text{an}, h)(O_U, F)} & h_{an*} E_{usu}(\Omega_{U^{an}/S^{an}}, F_b) \end{array}$$

Proof. Follows from Yoneda lemma. \square

By definition of the algebraic an analytic De Rahm realization functor, we have a natural transformation between them :

Definition 146. Let $S \in \text{SmVar}(\mathbb{C})$. Let $M \in \text{DA}_c(S)$ and $Q \in C(\text{Var}(\mathbb{C})^{sm}/S)$ projectively cofibrant such that $M = D(\mathbb{A}_S^1, et)(Q)$. We have the canonical transformation in $D_{\text{Ofil}, \mathcal{D}}(S^{an})$

$$\begin{aligned} T(\text{An}, \mathcal{F}_{an}^{GM})(M) : (\mathcal{F}_{GM}^S(M))^{an} &:= (e(S)_* \mathcal{H}om^\bullet(Q, E_{et}(\Omega_{/S}^\bullet, F_b)))^{an} \\ &\xrightarrow{T(an, \Omega_{/ \cdot})(Q)} e(S)_* \mathcal{H}om^\bullet(\text{An}_S^* Q, E_{et}(\Omega_{/S^{an}}^\bullet, F_b)) \\ &\xrightarrow{=} e(S)_* \mathcal{H}om^\bullet(\text{An}_S^* Q, E_{usu}(\Omega_{/S^{an}}^\bullet, F_b)) =: \mathcal{F}_{S, an}^{GM}(M) \end{aligned}$$

We give now the definition in the non smooth case : Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset J$, denote by $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ the projection. Consider, for $I \subset J \subset [1, \dots, l]$, resp. for each $I \subset [1, \dots, l]$, the following commutative diagrams in $\text{Var}(\mathbb{C})$

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ j_{IJ} \uparrow & & \uparrow p_{IJ} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}$$

We then have the following lemma

Lemma 22. The maps $T(an, \Omega_{/ \cdot})(L(i_{I*} j_I^* F))$ induce a morphism in $C_{\text{Ofil}, \mathcal{D}}(S/(\tilde{S}_I))$

$$\begin{aligned} (T(an, \Omega_{/ \cdot})(L(i_{I*} j_I^* F))) : (e(\tilde{S}_I)_* \mathcal{H}om^\bullet(L(i_{I*} j_I^* F), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)))^{an}, (u_{IJ}^q(F))^{an} \\ \rightarrow (e(\tilde{S}_I)_* \mathcal{H}om(\text{An}(\tilde{S}_I)^* L(i_{I*} j_I^* F), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)), u_{IJ}^q(F)) \end{aligned}$$

Proof. Obvious. \square

Definition 147. Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote $S_I = \cap_{i \in I} S_i$. We have then closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I = \prod_{i \in I} \tilde{S}_i$. Let $M \in \text{DA}_c(S)$ and $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}_S^1, et)(F)$. We have, by lemma 22, the canonical transformation in $D_{\text{Ofil}, \mathcal{D}, \infty}(S^{an})$

$$\begin{aligned} T(\text{An}, \mathcal{F}^{GM})(M) : (\mathcal{F}_S^{GM}(M))^{an} &:= (e(\tilde{S}_I)_* \mathcal{H}om^\bullet(L(i_{I*} j_I^* F), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)))^{an}, (u_{IJ}^q(F))^{an} \\ &\xrightarrow{T(an, \Omega_{/ \cdot})(L(i_{I*} j_I^* F))} (e(\tilde{S}_I)_* \mathcal{H}om(\text{An}(\tilde{S}_I)^* L(i_{I*} j_I^* F), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F)), u_{IJ}^q(F)) \\ &\xrightarrow{=} (e(\tilde{S}_I)_* \mathcal{H}om(\text{An}(\tilde{S}_I)^* L(i_{I*} j_I^* F), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F)), u_{IJ}^q(F)) =: \mathcal{F}_{S, an}^{GM}(M) \end{aligned}$$

The following proposition says this transformation map between $\mathcal{F}^{S, an}$ and $(\mathcal{F}_S^{FDR})^{an}$ is functorial in $S \in \text{Var}(\mathbb{C})$, hence define a commutative diagram of morphism of 2-functor :

Proposition 131. (i) Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{ps} S$ with, $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and ps the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We then have closed embedding $i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$ and $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I := g|_{T_I} : T_I \rightarrow S_I$. Then, for $M \in \text{DA}_c(S)$, the following diagram in $D_{\text{Ofil}, \mathcal{D}, \infty}(T^{an}/(Y^{an} \times \tilde{S}_I^{an}))$ commutes

$$\begin{array}{ccc} Rg^{*mod[-], \Gamma}(\mathcal{F}_S^{GM}(M)) \xrightarrow{Lg^{*mod[-], \Gamma}(T(\text{An}, \mathcal{F}_S^{FDR})(M))} Rg^{*mod[-], \Gamma}(\mathcal{F}_{S, an}^{GM}(M)) \\ \downarrow (T(g, \mathcal{F}^{GM})(M))^{an} & & \downarrow T(g, \mathcal{F}_{an}^{GM})(M) \\ (\mathcal{F}_T^{GM}(g^* M))^{an} & \xrightarrow{(T(\text{An}, \mathcal{F}_T^{GM})(g^* M))} & (\mathcal{F}_{T, an}^{GM}(g^* M)) \end{array}$$

(ii) Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, for $M, N \in \text{DA}_c(S)$, the following diagram in $D_{\text{Fil}, \mathcal{D}, \infty}(S^{an}/(\tilde{S}_I^{an}))$ commutes

$$\begin{array}{ccc}
(\mathcal{F}_S^{GM}(M) \otimes_{O_S} \mathcal{F}_S^{GM}(N)) \xrightarrow{T(\otimes, \mathcal{F}_S^{GM})(M, N)^{an}} & & (\mathcal{F}_S^{GM}(M \otimes N))^{an} \\
\downarrow = & & \downarrow T(\text{An}, \mathcal{F}_S^{GM})(M \otimes N) \\
(\mathcal{F}_S^{GM}(M))^{an} \otimes_{O_{S^{an}}} \mathcal{F}_S^{GM}(N)^{an} & & \\
\downarrow T(\text{An}, \mathcal{F}_S^{GM})(M) \otimes_{O_{S^{an}}} T(\text{An}, \mathcal{F}_S^{GM})(N) & & \downarrow \\
\mathcal{F}_{S, an}^{GM}(M) \otimes_{O_{S^{an}}} \mathcal{F}_{S, an}^{GM}(N) \xrightarrow{T(\otimes, \mathcal{F}_{an}^{GM})(M, N)} & & \mathcal{F}_{S, an}^{GM}(M \otimes N)
\end{array}$$

Proof. Immediate from definition. \square

Proposition 132. Let $f : X \rightarrow S$ a morphism with $S, X \in \text{Var}(\mathbb{C})$. Assume there exist a factorization

$$f : X \xrightarrow{l} Y \times S \xrightarrow{p} S$$

with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p the projection. Let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$.

(i) We have then the following commutative diagram in $D_{\text{Fil}, \mathcal{D}, \infty}(S^{an}/(\tilde{S}_I^{an}))$,

$$\begin{array}{ccc}
(\mathcal{F}_S^{GM}(M^{BM}(X/S)))^{an} \xrightarrow{T(\text{An}, \mathcal{F}_{an}^{FDR})(M^{BM}(X/S))} & & \mathcal{F}_{S, an}^{GM}(M^{BM}(X/S)) \\
\downarrow I^{GM}(X/S)^{an} & & \downarrow I^{GM}(X/S) \\
((p_{\tilde{S}_I} \Gamma_{X_I} E_{zar}(\Omega_{Y \times \tilde{S}_I / \tilde{S}_I}^\bullet))^{an} [d_Y], w_{IJ}(X/S)^{an}) \xrightarrow{(T_\omega^O(\text{an}, p_{\tilde{S}_I})^\gamma)} & & (p_{\tilde{S}_I} \Gamma_X E_{usu}(\Omega_{(Y \times \tilde{S}_I)^{an} / \tilde{S}_I^{an}}^\bullet) [d_Y], w_{IJ}(X^{an}/S^{an})) \\
\downarrow ((p_* T_\omega^O(\otimes, \gamma)(-))^{an}) & & \downarrow (p_* T_\omega^O(\otimes, \gamma)(-)) \\
(\int_f^{FDR}(\Gamma_{X_I} E_{zar}(O_{Y \times \tilde{S}_I}, F_b), x_{IJ}(X/S)))^{an} \xrightarrow{(T(\text{an}, \gamma)(-) \circ T^{\mathcal{D}mod}(\text{an}, f)(-))} & & \int_f^{FDR}(\Gamma_{X_I} E_{usu}(O_{(Y \times \tilde{S}_I)^{an}}, F_b), x_{IJ}(X/S))
\end{array}$$

(ii) We have then the following commutative diagram in $\text{PSh}_{\mathcal{D}^\infty \text{fil}}(S^{an}/(\tilde{S}_I^{an}))$,

$$\begin{array}{ccc}
J_S H^n(\mathcal{F}_S^{GM}(M^{BM}(X/S)))^{an} \xrightarrow{J_S(-) \circ H^n T(\text{An}, \mathcal{F}_{an}^{FDR})(M^{BM}(X/S))} & & H^n \mathcal{F}_{S, an}^{GM}(M^{BM}(X/S)) \\
\downarrow H^n(I^{GM}(X/S)^{an}) & & \downarrow H^n I^{GM}(X/S) \\
J_S H^n((p_{\tilde{S}_I} \Gamma_{X_I} E_{zar}(\Omega_{Y \times \tilde{S}_I / \tilde{S}_I}^\bullet))^{an} [d_Y], w_{IJ}(X/S)^{an}) \xrightarrow{H^n(T_\omega^O(\text{an}, p_{\tilde{S}_I})^\gamma)} & & H^n(p_{\tilde{S}_I} \Gamma_X E_{usu}(\Omega_{(Y \times \tilde{S}_I)^{an} / \tilde{S}_I^{an}}^\bullet) [d_Y], w_{IJ}(X^{an}/S^{an})) \\
\downarrow H^n((p_* T_\omega^O(\otimes, \gamma)(-))^{an}) & & \downarrow H^n(p_* T_\omega^O(\otimes, \gamma)(-)) \\
J_S H^n(\int_f^{FDR}(\Gamma_{X_I} E_{zar}(O_{Y \times \tilde{S}_I}, F_b), x_{IJ}(X/S)))^{an} \xrightarrow{H^n(T(\text{an}, \gamma)(-) \circ H^n T^{\mathcal{D}mod}(\text{an}, f)(-))} & & \int_f^{FDR}(\Gamma_{X_I} E_{usu}(O_{(Y \times \tilde{S}_I)^{an}}, F_b), x_{IJ}(X^{an}/S^{an}))
\end{array}$$

Proof. (i): Immediate from definition.

(ii): Follows from (i). \square

We deduce from proposition 132 and theorem 20 (GAGA for D-modules) the following :

Theorem 40. (i) Let $S \in \text{Var}(\mathbb{C})$. Then, for $M \in \text{DA}_c(S)$

$$\mathcal{J}_S(-) \circ H^n T(\text{An}, \mathcal{F}_{an}^{GM})(M) : J_S(H^n(\mathcal{F}_S^{GM}(M))^{an}) \xrightarrow{\sim} H^n \mathcal{F}_{S,an}^{GM}(M)$$

is an isomorphism in $\text{PSh}_{\mathcal{D}}(S^{an}/(\tilde{S}_I^{an}))$.

(ii) A relative version of Grothendieck GAGA theorem for De Rham cohomology Let $h : U \rightarrow S$ a smooth morphism with $S, U \in \text{SmVar}(\mathbb{C})$. Then,

$$\mathcal{J}_S(-) \circ J_S T_{\omega}^O(an, h) : J_S((R^n h_* \Omega_{U/S}^{\bullet})^{an}) \xrightarrow{\sim} R^n h_* \Omega_{U^{an}/S^{an}}^{\bullet}$$

is an isomorphism in $\text{PSh}_{\mathcal{D}}(S^{an})$.

Proof. (i): Follows from proposition 132(ii) and theorem 20 using a resolution by Corti-Hanamura motives.
(ii): Follows from (i) and lemma 21(ii). \square

6.3.2 The transformation map between the analytic filtered De Rham realization functor and the analytification of the filtered algebraic De Rham realization functor

Recall from section 2 that, for $S \in \text{Var}(\mathbb{C})$ we have the following commutative diagrams of sites

$$\begin{array}{ccccc}
\text{AnSp}(\mathbb{C})^2/S & \xrightarrow{\mu_S} & \text{AnSp}(\mathbb{C})^{2,pr}/S & & \\
\downarrow \text{Ans} & \searrow \rho_S & \downarrow \text{Ans} & \searrow \rho_S & \\
& & \text{AnSp}(\mathbb{C})^{2,sm}/S & \xrightarrow{\mu_S} & \text{AnSp}(\mathbb{C})^{2,smpr}/S \\
& & \downarrow \text{Ans} & & \downarrow \text{Ans} \\
\text{Var}(\mathbb{C})^2/S & \xrightarrow{\mu_S} & \text{Var}(\mathbb{C})^{2,smpr}/S & & \\
& \searrow \rho_S & \downarrow \text{Ans} & \searrow \rho_S & \\
& & \text{Var}(\mathbb{C})^2/S & \xrightarrow{\mu_S} & \text{Var}(\mathbb{C})^{2,smpr}/S
\end{array}$$

and

$$\begin{array}{ccccc}
\text{AnSp}(\mathbb{C})^{2,pr}/S & \xrightarrow{\text{Gr}_S^{12}} & \text{AnSp}(\mathbb{C})/S & & \\
\downarrow \text{Ans} & \searrow \rho_S & \downarrow \text{Ans} & \searrow \rho_S & \\
& & \text{AnSp}(\mathbb{C})^{2,smpr}/S & \xrightarrow{\text{Gr}_S^{12}} & \text{AnSp}(\mathbb{C})^{sm}/S \\
& & \downarrow \text{Ans} & & \downarrow \text{Ans} \\
\text{Var}(\mathbb{C})^{2,pr}/S & \xrightarrow{\text{Gr}_S^{12}} & \text{Var}(\mathbb{C})/S & & \\
& \searrow \rho_S & \downarrow \text{Ans} & \searrow \rho_S & \\
& & \text{Var}(\mathbb{C})^{2,sm}/S & \xrightarrow{\text{Gr}_S^{12}} & \text{Var}(\mathbb{C})^{sm}/S
\end{array} \tag{60}$$

and that for $f : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$ we have the following commutative diagrams of site,

$$\begin{array}{ccccc}
\text{AnSp}(\mathbb{C})^2/T^{an} & \xrightarrow{\text{An}_T} & \text{Var}(\mathbb{C})^2/T & & \\
\downarrow P(f) & \searrow \rho_T & \downarrow P(f) & \searrow \rho_T & \\
& \text{AnSp}(\mathbb{C})^{2,sm}/T^{an} & \xrightarrow{\text{An}_T} & \text{Var}(\mathbb{C})^{2,sm}/T & \\
& \downarrow P(f) & & \downarrow P(f) & \\
\text{AnSp}(\mathbb{C})^2/S^{an} & \xrightarrow{\text{An}_S} & \text{Var}(\mathbb{C})^2/S & & \\
& \searrow \rho_S & \searrow \rho_S & & \\
& \text{AnSp}(\mathbb{C})^{2,sm}/S^{an} & \xrightarrow{\text{An}_S} & \text{Var}(\mathbb{C})^{2,sm}/S & \\
& & & \downarrow P(f) &
\end{array}$$

Let $S \in \text{SmVar}(\mathbb{C})$. We have the canonical map in $C(\text{Var}(\mathbb{C})^{2,smpr}/S)$

$$\Omega_{/(S^{an}/S)}^{\Gamma,pr} : (\Omega_{/S}^{\bullet,\Gamma,pr}, F_{DR}) \rightarrow (\Omega_{/S^{an}}^{\bullet,\Gamma,pr}, F_{DR})$$

given by for $p : (Y \times S, Z) \rightarrow S$ the projection with $Y \in \text{SmVar}(\mathbb{C})$,

$$\begin{aligned}
& \Omega_{/(S^{an}/S)}^{\Gamma,pr}((Y \times S, Z)/S) : \Omega_{Y \times S/S}^{\bullet} \otimes_{O_{Y \times S}} \Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b) \\
& \xrightarrow{\Omega_{((Y \times S)^{an}/Y \times S)/(S^{an}/S)}(-)} \Omega_{(Y \times S)^{an}/S^{an}}^{\bullet} \otimes_{(O_{Y \times S})^{an}} (\Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b))^{an}
\end{aligned}$$

We have the following canonical transformation map given by the analytical functor:

Definition 148. Let $S \in \text{SmVar}(\mathbb{C})$. For $F \in C(\text{Var}(\mathbb{C})^{2,smpr}/S)$, we have the canonical transformation map in $C_{\mathcal{D}^\infty \text{fil}}(S^{an})$

$$\begin{aligned}
& T(an, \Omega_{/S}^{\Gamma,pr})(F) : \\
& (e(S)_* \mathcal{H}om^\bullet(F, E_{et}(\Omega_{/S}^{\bullet,\Gamma,pr}, F_{DR})))^{an} := O_{S^{an}} \otimes_{\text{an}_S^* O_S} \text{an}_S^*(e(S)_* \mathcal{H}om^\bullet(F, E_{et}(\Omega_{/S}^{\bullet,\Gamma,pr}, F_{DR}))) \\
& \xrightarrow{T(an,e)(-)} O_{S^{an}} \otimes_{\text{an}_S^* O_S} (e(S^{an})_* \text{An}_S^* \mathcal{H}om^\bullet(F, E_{et}(\Omega_{/S}^{\bullet,\Gamma,pr}, F_{DR}))) \\
& \xrightarrow{T(\text{An},hom)(-,-)} O_{S^{an}} \otimes_{\text{an}_S^* O_S} (e(S^{an})_* \mathcal{H}om^\bullet(\text{An}_S^* F, \text{An}_S^* E_{et}(\Omega_{/S}^{\bullet,\Gamma,pr}, F_{DR}))) \\
& \xrightarrow{\mathcal{H}om(\text{An}_S^* F, \text{An}_S^* E_{et}(\Omega_{/S^{an}}^{\Gamma,pr}, F_{DR})) \otimes m)} e(S^{an})_* \mathcal{H}om^\bullet(\text{An}_S^* F, E_{et}(\Omega_{/S^{an}}^{\bullet,\Gamma,pr,an}, F_{DR}))
\end{aligned}$$

By definition of the algebraic an analytic De Rahm realization functor, we have a natural transformation between them :

Definition 149. Let $S \in \text{SmVar}(\mathbb{C})$. Let $M \in \text{DA}_c(S)$ and $(F, W) \in C_{\text{fil}}(\text{Var}(\mathbb{C})^{sm}/S)$ such that $(M, W) = D(\mathbb{A}_S^1, et)(F, W)$. We have the canonical transformation map $\pi_S(D(\text{MHM}(S))) \subset D_{\mathcal{D}(1,0)\text{fil},\infty}(S^{an})$

$$\begin{aligned}
& T(\text{An}, \mathcal{F}_{an}^{FDR})(M) : (\mathcal{F}_{FDR}^S(M))^{an} := \\
& \left(\varinjlim_{r_{(X^*, D^*)/S}(L(F, W))} e(S)_* \mathcal{H}om^\bullet(\rho_{S^*} \text{Gr}_S^{12^*} R_{(X^*, D^*)/S}(\rho_S^* L(F, W)), E_{et}(\Omega_{/S}^{\bullet,\Gamma,pr}, F_{DR})) \right)^{an} \\
& \xrightarrow{=} \varinjlim_{r_{(X^*, D^*)/S}(LF)} (e(S)_* \mathcal{H}om^\bullet(\rho_{S^*} \text{Gr}_S^{12^*} R_{(X^*, D^*)/S}(\rho_S^* L(F, W)), E_{et}(\Omega_{/S}^{\bullet,\Gamma,pr}, F_{DR})) \right)^{an} \\
& \xrightarrow{T(an, \Omega_{/S}^{\Gamma,pr})(-)} \varinjlim_{r_{(X^*, D^*)/S}(LF)} e(S)_* \mathcal{H}om^\bullet(\text{An}_S^* \rho_{S^*} \text{Gr}_S^{12^*} R_{(X^*, D^*)/S}(\rho_S^* L(F, W)), E_{et}(\Omega_{/S^{an}}^{\bullet,\Gamma,pr,an}, F_{DR})) \\
& \xrightarrow{=} \varinjlim_{r_{(X^*, D^*)/S}(LF)} e(S)_* \mathcal{H}om^\bullet(\text{An}_S^* \rho_{S^*} \text{Gr}_S^{12^*} R_{(X^*, D^*)/S}(\rho_S^* L(F, W)), E_{usu}(\Omega_{/S^{an}}^{\bullet,\Gamma,pr,an}, F_{DR})) =: \mathcal{F}_{S,an}^{FDR}(M)
\end{aligned}$$

We give now the definition in the non smooth case : Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset J$, denote by $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ the projection. Consider, for $I \subset J \subset [1, \dots, l]$, resp. for each $I \subset [1, \dots, l]$, the following commutative diagrams in $\text{Var}(\mathbb{C})$

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ j_{IJ} \uparrow & & \uparrow p_{IJ} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array} .$$

We then have the following lemma

Lemma 23. *The maps $T(an, \Omega)(-)$ induce a morphism in $C_{\mathcal{D}^\infty(fil)}(S/(\tilde{S}_I))$*

$$\begin{aligned} & (T(an, \Omega_{/\tilde{S}_I}^{\Gamma, pr})(\rho_{\tilde{S}_I^*} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F))) : \\ & \left(\varinjlim_{r_{(X^*, D^*)/-}(-)} (e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\rho_{\tilde{S}_I^*} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})))^{an}, (u_{I,J}^q(F))^{an} \right) \\ \rightarrow & \left(\varinjlim_{r_{(X^*, D^*)/-}(-)} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}(\tilde{S}_I)^* \rho_{\tilde{S}_I^*} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})), u_{I,J}^q(F) \right) \end{aligned}$$

Proof. Obvious. □

Definition 150. *Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote $S_I = \cap_{i \in I} S_i$. We have then closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I = \prod_{i \in I} \tilde{S}_i$. Let $M \in \text{DA}_c(S)$ and $(F, W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ such that $(M, W) = D(\mathbb{A}_S^1, et)(F, W)$. We have, by lemma 23, the canonical transformation map in $\pi_S(D(\text{MHM}(S))) \subset D_{\mathcal{D}(1,0)fil,\infty}(S^{an})$*

$$\begin{aligned} & T(\text{An}, \mathcal{F}_S^{FDR})(M) : (\mathcal{F}_S^{FDR}(M))^{an} \xrightarrow{=} \\ & \left(\varinjlim_{r_{(X^*, D^*)/-}(-)} (e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\rho_{\tilde{S}_I^*} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^*(F, W))), \right. \\ & E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))^{an}, (u_{I,J}^q(F, W))^{an} \left. \xrightarrow{(T(an, \Omega_{/\tilde{S}_I}^{\Gamma, pr})(\rho_{\tilde{S}_I^*} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^*(F, W))))} \right. \\ & \left. \left(\varinjlim_{r_{(X^*, D^*)/-}(-)} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}(\tilde{S}_I)^* \rho_{\tilde{S}_I^*} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^*(F, W))), \right. \right. \\ & \left. \left. E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR}), u_{I,J}^q(F, W) \right) \right. \\ & \xrightarrow{=} \left(\varinjlim_{r_{(X^*, D^*)/-}(-)} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}(\tilde{S}_I)^* \rho_{\tilde{S}_I^*} \text{Gr}_{\tilde{S}_I}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^*(F, W))), \right. \\ & \left. E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR}), u_{I,J}^q(F, W) \right) \xrightarrow{=} \mathcal{F}_{S,an}^{FDR}(M) \end{aligned}$$

The following proposition says this transformation map between $\mathcal{F}^{S,an}$ and $(\mathcal{F}_S^{FDR})^{an}$ is functorial in $S \in \text{Var}(\mathbb{C})$, hence define a commutative diagram of morphism of 2-functor :

Proposition 133. *(i) Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with, $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We then have closed embedding $i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$ and $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I := g|_{T_I} : T_I \rightarrow S_I$.*

(i0) Then, for $M \in \text{DA}_c(S)$, the following diagram in $\pi_T(D(\text{MHM}(T^{an})) \subset D_{\mathcal{D}(1,0)\text{fil},\infty}(T^{an}/(Y^{an} \times \tilde{S}_I^{an}))$, see definition 121 and definition 142 commutes

$$\begin{array}{ccc} g_{\text{Hdg}}^{*\text{mod}}((\mathcal{F}_S^{\text{FDR}}(M))^{an}) = (g_{\text{Hdg}}^{*\text{mod}}(\mathcal{F}_S^{\text{FDR}}(M))) \xrightarrow{g_{\text{Hdg}}^{*\text{mod}}(T(An, \mathcal{F}_S^{\text{FDR}})(M))} g_{\text{Hdg}}^{*\text{mod}}(\mathcal{F}_{S,an}^{\text{FDR}}(M)) & & \\ \downarrow (T(g, \mathcal{F}^{\text{FDR}})(M))^{an} & & \downarrow T(g, \mathcal{F}_{an}^{\text{FDR}})(M) \\ (\mathcal{F}_T^{\text{FDR}}(g^*M))^{an} & \xrightarrow{T(An, \mathcal{F}_T^{\text{FDR}})(g^*M)} & \mathcal{F}_{T,an}^{\text{FDR}}(g^*M) \end{array}$$

(i1) Then, for $M \in \text{DA}_c(S)$, the following diagram in $\pi_T(D(\text{MHM}(T)) \subset D_{\mathcal{D}(1,0)\text{fil},\infty}(T^{an}/(Y^{an} \times \tilde{S}_I^{an}))$ commutes

$$\begin{array}{ccc} g_{\text{Hdg}}^{*\text{mod}}((\mathcal{F}_S^{\text{FDR}}(M))^{an}) = (g_{\text{Hdg}}^{*\text{mod}}(\mathcal{F}_S^{\text{FDR}}(M))) \xrightarrow{g_{\text{Hdg}}^{*\text{mod}}(T(An, \mathcal{F}_S^{\text{FDR}})(M))} g_{\text{Hdg}}^{*\text{mod}}(\mathcal{F}_{S,an}^{\text{FDR}}(M)) & & \\ \uparrow (T(g, \mathcal{F}^{\text{FDR}})(M))^{an} & & \uparrow T'(g, \mathcal{F}_{an}^{\text{FDR}})(M) \\ (\mathcal{F}_T^{\text{FDR}}(g^*M))^{an} & \xrightarrow{T(An, \mathcal{F}_T^{\text{FDR}})(g^*M)} & \mathcal{F}_{T,an}^{\text{FDR}}(g^*M) \end{array}$$

(i2) Then, for $M \in \text{DA}_c(T)$, the following diagram in $\pi_S(D(\text{MHM}(S^{an})) \subset D_{\mathcal{D}(1,0)\text{fil},\infty}(S^{an}/(\tilde{S}_I^{an}))$ commutes

$$\begin{array}{ccc} Rg_*^{\text{Hdg}}((\mathcal{F}_T^{\text{FDR}}(M))^{an}) = (Rg_*^{\text{Hdg}} * (\mathcal{F}_T^{\text{FDR}}(M))) \xrightarrow{Rg_*^{\text{Hdg}}(T(An, \mathcal{F}_S^{\text{FDR}})(M))} Rg_*^{\text{Hdg}}(\mathcal{F}_{S,an}^{\text{FDR}}(M)) & & \\ \uparrow (T_*(g, \mathcal{F}^{\text{FDR}})(M))^{an} & & \uparrow T_*(g, \mathcal{F}_{an}^{\text{FDR}})(M) \\ (\mathcal{F}_S^{\text{FDR}}(Rg_*M))^{an} & \xrightarrow{T(An, \mathcal{F}_S^{\text{FDR}})(Rg_*M)} & \mathcal{F}_{S,an}^{\text{FDR}}(Rg_*M) \end{array}$$

(i3) Then, for $M \in \text{DA}_c(T)$, the following diagram in $\pi_S(D(\text{MHM}(S^{an})) \subset D_{\mathcal{D}(1,0)\text{fil},\infty}(S^{an}/(\tilde{S}_I^{an}))$ commutes

$$\begin{array}{ccc} Rg_!^{\text{Hdg}}((\mathcal{F}_T^{\text{FDR}}(M))^{an}) = (Rg_!^{\text{Hdg}}!(\mathcal{F}_T^{\text{FDR}}(M))) \xrightarrow{Rg_!^{\text{Hdg}}(T(An, \mathcal{F}_S^{\text{FDR}})(M))} Rg_!^{\text{Hdg}}(\mathcal{F}_{S,an}^{\text{FDR}}(M)) & & \\ \downarrow (T_!(g, \mathcal{F}^{\text{FDR}})(M))^{an} & & \downarrow T_!(g, \mathcal{F}_{an}^{\text{FDR}})(M) \\ (\mathcal{F}_S^{\text{FDR}}(Rg_!M))^{an} & \xrightarrow{T(An, \mathcal{F}_S^{\text{FDR}})(Rg_!M)} & \mathcal{F}_{S,an}^{\text{FDR}}(Rg_!M) \end{array}$$

(ii) Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, for $M, N \in \text{DA}_c(S)$, the following diagram in $\pi_S(D(\text{MHM}(S)) \subset D_{\mathcal{D}(1,0)\text{fil},\infty}(S^{an}/(\tilde{S}_I^{an}))$ commutes

$$\begin{array}{ccc} (\mathcal{F}_S^{\text{FDR}}(M) \otimes_{\mathcal{O}_S} \mathcal{F}_S^{\text{FDR}}(N)) \xrightarrow{(T(\otimes, \mathcal{F}^{\text{FDR}})(M, N))^{an}} (\mathcal{F}_S^{\text{FDR}}(M \otimes N))^{an} & & \\ \downarrow = & & \downarrow \\ (\mathcal{F}_S^{\text{FDR}}(M))^{an} \otimes_{\mathcal{O}_{S^{an}}} (\mathcal{F}_S^{\text{FDR}}(N))^{an} & \xrightarrow{T(An, \mathcal{F}_S^{\text{FDR}})(M \otimes N)} & \\ \downarrow T(An, \mathcal{F}_S^{\text{FDR}})(M) \otimes_{\mathcal{O}_{S^{an}}} T(An, \mathcal{F}_S^{\text{FDR}})(N) & & \downarrow \\ \mathcal{F}_{S,an}^{\text{FDR}}(M) \otimes_{\mathcal{O}_{S^{an}}} \mathcal{F}_{S,an}^{\text{FDR}}(N) & \xrightarrow{(T(\otimes, \mathcal{F}_{an}^{\text{FDR}})(M, N))} & \mathcal{F}_{S,an}^{\text{FDR}}(M \otimes N) \end{array}$$

Proof. Immediate from definition. □

Proposition 134. *Let $f : X \rightarrow S$ a morphism with $S, X \in \text{Var}(\mathbb{C})$. Assume there exist a factorization*

$$f : X \xrightarrow{l} Y \times S \xrightarrow{p} S$$

with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p the projection. Let $S = \cup_i S_i$ an open affine cover and $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We have then the following commutative diagram in $D_{\mathcal{D}^\infty \text{fil}, \infty}(S^{an})$,

$$\begin{array}{ccc} (\mathcal{F}_S^{FDR}(M(X/S)))^{an} & \xrightarrow{T(\text{An}, \mathcal{F}_S^{FDR})(M(X/S))} & \mathcal{F}_{S, an}^{FDR}(M(X/S)) \\ \downarrow I(X/S) & & \downarrow I(X/S) \\ (Rf_!^{Hdg}(\Gamma_{X_I}^{\vee, Hdg}(O_{Y \times \tilde{S}_I}, F_b), x_{IJ}(X/S)))^{an} & \xrightarrow{T^{\mathcal{D}^{mod}(an, p_{\tilde{S}_I})}(-)} & Rf_!^{Hdg}((\Gamma_{X_I}^{\vee, Hdg}(O_{(Y \times \tilde{S}_I)^{an}}, F_b), x_{IJ}(X/S))) \end{array}$$

Proof. Immediate from definition. \square

We deduce from proposition 134 and theorem 20 (GAGA for D-modules) the following :

Theorem 41. *Let $S \in \text{Var}(\mathbb{C})$. For $M \in \text{DA}_c(S)$, the map in $\pi_S(D(\text{MHM}(S^{an}))) \subset D_{\mathcal{D}(1,0)\text{fil}, \infty}(S^{an})$*

$$T(\text{An}, \mathcal{F}^{FDR})(M) : (\mathcal{F}_S^{FDR}(M))^{an} \xrightarrow{\sim} \mathcal{F}_{S, an}^{FDR}(M)$$

given in definition 150 is an isomorphism.

Proof. Follows from proposition 134 and theorem 20. \square

We finish this subsection by the following easy proposition :

Proposition 135. *Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Let $M \in \text{DA}_c(S)$ and $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}_S^1, et)(F)$. Then the following diagram in $D_{\text{Ofil}, \mathcal{D}, \infty}(S/\tilde{S}_I)$ commutes*

$$\begin{array}{ccc} (F_S^{GM}(L\mathbb{D}_S M))^{an} & \xrightarrow{\mathcal{J} \circ (T(\mathcal{F}_S^{GM}, \mathcal{F}_S^{FDR})(M))^{an}} & J_S((\mathcal{F}_S^{FDR}(M))^{an}) \\ T(\text{An}, \mathcal{F}_S^{GM})(\mathbb{D}_S M) \downarrow & & \downarrow J_S(T(\text{An}, \mathcal{F}_S^{FDR})(M)) \\ \mathcal{F}_{S, an}^{GM}(L\mathbb{D}_S M) & \xrightarrow{T(\mathcal{F}_{S, an}^{GM}, \mathcal{F}_{S, an}^{FDR})(M)} & J_S(\mathcal{F}_{S, an}^{FDR}(M)) \end{array}$$

Proof. Immediate from definition. \square

7 The Hodge realization functor for relative motives

7.1 The Betti realization functor

We have two definition of the Betti realization functor which coincide at least for constructible motives, one given by [1] using the analytical functor and one given in [7] by composing the analytical functor with the forgetfull functor to the topological space of a complex analytic space wich is a CW complex (see also [20] for the absolute case) .

Definition 151. *Let $S \in \text{Var}(\mathbb{C})$.*

(i) *The Ayoub's Betti realization functor is*

$$\text{Bti}_S^* : \text{DA}(S) \rightarrow D(S^{an}), \quad M \in \text{DA}(S) \mapsto \text{Bti}_S^* M = \text{Re}(S^{an})_* \text{An}_S^* M = e(S^{an})_* \underline{\text{sing}}_{\mathbb{D}} \text{An}_S^* F$$

where $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$.

(ii) In [7], we define the Betti realization functor as

$$\widetilde{\text{Bti}}_S^* : \text{DA}(S) \rightarrow D(S^{an}) = D(S^{cw}), \quad M \mapsto \widetilde{\text{Bti}}_S^* M = \text{Re}(S^{cw})_* \widetilde{\text{Cw}}_S^* M = e(S^{cw})_* \underline{\text{sing}}_{\mathbb{D}}^* \widetilde{\text{Cw}}_S^* F$$

where $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$.

(iii) For the Corti-Hanamura weight structure on $\text{DA}^-(S)$, we have by functoriality of (i) the functor

$$\text{Bti}_S^* : \text{DA}^-(S) \rightarrow D_{fil, \infty}(S^{an}), \quad M \mapsto \text{Bti}_S^* M = e(S^{an})_* \underline{\text{sing}}_{\mathbb{D}}^* \text{An}_S^*(F, W)$$

where $(F, W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $(M, W) = D(\mathbb{A}^1, et)(F, W)$.

Note that by [7], An_S^* and $\widetilde{\text{Cw}}_S^*$ derive trivially.

Note that, by considering the explicit \mathbb{D}_S^1 local model for presheaves on $\text{AnSp}(\mathbb{C})^{sm}/S^{an}$, $\text{Bti}_S^*(\text{DA}^-(S)) \subset D^-(S^{an})$; by considering the explicit \mathbb{I}_S^1 local model for presheaves on CW^{sm}/S^{cw} , $\widetilde{\text{Bti}}_S^*(\text{DA}^-(S)) \subset D^-(S^{an})$.

Definition 152. Let $f : T \rightarrow S$ a morphism, with $T, S \in \text{Var}(\mathbb{C})$. We have, for $M \in \text{DA}(S)$, $(F, W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ such that $(M, W) = D(\mathbb{A}^1, et)(F, W)$, and an equivalence (\mathbb{A}^1, et) local $e : f^*(F, W) \rightarrow (F', W)$ with $(F', W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ such that $(f^*M, W) = D(\mathbb{A}^1, et)(F', W)$ the following canonical transformation map in $D_{fil}(T)$:

$$T(f, \text{Bti})(M) : f^* \text{Bti}_S^* M := f^* e(S^{an})_* \underline{\text{sing}}_{\mathbb{D}}^* \text{An}_S^*(F, W) \xrightarrow{T(f, e)(-)} e(T^{an})_* f^* \underline{\text{sing}}_{\mathbb{D}}^* \text{An}_S^*(F, W) \quad (61)$$

$$\xrightarrow{e(T^{an})_* T(f, e)(F, W)} e(T^{an})_* \underline{\text{sing}}_{\mathbb{D}}^* f^* \text{An}_T^*(F, W) \xrightarrow{=} e(T^{an})_* \underline{\text{sing}}_{\mathbb{D}}^* \text{An}_T^* f^*(F, W) \quad (62)$$

$$\xrightarrow{e(T^{an})_* \underline{\text{sing}}_{\mathbb{D}}^* \text{An}_T^* e} e(T^{an})_* \underline{\text{sing}}_{\mathbb{D}}^* \text{An}_T^*(F', W) =: \text{Bti}_T^* f^* M \quad (63)$$

Theorem 42. Let $f : X \rightarrow S$ a morphism, with $X, S \in \text{Var}(\mathbb{C})$. For $M \in \text{DA}_c(S)$,

$$T(f, \text{Bti})(M) : f^* \text{Bti}_S^*(M) \xrightarrow{\sim} \text{Bti}_X^* f^*(M)$$

is an isomorphism in $D_{fil}(X)$.

Proof. See [1]. □

Definition 153. • Let $f : X \rightarrow S$ a morphism, with $X, S \in \text{Var}(\mathbb{C})$. We have, for $M \in \text{DA}_c(X)$, the following transformation map

$$\begin{aligned} T_*(f, \text{Bti})(M) : \text{Bti}_S^*(Rf_* M) &\xrightarrow{\text{ad}(f^*, Rf_*)(\text{Bti}_S^*(f_* M))} Rf_* f^* \text{Bti}_S^*(Rf_* M) \\ &\xrightarrow{T(f, \text{Bti})(f_* M)} Rf_* \text{Bti}_X^*(f^* Rf_* M) \xrightarrow{\text{Bti}_X^*(\text{ad}(f^*, Rf_*)(M))} Rf_* \text{Bti}_X^*(M) \end{aligned}$$

Clearly, if $l : Z \hookrightarrow S$ is a closed embedding, then $T_*(l, \text{Bti})(M)$ is an isomorphism by theorem 42.

- Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have then, for $M \in \text{DA}_c(X)$, using theorem 42 for closed embeddings, the following transformation map

$$\begin{aligned} T_!(f, \mathcal{F}_{an}^{FDR})(M) : Rf_! \text{Bti}_X^*(M) &= R p_{S!} l_* \text{Bti}_X^*(M) \xrightarrow{T_*(l, \text{Bti})(M)^{-1}} R p_{S!} \text{Bti}(Y \times S)^*(l_* M) \\ &\xrightarrow{\text{Bti}(Y \times S)^* \text{ad}(L p_{S!}, p_S^*)(l_* M)} R p_{S!} \text{Bti}(Y \times S)^*(p_S^* L p_{S!} l_* M) \xrightarrow{T(p_S, \text{Bti})(p_{S!} l_* M)} \end{aligned}$$

$$R p_{S!} p_S^* \text{Bti}(Y \times S)^*(L p_{S!} l_* M) = R p_{S!} p_S^! \text{Bti}(Y \times S)^*(f_! M) \xrightarrow{\text{ad}(R p_{S!}, p_S^!)(-)} \text{Bti}(Y \times S)^*(f_! M)$$

Clearly, for $f : X \rightarrow S$ a proper morphism, with $X, S \in \text{Var}(\mathbb{C})$ we have, for $M \in \text{DA}_c(Y \times S)$, $T_!(f, \text{Bti})(M) = T_*(f, \text{Bti})(M)$.

- Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. We have, using the second point, for $M \in \text{DA}(S)$, the following transformation map

$$\begin{aligned} T^1(f, \text{Bti})(M) : \text{Bti}_X^*(f^!M) &\xrightarrow{\text{ad}(f, Rf^!)(\text{Bti}_X^*(f^!M))} f^!Rf_! \text{Bti}_X^*(f^!M) \\ &\xrightarrow{T_1(f, \text{Bti})(f^!M)} f^! \text{Bti}_S^*(f_!f^!M) \xrightarrow{\text{Bti}_S^*(\text{ad}(f, f^!)(M))} f^! \text{Bti}_S^*(M) \end{aligned}$$

Definition 154. Let $S \in \text{Var}(\mathbb{C})$. We have, for $M, N \in \text{DA}(S)$ and $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1, et)(F)$ and $N = D(\mathbb{A}^1, et)(G)$, the following transformation map in $D_{\mathcal{D}fil}(S)$

$$\begin{aligned} \text{Bti}_S^*M \otimes \text{Bti}_S^*N &:= (e(S)_* \underline{\text{sing}}_{\mathbb{D}^*} \text{An}_S^*F) \otimes (e(S)_* \underline{\text{sing}}_{\mathbb{D}^*} \text{An}_S^*G) \\ &\xrightarrow{T(\text{sing}_{\mathbb{D}^*}, \otimes)(\text{An}_S^*F, \text{An}_S^*G)} e(S)_* \underline{\text{sing}}_{\mathbb{D}^*} \text{An}_S^*(F \otimes G) =: \text{Bti}_S^*(M \otimes N) \end{aligned}$$

Theorem 43. (i) Let $f : X \rightarrow S$ a morphism, with $X, S \in \text{Var}(\mathbb{C})$. For $M \in \text{DA}_c(X)$,

$$T_1(f, \text{Bti})(M, W) : f_! \text{Bti}_X^*(M) \xrightarrow{\sim} \text{Bti}_S^*f_!M$$

is an isomorphism.

(ii) Let $f : X \rightarrow S$ a morphism, with $X, S \in \text{Var}(\mathbb{C})$. For $M \in \text{DA}_c(X)$,

$$T_*(f, \text{Bti})(M, W) : f_* \text{Bti}_X^*M \xrightarrow{\sim} \text{Bti}_S^*Rf_*M$$

is an isomorphism.

(iii) Let $f : X \rightarrow S$ a morphism, with $X, S \in \text{Var}(\mathbb{C})$. For $M \in \text{DA}_c(S)$,

$$T^1(f, \text{Bti})(M) : f^! \text{Bti}_S^*M \xrightarrow{\sim} \text{Bti}_X^*f^!M$$

is an isomorphism.

(iv) Let $S \in \text{Var}(\mathbb{C})$. For $M, N \in \text{DA}_c(S)$,

$$T(\otimes, \text{Bti})(M) : \text{Bti}_S^*M \otimes \text{Bti}_S^*N \xrightarrow{\sim} \text{Bti}_X^*(M \otimes N)$$

is an isomorphism.

Proof. See [1]. □

The main result on the Betti realization functor is the following

Theorem 44. (i) We have $\text{Bti}_S^* = \widetilde{\text{Bti}}_S^*$ on $\text{DA}^-(S)$

(ii) The canonical transformations $T(f, \text{Bti})$, for $f : T \rightarrow S$ a morphism in $\text{Var}(\mathbb{C})$, define a morphism of 2 functor

$$\text{Bti}_S^* : \text{DA}(\cdot) \rightarrow D(\cdot^{an}), \quad S \in \text{Var}(\mathbb{C}) \mapsto \text{Bti}_S^* : \text{DA}(S) \rightarrow D(S^{an})$$

which is a morphism of homotopic 2 functor.

Proof. (i): See [7]

(ii): Follows from theorem 42 and theorem 43. □

Remark 13. For $X \in \text{Var}(\mathbb{C})$, the quasi-isomorphisms

$$\mathbb{Z} \text{Hom}(\mathbb{D}_{et}^\bullet, X) \xrightarrow{\text{An}^*} \mathbb{Z} \text{Hom}(\mathbb{D}^n(0, 1), X^{an}) \xrightarrow{\text{Hom}(i, X^{cw})} \mathbb{Z} \text{Hom}([0, 1]^n, X^{cw}),$$

where,

$$\mathbb{D}_{et}^n := (e : U \rightarrow \mathbb{A}^n, \mathbb{D}^n(0, 1) \subset e(U)) \in \text{Fun}(\mathcal{V}_{\mathbb{A}^n}^{et}(\mathbb{D}^n(0, 1)), \text{Var}(\mathbb{C}))$$

is the system of etale neighborhood of the closed ball $\bar{\mathbb{D}}^n(0,1) \subset \mathbb{A}^n$, and $i : [0,1]^n \hookrightarrow \bar{\mathbb{D}}^n(0,1)$ is the closed embedding, shows that a closed singular chain $\alpha \in \mathbb{Z}\text{Hom}^n([0,1]^n, X^{cw})$, is homologue to a closed singular chain

$$\beta = \alpha + \partial\gamma = \tilde{\beta}|_{[0,1]^n} \in \mathbb{Z}\text{Hom}^n(\Delta^n, X^{cw})$$

which is the restriction by the closed embedding $[0,1]^n \hookrightarrow U^{cw} \xrightarrow{e} \mathbb{A}^n$, where $e : U \rightarrow \mathbb{A}^n$ an etale morphism with $U \in \text{Var}(\mathbb{C})$, of a complex algebraic morphism $\tilde{\beta} : U \rightarrow X$. Hence $\beta([0,1]^n) = \tilde{\beta}([0,1]^n) \subset X$ is the restriction of a real algebraic subset of dimension n in $\text{Res}_{\mathbb{R}}(X)$ (after restriction a scalar that is under the identification $\mathbb{C} \simeq \mathbb{R}^2$).

Definition 155. Let $S \in \text{Var}(\mathbb{C})$ The cohomological Betti realization functor is

$$\begin{aligned} \text{Bti}_S^\vee : \text{DA}(S) &\rightarrow D(S^{cw}), \\ M &\mapsto \text{Bti}_S^\vee(M) := R\mathcal{H}om(\text{Bti}_S^* M, \mathbb{Z}_{S^{cw}}) = R\mathcal{H}om(M, \text{Bti}_{S^*} \mathbb{Z}_{S^{cw}}) \end{aligned}$$

where for $\text{Bti}_{S^*} : K \in D(S^{cw}) \mapsto R\text{An}_{S^*} e(S^{an})^* K \in \text{DA}(S)$ is the right ajoin to Bti_S^* .

7.2 The Hodge realization functor for relative motives

Recall (see section 2) that for $S \in \text{Var}(\mathbb{C})$, we consider the dual functor

$$\mathbb{D}_S : C(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{sm}/S), F \mapsto \mathbb{D}_S F := \mathcal{H}om(F, E_{et}(\mathbb{Z}_S)).$$

Similarly, for $S \in \text{AnSp}(\mathbb{C})$, we consider the dual functor

$$\mathbb{D}_S : C(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{sm}/S), F \mapsto \mathbb{D}_S F := \mathcal{H}om(F, E_{usu}(\mathbb{Z}_S))$$

The filtered De Rham algebraic realization functor constructed in section 6 and on the other hand the Betti realization functor (see section 7.1) give the Hodge realization functor :

Definition 156. Let $S \in \text{SmVar}(\mathbb{C})$. We define the Hodge realization functor as

$$\begin{aligned} \mathcal{F}_S^{Hdg} &:= (\mathcal{F}_S^{FDR}, \text{Bti}_S^*) : \text{DA}_c(S) \rightarrow D_{\mathcal{D}(1,0)fil,\infty}(S) \times_I D_{fil}(S^{an}), \\ M &\mapsto \mathcal{F}_S^{Hdg}(M) := (\mathcal{F}_S^{FDR}(M), \text{Bti}_S^* M, \alpha(M)), \end{aligned}$$

where $\alpha(M)$ is the map in $D_{\mathbb{C}fil}(S^{an})$

$$\begin{aligned}
& \alpha(M) : (\text{Bti}_S^* M) \otimes \mathbb{C}_S := (e(S)_* \underline{\text{sing}}_{\mathbb{D}^*} \text{An}_S^* L(F, W)) \otimes \mathbb{C}_S \\
& \xrightarrow{s(e(S)_* \underline{\text{sing}}_{\mathbb{D}^*} \text{An}_S^* L(F, W))} DR(S)^{[-1]}(\mathcal{H}om(L\mathbb{D}_S L(e(S)_* \underline{\text{sing}}_{\mathbb{D}^*} \text{An}_S^* L(F, W)), E_{usu}(O_S))) \\
& \xrightarrow{\cong} DR(S)^{[-1]}(\mathcal{H}om(L\mathbb{D}_S L(e(S)_* \underline{\text{sing}}_{\mathbb{D}^*} \text{An}_S^* L(F, W)), e(S)_* E_{usu}(\Omega_{/S}^\bullet))) \\
& \xrightarrow{DR(S)^{[-1]}(\mathcal{H}om(T(e, \text{hom})(-, E(\mathbb{Z})), -))} DR(S)^{[-1]}(\mathcal{H}om(Le(S)_* \mathbb{D}_S L \underline{\text{sing}}_{\mathbb{D}^*} \text{An}_S^* L(F, W)), e(S)_* E_{usu}(\Omega_{/S}^\bullet))) \\
& \xrightarrow{DR(S)^{[-1]}(\mathcal{H}om(Le(S)_* \mathbb{D}_S \mathcal{C}(\text{An}_S^* L(F, W)), e(S)_* E_{usu}(\Omega_{/S}^\bullet)))} \\
& DR(S)^{[-1]}(\mathcal{H}om(Le(S)_* \mathbb{D}_S \text{An}_S^* L(F, W)), e(S)_* E_{usu}(\Omega_{/S}^\bullet))) \\
& \xrightarrow{DR(S)^{[-1]}(\mathcal{H}om(\mathcal{H}om(T(\text{An}, \text{hom})(-, E_{et}(\mathbb{Z})), e(S)_* E_{usu}(\Omega_{/S}^\bullet))))} \\
& DR(S)^{[-1]}(\mathcal{H}om(Le(S)_* \text{An}_S^* \mathbb{D}_S L(F, W), e(S)_* E_{usu}(\Omega_{/S}^\bullet))) \\
& \xrightarrow{DR(S)^{[-1]}(\mathcal{H}om(-, e(S)_* E_{usu}(\text{Gr}(\Omega_{/S^{an}})))^{-1}} \\
& DR(S)^{[-1]}(\mathcal{H}om(Le(S)_* \text{An}_S^* \mathbb{D}_S L(F, W)), e(S)_* \text{Gr}_{S^*}^{12} E_{usu}(\Omega_{/S^{an}}^{\bullet, \Gamma, pr, an}))) \\
& \xrightarrow{DR(S)^{[-1]}(T(e, \text{hom})(-, -)^{-1}} DR(S)^{[-1]}(e(S)_* \mathcal{H}om(L \text{An}_S^* \mathbb{D}_S L(F, W)), \text{Gr}_{S^*}^{12} E_{usu}(\Omega_{/S^{an}}^{\bullet, \Gamma, pr, an}))) \\
& \xrightarrow{DR(S)^{[-1]}(I(\text{Gr}_S^{12*}, \text{Gr}_{S^*}^{12})(-, -))} DR(S)^{[-1]}(e'(S)_* \mathcal{H}om(L \text{An}_S^* \text{Gr}_S^{12*} \mathbb{D}_S L(F, W)), E_{usu}(\Omega_{/S^{an}}^{\bullet, \Gamma, pr, an}))) \\
& \xrightarrow{DR(S)^{[-1]}(\mathcal{H}om(L \text{An}_S^* \text{Gr}_S^{12*} r_{(X^*, D^*)/S}(L(F, W)), E_{usu}(\Omega_{/S^{an}}^{\bullet, \Gamma, pr, an})))} \\
& DR(S)^{[-1]}(e'(S)_* \mathcal{H}om(\text{An}_S^* \rho_{S^*} \text{Gr}_S^{12*} R_{(X^*, D^*)/S}(\rho_S^* L(F, W)), E_{usu}(\Omega_{/S^{an}}^{\bullet, \Gamma, pr, an}))) \\
& \xrightarrow{\cong} DR(S)^{[-1]}(\mathcal{F}_{S, an}^{DR}(M, W)) \\
& \xrightarrow{DR(S)^{[-1]}(T(\text{An}, \mathcal{F}_S^{FDR})(M, W)^{-1}} DR(S)^{[-1]}((\mathcal{F}_S^{DR}(M, W))^{an})
\end{aligned}$$

where $(F, W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $(M, W) = D(\mathbb{A}^1, et)(F, W)$

- $s(K) : K \rightarrow DR(S)^{[-1]}(\mathcal{H}om(\mathbb{D}_S LK, E(O_S)))$ is the isomorphism of theorem 23,
- the map

$$\begin{aligned}
& T(e, \text{hom})(-, -) : e(S)_* \mathcal{H}om(L \text{An}_S^* \mathbb{D}_S L(F, W), \text{Gr}_{S^*}^{12} E_{usu}(\Omega_{/S^{an}}^{\bullet, \Gamma, pr, an}))) \\
& \rightarrow \mathcal{H}om(Le(S)_* \text{An}_S^* \mathbb{D}_S L(F, W), e(S)_* \text{Gr}_{S^*}^{12} E_{usu}(\Omega_{/S^{an}}^{\bullet, \Gamma, pr, an})))
\end{aligned}$$

is a filtered equivalence *usu* local by proposition 33 and proposition 121,

- $\mathcal{F}_S^{DR}(M) := o_{fil} \mathcal{F}_S^{FDR}(M) \in D_{\mathcal{D}0fil}(S)$, $\mathcal{F}_{S, an}^{DR}(M) := o_{fil} \mathcal{F}_{S, an}^{FDR}(M) \in D_{\mathcal{D}0fil}(S^{an})$,
- $T(\text{An}, \mathcal{F}_S^{FDR})(M) : (\mathcal{F}_S^{DR}(M))^{an} \xrightarrow{\cong} \mathcal{F}_{S, an}^{DR}(M)$ is an isomorphism by theorem 41.

We now give the definition in the non smooth case :

Definition 157. Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_i S_i$ an open cover such that there exists closed embedding $i_i : S \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We define the Hodge realization functor as

$$\begin{aligned}
\mathcal{F}_S^{Hdg} & := (\mathcal{F}_S^{FDR}, \text{Bti}_S^*) : \text{DA}(S) \rightarrow D_{\mathcal{D}(1,0)fil, \infty}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}), \\
M & \mapsto \mathcal{F}_S^{Hdg}(M) := (\mathcal{F}_S^{FDR}(M), \text{Bti}_S^* M, \alpha(M)),
\end{aligned}$$

where $\alpha(M)$ is the map in $D_{\mathbb{C}fil}(S^{an}/(\tilde{S}_I^{an}))$

$$\begin{aligned}
\alpha(M) : T(S/(\tilde{S}_I))((\text{Bti}_S^*(M, W)) \otimes \mathbb{C}_S) &:= (i_{I*}j_I^*((e(S)_{*\underline{\text{sing}}_{\mathbb{D}}^*} \text{An}_S^* L(F, W)) \otimes C_S), T^q(D_{IJ})(-)) \\
&\xrightarrow{s(e(S)_{*\underline{\text{sing}}_{\mathbb{D}}^*} \text{An}_S^* L(F, W))} \\
&DR(S)^{[-1]}(\text{Hom}(L\mathbb{D}_{\tilde{S}_I} L(i_{I*}j_I^*(e(S)_{*\underline{\text{sing}}_{\mathbb{D}}^*} \text{An}_S^* L(F, W)), E_{usu}(O_{\tilde{S}_I})), u_{IJ}(-)) \\
\rightleftharpoons DR(S)^{[-1]}(\text{Hom}(L\mathbb{D}_{\tilde{S}_I} L(i_{I*}j_I^*(e(S)_{*\underline{\text{sing}}_{\mathbb{D}}^*} \text{An}_S^* L(F, W)), e(\tilde{S}_I)_* E_{usu}(\Omega_{/\tilde{S}_I}^\bullet))), u_{IJ}(-)) \\
&\xrightarrow{DR(S)^{[-1]}(\text{Hom}(T(e, \text{hom})(-, E(\mathbb{Z})), -))} \\
&DR(S)^{[-1]}(\text{Hom}(Le(\tilde{S}_I)_* \mathbb{D}_{\tilde{S}_I} L(i_{I*}j_I^* \underline{\text{sing}}_{\mathbb{D}}^* \text{An}_S^* L(F, W)), e(\tilde{S}_I)_* E_{usu}(\Omega_{/\tilde{S}_I}^\bullet))) \\
&\xrightarrow{DR(S)^{[-1]}(\text{Hom}(Le(\tilde{S}_I)_* \mathbb{D}_{\tilde{S}_I} L(i_{I*}j_I^* c(\text{An}_S^* L(F, W)), e(\tilde{S}_I)_* E_{usu}(\Omega_{/\tilde{S}_I}^\bullet)))} \\
&DR(S)^{[-1]}(\text{Hom}(Le(\tilde{S}_I)_* \mathbb{D}_{\tilde{S}_I} L(i_{I*}j_I^* \text{An}_S^* L(F, W)), e(\tilde{S}_I)_* E_{usu}(\Omega_{/\tilde{S}_I}^\bullet))), u_{IJ}(-)) \\
&\xrightarrow{DR(S)^{[-1]}(\text{Hom}(\text{Hom}(T(\text{An}, \text{hom})(-, E_{et}(\mathbb{Z})) \circ \mathbb{D}_{\tilde{S}_I} T(\text{An}, i_I)(-), e(\tilde{S}_I)_* E_{usu}(\Omega_{/\tilde{S}_I}^\bullet))))} \\
&DR(S)^{[-1]}(\text{Hom}(Le(\tilde{S}_I)_* \text{An}_{\tilde{S}_I}^* \mathbb{D}_{\tilde{S}_I} L(i_{I*}j_I^*(F, W)), e(\tilde{S}_I)_* E_{usu}(\Omega_{/\tilde{S}_I}^\bullet))), u_{IJ}(-)) \\
&\xrightarrow{DR(S)^{[-1]}((\text{Hom}(-, e(\tilde{S}_I)_* E_{usu}(\text{Gr}(\Omega_{/S^{an}}))))^{-1})} \\
&DR(S)^{[-1]}(\text{Hom}(Le(\tilde{S}_I)_* \text{An}_{\tilde{S}_I}^* \mathbb{D}_{\tilde{S}_I} L(i_{I*}j_I^*(F, W)), e(\tilde{S}_I)_* \text{Gr}_{\tilde{S}_I^*}^{12} E_{usu}(\Omega_{/S^{an}}^{\bullet, \Gamma, pr, an}))), u_{IJ}(-)) \\
&\xrightarrow{DR(S)^{[-1]}(T(e, \text{hom})(-, -)^{-1})} \\
&DR(S)^{[-1]}(e(\tilde{S}_I)_* \text{Hom}(L \text{An}_{\tilde{S}_I}^* \mathbb{D}_{\tilde{S}_I} L(i_{I*}j_I^*(F, W)), \text{Gr}_{\tilde{S}_I^*}^{12} E_{usu}(\Omega_{/\tilde{S}_I^*}^{\bullet, \Gamma, pr, an}))), u_{IJ}(-)) \\
&\xrightarrow{DR(S)^{[-1]}(I(\text{Gr}_{\tilde{S}_I^*}^{12*}, \text{Gr}_{\tilde{S}_I^*}^{12})(-, -))} \\
&DR(S)^{[-1]}(e'(\tilde{S}_I)_* \text{Hom}(L \text{An}_{\tilde{S}_I}^* \text{Gr}_{\tilde{S}_I^*}^{12*} \mathbb{D}_{\tilde{S}_I} L(i_{I*}j_I^*(F, W)), E_{usu}(\Omega_{/\tilde{S}_I^*}^{\bullet, \Gamma, pr, an}))), u_{IJ}(-)) \\
&\xrightarrow{DR(S)^{[-1]}(\text{Hom}(\text{An}_{\tilde{S}_I}^* \text{Gr}_{\tilde{S}_I^*}^{12*} r_{(X^*, D^*)/\tilde{S}_I}(L(i_{I*}j_I^*(F, W))), E_{usu}(\Omega_{/\tilde{S}_I^*}^{\bullet, \Gamma, pr, an})))} \\
DR(S)^{[-1]}(e'(\tilde{S}_I)_* \text{Hom}(\text{An}_{\tilde{S}_I}^* \rho_{\tilde{S}_I^*} \text{Gr}_{\tilde{S}_I^*}^{12*} R_{(X^*, D^*)/\tilde{S}_I}(\rho_{\tilde{S}_I^*}^* L(i_{I*}j_I^*(F, W))), E_{usu}(\Omega_{/\tilde{S}_I^*}^{\bullet, \Gamma, pr, an}))), u_{IJ}(F)) \\
\rightleftharpoons DR(S)^{[-1]}(\mathcal{F}_{S, an}^{DR}(M, W)) \\
\xrightarrow{DR(S)^{[-1]}(T(\text{An}, \mathcal{F}_S^{FDR})(M, W)^{-1})} DR(S)^{[-1]}((\mathcal{F}_S^{DR}(M, W))^{an})
\end{aligned}$$

where $(F, W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $(M, W) = D(\mathbb{A}^1, et)(F, W)$

- $s(K) : K \rightarrow DR(S)^{[-1]}(\text{Hom}(\mathbb{D}_S LK, E(O_S)))$ is the isomorphism of theorem 23,
- the map $T(e, \text{hom})(-, -)$ is a filtered equivalence usu local by proposition 33 and proposition 121,
- $\mathcal{F}_S^{DR}(M) := o_{fil} \mathcal{F}_S^{FDR}(M) \in D_{\mathcal{D}0fil}(S)$, $\mathcal{F}_{S, an}^{DR}(M) := o_{fil} \mathcal{F}_{S, an}^{FDR}(M) \in D_{\mathcal{D}0fil}(S^{an})$,
- $T(\text{An}, \mathcal{F}_S^{FDR})(M) : (\mathcal{F}_S^{DR}(M))^{an} \xrightarrow{\sim} \mathcal{F}_{S, an}^{DR}(M)$ is an isomorphism by theorem 41.

We now give the functoriality with respect to the five operation using the De Rahm realization case and the Betti realization case :

Proposition 136. (i) Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume there exists a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p} S$, with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p the projection. Let $S = \cup_{i \in I} S_i$ an open cover and $i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, $\tilde{g}_I : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I = g|_{T_I} : T_I \rightarrow S_I$. Then, for $M \in DA_c(S)$, the following diagram

commutes :

$$\begin{array}{ccccc}
g^*(\alpha(M)) : g^* \text{Bti}_S^* M & \longrightarrow & DR(T)^{[-1]}(g_{Hdg}^{\text{mod}}(\mathcal{F}_{S,an}^{DR}(M))) & \longrightarrow & DR(T)^{[-1]}((g_{Hdg}^{\text{mod}} \mathcal{F}_S^{DR}(M))^{an}) \\
\downarrow T(g, \text{bti})(M) & & \downarrow DR(T)^{[-1]}(T(g, \mathcal{F}_{an}^{FDR}(M))) & & \downarrow DR(T)^{[-1]}((T(g, \mathcal{F}^{FDR}(M))^{an}) \\
\alpha(g^* M) : \text{Bti}_T^* g^* M & \longrightarrow & DR(T)^{[-1]}(\mathcal{F}_{T,an}^{DR}(g^* M)) & \longrightarrow & DR(T)^{[-1]}((\mathcal{F}_T^{DR}(g^* M))^{an})
\end{array}$$

see definition 100, definition 121, definition 142 and definition 152

(ii) Let $f : T \rightarrow S$ a morphism with $T, S \in \text{QPVar}(\mathbb{C})$. Then, for $M \in DA_c(T)$, the following diagram commutes :

$$\begin{array}{ccccc}
f_*(\alpha(M)) : Rf_* \text{Bti}_T^* M & \longrightarrow & DR(S)^{[-1]}(Rf_*^{Hdg} \mathcal{F}_{T,an}^{DR}(M)) & \longrightarrow & DR(S)^{[-1]}((Rf_*^{Hdg} \mathcal{F}_T^{DR}(M))^{an}) \\
\uparrow T_*(f, \text{bti})(M) & & \uparrow & & \uparrow DR(S)^{[-1]}((T_*(f, \mathcal{F}^{FDR}(M))^{an}) \\
\alpha(Rf_* M) : \text{Bti}_S^* Rf_* M & \longrightarrow & DR(S)^{[-1]}(\mathcal{F}_{S,an}^{DR}(Rf_* M)) & \longrightarrow & DR(S)^{[-1]}((\mathcal{F}_S^{DR}(Rf_* M))^{an})
\end{array}$$

see definition 100, definition 122, definition 143 and definition 153

(iii) Let $f : T \rightarrow S$ a morphism with $T, S \in \text{QPVar}(\mathbb{C})$. Then, for $M \in DA_c(T)$, the following diagram commutes :

$$\begin{array}{ccccc}
f_!(\alpha(M)) : Rf_! \text{Bti}_T^* M & \longrightarrow & DR(S)^{[-1]}(Rf_!^{Hdg} \mathcal{F}_{DR}^{T,an}(M)) & \longrightarrow & DR(S)^{[-1]}((Rf_!^{Hdg} \mathcal{F}_{DR}^T(M))^{an}) \\
\downarrow T_!(f, \text{bti})(M) & & \downarrow & & \downarrow DR(S)^{[-1]}((T_!(f, \mathcal{F}_{DR}(M))^{an}) \\
\alpha(Rf_! M) : \text{Bti}_S^* f_! M & \longrightarrow & DR(S)^{[-1]}((\mathcal{F}_{DR}^{S,an}(Rf_! M)) & \longrightarrow & DR(S)^{[-1]}((\mathcal{F}_{DR}^S(Rf_! M))^{an})
\end{array}$$

see definition 100, definition 122, definition 143 and definition 153

(iv) Let $f : T \rightarrow S$ a morphism with $T, S \in \text{QPVar}(\mathbb{C})$. Then, for $M \in DA_c(S)$, the following diagram commutes :

$$\begin{array}{ccccc}
f^!(\alpha(M)) : f^! \text{Bti}_S^* M & \longrightarrow & DR(T)^{[-1]}(f_{Hdg}^{\text{mod}}(\mathcal{F}_{S,an}^{DR}(M))) & \longrightarrow & DR(T)^{[-1]}((f_{Hdg}^{\text{mod}} \mathcal{F}_S^{DR}(M))^{an}) \\
\uparrow T^!(f, \text{bti})(M) & & \uparrow T^!(f, \mathcal{F}_{an}^{FDR}(M)) & & \uparrow DR^{[-1]}(T)((T^!(g, \mathcal{F}^{FDR}(M))^{an}) \\
\alpha(f^! M) : \text{Bti}_T^* f^! M & \longrightarrow & DR(T)^{[-1]}(\mathcal{F}_{T,an}^{DR}(f^! M)) & \longrightarrow & DR(T)^{[-1]}((\mathcal{F}_T^{DR}(f^! M))^{an})
\end{array}$$

see definition 100, definition 122, definition 143 and definition 153

(v) Let $S \in \text{Var}(\mathbb{C})$. Then, for $M, N \in DA_c(S)$, the following diagram commutes :

$$\begin{array}{ccc}
\text{Bti}_S^* M \otimes \text{Bti}_S^* N & \xrightarrow{\alpha(M) \otimes \alpha(N)} & DR(S)((\mathcal{F}_S^{DR}(M) \otimes_{\mathcal{O}_S} \mathcal{F}_S^{DR}(N))^{an}) \\
\downarrow T(\otimes, \text{bti})(M, N) & & \downarrow DR(S)((T(\otimes, \mathcal{F}^{DR})(M, N))^{an}) \\
\text{Bti}_S^*(M \otimes N) & \xrightarrow{(\alpha(M \otimes N))} & DR(S)((\mathcal{F}_{DR}^S(M \otimes N))^{an})
\end{array}$$

see definition 124 and definition 154.

Proof. (i): The commutativity of the right square is given by applying the functor $DR(T)^{[-1]}$ to the commutative diagram

$$\begin{array}{ccc}
g_{Hdg}^{\text{mod}}((\mathcal{F}_S^{FDR}(M))^{an}) = (g_{Hdg}^{\text{mod}}(\mathcal{F}_S^{FDR}(M))) \xrightarrow{g_{Hdg}^{\text{mod}}(T(An, \mathcal{F}_S^{FDR})(M))} g_{Hdg}^{\text{mod}}(\mathcal{F}_{S,an}^{FDR}(M)) \\
\downarrow T(g, \mathcal{F}^{FDR})(M)^{an} & & \downarrow T(g, \mathcal{F}_{an}^{FDR})(M) \\
(\mathcal{F}_T^{FDR}(g^* M))^{an} & \xrightarrow{T(An, \mathcal{F}_T^{FDR})(g^* M)} & \mathcal{F}_{T,an}^{FDR}(g^* M)
\end{array}$$

given in proposition 133(i0). On the other hand, the commutativity of the left square follows from the following commutative diagram :

$$\begin{array}{ccc}
DR(T)^{[-]}(\Gamma_{T_I}^{\vee, Hdg} \tilde{g}_I^{*mod} e'(\tilde{S}_I) * \mathcal{H}om(\text{An}_{\tilde{S}_I}^* \rho_{\tilde{S}_I} R_{(-,-)/-}^{CH}(\rho_{\tilde{S}_I}^* Li_{I*} j_I^* F)), & & \\
(\tilde{g}_I^* i_{I*} j_I^* e(S) * \text{An}_S^* LF, \tilde{g}_I^* u_{IJ}(-)) \longrightarrow & & E_{usu}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr, an}), \tilde{g}_I^{*mod}(u_{IJ}(F))) \\
\downarrow T(e, g)(F) & & \downarrow DR(T)^{[-]}(T(g, \mathcal{F}_{an}^{DR})(M)) \\
(i'_{I*} j_I^* e(T) * \text{An}_T^* g^* F, u_{IJ}(-)) \longrightarrow & & DR(T)^{[-]}(e'(Y \times \tilde{S}_I) * \mathcal{H}om(\text{An}_{Y \times \tilde{S}_I}^* \rho_{\tilde{S}_I} R_{(-,-)/-}^{CH}(\rho_{Y \times \tilde{S}_I}^* \tilde{g}_I^* Li_{I*} j_I^* F)), \\
& & E_{usu}(\Omega_{Y \times \tilde{S}_I}^{\bullet}), u_{IJ}^q(g^* F))
\end{array}$$

(ii): Follows from (i) by adjonction.

(iii): The closed embedding case is given by (ii) and the smooth projection case follows from (i) by adjonction.

(iv): Follows from (iii) by adjonction.

(v): Obvious

□

We can now state the following key proposition and the main theorem:

Proposition 137. *Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$, X quasi-projective. Consider a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y = \mathbb{P}^{N.o} \subset \mathbb{P}^N$ an open subset, l a closed embedding and p_S the projection. Let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Recall that $S_I := \cap_{i \in I} S_i$, $X_I = f^{-1}(S_I)$, and $\tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. Then, using proposition 136(iii), the maps of definition 122 and definition 153 gives an isomorphism in $D_{\mathcal{D}(1,0), \text{fil}}(S/(\tilde{S}_I)) \times_I D(S^{an})$*

$$\begin{aligned}
& (T_!(f, \mathcal{F}^{FDR})(\mathbb{Z}(X/X)), T_!(f, \text{Bti})(\mathbb{Z}(X/X))) : \\
& \mathcal{F}_S^{Hdg}(M^{BM}(X/S)) := (\mathcal{F}_S^{FDR}(Rf_! \mathbb{Z}(X/X)), \text{Bti}_S^* Rf_! \mathbb{Z}(X/X), \alpha(Rf_! \mathbb{Z}(X/X))) \\
& \xrightarrow{\sim} (Rf_{Hdg!}(\Gamma_{X_I}^{\vee, Hdg}(O_{Y \times \tilde{S}_I}, F_b), x_{IJ}(X/S)), Rf_! \mathbb{Z}_{X^{an}}, f_!(\alpha(X/(Y \times \tilde{S}_I)))) =: Rf_{Hdg} \mathbb{Z}_X^{Hdg}
\end{aligned}$$

Proof. Follows from proposition 136(iii), theorem 33(i) and theorem 43(i). □

The main theorem of this article is the following :

Theorem 45. (i) *For $S \in \text{Var}(\mathbb{C})$, we have $\mathcal{F}_S^{Hdg}(\text{DA}_c(S)) \subset D(\text{MHM}(S))$.*

(ii) *The Hodge realization functor $\mathcal{F}_{Hdg}(-)$ define a morphism of 2-functor on $\text{Var}(\mathbb{C})$*

$$\mathcal{F}_-^{Hdg} : \text{Var}(\mathbb{C}) \rightarrow (\text{DA}_c(-) \rightarrow D(\text{MHM}(-)))$$

whose restriction to $\text{QPVar}(\mathbb{C})$ is an homotopic 2-functor in sense of Ayoub. More precisely,

(ii0) *for $g : T \rightarrow S$ a morphism, with $T, S \in \text{QPVar}(\mathbb{C})$, and $M \in \text{DA}_c(S)$, the the maps of definition 121 and of definition 152 induce an isomorphism in $D(\text{MHM}(T))$*

$$\begin{aligned}
& T(g, \mathcal{F}^{Hdg})(M) := (T(g, \mathcal{F}^{FDR})(M), T(g, \text{bti})(M)) : \\
& g^* \mathcal{F}_S^{Hdg}(M) := (g_{Hdg}^{*mod} \mathcal{F}_S^{FDR}(M), g^* \text{Bti}_S(M), g^*(\alpha(M))) \\
& \xrightarrow{\sim} (\mathcal{F}_T^{FDR}(g^* M), \text{Bti}_T^*(g^* M), \alpha(g^* M)) =: \mathcal{F}_T^{Hdg}(g^* M),
\end{aligned}$$

(ii1) *for $f : T \rightarrow S$ a morphism, with $T, S \in \text{QPVar}(\mathbb{C})$, and $M \in \text{DA}_c(T)$, the maps of definition 122 and of definition 153 induce an isomorphism in $D(\text{MHM}(S))$*

$$\begin{aligned}
& T_*(f, \mathcal{F}^{Hdg})(M) := (T_*(f, \mathcal{F}^{FDR})(M), T_*(f, \text{bti})(M)) : \\
& Rf_{Hdg*} \mathcal{F}_T^{Hdg}(M) := (Rf_*^{Hdg} \mathcal{F}_T^{FDR}(M), Rf_* \text{Bti}_S(M), f_*(\alpha(M))) \\
& \xrightarrow{\sim} (\mathcal{F}_S^{FDR}(Rf_* M), \text{Bti}_S^*(Rf_* M), \alpha(Rf_* M)) =: \mathcal{F}_S^{Hdg}(Rf_* M),
\end{aligned}$$

(ii2) for $f : T \rightarrow S$ a morphism, with $T, S \in \text{QPVar}(\mathbb{C})$, and $M \in \text{DA}_c(T)$, the maps of definition 122 and of definition 153 induce an isomorphism in $D(\text{MHM}(S))$

$$\begin{aligned} T_!(f, \mathcal{F}^{\text{Hdg}})(M) &:= (T_!(f, \mathcal{F}^{\text{FDR}})(M), T_!(f, \text{bti})(M)) : \\ Rf_{! \text{Hdg}} \mathcal{F}_T^{\text{Hdg}}(M) &:= (Rf_{! \text{Hdg}} \mathcal{F}_T^{\text{FDR}}(M), Rf_{! \text{Hdg}} \text{Bti}_S^*(M), f_!(\alpha(M))) \\ &\xrightarrow{\sim} (\mathcal{F}_S^{\text{FDR}}(Rf_!M), \text{Bti}_S^*(Rf_!M), \alpha(f_!M)) =: \mathcal{F}_T^{\text{Hdg}}(f_!M), \end{aligned}$$

(ii3) for $f : T \rightarrow S$ a morphism, with $T, S \in \text{QPVar}(\mathbb{C})$, and $M \in \text{DA}_c(S)$, the maps of definition 122 and of definition 153 induce an isomorphism in $D(\text{MHM}(T))$

$$\begin{aligned} T^!(f, \mathcal{F}^{\text{Hdg}})(M) &:= (T^!(f, \mathcal{F}^{\text{FDR}})(M), T^!(f, \text{bti})(M)) : \\ f^{*\text{Hdg}} \mathcal{F}_S^{\text{Hdg}}(M) &:= (f^{*\text{Hdg}} \mathcal{F}_S^{\text{FDR}}(M), f^! \text{Bti}_S(M), f^!(\alpha(M))) \\ &\xrightarrow{\sim} (\mathcal{F}_T^{\text{FDR}}(f^!M), \text{Bti}_T^*(f^!M), \alpha(f^!M)) =: \mathcal{F}_T^{\text{Hdg}}(f^!M), \end{aligned}$$

(ii4) for $S \in \text{Var}(\mathbb{C})$, and $M, N \in \text{DA}_c(S)$, the maps of definition 124 and of definition 154 induce an isomorphism in $D(\text{MHM}(S))$

$$\begin{aligned} T(\otimes, \mathcal{F}^{\text{Hdg}})(M, N) &:= (T(\otimes, \mathcal{F}_S^{\text{FDR}})(M, N), T(\otimes, \text{bti})(M, N)) : \\ (\mathcal{F}_S^{\text{FDR}}(M) \otimes_{\mathcal{O}_S}^L \mathcal{F}_S^{\text{FDR}}(N), \text{Bti}_S(M) \otimes \text{Bti}_S(N), \alpha(M) \otimes \alpha(N)) \\ &\xrightarrow{\sim} \mathcal{F}_S^{\text{Hdg}}(M \otimes N) := (\mathcal{F}_S^{\text{FDR}}(M \otimes N), \text{Bti}_S(M \otimes N), \alpha(M \otimes N)). \end{aligned}$$

(iii) For $S \in \text{Var}(\mathbb{C})$, the following diagram commutes :

$$\begin{array}{ccc} \text{Var}(\mathbb{C})/S & \xrightarrow{\text{MH}(S)} & D(\text{MHM}(S)) \\ \downarrow M(S) & & \downarrow \iota^S \\ \text{DA}(S) & \xrightarrow{\mathcal{F}_S^{\text{Hdg}}} & D_{\mathcal{D} \text{fil}, \infty}(S/\tilde{S}_I) \times_I D_{\text{fil}}(S^{\text{an}}) \end{array}$$

Proof. (i): Let $M \in \text{DA}_c(S)$. There exist a generalized distinguish triangle in $\text{DA}(S)$

$$M \rightarrow M(X_0/S)[d_0] \rightarrow \cdots \rightarrow M(X_m/S)[d_m],$$

with $f_n : X_n \rightarrow S$ morphisms and $X_n \in \text{QPVar}(\mathbb{C})$. This gives the following generalized distinguish triangle in $D_{\mathcal{D} \text{fil}}(S) \times_I D(S^{\text{an}})$

$$\mathcal{F}_S^{\text{Hdg}}(M) \rightarrow \mathcal{F}_S^{\text{Hdg}}(M(X_0/S))[d_0] \rightarrow \cdots \rightarrow \mathcal{F}_S^{\text{Hdg}}(M(X_m/S))[d_m],$$

On the other hand, by proposition 137, we have

$$\mathcal{F}_S^{\text{Hdg}}(M(X_n/S)) \xrightarrow{\sim} Rf_{* \text{Hdg}} \mathbb{Z}_X^{\text{Hdg}} \in D(\text{MHM}(S))$$

(ii0): Follows from theorem 32, proposition 136(i) and theorem 42.

(ii1): Follows from theorem 33(ii), proposition 136(ii), and theorem 43(ii).

(ii2): Follows from theorem 33(i), proposition 136(iii), and theorem 43(i).

(ii3): Follows from theorem 33(iii), proposition 136(iv), and theorem 43(iii).

(ii4): Follows from theorem 34, proposition 136(v) and theorem 43(iv).

(iii): By (ii), for $g : X'/S \rightarrow X/S$ a morphism, with $X', X, S \in \text{Var}(\mathbb{C})$ and $X/S = f : X \rightarrow S$, $X'/S = f' : X' \rightarrow S$, we have by adjonction the following commutative diagram

$$\begin{array}{ccc} \mathcal{F}_S^{\text{Hdg}}(M(X'/S) = f'_! f'^! \mathbb{Z}_S) = f_! g^! g^! f'^! \mathbb{Z}_S & \xrightarrow{\mathcal{F}_S^{\text{Hdg}}(M(X'/S)(g)=f_! \text{ad}(g_!, g^!)(f'^! \mathbb{Z}_S))} & \mathcal{F}_S^{\text{Hdg}}(M(X/S) = f_! f^! \mathbb{Z}_S) . \\ \sim \downarrow & & \downarrow \sim \\ \text{MH}(X'/S) = f'_! f'^! \mathbb{Z}_S^{\text{Hdg}} = f_! g^! g^! f'^! \mathbb{Z}_S^{\text{Hdg}} & \xrightarrow{f_! \text{ad}(g_!, g^!)(f'^! \mathbb{Z}_S^{\text{Hdg}})} & \text{MH}(X/S) = f_! f^! \mathbb{Z}_S^{\text{Hdg}} \end{array}$$

This proves (iii). \square

The theorem 45 gives immediately the following :

Corollary 10. *Let $f : X \rightarrow S$, $f' : X' \rightarrow S$ morphisms, with $X, X', S \in \text{Var}(\mathbb{C})$. Let $\bar{S} \in \text{PVar}(\mathbb{C})$ a compactification of S . Let $\bar{X}, \bar{X}' \in \text{PVar}(\mathbb{C})$ compactification of X and X' respectively, such that f (resp. f') extend to a morphism $\bar{f} : \bar{X} \rightarrow \bar{S}$, resp. $\bar{f}' : \bar{X}' \rightarrow \bar{S}$. Denote $D = \bar{X} \setminus X$ and $D' = \bar{X}' \setminus X'$ and $E = (D \times_{\bar{S}} \bar{X}') \cup (\bar{X} \times_{\bar{S}} D')$. We have the following commutative digram*

$$\begin{array}{ccc}
RHom^\bullet(M((\bar{X}, D)/\bar{S}), M((\bar{X}', D')/\bar{S})^\vee(d)) & \xrightarrow{\mathcal{F}_S^{FDR}} & RHom^\bullet(f_! \mathbb{Z}_X^{Hdg}, f'_! \mathbb{Z}_{X'}^{Hdg}) \\
\downarrow RI() & & \downarrow RI() \\
RHom^\bullet(M(\text{pt}), M(\bar{X} \times_{\bar{S}} \bar{X}', E)(d)) & \xrightarrow{\mathcal{F}_{FDR}^{\text{pt}}} & RHom^\bullet(\mathbb{Z}_{\text{pt}}^{Hdg}, a_{X \times_S X'} \mathbb{Z}_{X \times_S X'}^{Hdg}) \\
\downarrow l & & \downarrow l \\
\mathcal{Z}^d(\bar{X} \times_{\bar{S}} \bar{X}', E, \bullet) & \xrightarrow{\mathcal{R}_{\bar{X} \times_{\bar{S}} \bar{X}'}} & C_{2d_X - 2d + \bullet}^D(\bar{X} \times_{\bar{S}} \bar{X}', E, Z(d))
\end{array}$$

Proof. The upper square of this diagram follows from theorem 45(ii) and the following isomorphism :

- $\text{ad}(j_!, j^*)(\mathbb{Z}_{\bar{X}}) : M(X/\bar{S}) \xrightarrow{\sim} \text{Cone}(M(\bar{X}/\bar{S}) \xrightarrow{\text{ad}(i^*, i_*)(\mathbb{Z}_{\bar{X}})} M(D/\bar{S})[-1]) =: M((\bar{X}, D)/\bar{S})$
- $\text{ad}(j'_!, j'^*)(\mathbb{Z}_{\bar{X}'}) : M(X'/\bar{S}) \xrightarrow{\sim} \text{Cone}(M(\bar{X}'/\bar{S}) \xrightarrow{\text{ad}(i'^*, i'_*)(\mathbb{Z}_{\bar{X}'})} M(D'/\bar{S})[-1]) =: M((\bar{X}', D')/\bar{S})$

where $i : D \hookrightarrow \bar{X}$, $i' : D' \hookrightarrow \bar{X}'$ denote the closed embeddings and $j : X \hookrightarrow \bar{X}$, $j' : X' \hookrightarrow \bar{X}'$ the open embeddings. On the other side, the lower square follows from the absolute case. □

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